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Orlicz–John ellipsoids ☆

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The Orlicz–John ellipsoids, which are in the framework of the booming Orlicz Brunn–Minkowski theory, are introduced for the first time. It turns out that they are generalizations of the classical John ellipsoid and the evolved L_p John ellipsoids. The analog of Ball's volume-ratio inequality is established for the new Orlicz–John ellipsoids. The connection between the isotropy of measures and the characterization of Orlicz–John ellipsoids is demonstrated.

1. Introduction

A fundamental tool in convex geometry and Banach space geometry is the well-known *John ellipsoid*, which was originally introduced by Fritz John [26]. For each convex body (compact convex subset with nonempty interior) K in the Euclidean n-space $\mathbb{R}^n$, its John ellipsoid $\mathrm{J}K$ is the unique ellipsoid of maximal volume contained in K. For more information about the John ellipsoid, one can refer to [3,19,21,27] and the references within.

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The John ellipsoid is within the classical Brunn–Minkowski theory and is extremely useful in convex geometry, Banach space geometry and PDEs (see, e.g., [1,2,16,28,47]). One important result concerning the John ellipsoid is Ball's volume-ratio inequality, which states that: if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$\frac{|K|}{|\mathrm{J}K|} \leq \frac{2^n}{\omega_n}, \tag{1.1}$$

with equality if and only if K is a parallelotope. Here, $|\cdot|$ denotes n-dimensional volume and $\omega_n = \pi^{n/2}/\Gamma(1+\frac{n}{2})$ denotes the volume of the unit ball, B, in $\mathbb{R}^n$. The fact that there is equality in (1.1) only for parallelotopes was established by Barthe [4].

In 2005, the classical John ellipsoid had evolved into the L_p John ellipsoids under the impetus of Lutwak, Yang and Zhang [38]. In retrospect, it is interesting that it took nearly a decade for the L_p John ellipsoids to be discovered, after the emergence of the L_p Brunn–Minkowski theory initiated by Lutwak [32,33]. During the last two decades, the L_p Brunn–Minkowski theory has achieved great developments and expanded rapidly (see, e.g., [7–9,11,12,22–24,29–38,42,49,51–53,55–58]).

Suppose $p \in (0,\infty]$ and K is a convex body in $\mathbb{R}^n$ with the origin in its interior. Amongst all origin-symmetric ellipsoids E, the unique ellipsoid that solves the constrained maximization problem

$$\max_E \left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}} \quad \text{subject to} \quad \overline{V}_p(K,E) \leq 1$$

is called the L_p *John ellipsoid* [38] of K and denoted by E_pK. Here

$$\overline{V}_p(K,E) = \left[\int_{S^{n-1}} \left(\frac{h_E}{h_K}\right)^p d\overline{V}_K\right]^{\frac{1}{p}}, \quad 0 < p < \infty,$$

is the normalized L_p mixed volume of K and E; S^{n-1} is the unit sphere in $\mathbb{R}^n$; h_K and h_E are the support functions of K and E, respectively; $\overline{V}_K$ is the normalized cone-volume measure of K. For $p=\infty$, we define $\overline{V}_\infty(K,E) = \sup\{h_E(u)/h_K(u) : u \in \operatorname{supp}\overline{V}_K\}$. Note that the cone-volume measure has been appeared and investigated widely in various contexts recently (see e.g., [5,7,8,18,24,25,30,31,43,45,52,53,57]).

In general, the L_p John ellipsoid E_pK is not contained in K (except when $p=\infty$). However, when $1 \leq p \leq \infty$, it has $|\mathrm{E}_pK| \leq |K|$. If K is an origin-symmetric convex body in $\mathbb{R}^n$ and $0 < p \leq \infty$, then the L_p version [38] of Ball's volume-ratio inequality

$$\frac{|K|}{|\mathrm{E}_pK|} \leq \frac{2^n}{\omega_n}$$

still holds, with equality if and only if K is a parallelotope.

The L_p John ellipsoids provide a unified treatment for several fundamental objects in convex geometry. If the John point of K, i.e., the center of $\mathrm{J}K$, is at the origin, then $\mathrm{E}_\infty K$ is precisely the classical John ellipsoid $\mathrm{J}K$. The L_2 John ellipsoid $\mathrm{E}_2 K$ is the new ellipsoid $\Gamma_{-2}K$ previously found by Lutwak, Yang and Zhang in [34], which is now called the *LYZ ellipsoid* and is in some sense dual to the *Legendre ellipsoid of inertia* in classical mechanics [41]. The L_1 John ellipsoid $\mathrm{E}_1 K$ is the so-called *Petty ellipsoid*. The volume-normalized Petty ellipsoid is obtained by minimizing the surface area of K under $\mathrm{SL}(n)$ transformations of K. See Petty [46] and also Giannopoulos and Papadimitrakis [17].

Beginning with the ground-breaking articles of Lutwak, Yang, Zhang and Harbel [22,39,40], a more wide extension of the L_p Brunn–Minkowski theory, called the Orlicz Brunn–Minkowski theory, emerged out three years ago. In these articles, the fundamental notions of the L_p projection body and the L_p centroid body were extended to an Orlicz setting (see also [10,59]). It represents a generalization of the L_p Brunn–Minkowski theory, analogous to the way that Orlicz spaces generalize L_p spaces [48]. Very recently, one essential obstacle in the development of Orlicz Brunn–Minkowski theory, what is the lack of a notion corresponding to L_p addition, has been smoothed by Gardner, Hug and Weil [14,15].

In view of the fundamental importance of the John ellipsoid in convex geometry, we are tempted to consider the naturally posed problem in the booming Orlicz Brunn–Minkowski theory: what is the Orlicz extension of the L_p John ellipsoid? Our main task in this paper is to demonstrate this existence of such an Orlicz analogue.

For this aim, we consider convex $\varphi : [0,\infty) \to [0,\infty)$, that is strictly increasing and satisfies $\varphi(0) = 0$. For convex bodies K, L in $\mathbb{R}^n$ with the origin in their interiors, the *normalized Orlicz mixed volume* of K and L regarding φ, $\overline{V}_\varphi(K,L)$, is defined by

$$\overline{V}_\varphi(K,L) = \varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K}\right) d\overline{V}_K\right).$$

Inspired by Lutwak, Yang and Zhang's work on L_p John ellipsoids [38], we focus on

Problem S_φ. Suppose K is a convex body in $\mathbb{R}^n$ with the origin in its interior. Find an ellipsoid E, amongst all origin-symmetric ellipsoids, which solves the following constrained maximization problem:

$$\max_E |E| \quad \text{subject to} \quad \overline{V}_\varphi(K,E) \le 1.$$

In Section 4, we prove that there exists a unique ellipsoid which solves Problem S_φ. It is called the *Orlicz–John ellipsoid* of K, and denoted by $\mathrm{E}_\varphi K$. Note that if $\varphi(t) = t^p$, $1 \le p < \infty$, then the Orlicz–John ellipsoid $\mathrm{E}_\varphi K$ precisely turns out to be the L_p John ellipsoid $\mathrm{E}_p K$.

An important feature on the family of L_p John ellipsoids is that $\mathrm{E}_p K$ is continuous in $p \in (0, \infty]$. In Section 5, we show that the Orlicz–John ellipsoid $\mathrm{E}_\varphi K$ is jointly continuous in φ and K. In Section 6, we prove that as $p \to \infty$, the Orlicz–John ellipsoid $\mathrm{E}_{\varphi^p} K$ approaches to $\mathrm{E}_\infty K$. This insight throws light on a connection between Orlicz–John ellipsoids and the classical John ellipsoid. The Orlicz version of Ball's volume-ratio inequality is established in Section 7. Finally, we provide a characterization of the Orlicz–John ellipsoid, which is closely related to the isotropy of measures.

2. Preliminaries

For quick reference we recall some basic results from the Brunn–Minkowski theory. Good references are Gardner [13], Gruber [20], Schneider [50], and Thompson [54].

The setting will be Euclidean n-space $\mathbb{R}^n$. As usual, $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$.

In addition to its denoting absolute value, without confusion we will use $|\cdot|$ to denote the standard Euclidean norm on $\mathbb{R}^n$, often to denote n-dimensional volume, and on occasion to denote the absolute value of the determinant of an $n \times n$ matrix.

For $x \in \mathbb{R}^n$, let $\langle x \rangle = |x|^{-1} x$, whenever $x \neq 0$.

Throughout, $\mathcal{E}^n$ is used exclusively to denote the class of origin-symmetric ellipsoids in $\mathbb{R}^n$. We write $\mathcal{K}_o^n$ for the set of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. The support function of a convex body $K \in \mathcal{K}_o^n$, h_K, is defined for all $x \in \mathbb{R}^n$ by $h_K(x) = \max\{x \cdot y : y \in K\}$. If $T \in \mathrm{GL}(n)$, then for the support function of the image $TK = \{Tx : x \in K\}$, we obviously have

$$h_{TK}(x) = h_K\big(T^t x\big), \tag{2.1}$$

where T^t denotes the transpose of T.

The set $\mathcal{K}_o^n$ is often equipped with the Hausdorff metric δ_H, which is defined for $K_1, K_2 \in \mathcal{K}_o^n$ by $\delta_H(K_1, K_2) = \max_{S^{n-1}} |h_{K_1} - h_{K_2}|$.

The classical Aleksandrov–Fenchel–Jessen *surface area measure*, S_K, of the convex body K can be defined as the unique Borel measure on S^{n-1} such that

$$\int_{S^{n-1}} f(u) dS_K(u) = \int_{\partial K} f\big(\gamma_K(y)\big) d\mathcal{H}^{n-1}(y)$$

for each continuous $f : S^{n-1} \to \mathbb{R}$, where $\gamma_K(y)$ is the outer unit normal of ∂K at $y \in \partial K$. Recall that γ_K exists almost everywhere for $y \in \partial K$ with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on ∂K.

The *cone-volume measure*, V_K, of the convex body K is a Borel measure on S^{n-1} defined for a Borel set $\omega \subseteq S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n} \int_\omega h_K dS_K.$$

It is convenient to use the *normalized* cone-volume measure $\overline{V}_K = \frac{V_K}{|K|}$, of K. Observe that $\overline{V}_K$ is a probability measure on S^{n-1}. Also, $\overline{V}_K$ is GL(n)-invariant, i.e., for $T \in \mathrm{GL}(n)$ and a Borel subset $\omega \subseteq S^{n-1}$, it yields

$$\overline{V}_{T^t K}(\omega) = \overline{V}_K\big(\langle T\omega\rangle\big), \tag{2.2}$$

where $\langle T\omega\rangle = \{\langle Tu\rangle : u \in \omega\}$.

The projection body, ΠK, of the convex body $K \in \mathcal{K}_o^n$, is a convex body in $\mathcal{K}_o^n$ whose support function is defined for $u \in S^{n-1}$ by

$$h_{\Pi K}(u) = \frac{1}{2}\int\limits_{S^{n-1}} |u\cdot v| dS_K(v).$$

According to the definitions of ΠK, S_K and $\overline{V}_K$, it immediately yields that for $u \in S^{n-1}$,

$$\frac{2h_{\Pi K}(u)}{n|K|} = \int\limits_{S^{n-1}} \frac{|u\cdot v|}{h_K(v)} d\overline{V}_K(v).$$

A finite positive Borel measure μ on S^{n-1} is said to be *isotropic* if

$$\int\limits_{S^{n-1}} (u\cdot v)^2 d\mu(u) = \frac{|\mu|}{n},$$

for all $v \in S^{n-1}$. Here, $|\mu|$ denotes the total mass of μ. For nonzero $x \in \mathbb{R}^n$, the notation $x\otimes x$ represents the rank 1 linear operator on $\mathbb{R}^n$ that takes y to $(x\cdot y)x$. It immediately gives that

$$\mathrm{tr}(x\otimes x) = |x|^2.$$

Equivalently, μ is isotropic if

$$\int\limits_{S^{n-1}} u\otimes u d\mu(u) = \frac{|\mu|}{n} I_n,$$

where I_n denotes the identity operator on $\mathbb{R}^n$. For more information about the importance of isotropy, one can refer to [6,16,17,41,44].

Let Φ be the class of convex functions $\varphi : [0,\infty) \to [0,\infty)$ that are strictly increasing and satisfy $\varphi(0) = 0$. We say that a sequence $\{\varphi_i\}_{i\in\mathbb{N}} \subset \Phi$ is such that $\varphi_i \to \varphi_0 \in \Phi$, provided

$$|\varphi_i - \varphi_0|_I := \max_{t\in I}\big|\varphi_i(t) - \varphi_0(t)\big| \to 0,$$

for each compact interval $I \subset [0,\infty)$.

Let μ be a finite Borel measure on S^{n-1}. For a continuous function $f : S^{n-1} \to [0, \infty)$, the Orlicz norm $\|f\|_\varphi$ of f is defined by

$$\|f\|_\varphi = \inf\left\{\lambda > 0 : \frac{1}{|\mu|} \int_{S^{n-1}} \varphi\left(\frac{f}{\lambda}\right) d\mu \le \varphi(1)\right\}.$$

The following Lemma 2.1 was previously proved in [22]. Lemma 2.2 is explicitly appeared in the proof of Lemma 2.1, which will be used frequently throughout this paper.

Lemma 2.1. *Suppose μ is a finite Borel measure on S^{n-1} and the function $f : S^{n-1} \to [0, \infty)$ is continuous and such that $\mu(\{f \ne 0\}) > 0$. Then the Orlicz norm $\|f\|_\varphi$ is positive and*

$$\|f\|_\varphi = \lambda_0 \quad \Longleftrightarrow \quad \frac{1}{|\mu|} \int_{S^{n-1}} \varphi\left(\frac{f}{\lambda_0}\right) d\mu = \varphi(1).$$

Lemma 2.2. *Suppose μ is a finite Borel measure on S^{n-1} and the function $f : S^{n-1} \to [0, \infty)$ is continuous and such that $\mu(\{f \ne 0\}) > 0$. Then the function*

$$\psi(\lambda) := \int_{S^{n-1}} \varphi\left(\frac{f}{\lambda}\right) d\mu, \quad \lambda \in (0, \infty),$$

has the following properties:

(1) *ψ is continuous and strictly decreasing in $(0, \infty)$;*
(2) *$\lim_{\lambda \to 0^+} \psi(\lambda) = \infty$;*
(3) *$\lim_{\lambda \to \infty} \psi(\lambda) = 0$;*
(4) *$0 < \psi^{-1}(a) < \infty$ for each $a \in (0, \infty)$.*

For $K, L \in \mathcal{K}_o^n$ and $\varepsilon \ge 0$, we define the function $h_{\varphi,\varepsilon} : \mathbb{R}^n \to [0, \infty)$ as

$$h_{\varphi,\varepsilon}(x) = \inf\left\{\lambda > 0 : \varphi\left(\frac{h_K(x)}{\lambda}\right) + \varepsilon\varphi\left(\frac{h_L(x)}{\lambda}\right) \le \varphi(1)\right\}.$$

Observe that $h_{\varphi,\varepsilon}$ is both sublinear and positive when $x \ne 0$. Hence, there exists a unique convex body $K +_{\varphi,\varepsilon} L \in \mathcal{K}_o^n$ such that whose support function is precisely $h_{\varphi,\varepsilon}$. According to Lemmas 8.2 and 8.4 in [15], it gives that

$$K +_{\varphi,\varepsilon} L \to K, \quad \text{as } \varepsilon \to 0^+,$$

and

$$\lim_{\varepsilon \to 0^+} \frac{h_{K +_{\varphi,\varepsilon} L}(u) - h_K(u)}{\varepsilon} = \frac{h_K(u)}{\varphi'_-(1)} \varphi\left(\frac{h_L(u)}{h_K(u)}\right), \tag{2.3}$$

uniformly for $u \in S^{n-1}$, where $\varphi'_-(1)$ denotes the left derivative of φ at 1. Note that $h(\varepsilon, u) = h_{\varphi,\varepsilon}(u) : [0, \infty) \times S^{n-1} \to [0, \infty)$ is jointly continuous in ε and u (see Lemma A.1 in Appendix A for details). So, by (2.3) and Aleksandrov's variational principle (see Lemma 3.1 in [8] or Lemma 8.3 in [15]), it yields that

$$\lim_{\varepsilon\to 0^+} \frac{|K +_{\varphi,\varepsilon} L| - |K|}{\varepsilon} = \int_{S^{n-1}} \lim_{\varepsilon\to 0^+} \frac{h_{K+_{\varphi,\varepsilon}L}(u) - h_K(u)}{\varepsilon} dS_K(u)$$
$$= \frac{n}{\varphi'_-(1)} \int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K}\right) dV_K.$$

Now, we are in the position to give the definition of Orlicz mixed volume.

Definition 2.3. Let $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. The geometric quantity

$$V_\varphi(K, L) = \int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K}\right) dV_K$$

is called the *Orlicz mixed volume* of K and L regarding φ. The *normalized Orlicz mixed volume* $\overline{V}_\varphi(K, L)$, of K and L regarding φ, is defined by

$$\overline{V}_\varphi(K, L) = \varphi^{-1}\left(\frac{V_\varphi(K, L)}{|K|}\right) = \varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K}\right) d\overline{V}_K\right).$$

If $\varphi(t) = t^p$, $1 \le p < \infty$, then $V_\varphi(K, L)$ and $\overline{V}_\varphi(K, L)$ reduce to the L_p mixed volume $V_p(K, L)$ and the normalized L_p mixed volume $\overline{V}_p(K, L)$ used in [38], respectively.

Lemma 2.4. *Suppose $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then*

(1) $\overline{V}_\varphi(K, K) = 1$.
(2) $V_\varphi(K, K) = \varphi(1)|K|$.
(3) $\overline{V}_\varphi(K, \lambda L) = \overline{V}_\varphi(\lambda^{-1}K, L)$, *for all* $\lambda > 0$.
(4) $\overline{V}_\varphi(K, TL) = \overline{V}_\varphi(T^{-1}K, L)$, *for all* $T \in \mathrm{GL}(n)$.
(5) $V_\varphi(K, TL) = |T| V_\varphi(T^{-1}K, L)$, *for all* $T \in \mathrm{GL}(n)$.

Proof. From Definition 2.3, it immediately gives (1) and (2).

Combining Definition 2.3 with the facts that

$$\frac{h_{\lambda L}}{h_K} = \frac{h_L}{h_{\lambda^{-1}K}} \quad \text{and} \quad \overline{V}_K = \overline{V}_{\lambda^{-1}K},$$

it yields (3) directly.

From (2.1), (2.2) and Definition 2.3, it follows that

$$\begin{aligned}\overline{V}_\varphi(K,TL) &= \varphi^{-1}\left(\int\limits_{S^{n-1}} \varphi\left(\frac{h_L(T^t u)}{h_K(T^{-t}T^t u)}\right) d\overline{V}_K(u)\right)\\ &= \varphi^{-1}\left(\int\limits_{S^{n-1}} \varphi\left(\frac{h_L(\langle T^t u\rangle)}{h_{T^{-1}K}(\langle T^t u\rangle)}\right) d\overline{V}_{T^{-1}K}\left(\langle T^t u\rangle\right)\right)\\ &= \overline{V}_\varphi\left(T^{-1}K,L\right).\end{aligned}$$

From Definition 2.3 and (4), it follows that

$$\varphi^{-1}\left(\frac{V_\varphi(K,TL)}{|K|}\right) = \varphi^{-1}\left(\frac{V_\varphi(T^{-1}K,L)}{|T^{-1}K|}\right),$$

which yields (5) directly. □

Following GHW [15], we introduce the important geometric quantity $\widehat{V}_\varphi(K,L)$.

Definition 2.5. Let $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. The geometric quantity

$$\widehat{V}_\varphi(K,L) = \inf\left\{\lambda > 0 : \int\limits_{S^{n-1}} \varphi\left(\frac{h_L}{\lambda h_K}\right) d\overline{V}_K \le \varphi(1)\right\}$$

is called the *quasi-Orlicz mixed volume* of K and L regarding φ.

If $\varphi(t) = t^p$, $1 \le p < \infty$, then $\widehat{V}_\varphi(K,L)$ also reduces to the normalized L_p mixed volume $\overline{V}_p(K,L)$. Since

$$\widehat{V}_\varphi(K,L) = \left\|\frac{h_L}{h_K}\right\|_\varphi,$$

where $\|\cdot\|_\varphi$ is the Orlicz norm with respect to the measure $\overline{V}_K$, we obtain the following lemma immediately.

Lemma 2.6. *Suppose $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then*

(1) $\widehat{V}_\varphi(K,K) = 1$.
(2) $\widehat{V}_\varphi(K,\lambda L) = \widehat{V}_\varphi(\lambda^{-1}K,L) = \lambda \widehat{V}_\varphi(K,L)$, *for all* $\lambda > 0$.
(3) $\widehat{V}_\varphi(TK,L) = \widehat{V}_\varphi(K,T^{-1}L)$, *for all* $T \in \mathrm{GL}(n)$.

In what follows, we provide a simple identity, which will be used frequently.

Lemma 2.7. *Suppose $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then*

$$\overline{V}_\varphi\left(K, \frac{L}{\widehat{V}_\varphi(K,L)}\right) = 1.$$

Proof. From Definition 2.3, Definition 2.5, and then Lemma 2.1, it follows that

$$\varphi\big(\overline{V}_\varphi\big(K, \widehat{V}_\varphi(K,L)^{-1}L\big)\big) = \int_{S^{n-1}} \varphi\left(\frac{h_L}{\widehat{V}_\varphi(K,L)h_K}\right) d\overline{V}_K = \varphi(1).$$

Thus,

$$\overline{V}_\varphi\left(K, \frac{L}{\widehat{V}_\varphi(K,L)}\right) = 1,$$

as desired. □

3. The continuity of Orlicz mixed volumes

In this section, we show the continuity of the functionals $\overline{V}_\varphi(K,L)$ and $\widehat{V}_\varphi(K,L)$ with respect to φ, K and L, which will be needed in Section 5.

Theorem 3.1. *Suppose $K, K_i, L, L_j \in \mathcal{K}_o^n$ and $\varphi, \varphi_k \in \Phi$, where $i, j, k \in \mathbb{N}$. If $K_i \to K$, $L_j \to L$ and $\varphi_k \to \varphi$, then*

$$\lim_{i,j,k\to\infty} V_{\varphi_k}(K_i, L_j) = V_\varphi(K,L),$$

and

$$\lim_{i,j,k\to\infty} \overline{V}_{\varphi_k}(K_i, L_j) = \overline{V}_\varphi(K,L).$$

Proof. Let

$$c_m = \frac{\inf(\{\min_{S^{n-1}} h_L\} \cup \{\min_{S^{n-1}} h_{L_j} : j \in \mathbb{N}\})}{\sup(\{\max_{S^{n-1}} h_K\} \cup \{\max_{S^{n-1}} h_{K_i} : i \in \mathbb{N}\})},$$

and

$$c_M = \frac{\sup(\{\max_{S^{n-1}} h_L\} \cup \{\max_{S^{n-1}} h_{L_j} : j \in \mathbb{N}\})}{\inf(\{\min_{S^{n-1}} h_K\} \cup \{\min_{S^{n-1}} h_{K_i} : i \in \mathbb{N}\})}.$$

First, we prove

$$0 < c_m \le c_M < \infty.$$

From the definition of Hausdorff metric, we know that $K_i \to K$ and $L_j \to L$ are equivalent to $h_{K_i} \to h_K$ and $h_{L_j} \to h_L$ uniformly on S^{n-1}, respectively. Furthermore, since h_K, h_L, h_{K_i}, and $h_{L_j}, i, j \in \mathbb{N}$, are strictly positive on S^{n-1}, it follows that there exists an $N_0 \in \mathbb{N}$, such that for all $i, j > N_0$ and $u \in S^{n-1}$,

$$\min_{S^{n-1}} h_{\frac{K}{2}} \le h_{K_i}(u) \le \max_{S^{n-1}} h_{2K} \quad \text{and} \quad \min_{S^{n-1}} h_{\frac{L}{2}} \le h_{L_j}(u) \le \max_{S^{n-1}} h_{2L}.$$

Let

$$b_m = \min\Big\{\min_{S^{n-1}} h_{\frac{K}{2}}, \min_{S^{n-1}} h_{\frac{L}{2}}, \min_{1\le i\le N_0} \min_{S^{n-1}} h_{K_i}, \min_{1\le j\le N_0} \min_{S^{n-1}} h_{L_j}\Big\},$$

and

$$b_M = \max\Big\{\max_{S^{n-1}} h_{2K}, \max_{S^{n-1}} h_{2L}, \max_{1\le i\le N_0} \max_{S^{n-1}} h_{K_i}, \max_{1\le j\le N_0} \max_{S^{n-1}} h_{L_j}\Big\}.$$

Then, $0 < b_m < b_M < \infty$. Meanwhile, we have

$$\begin{aligned} b_m B \subseteq K \subseteq b_M B, &\qquad b_m B \subseteq K_i \subseteq b_M B, \quad \text{for } i \in \mathbb{N}, \\ b_m B \subseteq L \subseteq b_M B, &\qquad b_m B \subseteq L_j \subseteq b_M B, \quad \text{for } j \in \mathbb{N}. \end{aligned}$$

Thus, by the definitions of c_m and c_M, it yields

$$0 < \frac{b_m}{b_M} \le c_m \le c_M \le \frac{b_M}{b_m} < \infty,$$

as desired.

Next, we prove

$$\lim_{i,j,k\to\infty} V_{\varphi_k}(K_i, L_j) = V_\varphi(K, L).$$

Let $\varepsilon > 0$. Three observations are in order.

Firstly, since $\{\varphi_k\}$ converges uniformly to φ on $[c_m, c_M]$, by $c_m \le \frac{h_{L_j}(u)}{h_{K_i}(u)} \le c_M$ for all $u \in S^{n-1}$, there exists an $N_1 \in \mathbb{N}$, such that for all $k \ge N_1$,

$$\left|\varphi_k\left(\frac{h_{L_j}(u)}{h_{K_i}(u)}\right) - \varphi\left(\frac{h_{L_j}(u)}{h_{K_i}(u)}\right)\right| < \frac{\varepsilon}{3} \tag{3.1}$$

holds independently of i and j and uniformly for $u \in S^{n-1}$.

Secondly, there exists an $N_2 \in \mathbb{N}$, such that

$$\left|\varphi\left(\frac{h_{L_j}(u)}{h_{K_i}(u)}\right) - \varphi\left(\frac{h_L(u)}{h_K(u)}\right)\right| < \frac{\varepsilon}{3} \tag{3.2}$$

holds uniformly for $u \in S^{n-1}$ and for all $i, j \geq N_2$. Indeed, since the convex function φ is Lipschitzian on $[c_m, c_M]$, there exists a constant $C > 0$, such that for all $u \in S^{n-1}$,

$$\left|\varphi\left(\frac{h_{L_j}(u)}{h_{K_i}(u)}\right) - \varphi\left(\frac{h_L(u)}{h_K(u)}\right)\right| \leq C\left|\frac{h_{L_j}(u)}{h_{K_i}(u)} - \frac{h_L(u)}{h_K(u)}\right| \leq C \cdot \frac{\delta_H(L_j, L)\max_{S^{n-1}} h_K + \delta_H(K_i, K)\max_{S^{n-1}} h_L}{\min_{S^{n-1}} h_{K_i} \cdot \min_{S^{n-1}} h_K}.$$

Thirdly, since the measure sequence $\{\overline{V}_{K_i}\}$ weakly converges to $\overline{V}_K$, there exists an $N_3 \in \mathbb{N}$, such that for all $i \geq N_3$,

$$\left|\int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K}\right) d\overline{V}_{K_i} - \int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K}\right) d\overline{V}_K\right| < \frac{\varepsilon}{3}. \tag{3.3}$$

In terms of (3.1), (3.2) and (3.3), it follows that for all $i, j, k \geq \max\{N_1, N_2, N_3\}$,

$$\begin{aligned}
&\left|\int_{S^{n-1}} \varphi_k\left(\frac{h_{L_j}}{h_{K_i}}\right) d\overline{V}_{K_i} - \int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K}\right) d\overline{V}_K\right| \\
&\leq \int_{S^{n-1}} \left|\varphi_k\left(\frac{h_{L_j}}{h_{K_i}}\right) - \varphi\left(\frac{h_{L_j}}{h_{K_i}}\right)\right| d\overline{V}_{K_i} \\
&\quad + \int_{S^{n-1}} \left|\varphi\left(\frac{h_{L_j}}{h_{K_i}}\right) - \varphi\left(\frac{h_L}{h_K}\right)\right| d\overline{V}_{K_i} \\
&\quad + \left|\int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K}\right) d\overline{V}_{K_i} - \int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K}\right) d\overline{V}_K\right| \\
&< \varepsilon.
\end{aligned}$$

That is,

$$\lim_{i,j,k\to\infty} \frac{V_{\varphi_k}(K_i, L_j)}{|K_i|} = \frac{V_\varphi(K, L)}{|K|}.$$

Combined with the fact $|K_i| \to |K|$, it yields the first conclusion.

Finally, we proceed to prove

$$\lim_{i,j,k\to\infty} \overline{V}_{\varphi_k}(K_i, L_j) = \overline{V}_\varphi(K, L).$$

Let

$$a_m = \inf\left(\{\phi(c_m)\} \cup \{\phi_k(c_m) : k \in \mathbb{N}\}\right),$$

and

$$a_M = \sup\big(\{\phi(c_M)\} \cup \{\phi_k(c_M) : k \in \mathbb{N}\}\big).$$

Then, $0 < a_m \le a_M < \infty$.

For brevity, let

$$a_{i,j,k} = \frac{V_{\varphi_k}(K_i, L_j)}{|K_i|} \quad \text{and} \quad a = \frac{V_\varphi(K, L)}{|K|}.$$

Thus, to show the desired limit, it suffices to show

$$\lim_{i,j,k\to\infty} {\varphi_k}^{-1}(a_{i,j,k}) = \varphi^{-1}(a).$$

Since

$$\varphi_l \to \varphi \quad \Rightarrow \quad \varphi_l \to \varphi, \quad \text{uniformly on } [c_m, c_M],$$

it implies

$${\varphi_l}^{-1} \to \varphi^{-1}, \quad \text{uniformly on } [a_m, a_M].$$

Note that

$$a_{i,j,k}, a \in [a_m, a_M], \quad \text{for each } i, j, k.$$

So, for any $\varepsilon' > 0$, there exists an $N_4 \in \mathbb{N}$, such that for all $l \ge N_4$,

$$\big|{\varphi_l}^{-1}(a_{i,j,k}) - \varphi^{-1}(a_{i,j,k})\big| < \frac{\varepsilon'}{2} \tag{3.4}$$

holds independently of i, j and k. By the first conclusion and the continuity of φ^{-1} on $[a_m, a_M]$, there exists an $N_5 \in \mathbb{N}$, such that for all $i, j, k > N_5$,

$$\big|\varphi^{-1}(a_{i,j,k}) - \varphi^{-1}(a)\big| < \frac{\varepsilon'}{2}. \tag{3.5}$$

In terms of (3.4) and (3.5), it implies that for $i, j, k \ge \max\{N_4, N_5\}$,

$$\big|{\varphi_k}^{-1}(a_{i,j,k}) - \varphi^{-1}(a)\big| < \varepsilon'.$$

This completes the proof. □

Theorem 3.2. *Suppose $K, K_i, L, L_j \in \mathcal{K}_o^n$ and $\varphi, \varphi_k \in \Phi$, where $i, j, k \in \mathbb{N}$. If $K_i \to K$, $L_j \to L$ and $\varphi_k \to \varphi$, then*

$$\lim_{i,j,k\to\infty} \widehat{V}_{\varphi_k}(K_i, L_j) = \widehat{V}_\varphi(K, L).$$

Proof. Let c_m and c_M be the numbers in the proof of Theorem 3.1. For each i, j, k, let

$$\psi_{(i,j,k)}(\lambda) = \int_{S^{n-1}} \varphi_k\left(\frac{h_{L_j}}{\lambda h_{K_i}}\right) d\overline{V}_{K_i},$$

$$\psi(\lambda) = \int_{S^{n-1}} \varphi\left(\frac{h_L}{\lambda h_K}\right) d\overline{V}_K,$$

$$\psi_m^{(k)}(\lambda) = \varphi_k\left(\frac{c_m}{\lambda}\right),$$

$$\psi_M^{(k)}(\lambda) = \varphi_k\left(\frac{c_M}{\lambda}\right),$$

where $\lambda \in (0, \infty)$.

For brevity, let

$$\lambda_0 = \widehat{V}_\varphi(K, L) \quad \text{and} \quad \lambda_{(i,j,k)} = \widehat{V}_{\varphi_k}(K_i, L_j).$$

From

$$\psi_m^{(k)} \le \psi \le \psi_M^{(k)}, \qquad \psi_m^{(k)} \le \psi_{(i,j,k)} \le \psi_M^{(k)}, \quad \text{for } i, j, k \in \mathbb{N},$$

and Lemma 2.2, it yields that

$$c_m = \left(\psi_m^{(k)}\right)^{-1}\left(\varphi_k(1)\right) \le \left(\psi_{(i,j,k)}\right)^{-1}\left(\varphi_k(1)\right) \le \left(\psi_M^{(k)}\right)^{-1}\left(\varphi_k(1)\right) = c_M.$$

That is,

$$c_m \le \lambda_{(i,j,k)} \le c_M, \quad \text{for } i, j, k \in \mathbb{N}.$$

Thus, to show that the sequence $\{\lambda_{(i,j,k)}\}_{i,j,k}$ converges to λ_0 as $i, j, k \to \infty$, it suffices to show each convergent subsequence $\{\lambda_{(i_p,j_q,k_r)}\}_{p,q,r\in\mathbb{N}}$ converges to λ_0 when $i_p, j_q, k_r \to \infty$.

Assume that $\lim_{p,q,r\to\infty} \lambda_{(i_p,j_q,k_r)} = {\lambda_0}'$. From Lemma 2.7 and Theorem 3.1, it follows that

$$1 = \lim_{p,q,r\to\infty} \overline{V}_{\varphi_{k_r}}\left(K_{i_p}, \frac{L_{j_q}}{\lambda_{(i_p,j_q,k_r)}}\right) = \overline{V}_\varphi\left(K, \frac{L}{{\lambda_0}'}\right),$$

which in turn gives $\psi(\lambda_0{}') = \varphi(1)$. Lemma 2.2 guarantees that $\lambda_0{}' = \psi^{-1}(\varphi(1)) = \lambda_0$. This completes the proof. $\square$

4. Orlicz–John ellipsoids

In this section, we focus on the main Problem S_φ posed in Section 1.

Problem S_φ. Given a convex body K in $\mathbb{R}^n$ that contains the origin in its interior, find an ellipsoid E, amongst all origin-symmetric ellipsoids, which solves the following constrained maximization problem:

$$\max_E |E| \quad \text{subject to} \quad \overline{V}_\varphi(K, E) \leq 1.$$

Lemma 4.1. *There exists a solution to Problem S_φ.*

Proof. Given an ellipsoid $E \in \mathcal{E}^n$, we use d_E to denote its maximal principal radius. There exists a $v_E \in S^{n-1}$ such that $d_E|v_E \cdot u| \leq h_E(u)$ for all $u \in S^{n-1}$. From the strict monotonicity and convexity of φ, together with Jensen's inequality, it follows that

$$\begin{aligned}
\frac{2d_E}{n|K|} \min_{S^{n-1}} h_{\Pi K} &\leq \frac{2d_E}{n|K|} h_{\Pi K}(v_E) \\
&= \int_{S^{n-1}} \frac{d_E|u \cdot v_E|}{h_K(u)} d\overline{V}_K(u) \\
&\leq \varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{d_E|u \cdot v_E|}{h_K(u)}\right) d\overline{V}_K(u)\right) \\
&\leq \varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{h_E}{h_K}\right) d\overline{V}_K\right) \\
&= \overline{V}_\varphi(K, E).
\end{aligned}$$

Let $\mathcal{E}_\varphi = \{E \in \mathcal{E}^n : \overline{V}_\varphi(K, E) \leq 1\}$. Then, the above inequalities yield that

$$d_E \leq \frac{n|K|}{2\min_{S^{n-1}} h_{\Pi K}},$$

for all $E \in \mathcal{E}_\varphi$. Thus, the set $\mathcal{E}_\varphi$ is bounded in the metric space $(\mathcal{E}^n, \delta_H)$. According to Theorem 3.1, the functional $\overline{V}_\varphi(K, \cdot)$ is continuous. So, $\mathcal{E}_\varphi$ is also closed. By the Blaschke selection theorem, each maximizing sequence of ellipsoids for Problem S_φ has a convergent subsequence whose limit is still in $\mathcal{E}_\varphi$. Therefore, a solution to Problem S_φ exists. $\square$

Theorem 4.2. *There exists a unique solution to Problem S_φ.*

Proof. We argue by contradiction. Assume that there are two different solutions E_1 and E_2 to Problem S_φ. Let $E_1 = T_1 B$ and $E_2 = T_2 B$, where $T_1, T_2 \in \mathrm{GL}(n)$. Then, $\det(T_1) = \det(T_2)$ and $\varphi(\overline{V}_\varphi(K, E_i)) \leq \varphi(1)$, for $i = 1, 2$.

Since each $T \in \mathrm{GL}(n)$ can be represented in the form $T = PQ$, where P is symmetric, positive definite and Q is orthogonal, we may assume that T_1 and T_2 are symmetric and positive definite. Then $T_1 \neq \lambda T_2$, for all $\lambda > 0$. From the Minkowski inequality for positive definite matrices, it gives that

$$\det\left(\frac{T_1}{2} + \frac{T_2}{2}\right)^{\frac{1}{n}} > \frac{1}{2}\det(T_1)^{\frac{1}{n}} + \frac{1}{2}\det(T_2)^{\frac{1}{n}}.$$

Let $E_3 = \frac{T_1+T_2}{2} B$. Then we have

$$|E_3| > |E_1| = |E_2|. \tag{4.1}$$

From (2.1) and the triangle inequality, it yields that for all $u \in S^{n-1}$,

$$h_{E_3}(u) = \left|\frac{T_1^t + T_2^t}{2} u\right| \leq \frac{|T_1^t u| + |T_2^t u|}{2} = \frac{h_{E_1}(u) + h_{E_2}(u)}{2}. \tag{4.2}$$

Now, from Definition 2.3, the monotonicity of φ together with (4.2), and the convexity of φ, it follows that

$$\begin{aligned}
\varphi\big(\overline{V}_\varphi(K, E_3)\big) &= \int_{S^{n-1}} \varphi\left(\frac{h_{E_3}}{h_K}\right) d\overline{V}_K \\
&\leq \int_{S^{n-1}} \varphi\left(\frac{h_{E_1} + h_{E_2}}{2h_K}\right) d\overline{V}_K \\
&\leq \frac{1}{2}\varphi\big(\overline{V}_\varphi(K, E_1)\big) + \frac{1}{2}\varphi\big(\overline{V}_\varphi(K, E_2)\big) \\
&\leq \varphi(1).
\end{aligned}$$

That is, E_3 satisfies the constraint $\overline{V}_\varphi(K, E_3) \leq 1$. Then, it will result in

$$|E_3| \leq |E_1| = |E_2|,$$

which contradicts (4.1). □

Up to now, we proved the existence and uniqueness of the solution to Problem S_φ. In light of the close connection of the functionals $\overline{V}_\varphi(K, E)$ and $\widehat{V}_\varphi(K, E)$, we can give an alternative formulation to Problem S_φ.

Lemma 4.3. *The following propositions hold:*

(1) $\max_{\{E\in\mathcal{E}^n:\overline{V}_\varphi(K,E)\le 1\}}|E| = \max_{\{E\in\mathcal{E}^n:\overline{V}_\varphi(K,E)=1\}}|E|$.
(2) $\{E\in\mathcal{E}^n : \overline{V}_\varphi(K,E)=1\} = \{E\in\mathcal{E}^n : \widehat{V}_\varphi(K,E)=1\}$.
(3) $\max_{\{E\in\mathcal{E}^n:\widehat{V}_\varphi(K,E)=1\}}|E| = \max_{\{E\in\mathcal{E}^n:\widehat{V}_\varphi(K,E)\le 1\}}|E|$.

Proof. To prove (1), it suffices to prove $E_1\in\mathcal{E}^n$ cannot be a solution to Problem S_φ if $\overline{V}_\varphi(K,E_1)<1$. Indeed, Lemmas 2.2 and 2.7 imply

$$0<\widehat{V}_\varphi(K,E_1)<1,$$

whenever $\overline{V}_\varphi(K,E_1)<1$. Thus,

$$\bigl|\widehat{V}_\varphi(K,E_1)^{-1}E_1\bigr|>|E_1|.$$

On the other hand, Lemma 2.7 guarantees that

$$\widehat{V}_\varphi(K,E_1)^{-1}E_1\in\bigl\{E\in\mathcal{E}^n:\overline{V}_\varphi(K,E)\le 1\bigr\}.$$

Therefore, (1) holds.

For an ellipsoid $E\in\mathcal{E}^n$, Lemma 2.7 guarantees that

$$\overline{V}_\varphi(K,E)=1 \quad\Longleftrightarrow\quad \widehat{V}_\varphi(K,E)=1,$$

which immediately yields (2).

Let E_2 be an ellipsoid in $\mathcal{E}^n$, such that $\widehat{V}_\varphi(K,E_2)<1$. Lemma 2.6 implies that

$$\widehat{V}_\varphi\bigl(K,\widehat{V}_\varphi(K,E_2)^{-1}E_2\bigr)=1.$$

Consequently,

$$\widehat{V}_\varphi(K,E_2)^{-1}E_2\in\bigl\{E\in\mathcal{E}^n:\widehat{V}_\varphi(K,E)\le 1\bigr\}.$$

However,

$$\bigl|\widehat{V}_\varphi(K,E_2)^{-1}E_2\bigr|>|E_2|.$$

From propositions (1), (2) and Lemma 4.1, it yields (3). □

At this stage, we can reformulate Problem S_φ as the following: given a convex body $K\in\mathcal{K}_o^n$, find an ellipsoid E, amongst all origin-symmetric ellipsoids, which solves the constrained maximization problem:

$$\max_E\frac{|E|}{\omega_n}\quad\text{subject to}\quad\widehat{V}_\varphi(K,E)\le 1.$$

Following the routine of LYZ [38], we can similarly propose Problem $\overline{S}_\varphi$, which is in some sense dual to Problem S_φ.

Problem $\overline{S}_\varphi$. Given a convex body $K \in \mathcal{K}_o^n$, find an ellipsoid E, amongst all origin-symmetric ellipsoids, which solves the following constrained minimization problem:

$$\min_E \widehat{V}_\varphi(K,E) \quad \text{subject to} \quad \frac{|E|}{\omega_n} \geq 1.$$

If $\varphi(t) = t^p$, $1 \leq p < \infty$, then Problem $\overline{S}_\varphi$ turns to Problem $\overline{S}_p$, which was originally discussed by LYZ in [38]. In what follows, we show an interesting fact that the solutions to Problem S_φ and Problem $\overline{S}_\varphi$ only differ by a scale factor.

Theorem 4.4. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$.*

(1) *Let E_M be the unique solution to Problem S_φ, then*

$$\left(\frac{\omega_n}{|E_M|}\right)^{\frac{1}{n}} E_M$$

is a solution to Problem $\overline{S}_\varphi$.

(2) *If E_m is a solution to Problem $\overline{S}_\varphi$, then*

$$\widehat{V}_\varphi(K,E_m)^{-1} E_m$$

is a solution to Problem S_φ.

Consequently, there exists a unique solution to Problem $\overline{S}_\varphi$.

Proof. (1) For any $E \in \{E \in \mathcal{E}^n : |E| \geq \omega_n\}$, it obviously has

$$\widehat{V}_\varphi(K,E)^{-1} E \in \big\{E \in \mathcal{E}^n : \widehat{V}_\varphi(K,E) \leq 1\big\}.$$

So,

$$|E_M| \geq |\widehat{V}_\varphi(K,E)^{-1} E|.$$

According to Lemma 4.3(1), we have $\widehat{V}_\varphi(K,E_M) = 1$. Hence,

$$\widehat{V}_\varphi(K,E) \geq \left(\frac{|E|}{|E_M|}\right)^{\frac{1}{n}} \geq \left(\frac{\omega_n}{|E_M|}\right)^{\frac{1}{n}} = \widehat{V}_\varphi\left(K, \left(\frac{\omega_n}{|E_M|}\right)^{\frac{1}{n}} E_M\right).$$

Added that $(\frac{\omega_n}{|E_M|})^{\frac{1}{n}} E_M \in \{E \in \mathcal{E}^n : |E| \geq \omega_n\}$, it implies that the ellipsoid $(\frac{\omega_n}{|E_M|})^{\frac{1}{n}} E_M$ is a solution to Problem $\overline{S}_\varphi$.

(2) For any $E \in \{E \in \mathcal{E}^n : \widehat{V}_\varphi(K, E) \le 1\}$, we have

$$\left(\frac{\omega_n}{|E|}\right)^{\frac{1}{n}} E \in \left\{E \in \mathcal{E}^n : |E| \ge \omega_n\right\}.$$

So,

$$\widehat{V}_\varphi\left(K, \left(\frac{\omega_n}{|E|}\right)^{\frac{1}{n}} E\right) = \left(\frac{\omega_n}{|E|}\right)^{\frac{1}{n}} \widehat{V}_\varphi(K, E) \ge \widehat{V}_\varphi(K, E_m).$$

By the positive homogeneity of $\widehat{V}_\varphi(K, \lambda E)$ in λ, it guarantees that $|E_m| = \omega_n$. Hence, the above inequality can be rewritten as

$$\left|\widehat{V}_\varphi(K, E_m)^{-1} E_m\right| \ge \left|\widehat{V}_\varphi(K, E)^{-1} E\right| \ge |E|.$$

This completes the proof. □

In light of Theorem 4.2 and Theorem 4.4, we introduce a family of ellipsoids in the framework of Orlicz Brunn–Minkowski theory, which is an extension of LYZ's L_p John ellipsoids.

Definition 4.5. Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Amongst all origin-symmetric ellipsoids, the unique ellipsoid that solves the constrained maximization problem

$$\max_E |E| \quad \text{subject to} \quad \overline{V}_\varphi(K, E) \le 1$$

is called the *Orlicz–John ellipsoid* of K regarding φ and is denoted by $\mathrm{E}_\varphi K$. Amongst all origin-symmetric ellipsoids, the unique ellipsoid that solves the constrained minimization problem

$$\min_E \widehat{V}_\varphi(K, E) \quad \text{subject to} \quad |E| = \omega_n$$

is called the *normalized Orlicz–John ellipsoid* of K regarding φ and is denoted by $\overline{\mathrm{E}}_\varphi K$.

If $\varphi(t) = t^p$, $1 \le p < \infty$, then the ellipsoids $\mathrm{E}_\varphi K$ and $\overline{\mathrm{E}}_\varphi K$ precisely turn out to be the L_p John ellipsoid $\mathrm{E}_p K$ and the normalized L_p John ellipsoid $\overline{\mathrm{E}}_p K$, respectively.

From Definition 4.5 and Lemma 2.4, we obtain the following result.

Lemma 4.6. *Suppose $K \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $T \in \mathrm{GL}(n)$. Then*

$$T\mathrm{E}_\varphi K = \mathrm{E}_\varphi T K.$$

Obviously, $\mathrm{E}_\varphi B = B$, and from Lemma 4.6 we see that if E is an origin-symmetric ellipsoid, then

$$\mathrm{E}_\varphi E = E.$$

Observe that for all $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $s > 0$,

$$\overline{V}_{s\varphi}(K, L) = (s\varphi)^{-1}\left(\int_{S^{n-1}} s\varphi\left(\frac{h_L}{h_K}\right) d\overline{V}_K\right) = \overline{V}_\varphi(K, L).$$

Hence, we immediately obtain the following.

Lemma 4.7. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then for all $s > 0$,*

$$\mathrm{E}_{s\varphi} K = \mathrm{E}_\varphi K.$$

5. The continuity of Orlicz–John ellipsoids

In this section, we prove the continuity of $\mathrm{E}_\varphi K$ with respect to φ and K.

Theorem 5.1. *Suppose $K, K_i \in \mathcal{K}_o^n$ and $\varphi, \varphi_j \in \Phi$, where $i, j \in \mathbb{N}$. If $K_i \to K$ and $\varphi_j \to \varphi$, then*

$$\lim_{i,j\to\infty} \mathrm{E}_{\varphi_j} K_i = \mathrm{E}_\varphi K.$$

If $K_i \in \mathcal{K}_o^n$, $K_i \to K \in \mathcal{K}_o^n$, then there exist r_m and r_M, $0 < r_m \le r_M < \infty$, such that

$$r_m B \subseteq K \subseteq r_M B \quad \text{and} \quad r_m B \subseteq K_i \subseteq r_M B,$$

for all $i \in \mathbb{N}$.

Throughout this section, let

$$c = \frac{n r_M^n \omega_n}{2 r_m^n \omega_{n-1}}$$

and

$$\mathcal{E} = \big\{E \in \mathcal{E}^n : |E| = \omega_n \text{ and } E \subseteq cB\big\}.$$

To prove Theorem 5.1, several lemmas are in order.

Lemma 5.2. *All the ellipsoids $\overline{\mathrm{E}}_{\varphi_j} K_i$, $\overline{\mathrm{E}}_\varphi K$, $\overline{\mathrm{E}}_{\varphi_j} K$ and $\overline{\mathrm{E}}_\varphi K_i$ are in $\mathcal{E}$.*

Proof. Since four assertions can be proved similarly, we only give the proof on $\overline{\mathrm{E}}_{\varphi_j} K_i$.

According to the definition of normalized Orlicz–John ellipsoids, we have

$$|\overline{\mathrm{E}}_{\varphi_j} K_i| = \omega_n.$$

For $E \in \mathcal{E}^n$, let d_E denote its maximal principal radius. Recall the fact (appeared in the proof of Lemma 4.1) that

$$d_E \le \frac{n|K_i|}{2\min_{S^{n-1}} h_{\Pi K_i}}, \tag{5.1}$$

whenever $\overline{V}_\varphi(K_i, E) \le 1$.

For all $i \in \mathbb{N}$, the following three facts can be observed.

$$h_{\Pi K_i}(u) \ge h_{\Pi (r_m B)} = r_m^{n-1}\omega_{n-1}, \quad \text{for all } u \in S^{n-1}. \tag{5.2}$$

$$|K_i| \le r_M^n \omega_n. \tag{5.3}$$

$$|\mathrm{E}_\infty K_i| \ge \big|\mathrm{E}_\infty (r_m B)\big| = r_m^n \omega_n. \tag{5.4}$$

From Theorem 4.4(1), (5.1), (5.2) together with (5.3), and (5.4) together with (5.2), we have

$$\begin{aligned} d_{\overline{\mathrm{E}}_{\varphi_j} K_i} &= d_{(\frac{\omega_n}{|\mathrm{E}_{\varphi_j} K_i|})^{\frac{1}{n}} \mathrm{E}_{\varphi_j} K_i} \\ &= \left(\frac{\omega_n}{|\mathrm{E}_{\varphi_j} K_i|}\right)^{\frac{1}{n}} d_{\mathrm{E}_{\varphi_j} K_i} \\ &\le \left(\frac{\omega_n}{|\mathrm{E}_{\varphi_j} K_i|}\right)^{\frac{1}{n}} \frac{n|K_i|}{2\min_{S^{n-1}} h_{\Pi K_i}} \\ &\le \left(\frac{\omega_n}{|\mathrm{E}_\infty K_i|}\right)^{\frac{1}{n}} \frac{n r_M^n \omega_n}{2\min\limits_{S^{n-1}} h_{\Pi (r_m B)}} \\ &\le \left(\frac{\omega_n}{r_m^n \omega_n}\right)^{\frac{1}{n}} \frac{n r_M^n \omega_n}{2 r_m^{n-1} \omega_{n-1}} \\ &= c. \end{aligned}$$

Hence,

$$\overline{\mathrm{E}}_{\varphi_j} K_i \in \mathcal{E}.$$

Note that we used Theorem 7.2 from the first inequality to the second. □

From the compactness of the sets $\{K \in \mathcal{K}_o^n : r_m B \subseteq K \subseteq r_M B\}$ and $\mathcal{E}$, Theorem 3.1, and Theorem 3.2, we immediately obtain

Lemma 5.3. *The limits*

$$\lim_{i,j\to\infty} \widehat{V}_{\varphi_j}(K_i, E) = \widehat{V}_\varphi(K, E)$$

and

$$\lim_{i,j\to\infty} \overline{V}_{\varphi_j}(K_i, E) = \overline{V}_\varphi(K, E)$$

are both uniform in $E \in \mathcal{E}$.

Lemma 5.4. $\lim_{i,j\to\infty} \widehat{V}_{\varphi_j}(K_i, \overline{\mathrm{E}}_{\varphi_j} K_i) = \widehat{V}_\varphi(K, \overline{\mathrm{E}}_\varphi K)$.

Proof. From Definition 4.5, Lemma 5.2, Lemma 5.3, and Theorem 3.2, we have

$$\begin{aligned}\lim_{i,j\to\infty} \widehat{V}_{\varphi_j}(K_i, \overline{\mathrm{E}}_{\varphi_j} K_i) &= \lim_{i,j\to\infty} \min_{E\in\mathcal{E}} \widehat{V}_{\varphi_j}(K_i, E) \\ &= \min_{E\in\mathcal{E}} \lim_{i,j\to\infty} \widehat{V}_{\varphi_j}(K_i, E) \\ &= \min_{E\in\mathcal{E}} \widehat{V}_\varphi(K, E) \\ &= \widehat{V}_\varphi(K, \overline{\mathrm{E}}_\varphi K). \qquad \square\end{aligned}$$

Lemma 5.5. $\lim_{i,j\to\infty} \overline{\mathrm{E}}_{\varphi_j} K_i = \overline{\mathrm{E}}_\varphi K$.

Proof. We argue by contradiction and assume the conclusion to be false. From the definition and the compactness of $\mathcal{E}$, the Blaschke selection theorem and our assumption, there exists a convergent subsequence $\{\overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p}\}_{p,q\in\mathbb{N}}$, such that

$$\lim_{p,q\to\infty} \overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p} = E_0 \neq \overline{\mathrm{E}}_\varphi K, \tag{5.5}$$

where $K_{i_p} \to K$, as $p \to \infty$ and $\varphi_{j_q} \to \varphi$, as $q \to \infty$.

First, we show that $E_0 \in \mathcal{K}_o^n$. Indeed, since the volume functional is continuous with respect to the Hausdorff metric δ_H and $|\overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p}| = \omega_n$ for each $p, q \in \mathbb{N}$, we have $|E_0| = \omega_n$. So, E_0 is a convex body. Since the convergence of $\{\overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p}\}_{p,q}$ is equivalent to the uniform convergence of $\{h_{\overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p}}\}_{p,q}$ on S^{n-1} and $h_{\overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p}}(u) = h_{\overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p}}(-u)$ for all $u \in S^{n-1}$, we have $h_{E_0}(u) = h_{E_0}(-u)$ for all $u \in S^{n-1}$. Hence, $E_0 \in \mathcal{K}_o^n$. Therefore, E_0 is a non-degenerated origin-symmetric ellipsoid.

Now, from Theorem 3.2 and Lemma 5.4, it follows that

$$\begin{aligned}\widehat{V}_\varphi\Big(K, \lim_{p,q\to\infty} \overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p}\Big) &= \lim_{p,q\to\infty} \widehat{V}_\varphi(K, \overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p}) \\ &= \lim_{p,q\to\infty} \lim_{k\to\infty} \widehat{V}_{\varphi_k}(K, \overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p})\end{aligned}$$

$$= \lim_{p,q,k\to\infty} \widehat{V}_{\varphi_k}(K_{i_p}, \overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p})$$

$$= \lim_{p,q\to\infty} \widehat{V}_{\varphi_{j_q}}(K_{i_p}, \overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p})$$

$$= \widehat{V}_\varphi(K, \overline{\mathrm{E}}_\varphi K).$$

Since the solution to Problem $\overline{S}_\varphi$ is unique, we have

$$\lim_{p,q\to\infty} \overline{\mathrm{E}}_{\varphi_{j_q}} K_{i_p} = \overline{\mathrm{E}}_\varphi K,$$

which contradicts (5.5). □

Proof of Theorem 5.1. From Lemma 5.4 and Lemma 5.5, together with the identity

$$\mathrm{E}_\varphi K = \frac{\overline{\mathrm{E}}_\varphi K}{\widehat{V}_\varphi(K, \overline{\mathrm{E}}_\varphi K)},$$

Theorem 5.1 can be derived immediately. □

From Theorem 5.1, we obtain the following two corollaries directly.

Corollary 5.6. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi, \varphi_i \in \Phi$, where $i \in \mathbb{N}$. If $\varphi_i \to \varphi$, then*

$$\lim_{i\to\infty} \mathrm{E}_{\varphi_i} K = \mathrm{E}_\varphi K.$$

Corollary 5.7. *Suppose $K, K_i \in \mathcal{K}_o^n$ and $\varphi \in \Phi$, where $i \in \mathbb{N}$. If $K_i \to K$, then*

$$\lim_{i\to\infty} \mathrm{E}_\varphi K_i = \mathrm{E}_\varphi K.$$

6. A common limit position

In this section, we show a connection linking the Orlicz–John ellipsoids and the classical John ellipsoid.

First, from Theorem 5.1, we have

Corollary 6.1. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. If $p_0 \in [1, \infty)$, then*

$$\lim_{p\to p_0} \mathrm{E}_{\varphi^p} K = \mathrm{E}_{\varphi^{p_0}} K.$$

If $\varphi(t) = t$, then $\{\mathrm{E}_{\varphi^p} K\}_{1\le p<\infty}$ is precisely the continuous family $\{\mathrm{E}_p K\}_{1\le p<\infty}$ of LYZ's L_p John ellipsoids.

In [38], it is proved that $\lim_{p\to\infty} \mathrm{E}_p K = \mathrm{E}_\infty K$. In what follows, we establish an Orlicz version of this limit. Its thrust is that, in effect, Orlicz–John ellipsoids can approach to the classical John ellipsoid.

Recall that for $K, L \in \mathcal{K}_o^n$, the normalized L_∞ mixed volume $\overline{V}_\infty(K, L)$ [38] of K and L is defined by

$$\overline{V}_\infty(K, L) = \sup\left\{\frac{h_L(u)}{h_K(u)} : u \in \operatorname{supp} \overline{V}_K\right\}.$$

To prove Theorem 6.3, it is convenient to extend the domain $\mathcal{K}_o^n$ of $\overline{V}_\infty(K, \cdot)$ and $\overline{V}_\varphi(K, \cdot)$ to $\mathcal{K}^n$, the class of compact convex sets in $\mathbb{R}^n$ containing the origin.

Lemma 6.2. *Suppose $K \in \mathcal{K}_o^n$, $L \in \mathcal{K}^n$, $\varphi \in \Phi$ and $p \in [1, \infty)$. Then*

(1) *$\overline{V}_{\varphi^p}(K, L)$ is increasing and bounded from above in p.*
(2) *$\lim_{p\to\infty} \overline{V}_{\varphi^p}(K, L) = \overline{V}_\infty(K, L)$.*

Proof. From the definition of Orlicz mixed volumes, it follows that

$$\begin{aligned}\overline{V}_{\varphi^p}(K, L) &= \left(\varphi^p\right)^{-1}\left(\int\limits_{S^{n-1}} \varphi^p\left(\frac{h_L}{h_K}\right) d\overline{V}_K\right) \\ &= \varphi^{-1}\left(\left(\int\limits_{S^{n-1}} \varphi^p\left(\frac{h_L}{h_K}\right) d\overline{V}_K\right)^{\frac{1}{p}}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{p\to\infty} \overline{V}_{\varphi^p}(K, L) &= \varphi^{-1}\left(\sup_{\operatorname{supp} \overline{V}_K} \varphi\left(\frac{h_L}{h_K}\right)\right) \\ &= \varphi^{-1}\left(\varphi\left(\sup_{\operatorname{supp} \overline{V}_K} \frac{h_L}{h_K}\right)\right) \\ &= \overline{V}_\infty(K, L).\end{aligned}$$

Since $\varphi^{-1}(\cdot)$ is increasing, by Jensen's inequality, we have

$$\overline{V}_{\varphi^q}(K, L) \le \overline{V}_{\varphi^p}(K, L) \le \lim_{p\to\infty} \overline{V}_{\varphi^p}(K, L) = \overline{V}_\infty(K, L),$$

for $1 \le q < p < \infty$. □

Theorem 6.3. *Suppose $K \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $1 \le p < \infty$. Then*

$$\lim_{p\to\infty} \mathrm{E}_{\varphi^p} K = \mathrm{E}_\infty K.$$

Proof. For brevity, we use E_p, $F_p(\cdot)$ and $F_\infty(\cdot)$ to denote $\mathrm{E}_{\varphi^p} K$, $\overline{V}_{\varphi^p}(K, \cdot)$ and $\overline{V}_\infty(K, \cdot)$, respectively.

Write $\mathcal{F}$ for the set of origin-symmetric ellipsoids contained in $\frac{n}{2}|K|(\min_{S^{n-1}} h_{\Pi K})^{-1}B$. Keep in mind that an ellipsoid in $\mathcal{F}$ may be degenerate. Thus, $(\mathcal{F}, \delta_H)$ is compact. Also, from the proof of Lemma 4.1, we know that $\mathrm{E}_p, \mathrm{E}_\infty K \in \mathcal{F}$.

First, we show

$$\lim_{p\to\infty} F_p(E) = F_\infty(E), \quad \text{uniformly in } E \in \mathcal{F}. \tag{6.1}$$

We argue by contradiction and assume (6.1) to be false. From this assumption and Lemma 6.2, there exist an $\varepsilon_0 > 0$, a sequence $\{p_i\}_{i\in\mathbb{N}}$ strictly increasing to ∞, and a sequence $\{E_i\}_{i\in\mathbb{N}} \subset \mathcal{F}$, such that

$$F_{p_i}(E_i) < F_\infty(E_i) - \varepsilon_0, \quad \text{for } i \in \mathbb{N}.$$

Thus, these inequalities together with Lemma 6.2(1) yield that

$$F_p(E_i) < F_\infty(E_i) - \varepsilon_0, \quad \text{for } p \le p_i. \tag{6.2}$$

Meanwhile, by the compactness of $\mathcal{F}$, there exists a convergent subsequence $\{E_{i_l}\}_l$ of $\{E_i\}$, namely, $E_{i_l} \to E_0 \in \mathcal{F}$. Since F_p and F_∞ are continuous on $\mathcal{F}$, replacing E_i by E_{i_l} in (6.2) and letting $l \to \infty$, we have

$$F_p(E_0) \le F_\infty(E_0) - \varepsilon_0, \quad \text{for } p \in [1,\infty),$$

which contradicts the fact (as shown by Lemma 6.2(2)), that $F_p \to F_\infty$ pointwise on $\mathcal{F}$ as $p \to \infty$. Hence, the uniform convergence in (6.1) holds.

Now, we are in the position to finish the proof of this theorem. It suffices to prove that for each sequence $\{\mathrm{E}_{p_j}\}_{j\in\mathbb{N}}$, where $p_j \to \infty$ as $j \to \infty$, we have

$$\lim_{j\to\infty} \mathrm{E}_{p_j} = \mathrm{E}_\infty K. \tag{6.3}$$

We argue by contradiction again and assume (6.3) to be false. From the compactness of $\mathcal{F}$ and the assumption, there exists a subsequence $\{\mathrm{E}_{p_{j_k}}\}_k$ of $\{\mathrm{E}_{p_j}\}_{j\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathrm{E}_{p_{j_k}} = E' \in \mathcal{F} \quad \text{and} \quad E' \ne \mathrm{E}_\infty K. \tag{6.4}$$

But now, the uniform convergence in (6.1), and the continuity of F_∞ on $\mathcal{F}$ together with (6.4), necessarily lead to that

$$\lim_{\substack{q\to\infty \\ k\to\infty}} F_q(\mathrm{E}_{p_{j_k}}) = F_\infty(E'). \tag{6.5}$$

Indeed, by the uniform convergence in (6.1) and the continuity of F_∞ on $\mathcal{F}$, for given $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$, such that

$$\left|F_q(\mathrm{E}_{p_{j_k}}) - F_\infty(\mathrm{E}_{p_{j_k}})\right| \le \frac{\varepsilon}{2}, \quad \text{for } q \ge N_1 \text{ and } k \in \mathbb{N},$$

and

$$\left|F_\infty(\mathrm{E}_{p_{j_k}}) - F_\infty\left(E'\right)\right| \le \frac{\varepsilon}{2}, \quad \text{for } k \ge N_2.$$

Thus, from the triangle inequality, it yields

$$\left|F_q(\mathrm{E}_{p_{j_k}}) - F_\infty\left(E'\right)\right| \le \varepsilon, \quad \text{for } q, k \ge \max\{N_1, N_2\}.$$

Therefore, (6.5) is confirmed.

Now, from (6.5), the definitions of $F_{p_{j_k}}$ and $\mathrm{E}_{p_{j_k}}$, and the definition of Orlicz–John ellipsoids, we have

$$\begin{aligned} F_\infty\left(E'\right) &= \lim_{\substack{q\to\infty \\ k\to\infty}} F_q(\mathrm{E}_{p_{j_k}}) \\ &= \lim_{k\to\infty} F_{p_{j_k}}(\mathrm{E}_{p_{j_k}}) \\ &= \lim_{k\to\infty} \overline{V}_{\varphi^{p_{j_k}}}(K, \mathrm{E}_{\varphi^{p_{j_k}}}K) \\ &= 1. \end{aligned}$$

That is, $F_\infty(E') = 1$. Along this passage further, from the definition of F_∞ and the implication

$$\overline{V}_\infty(K, L) = 1 \quad \Longrightarrow \quad L \subseteq K, \quad \text{for } L \in \mathcal{K}^n,$$

we conclude that $E' \subseteq K$. Recall that $\mathrm{E}_\infty K$ is the unique origin-symmetric ellipsoid of maximal volume contained in K. Thus, the assumption $E' \ne \mathrm{E}_\infty K$ in (6.4) implies that

$$\left|E'\right| < |\mathrm{E}_\infty K|. \tag{6.6}$$

On the other hand, by the definition of $\mathrm{E}_{p_{j_k}}$ and Theorem 7.2, we have

$$|\mathrm{E}_{p_{j_k}} K| \ge |\mathrm{E}_\infty K|, \quad \text{for } k \in \mathbb{N}.$$

Thus, from that $\mathrm{E}_{p_{j_k}} \to E'$ and the continuity of volume functional, we obtain

$$\left|E'\right| \ge |\mathrm{E}_\infty K|,$$

which contradicts (6.6). This completes the proof. □

Note that if the John point of convex body K is at the origin, then the Orlicz–John ellipsoid $\mathrm{E}_{\varphi^p} K$ converges to the classical John ellipsoid $\mathrm{J}K$ as $p \to \infty$.

7. Volume ratio inequalities

In general, the Orlicz–John ellipsoid $\mathrm{E}_\varphi K$ is not contained in K. However, the volume functional over the class of Orlicz–John ellipsoids is bounded.

Lemma 7.1. *Suppose $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $1 \le p < q < \infty$. Then*

$$\overline{V}_1(K,L) \le \overline{V}_\varphi(K,L) \le \overline{V}_{\varphi^p}(K,L) \le \overline{V}_{\varphi^q}(K,L) \le \overline{V}_\infty(K,L).$$

Proof. By Lemma 6.2, we only need to prove the left inequality.

From the convexity of φ and Jensen's inequality, we have

$$\varphi\big(\overline{V}_1(K,L)\big) = \varphi\bigg(\int_{S^{n-1}} \frac{h_L}{h_K} d\overline{V}_K\bigg) \le \int_{S^{n-1}} \varphi\bigg(\frac{h_L}{h_K}\bigg) d\overline{V}_K = \varphi\big(\overline{V}_\varphi(K,L)\big).$$

Since φ^{-1} is increasing, the left inequality is derived. □

Theorem 7.2. *Suppose $K \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $1 \le p < q < \infty$. Then*

$$|\mathrm{E}_\infty K| \le |\mathrm{E}_{\varphi^q} K| \le |\mathrm{E}_{\varphi^p} K| \le |\mathrm{E}_\varphi K| \le |\mathrm{E}_1 K|.$$

Proof. From Lemma 7.1, it follows that

$$\begin{aligned} \big\{E \in \mathcal{E}^n : \overline{V}_\infty(K,E) \le 1\big\} &\subseteq \big\{E \in \mathcal{E}^n : \overline{V}_{\varphi^q}(K,E) \le 1\big\} \\ &\subseteq \big\{E \in \mathcal{E}^n : \overline{V}_{\varphi^p}(K,E) \le 1\big\} \\ &\subseteq \big\{E \in \mathcal{E}^n : \overline{V}_\varphi(K,E) \le 1\big\} \\ &\subseteq \big\{E \in \mathcal{E}^n : \overline{V}_1(K,E) \le 1\big\}. \end{aligned}$$

From the above inclusions and the definition of Orlicz–John ellipsoids, it immediately yields the desired conclusion. □

Theorem 7.2 shows that $|\mathrm{E}_\varphi K|$ is dominated by the volumes of the Petty ellipsoid $\mathrm{E}_1 K$ from above and the L_∞ John ellipsoid $\mathrm{E}_\infty K$ (or the John ellipsoid $\mathrm{J}K$) from below, respectively.

The following lemma was proved in [15], which can be regarded as the Orlicz version of Minkowski's inequality.

Lemma 7.3. *Suppose $K, L \in \mathcal{K}_o^n$, and $\varphi \in \Phi$. Then*

$$\overline{V}_\varphi(K,L) \ge \bigg(\frac{|L|}{|K|}\bigg)^{\frac{1}{n}},$$

and

$$\widehat{V}_\varphi(K,L) \ge \left(\frac{|L|}{|K|}\right)^{\frac{1}{n}}.$$

If φ is strictly convex, each equality holds if and only if K and L are dilates.

Theorem 7.4. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then*

$$|\mathrm{E}_\varphi K| \le |K|.$$

If φ is strictly convex, the equality holds if and only if $K \in \mathcal{E}^n$.

Proof. From Lemma 7.3, we have

$$\overline{V}_\varphi(K, \mathrm{E}_\varphi K) \ge \left(\frac{|\mathrm{E}_\varphi K|}{|K|}\right)^{\frac{1}{n}}.$$

If φ is strictly convex, the equality holds if and only if $K \in \mathcal{E}^n$. In view of the fact

$$\overline{V}_\varphi(K, \mathrm{E}_\varphi K) = 1,$$

the desired inequality is obtained. □

It is interesting that Ball's volume-ratio inequality still holds for Orlicz–John ellipsoids.

Theorem 7.5. *Suppose $K \in \mathcal{K}_o^n$ is origin-symmetric and $\varphi \in \Phi$. Then*

$$\frac{|K|}{|\mathrm{E}_\varphi K|} \le \frac{2^n}{\omega_n},$$

with equality if and only if K is a parallelotope.

Proof. From Theorem 7.2, Ball's volume-ratio inequality, and the fact $\mathrm{E}_\infty K = \mathrm{J}K$, we have

$$\frac{|K|}{|\mathrm{E}_\varphi K|} \le \frac{|K|}{|\mathrm{J}K|} \le \frac{2^n}{\omega_n}.$$

In the rest, we prove the equality condition.

If $\frac{|K|}{|\mathrm{E}_\varphi K|} = \frac{2^n}{\omega_n}$, then

$$\frac{|K|}{|\mathrm{J}K|} = \frac{2^n}{\omega_n}.$$

From the equality condition of Ball's volume-ratio inequality, it implies that K is a parallelotope.

Conversely, since $\frac{|K|}{|\mathrm{E}_\varphi K|}$ is $\mathrm{GL}(n)$-invariant by Lemma 4.6, we may assume that $K = [-1,1]^n$. Then

$$\bar{\mathrm{E}}_1 K = B \quad \text{and} \quad \overline{V}_\varphi(K,B) = \overline{V}_1(K,B) = 1.$$

Let E be an origin-symmetric ellipsoid with volume ω_n. From Lemma 7.1, we have

$$\overline{V}_\varphi(K,E) \geq \overline{V}_1(K,E) \geq \overline{V}_1(K,\bar{\mathrm{E}}_1 K) = \overline{V}_1(K,B) = 1.$$

That is, $\overline{V}_\varphi(K,E) \geq 1$. Then from Lemma 2.2, it follows that

$$\widehat{V}_\varphi(K,E) \geq 1. \tag{7.1}$$

From Lemma 2.7, $\overline{V}_\varphi(K,B) = 1$, if and only if

$$\widehat{V}_\varphi(K,B) = 1. \tag{7.2}$$

Combining (7.1) with (7.2), it gives

$$\widehat{V}_\varphi(K,E) \geq \widehat{V}_\varphi(K,B).$$

So, according to the definition of $\bar{\mathrm{E}}_\varphi K$, it follows that

$$\bar{\mathrm{E}}_\varphi K = B.$$

Thus,

$$\mathrm{E}_\varphi K = \widehat{V}_\varphi(K,\bar{\mathrm{E}}_\varphi K)^{-1}\bar{\mathrm{E}}_\varphi K = B.$$

Consequently,

$$\frac{|K|}{|\mathrm{E}_\varphi K|} = \frac{2^n}{\omega_n}. \qquad \square$$

8. A characterization of Orlicz–John ellipsoids

In this section, we show the sufficient and necessary conditions, which characterize the solution to Problem S_φ and establish a connection with the isotropy of measures.

Definition 8.1. Given a convex body $K \in \mathcal{K}_o^n$ and a function $\varphi \in \Phi \cap C^1(0,\infty)$, define the *Orlicz surface area measure* $S_\varphi(K,\cdot)$ of K regarding φ by

$$S_\varphi(K,\omega) = \frac{1}{\varphi'(1)} \int\limits_\omega \varphi'(1/h_K)dS_K$$

for a Borel subset $\omega \subseteq S^{n-1}$.

If $\varphi(t) = t^p$, $p \geq 1$, then $S_\varphi(K,\cdot)$ reduces to the L_p surface area measure $S_p(K,\cdot)$. That is,

$$S_p(K,\omega) = \int\limits_\omega h_K^{1-p}dS_K.$$

For the L_p surface area measure $S_p(K,\cdot)$, we know that $S_p(\lambda K,\cdot) = \lambda^{n-p}S_p(K,\cdot)$, $\lambda > 0$. However, this property does not always hold in the Orlicz setting.

The next lemma plays a crucial role to characterize Orlicz–John ellipsoids, since the normalized Orlicz–John ellipsoid $\overline{\mathrm{E}}_\varphi K$ solves the following extremal problem for certain $\mathrm{GL}(n)$-image of K.

Lemma 8.2. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi \cap C^1(0,\infty)$. Then modulo orthogonal transformations, there exists a unique solution to the following constrained minimization problem:*

$$\min\bigl\{V_\varphi(K,TB) : T \in \mathrm{SL}(n)\bigr\}.$$

Moreover, the identity operator I_n is the solution, if and only if $S_\varphi(K,\cdot)$ is isotropic on S^{n-1}.

Note that we can prove the existence and uniqueness of the solution by using similar arguments in the proof of Lemma 4.1 and Theorem 4.2. But to avoid the redundancy, we take Gruber's techniques, which were originally practiced in [21] and [19]. The advantage is that the sufficient condition in the second statement can also be proved along the way. In contrast, we prove the necessity by variational method.

Proof. It is known that each $A \in \mathrm{GL}(n)$ can be represented in the form $A = TQ$, where T is symmetric and positive definite, and Q is orthogonal. As Gruber and Schuster [21] observed, a symmetric matrix $T = (t_{ij})_{n\times n}$ is identified with a point in $\mathbb{R}^{\frac{1}{2}n(n+1)}$, with the coordinate

$$(t_{11},\cdots,t_{1n},t_{22},\cdots,t_{2n},\cdots,t_{nn}).$$

Thus the set of all symmetric and positive definite matrices of order n is represented by an open convex cone $\mathcal{P} \subseteq \mathbb{R}^{\frac{1}{2}n(n+1)}$ with apex at the origin. Moreover, the set

$$\mathcal{D} = \bigl\{T \in \mathcal{P} : \det(T) \geq 1\bigr\}$$

is a closed, strictly convex set with nonempty interior in $\mathcal{P}$ and has a smooth boundary.

For $T \in \mathcal{P}$, let

$$F(T) = V_\varphi\big(K, T^t B\big) = \int_{S^{n-1}} \varphi\left(\frac{|Tu|}{h_K(u)}\right) dV_K(u),$$

and

$$\mathcal{T}_a = \big\{T \in \mathcal{P} : F(T) \le a\big\},$$

where $a > 0$.

Now, we prove the existence and uniqueness of the solution.

Three observations are in order. Firstly, $\mathcal{T}_a$ is a convex set with nonempty interior. Secondly, $\mathcal{T}_{a_1} \subset \mathcal{T}_{a_2}$, whenever $0 < a_1 < a_2$. Finally, $\mathcal{T}_a \cap \mathcal{D} = \emptyset$, whenever $a > 0$ is sufficiently small. In contrast, if $a > 0$ is sufficiently big, then $\operatorname{int}(\mathcal{T}_a \cap \mathcal{D}) \neq \emptyset$. Consequently, there exists a unique $a_0 > 0$ such that the convex sets $\mathcal{T}_{a_0}$ and $\mathcal{D}$ enjoy a unique common boundary point T_0. In other words, for any $T \in \partial\mathcal{D}$ (i.e., symmetric, positive definite $T \in \mathrm{SL}(n)$),

$$F(T) \ge F(T_0), \tag{8.1}$$

with equality if and only if $T = T_0$. This proves the unique existence of the solution.

Next, we prove the sufficient condition in the second statement.

Suppose that $S_\varphi(K, \cdot)$ is isotropic. It suffices to prove $a_0 = F(I_n)$ and $T_0 = I_n$.

It is clear that $\partial\mathcal{T}_{F(I_n)}$ is given by the equation $F(T) = F(I_n)$, and $F(T)$ is smooth in a neighborhood of I_n. Then,

$$\begin{aligned}
\frac{\partial}{\partial t_{ij}}\bigg|_{T=I_n} F(T) &= \frac{\partial}{\partial t_{ij}}\bigg|_{T=I_n} \int_{S^{n-1}} \varphi\left(\frac{|Tu|}{h_K(u)}\right) dV_K(u) \\
&= \int_{S^{n-1}} \varphi'\left(\frac{1}{h_K(u)}\right) \frac{(e_i \cdot u)(e_j \cdot u)}{h_K(u)} dV_K(u) \\
&= \frac{\varphi'(1)}{n} e_i \cdot \left(\int_{S^{n-1}} u \otimes u dS_\varphi(K, u)\right) e_j \\
&= \frac{\varphi'(1)|S_\varphi(K, \cdot)|}{n^2} e_i \cdot I_n e_j \\
&= \frac{\varphi'(1)|S_\varphi(K, \cdot)|}{n^2} \delta_{ij},
\end{aligned}$$

where $e_1, \cdots, e_n$ is an orthonormal basis of $\mathbb{R}^n$ and δ_{ij} is the Kronecker symbols. Thus, I_n is an outer normal vector of $\mathcal{T}_{F(I_n)}$ at the boundary point I_n.

Since

$$\frac{\partial}{\partial t_{ij}}\bigg|_{T=I_n} \det(T) = \delta_{ij},$$

I_n is also an inner normal vector of $\mathcal{D}$ at the boundary point I_n. In view of the convexity of the sets $\mathcal{T}_{F(I_n)}$ and $\mathcal{D}$, it is concluded that the tangent hyperplane of $\mathcal{D}$ at I_n separates $\mathcal{T}_{F(I_n)}$ and $\mathcal{D}$. Hence, $\mathcal{T}_{F(I_n)} \cap \mathcal{D} = \{I_n\}$.

So, for each $T \in \partial\mathcal{D}$, $T \neq I_n$, it gives

$$F(T) > F(I_n).$$

Combining this inequality with (8.1), we conclude that $a_0 = F(I_n)$, and therefore $T_0 = I_n$.

Finally, we prove the necessity.

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Choose $\varepsilon_0 > 0$ sufficiently small so that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the matrix $I_n + \varepsilon L$ is invertible. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, define

$$L_\varepsilon = \frac{I_n + \varepsilon L}{|I_n + \varepsilon L|^{\frac{1}{n}}}.$$

Then $L_\varepsilon \in \mathrm{SL}(n)$. The condition that I_n is the unique solution implies that $V_\varphi(K, L_\varepsilon^t B) \geq V_\varphi(K, L_0^t B)$ for all ε.

Note that

$$V_\varphi\big(K, L_\varepsilon^t B\big) = \int_{S^{n-1}} \varphi\bigg(\frac{(1 + 2\varepsilon u \cdot Lu + \varepsilon^2 Lu \cdot Lu)^{\frac{1}{2}}}{|I_n + \varepsilon L|^{\frac{1}{n}} h_K(u)}\bigg) dV_K(u).$$

From the smoothness of φ and smoothness of $|L_\varepsilon u|$ in ε, it implies that the integrand depends smoothly on ε. So, we have

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} V_\varphi\big(K, L_\varepsilon^t B\big) = 0.$$

Calculating it directly, we have

$$\begin{aligned}
0 &= \int_{S^{n-1}} \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \varphi\bigg(\frac{(1 + 2\varepsilon u \cdot Lu + \varepsilon^2 Lu \cdot Lu)^{\frac{1}{2}}}{|I_n + \varepsilon L|^{\frac{1}{n}} h_K(u)}\bigg) dV_K(u) \\
&= \frac{1}{n} \int_{S^{n-1}} \varphi'\bigg(\frac{1}{h_K(u)}\bigg)\bigg(-\frac{\operatorname{tr} L}{n} + u \cdot Lu\bigg) dS_K(u) \\
&= \frac{\varphi'(1)}{n} \int_{S^{n-1}} \bigg(-\frac{\operatorname{tr} L}{n} + u \cdot Lu\bigg) dS_\varphi(K, u).
\end{aligned}$$

Let $L = v \otimes v$ for $v \in S^{n-1}$. Using the facts $\operatorname{tr}(v \otimes v) = 1$ and $u \cdot (v \otimes v)u = (v \cdot u)^2$, it gives

$$\frac{|S_\varphi(K,\cdot)|}{n} = \int_{S^{n-1}} (u \cdot v)^2 dS_\varphi(K,u).$$

Thus, the measure $S_\varphi(K,\cdot)$ is isotropic. □

Corollary 8.3. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi \cap C^1(0,\infty)$.*

(1) *There exists an $\mathrm{SL}(n)$ transformation T, such that $S_\varphi(TK,\cdot)$ is isotropic.*
(2) *If $T_1, T_2 \in \mathrm{SL}(n)$ and $S_\varphi(T_1K,\cdot)$, $S_\varphi(T_2K,\cdot)$ are both isotropic, then there exists an orthogonal O such that $T_2 = OT_1$.*

If $\varphi(t) = t^p$, $p \geq 1$, Corollary 8.3 reduces to the isotropy of the L_p surface area measures, which was essentially proved in [38] by Lutwak, Yang and Zhang.

Using Lemma 8.2, we provide a characterization of Orlicz–John ellipsoids.

Theorem 8.4. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi \cap C^1(0,\infty)$. Then*

$$\mathrm{E}_\varphi K = \widehat{V}_\varphi(K,B)^{-1}B,$$

if and only if $S_\varphi(\widehat{V}_\varphi(K,B)K,\cdot)$ is isotropic on S^{n-1}.

If the normalized Orlicz–John ellipsoid is an ellipsoid, say, $\overline{\mathrm{E}}_\varphi K = TB$, $T \in \mathrm{SL}(n)$, then from Theorem 4.4(2), Lemma 4.6 together with Lemma 2.6(3), and Theorem 8.4, we have

$$\begin{aligned} \overline{\mathrm{E}}_\varphi K = TB \quad &\Longleftrightarrow \quad \mathrm{E}_\varphi K = \widehat{V}_\varphi(K,TB)^{-1}TB \\ &\Longleftrightarrow \quad \mathrm{E}_\varphi\big(\widehat{V}_\varphi\big(T^{-1}K,B\big)T^{-1}K\big) = B \\ &\Longleftrightarrow \quad S_\varphi\big(\widehat{V}_\varphi\big(T^{-1}K,B\big)T^{-1}K,\cdot\big) \quad \text{is isotropic.} \end{aligned}$$

Proof. From Lemma 8.2, the measure $S_\varphi(\lambda K,\cdot)$ is isotropic if and only if

$$\overline{V}_\varphi(\lambda K,B) = \min\big\{\overline{V}_\varphi(\lambda K,TB) : T \in \mathrm{SL}(n)\big\}, \quad \lambda > 0.$$

Let $\lambda = \widehat{V}_\varphi(K,B)$. For $T \in \mathrm{SL}(n)$, from Lemma 2.4(3) together with Lemma 2.7, Lemma 2.7 again, the property that $\overline{V}_\varphi(K,\lambda^{-1}TB)$ is strictly decreasing in $\lambda \in (0,\infty)$, the definition of $\overline{\mathrm{E}}_\varphi K$, and Theorem 4.4(2), we have

$$S_\varphi\big(\widehat{V}_\varphi(K,B)K,\cdot\big) \text{ is isotropic} \quad \Longleftrightarrow \quad \overline{V}_\varphi\big(\widehat{V}_\varphi(K,B)K,B\big) \leq \overline{V}_\varphi\big(\widehat{V}_\varphi(K,B)K,TB\big)$$

$$\begin{aligned}
&\iff && 1 \le \overline{V}_\varphi\left(K, \frac{TB}{\widehat{V}_\varphi(K,B)}\right) \\
&\iff && \overline{V}_\varphi\left(K, \frac{TB}{\widehat{V}_\varphi(K,TB)}\right) \le \overline{V}_\varphi\left(K, \frac{TB}{\widehat{V}_\varphi(K,B)}\right) \\
&\iff && \widehat{V}_\varphi(K,TB) \ge \widehat{V}_\varphi(K,B) \\
&\iff && \overline{\mathrm{E}}_\varphi K = B \\
&\iff && \mathrm{E}_\varphi K = \widehat{V}_\varphi(K,B)^{-1}B.
\end{aligned}$$

This completes the proof. □

Let $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi \cap C^1(0,\infty)$. The constrained minimization problem posed in Lemma 8.2 can be restated as:

$$\min_{E \in \mathcal{E}^n} \overline{V}_\varphi(K,E) \quad \text{subject to} \quad |E| = \omega_n. \tag{8.2}$$

From Lemma 8.2 and Theorem 8.4, this problem and Problem $\overline{S}_\varphi$ for K have the identical solution, if and only if there is a $T \in \mathrm{SL}(n)$ such that $S_\varphi(TK,\cdot)$ and $S_\varphi(\widehat{V}_\varphi(TK,B)TK,\cdot)$ are both isotropic on S^{n-1}.

We point out that for different dilations $\lambda_1 K$ and $\lambda_2 K$, $\lambda_1, \lambda_2 > 0$, problems (8.2) do not generally have the identical solution. By contrast, the homogeneity of $\widehat{V}_\varphi(\lambda K, L)$ in $\lambda \in (0,\infty)$ guarantees that all Problems $\overline{S}_\varphi$ for λK in $\lambda \in (0,\infty)$ have the identical solution.

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Appendix A

For $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$, define the function $h : [0,\infty) \times S^{n-1} \to [0,\infty)$ as

$$h(\varepsilon,u) = \inf\left\{\lambda > 0 : \varphi\left(\frac{h_K(u)}{\lambda}\right) + \varepsilon\varphi\left(\frac{h_L(u)}{\lambda}\right) \le \varphi(1)\right\}.$$

Lemma A.1. *The function $h(\varepsilon,u)$ is continuous in $(\varepsilon,u) \in [0,\infty) \times S^{n-1}$.*

Proof. Suppose $(\varepsilon_0,u_0), (\varepsilon_j,u_j) \in [0,\infty) \times S^{n-1}$, $j \in \mathbb{N}$, and $(\varepsilon_j,u_j) \to (\varepsilon_0,u_0)$ as $j \to \infty$. What follows aim to show $\lim_{j\to\infty} h(\varepsilon_j,u_j) = h(\varepsilon_0,u_0)$.

First, we prove the sequence $\{h(\varepsilon_j, u_j)\}_{j\in\mathbb{N}}$ is bounded.

Since $\varepsilon_j, \varepsilon_0 \in [0,\infty), \varepsilon_j \to \varepsilon_0$, there exists a number $\varepsilon_M > 0$ such that $\{\varepsilon_j\}_j \subset [0, \varepsilon_M]$. From $K, L \in \mathcal{K}_o^n$, there exist positive r_m and r_M, such that

$$r_m B \subseteq K \subseteq r_M B \quad \text{and} \quad r_m B \subseteq L \subseteq r_M B.$$

Since φ is strictly increasing in $(0,\infty)$, for $\lambda > 0$, $u \in S^{n-1}$ and $\varepsilon \in [0, \varepsilon_M]$, we have

$$\varphi\left(\frac{r_m}{\lambda}\right) \le \varphi\left(\frac{h_K(u)}{\lambda}\right) + \varepsilon\varphi\left(\frac{h_L(u)}{\lambda}\right) \le \varphi\left(\frac{r_M}{\lambda}\right) + \varepsilon_M\varphi\left(\frac{r_M}{\lambda}\right).$$

Recall that $\varphi \in \Phi$. So, $\varphi(\frac{r_m}{\lambda})$, $\varphi(\frac{h_K(u)}{\lambda}) + \varepsilon\varphi(\frac{h_L(u)}{\lambda})$ and $\varphi(\frac{r_M}{\lambda}) + \varepsilon_M\varphi(\frac{r_M}{\lambda})$ satisfy the following: 1) they are strictly decreasing and continuous in $\lambda \in (0,\infty)$; 2) they tend to ∞ as $\lambda \to 0^+$; 3) they converge to 0 as $\lambda \to \infty$. Combining these properties with the above inequalities and the definition of $h(\varepsilon, u)$, it yields that

$$r_m \le h(\varepsilon_0, u_0) \le R \quad \text{and} \quad r_m \le h(\varepsilon_j, u_j) \le R, \quad \forall j \in \mathbb{N},$$

where R is the unique positive number such that

$$\varphi\left(\frac{r_M}{R}\right) + \varepsilon_M\varphi\left(\frac{r_M}{R}\right) = \varphi(1).$$

With the boundedness of the sequence $\{h(\varepsilon_j, u_j)\}_{j\in\mathbb{N}}$ in hand, we proceed to complete the proof. It suffices to prove that any convergent subsequence $\{h(\varepsilon_{j_k}, u_{j_k})\}_k$ of $\{h(\varepsilon_j, u_j)\}$ converges to $h(\varepsilon_0, u_0)$. Assume

$$\lim_{k\to\infty} h(\varepsilon_{j_k}, u_{j_k}) = h_0.$$

From the facts that $\varepsilon_{j_k} \to \varepsilon_0$ and $u_{j_k} \to u_0$, the continuity of φ, h_K and h_L, together with our assumption, we have

$$\begin{aligned}
&\lim_{k\to\infty}\left[\varphi\left(\frac{h_K(u_{j_k})}{h(\varepsilon_{j_k}, u_{j_k})}\right) + \varepsilon_{j_k}\varphi\left(\frac{h_L(u_{j_k})}{h(\varepsilon_{j_k}, u_{j_k})}\right)\right] \\
&\quad = \lim_{k\to\infty}\varphi\left(\frac{h_K(u_{j_k})}{h(\varepsilon_{j_k}, u_{j_k})}\right) + \lim_{k\to\infty}\varepsilon_{j_k}\varphi\left(\frac{h_L(u_{j_k})}{h(\varepsilon_{j_k}, u_{j_k})}\right) \\
&\quad = \varphi\left(\frac{\lim_{k\to\infty} h_K(u_{j_k})}{\lim_{k\to\infty} h(\varepsilon_{j_k}, u_{j_k})}\right) + \varepsilon_0\varphi\left(\frac{\lim_{k\to\infty} h_L(u_{j_k})}{\lim_{k\to\infty} h(\varepsilon_{j_k}, u_{j_k})}\right) \\
&\quad = \varphi\left(\frac{h_K(u_0)}{h_0}\right) + \varepsilon_0\varphi\left(\frac{h_L(u_0)}{h_0}\right).
\end{aligned}$$

Observe that for each j,

$$\varphi\left(\frac{h_K(u_{j_k})}{h(\varepsilon_{j_k}, u_{j_k})}\right) + \varepsilon_{j_k}\varphi\left(\frac{h_L(u_{j_k})}{h(\varepsilon_{j_k}, u_{j_k})}\right) = \varphi(1).$$

Thus,

$$\varphi\left(\frac{h_K(u_0)}{h_0}\right) + \varepsilon_0\varphi\left(\frac{h_L(u_0)}{h_0}\right) = \varphi(1).$$

Note that $h(\varepsilon_0, u_0)$ is the unique positive number which solves the equation

$$\varphi\left(\frac{h_K(u_0)}{\lambda}\right) + \varepsilon_0\varphi\left(\frac{h_L(u_0)}{\lambda}\right) = \varphi(1), \quad \lambda > 0.$$

Therefore, we conclude that $h_0 = h(\varepsilon_0, u_0)$. This completes the proof. □

References

[1] K. Ball, Volumes of sections of cubes and related problems, in: J. Lindenstrauss, V.D. Milman (Eds.), Geometric Aspects of Functional Analysis, in: Lecture Notes in Math., vol. 1376, Springer, Berlin, 1989, pp. 251–260.

[2] K. Ball, Volumes ratios and a reverse isoperimetric inequality, J. Lond. Math. Soc. 44 (1991) 351–359.

[3] K. Ball, Ellipsoids of maximal volume in convex bodies, Geom. Dedicata 41 (1992) 241–250.

[4] F. Barthe, On a reverse form of the Brascamp–Lieb inequality, Invent. Math. 134 (1998) 335–361.

[5] F. Barthe, O. Guedon, S. Mendelson, A. Naor, A probabilistic approach to the geometry of the l_p^n ball, Ann. Probab. 33 (2005) 480–513.

[6] J. Bastero, M. Romance, Positions of convex bodies associated to extremal problems and isotropic measures, Adv. Math. 184 (2004) 64–88.

[7] K. Böröczky, E. Lutwak, D. Yang, G. Zhang, The log-Brunn–Minkowski inequality, Adv. Math. 231 (2012) 1974–1997.

[8] K. Böröczky, E. Lutwak, D. Yang, G. Zhang, The logarithmic Minkowski problem, J. Amer. Math. Soc. 26 (2013) 831–852.

[9] S. Campi, P. Gronchi, The L^p-Busemann–Petty centroid inequality, Adv. Math. 167 (2002) 128–141.

[10] F. Chen, J. Zhou, C. Yang, On the reverse Orlicz Busemann–Petty centroid inequality, Adv. in Appl. Math. 47 (2011) 820–828.

[11] K.S. Chou, X.J. Wang, The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry, Adv. Math. 205 (2006) 33–83.

[12] B. Fleury, O. Guédon, G.A. Paouris, A stability result for mean width of L_p-centroid bodies, Adv. Math. 214 (2007) 865–877.

[13] R.J. Gardner, Geometric Tomography, Cambridge University Press, Cambridge, 2006.

[14] R.J. Gardner, D. Hug, W. Weil, Operations between sets in geometry, J. Eur. Math. Soc. 15 (2013) 2297–2352.

[15] R.J. Gardner, D. Hug, W. Weil, The Orlicz–Brunn–Minkowski theory: a general framework, additions, and inequalities, J. Differential Geom. 97 (2014) 427–476.

[16] A.A. Giannopoulos, V.D. Milman, Extremal problems and isotropic positions of convex bodies, Israel J. Math. 117 (2000) 29–60.

[17] A.A. Giannopoulos, M. Papadimitrakis, Isotropic surface area measures, Mathematika 46 (1999) 1–13.

[18] M. Gromov, V.D. Milman, Generalization of the spherical isoperimetric inequality for uniformly convex Banach spaces, Compos. Math. 62 (1987) 263–282.

[19] P.M. Gruber, John and Loewner ellipsoids, Discrete Comput. Geom. 46 (2011) 776–788.

[20] P.M. Gruber, Convex and Discrete Geometry, Springer, Berlin, 2007.
[21] P.M. Gruber, F.E. Schuster, An arithmetic proof of John's ellipsoid theorem, Arch. Math. 85 (2005) 82–88.
[22] C. Haberl, E. Lutwak, D. Yang, G. Zhang, The even Orlicz Minkowski problem, Adv. Math. 224 (2010) 2485–2510.
[23] C. Haberl, F. Schuster, General L_p affine isoperimetric inequalities, J. Differential Geom. 83 (2009) 1–26.
[24] B. He, G. Leng, K. Li, Projection problems for symmetric polytopes, Adv. Math. 207 (2006) 73–90.
[25] M. Henk, E. Linke, Cone-volume measures of polytopes, Adv. Math. 253 (2014) 50–62.
[26] F. John, Extremum problems with inequalities as subsidiary conditions, in: Studies and Essays Presented to R. Courant on His 60th Birthday, Interscience Publishers, Inc., New York, 1948, pp. 187–204.
[27] B. Klartag, On John-type ellipsoids, in: V.D. Milman, G. Schechtman (Eds.), Geometric Aspects of Functional Analysis, in: Lecture Notes in Math., vol. 1850, Springer, Berlin, 2004, pp. 149–158.
[28] D. Lewis, Ellipsoids defined by Banach ideal norms, Mathematika 26 (1979) 18–29.
[29] M. Ludwig, Ellipsoids and matrix-valued valuations, Duke Math. J. 119 (2003) 159–188.
[30] M. Ludwig, General affine surface areas, Adv. Math. 224 (2010) 2346–2360.
[31] M. Ludwig, M. Reitzner, A classification of SL(n) invariant valuations, Ann. of Math. 172 (2010) 1219–1267.
[32] E. Lutwak, The Brunn–Minkowski–Firey theory. I. Mixed volumes and the Minkowski problem, J. Differential Geom. 38 (1993) 131–150.
[33] E. Lutwak, The Brunn–Minkowski–Firey theory. II. Affine and geominimal surface areas, Adv. Math. 118 (1996) 244–294.
[34] E. Lutwak, D. Yang, G. Zhang, A new ellipsoid associated with convex bodies, Duke Math. J. 104 (2000) 375–390.
[35] E. Lutwak, D. Yang, G. Zhang, L_p affine isoperimetric inequalities, J. Differential Geom. 56 (2000) 111–132.
[36] E. Lutwak, D. Yang, G. Zhang, A new affine invariant for polytopes and Schneider's projection problem, Trans. Amer. Math. Soc. 353 (2001) 1767–1779.
[37] E. Lutwak, D. Yang, G. Zhang, On the L_p-Minkowski problem, Trans. Amer. Math. Soc. 356 (2004) 4359–4370.
[38] E. Lutwak, D. Yang, G. Zhang, L_p John ellipsoids, Proc. Lond. Math. Soc. 90 (2005) 497–520.
[39] E. Lutwak, D. Yang, G. Zhang, Orlicz projection bodies, Adv. Math. 223 (2010) 220–242.
[40] E. Lutwak, D. Yang, G. Zhang, Orlicz centroid bodies, J. Differential Geom. 84 (2010) 365–387.
[41] V.D. Milman, A. Pajor, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space, in: J. Lindenstrauss, V.D. Milman (Eds.), Geometric Aspects of Functional Analysis, in: Lecture Notes in Math., vol. 1376, Springer, New York, 1989, pp. 64–104.
[42] I. Molchanov, Convex and star-shaped sets associated with multivariate stable distributions. I. Moments and densities, J. Multivariate Anal. 100 (2009) 2195–2213.
[43] A. Naor, The surface measure and cone measure on the sphere of l_p^n, Trans. Amer. Math. Soc. 359 (2007) 1045–1079.
[44] G. Paouris, Concentration of mass on convex bodies, Geom. Funct. Anal. 16 (2006) 1021–1049.
[45] G. Paouris, E. Werner, Relative entropy of cone-volumes and L_p centroid bodies, Proc. Lond. Math. Soc. 104 (2012) 253–286.
[46] C.M. Petty, Surface area of a convex body under affine transformations, Proc. Amer. Math. Soc. 12 (1961) 824–828.
[47] G. Pisier, The Volume of Convex Bodies and Banach Geometry, Cambridge University Press, Cambridge, 1989.
[48] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Monogr. Textbooks Pure Appl. Math., vol. 146, Marcel Dekker, New York, 1991.
[49] D. Ryabogin, A. Zvavitch, The Fourier transform and Firey projections of convex bodies, Indiana Univ. Math. J. 53 (2004) 667–682.
[50] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Cambridge University Press, Cambridge, 2014.
[51] C. Schütt, E. Werner, Surface bodies and p-affine surface area, Adv. Math. 187 (2004) 98–145.
[52] A. Stancu, The discrete planar L_0-Minkowski problem, Adv. Math. 167 (2002) 160–174.
[53] A. Stancu, Centro-affine invariants for smooth convex bodies, Int. Math. Res. Not. 180 (2012) 2289–2320.

[54] A.C. Thompson, Minkowski Geometry, Encyclopedia Math. Appl., vol. 63, Cambridge University Press, Cambridge, 1996.
[55] E.M. Werner, Rényi divergence and L_p-affine surface area for convex bodies, Adv. Math. 230 (2012) 1040–1059.
[56] E.M. Werner, D. Ye, New L_p affine isoperimetric inequalities, Adv. Math. 218 (2008) 762–780.
[57] G. Xiong, Extremum problems for the cone volume functional of convex polytopes, Adv. Math. 225 (2010) 3214–3228.
[58] V. Yaskin, M. Yaskina, Centroid bodies and comparison of volumes, Indiana Univ. Math. J. 55 (2006) 1175–1194.
[59] G. Zhu, The Orlicz centroid inequality for star bodies, Adv. in Appl. Math. 48 (2012) 432–445.