The L_p Minkowski Problem for the Electrostatic $\mathfrak{p}$-capacity for $\mathfrak{p} \geqslant n$

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ABSTRACT. Sufficient conditions are given for the existence of solutions to the discrete L_p Minkowski problem for $\mathfrak{p}$-capacity when $0 < p < 1$ and $\mathfrak{p} \geqslant n$.

1. INTRODUCTION

The setting for this article is the Euclidean n-space $\mathbb{R}^n$. A *convex body* in $\mathbb{R}^n$ is a compact convex set with nonempty interior. A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$, provided it has positive *volume* (i.e., n-dimensional Lebesgue measure). The Brunn-Minkowski theory of convex bodies, also called the mixed volume theory, which was developed by Minkowski, Aleksandrov, Fenchel, and many others, centers around the study of geometric functionals of convex bodies and the differentials of these functionals. Usually, the differentials of these functionals produce *new* geometric measures. This theory depends heavily on analytic tools such as the cosine transform on the unit sphere $\mathbb{S}^{n-1}$ and Monge-Ampère type equations.

A Minkowski problem is a *characterization* problem for a geometric measure generated by convex bodies: it asks for necessary and sufficient conditions in order for a given measure to arise as the measure generated by a convex body. The solution to a Minkowski problem, in general, amounts to solving a fully nonlinear partial differential equation. The study of Minkowski problems has a long history and strong influence on both the Brunn-Minkowski theory and fully nonlinear partial differential equations. (For details, see, e.g., [39, Chapter 8].)

The L_p Minkowski problem for volume developed in the 1990s, and through intensive investigation significantly generalized the classical Minkowski problem.

1.1. ***L_p surface area measures and the L_p Minkowski problem for volume.*** The L_p Brunn-Minkowski theory (see, e.g., [39, Sections 9.1 and 9.2]) is an

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extension of the classical Brunn-Minkowski theory, in which the L_p surface area measure introduced by Lutwak [33] is one of the most fundamental notions.

Let K be a convex body in $\mathbb{R}^n$ with the origin in its interior and $p \in \mathbb{R}$. Its L_p *surface area measure* $S_p(K,\cdot)$ is a finite Borel measure on $\mathbb{S}^{n-1}$, defined for Borel $\omega \subseteq \mathbb{S}^{n-1}$ by

$$S_p(K,\omega) = \int_{x \in g_K^{-1}(\omega)} (x \cdot g_K(x))^{1-p} \, d\mathcal{H}^{n-1}(x),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure; also, $g_K : \partial' K \to \mathbb{S}^{n-1}$ is the Gauss map defined on the set $\partial' K$ of those points of ∂K that have a unique outer unit normal vector. Alternatively, $S_p(K,\cdot)$ can be defined by

$$S_p(K,\omega) = \int_\omega h_K^{1-p}(u) \, dS(K,u),$$

where $h_K : \mathbb{R}^n \to \mathbb{R}$, $h_K(x) = \max\{x \cdot y : y \in K\}$ is the *support function* of K; $dS(K,\cdot)$ is the classical *surface area measure* of K.

Tracing the source, the L_p surface area measure resulted from the differential of the volume functional of L_p combinations of convex bodies.

In 1962, Firey [24] introduced the notion of L_p *sum* of convex bodies. Let K, L be convex bodies with the origin in their interiors and $1 \leqslant p < \infty$. Their L_p sum $K +_p L$ is the compact convex set that has support function $h_{K+_pL} = (h_K^p + h_L^p)^{1/p}$. For $t > 0$, the L_p *scalar multiplication* $t \cdot_p K$ is the set $t^{1/p}K$. Note that $K +_1 L = K + L = \{x + y : x \in K,\ y \in L\}$ is the *Minkowski sum* of K and L.

Using the L_p combination, Lutwak [33] established the following L_p variational formula:

$$\frac{dV(K +_p t \cdot_p L)}{dt}\bigg|_{t=0^+} = \frac{1}{p} \int_{\mathbb{S}^{n-1}} h_L^p(u) \, dS_p(K,u),$$

where V is the n-dimensional volume. When $p = 1$, it reduces to the celebrated *Aleksandrov variational formula*

$$\frac{dV(K + tL)}{dt}\bigg|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(u) \, dS(K,u). \tag{1.1}$$

The integral in (1.1), divided by the factor n, is called the *first mixed volume* $V_1(K,L)$ of K and L. That is, $V_1(K,L) = (1/n)\int_{\mathbb{S}^{n-1}} h_L(u) \, dS(K,u)$, which is a generalization of the well-known volume formula

$$V(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u) \, dS(K,u). \tag{1.2}$$

THE L_p MINKOWSKI PROBLEM FOR VOLUME *Suppose that μ is a finite Borel measure on $\mathbb{S}^{n-1}$ and $p \in \mathbb{R}$. What are the necessary and sufficient conditions on μ such that μ is the L_p surface area measure $S_p(K, \cdot)$ of a convex body K in $\mathbb{R}^n$?*

The L_1 Minkowski problem is precisely the classical Minkowski problem. More than a century ago, Minkowski himself [36] solved this problem when the given measure is either discrete or has a continuous density. Aleksandrov [3, 4] and Fenchel-Jessen [23] independently solved the problem for arbitrary measures. The L_0 Minkowski problem is also called the logarithmic Minkowski problem. In [10], the authors posed the subspace concentration condition and completely solved the even logarithmic Minkowski problem. Additional references regarding the logarithmic Minkowski problem can be found in, for instance, [7–9, 11, 18, 40–42, 45].

For $0 < p < 1$, the L_p Minkowski problem was essentially solved by Chen, Li, and Zhu [19] (see also [12, 46] for more details). It is worth mentioning that in the very recent work [6], the authors discussed the case $-n < p < 1$ for an absolutely continuous measure and provided an almost-optimal sufficient condition for the case $0 < p < 1$.

Since, for strictly convex bodies with smooth boundaries the density of the surface area measure with respect to the Lebesgue measure is just the reciprocal of the Gauss curvature of closed convex hypersurface, analytically, the classical Minkowski problem is equivalent to solving a Monge-Ampère equation. Establishing the regularity of the solution is difficult and has led to a long series of highly influential works (see, e.g., Lewy [32], Nirenberg [37], Cheng-Yau [20], Pogorelov [38], Caffarelli [13, 14]).

By now, the L_p Minkowski problem for volume has been investigated and achieved great developments (see, e.g., [5, 10, 17, 21, 27, 28, 33, 35, 40, 45, 46]). Its solutions have been applied to establish sharp affine isoperimetric inequalities, such as the affine Moser-Trudinger and the affine Morrey-Sobolev inequalities, the affine L_p Sobolev-Zhang inequality, and so on (see, e.g., [34, 44] for more details).

1.2. L_p $\mathfrak{p}$-capacitary measures; the L_p Minkowski problem for $\mathfrak{p}$-capacity. Without a doubt, the Minkowski problem for electrostatic $\mathfrak{p}$-capacity is an extremely important variant among Minkowski problems. Recall that for $1 < \mathfrak{p} < n$, the electrostatic $\mathfrak{p}$-capacity of a compact set K in $\mathbb{R}^n$ is defined by

$$C_{\mathfrak{p}}(K) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla u|^{\mathfrak{p}} \, \mathrm{d}x : u \in C_c^{\infty}(\mathbb{R}^n) \text{ and } u \geqslant \chi_K \right\}, \tag{1.3}$$

where $C_c^{\infty}(\mathbb{R}^n)$ denotes the set of smooth functions with compact supports, and χ_K is the characteristic function of K. The quantity $C_2(K)$ is the classical electrostatic (or Newtonian) capacity of K.

For convex bodies K and L, via the variation of capacity functional $C_2(K)$, there is the classical Hadamard variational formula

$$\frac{\mathrm{d}C_2(K + tL)}{\mathrm{d}t}\bigg|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(u) \, \mathrm{d}\mu_2(K, u) \tag{1.4}$$

and its special case, the Poincaré capacity formula

$$C_2(K) = \frac{1}{n-2}\int_{\mathbb{S}^{n-1}} h_K(u)\,\mathrm{d}\mu_2(K,u). \tag{1.5}$$

Here, $\mu_2(K,\cdot)$ is called the *electrostatic capacitary measure* of K.

In his celebrated article [29], Jerison pointed out the resemblance between the Poincaré formula (1.5) and the volume formula (1.2) and also a resemblance between their variational formulas (1.4) and (1.1). Thus, he initiated a study of the Minkowski problem for electrostatic capacity: *Given a finite Borel measure μ on $\mathbb{S}^{n-1}$, what are the necessary and sufficient conditions on μ such that μ is the electrostatic capacitary measure $\mu_2(K,\cdot)$ of a convex body K in $\mathbb{R}^n$?*

Jerison [29] solved, in full generality, the above Minkowski problem. He also proved the necessary and sufficient conditions for existence of a solution, which are unexpectedly *identical* to the corresponding conditions in the classical Minkowski problem. Uniqueness was settled by Caffarelli, Jerison, and Lieb [16], while the regularity part of the proof depends on the ideas of Caffarelli [15] for the regularity of solutions to the Monge-Ampère equation.

Jerison's work inspired much subsequent research on this topic. In [22], the authors extended Jerison's work to the electrostatic $\mathfrak{p}$-capacity, $1 < \mathfrak{p} < n$, and established the Hadamard $\mathfrak{p}$-capacitary variational formula

$$\frac{\mathrm{d}C_{\mathfrak{p}}(K+tL)}{\mathrm{d}t}\bigg|_{t=0^+} = (\mathfrak{p}-1)\int_{\mathbb{S}^{n-1}} h_L(u)\,\mathrm{d}\mu_{\mathfrak{p}}(K,u),$$

and therefore the Poincaré $\mathfrak{p}$-capacity formula

$$C_{\mathfrak{p}}(K) = \frac{\mathfrak{p}-1}{n-\mathfrak{p}}\int_{\mathbb{S}^{n-1}} h_K(u)\,\mathrm{d}\mu_{\mathfrak{p}}(K,u).$$

Here, the new measure $\mu_{\mathfrak{p}}(K,\cdot)$ is called the *electrostatic $\mathfrak{p}$-capacitary measure* of K. Naturally, the Minkowski problem for $\mathfrak{p}$-capacity was posed [22]: *Given a finite Borel measure μ on $\mathbb{S}^{n-1}$, what are the necessary and sufficient conditions on μ such that μ is the $\mathfrak{p}$-capacitary measure $\mu_{\mathfrak{p}}(K,\cdot)$ of a convex body K in $\mathbb{R}^n$?*

In [22], the authors proved the uniqueness of the solution when $1 < \mathfrak{p} < n$, and existence and regularity when $1 < \mathfrak{p} < 2$. Very recently, the existence for $2 < \mathfrak{p} < n$ was solved by M. Akman, J. Gong, J. Hineman, J. Lewis, and A. Vogel [1].

Inspired by the developed L_p Minkowski problem for volume, D. Zou and G. Xiong [47] initiated research on the following *L_p Minkowski problem for the $\mathfrak{p}$-capacitary measure*.

Let $p \in \mathbb{R}$ and $1 < \mathfrak{p} < n$. For a convex body K in $\mathbb{R}^n$ with the origin in its interior, its *L_p $\mathfrak{p}$-capacitary measure* $\mu_{p,\mathfrak{p}}(K,\cdot)$ is a finite Borel measure on $\mathbb{S}^{n-1}$, defined for Borel $\omega \subseteq \mathbb{S}^{n-1}$ by

$$\mu_{p,\mathfrak{p}}(K,\omega) = \int_{\omega} h_K^{1-p}(u)\,\mathrm{d}\mu_{\mathfrak{p}}(K,u).$$

Much as with the L_p surface area measure $S_p(K,\cdot)$, $\mu_{p,\mathfrak{p}}(K,\cdot)$ also resulted from the variation of the $\mathfrak{p}$-capacity functional of the L_p sum of convex bodies. Specifically, if K, L are convex bodies in $\mathbb{R}^n$ with the origin in their interiors, then for $1 \leqslant p < \infty$,

$$\frac{\mathrm{d}C_{\mathfrak{p}}(K +_p t \cdot_p L)}{\mathrm{d}t}\bigg|_{t=0^+} = \frac{\mathfrak{p}-1}{p}\int_{\mathbb{S}^{n-1}} h_L^p(u)\,\mathrm{d}\mu_{p,\mathfrak{p}}(K,u).$$

THE L_p MINKOWSKI PROBLEM FOR $\mathfrak{p}$-CAPACITY *Suppose μ is a finite Borel measure on $\mathbb{S}^{n-1}$, $p \in \mathbb{R}$, and $1 < \mathfrak{p} < n$. What are the necessary and sufficient conditions on μ such that μ is the L_p $\mathfrak{p}$-capacitary measure $\mu_{p,\mathfrak{p}}(K,\cdot)$ of a convex body K in $\mathbb{R}^n$?*

In [47], Zou and Xiong *completely* solved the L_p Minkowski problem for $\mathfrak{p}$-capacity for $p > 1$ and $1 < \mathfrak{p} < n$. It is striking that the conditions for the existence and uniqueness of the solution are also unexpectedly *identical* to the corresponding conditions in the L_p Minkowski problem for *volume* for $p > 1$. Very recently, G. Xiong, J. Xiong and L. Xu [43] solved the discrete measure case for $0 < p < 1$ and $1 < \mathfrak{p} < 2$.

In this article, we aim to investigate the L_p Minkowski problem for $\mathfrak{p}$-capacity for $0 < p < 1$ and $\mathfrak{p} \geqslant n$. The first and foremost thing is to extend the index $\mathfrak{p}$ involved in the $\mathfrak{p}$-capacity $C_{\mathfrak{p}}(K)$ and the $\mathfrak{p}$-capacitary measure $\mu_{p,\mathfrak{p}}(K,\cdot)$ of convex body K ro $\mathfrak{p} \geqslant n$. Luckily, this difficulty is smoothed by M. Akman, J. Lewis, O. Saari, and A. Vogel [2].

Recall that beyond using the infimum (1.3), the $\mathfrak{p}$-capacity $C_{\mathfrak{p}}(K)$, $1 < \mathfrak{p} < n$, can be equivalently defined via the solutions to the $\mathfrak{p}$-Laplace equation. For a convex body K in $\mathbb{R}^n$, let $U = U_K$ be the unique solution to the the boundary value problem of $\mathfrak{p}$-Laplace equation

$$\begin{cases} \nabla\cdot(|\nabla u|^{\mathfrak{p}-2}\nabla u) = 0 & \text{in } \mathbb{R}^n \setminus K,\\ u = 1 & \text{on } \partial K,\\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

Then,

$$C_{\mathfrak{p}}(K)^{1/(\mathfrak{p}-1)} = -\lim_{|x|\to\infty}\frac{U(x)}{F(x)},$$

where

$$F(x) = (n\omega_n)^{1/(1-\mathfrak{p})}\left(\frac{\mathfrak{p}-1}{\mathfrak{p}-n}\right)|x|^{(\mathfrak{p}-n)/(\mathfrak{p}-1)}$$

is the fundamental solution to the $\mathfrak{p}$-Laplace equation.

Following this clue, the authors [2] proved that for a convex body K in $\mathbb{R}^n$, there exists a unique solution $U = U_K$ to the boundary value problem of $\mathfrak{p}$-Laplace

equation for $\mathfrak{p} \geqslant n$:

$$\begin{cases} \nabla \cdot (|\nabla u|^{\mathfrak{p}-2}\nabla u) = 0 & \text{in } \mathbb{R}^n \setminus K, \\ u = 0 & \text{on } \partial K, \\ u(x) = F(x) + a + o(1) & \text{as } |x| \to \infty, \end{cases}$$

where $a \in \mathbb{R}$ is uniquely determined by K, and

$$F(x) = \begin{cases} (n\omega_n)^{1/(1-\mathfrak{p})} \left(\dfrac{\mathfrak{p}-1}{\mathfrak{p}-n}\right) |x|^{(\mathfrak{p}-n)/(\mathfrak{p}-1)} & \text{when } \mathfrak{p} > n, \\ (n\omega_n)^{1/(1-n)} \log|x| + 1 & \text{when } \mathfrak{p} = n \end{cases}$$

is the fundamental solution. Then, they defined the $\mathfrak{p}$-*capacity* $C_{\mathfrak{p}}(K)$ of K by

$$C_{\mathfrak{p}}(K) = \begin{cases} (-a)^{\mathfrak{p}-1} & \text{when } \mathfrak{p} > n, \\ \exp\left(-\dfrac{a}{(n\omega_n)^{1/(1-n)}}\right) & \text{when } \mathfrak{p} = n. \end{cases}$$

They [2] also established the following Hadamard variational formula: For $\mathfrak{p} > n$,

$$\left.\frac{\mathrm{d}C_{\mathfrak{p}}(K+tL)}{\mathrm{d}t}\right|_{t=0^+} = (\mathfrak{p}-1)C_{\mathfrak{p}}(K)^{(\mathfrak{p}-2)/(\mathfrak{p}-1)} \int_{\mathbb{S}^{n-1}} h_L(u)\,\mathrm{d}\mu_{\mathfrak{p}}(K,u);$$

for $\mathfrak{p} = n$,

$$\left.\frac{\mathrm{d}C_n(K+tL)}{\mathrm{d}t}\right|_{t=0^+} = (n\omega_n)^{1/(n-1)} C_n(K) \int_{\mathbb{S}^{n-1}} h_L(u)\,\mathrm{d}\mu_n(K,u).$$

Henceforth, for $\mathfrak{p} \geqslant n$, the $\mathfrak{p}$-*capacitary measure* $\mu_{\mathfrak{p}}(K,\cdot)$ of K emerged. Naturally, Akman, Lewis, Saari, and Vogel [2] posed the Minkowski problem for $\mathfrak{p}$-capacity for $\mathfrak{p} \geqslant n$, and solved the existence and uniqueness.

In 2020, Zou and Xiong [47] developed the L_p Minkowski problem for $\mathfrak{p}$-capacity for $1 < \mathfrak{p} < n$. We now extend the $\mathfrak{p}$-index of the L_p $\mathfrak{p}$-capacitary measure $\mu_{p,\mathfrak{p}}(K,\cdot) = h_K^{1-p}\mu_{\mathfrak{p}}(K,\cdot)$ to $\mathfrak{p} \geqslant n$, and solve its associated L_p Minkowski problem.

Theorem 1.1. *Suppose μ is a finite discrete Borel measure on $\mathbb{S}^{n-1}$, $0 < p < 1$, and $\mathfrak{p} > n$. If the support set of μ is in general position, then there exists a polytope P containing the origin in its interior, such that $\mu_{p,\mathfrak{p}}(P,\cdot) = c\mu$, where*

$$c = \begin{cases} 1 & \text{if } p \neq \dfrac{\mathfrak{p}-n}{\mathfrak{p}-1}, \\ C_{\mathfrak{p}}(P)^{1/(\mathfrak{p}-1)} & \text{if } p = \dfrac{\mathfrak{p}-n}{\mathfrak{p}-1}. \end{cases}$$

Theorem 1.2. *Suppose that μ is a finite discrete Borel measure on $\mathbb{S}^{n-1}$ and $0 < p < 1$. If the support set of μ is in general position, then there exists a polytope P containing the origin in its interior, such that $\mu_{p,n}(P, \cdot) = \mu$.*

Recall that a finite set Ω of unit vectors in $\mathbb{R}^n$ is said to be in *general position*, if Ω is not contained in a closed hemisphere of $\mathbb{S}^{n-1}$ and any n elements of Ω are linearly independent. Károlyi and Lovász [30] first studied the class of polytopes whose facet normals are in general position. In practice, this kind of polytopes are very important, since any convex body can be approximated by polytopes whose facet normals are in general position.

It is worth mentioning that unlike the technical condition posed in [43] that the given measure does not have a pair of antipodal point masses, we solve the existence of the solutions to the discrete L_p capacitary Minkowski problem for $0 < p < 1$ and $\mathfrak{p} \geqslant n$ under the more geometric condition that the support set of the given measure is in general position.

This article is organized as follows. In Section 2, we collect some basic facts on convex bodies and the $\mathfrak{p}$-capacity for $\mathfrak{p} \geqslant n$. In Section 3, we establish the Hadamard variational formula for $\mathfrak{p}$-capacity of Wulff shapes. In Section 4, we study an extremal problem under translation transforms. After clarifying the relationship between two dual extremal problems and our concerned Minkowski problem in Section 5, we present the proof of the main results in Section 6.

2. Preliminaries

2.1. Basics of convex bodies. For quick reference, we collect here some basic facts on convex bodies. Good references are the books by Gardner [25], Gruber [26], and Schneider [39].

Write $x \cdot y$ for the standard inner product of $x, y \in \mathbb{R}^n$. Let B be the standard unit ball of $\mathbb{R}^n$. Denote by $\mathcal{K}^n$ the set of convex bodies in $\mathbb{R}^n$, and by $\mathcal{K}^n_o$ the set of convex bodies with the origin o in their interiors.

Note that $\mathcal{K}^n$ is often equipped with the *Hausdorff metric* δ_H, defined for compact convex sets K, L by $\delta_H(K, L) = \max\{|h_K(u) - h_L(u)| : u \in \mathbb{S}^{n-1}\}$.

Write $\operatorname{int} K$ and ∂K for the interior and boundary of a set K, respectively.

For $u \in \mathbb{S}^{n-1}$, the *support hyperplane* $H(K, u)$ of $K \in \mathcal{K}^n$ is defined by

$$H(K, u) = \{x \in \mathbb{R}^n : x \cdot u = h(K, u)\}.$$

The *half-space* $H^-(K, u)$ in the direction u is defined by

$$H^-(K, u) = \{x \in \mathbb{R}^n : x \cdot u \leqslant h(K, u)\}.$$

The *support set* $F(K, u)$ of $K \in \mathcal{K}^n$ in the direction u is defined by $F(K, u) = K \cap H(K, u)$.

Suppose the unit vectors $u_1, \ldots, u_N$, $N \geqslant n + 1$, are not concentrated on any closed hemisphere of $\mathbb{S}^{n-1}$. Let $P(u_1, \ldots, u_N)$ be the set with $P \in P(u_1, \ldots, u_N)$,

if the polytope

$$P = \bigcap_{k=1}^{N} H^-(P, u_k) = \bigcap_{k=1}^{N} \{x \in \mathbb{R}^n : x \cdot u_k \leqslant h_P(u_k)\}.$$

Obviously, for $P \in P(u_1, \ldots, u_N)$, P has at most N facets (i.e., $(n-1)$-dimensional faces), and the set of outer normal unit vectors of P is a subset of $\{u_1, \ldots, u_N\}$.

For a set $A \subseteq \mathbb{R}^n$, the set of all linear combinations and positive combinations of any finitely many elements of A is called the *linear hull* and *positive hull* of A, and is denoted by $\operatorname{lin} A$ and $\operatorname{pos} A$, respectively.

The following lemma will be needed. One can refer to [39, Theorem 1.8.8] for its proof.

Lemma 2.1. *The convergence* $\lim_{i\to\infty} K_i = K$ *in* $\mathcal{K}^n$ *is equivalent to the following conditions taken together:*

(1) *Each point in* K *is the limit of a sequence* $\{x_i\}$ *with* $x_i \in K_i$ *for* $i \in \mathbb{N}$.
(2) *The limit of any convergent sequence* $\{x_{i_j}\}$ *with* $x_{i_j} \in K_{i_j}$ *for* $j \in \mathbb{N}$ *belongs to* K.

2.2. $\mathfrak{p}$***-capacity and*** $\mathfrak{p}$***-capacitary measure.*** Let K be a compact convex set in $\mathbb{R}^n$ and $\mathfrak{p} \geqslant n$. By the definition of $\mathfrak{p}$-capacity, $C_\mathfrak{p}(K)$ is positively homogeneous of degree $(\mathfrak{p}-n)$ for $\mathfrak{p} > n$, that is, $C_\mathfrak{p}(sE) = s^{\mathfrak{p}-n} C_\mathfrak{p}(E)$, for $s > 0$; and C_n is positively homogeneous of degree 1, that is, $C_n(sE) = sC_n(E)$, for $s > 0$.

In [2], Akman, Lewis, Saari, and Vogel proved that the functional $C_\mathfrak{p}$ is translation invariant, that is, $C_\mathfrak{p}(K + x) = C_\mathfrak{p}(K)$, for $x \in \mathbb{R}^n$. They also proved that if a sequence of compact convex sets $\{K_i\}_{i=1}^{\infty}$ converges to a compact set K, then either K is a single point (in which case $\lim_{i\to\infty} C_\mathfrak{p}(K_i) = 0$) or $\lim_{i\to\infty} C_\mathfrak{p}(K_i) = C_\mathfrak{p}(K) > 0$.

For a convex body K, ∇U has non-tangential limits $\mathcal{H}^{n-1}$-almost everywhere on ∂K and $|\nabla U| \in L^\mathfrak{p}(\partial\Omega, \mathcal{H}^{n-1})$ (see, e.g., [31, Theorem 1]). Hence, the $\mathfrak{p}$-*capacitary measure* $\mu_\mathfrak{p}(K, \cdot)$ of K can be defined, for Borel set $\omega \subseteq \mathbb{S}^{n-1}$, by

$$\mu_\mathfrak{p}(K, \omega) = \int_{x \in g_K^{-1}(\omega)} |\nabla U(x)|^\mathfrak{p} \, \mathrm{d}\mathcal{H}^{n-1}(x),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure; $g_K : \partial' K \to \mathbb{S}^{n-1}$ is the Gauss map defined on the set $\partial' K$ of those points of ∂K that have a unique outer unit normal vector. Here, $\mu_\mathfrak{p}(K, \cdot)$ is absolutely continuous with respect to the surface area measure S_K. For $\mathfrak{p} > n$, it is positively homogeneous of degree $((\mathfrak{p}-n)/(\mathfrak{p}-1)-1)$, that is,

$$\mu_\mathfrak{p}(sK, \cdot) = s^{(\mathfrak{p}-n)/(\mathfrak{p}-1)-1} \mu_\mathfrak{p}(K, \cdot), \quad \text{for } s > 0;$$

and μ_n is positively homogeneous of degree -1, that is,

$$\mu_n(sK, \cdot) = s^{-1}\mu_n(K, \cdot), \quad \text{for } s > 0.$$

Let $p \in \mathbb{R}$ and $\mathfrak{p} \geqslant n$. For $K \in \mathcal{K}_o^n$, its L_p $\mathfrak{p}$-*capacitary measure* $\mu_{p,\mathfrak{p}}(K, \cdot)$ is a finite Borel measure on $\mathbb{S}^{n-1}$, defined for Borel $\omega \subseteq \mathbb{S}^{n-1}$ by

$$\mu_{p,\mathfrak{p}}(K, \omega) = \int_\omega h_K^{1-p}(u)\, \mathrm{d}\mu_{\mathfrak{p}}(K, u).$$

Also, $\mu_{p,\mathfrak{p}}$ is positively homogeneous of degree $((\mathfrak{p} - n)/(\mathfrak{p} - 1 - p))$, that is,

$$\mu_{p,\mathfrak{p}}(sK, \cdot) = s^{(\mathfrak{p}-n)/(\mathfrak{p}-1)-p}, \quad \text{for } s > 0;$$

and $\mu_{p,n}$ is positively homogeneous of degree $-p$, i.e.,

$$\mu_{p,n}(sK, \cdot) = s^{-p}\mu_{p,n}(K, \cdot), \quad \text{for } s > 0.$$

Lemma 2.2. *Let K, L be convex bodies in $\mathbb{R}^n$ and $\lambda \in (0, 1)$. Then,*

$$C_{\mathfrak{p}}((1-\lambda)K + \lambda L)^{1/(\mathfrak{p}-n)} \geqslant (1-\lambda)C_{\mathfrak{p}}(K)^{1/(\mathfrak{p}-n)} + \lambda C_{\mathfrak{p}}(L)^{1/(\mathfrak{p}-n)}, \quad \text{for } \mathfrak{p} > n, \tag{2.1}$$

and

$$C_n((1-\lambda)K + \lambda L) \geqslant (1-\lambda)C_n(K) + \lambda C_n(L). \tag{2.2}$$

Equality in (2.1) *or* (2.2) *holds if and only if K and L are homothetic.*

Lemma 2.3. *Let K, L be convex bodies in $\mathbb{R}^n$. Then,*

$$\lim_{t\to 0^+} \frac{C_{\mathfrak{p}}(K + tL) - C_{\mathfrak{p}}(K)}{t} = (\mathfrak{p} - 1)C_{\mathfrak{p}}(K)^{(\mathfrak{p}-2)/(\mathfrak{p}-1)} \int_{\mathbb{S}^{n-1}} h_L(u)\, \mathrm{d}\mu_{\mathfrak{p}}(K, u), \qquad \text{for } \mathfrak{p} > n, \tag{2.3}$$

and

$$\lim_{t\to 0^+} \frac{C_n(K + tL) - C_n(K)}{t} = (n\omega_n)^{1/(n-1)} C_n(K) \int_{\mathbb{S}^{n-1}} h_L(u)\, \mathrm{d}\mu_n(K, u). \tag{2.4}$$

One can refer to [2] for the proof of the above two lemmas.

By Lemma 2.3 and the positive homogeneity of $C_{\mathfrak{p}}(K)$, it follows that

$$C_{\mathfrak{p}}(K)^{1/(\mathfrak{p}-1)} = \frac{\mathfrak{p} - 1}{\mathfrak{p} - n} \int_{\mathbb{S}^{n-1}} h_K(u)\, \mathrm{d}\mu_{\mathfrak{p}}(K, u), \tag{2.5}$$

and

$$(n\omega_n)^{1/(1-n)} = \int_{\mathbb{S}^{n-1}} h_K(u)\,\mathrm{d}\mu_n(K,u). \tag{2.6}$$

Let K, L be convex bodies in $\mathbb{R}^n$. Define

$$C_{\mathfrak{p}}(K,L) = \frac{\mathfrak{p}-1}{\mathfrak{p}-n} C_{\mathfrak{p}}(K)^{(\mathfrak{p}-2)/(\mathfrak{p}-1)} \int_{\mathbb{S}^{n-1}} h_L(u)\,\mathrm{d}\mu_{\mathfrak{p}}(K,u), \qquad \text{for } \mathfrak{p} > n \tag{2.7}$$

and

$$C_n(K,L) = (n\omega_n)^{1/(n-1)} C_n(K) \int_{\mathbb{S}^{n-1}} h_L(u)\,\mathrm{d}\mu_n(K,u).$$

Obviously, $C_{\mathfrak{p}}(K,K) = C_{\mathfrak{p}}(K)$.

Theorem 2.4. *Let K, L be convex bodies in $\mathbb{R}^n$. Then,*

$$C_{\mathfrak{p}}(K,L) \geqslant C_{\mathfrak{p}}(K)^{1-1/(\mathfrak{p}-n)} C_{\mathfrak{p}}(L)^{1/(\mathfrak{p}-n)}, \quad \text{for } \mathfrak{p} > n, \tag{2.8}$$

and

$$C_n(K,L) \geqslant C_n(L). \tag{2.9}$$

Equality in (2.8) *or* (2.9) *holds if and only if K and L are homothetic.*

Proof. We show (2.8). (2.9) is proved similarly. For $t \geqslant 0$, define

$$f(t) = C_{\mathfrak{p}}(K + tL)^{1/(\mathfrak{p}-n)} - C_{\mathfrak{p}}(K)^{1/(\mathfrak{p}-n)} - tC_{\mathfrak{p}}(L)^{1/(\mathfrak{p}-n)}.$$

Then, f is nonnegative and concave on $[0,\infty)$.

Indeed, from the definition of f and (2.1), it follows that for $t_1, t_2 \geqslant 0$ and $0 < \lambda < 1$,

$$\begin{aligned} f((1-\lambda)t_1 + \lambda t_2) &= C_{\mathfrak{p}}((1-\lambda)(K+t_1L) + \lambda(K+t_2L))^{1/(\mathfrak{p}-n)} \\ &\quad - C_{\mathfrak{p}}(K)^{1/(\mathfrak{p}-n)} - ((1-\lambda)t_1 + \lambda t_2) C_{\mathfrak{p}}(L)^{1/(\mathfrak{p}-n)} \\ &\geqslant (1-\lambda) C_{\mathfrak{p}}(K+t_1L)^{1/(\mathfrak{p}-n)} + \lambda C_{\mathfrak{p}}(K+t_2L)^{1/(\mathfrak{p}-n)} \\ &\quad - (1-\lambda) C_{\mathfrak{p}}(K)^{1/(\mathfrak{p}-n)} - \lambda C_{\mathfrak{p}}(K)^{1/(\mathfrak{p}-n)} \\ &\quad - (1-\lambda) t_1 C_{\mathfrak{p}}(L)^{1/(\mathfrak{p}-n)} - \lambda t_2 C_{\mathfrak{p}}(L)^{1/(\mathfrak{p}-n)} \\ &= (1-\lambda) f(t_1) + \lambda f(t_2), \end{aligned}$$

as desired.

From (2.3) and (2.7), it follows that

$$\lim_{t\to 0^+} \frac{f(t) - f(0)}{t} = C_{\mathfrak{p}}(K)^{1/(\mathfrak{p}-n)-1} C_{\mathfrak{p}}(K,L) - C_{\mathfrak{p}}(L)^{1/(\mathfrak{p}-n)} \geqslant 0.$$

Since f is nonnegative and concave, it follows that if the equality holds on the right side, then f must be linear. Thus, K and L are homothetic. ∎

3. The Hadamard Variational Formula for $\mathfrak{p}$-capacity

In this section, we prove the Hadamard variational formula for $\mathfrak{p}$-capacity for Wulff shapes, which will be used in Section 5.

3.1. Wulff shapes. Let $\Omega \subseteq \mathbb{S}^{n-1}$ denote a closed set that is not contained in any closed hemisphere of $\mathbb{S}^{n-1}$. Let $h : \Omega \to (0, \infty)$ be continuous. The *Wulff shape* $[h] \in \mathcal{K}_o^n$, also called the *Aleksandrov body*, determined by h, is the convex body

$$[h] = \{x \in \mathbb{R}^n : x \cdot u \leqslant h(u) \text{ for all } u \in \Omega\}.$$

From the definition of support function, it follows that

$$h_{[h]} \leqslant h \quad \text{on } \Omega. \tag{3.1}$$

For Wulff shape $[h]$, we have

$$S_{[h]}(\mathbb{S}^{n-1} \setminus \Omega) = 0 \quad \text{and} \quad S_{[h]}(\{u \in \Omega : h_{[h]}(u) < h(u)\}) = 0.$$

One can refer to [39, Lemma 7.5.1] for their proofs. These, together with (2.5), (2.7), and the fact that $\mu_{\mathfrak{p}}([h], \cdot)$ is absolutely continuous with respect to $S_{[h]}$, yield that

$$C_{\mathfrak{p}}([h])^{1/(\mathfrak{p}-1)} = \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \int_\Omega h(u)\, \mathrm{d}\mu_{\mathfrak{p}}([h], u), \quad \text{for } \mathfrak{p} > n, \tag{3.2}$$

and

$$(n\omega_n)^{1/(1-n)} = \int_\Omega h(u)\, \mathrm{d}\mu_n([h], u).$$

Let $L \in \mathcal{K}^n$. Then,

$$C_{\mathfrak{p}}([h], L) = \frac{\mathfrak{p}-1}{\mathfrak{p}-n} C_{\mathfrak{p}}([h])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)} \int_\Omega h_L(u)\, \mathrm{d}\mu_{\mathfrak{p}}([h], u), \quad \text{for } \mathfrak{p} > n, \tag{3.3}$$

and

$$C_n([h], L) = (n\omega_n)^{1/(n-1)} C_n([h]) \int_\Omega h_L(u)\, \mathrm{d}\mu_n([h], u).$$

The *Aleksandrov convergence lemma* reads: If a sequence of continuous functions $h_i : \Omega \to (0, \infty)$ converges uniformly to $h : \Omega \to (0, \infty)$, then $[h_i]$ converges to $[h]$ in $\mathcal{K}_o^n$. (See, e.g., [39, Lemma 7.5.2], for its proof.)

3.2. The Hadamard variational formula for $\mathfrak{p}$-capacity.

Lemma 3.1. *Suppose that $\Omega \subseteq \mathbb{S}^{n-1}$ is a closed set which is not contained in any closed hemisphere of $\mathbb{S}^{n-1}$, and $I \subseteq \mathbb{R}$ is an interval containing both 0 and some positive number. Assume that $h_t(u) = h(t,u) : I \times \Omega \to (0,\infty)$ is continuous, such that the convergence*

$$h'_+(0,u) = \lim_{t\to 0^+} \frac{h(t,u) - h(0,u)}{t} \tag{3.4}$$

is uniform on Ω. Then,

$$\begin{aligned} &\lim_{t\to 0^+} \frac{C_{\mathfrak{p}}([h_t]) - C_{\mathfrak{p}}([h_0])}{t} \\ &\quad = (\mathfrak{p}-1) C_{\mathfrak{p}}([h_0])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)} \int_\Omega h'_+(0,u)\, \mathrm{d}\mu_{\mathfrak{p}}([h_0],u), \\ &\qquad \text{for } \mathfrak{p} > n, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\lim_{t\to 0^+} \frac{C_n([h_t]) - C_n([h_0])}{t} \\ &\quad = (n\omega_n)^{1/(n-1)} C_n([h_0]) \int_\Omega h'_+(0,u)\, \mathrm{d}\mu_n([h_0],u). \end{aligned} \tag{3.6}$$

Proof. We prove (3.5), and (3.6) is proved similarly.

The uniform convergence of (3.4) implies that $h_t \to h_0$, uniformly on Ω. By the Aleksandrov convergence lemma, we have that $\lim_{t\to 0^+}[h_t] = [h_0]$. Thus, $C_{\mathfrak{p}}([h_t]) \to C_{\mathfrak{p}}([h_0])$, and $\mu_{\mathfrak{p}}([h_t],\cdot) \to \mu_{\mathfrak{p}}([h_0],\cdot)$, weakly, as $t \to 0^+$. Since the convergence

$$\lim_{t\to 0^+} \frac{h(t,u) - h(0,u)}{t}$$

is uniform on Ω, it follows that

$$\lim_{t\to 0^+} \int_\Omega \frac{h_t(u) - h_0(u)}{t}\, \mathrm{d}\mu_{\mathfrak{p}}([h_t],u) = \int_\Omega h'_+(0,u)\, \mathrm{d}\mu_{\mathfrak{p}}([h_0],u). \tag{3.7}$$

On one hand, from (3.2), (3.3), the fact that $\lim_{t\to 0^+} C_{\mathfrak{p}}([h_t]) = C_{\mathfrak{p}}([h_0])$, and inequality (3.1) for $h = h_0$, it follows that

$$\begin{aligned} &\liminf_{t\to 0^+} \frac{C_{\mathfrak{p}}([h_t]) - C_{\mathfrak{p}}([h_t],[h_0])}{t} \\ &\quad = \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \liminf_{t\to 0^+} C_{\mathfrak{p}}([h_t])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)} \int_\Omega \frac{h_t(u) - h_{[h_0]}(u)}{t}\, \mathrm{d}\mu_{\mathfrak{p}}([h_t],u) \\ &\quad \geqslant \frac{\mathfrak{p}-1}{\mathfrak{p}-n} C_{\mathfrak{p}}([h_0])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)} \liminf_{t\to 0^+} \int_\Omega \frac{h_t(u) - h_0(u)}{t}\, \mathrm{d}\mu_{\mathfrak{p}}([h_t],u). \end{aligned}$$

This, combined with (3.7), yields that

$$\begin{aligned}(3.8)\qquad &\liminf_{t\to 0^+}\frac{C_{\mathfrak{p}}([h_t])-C_{\mathfrak{p}}([h_t],[h_0])}{t}\\ &\quad\geqslant \frac{\mathfrak{p}-1}{\mathfrak{p}-n}C_{\mathfrak{p}}([h_0])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)}\int_{\Omega}h'_+(0,u)\,\mathrm{d}\mu_{\mathfrak{p}}([h_0],u).\end{aligned}$$

For brevity, let

$$M=\frac{\mathfrak{p}-1}{\mathfrak{p}-n}C_{\mathfrak{p}}([h_0])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)}\int_{\Omega}h'_+(0,u)\,\mathrm{d}\mu_{\mathfrak{p}}([h_0],u).$$

From (3.8) and (2.8), it follows that

$$\begin{aligned}M&\leqslant\liminf_{t\to 0^+}\frac{C_{\mathfrak{p}}([h_t])-C_{\mathfrak{p}}([h_t],[h_0])}{t}\\ &\leqslant\liminf_{t\to 0^+}\frac{C_{\mathfrak{p}}([h_t])-C_{\mathfrak{p}}([h_t])^{1-1/(\mathfrak{p}-n)}C_{\mathfrak{p}}([h_0])^{1/(\mathfrak{p}-n)}}{t}.\end{aligned}$$

Since $\lim_{t\to 0^+}C_{\mathfrak{p}}([h_t])=C_{\mathfrak{p}}([h_0])$, we have

$$(3.9)\qquad M\leqslant C_{\mathfrak{p}}([h_0])^{1-1/(\mathfrak{p}-n)}\liminf_{t\to 0^+}\frac{C_{\mathfrak{p}}([h_t])^{1/(\mathfrak{p}-n)}-C_{\mathfrak{p}}([h_0])^{1/(\mathfrak{p}-n)}}{t}.$$

On the other hand, from (3.3), (3.2), the inequality (3.1) for $h=h_t$, and the uniform convergence in (3.4), it follows that

$$\begin{aligned}&\limsup_{t\to 0^+}\frac{C_{\mathfrak{p}}([h_0],[h_t])-C_{\mathfrak{p}}([h_0])}{t}\\ &\quad=\frac{\mathfrak{p}-1}{\mathfrak{p}-n}C_{\mathfrak{p}}([h_0])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)}\limsup_{t\to 0^+}\int_{\Omega}\frac{h_{[h_t]}(u)-h_0(u)}{t}\,\mathrm{d}\mu_{\mathfrak{p}}([h_0],u)\\ &\quad\leqslant\frac{\mathfrak{p}-1}{\mathfrak{p}-n}C_{\mathfrak{p}}([h_0])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)}\limsup_{t\to 0^+}\int_{\Omega}\frac{h_t(u)-h_0(u)}{t}\,\mathrm{d}\mu_{\mathfrak{p}}([h_0],u)\\ &\quad=\frac{\mathfrak{p}-1}{\mathfrak{p}-n}C_{\mathfrak{p}}([h_0])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)}\int_{\Omega}h'_+(0,u)\,\mathrm{d}\mu_{\mathfrak{p}}([h_0],u)=M.\end{aligned}$$

This, combined with (2.8), yields that

$$\begin{aligned}M&\geqslant\limsup_{t\to 0^+}\frac{C_{\mathfrak{p}}([h_0],[h_t])-C_{\mathfrak{p}}([h_0])}{t}\\ &\geqslant\limsup_{t\to 0^+}\frac{C_{\mathfrak{p}}([h_0])^{1-1/(\mathfrak{p}-n)}C_{\mathfrak{p}}([h_t])^{1/(\mathfrak{p}-n)}-C_{\mathfrak{p}}([h_0])}{t},\end{aligned}$$

and therefore

$$M \geqslant C_{\mathfrak{p}}([h_0])^{1-1/(\mathfrak{p}-n)} \limsup_{t\to 0^+} \frac{C_{\mathfrak{p}}([h_t])^{1/(\mathfrak{p}-n)} - C_{\mathfrak{p}}([h_0])^{1/(\mathfrak{p}-n)}}{t}. \tag{3.10}$$

Combining (3.9) with (3.10), it follows that

$$M = C_{\mathfrak{p}}([h_0])^{1-1/(\mathfrak{p}-n)} \lim_{t\to 0^+} \frac{C_{\mathfrak{p}}([h_t])^{1/(\mathfrak{p}-n)} - C_{\mathfrak{p}}([h_0])^{1/(\mathfrak{p}-n)}}{t}.$$

Thus,

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{C_{\mathfrak{p}}([h_t]) - C_{\mathfrak{p}}([h_0])}{t} \\
&\quad = \lim_{t\to 0^+} \frac{(C_{\mathfrak{p}}([h_t])^{1/(\mathfrak{p}-n)})^{\mathfrak{p}-n} - (C_{\mathfrak{p}}([h_0])^{1/(\mathfrak{p}-n)})^{\mathfrak{p}-n}}{t} \\
&\quad = (\mathfrak{p}-n) C_{\mathfrak{p}}([h_0])^{1-1/(\mathfrak{p}-n)} \lim_{t\to 0^+} \frac{C_{\mathfrak{p}}([h_t])^{1/(\mathfrak{p}-n)} - C_{\mathfrak{p}}([h_0])^{1/(\mathfrak{p}-n)}}{t} \\
&\quad = (\mathfrak{p}-n) M,
\end{aligned}$$

as desired. □

Theorem 3.2. *Suppose that $\Omega \subseteq \mathbb{S}^{n-1}$ is a closed set which is not contained in any closed hemisphere of $\mathbb{S}^{n-1}$, and $I \subseteq \mathbb{R}$ is an interval containing 0 in its interior. Assume that $h_t(u) = h(t,u) : I \times \Omega \to (0,\infty)$ is continuous, such that the convergence*

$$h'(0,u) = \lim_{t\to 0} \frac{h(t,u) - h(0,u)}{t}$$

is uniform on Ω. Then,

$$\left.\frac{\mathrm{d}C_{\mathfrak{p}}([h_t])}{\mathrm{d}t}\right|_{t=0} = (\mathfrak{p}-1) C_{\mathfrak{p}}([h_0])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)} \int_{\Omega} h'(0,u)\, \mathrm{d}\mu_{\mathfrak{p}}([h_0],u), \quad \text{for } \mathfrak{p} > n, \tag{3.11}$$

and

$$\left.\frac{\mathrm{d}C_n([h_t])}{\mathrm{d}t}\right|_{t=0} = (n\omega_n)^{1/(n-1)} C_n([h_0]) \int_{\Omega} h'(0,u)\, \mathrm{d}\mu_n([h_0],u). \tag{3.12}$$

Proof. We prove (3.11), and (3.12) is proved similarly.
It suffices to show that

$$\begin{aligned}
&\lim_{t\to 0^-} \frac{C_{\mathfrak{p}}([h_t]) - C_{\mathfrak{p}}([h_0])}{t} \\
&\quad = (\mathfrak{p}-1) C_{\mathfrak{p}}([h_0])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)} \int_{\Omega} h'(0,u)\, \mathrm{d}\mu_{\mathfrak{p}}([h_0],u).
\end{aligned}$$

For this aim, define $\tilde{h}(t,u) : -I \times \Omega \to (0,\infty)$ by $\tilde{h}(t,u) = h(-t,u)$. Then $[\tilde{h}_{-t}] = [h_t]$, $[\tilde{h}_0] = [h_0]$ and $\tilde{h}'(0,u) = -h'(0,u)$. By Lemma 3.1, it follows that

$$\begin{aligned}\lim_{t\to 0^-} & \frac{C_{\mathfrak{p}}([h_t]) - C_{\mathfrak{p}}([h_0])}{-t} \\ &= \lim_{t\to 0^+} \frac{C_{\mathfrak{p}}([\tilde{h}_t]) - C_{\mathfrak{p}}([\tilde{h}_0])}{t} \\ &= -(\mathfrak{p}-1)C_{\mathfrak{p}}([h_0])^{(\mathfrak{p}-2)/(\mathfrak{p}-1)} \int_\Omega h'(0,u)\,\mathrm{d}\mu_{\mathfrak{p}}([h_0],u),\end{aligned}$$

as desired. □

4. An Extremal Problem for $F_p(Q,x)$ Under Translation Transforms

Suppose that $c_1,\dots,c_N \in (0,\infty)$ and the unit vectors $u_1,\dots,u_N$ are not concentrated on any closed hemisphere of $\mathbb{S}^{n-1}$. Let

$$\mu = \sum_{k=1}^N c_k \delta_{u_k}(\cdot)$$

be the discrete measure on $\mathbb{S}^{n-1}$, where δ_u denotes the Dirac measure.

Recall that $P(u_1,\dots,u_N)$ is the set of polytopes whose facet normals are in the set $\{u_1,\dots,u_N\}$. Let $Q \in P(u_1,\dots,u_N)$ and $0 < p < 1$. For $\mathfrak{p} > n$, define

$$\begin{aligned}F_p(Q,x) &= \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{k=1}^N c_k (h_Q(u_k) - x\cdot u_k)^p \\ &= \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \int_{\mathbb{S}^{n-1}} (h_Q(u) - x\cdot u)^p \,\mathrm{d}\mu(u);\end{aligned} \tag{4.1}$$

and for $\mathfrak{p} = n$, define

$$\begin{aligned}F_p(Q,x) &= (n\omega_n)^{1/(n-1)} \sum_{k=1}^N c_k (h_Q(u_k) - x\cdot u_k)^p \\ &= (n\omega_n)^{1/(n-1)} \int_{\mathbb{S}^{n-1}} (h_Q(u) - x\cdot u)^p \,\mathrm{d}\mu(u).\end{aligned}$$

The following lemma shows there exists a unique point $x_Q \in \operatorname{int} Q$ such that $F_p(Q,x)$ attains its maximum.

Lemma 4.1. *Let the polytope $Q \in P(u_1,\dots,u_N)$. Then, there exists a unique point $x_Q \in \operatorname{int} Q$ such that $F_p(Q,x_Q) = \max_{x\in Q} F_p(Q,x)$. Moreover,*

$$\sum_{k=1}^N c_k (h_Q(u_k) - x_Q\cdot u_k)^{p-1} u_k = o. \tag{4.2}$$

Proof. We prove the case $\mathfrak{p} > n$. The $\mathfrak{p} = n$ case is proved similarly.

First, we prove the uniqueness of the maximal point. Assume $x_1, x_2 \in \operatorname{int} Q$ and

$$F_p(Q, x_1) = F_p(Q, x_2) = \max_{x \in Q} F_p(Q, x).$$

By (4.1), the concavity of t^p for $0 < p < 1$ together with the Jensen inequality and the above assumption, it follows that

$$\begin{aligned}
&F_p\left(Q, \frac{1}{2}(x_1 + x_2)\right) \\
&= \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{k=1}^{N} c_k \left(h_Q(u_k) - \frac{1}{2}(x_1 + x_2) \cdot u_k\right)^p \\
&= \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{k=1}^{N} c_k \left(\frac{1}{2}(h_Q(u_k) - x_1 \cdot u_k) + \frac{1}{2}(h_Q(u_k) - x_2 \cdot u_k)\right)^p \\
&\geqslant \frac{\mathfrak{p}-1}{2(\mathfrak{p}-n)} \sum_{k=1}^{N} c_k (h_Q(u_k) - x_1 \cdot u_k)^p + \frac{\mathfrak{p}-1}{2(\mathfrak{p}-n)} \sum_{k=1}^{N} c_k (h_Q(u_k) - x_2 \cdot u_k)^p \\
&= \frac{1}{2} F_p(Q, x_1) + \frac{1}{2} F_p(Q, x_2) \\
&= \max_{x \in Q} F_p(Q, x).
\end{aligned}$$

Since Q is convex, $\frac{1}{2}(x_1 + x_2) \in Q$, and the equality in the fourth line holds. By the equality condition of the Jensen inequality, we have

$$h_Q(u_k) - x_1 \cdot u_k = h_Q(u_k) - x_2 \cdot u_k, \quad \text{for } k = 1, \ldots, N.$$

That is,

$$x_1 \cdot u_k = x_2 \cdot u_k, \quad \text{for } k = 1, \ldots, N.$$

Since the unit vectors $u_1, \ldots, u_N$ are not concentrated on any closed hemisphere, it follows that $x_1 = x_2$, which proves the uniqueness.

Second, we prove the existence of the maximal point. Since $F_p(Q, x)$ is continuous in $x \in Q$ and Q is compact, so $F_p(Q, x)$ attains its maximum at a point of Q, say, x_Q. In the following, we prove that $x_Q \in \operatorname{int} Q$.

Assume $x_Q \in \partial Q$. Fix $y_0 \in \operatorname{int} Q$. Let $u_0 = (y_0 - x_Q)/|y_0 - x_Q|$. Then, for sufficiently small $\delta > 0$, it follows that $x_Q + \delta u_0 \in \operatorname{int} Q$. Next, we aim to show that

$$\begin{aligned}
&\frac{\mathfrak{p}-n}{\mathfrak{p}-1}(F_p(Q, x_Q + \delta u_0) - F_p(Q, x_Q)) \\
&\qquad = \sum_{k=1}^{N} c_k (h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k)^p - \sum_{k=1}^{N} c_k (h_Q(u_k) - x_Q \cdot u_k)^p
\end{aligned}$$

is positive, which will contradict the maximality of F_p at x_Q. Consequently, we have $x_Q \in \operatorname{int} Q$.

To this end, we divide $\{u_1, \ldots, u_N\}$ into two parts and let

$$U_1 = \{u_k : x_Q \cdot u_k = h_Q(u_k),\ k \in \{1, \ldots, N\}\},$$
$$U_2 = \{u_k : x_Q \cdot u_k < h_Q(u_k),\ k \in \{1, \ldots, N\}\}.$$

Then,

$$\begin{aligned}
&\frac{\mathfrak{p}-n}{\mathfrak{p}-1}(F_p(Q, x_Q + \delta u_0) - F_p(Q, x_Q)) \\
&\quad = \sum_{u_k \in U_1 \cup U_2} c_k[(h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k)^p - (h_Q(u_k) - x_Q \cdot u_k)^p] \\
&\quad = \sum_{u_k \in U_1} c_k(-\delta u_0 \cdot u_k)^p \\
&\qquad + \sum_{u_k \in U_2} c_k[(h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k)^p - (h_Q(u_k) - x_Q \cdot u_k)^p] \\
&\quad \geqslant \sum_{u_k \in U_1} c_k(-\delta u_0 \cdot u_k)^p \\
&\qquad - \sum_{u_k \in U_2} c_k\,|\,(h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k)^p - (h_Q(u_k) - x_Q \cdot u_k)^p\,|.
\end{aligned}$$

Since $x_Q \in \partial Q$, there exists a $u_{i_0} \in \{u_1, \ldots, u_N\}$ such that $x_Q \cdot u_{i_0} = h_Q(u_{i_0})$. Thus, U_1 is nonempty. Since $x_Q + \delta u_0 \in \operatorname{int} Q$ for sufficiently small $\delta > 0$, it follows that for any $u_k \in U_1$,

$$-\delta u_0 \cdot u_k = h_Q(u_k) - (x_Q + \delta u_0) \cdot u_k > 0.$$

Thus,

$$\sum_{u_k \in U_1} c_k(-\delta u_0 \cdot u_k)^p > 0. \tag{4.3}$$

Let $C = \min_{u_k \in U_2}(h_Q(u_k) - x_Q \cdot u_k)$. Then, for any $u_k \in U_2$, it follows that

$$h_Q(u_k) - x_Q \cdot u_k \geqslant \min_{u_k \in U_2}(h_Q(u_k) - x_Q \cdot u_k) = C > 0.$$

Thus, for sufficiently small $\delta > 0$, we have

$$h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k \geqslant \frac{C}{2} > 0.$$

By the concavity of t^p for $0 < p < 1$, we have

$$|(h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k)^p - (h_Q(u_k) - x_Q \cdot u_k)^p| \leqslant p\left(\frac{C}{2}\right)^{p-1} |-\delta u_0 \cdot u_k|.$$

Thus,

$$\sum_{u_k\in U_2} c_k|(h_Q(u_k)-x_Q\cdot u_k-\delta u_0\cdot u_k)^p-(h_Q(u_k)-x_Q\cdot u_k)^p| \leqslant \delta p\left(\frac{C}{2}\right)^{p-1}\sum_{u_k\in U_2} c_k|u_0\cdot u_k|. \tag{4.4}$$

Thus, according to (4.3) and (4.4), if follows that

$$\begin{aligned}&\frac{\mathfrak{p}-n}{\mathfrak{p}-1}(F_p(Q,x_Q+\delta u_0)-F_p(Q,x_Q))\\ &\quad\geqslant \sum_{u_k\in U_1} c_k(-\delta u_0\cdot u_k)^p-\delta p\left(\frac{C}{2}\right)^{p-1}\sum_{u_k\in U_2} c_k|u_0\cdot u_k|\\ &\quad=\delta^p\left\{\sum_{u_k\in U_1} c_k(-u_0\cdot u_k)^p-\delta^{1-p}p\left(\frac{C}{2}\right)^{p-1}\sum_{u_k\in U_2} c_k|u_0\cdot u_k|\right\}>0,\end{aligned}$$

for sufficiently small $\delta>0$. So $x_Q\in \operatorname{int} Q$. The existence is proved.

Finally, we prove (4.2). Since $F_p(Q,x)$ attains its maximum at the interior point x_Q, we have

$$0=\frac{\partial F_p(Q,x)}{\partial x_i}\Big|_{x=x_Q}=\frac{\mathfrak{p}-1}{\mathfrak{p}-n}\sum_{k=1}^{N} c_k p(h_Q(u_k)-x_Q\cdot u_k)^{p-1}(-u_{k,i}),$$

for $i=1,\dots,n$, where $x=(x_1,\dots,x_n)^T$ and $u_k=(u_{k,1},\dots,u_{k,n})^T$. That is,

$$\sum_{k=1}^{N} c_k(h_Q(u_k)-x_Q\cdot u_k)^{p-1}u_k=o,$$

as desired. □

From now on, we shall use x_Q to denote the maximal point of the function $F_p(Q,x)$ on Q, and call it the *maximal translation point*.

Lemma 4.2. *Suppose that the polytope $Q_i\in P(u_1,\dots,u_N)$, and also that $Q_i\to Q\in P(u_1,\dots,u_N)$, as $i\to\infty$. Then, we have that $x_{Q_i}\to x_Q$, and $F_p(Q_i,x_{Q_i})\to F_p(Q,x_Q)$, as $i\to\infty$.*

Proof. Since $Q_i\to Q$, it follows that for sufficiently large i, $x_{Q_i}\in Q_i\subseteq Q+B$. Thus, $\{x_{Q_i}\}_i$ is a bounded sequence.

Let $\{x_{Q_{i_j}}\}_j$ be a convergent subsequence of $\{x_{Q_i}\}_i$. Assume that $x_{Q_{i_j}}\to x'$, but $x'\neq x_Q$. By Lemma 2.1, it follows that $x'\in Q$. Hence, we have that $F_p(Q,x')<F_p(Q,x_Q)$. By the continuity of $F_p(Q,x)$ in Q and x,

$$\lim_{j\to\infty} F_p(Q_{i_j},x_{Q_{i_j}})=F_p(Q,x').$$

Meanwhile, by Lemma 2.1, for $x_Q \in Q$, there exists a $y_{i_j} \in Q_{i_j}$ such that $y_{i_j} \to x_Q$. Hence, $\lim_{j\to\infty} F_p(Q_{i_j}, y_{i_j}) = F_p(Q, x_Q)$. Thus,

$$\lim_{j\to\infty} F_p(Q_{i_j}, x_{Q_{i_j}}) < \lim_{j\to\infty} F_p(Q_{i_j}, y_{i_j}). \tag{4.5}$$

However, for any Q_{i_j}, we have $F_p(Q_{i_j}, x_{Q_{i_j}}) \geqslant F_p(Q_{i_j}, y_{i_j})$. Thus,

$$\lim_{j\to\infty} F_p(Q_{i_j}, x_{Q_{i_j}}) \geqslant \lim_{j\to\infty} F_p(Q_{i_j}, y_{i_j}),$$

which contradicts (4.5). Thus, $x_{Q_{i_j}} \to x_Q$, and so $x_{Q_i} \to x_Q$. By the continuity of F_p, it follows that $F_p(Q_i, x_{Q_i}) \to F_p(Q, x_Q)$, and this completes the proof. □

Lemma 4.3. *Let $Q \in P(u_1, \dots, u_N)$. Then,*

(1) $F_p(Q + y, x_{Q+y}) = F_p(Q, x_Q)$, *for* $y \in \mathbb{R}^n$;
(2) $F_p(\lambda Q, x_{\lambda Q}) = \lambda^p F_p(Q, x_Q)$, *for* $\lambda > 0$.

Proof. We prove the case $\mathfrak{p} > n$. The $\mathfrak{p} = n$ case is proved similarly. From (4.1), it follows that

$$\begin{aligned}
F_p(Q + y, x_{Q+y}) &= \max_{z\in Q+y} F_p(Q + y, z) \\
&= \max_{z-y\in Q} \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{k=1}^{N} c_k (h_{Q+y}(u_k) - z \cdot u_k)^p \\
&= \max_{z-y\in Q} \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{k=1}^{N} c_k (h_Q(u_k) - (z - y) \cdot u_k)^p \\
&= \max_{x\in Q} \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{k=1}^{N} c_k (h_Q(u_k) - x \cdot u_k)^p \\
&= F_p(Q, x_Q).
\end{aligned}$$

Similarly,

$$\begin{aligned}
F_p(\lambda Q, x_{\lambda Q}) &= \max_{z\in\lambda Q} F_p(\lambda Q, z) \\
&= \max_{(z/\lambda)\in Q} \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{k=1}^{N} c_k (h_{\lambda Q}(u_k) - z \cdot u_k)^p \\
&= \lambda^p \max_{(z/\lambda)\in Q} \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{k=1}^{N} c_k \left(h_Q(u_k) - \frac{z}{\lambda} \cdot u_k\right)^p \\
&= \lambda^p \max_{x\in Q} \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{k=1}^{N} c_k (h_Q(u_k) - x \cdot u_k)^p \\
&= \lambda^p F_p(Q, x_Q),
\end{aligned}$$

as desired. □

5. TWO DUAL EXTREMAL PROBLEMS FOR $\mathfrak{p}$-CAPACITY

Suppose μ is the discrete measure on $\mathbb{S}^{n-1}$ such that

$$\mu = \sum_{k=1}^{N} c_k \delta_{u_k}(\cdot),$$

where $N \geqslant n+1$, $c_k > 0$ and $u_1, \ldots, u_N$ are not concentrated on any closed hemisphere.

Recall that $P(u_1, \ldots, u_N)$ is the set of polytopes whose facet normals are in the set $\{u_1, \ldots, u_N\}$. Let $Q \in P(u_1, \ldots, u_N)$ and $0 < p < 1$. For $\mathfrak{p} > n$,

$$F_p(Q, x_Q) = \frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{k=1}^{N} c_k (h_Q(u_k) - x_Q \cdot u_k)^p,$$

and for $\mathfrak{p} = n$,

$$F_p(Q, x_Q) = (n\omega_n)^{1/(n-1)} \sum_{k=1}^{N} c_k (h_Q(u_k) - x_Q \cdot u_k)^p.$$

Here, x_Q is the maximal translation point.

To prove the main results of this article, we start from the following extremal problems, which are closely connected with our concerned L_p Minkowski problem for $\mathfrak{p}$-capacity.

Problem 1. *Among all the polytopes $Q \in P(u_1, \ldots, u_N)$, find one to solve the constrained minimization problem* $\inf_Q F_p(Q, x_Q)$ *subject to* $C_{\mathfrak{p}}(Q) = 1$.

Problem 2. *Among all the polytopes $Q \in P(u_1, \ldots, u_N)$, find one to solve the constrained maximization problem* $\sup_Q C_{\mathfrak{p}}(Q)$ *subject to* $F_p(Q, x_Q) = 1$.

The two lemmas below show the duality between Problem 1 and Problem 2.

Lemma 5.1. *Suppose $\mathfrak{p} > n$. Then, the following assertions hold:*

(1) *If the polytope $\bar{P}$ solves Problem 1, then the polytope $P = F_p(\bar{P}, x_{\bar{P}})^{-1/p}\bar{P}$ solves Problem 2.*
(2) *If the polytope P solves Problem 2, then the polytope $\bar{P} = C_{\mathfrak{p}}(P)^{-1/(\mathfrak{p}-n)}P$ solves Problem 1.*

Proof. (1) Assume $\bar{P}$ solves Problem 1. Let $Q \in P(u_1, \ldots, u_N)$ such that $F_p(Q, x_Q) = 1$. Let $\bar{Q} = C_{\mathfrak{p}}(Q)^{-1/(\mathfrak{p}-n)}Q$. From the positive homogeneity of degree $(\mathfrak{p}-n)$ of $C_{\mathfrak{p}}$, the fact that $C_{\mathfrak{p}}(\bar{P}) = 1$, the assumption together with the fact that $C_{\mathfrak{p}}(\bar{Q}) = 1$, Lemma 4.3 (2), and finally the fact that $F_p(Q, x_Q) = 1$, we have

$$\begin{aligned} C_{\mathfrak{p}}(P) &= C_{\mathfrak{p}}(\bar{P}) F_p(\bar{P}, x_{\bar{P}})^{-(\mathfrak{p}-n)/p} = F_p(\bar{P}, x_{\bar{P}})^{-(\mathfrak{p}-n)/p} \\ &\geqslant F_p(\bar{Q}, x_{\bar{Q}})^{-(\mathfrak{p}-n)/p} = C_{\mathfrak{p}}(Q) F_p(Q, x_Q)^{-(\mathfrak{p}-n)/p} = C_{\mathfrak{p}}(Q). \end{aligned}$$

Thus, P solves Problem 2.

(2) Assume P solves Problem 2. Let $\bar{Q} \in P(u_1, \ldots, u_N)$ such that $C_{\mathfrak{p}}(\bar{Q}) = 1$. Let $Q = F_p(\bar{Q}, x_{\bar{Q}})^{-1/p}\bar{Q}$. By Lemma 4.3 (2), it follows that $F_p(Q, x_Q) = 1$. From Lemma 4.3 (2), the fact that $F_p(P, x_P) = 1$, the assumption that $F_p(Q, x_Q) = 1$, the positive homogeneity of degree $(\mathfrak{p} - n)$ of $C_{\mathfrak{p}}$, and finally that $C_{\mathfrak{p}}(\bar{Q}) = 1$, we have

$$\begin{aligned} F_p(\bar{P}, x_{\bar{P}}) &= F_p(P, x_P)C_{\mathfrak{p}}(P)^{-p/(\mathfrak{p}-n)} = C_{\mathfrak{p}}(P)^{-p/(\mathfrak{p}-n)} \\ &\leqslant C_{\mathfrak{p}}(Q)^{-p/(\mathfrak{p}-n)} = F_p(\bar{Q}, x_{\bar{Q}})C_{\mathfrak{p}}(\bar{Q})^{-p/(\mathfrak{p}-n)} = F_p(\bar{Q}, x_{\bar{Q}}). \end{aligned}$$

Thus, $\bar{P}$ solves Problem 1. ◻

Similarly, we can prove the following results for $\mathfrak{p} = n$.

Lemma 5.2. *Suppose $\mathfrak{p} = n$. Then, the following assertions hold:*

(1) *If the polytope $\bar{P}$ solves Problem 1, then the polytope $P = F_p(\bar{P}, x_{\bar{P}})^{-1/p}\bar{P}$ solves Problem 2.*
(2) *If the polytope P solves Problem 2, then the polytope $\bar{P} = C_n(P)^{-1}P$ solves Problem 1.*

The following is the normalized L_p Minkowski problem for $\mathfrak{p}$-capacity for $\mathfrak{p} > n$.

Problem 3. *Suppose $\mathfrak{p} > n$. Among all the polytopes containing the origin in their interiors, find a polytope P such that $\dfrac{\mu_{p,\mathfrak{p}}(P, \cdot)}{C_{\mathfrak{p}}(P)^{1/(\mathfrak{p}-1)}} = \mu$.*

Essentially, Problem 3 is to find a polytope P so that its normalized L_p $\mathfrak{p}$-capacitary measure

$$C_{\mathfrak{p}}(P)^{-1/(\mathfrak{p}-1)}\mu_{p,\mathfrak{p}}(P, \cdot) = C_{\mathfrak{p}}(P)^{-1/(\mathfrak{p}-1)}h_P^{1-p}(\cdot)\mu_{\mathfrak{p}}(P, \cdot)$$

is the given measure μ.

The following lemma presents relations between Problem 2 and Problem 3 for $\mathfrak{p} > n$.

Lemma 5.3. *Suppose that P is a polytope with outer unit normal vectors $u_1, \ldots, u_N$, and $\mathfrak{p} > n$. If P solves Problem 2 and $x_P = o$, then P solves Problem 3.*

Proof. For $\delta_1, \ldots, \delta_N \in \mathbb{R}$, choose $|t| > 0$ small enough so that the polytope P_t defined by

$$P_t = \{x : x \cdot u_k \leqslant h_P(u_k) + t\delta_k,\ k = 1, \ldots, N\}$$

has exactly N facets. Let $\alpha(t)P_t = F_p(P_t, x_{P_t})^{-1/p}P_t$. Then, $F_p(\alpha(t)P_t, x_{\alpha(t)P_t}) = 1$, $\alpha(t)P_t \in P_N(u_1, \ldots, u_N)$, $P_t \to P$, and $\alpha(t)P_t \to P$, as $t \to 0$.

For brevity, let $x(t) = x_{P_t}$. By (4.2) of Lemma 4.1, it follows that

$$\sum_{k=1}^{N} c_k (h_{P_t}(u_k) - x(t) \cdot u_k)^{p-1} u_{k,i} = 0, \quad \text{for } i = 1, \dots, n,$$

where $u_k = (u_{k,1}, \dots, u_{k,n})^T$. Let $t = 0$. Then, $P_0 = P$, $x(0) = o$, and

$$\sum_{k=1}^{N} c_k h_P^{p-1}(u_k) u_{k,i} = 0, \quad \text{for } i = 1, \dots, n. \tag{5.1}$$

We first show that $x'(t)|_{t=0}$ exists. Let

$$y_i(t, x_1, \dots, x_n) = \sum_{k=1}^{N} c_k [h_{P_t}(u_k) - (x_1 u_{k,1} + \cdots + x_n u_{k,n})]^{p-1} u_{k,i},$$

for $i = 1, \dots, n$. Then,

$$\left. \frac{\partial y_i}{\partial x_j} \right|_{(0,\dots,0)} = \sum_{k=1}^{N} (1-p) c_k h_P^{p-2}(u_k) u_{k,i} u_{k,j}.$$

Thus,

$$\left(\left. \frac{\partial y}{\partial x} \right|_{(0,\dots,0)} \right)_{n \times n} = \sum_{k=1}^{N} (1-p) c_k h_P^{p-2}(u_k) u_k u_k^T.$$

Since $u_1, \dots, u_N$ are not concentrated on any closed hemisphere, for $x \in \mathbb{R}^n$ with $x \neq o$, there exists a $u_{i_0} \in \{u_1, \dots, u_N\}$ such that $u_{i_0} \cdot x \neq 0$. Thus,

$$\begin{aligned} x^T \Big(\sum_{k=1}^{N} (1-p) c_k h_P^{p-2}(u_k) u_k u_k^T \Big) x \\ &= \sum_{k=1}^{N} (1-p) c_k h_P^{p-2}(u_k) (x \cdot u_k)^2 \\ &\geqslant (1-p) c_{i_0} h_P^{p-2}(u_{i_0}) (x \cdot u_{i_0})^2 > 0, \end{aligned}$$

which implies that

$$(\partial y / (\partial x)|_{(0,\dots,0)})_{n \times n}$$

is positively definite. By the implicit function theorem, it follows that $x'(t)|_{t=0} = x'(0) = (x_1'(0), \dots, x_n'(0))$ exists.

Now, we can finish the proof. Since $C_{\mathfrak{p}}$ attains its maximum at the polytope P, from (3.11), $F_p(P, x(0)) = 1$, $\alpha(0) = 1$, $h_{P_t}(u_k) = h_P(u_k) + t\delta_k$ for $k = 1, \dots, N$, (5.1), and (2.5), we have

$$
\begin{aligned}
0 &= \frac{1}{\mathfrak{p}-1} C_{\mathfrak{p}}(P)^{-(\mathfrak{p}-2)/(\mathfrak{p}-1)} \frac{\mathrm{d}C_{\mathfrak{p}}(\alpha(t)P_t)}{\mathrm{d}t}\bigg|_{t=0} \\
&= \sum_{j=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t} h_{\alpha(t)P_t}(u_j)\bigg|_{t=0} \mu_{\mathfrak{p}}(P,\{u_j\}) \\
&= \sum_{j=1}^{N} \bigg[-\frac{1}{p} F_p(P,x(0))^{-1/p-1} \frac{\mathrm{d}}{\mathrm{d}t} F_p(P_t,x(t))\bigg|_{t=0} h_P(u_j) \\
&\qquad\qquad + \alpha(0) \frac{\mathrm{d}}{\mathrm{d}t} h_{P_t}(u_j)\bigg|_{t=0} \bigg] \mu_{\mathfrak{p}}(P,\{u_j\}) \\
&= \sum_{j=1}^{N} \bigg[-\frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{k=1}^{N} c_k (h_{P_t}(u_k) - x(t)\cdot u_k)^p \Big)\bigg|_{t=0} h_P(u_j) \\
&\qquad\qquad + \delta_j \bigg] \mu_{\mathfrak{p}}(P,\{u_j\}) \\
&= \sum_{j=1}^{N} \bigg[-\frac{\mathfrak{p}-1}{\mathfrak{p}-n} \Big(\sum_{k=1}^{N} c_k h_P^{p-1}(u_k)(\delta_k - x'(0)\cdot u_k) \Big) h_P(u_j) \\
&\qquad\qquad + \delta_j \bigg] \mu_{\mathfrak{p}}(P,\{u_j\}) \\
&= \sum_{j=1}^{N} \bigg[-\frac{\mathfrak{p}-1}{\mathfrak{p}-n} \Big(\sum_{k=1}^{N} c_k h_P^{p-1}(u_k)\delta_k \\
&\qquad\qquad - x'(0)\cdot \Big(\sum_{k=1}^{N} c_k h_P^{p-1}(u_k) u_k \Big) \Big) h_P(u_j) + \delta_j \bigg] \mu_{\mathfrak{p}}(P,\{u_j\}) \\
&= \bigg(-\frac{\mathfrak{p}-1}{\mathfrak{p}-n} \sum_{j=1}^{N} h_P(u_j)\mu_{\mathfrak{p}}(P,\{u_j\}) \bigg) \sum_{k=1}^{N} c_k h_P^{p-1}(u_k)\delta_k \\
&\qquad\qquad + \sum_{j=1}^{N} \mu_{\mathfrak{p}}(P,\{u_j\})\delta_j \\
&= \sum_{j=1}^{N} (-C_{\mathfrak{p}}(P)^{1/(\mathfrak{p}-1)} h_P^{p-1}(u_j)c_j + \mu_{\mathfrak{p}}(P,\{u_j\}))\delta_j.
\end{aligned}
$$

Since $\delta_1,\dots,\delta_N$ are arbitrary real numbers, we have

$$\mu_{\mathfrak{p}}(P,\{u_j\}) = C_{\mathfrak{p}}(P)^{1/(\mathfrak{p}-1)} h_P^{p-1}(u_j)c_j, \quad \text{for } j = 1,\dots,N.$$

In light of the fact that P is n-dimensional and o is in its interior, it follows that $C_{\mathfrak{p}}(P) > 0$ and $h_P(u_j) > 0$. Therefore,

$$\frac{h_P^{1-p}(u_j)\mu_{\mathfrak{p}}(P,\{u_j\})}{C_{\mathfrak{p}}(P)^{1/(\mathfrak{p}-1)}} = c_j, \quad \text{for } j = 1,\dots,N.$$

That is, $\dfrac{\mu_{p,\mathfrak{p}}(P,\cdot)}{C_{\mathfrak{p}}(P)^{1/(\mathfrak{p}-1)}} = \mu$. Thus, P solves Problem 3. □

Recall that our original concerned L_p Minkowski problem for $\mathfrak{p}$-capacity is the following.

Problem 4. *Among all the polytopes containing the origin in their interiors, find a polytope P such that $\mu_{p,\mathfrak{p}}(P,\cdot) = \mu$.*

The next lemma shows the equivalence between Problem 4 and Problem 3 for $\mathfrak{p} > n$.

Lemma 5.4. *Suppose that P is a polytope with outer unit normal vectors $u_1, \dots, u_N$. Then, the following assertions hold for $\mathfrak{p} > n$:*

(1) *Assume that $p \neq (\mathfrak{p}-n)/(\mathfrak{p}-1)$. If P solves Problem 3, then we have that $\bar{P} = C_{\mathfrak{p}}(P)^{1/(p(\mathfrak{p}-1)-(\mathfrak{p}-n))}P$ solves Problem 4.*

(2) *If $\bar{P}$ solves Problem 4, then $P = C_{\mathfrak{p}}(\bar{P})^{-1/(p(\mathfrak{p}-1))}\bar{P}$ solves Problem 3.*

Proof. (1) From the positive homogeneity of degree $((\mathfrak{p}-n)/(\mathfrak{p}-1)-p)$ of $\mu_{p,\mathfrak{p}}$ and the assumption that $C_{\mathfrak{p}}(P)^{-1/(\mathfrak{p}-1)}\mu_{p,\mathfrak{p}}(P,\cdot) = \mu$, it follows that

$$\begin{aligned}\mu_{p,\mathfrak{p}}(\bar{P},\cdot) &= \mu_{p,\mathfrak{p}}(C_{\mathfrak{p}}(P)^{1/(p(\mathfrak{p}-1)-(\mathfrak{p}-n))}P,\cdot)\\ &= C_{\mathfrak{p}}(P)^{-1/(\mathfrak{p}-1)}\mu_{p,\mathfrak{p}}(P,\cdot) = \mu.\end{aligned}$$

Thus, $\bar{P}$ solves Problem 3.

(2) From the positive homogeneity of degree $((\mathfrak{p}-n)/(\mathfrak{p}-1)-p)$ of $\mu_{p,\mathfrak{p}}$, the positive homogeneity of degree $(\mathfrak{p}-n)$ of $C_{\mathfrak{p}}$, and the assumption $\mu_{p,\mathfrak{p}}(P,\cdot) = \mu$, it follows that

$$\begin{aligned}&C_{\mathfrak{p}}(P)^{-1/(\mathfrak{p}-1)}\mu_{p,\mathfrak{p}}(P,\cdot)\\ &\quad= C_{\mathfrak{p}}(C_{\mathfrak{p}}(\bar{P})^{-1/(p(\mathfrak{p}-1))}\bar{P})^{-1/(\mathfrak{p}-1)}\mu_{p,\mathfrak{p}}(C_{\mathfrak{p}}(\bar{P})^{-1/(p(\mathfrak{p}-1))}\bar{P},\cdot)\\ &\quad= \mu_{p,\mathfrak{p}}(\bar{P},\cdot) = \mu.\end{aligned}$$

Thus, P solves Problem 4. □

The following lemma presents relations between Problem 2 and Problem 4 for $\mathfrak{p} = n$.

Lemma 5.5. *Suppose that P is a polytope with outer unit normal vectors $u_1, \dots, u_N$, and $\mathfrak{p} = n$. If P solves Problem 2 and $x_P = o$, then P solves Problem 4.*

Proof. For $\delta_1, \dots, \delta_N \in \mathbb{R}$, choose $|t| > 0$ small enough so that the polytope P_t defined by $P_t = \{x : x \cdot u_k \leqslant h_P(u_k) + t\delta_k,\ k = 1, \dots, N\}$ has exactly N facets. Also, let $\alpha(t)P_t = F_p(P_t, x_{P_t})^{-1/p}P_t$. Then, $F_p(\alpha(t)P_t, x_{\alpha(t)P_t}) = 1$, $\alpha(t)P_t \in P_N(u_1, \dots, u_N)$, $P_t \to P$, and $\alpha(t)P_t \to P$, as $t \to 0$.

For brevity, let $x(t) = x_{P_t}$. As the proof of Lemma 5.3, we can prove that $x'(0)$ exists, and

$$\sum_{k=1}^{N} c_k h_P^{p-1}(u_k) u_{k,i} = 0, \quad \text{for } i = 1, \dots, n. \tag{5.2}$$

Now, we can finish the proof. Since $C_{\mathfrak{p}}$ attains its maximum at the polytope P, by equations (3.12), (5.2), and (2.6), and by the fact that $F_p(P, x(0)) = 1$, that $\alpha(0) = 1$, and that $h_{P_t}(u_k) = h_P(u_k) + t\delta_k$ for $k = 1, \dots, N$, we have

$$\begin{aligned}
0 &= (n\omega_n)^{1/(1-n)} C_n(P)^{-1} \frac{\mathrm{d}C_n(\alpha(t)P_t)}{\mathrm{d}t}\Big|_{t=0} \\
&= \sum_{j=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t} h_{\alpha(t)P_t}(u_j)\Big|_{t=0} \mu_n(P, \{u_j\}) \\
&= \sum_{j=1}^{N} \Big[-\frac{1}{p} F_p(P, x(0))^{-1/p-1} \frac{\mathrm{d}}{\mathrm{d}t} F_p(P_t, x(t))\Big|_{t=0} h_P(u_j) \\
&\qquad + \alpha(0) \frac{\mathrm{d}}{\mathrm{d}t} h_{P_t}(u_j)\Big|_{t=0} \Big] \mu_n(P, \{u_j\}) \\
&= \sum_{j=1}^{N} \Big[-\frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}t} \Big((n\omega_n)^{1/(n-1)} \sum_{k=1}^{N} c_k (h_{P_t}(u_k) - x(t) \cdot u_k)^p \Big)\Big|_{t=0} h_P(u_j) \\
&\qquad + \delta_j \Big] \mu_n(P, \{u_j\}) \\
&= \sum_{j=1}^{N} \Big[-(n\omega_n)^{1/(n-1)} \Big(\sum_{k=1}^{N} c_k h_P^{p-1}(u_k)(\delta_k - x'(0) \cdot u_k) \Big) h_P(u_j) \\
&\qquad + \delta_j \Big] \mu_n(P, \{u_j\}) \\
&= \sum_{j=1}^{N} \Big[-(n\omega_n)^{1/(n-1)} \Big(\sum_{k=1}^{N} c_k h_P^{p-1}(u_k)\delta_k \\
&\qquad - x'(0) \cdot \Big(\sum_{k=1}^{N} c_k h_P^{p-1}(u_k) u_k \Big) \Big) h_P(u_j) + \delta_j \Big] \mu_n(P, \{u_j\}) \\
&= \Big(-(n\omega_n)^{1/(n-1)} \sum_{j=1}^{N} h_P(u_j) \mu_n(P, \{u_j\}) \Big) \sum_{k=1}^{N} c_k h_P^{p-1}(u_k)\delta_k \\
&\qquad + \sum_{j=1}^{N} \mu_n(P, \{u_j\})\delta_j \\
&= \sum_{j=1}^{N} (-h_P^{p-1}(u_j) c_j + \mu_n(P, \{u_j\}))\delta_j.
\end{aligned}$$

Since $\delta_1, \dots, \delta_N$ are arbitrary real numbers, we have

$$\mu_n(P, \{u_j\}) = h_P^{p-1}(u_j)c_j, \quad \text{for } j = 1, \dots, N.$$

As P is n-dimensional and o is in its interior, it follows that $h_P(u_j) > 0$. Therefore,

$$h_P^{1-p}(u_j)\mu_n(P, \{u_j\}) = c_j, \quad \text{for } j = 1, \dots, N.$$

That is, $\mu_{p,n}(P, \cdot) = \mu$. Thus, P solves Problem 4. □

6. Existence of Solutions to the L_p Minkowski Problem for $\mathfrak{p}$-capacity

Throughout this section, let $0 < p < 1$ and $\mathfrak{p} \geqslant n$. Recall that μ is the discrete measure on $\mathbb{S}^{n-1}$ such that

$$\mu = \sum_{k=1}^{N} c_k \delta_{u_k}(\cdot),$$

where $N \geqslant n+1$, $c_k > 0$, and $u_1, \dots, u_N$ are not concentrated on any closed hemisphere.

Lemma 6.1. *Suppose the polytope P solves Problem 1. Then, P has exactly N facets whose outer unit normal vectors are $u_1, \dots, u_N$.*

Proof. We prove the case $\mathfrak{p} > n$. The $\mathfrak{p} = n$ case is proved similarly.

By the translation invariance of $C_{\mathfrak{p}}$ and Lemma 4.3 (1), it follows that any translation of P also solves Problem 2. Thus, we assume that $x_P = o$. We argue by contradiction.

Assume that $u_{i_0} \in \{u_1, \dots, u_N\}$, but the support set $F(P, u_{i_0}) = P \cap H(P, u_{i_0})$ is not a facet of P. Fix $\delta > 0$, and let

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \leqslant h_P(u_{i_0}) - \delta\}$$

and

$$\tau P_\delta = \tau(\delta)P_\delta = C_{\mathfrak{p}}(P_\delta)^{-1/(\mathfrak{p}-n)}P_\delta.$$

Then, $C_{\mathfrak{p}}(\tau P_\delta) = 1$ and $\tau P_\delta \to P$, as $\delta \to 0^+$. By Lemma 4.2, it follows that

$$x_{P_\delta} \to x_P = o \in \operatorname{int} P \quad \text{as } \delta \to 0^+.$$

Thus, for sufficiently small $\delta > 0$, we can assume that $x_{P_\delta} \in \operatorname{int} P$ and

$$h_P(u_k) - x_{P_\delta} \cdot u_k > \delta > 0, \quad \text{for } k = 1, \dots, N.$$

In the following, we show $F_p(\tau P_\delta, x_{\tau P_\delta}) < F_p(P, o)$, which contradicts the fact that $F_p(P, o)$ is the minimum. Since

$$\begin{aligned}\frac{\mathfrak{p}-n}{\mathfrak{p}-1}F_p(\tau P_\delta, x_{\tau P_\delta}) &= \tau^p \sum_{k=1}^{N} c_k(h_{P_\delta}(u_k) - x_{P_\delta}\cdot u_k)^p \\ &= \tau^p\Big(\sum_{k=1}^{N} c_k(h_P(u_k) - x_{P_\delta}\cdot u_k)^p\Big) \\ &\quad - \tau^p c_{i_0}(h_P(u_{i_0}) - x_{P_\delta}\cdot u_{i_0})^p \\ &\quad + \tau^p c_{i_0}(h_{P_\delta}(u_{i_0}) - x_{P_\delta}\cdot u_{i_0})^p \\ &= \frac{\mathfrak{p}-n}{\mathfrak{p}-1}F_p(P, x_{P_\delta}) + G(\delta),\end{aligned}$$

where

$$\begin{aligned}G(\delta) = (\tau^p-1)\Big(\sum_{k=1}^{N} c_k(h_P(u_k) - x_{P_\delta}\cdot u_k)^p\Big) \\ + c_{i_0}\tau^p[(h_P(u_{i_0}) - x_{P_\delta}\cdot u_{i_0} - \delta)^p - (h_P(u_{i_0}) - x_{P_\delta}\cdot u_{i_0})^p].\end{aligned}$$

By showing $G(\delta) < 0$, then $F_p(\tau P_\delta, x_{\tau P_\delta}) < F_p(P, x_{P_\delta}) \leqslant F_p(P, o)$, as desired.

Since $0 < h_P(u_{i_0}) - x_{P_\delta}\cdot u_{i_0} - \delta < h_P(u_{i_0}) - x_{P_\delta}\cdot u_{i_0} < d_0$, where d_0 is the diameter of P, by the concavity of t^p on $[0,\infty)$ for $0 < p < 1$, it follows that

$$(h_P(u_{i_0}) - x_{P_\delta}\cdot u_{i_0} - \delta)^p - (h_P(u_{i_0}) - x_{P_\delta}\cdot u_{i_0})^p < (d_0-\delta)^p - d_0^p.$$

Hence,

$$\begin{aligned}G(\delta) &< (\tau^p-1)\Big(\sum_{k=1}^{N} c_k(h_P(u_k) - x_{P_\delta}\cdot u_k)^p\Big) + c_{i_0}\tau^p[(d_0-\delta)^p - d_0^p] \\ &= \tau^p[(d_0-\delta)^p - d_0^p] \\ &\quad\times\left(c_{i_0} + \frac{\tau^p-1}{(d_0-\delta)^p - d_0^p}\frac{1}{\tau^p}\sum_{k=1}^{N} c_k(h_P(u_k) - x_{P_\delta}\cdot u_k)^p\right).\end{aligned}$$

From the variational formula for $\mathfrak{p}$-capacity (3.11), it follows that

$$\begin{aligned}\lim_{\delta\to 0^+}\frac{\tau^p-1}{(d_0-\delta)^p - d_0^p} &= \lim_{\delta\to 0^+}\frac{(C_{\mathfrak{p}}(P_\delta))^{-p/(\mathfrak{p}-n)} - 1}{(d_0-\delta)^p - d_0^p} \\ &= \frac{-\dfrac{p(\mathfrak{p}-1)}{\mathfrak{p}-n}\displaystyle\sum_{k=1}^{N}\mu_{\mathfrak{p}}(P,\{u_k\})h'(u_k,0)}{-pd_0^{p-1}} \\ &= \frac{\mathfrak{p}-1}{\mathfrak{p}-n}\frac{\displaystyle\sum_{k=1}^{N}\mu_{\mathfrak{p}}(P,\{u_k\})h'(u_k,0)}{d_0^{p-1}}.\end{aligned}$$

Here, $h'(u_k,0) = \lim_{\delta\to 0^+}((h_{P_\delta}(u_k) - h_P(u_k))/\delta)$.

Assume $\mu_{\mathfrak{p}}(P,\{u_k\}) \neq 0$, for some k. Since $\mu_{\mathfrak{p}}(P,\cdot)$ is absolutely continuous with respect to the surface measure $S(P,\cdot)$, it follows that P has a facet with normal vector u_k. By the definition of P_δ, we have $h_{P_\delta}(u_k) = h_P(u_k)$, for sufficiently small $\delta > 0$. Thus, $h'(u_k, 0) = 0$ and

$$\sum_{k=1}^{N} \mu_{\mathfrak{p}}(P,\{u_k\}) h'(u_k, 0) = 0.$$

Therefore,

$$\lim_{\delta\to 0^+} \frac{\tau^p - 1}{(d_0-\delta)^p - d_0^p} = 0.$$

Combining $(d_0-\delta)^p - d_0^p < 0$, $c_{i_0} > 0$ and

$$\frac{1}{\tau^p}\sum_{k=1}^{N} c_k (h_P(u_k) - x_{P_\delta}\cdot u_k)^p \to \sum_{k=1}^{N} c_k h_P^p(u_k) > 0, \quad \text{as } \delta \to 0^+,$$

we have that for sufficiently small $\delta > 0$, $G(\delta) < 0$.

Consequently, P has exactly N facets. This completes the proof. □

Corollary 6.2. *Suppose that the polytope P solves Problem 2. Then, P has exactly N facets whose outer unit normal vectors are $u_1,\dots,u_N$.*

Lemma 6.3. *If the support set of μ is in general position, then there exists a polytope P solving Problem 2.*

Proof. We prove the case $\mathfrak{p} > n$. The $\mathfrak{p} = n$ case is proved similarly.

Take a maximizing sequence $\{P_i\}_i$ for Problem 2, such that

$$P_i \in P(u_1,\dots,u_N), \quad x_{P_i} = o, \quad F_p(P_i, x_{P_i}) = 1,$$

and

$$\lim_{i\to\infty} C_{\mathfrak{p}}(P_i) = \sup\{C_{\mathfrak{p}}(Q) : Q \in P(u_1,\dots,u_N),\ F_p(Q, x_Q) = 1\}.$$

First, we claim $\{P_i\}_i$ is bounded. Since $x_{P_i} = o$, by the definition of F_p, it follows that

$$\frac{\mathfrak{p}-1}{\mathfrak{p}-n}\sum_{k=1}^{N} c_k h_{P_i}^p(u_k) = F_p(P_i, o) = F_p(P_i, x_{P_i}) = 1.$$

Hence, for any i,

$$h_{P_i}(u_k) \leqslant \left(\frac{\mathfrak{p}-n}{(\mathfrak{p}-1)\min\limits_{1\leqslant k\leqslant N} c_k}\right)^{1/p} < \infty, \quad \text{for } k = 1,\dots,N,$$

which implies that $\{P_i\}_i$ is bounded.

By the Blaschke selection theorem, there exists a convergent subsequence $\{P_{i_j}\}_j$ of $\{P_i\}_i$ such that $P_{i_j} \to P$, as $j \to \infty$.

Next, we prove that P is n dimensional.

Assume $\dim P \leqslant n-1$, then there exists a $u_0 \in \mathbb{S}^{n-1}$ such that $P \subseteq u_0^{\perp}$. Thus,

$$h_P(u_0) = h_P(-u_0) = 0. \tag{6.1}$$

In the following, we show that P is necessarily the single point set $\{o\}$.

Since P is a polytope, from (6.1), we can find vertices x_1, x_2 of P such that

$$0 = h_P(u_0) = \max\{x \cdot u_0 : x \in P\} = x_1 \cdot u_0$$

and

$$0 = h_P(-u_0) = \max\{x \cdot (-u_0) : x \in P\} = x_2 \cdot (-u_0).$$

Among the set of support sets $\{F(P, u_i),\ i = 1, \dots, N\}$ of P, we pick up all the support sets, namely, $F(P, u_{i_j})$, $j = 1, \dots, \ell$, such that $x_1 \in F(P, u_{i_j})$. Then, $u_0 \in \operatorname{pos}\{u_{i_1}, \dots, u_{i_\ell}\}$. Without loss of generality, let

$$u_0 = \sum_{j=1}^{\ell} \alpha_j u_{i_j}, \quad \alpha_j > 0.$$

Hence,

$$\begin{aligned} 0 = h_P(u_0) = x_1 \cdot u_0 &= x_1 \cdot \Big(\sum_{j=1}^{\ell} \alpha_j u_{i_j} \Big) \\ &= \sum_{j=1}^{\ell} \alpha_j (x_1 \cdot u_{i_j}) = \sum_{j=1}^{\ell} \alpha_j h_P(u_{i_j}). \end{aligned}$$

Thus,

$$h_P(u_{i_j}) = 0, \quad j = 1, \dots, \ell. \tag{6.2}$$

Similarly, let $-u_0 = \sum_{j=1}^{\ell'} \beta_j u_{k_j}$, $\beta_j > 0$. Then,

$$h_P(u_{k_j}) = 0, \quad j = 1, \dots, \ell'. \tag{6.3}$$

Since

$$o = u_0 + (-u_0) = \sum_{j=1}^{\ell} \alpha_j u_{i_j} + \sum_{j=1}^{\ell'} \beta_j u_{k_j} = \sum_{j=1}^{q} \gamma_j u_{p_j}, \tag{6.4}$$

where $\gamma_j > 0$, we have

$$q \leqslant \ell' + \ell \qquad \text{and} \qquad \{u_{p_j}\}_{j=1}^{q} = \{u_{i_j}\}_{j=1}^{\ell} \cup \{u_{k_j}\}_{j=1}^{\ell'} \subseteq \{u_1, \dots, u_N\},$$

which implies that $u_{p_1},\dots,u_{p_q}$ are linearly dependent. Since $u_1,\dots,u_N$ are in general position, it follows that $q \geqslant n+1$ and therefore $\operatorname{lin}\{u_{p_1},\dots,u_{p_q}\} = \mathbb{R}^n$. Since the γ_j involved in (6.4) are all positive, it follows that $u_{p_1},\dots,u_{p_q}$ are not concentrated on any closed hemisphere of $\mathbb{S}^{n-1}$. For any $x \in P$, from (6.2) and (6.3), it follows that

$$x \cdot u_{p_j} \leqslant h_P(u_{p_j}) = 0, \quad \text{for } j = 1,\dots,q,$$

which implies that $x = o$, and therefore $P = \{o\}$. Thus, $\lim_{i\to\infty} C_{\mathfrak{p}}(P_i) = 0$. However,

$$\begin{aligned}\lim_{i\to\infty} C_{\mathfrak{p}}(P_i) &= \sup\{C_{\mathfrak{p}}(Q) : Q \in P(u_1,\dots,u_N),\ F_p(Q,x_Q) = 1\}\\ &\geqslant C_{\mathfrak{p}}(\tau Q_0) > 0,\end{aligned}$$

where $Q_0 = \{x : x\cdot u_k \leqslant 1,\ k = 1,\dots,N\}$ and $\tau > 0$ satisfies $F_p(\tau Q_0, x_{\tau Q_0}) = 1$. This is a contradiction.

Consequently, P is n dimensional, and this completes the proof. □

We can now conclude the proof of Theorem 1.1 and Theorem 1.2.

Theorem 1.1. *Suppose μ is a finite discrete Borel measure on $\mathbb{S}^{n-1}$, $0 < p < 1$, and $\mathfrak{p} > n$. If the support set of μ is in general position, then there exists a polytope P containing the origin in its interior, such that $\mu_{p,\mathfrak{p}}(P,\cdot) = c\mu$, where*

$$c = \begin{cases} 1 & \text{if } p \neq \dfrac{\mathfrak{p}-n}{\mathfrak{p}-1},\\[2ex] C_{\mathfrak{p}}(P)^{1/(\mathfrak{p}-1)} & \text{if } p = \dfrac{\mathfrak{p}-n}{\mathfrak{p}-1}. \end{cases}$$

Proof. For the discrete measure μ, by Lemma 6.3, there exists a polytope Q_0 which solves Problem 2. That is, $F_p(Q_0, x_{Q_0}) = 1$ and

$$C_{\mathfrak{p}}(Q_0) = \sup\{C_{\mathfrak{p}}(Q) : Q \in P(u_1,\dots,u_N),\ F_p(Q,x_Q) = 1\}.$$

By the translation invariance of $C_{\mathfrak{p}}$ and Lemma 4.3 (1), $P_0 = Q_0 - x_{Q_0}$ is still the solution to Problem 2 and $x_{P_0} = o$. Combining this with Corollary 6.2 and Lemma 5.3, we have

$$C_{\mathfrak{p}}(P_0)^{-1/(\mathfrak{p}-1)}\mu_{p,\mathfrak{p}}(P_0,\cdot) = \mu.$$

If $p \neq (\mathfrak{p}-n)/(\mathfrak{p}-1)$, by Lemma 5.4 (1), we have

$$\mu_{p,\mathfrak{p}}(C_{\mathfrak{p}}(P_0)^{1/(p(\mathfrak{p}-1)-(\mathfrak{p}-n))}P_0,\cdot) = \mu.$$

That is, $P = C_{\mathfrak{p}}(P_0)^{1/(p(\mathfrak{p}-1)-(\mathfrak{p}-n))}P_0$ is the desired solution.

If $p = (\mathfrak{p}-n)/(\mathfrak{p}-1)$, then $P = P_0$ is the desired solution. □

Theorem 1.2. *Suppose that μ is a finite discrete Borel measure on $\mathbb{S}^{n-1}$ and $0 < p < 1$. If the support set of μ is in general position, then there exists a polytope P containing the origin in its interior, such that $\mu_{p,n}(P, \cdot) = \mu$.*

Proof. For the discrete measure μ, by Lemma 6.3, there exists a polytope Q_0 which solves Problem 2; that is, $F_p(Q_0, x_{Q_0}) = 1$ and

$$C_n(Q_0) = \sup\{C_n(Q) : Q \in P(u_1, \dots, u_N), F_p(Q, x_Q) = 1\}.$$

By the translation invariance of C_n and Lemma 4.3 (1), $P = Q_0 - x_{Q_0}$ is still the solution to Problem 2 and $x_P = o$. Combining this with Corollary 6.2 and Lemma 5.5, we have $\mu_{p,n}(P, \cdot) = \mu$, which completes the proof. □

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