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Orlicz–Legendre Ellipsoids

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Abstract The Orlicz–Legendre ellipsoids, which are in the framework of emerging dual Orlicz Brunn–Minkowski theory, are introduced for the first time. They are in some sense dual to the recently found Orlicz–John ellipsoids, and have largely generalized the classical Legendre ellipsoid of inertia. Several new affine isoperimetric inequalities are established. The connection between the characterization of Orlicz–Legendre ellipsoids and isotropy of measures is demonstrated.

Keywords Orlicz Brunn–Minkowski theory · Legendre ellipsoid · Löwner ellipsoid · Isotropy

Mathematics Subject Classification 52A40

1 Introduction

Corresponding to each body in Euclidean n-space $\mathbb{R}^n$, there is a unique ellipsoid with the following property: The moment of inertia of the ellipsoid and the moment of inertia of the body are the same about *every* 1-dimensional subspace of $\mathbb{R}^n$. This ellipsoid is called the *Legendre ellipsoid* of the body. The Legendre ellipsoid is a well-known concept from classical mechanics, and is closely related to the long-standing unsolved maximal slicing problem. See, e.g., Lindenstrauss and Milman [27], and Milman and Pajor [46].

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The Legendre ellipsoid is an object in the *dual* Brunn–Minkowski theory, which was originated by Lutwak [31] and achieved great development since 1980s. See, e.g., [9–11,32,33,55,56]. It is remarkable that for each convex body (compact convex subset with non-empty interior) K in $\mathbb{R}^n$, Lutwak, Yang and Zhang [36] introduced a new ellipsoid by using the notion of L_2-curvature, which is now called the *LYZ ellipsoid* and is precisely the dual analogue of the Legendre ellipsoid.

Following LYZ [36], we write $\Gamma_2 K$ and $\Gamma_{-2} K$ for the Legendre ellipsoid and LYZ ellipsoid, respectively. In [39], LYZ extended the domain of Γ_{-2} to star-shaped sets and showed the relationship between the two ellipsoids: If K is a star-shaped set, then $\Gamma_{-2} K \subset \Gamma_2 K$, with equality if and only if K is an ellipsoid centered at the origin. This inclusion is the geometric analogue of one of the basic inequalities in information theory: the *Cramer–Rao inequality*. When viewed as suitably normalized matrix-valued operators on the space of convex bodies, it was proved by Ludwig [28] that the Legendre ellipsoid and the LYZ ellipsoid are the only linearly invariant operators that satisfy the inclusion-exclusion principle. The Legendre ellipsoid also has applications in Finsler geometry [45].

In the geometry of convex bodies, many extremal problems of an affine nature often have ellipsoids as extremal bodies. Besides the above-mentioned Legendre ellipsoid and LYZ ellipsoid, the *John ellipsoid* $\mathrm{J}K$ [24] and the *Löwner ellipsoid* $\mathrm{L}K$ are of fundamental importance. Since the object considered in this paper is *dual* to the John ellipsoid, in what follows, we recall the John ellipsoid in detail.

Associated with each convex body K in $\mathbb{R}^n$, its John ellipsoid $\mathrm{J}K$ is the unique ellipsoid of maximal volume contained in K. The John ellipsoid has many applications in convex geometry, functional analysis, PDEs, etc. Particularly, by combining the isotropic characterization of the John ellipsoid and the celebrated Brascamp–Lieb inequality, it has a powerful effect on attacking reverse isoperimetric problems. See, e.g., [1–3,40–42].

Since 2005, the family of John ellipsoids has expanded rapidly, and experienced the L_p stage [41] and the very recent Orlicz stage [59]. It is interesting that with the expansion of the family, several ellipsoids, including the LYZ ellipsoid, are found to be close relatives of the John ellipsoid. We do a bit of review on this point.

Motivated by the study of geometry of L_p Brunn–Minkowski theory (see, e.g., [34,35,37]), LYZ [41] introduced a family of ellipsoids, called the L_p John ellipsoids $\mathrm{E}_p K$, $p > 0$. It is striking that the bodies $\mathrm{E}_p K$ form a *spectrum* linking several fundamental objects in convex geometry: If the John point of K, i.e., the center of $\mathrm{J}K$, is at the origin, then $\mathrm{E}_\infty K$ is precisely the classical John ellipsoid $\mathrm{J}K$. The L_2 John ellipsoid $\mathrm{E}_2 K$ is just the LYZ ellipsoid. The L_1 John ellipsoid $\mathrm{E}_1 K$ is the so-called *Petty ellipsoid*. The volume-normalized Petty ellipsoid is obtained by minimizing the surface area of K under $\mathrm{SL}(n)$ transformations of K [14,47].

Throughout this paper, we consider convex $\varphi : [0, \infty) \to [0, \infty)$ that is strictly increasing and satisfies $\varphi(0) = 0$. Along the line of extension, the authors of this paper originally introduced the *Orlicz–John ellipsoids* [59] $\mathrm{E}_\varphi K$ for each convex body K with the origin in its interior, in the framework of booming Orlicz Brunn–Minkowski theory (see, e.g., [12,13,21,29,43,44]). The new Orlicz–John ellipsoids $\mathrm{E}_\varphi K$ generalize LYZ's L_p John ellipsoids $\mathrm{E}_p K$ to the Orlicz setting, analogous to the way that Orlicz norms [50] generalize L_p norms. Indeed, if $\varphi(t) = t^p$, $1 \le p < \infty$, then $\mathrm{E}_\varphi K$ precisely turns to the L_p John ellipsoid $\mathrm{E}_p K$. If $p \to \infty$, then $\mathrm{E}_{\varphi^p} K$ approaches $\mathrm{E}_\infty K$.

The *Löwner ellipsoid* $\mathrm{L}K$ is the unique ellipsoid of minimal volume containing K, which is investigated widely in the field of convex geometry and local theory of Banach spaces. We refer to, e.g., [1,2,14,17–20,24–27,30,49,58].

As LYZ [39] pointed out, there is in fact a "dictionary" correspondence between the Brunn–Minkowski theory [51] and its dual [31]. In retrospect, the John ellipsoid, LYZ ellipsoid and Petty ellipsoid are objects within the Brunn–Minkowski theory; while the Legendre ellipsoid and Löwner ellipsoid are objects within the dual Brunn–Minkowski theory. Along the idea of dictionary relation, we are tempted to consider the naturally posed problem: What is the dual analogue of the newly found Orlicz–John ellipsoid?

One of the main tasks in this paper is to demonstrate this existence of such a *dual* analogue of the Orlicz–John ellipsoid. Incidentally, it precisely acts as the *spectrum* linking the Legendre ellipsoid and Löwner ellipsoid. So, this paper is a sequel of [59].

For star bodies K, L in $\mathbb{R}^n$, define the *normalized dual Orlicz mixed volume* $\bar{\tilde{V}}_\varphi(K, L)$ of K and L with respect to φ by

$$\bar{\tilde{V}}_\varphi(K, L) = \varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{\rho_K}{\rho_L}\right) dV_K^*\right).$$

Here, S^{n-1} is the unit sphere in $\mathbb{R}^n$; ρ_K and ρ_L are the radial functions of K and L, respectively; V_K^* is the normalized dual conical measure of K, defined by

$$dV_K^* = \frac{\rho_K^n}{nV(K)} dS,$$

where S is the spherical Lebesgue measure on S^{n-1}.

Enlightened by our work on Orlicz–John ellipsoids [59], we focus on

Problem $\tilde{S}_\varphi$ Suppose K is a star body in $\mathbb{R}^n$. Find an ellipsoid E, among all origin-symmetric ellipsoids, which solves the following constrained minimization problem:

$$\min_E V(E) \quad \text{subject to} \quad \bar{\tilde{V}}_\varphi(K, E) \le 1.$$

In Sect. 4, we prove that there exists a unique ellipsoid which solves the above minimization problem. It is called the *Orlicz–Legendre ellipsoid* of K with respect to φ, and denoted by $\mathrm{L}_\varphi K$. If $\varphi(t) = t^2$, then $\mathrm{L}_\varphi K$ is precisely the Legendre ellipsoid $\Gamma_2 K$.

It is interesting that the Orlicz–Legendre ellipsoid mirrors the Orlicz–John ellipsoid. Similar to the important property of the Orlicz–John ellipsoid $\mathrm{E}_\varphi K$, in Sect. 5 we show that the Orlicz–Legendre ellipsoid $\mathrm{L}_\varphi K$ is jointly continuous in φ and K.

In Sect. 6, we establish a characterization of Orlicz–Legendre ellipsoids, which is closely related to the isotropy of measures.

In general, Orlicz–Legendre ellipsoids $\mathrm{L}_\varphi K$ do not contain K. In Sect. 7, we prove that: If K is a star body (about the origin) in $\mathbb{R}^n$, then

$$V(\mathrm{L}_\varphi K) \geq V(K),$$

with equality if and only if K is an ellipsoid centered at the origin.

If $\varphi(t) = t^2$, it reduces to the celebrated inequality: $V(\Gamma_2 K) \geq V(K)$, which goes back to Blaschke [6], John [23], Milman and Pajor [46], Petty [48], and also LYZ [36].

2 Preliminaries

2.1 Notation

The setting will be the Euclidean n-space $\mathbb{R}^n$. As usual, $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$, and V denotes the n-dimensional volume.

In addition to its denoting absolute value, without confusion we often use $|\cdot|$ to denote the standard Euclidean norm, on occasion the total mass of a measure, and the absolute value of the determinant of an $n \times n$ matrix.

For a continuous real function f defined on S^{n-1}, write $\|f\|_\infty$ for the L_∞ norm of f. Let $\mathscr{L}^n$ denote the space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$. For $T \in \mathscr{L}^n$, T^t and $\|T\|$ denote the transpose and norm of T, respectively.

A finite positive Borel measure μ on S^{n-1} is said to be *isotropic* if

$$\frac{n}{|\mu|} \int_{S^{n-1}} (u \cdot v)^2 d\mu(u) = 1, \quad \text{for all } v \in S^{n-1}.$$

For nonzero $x \in \mathbb{R}^n$, the notation $x \otimes x$ represents the rank 1 linear operator on $\mathbb{R}^n$ that takes y to $(x \cdot y)x$. It immediately gives

$$\mathrm{tr}(x \otimes x) = |x|^2.$$

Equivalently, μ is isotropic if

$$\frac{n}{|\mu|} \int_{S^{n-1}} u \otimes u d\mu(u) = I_n,$$

where I_n denotes the identity operator on $\mathbb{R}^n$. For more information on the isotropy of measures, we refer to [5,14,15,46].

2.2 Orlicz Norms

Throughout this paper, Φ denotes the class of convex functions $\varphi : [0, \infty) \to [0, \infty)$ that are strictly increasing and satisfy $\varphi(0) = 0$.

We say a sequence $\{\varphi_i\}_{i\in\mathbb{N}} \subset \Phi$ is such that $\varphi_i \to \varphi_0 \in \Phi$, provided

$$|\varphi_i - \varphi_0|_I = \max_{t\in I} |\varphi_i(t) - \varphi_0(t)| \to 0,$$

for each compact interval $I \subset [0, \infty)$.

Let μ be a finite positive Borel measure on S^{n-1}. For a continuous function $f : S^{n-1} \to [0, \infty)$, the *Orlicz norm* $\|f : \mu\|_\varphi$ of f is defined by

$$\|f : \mu\|_\varphi = \inf\left\{\lambda > 0 : \frac{1}{|\mu|}\int_{S^{n-1}} \varphi\left(\frac{f}{\lambda}\right) d\mu \le \varphi(1)\right\}.$$

If $\varphi(t) = t^p$, $1 \le p < \infty$, then $\|f : \mu\|_\varphi$ is just the classical L_p norm. According to the context, without confusion we write $\|f\|_\varphi$ for $\|f : \mu\|_\varphi$.

Lemma 2.1 was previously proved in [21], which will be used frequently.

Lemma 2.1 *Suppose μ is a finite positive Borel measure on S^{n-1} and the function $f : S^{n-1} \to [0, \infty)$ is continuous and such that $\mu(\{f \ne 0\}) > 0$. Then the function*

$$\psi(\lambda) = \int_{S^{n-1}} \varphi\left(\frac{f}{\lambda}\right) d\mu \quad \lambda \in (0, \infty),$$

has the following properties:

(1) ψ *is continuous and strictly decreasing in* $(0, \infty)$*;*
(2) $\lim_{\lambda\to 0^+} \psi(\lambda) = \infty$*;*
(3) $\lim_{\lambda\to\infty} \psi(\lambda) = 0$*;*
(4) $0 < \psi^{-1}(a) < \infty$ *for each* $a \in (0, \infty)$.

Consequently, the Orlicz norm $\|f\|_\varphi$ is strictly positive. Moreover,

$$\|f\|_\varphi = \lambda_0 \iff \frac{1}{|\mu|}\int_{S^{n-1}} \varphi\left(\frac{f}{\lambda_0}\right) d\mu = \varphi(1).$$

2.3 Convex Bodies and Star Bodies

The *support function* h_K of a compact convex set K in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \quad \text{for } x \in \mathbb{R}^n.$$

For $T \in \mathrm{GL}(n)$, the support function of the image $TK = \{Tx : x \in K\}$ is given by

$$h_{TK}(x) = h_K(T^t x).$$

As usual, a *body* is a compact set with non-empty interior. Write $\mathcal{K}_o^n$ for the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. $\mathcal{K}_o^n$ is often equipped with the *Hausdorff metric* δ_H, which is defined by

$$\delta_H(K_1, K_2) = \|h_{K_1} - h_{K_2}\|_\infty, \quad \text{for } K_1, K_2 \in \mathcal{K}_o^n.$$

Next, we turn to some basics on star bodies.

A set $K \subseteq \mathbb{R}^n$ is *star-shaped*, if $\lambda x \in K$ for any $(\lambda, x) \in [0, 1] \times K$. For a non-empty, compact and star-shaped set K in $\mathbb{R}^n$, its *radial function* ρ_K is defined by

$$\rho_K(x) = \sup\{\lambda \geq 0 : \lambda x \in K\}, \quad \text{for } x \in \mathbb{R}^n \backslash \{o\}.$$

It is easily seen that ρ_K is homogeneous of degree -1. For $T \in \mathrm{GL}(n)$, we obviously have

$$\rho_{TK}(x) = \rho_K(T^{-1}x). \tag{2.1}$$

A star-shaped set K is called a *star body* about the origin o, if $o \in \mathrm{int}K$ and ρ_K is continuous on S^{n-1}. Write $\mathcal{S}_o^n$ for the class of star bodies about the origin o in $\mathbb{R}^n$. $\mathcal{S}_o^n$ is often equipped with the *dual metric* $\tilde{\delta}_H$, which is defined by

$$\tilde{\delta}_H(K_1, K_2) = \|\rho_{K_1} - \rho_{K_2}\|_\infty, \quad \text{for } K_1, K_2 \in \mathcal{S}_o^n.$$

The *dual conical measure* $\tilde{V}_K$ of a star body $K \in \mathcal{S}_o^n$ is a Borel measure on S^{n-1} defined by

$$d\tilde{V}_K = \frac{\rho_K^n}{n} dS.$$

It is convenient to use its normalization V_K^*, given by $V_K^* = \frac{\tilde{V}_K}{V(K)}$. Observe that V_K^* was first introduced by LYZ [44] to define Orlicz centroid bodies. Note that the dual conical measure differs from the cone-volume measure (see, e.g., [7,8,21,22,38,43,52,53,60]), but both are outgrowth from the cone measure (see, e.g., [4,16]).

Note that for each Borel subset $\omega \subseteq S^{n-1}$, we also have

$$\tilde{V}_K(\omega) = V\left(K \cap \{su : s \geq 0 \text{ and } u \in \omega\}\right).$$

Thus, it follows that

$$\tilde{V}_{TK}(\omega) = \tilde{V}_K(\langle T^{-1}\omega \rangle), \quad \text{for } T \in \mathrm{SL}(n), \tag{2.2}$$

where $\langle T^{-1}\omega \rangle = \{\frac{T^{-1}u}{|T^{-1}u|} : u \in \omega\}$.

For $K \in \mathcal{K}_o^n$, its *polar body* K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \text{ for } \ y \in K\}.$$

For $K \in \mathcal{K}_o^n$, we have

$$\rho_{K^*}(u) = \frac{1}{h_K(u)} \quad \text{and} \quad h_{K^*}(u) = \frac{1}{\rho_K(u)}, \quad \text{for } u \in S^{n-1}, \tag{2.3}$$

and

$$(TK)^* = T^{-t}K^*, \quad \text{for } T \in \mathrm{GL}(n). \tag{2.4}$$

2.4 Ellipsoids and Linear Operators

Throughout, $\mathcal{E}^n$ is used exclusively to denote the class of n-dimensional origin-symmetric ellipsoids in $\mathbb{R}^n$.

For $E \in \mathcal{E}^n$, let $d(E)$ denote its maximal principal radius. Two facts are in order. First, $T \in \mathscr{L}^n$ is non-degenerated, if and only if the ellipsoid TB is non-degenerated. Second, for $T \in \mathscr{L}^n$, since

$$\|T\| = \max_{u \in S^{n-1}} |Tu| = \max_{u \in S^{n-1}} |T^t u| = \left\|T^t\right\|,$$

it follows that

$$d(TB) = \max_{u \in S^{n-1}} h_{TB}(u) = \max_{u \in S^{n-1}} |T^t u| = \max_{u \in S^{n-1}} |Tu| = \max_{u \in S^{n-1}} h_{T^t B}(u) = d(T^t B).$$

Let

$$d_n(T_1, T_2) = \|T_1 - T_2\|, \quad \text{for } T_1, T_2 \in \mathscr{L}^n.$$

Then the metric space $(\mathscr{L}^n, d_n)$ is complete. Since $\mathscr{L}^n$ is of finite dimension, a set in $(\mathscr{L}^n, d_n)$ is compact if and only if it is bounded and closed.

We conclude this section with the following basic known facts, which will be used in Sects. 4 and 5.

Lemma 2.2 *Suppose* $\{T_j\}_{j \in \mathbb{N}} \subset \mathrm{SL}(n)$. *Then*

$$\|T_j\| \to \infty \quad \Longleftrightarrow \quad \|T_j^{-1}\| \to \infty.$$

Thus, $\{T_j\}_{j \in \mathbb{N}}$ *is bounded from above, if and only if* $\{T_j^{-1}\}_{j \in \mathbb{N}}$ *is bounded from above.*

Lemma 2.3 *Suppose* $E_0 \in \mathcal{E}^n$, $\{E_j\}_{j \in \mathbb{N}} \subset \mathcal{E}^n$ *and* $V(E_j) = a$, $\forall j \in \mathbb{N}$, $a > 0$. *Then* $E_j \to E_0$ *with respect to* δ_H, *if and only if* $E_j \to E_0$ *with respect to* $\tilde{\delta}_H$.

3 Dual Orlicz Mixed Volumes

In order to define Orlicz–Legendre ellipsoids, we make some necessary preparations.

Definition 3.1 Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. The geometric quantity

$$\tilde{V}_\varphi(K, L) = \int_{S^{n-1}} \varphi\left(\frac{\rho_K}{\rho_L}\right) d\tilde{V}_K$$

is called the *dual Orlicz mixed volume* of K and L with respect to φ. The quantity

$$\bar{\tilde{V}}_{\varphi}(K, L) = \varphi^{-1}\left(\frac{\tilde{V}_{\varphi}(K, L)}{V(K)}\right) = \varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{\rho_K}{\rho_L}\right) dV_K^*\right)$$

is called the *normalized dual Orlicz mixed volume* of K and L with respect to φ.

It is noted that dual Orlicz mixed volumes were previously introduced in [57].

Obviously, $\tilde{V}_{\varphi}(K, K) = \varphi(1)V(K)$, and $\bar{\tilde{V}}(K, K) = 1$. If $\varphi(t) = t^p$, $1 \le p < \infty$, then $\tilde{V}_{\varphi}(K, L)$ reduces to the classical dual mixed volume

$$\tilde{V}_{-p}(K, L) = \int_{S^{n-1}} \left(\frac{\rho_K}{\rho_L}\right)^p d\tilde{V}_K,$$

and $\bar{\tilde{V}}_{\varphi}(K, L)$ reduces to normalized dual mixed volume [54]

$$\bar{\tilde{V}}_{-p}(K, L) = \left[\frac{\tilde{V}_p(K, L)}{V(K)}\right]^{\frac{1}{p}} = \left(\int_{S^{n-1}} \left(\frac{\rho_K}{\rho_L}\right)^p dV_K^*\right)^{\frac{1}{p}}.$$

Combining Definition 3.1 with (2.1) and (2.2), we have the following.

Lemma 3.2 *Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then $\tilde{V}_{\varphi}(TK, L) = |T|\tilde{V}_{\varphi}(K, T^{-1}L)$, for $T \in \mathrm{GL}(n)$.*

Along with the functional $\tilde{V}_{\varphi}(K, L)$, we introduce

Definition 3.3 Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$; define

$$O_{\varphi}(K, L) = \left\|\frac{\rho_K}{\rho_L} : \tilde{V}_K\right\|_{\varphi} = \inf\left\{\lambda > 0 : \varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{\rho_K}{\lambda\rho_L}\right) dV_K^*\right) \le 1\right\}.$$

Obviously, $O_{\varphi}(K, K) = 1$. If $\varphi(t) = t^p$, $1 \le p < \infty$, then $O_{\varphi}(K, L) = \bar{\tilde{V}}_{-p}(K, L)$.

From Definition 3.3 and Definition 3.1, we have

$$\begin{aligned} O_{\varphi}(K, L) &= \inf\left\{\lambda > 0 : \frac{\tilde{V}_{\varphi}(K, \lambda L)}{V(K)} \le \varphi(1)\right\} \\ &= \inf\left\{\lambda > 0 : \bar{\tilde{V}}_{\varphi}(K, \lambda L) \le 1\right\}. \end{aligned}$$

Combining this with Lemma 3.2, we immediately obtain

Lemma 3.4 *Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then*

(1) $O_{\varphi}(TK, L) = O_{\varphi}(K, T^{-1}L)$, *for all* $T \in \mathrm{GL}(n)$.
(2) $O_{\varphi}(\lambda K, L) = O_{\varphi}(K, \lambda^{-1}L) = \lambda O_{\varphi}(K, L)$, *for all* $\lambda > 0$.

The next lemma provides a simple but powerful identity.

Lemma 3.5 *Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then*

$$\tilde{\bar{V}}_\varphi(K, O_\varphi(K, L)L) = 1.$$

Consequently, there is the following equivalence

$$\tilde{\bar{V}}_\varphi(K, L) = 1 \quad \Longleftrightarrow \quad O_\varphi(K, L) = 1.$$

Proof From Definition 3.1, Definition 3.3, together with Lemma 2.1, it follows that

$$\varphi\left(\tilde{\bar{V}}_\varphi(K, O_\varphi(K, L)L)\right) = \int_{S^{n-1}} \varphi\left(\frac{\rho_K}{O_\varphi(K, L)\rho_K}\right) dV_K^* = \varphi(1).$$

Thus, $\tilde{\bar{V}}_\varphi(K, O_\varphi(K, L)L) = 1$. By Lemma 2.1 again, the desired equivalence follows. □

What follows establishes the dual Orlicz Minkowski inequalities.

Lemma 3.6 *Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then*

$$\tilde{\bar{V}}_\varphi(K, L) \geq \left(\frac{V(K)}{V(L)}\right)^{\frac{1}{n}}, \tag{3.1}$$

and

$$O_\varphi(K, L) \geq \left(\frac{V(K)}{V(L)}\right)^{\frac{1}{n}}. \tag{3.2}$$

Each equality holds in the above inequalities if and only if K and L are dilates.

Proof From Definition 3.1, the fact that φ^{-1} is strictly increasing in $(0, \infty)$ together with the convexity of φ and Jensen's inequality, the definition of V_K^*, and Hölder's inequality, we have

$$\begin{aligned}
\tilde{\bar{V}}_\varphi(K, L) &= \varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{\rho_K}{\rho_L}\right) dV_K^*\right) \\
&\geq \frac{1}{nV(K)} \int_{S^{n-1}} \frac{\rho_K^{n+1}}{\rho_L} dS \\
&\geq \frac{1}{V(K)} \left(\frac{1}{n}\int_{S^{n-1}} \rho_K^{(n+1)\cdot\frac{n}{n+1}} dS\right)^{\frac{n+1}{n}} \left(\frac{1}{n}\int_{S^{n-1}} \rho_L^{-1\cdot(-n)} dS\right)^{-\frac{1}{n}} \\
&= \left(\frac{V(K)}{V(L)}\right)^{\frac{1}{n}}.
\end{aligned}$$

By the equality condition of Hölder's inequality, the equality in the fourth line occurs only if ρ_K/ρ_L is a positive constant on S^{n-1}. Thus, the equality holds in (3.1) only if K and L are dilates. Conversely, if $K = sL$ for some $s > 0$, then $\bar{\tilde{V}}_\varphi(K, L) = s = (V(K)/V(L)))^{1/n}$.

By Lemma 3.5 with inequality (3.1), we can derive inequality (3.2) directly. □

The next lemma is crucial to proving the continuity of the functionals $\tilde{V}_\varphi(K, L)$, $\bar{\tilde{V}}_\varphi(K, L)$ and $O_\varphi(K, L)$ in (K, L, φ).

Lemma 3.7 *Suppose f_i, f are strictly positive and continuous functions on S^{n-1}; $\varphi_k, \varphi \in \Phi$; μ_l, μ are Borel probability measures on S^{n-1}; $i, k, l \in \mathbb{N}$. If $f_i \to f$ uniformly, $\varphi_k \to \varphi$ uniformly on each compact set, and $\mu_l \to \mu$ weakly, then*

$$\int_{S^{n-1}} \varphi_k(f_i)\, d\mu_l \to \int_{S^{n-1}} \varphi(f)\, d\mu, \tag{3.3}$$

$$\varphi_k^{-1}\left(\int_{S^{n-1}} \varphi_k(f_i)\, d\mu_l\right) \to \varphi^{-1}\left(\int_{S^{n-1}} \varphi(f)\, d\mu\right), \tag{3.4}$$

and

$$\|f_i : \mu_l\|_{\varphi_k} \to \|f : \mu\|_\varphi. \tag{3.5}$$

Proof Since $f_i \to f$ uniformly, there exists an $N_0 \in \mathbb{N}$, such that

$$\frac{1}{2} \min_{u \in S^{n-1}} f(u) \le f_i \le 2 \max_{u \in S^{n-1}} f(u), \quad \text{for } i > N_0.$$

Let

$$c_m = \min\left\{\frac{1}{2} \min_{u \in S^{n-1}} f(u),\ \min_{u \in S^{n-1}} f_i(u),\ \text{with } i \le N_0\right\},$$

and

$$c_M = \max\left\{2 \max_{u \in S^{n-1}} f(u),\ \max_{u \in S^{n-1}} f_i(u),\ \text{with } i \le N_0\right\}.$$

So, by the strict positivity and continuity of f_i and f, we have

$$f(u), f_i(u) \in [c_m, c_M] \subset (0, \infty), \quad \text{for } u \in S^{n-1} \text{ and } i \in \mathbb{N}. \tag{3.6}$$

From the continuity and uniform convergence of f_i and φ_k on $[c_m, c_M]$, it follows that as $i, k \to \infty$,

$$\varphi_k(f_i) \to \varphi(f), \quad \text{uniformly on } S^{n-1}.$$

Added that $\mu_l \to \mu$ weakly as $l \to \infty$, one immediately concludes (3.3) and (3.4).

Finally, we conclude to show (3.5).

From (3.6) together with the strict monotonicity of φ and φ^{-1}, Lemma 2.1, and (3.6) together with the strict monotonicity of φ and φ^{-1} again, it follows that

$$\frac{c_m}{\|f_i:\mu_l\|_{\varphi_k}} \le \varphi_k^{-1}\left(\int_{S^{n-1}} \varphi_k\left(\frac{f_i}{\|f_i:\mu_l\|_{\varphi_k}}\right)d\mu_l\right) = 1 \le \frac{c_M}{\|f_i:\mu_l\|_{\varphi_k}},$$

which immediately gives

$$c_m \le \|f_i:\mu_l\|_{\varphi_k} \le c_M, \quad \text{for } i,k,l \in \mathbb{N}.$$

Since $\left\{\|f_i:\mu_l\|_{\varphi_k} : i,k,l \in \mathbb{N}\right\}$ is bounded, to prove (3.5), it suffices to prove that each convergent subsequence $\{\|f_{i_p}:\mu_{l_r}\|_{\varphi_{k_q}}\}_{p,q,r\in\mathbb{N}}$ of $\left\{\|f_i:\mu_l\|_{\varphi_k} : i,k,l \in \mathbb{N}\right\}$ necessarily converges to $\|f:\mu\|_\varphi$, as $i_p, k_q, l_r \to \infty$.

Assume $\lim\limits_{p,q,r\to\infty} \left\|f_{i_p}:\mu_{l_r}\right\|_{\varphi_{k_q}} = \lambda_0$. Then, $\frac{f_{i_p}}{\left\|f_{i_p}:\mu_{l_r}\right\|_{\varphi_{k_q}}} \to \frac{f}{\lambda_0}$ uniformly on S^{n-1}. Hence, by (3.4), we have

$$\lim_{p,q,r\to\infty} \varphi_{k_q}^{-1}\left(\int_{S^{n-1}} \varphi_{k_q}\left(\frac{f_{i_p}}{\left\|f_{i_p}:\mu_{l_r}\right\|_{\varphi_{k_q}}}\right)d\mu_{l_r}\right) = \varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{f}{\lambda_0}\right)d\mu\right).$$

Meanwhile, since

$$\varphi_{k_q}^{-1}\left(\int_{S^{n-1}} \varphi_{k_q}\left(\frac{f_{i_p}}{\left\|f_{i_p}:\mu_{l_r}\right\|_{\varphi_{k_q}}}\right)d\mu_{l_r}\right) = 1, \quad \text{for each } (p,q,r),$$

it yields that $\varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{f}{\lambda_0}\right)d\mu\right) = 1$. From Lemma 2.1, it follows that $\lambda_0 = \|f:\mu\|_\varphi$, which concludes (3.5). □

Using Lemma 3.7, we immediately obtain

Lemma 3.8 *Suppose $K, K_i, L, L_j \in \mathcal{S}_o^n$ and $\varphi, \varphi_k \in \Phi$, $i, j, k \in \mathbb{N}$. If $K_i \to K$, $L_j \to L$ and $\varphi_k \to \varphi$; then*

$$\lim_{i,j,k\to\infty} \tilde{V}_{\varphi_k}(K_i, L_j) = \tilde{V}_\varphi(K, L),$$

$$\lim_{i,j,k\to\infty} \bar{\tilde{V}}_{\varphi_k}(K_i, L_j) = \bar{\tilde{V}}_\varphi(K, L),$$

and

$$\lim_{i,j,k\to\infty} O_{\varphi_k}(K_i, L_j) = O_\varphi(K, L).$$

Proof That $K_i \to K$ and $L_j \to L$ yields ρ_{K_i}/ρ_{L_j} and ρ_K/ρ_L are strictly positive continuous on S^{n-1}; $\rho_{K_i}/\rho_{L_j} \to \rho_K/\rho_L$ uniformly on S^{n-1}; $\tilde{V}_{K_i} \to \tilde{V}_K$ weakly, and $V^*_{K_i} \to V^*_K$ weakly. Combining these facts and applying Lemma 3.7, it yields the desired limits. □

Recall that

$$\bar{\tilde{V}}_{-1}(K, L) = \int_{S^{n-1}} \frac{\rho_K}{\rho_L} dV^*_K, \quad \text{for } K, L \in \mathcal{S}^n_o.$$

The next lemma will be used in Sect. 5.

Lemma 3.9 *Suppose $K, L \in \mathcal{S}^n_o$, $\varphi \in \Phi$ and $p \in [1, \infty)$. Then for $1 < p < q < \infty$,*

$$\bar{\tilde{V}}_{-1}(K, L) \le O_\varphi(K, L) \le O_{\varphi^p}(K, L) \le O_{\varphi^q}(K, L) \le \left\| \frac{\rho_K}{\rho_L} \right\|_\infty = \lim_{p\to\infty} O_{\varphi^p}(K, L).$$

Proof For $\lambda \in (0, \infty)$, let $g_p(\lambda) = \left(\int_{S^{n-1}} \varphi\left(\frac{\rho_K}{\lambda\rho_L}\right)^p dV^*_K\right)^{1/p}, \quad 1 \le p < \infty.$

From the definition of $\bar{\tilde{V}}_{-1}(K, \lambda L)$ together with the convexity of φ and Jensen's inequality, Hölder's inequality, and finally the fact that $\lim_{p\to\infty} g_p(\lambda) = \varphi\left(\frac{1}{\lambda}\|\frac{\rho_K}{\rho_L}\|_\infty\right)$, it follows that

$$\varphi\left(\bar{\tilde{V}}_{-1}(K, \lambda L)\right) \le g_1(\lambda) \le g_p(\lambda) \le g_q(\lambda) \le \varphi\left(\frac{1}{\lambda}\left\|\frac{\rho_K}{\rho_L}\right\|_\infty\right), \quad \text{for } 1 < p < q < \infty.$$

Thus, from the definitions of $O_{\varphi^p}(K, L)$ and $g_p(\lambda)$, and the monotonicity of φ^{-1}, it yields the desired inequalities except the equality on the right.

Finally, we proceed to show $\left\|\frac{\rho_K}{\rho_L}\right\|_\infty = \lim_{p\to\infty} O_{\varphi^p}(K, L)$.

Take $\{p_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ with $\lim_{j\to\infty} p_j = \infty$. For brevity, let $\lambda_\infty = \|\frac{\rho_K}{\rho_L}\|_\infty$, $\lambda_j = O_{\varphi^{p_j}}(K, L)$, and $g_\infty(\lambda) = \varphi\left(\frac{1}{\lambda}\|\frac{\rho_K}{\rho_L}\|_\infty\right)$. Recall that g_{p_j} and g_∞ are positive and continuous on $[\lambda_1, \lambda_\infty]$, and g_{p_j} increases to g_∞ pointwise on $[\lambda_1, \lambda_\infty]$. By Dini's theorem, $g_{p_j} \to g_\infty$, uniformly on $[\lambda_1, \lambda_\infty]$. Consequently,

$$\lim_{j\to\infty} g_j(\lambda_j) = \left(\lim_{j\to\infty} g_j\right)\left(\lim_{j\to\infty} \lambda_j\right) = g_\infty\left(\lim_{j\to\infty} \lambda_j\right).$$

Added that g_∞ is strictly decreasing on $[\lambda_1, \lambda_\infty]$, and $g_{p_j}(\lambda_j) = \varphi(1)$ for each j, we obtain

$$\lim_{j\to\infty} \lambda_j = \lambda_\infty,$$

which completes the proof. □

4 Orlicz–Legendre Ellipsoids

Let $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. For any $T \in \mathrm{SL}(n)$, by Lemma 3.6 it gives

$$\bar{\bar{\tilde{V}}}_{\varphi}(K, TB) \geq \left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}} \quad \text{and} \quad O_{\varphi}(K, TB) \geq \left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}}.$$

In view of the intimate connection between $\bar{\bar{\tilde{V}}}_{\varphi}$ and O_{φ}, to find the so-called Orlicz–Legendre ellipsoids, we also consider the following three problems, which are closely related to our originally posed Problem $\tilde{S}_{\varphi}$.

Problem P_1 Find an ellipsoid E, among all origin-symmetric ellipsoids, which solves the constrained minimization problem

$$\min_{E} \bar{\bar{\tilde{V}}}_{\varphi}(K, E) \quad \text{subject to} \quad V(E) \leq \omega_n.$$

Problem P_2 Find an ellipsoid E, among all origin-symmetric ellipsoids, which solves the constrained minimization problem

$$\min_{E} O_{\varphi}(K, E) \quad \text{subject to} \quad V(E) \leq \omega_n.$$

The homogeneity of the volume functional and Orlicz norm prompts us to consider the following Problem P_3, which is in some sense dual to Problem P_2.

Problem P_3 Find an ellipsoid E, among all origin-symmetric ellipsoids, which solves the constrained maximization problem

$$\min_{E} V(E) \quad \text{subject to} \quad O_{\varphi}(K, E) \leq 1.$$

For convenient comparison, we restate Problem $\tilde{S}_{\varphi}$ as the following.

Problem $\tilde{S}_{\varphi}$ Find an ellipsoid E, among all origin-symmetric ellipsoids, which solves the constrained maximization problem

$$\min_{E} V(E) \quad \text{subject to} \quad \bar{\bar{\tilde{V}}}_{\varphi}(K, E) \leq 1.$$

This section is organized as follows. Lemma 4.2 and Theorem 4.3 demonstrate the existence and uniqueness of the solution to P_1, respectively. The connection between P_1 and P_2 is established by Lemma 4.4. Theorem 4.5 demonstrates the existence and uniqueness of a solution to P_2. Theorem 4.6 shows that the solutions to P_2 and P_3 only differ by a scalar factor. Lemma 4.7 reveals that P_3 and $\tilde{S}_{\varphi}$ are essentially identical. Therefore, the notion of Orlicz–Legendre ellipsoids is ready to come out.

From Definition 3.1 together with the fact that φ^{-1} is strictly increasing in $(0, \infty)$, the objective functional in P_1 can be replaced by $\tilde{V}_{\varphi}(K, E)$.

Lemma 4.1 *Suppose $T \in \mathrm{SL}(n)$. Then*

$$\lim_{\substack{T\in \mathrm{SL}(n)\\ \|T\|\to\infty}} \tilde{V}_\varphi(K, TB) = \infty \quad \text{and} \quad \lim_{\substack{T\in \mathrm{SL}(n)\\ \|T\|\to\infty}} O_\varphi(K, TB) = \infty.$$

Proof Let $T \in \mathrm{SL}(n)$ and $r_K = \min_{S^{n-1}} \rho_K$. Then $r_K B \subseteq K$. For $|\alpha| > n$, it is known that

$$\lim_{\substack{T\in \mathrm{SL}(n)\\ \|T\|\to\infty}} \int_{S^{n-1}} |Tu|^\alpha dS(u) = \infty. \tag{4.1}$$

From the definition of $\tilde{V}_\varphi(K, TB)$ together with the convexity of φ and Jensen's inequality, the facts

$$\int_{S^{n-1}} \frac{\rho_K^{n+1}}{\rho_{TB}} dS = \int_{S^{n-1}} \rho_{T^{-1}K}^{n+1} dS \quad \text{and} \quad T^{-1}K \supseteq T^{-1}(r_K B)$$

together with the monotonicity of φ, and finally (2.1), it follows that

$$\begin{aligned}
\frac{\tilde{V}_\varphi(K, TB)}{V(K)} &\geq \varphi\left(\frac{1}{nV(K)} \int_{S^{n-1}} \frac{\rho_K^{n+1}}{\rho_{TB}} dS\right) \\
&\geq \varphi\left(\frac{1}{nV(K)} \int_{S^{n-1}} \rho_{T^{-1}(r_K B)}^{n+1} dS\right) \\
&= \varphi\left(\frac{r_K^{n+1}}{nV(K)} \int_{S^{n-1}} |Tu|^{n+1} dS(u)\right).
\end{aligned}$$

Thus, by the monotonicity of φ and (4.1), it concludes the first limit.

To prove the second limit, we argue by contradiction and assume it to be false. Then, there is a constant $c \in (0, \infty)$ such that $\sup\left\{O_\varphi(K, T_j B) : j \in \mathbb{N}\right\} < c$, for any sequence $\{T_j\}_j \subset \mathbb{N}$ with $\lim_{j\to\infty} \|T_j\| = \infty$. By using arguments similar to those above, we can show that for each $j \in \mathbb{N}$,

$$\frac{\tilde{V}_\varphi(K, O_\varphi(K, T_j B)T_j B)}{V(K)} \geq \varphi\left(\frac{r_K^{n+1}}{ncV(K)} \int_{S^{n-1}} |T_j u|^{n+1} dS(u)\right).$$

Combining this with (4.1), we have

$$\lim_{j\to\infty} \frac{\tilde{V}_\varphi(K, O_\varphi(K, T_j B)T_j B)}{V(K)} = \infty.$$

However, this obviously contradicts Lemma 3.5. $\square$

Now, using Lemma 4.1, we can prove the existence of a solution to problem P_1.

Lemma 4.2 *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then there exists a solution to* P_1.

Proof Note that any $E \in \mathcal{E}^n$ with $V(E) < \omega_n$ cannot be a solution to P_1. Hence, Problem P_1 can be equivalently restated as

$$\inf\left\{\tilde{V}_\varphi(K, TB) : T \in \mathrm{SL}(n)\right\}.$$

Observe that the infimum exists, since

$$V(K)\varphi\left(\left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}}\right) \le \inf\left\{\tilde{V}_\varphi(K, TB) : T \in \mathrm{SL}(n)\right\} \le \tilde{V}_\varphi(K, B) < \infty,$$

where the left inequality follows from Lemma 3.6 and Definition 3.1.

Let

$$\mathcal{T} = \left\{T \in \mathrm{SL}(n) : \tilde{V}_\varphi(K, TB) \le \tilde{V}_\varphi(K, B)\right\}.$$

From the definition of d_n (see Sect. 2.4) and Lemma 3.8, $\tilde{V}_\varphi(K, TB)$ is continuous in $T \in (\mathrm{SL}(n), d_n)$. Thus, the set $\mathcal{T}$ is closed in $(\mathrm{SL}(n), d_n)$. Meanwhile, the definition of $\mathcal{T}$ and Lemma 4.1 guarantee that $\mathcal{T}$ is bounded in $(\mathrm{SL}(n), d_n)$. Hence, $\mathcal{T}$ is compact.

The continuity of $\tilde{V}_\varphi(K, TB)$ on $(\mathcal{T}, d_n)$ implies that there exists a $T_0 \in \mathcal{T}$ such that

$$\tilde{V}_\varphi(K, T_0 B) = \min\{\tilde{V}_\varphi(K, TB) : T \in \mathcal{T}\} = \inf\{\tilde{V}_\varphi(K, TB) : T \in \mathrm{SL}(n)\},$$

which completes the proof. □

Theorem 4.3 *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then, modulo orthogonal transformations, there exists a unique* $\mathrm{SL}(n)$ *transformation solving the extremal problem*

$$\min\left\{\tilde{V}_\varphi(K, TB) : T \in \mathrm{SL}(n)\right\}.$$

Equivalently, there exists a unique solution to Problem P_1.

Proof Lemma 4.2 shows the existence. We prove the uniqueness by contradiction.

Assume that $T_1, T_2 \in \mathrm{SL}(n)$ both solve the considered minimization problem. Let $E_1 = T_1 B$, $E_2 = T_2 B$. It is known that each $T \in \mathrm{SL}(n)$ can be represented in the form $T = PQ$, where P is symmetric, positive definite and Q is orthogonal. So, without loss of generality, we may assume that T_1, T_2 are symmetric and positive definite.

By the Minkowski inequality for symmetric and positive definite matrices, we have

$$\det\left(\frac{T_1^{-1} + T_2^{-1}}{2}\right)^{\frac{1}{n}} > \frac{1}{2}\det\left(T_1^{-1}\right)^{\frac{1}{n}} + \frac{1}{2}\det\left(T_2^{-1}\right)^{\frac{1}{n}} = 1.$$

Let

$$T_3^{-1} = \det\left(\frac{T_1^{-1}+T_2^{-1}}{2}\right)^{-\frac{1}{n}} \frac{T_1^{-1}+T_2^{-1}}{2}.$$

Then $T_3 \in \mathrm{SL}(n)$ is symmetric. Moreover, for all $u \in S^{n-1}$, we have

$$h_{T_3^{-1}B}(u) < h_{\frac{T_1^{-1}+T_2^{-1}}{2}B}(u) \leq \frac{1}{2}h_{T_1^{-1}B}(u) + \frac{1}{2}h_{T_1^{-1}B}(u).$$

Thus, from the fact that φ is strictly increasing and convex in $[0, \infty)$, it implies that

$$\varphi\left(\rho_K h_{T_3^{-1}B}\right) < \frac{1}{2}\varphi\left(\rho_K h_{T_1^{-1}B}\right) + \frac{1}{2}\varphi\left(\rho_K h_{T_2^{-1}B}\right).$$

Hence, letting $E_3 = T_3 B$ and using the fact

$$\tilde{V}_\varphi(K, E_i) = \int_{S^{n-1}} \varphi\left(\rho_K h_{T_i^{-1}B}\right) d\tilde{V}_K, \quad i = 1, 2, 3,$$

it gives

$$\tilde{V}_\varphi(K, E_3) < \tilde{V}_\varphi(K, E_1) = \tilde{V}_\varphi(K, E_2).$$

However, from the fact that $T_3 \in \mathrm{SL}(n)$ and the assumption on E_1 and E_2, we also have

$$\tilde{V}_\varphi(K, E_3) \geq \tilde{V}_\varphi(K, E_1) = \tilde{V}_\varphi(K, E_2),$$

which contradicts the above. This completes the proof. □

Lemma 4.4 *Suppose $E_0 \in \mathcal{E}^n$ and $V(E_0) = \omega_n$. Then, for any $T \in \mathrm{SL}(n)$,*

$$\tilde{V}_\varphi\left(K, O_\varphi(K, E_0)E_0\right) \leq \tilde{V}_\varphi\left(K, O_\varphi(K, E_0)TE_0\right)$$

if and only if

$$O_\varphi(K, E_0) \leq O_\varphi(K, TE_0).$$

Proof From Definition 3.1 together with the strict monotonicity of φ^{-1}, Lemma 3.5, and Lemma 2.1 together with Definition 3.3, it follows that

$$\begin{aligned}
&\tilde{V}_\varphi\left(K, O_\varphi(K, E_0)E_0\right) \le \tilde{V}_\varphi\left(K, O_\varphi(K, E_0)TE_0\right)\\
&\quad\Longleftrightarrow\quad \bar{\bar{\tilde{V}}}_\varphi\left(K, O_\varphi(K, E_0)E_0\right) \le \bar{\bar{\tilde{V}}}_\varphi\left(K, O_\varphi(K, E_0)TE_0\right)\\
&\quad\Longleftrightarrow\quad 1 \le \bar{\bar{\tilde{V}}}_\varphi\left(K, O_\varphi(K, E_0)TE_0\right)\\
&\quad\Longleftrightarrow\quad \bar{\bar{\tilde{V}}}_\varphi\left(K, O_\varphi(K, TE_0)TE_0\right) \le \bar{\bar{\tilde{V}}}_\varphi\left(K, O_\varphi(K, E_0)TE_0\right)\\
&\quad\Longleftrightarrow\quad \tilde{V}_\varphi\left(K, O_\varphi(K, TE_0)TE_0\right) \le \tilde{V}_\varphi\left(K, O_\varphi(K, E_0)TE_0\right)\\
&\quad\Longleftrightarrow\quad O_\varphi(K, E_0) \le O_\varphi(K, TE_0),
\end{aligned}$$

as desired. □

From Theorem 4.3 and Lemma 4.4, we can prove the following

Theorem 4.5 *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then there exists a unique solution to Problem* P_2.

Proof First, we prove the existence of a solution to problem P_2. Note that the constraint condition in P_2 can be turned into $V(E) = \omega_n$.

Let $\lambda_0 = \inf\left\{O_\varphi(K, TB) : T \in \mathrm{SL}(n)\right\}$. From Lemma 3.6, we have

$$0 < \left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}} \le \lambda_0 \le O_\varphi(K, B) < \infty.$$

Similar to the proof of Lemma 4.2, we can show the set

$$\{T \in \mathrm{SL}(n) : O_\varphi(K, TB) \le O_\varphi(K, B)\}$$

is compact. Combining this with the continuity of $O_\varphi(K, TB)$, the existence of solution to P_2 is demonstrated.

Now, we proceed to prove the uniqueness.

Assume ellipsoid E_0 is a solution to P_2. Then, $O_\varphi(K, E_0) \le O_\varphi(K, TE_0)$, for $T \in \mathrm{SL}(n)$. Thus, by Lemma 4.4, it follows that

$$\tilde{V}_\varphi\left(K, O_\varphi(K, E_0)E_0\right) \le \tilde{V}_\varphi\left(K, O_\varphi(K, E_0)TE_0\right), \quad \text{for } T \in \mathrm{SL}(n).$$

Thus, E_0 is a solution to Problem P_1 for star body $\lambda_0^{-1}K$. Hence, by Theorem 4.3, the solution to P_2 is unique. □

Theorem 4.6 *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then*

(1) *If E_0 is the unique solution to Problem* P_2*, then $O_\varphi(K, E_0)E_0$ is a solution to Problem* P_3.

(2) *If E_1 is a solution to Problem* P_3*, then $\left(\frac{\omega_n}{V(E_1)}\right)^{\frac{1}{n}} E_1$ is a solution to Problem* P_2.

Consequently, there exists a unique solution to Problem P_3.

Proof (1) Let $E \in \mathcal{E}^n$ with $O_\varphi(K, E) \leq 1$. Since $\left(\frac{\omega_n}{V(E)}\right)^{\frac{1}{n}} E$ satisfies the constraint condition of P_2, by Lemma 3.4 (2), the fact $V(E_0) = \omega_n$, and the assumption $O_\varphi(K, E) \leq 1$, we have

$$\begin{aligned} V\left(O_\varphi(K, E_0)E_0\right) &= O_\varphi(K, E_0)^n V(E_0) \\ &\leq O_\varphi\left(K, \left(\frac{\omega_n}{V(E)}\right)^{\frac{1}{n}} E\right)^n V(E_0) \\ &= \frac{V(E)}{\omega_n} O_\varphi(K, E)^n V(E_0) \\ &= V(E) O_\varphi(K, E)^n \\ &\leq V(E), \end{aligned}$$

which shows that $O_\varphi(K, E_0)E_0$ solves Problem P_3.

(2) Note that a solution E_1 to P_3 must satisfy $O_\varphi(K, E_1) = 1$.

Let $E' \in \mathcal{E}^n$ with $V(E') \leq \omega_n$. By Lemma 3.4 (2), $O_\varphi\left(K, O_\varphi(K, E')E'\right) = 1$. Thus $O_\varphi(K, E')E'$ satisfies the constraint condition of Problem P_3. Since E_1 is a solution to Problem P_3, it follows that

$$V\left(O_\varphi(K, E')E'\right) \geq V(E_1).$$

So, by the assumption $V(E') \leq \omega_n$, the fact $O_\varphi(K, E_1) = 1$ and Lemma 3.4 (2), we have

$$O_\varphi(K, E') \geq \left(\frac{V(E_1)}{V(E')}\right)^{\frac{1}{n}} \geq \left(\frac{V(E_1)}{\omega_n}\right)^{\frac{1}{n}} = O_\varphi\left(K, \left(\frac{\omega_n}{V(E_1)}\right)^{\frac{1}{n}} E_1\right),$$

which shows that $\left(\frac{\omega_n}{V(E_1)}\right)^{\frac{1}{n}} E_1$ solves Problem P_2. □

Lemma 4.7 *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then Problems P_3 and $\tilde{S}_\varphi$ have the same solution.*

Proof From the facts

$$\min_{\{E \in \mathcal{E}^n : O_\varphi(K,E) \leq 1\}} V(E) = \min_{\{E \in \mathcal{E}^n : O_\varphi(K,E) = 1\}} V(E)$$

and

$$\min_{\left\{E \in \mathcal{E}^n : \bar{\tilde{V}}_\varphi(K,E) \leq 1\right\}} V(E) = \min_{\left\{E \in \mathcal{E}^n : \bar{\tilde{V}}_\varphi(K,E) = 1\right\}} V(E)$$

together with the implication that for $E \in \mathcal{E}^n$,

$$O_\varphi(K, E) = 1 \iff \bar{\tilde{V}}_\varphi(K, E) = 1,$$

it follows that an ellipsoid $E \in \mathcal{E}^n$ solves Problem P_3, if and only if it solves Problem $\tilde{S}_\varphi$. By Theorem 4.6, it concludes the proof. □

For different dilations $\lambda_1 K$ and $\lambda_2 K$, $\lambda_1, \lambda_2 > 0$, Problems P_1 do not generally have the identical solution. By contrast, the homogeneity of $O_\varphi(\lambda K, L)$ in $\lambda \in (0, \infty)$ guarantees that all Problems P_2 for λK in $\lambda \in (0, \infty)$ have the identical unique solution. Problems P_3 and $\tilde{S}_\varphi$ are identical, and Problem P_3 is the dual problem of P_2. Thus, Problem $\tilde{S}_\varphi$ is not the dual problem of P_1 in general.

In view of Theorem 4.5, Theorem 4.6 and Lemma 4.7, we are in position to introduce a family of ellipsoids in the framework of dual Orlicz Brunn–Minkowski theory, which are extensions of Legendre ellipsoids.

Definition 4.8 Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Among all origin-symmetric ellipsoids E, the unique ellipsoid that solves the constrained minimization problem

$$\min_E V(E) \quad \text{subject to} \quad \bar{\tilde{V}}_\varphi(K, E) \leq 1$$

is called the *Orlicz–Legendre ellipsoid* of K with respect to φ, and is denoted by $\mathrm{L}_\varphi K$.

Among all origin-symmetric ellipsoids E, the unique ellipsoid that solves the constrained minimization problem

$$\min_E O_\varphi(K, E) \quad \text{subject to} \quad V(E) = \omega_n$$

is called the *normalized Orlicz–Legendre ellipsoid* of K with respect to φ, and is denoted by $\overline{\mathrm{L}}_\varphi K$.

For the polar of $\mathrm{L}_\varphi K$ or $\overline{\mathrm{L}}_\varphi K$, we write $\mathrm{L}_\varphi^* K$ or $\overline{\mathrm{L}}_\varphi^* K$, rather than $(\mathrm{L}_\varphi K)^*$ or $(\overline{\mathrm{L}}_\varphi K)^*$.

If $\varphi(t) = t^p$, $1 \leq p < \infty$, we write $\mathrm{L}_\varphi K$ and $\overline{\mathrm{L}}_\varphi K$ for $\mathrm{L}_p K$ and $\overline{\mathrm{L}}_p K$, respectively. Especially, $\mathrm{L}_2 K$ is precisely the Legendre ellipsoid $\Gamma_2 K$.

We observe that for the case $\varphi(t) = t^p$, Problems P_1 and P_2 are identical, and were previously solved by Bastero and Romance [5]. Yu [54] introduced the ellipsoids $\mathrm{L}_p K$ for convex bodies containing the origin in their interiors.

From Theorem 4.6, it is obvious that

$$\mathrm{L}_\varphi K = O_\varphi(K, \overline{\mathrm{L}}_\varphi K)\overline{\mathrm{L}}_\varphi K \quad \text{and} \quad \overline{\mathrm{L}}_\varphi K = \left(\frac{\omega_n}{V(\mathrm{L}_\varphi K)}\right)^{\frac{1}{n}} \mathrm{L}_\varphi K. \tag{4.2}$$

Definition 4.8 combined with inequality (3.1) shows that for any $E \in \mathcal{E}^n$,

$$\mathrm{L}_\varphi E = E.$$

From Definition 4.8 and Lemma 3.4, we easily know that the operator L_φ intertwines with elements of $\mathrm{GL}(n)$.

Lemma 4.9 *Suppose $K \in \mathcal{S}^n$ and $\varphi \in \Phi$. Then for any $T \in \mathrm{GL}(n)$,*

$$\mathrm{L}_\varphi(TK) = T(\mathrm{L}_\varphi K).$$

To connect Orlicz–Legendre ellipsoids with the Löwner ellipsoid, it is necessary to introduce the following.

Definition 4.10 Suppose $K \in \mathcal{S}_o^n$. The unique origin-symmetric ellipsoid of minimal volume containing K is denoted by $\mathrm{L}_\infty K$.

Among all origin-symmetric ellipsoids E, the unique ellipsoid that uniquely solves the constrained minimization problem

$$\min_E \left\| \frac{\rho_K}{\rho_E} \right\|_\infty \quad \text{subject to} \quad V(E) \le \omega_n$$

is denoted by $\overline{\mathrm{L}}_\infty K$.

The notion of $\mathrm{L}_\infty K$ is well defined, since $(\mathrm{L}_\infty K)^*$ is the L_∞ John ellipsoid $\mathrm{E}_\infty\,(\mathrm{conv}K)^*$, i.e., the unique origin-symmetric ellipsoid of maximal volume contained in $(\mathrm{conv}K)^*$. The extremal problems involved in the definition are dual to each other. Thus, we have

$$\mathrm{L}_\infty K = \left(\mathrm{E}_\infty\,(\mathrm{conv}K)^*\right)^* \quad \text{and} \quad \overline{\mathrm{L}}_\infty K = \left(\frac{\omega_n}{V(\mathrm{L}_\infty K)}\right)^{\frac{1}{n}} \mathrm{L}_\infty K.$$

Here, $\mathrm{conv}K$ denotes the convex hull of K.

For a convex body $K \in \mathcal{K}_o^n$, if the John point of K^* is at the origin, then $(\mathrm{L}_\infty K)^*$ is precisely the John ellipsoid $\mathrm{J}(K^*)$ of K^*. If K is an origin-symmetric star body in $\mathbb{R}^n$, then $\mathrm{L}_\infty K$ is precisely the Löwner ellipsoid of K.

5 The Continuity of Orlicz–Legendre Ellipsoids

In this section, we aim to show the continuity of Orlicz–Legendre ellipsoids $\mathrm{L}_\varphi K$ with respect to φ and K.

Throughout this section, we suppose $\varphi \in \Phi$, $K, K_i \in \mathcal{S}_o^n$, $\varphi, \varphi_j \in \Phi$, $i, j \in \mathbb{N}$, and $K_i \to K$ and $\varphi_j \to \varphi$. It is easily seen that there exist positive r_m and r_M such that

$$r_m B \subseteq K \subseteq r_M B \quad \text{and} \quad r_m B \subseteq K_i \subseteq r_M B \quad \text{for each } i \in \mathbb{N}.$$

Lemma 5.1 $\sup\limits_{i,j\in\mathbb{N}} \left\{ d\left(\overline{\mathrm{L}}_\varphi K\right), d\left(\overline{\mathrm{L}}_\varphi K_i\right), d\left(\overline{\mathrm{L}}_{\varphi_j} K\right), d\left(\overline{\mathrm{L}}_{\varphi_j} K_i\right) \right\} < \infty.$

Proof Let $E \in \mathcal{E}^n$. First, we prove the implication

$$O_\varphi(K, E) \le 1 \quad \Longrightarrow \quad d(E^*) \le \frac{n\omega_n}{2r_m\omega_{n-1}} \varphi^{-1}\left(\left(\frac{r_M}{r_m}\right)^n \varphi(1)\right). \tag{5.1}$$

Assume $O_\varphi(K,E)\le 1$. From the definition of $O_\varphi(K,E)$ together with Lemma 2.1 and Lemma 3.5, the definition of $\tilde{V}_\varphi(K,E)$ together with the fact $r_mB\subseteq K\subseteq r_MB$ and the monotonicity of φ, the convexity of φ together with Jensen's inequality, the monotonicity of φ again together with (2.3) and the fact $h_{E^*}(u)\ge d(E^*)|v_{E^*}\cdot u|$ for $u\in S^{n-1}$, and finally Cauchy's projection formula, it follows that

$$\begin{aligned}\varphi(1)&\ge\frac{\tilde{V}_\varphi(K,E)}{V(K)}\\&\ge\left(\frac{r_m}{r_M}\right)^n\frac{1}{n\omega_n}\int_{S^{n-1}}\varphi\left(\frac{r_m}{\rho_E}\right)dS\\&\ge\left(\frac{r_m}{r_M}\right)^n\varphi\left(\frac{1}{n\omega_n}\int_{S^{n-1}}\frac{r_m}{\rho_E}dS\right)\\&\ge\left(\frac{r_m}{r_M}\right)^n\varphi\left(\frac{r_m}{n\omega_n}\int_{S^{n-1}}d(E^*)|v_{E^*}\cdot u|dS(u)\right)\\&=\left(\frac{r_m}{r_M}\right)^n\varphi\left(\frac{2r_m\omega_{n-1}}{n\omega_n}d(E^*)\right),\end{aligned}$$

which, together with the monotonicity of φ^{-1}, immediately yields (5.1).

Since that $\varphi_j\to\varphi$ implies $\varphi_j(1)\to\varphi(1)$ and $\varphi_j^{-1}\to\varphi^{-1}$, it follows that

$$\varphi_j^{-1}\left(\left(\frac{r_M}{r_m}\right)^n\varphi_j(1)\right)\to\varphi^{-1}\left(\left(\frac{r_M}{r_m}\right)^n\varphi(1)\right),$$

and therefore

$$\sup_{j\in\mathbb{N}}\left\{\varphi^{-1}\left(\left(\frac{r_M}{r_m}\right)^n\varphi(1)\right),\varphi_j^{-1}\left(\left(\frac{r_M}{r_m}\right)^n\varphi_j(1)\right)\right\}<\infty.$$

This, as well as (5.1), yields

$$\sup_{i,j\in\mathbb{N}}\left\{d\left(\mathrm{L}_\varphi^*K\right),d\left(\mathrm{L}_\varphi^*K_i\right),d\left(\mathrm{L}_{\varphi_j}^*K\right),d\left(\mathrm{L}_{\varphi_j}^*K_i\right)\right\}<\infty. \tag{5.2}$$

From (4.2), the inequality $V(\mathrm{L}_\varphi K)\le V(\mathrm{L}_\infty K)$ (which will be given by Theorem 7.2) together with Definition 4.10, we have

$$d\left(\overline{\mathrm{L}}_\varphi^*K\right)\le r_Md\left(\mathrm{L}_\varphi^*K\right). \tag{5.3}$$

Observe that (5.3) also holds when φ is replaced by φ_j or K is replaced by K_i. Thus, from (5.2), it follows that

$$\sup_{i,j\in\mathbb{N}} \left\{ d\left(\overline{\mathrm{L}}_{\varphi}^{*} K\right), d\left(\overline{\mathrm{L}}_{\varphi}^{*} K_i\right), d\left(\overline{\mathrm{L}}_{\varphi_j}^{*} K\right), d\left(\overline{\mathrm{L}}_{\varphi_j}^{*} K_i\right) \right\} < \infty,$$

which, together with Lemma 2.2, concludes the desired lemma. □

Now, from Lemma 5.1, there exists a constant $R \in (0, \infty)$ such that all the ellipsoids $\overline{\mathrm{L}}_{\varphi} K, \overline{\mathrm{L}}_{\varphi_j} K, \overline{\mathrm{L}}_{\varphi} K_i$ and $\overline{\mathrm{L}}_{\varphi_j} K_i$ are in the set

$$\mathcal{E}_R = \left\{ E \in \mathcal{E}^n : V(E) = \omega_n \text{ and } E \subseteq RB \right\}.$$

From the compactness of the sets $\mathcal{E}_R$ and $\{K \in \mathcal{S}_o^n : r_m B \subseteq K \subseteq r_M B\}$, together with Lemma 3.8, we immediately obtain:

Lemma 5.2 *The limit* $\lim_{i,j\to\infty} O_{\varphi_j}(K_i, E) = O_{\varphi}(K, E)$ *is uniform in* $E \in \mathcal{E}_R$.

Lemma 5.3 $\lim_{i,j\to\infty} O_{\varphi_j}(K_i, \overline{\mathrm{L}}_{\varphi_j} K_i) = O_{\varphi}(K, \overline{\mathrm{L}}_{\varphi} K)$.

Proof From Definition 4.8 and Lemma 5.2, we have

$$\begin{aligned}
\lim_{i,j\to\infty} O_{\varphi_j}(K_i, \overline{\mathrm{L}}_{\varphi_j} K_i) &= \lim_{i,j\to\infty} \min_{E\in\mathcal{E}_R} O_{\varphi_j}(K_i, E) \\
&= \min_{E\in\mathcal{E}_R} \lim_{i,j\to\infty} O_{\varphi_j}(K_i, E) \\
&= \min_{E\in\mathcal{E}_R} O_{\varphi}(K, E) \\
&= O_{\varphi}(K, \overline{\mathrm{L}}_{\varphi} K),
\end{aligned}$$

as desired. □

Lemma 5.4 $\lim_{i,j\to\infty} \overline{\mathrm{L}}_{\varphi_j} K_i = \overline{\mathrm{L}}_{\varphi} K$.

Proof From the compactness of $\mathcal{E}_R$, to prove the lemma, it suffices to prove that any convergent subsequence $\{\overline{\mathrm{L}}_{\varphi_{j_q}} K_{i_p}\}_{p,q\in\mathbb{N}}$ of $\{\overline{\mathrm{L}}_{\varphi_j} K_i\}$ must converge to $\overline{\mathrm{L}}_{\varphi} K$.

From Lemma 3.8 and Lemma 5.3, it follows that

$$\begin{aligned}
O_{\varphi}(K, \lim_{p,q\to\infty} \overline{\mathrm{L}}_{\varphi_{j_q}} K_{i_p}) &= \lim_{p,q\to\infty} O_{\varphi}(K, \overline{\mathrm{L}}_{\varphi_{j_q}} K_{i_p}) \\
&= \lim_{p,q\to\infty} O_{\varphi_{j_q}}(K_{i_p}, \overline{\mathrm{L}}_{\varphi_{j_q}} K_{i_p}) \\
&= O_{\varphi}(K, \overline{\mathrm{L}}_{\varphi} K),
\end{aligned}$$

which, together with the uniqueness of $\overline{\mathrm{L}}_{\varphi} K$, implies $\lim_{p,q\to\infty} \overline{\mathrm{L}}_{\varphi_{j_q}} K_{i_p} = \overline{\mathrm{L}}_{\varphi} K$. □

Theorem 5.5 *Suppose* $K, K_i \in \mathcal{S}_o^n$ *and* $\varphi, \varphi_j \in \Phi$, $i, j \in \mathbb{N}$. *If* $K_i \to K$ *and* $\varphi_j \to \varphi$, *then*

$$\lim_{i,j\to\infty} \mathrm{L}_{\varphi_j} K_i = \mathrm{L}_{\varphi} K.$$

Proof From Lemma 5.3, Lemma 5.4, together with the identity

$$\mathrm{L}_\varphi K = O_\varphi(K, \overline{\mathrm{L}}_\varphi K)\overline{\mathrm{L}}_\varphi K,$$

the desired limit is immediately derived. □

Note that $\mathrm{L}_{\varphi^p} K$ and therefore $\mathrm{L}_p K$ are continuous in $(K, p) \in \mathcal{S}_o^n \times [1, \infty)$. We observe that although Yu et al. [54] first introduced the notion of L_p Legendre ellipsoids, they did not consider the above continuity at all. We proceed to show the following.

Theorem 5.6 *Suppose* $K \in \mathcal{S}_o^n$ *and* $\varphi \in \Phi$. *Then* $\lim_{p\to\infty} \mathrm{L}_{\varphi^p} K = \mathrm{L}_\infty K$.

Proof By Lemma 5.1, there exists a constant $C \in (0, \infty)$ such that $\overline{\mathrm{L}}_\infty K$ and all the Orlicz–Legendre ellipsoids $\overline{\mathrm{L}}_{\varphi^p} K$ are in the set

$$\mathcal{F} = \{E \in \mathcal{E}^n : V(E) = \omega_n \text{ and } E \subseteq CB\}.$$

For $E \in \mathcal{F}$, let $f_p(E) = O_{\varphi^p}(K, E)$ for $p \in [1, \infty)$, and let $f_\infty(E) = \|\rho_K/\rho_E\|_\infty$. By Lemma 3.9, the sequence $\{f_j\}$ of continuous functionals is increasing pointwise to f_∞ on the compact set $\mathcal{F}$, and therefore by Dini's theorem,

$$f_j \to f_\infty, \quad \text{uniformly on } \mathcal{F}. \tag{5.4}$$

The theorem will be obtained after the following steps.

First, using (5.4) and an argument similar to the proofs of Lemmas 5.3 and 5.4, it will establish

$$\lim_{p\to\infty} \overline{\mathrm{L}}_{\varphi^p} K = \overline{\mathrm{L}}_\infty K. \tag{5.5}$$

Second, using the definition of f_p, (5.4), (5.5), and the definition of f_∞, it will establish

$$\lim_{p\to\infty} O_{\varphi^p}(K, \overline{\mathrm{L}}_{\varphi^p} K) = \left\| \frac{\rho_K}{\rho_{\overline{\mathrm{L}}_\infty K}} \right\|_\infty. \tag{5.6}$$

Finally, from the identities

$$O_{\varphi^p}(K, \overline{\mathrm{L}}_{\varphi^p} K)\overline{\mathrm{L}}_{\varphi^p} K = \mathrm{L}_{\varphi^p} K \quad \text{and} \quad \left\| \frac{\rho_K}{\rho_{\overline{\mathrm{L}}_\infty K}} \right\|_\infty \overline{\mathrm{L}}_\infty K = \mathrm{L}_\infty K,$$

together with (5.5) and (5.6), it concludes the desired limit. □

6 A Characterization of Orlicz–Legendre Ellipsoids

In this section, we establish a connection linking the characterization of Orlicz–Legendre ellipsoids and the isotropy of measures.

Definition 6.1 Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi \cap C^1[0, \infty)$, the Borel measure $\mu_\varphi(K, \cdot)$ on S^{n-1} is defined by

$$d\mu_\varphi(K, \cdot) = \varphi'(\rho_K)\, \rho_K^{n+1} dS.$$

The next theorem characterizes the Orlicz–Legendre ellipsoid $\mathrm{L}_\varphi K$.

Theorem 6.2 *Suppose $K \in \mathcal{S}_o^n$, $\varphi \in \Phi \cap C^1[0, \infty)$, and $T \in \mathrm{GL}(n)$. Then, $\mathrm{L}_\varphi K = TB$, if and only if $\mu_\varphi\left(O_\varphi(T^{-1}K, B)^{-1}T^{-1}K, \cdot\right)$ is isotropic on S^{n-1}.*

Proof In terms of Lemma 4.9, without loss of generality, we may assume $T = I_n$. Then, $\overline{\mathrm{L}}_\varphi K = B$. Let $K' = O_\varphi(K, B)^{-1}K$. Note that by Lemma 4.4,

$$\overline{\mathrm{L}}_\varphi K = B \quad \Longleftrightarrow \quad \tilde{V}_\varphi(K', B) = \min_{T \in \mathrm{SL}(n)} \tilde{V}_\varphi(K', TB).$$

Hence, to prove the desired equivalence, it suffices to prove that

$$\tilde{V}_\varphi(K', B) = \min_{T \in \mathrm{SL}(n)} \tilde{V}_\varphi(K', TB) \quad \Longleftrightarrow \quad \mu_\varphi\left(K', \cdot\right) \text{ is isotropic on } S^{n-1}.$$

First, we assume $\tilde{V}_\varphi(K', B) = \min_{T \in \mathrm{SL}(n)} \tilde{V}_\varphi(K', TB)$, and prove the necessity by the variational method.

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Choose $\varepsilon_0 > 0$ sufficiently small so that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the matrix $I_n + \varepsilon L$ is invertible. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, define

$$L_\varepsilon = \frac{I_n + \varepsilon L}{|I_n + \varepsilon L|^{\frac{1}{n}}}.$$

Then $L_\varepsilon \in \mathrm{SL}(n)$. Our assumption implies that for all ε,

$$\tilde{V}_\varphi(K', L_\varepsilon^{-1}B) \geq \tilde{V}_\varphi(K', B).$$

The fact $\frac{1}{\rho_{L_\varepsilon^{-1}B}(u)} = h_{L_\varepsilon^t B}(u)$ for $u \in S^{n-1}$, together with the definition of $\tilde{V}_\varphi(K', L_\varepsilon^{-1}B)$, gives

$$\tilde{V}_\varphi(K', L_\varepsilon^{-1}B) = \int_{S^{n-1}} \varphi\left(\rho_{K'}(u) \frac{(1 + 2\varepsilon u \cdot Lu + \varepsilon^2 Lu \cdot Lu)^{\frac{1}{2}}}{|I_n + \varepsilon L|^{\frac{1}{n}}}\right) d\tilde{V}_{K'}(u).$$

From the smoothness of φ and $|L_\varepsilon u|$ in ε, the integrand depends smoothly on ε. Thus,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \tilde{V}_\varphi(K', L_\varepsilon^{-1}B) = 0.$$

Calculating it directly, we have

$$0=\frac{1}{n}\int_{S^{n-1}}\left(-\frac{\operatorname{tr}L}{n}+u\cdot Lu\right)d\mu_{\varphi}(K',u).$$

Let $v\in S^{n-1}$ and $L=v\otimes v$. Using the facts $\operatorname{tr}(v\otimes v)=1$ and $u\cdot(v\otimes v)u=(u\cdot v)^2$, it gives

$$\int_{S^{n-1}}(u\cdot v)^2 d\mu_{\varphi}(K',u)=\frac{|\mu_{\varphi}(K',\cdot)|}{n}.$$

Thus, $\mu_{\varphi}(K',\cdot)$ is isotropic on S^{n-1}.

Conversely, suppose that $\mu_{\varphi}(K',\cdot)$ is isotropic on S^{n-1}. We prove that if $E\in\mathcal{E}^n$ and $V(E)=\omega_n$, then

$$\tilde{V}_{\varphi}(K',E)\geq\tilde{V}_{\varphi}(K',B).$$

The proof will be completed after two steps.

First, for $a=(a_1,\dots,a_n)\in[0,\infty)^n$, define

$$F(a)=\int_{S^{n-1}}\varphi\left(\rho_{K'}(u)\right)|\operatorname{diag}(a_1,\dots,a_n)u|d\tilde{V}_{K'}(u),$$

where $\operatorname{diag}(a_1,\dots,a_n)$ denotes the $n\times n$ diagonal matrix with diagonal elements $a_1,\dots,a_n$. We aim to show that

$$F(a)\geq F(e),\quad\text{whenever }\prod_{j=1}^{n}a_j=1.\tag{6.1}$$

Here, e denotes the point $(1,\dots,1)$.

From the smoothness of φ and $|\operatorname{diag}(a_1,\dots,a_n)u|$ in $(a_1,\dots,a_n)$, we have

$$\left.\frac{\partial}{\partial a_j}\right|_{a=e}F(a)=\int_{S^{n-1}}u_j^2\varphi'\left(\rho_{K'}(u)\right)\rho_{K'}(u)d\tilde{V}_{K'}(u),$$

where $(u_1,\dots,u_n)$ denotes the coordinates of $u\in S^{n-1}$. From the isotropy of $\mu_{\varphi}(K',\cdot)$, it follows that

$$\left.\frac{\partial}{\partial a_j}\right|_{a=e}F(a)=\frac{|\mu_{\varphi}(K',\cdot)|}{n}.$$

Thus,

$$\nabla F(e)=\frac{|\mu_{\varphi}(K',\cdot)|}{n}e.\tag{6.2}$$

It can be checked that the function $F : [0,\infty)^n \to [0,\infty)$ is continuous and convex, and $F(\lambda a)$ is strictly increasing in $\lambda \in [0,\infty)$, for $a \in (0,\infty)^n$. Thus, $F^{-1}([0, F(e)])$ is a convex body. Its boundary is given by the equation $F(a) = F(e)$ with $a \in [0,\infty)^n$, so (6.2) implies the vector e is an outer normal of the convex body $F^{-1}([0, F(e)])$ at the boundary point e.

Consequently, $F^{-1}([0, F(e)]) \subset \{a \in \mathbb{R}^n : a \cdot e \le n\}$. That is to say, for all $a \in [0,\infty)^n$, if $F(a) \le F(e)$, then $a\cdot e \le n$. In contrast, for all $b = (b_1, \dots, b_n) \in (0,\infty)^n$ with $b_1 \cdots b_n = 1$, the AM–GM inequality yields that $b \cdot e \ge n$, with equality if and only if $b = e$. Hence, (6.1) is derived.

Finally, represent each $T \in \mathrm{SL}(n)$ into the form $T = O_1^{-1} A O_2$, where O_1, O_2 are $n \times n$ orthogonal matrices, and $A = \mathrm{diag}(a_1, \dots, a_n)$ is diagonal and positive definite with $a_1 a_2 \cdots a_n = 1$. Note that $\tilde{V}_\varphi(K', TB) = \tilde{V}_\varphi(O_1 K', AB)$, and $\mu_\varphi(O_1 K', \cdot)$ is isotropic on S^{n-1}. So, applying (6.1) to the body $O_1 K'$, it gives

$$\tilde{V}_\varphi(K', TB) \ge \tilde{V}_\varphi(K', B),$$

which concludes the desired sufficiency. □

Corollary 6.3 *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi \cap C^1[0,\infty)$. Then, modulo orthogonal transformations, there exists an* $\mathrm{SL}(n)$ *transformation T such that the measure $\mu_\varphi(TK, \cdot)$ is isotropic on S^{n-1}.*

7 Volume Ratio Inequalities

In general, the Orlicz–Legendre ellipsoid $\mathrm{L}_\varphi K$ does not contain K. However, we show that the volume functional over the class of Orlicz–Legendre ellipsoids of K is bounded by $V(\mathrm{L}_1 K)$ from below and by $V(\mathrm{L}_\infty K)$ from above.

Theorem 7.1 *Suppose $K \in \mathcal{S}_o^n$, $\varphi \in \Phi$ and $1 \le p < q < \infty$. Then*

$$V(\mathrm{L}_1 K) \le V(\mathrm{L}_\varphi K) \le V(\mathrm{L}_{\varphi^p} K) \le V(\mathrm{L}_{\varphi^q} K) \le V(\mathrm{L}_\infty K).$$

Proof From Lemma 3.9, it follows that

$$\begin{aligned}\left\{E \in \mathcal{E}^n : \left\|\frac{\rho_K}{\rho_E} : V_K^*\right\|_1 \le 1\right\} &\supseteq \left\{E \in \mathcal{E}^n : \left\|\frac{\rho_K}{\rho_E} : V_K^*\right\|_\varphi \le 1\right\} \\ &\supseteq \left\{E \in \mathcal{E}^n : \left\|\frac{\rho_K}{\rho_E} : V_K^*\right\|_{\varphi^p} \le 1\right\} \\ &\supseteq \left\{E \in \mathcal{E}^n : \left\|\frac{\rho_K}{\rho_E} : V_K^*\right\|_{\varphi^q} \le 1\right\} \\ &\supseteq \left\{E \in \mathcal{E}^n : \left\|\frac{\rho_K}{\rho_E}\right\|_\infty \le 1\right\}.\end{aligned}$$

From the above inclusions and the definition of Orlicz–Legendre ellipsoids, the desired inequalities are obtained. □

Theorem 7.2 *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then*

$$V(\mathrm{L}_\varphi K) \geq V(K),$$

with equality if and only if $K \in \mathcal{E}^n$.

Proof From Lemma 3.6 and the fact that $\mathrm{L}_\varphi E = E$ for any $E \in \mathcal{E}^n$, it follows that

$$O_\varphi\left(K, \mathrm{L}_\varphi K\right) \geq \left(\frac{V(K)}{V(\mathrm{L}_\varphi K)}\right)^{\frac{1}{n}},$$

with equality if and only if $K \in \mathcal{E}^n$. Combining this with the fact $1 = O_\varphi\left(K, \mathrm{L}_\varphi K\right)$, it establishes the desired inequality. □

If $\varphi(t) = t^p$, $1 \leq p < \infty$, then Theorem 7.2 implies that $V(\mathrm{L}_p K) \geq V(K)$, and in particular that $V(\Gamma_2 K) \geq V(K)$.

A classical result on John's ellipsoid is Ball's volume ratio inequality [1,2], which states: if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$\frac{V(K)}{V(\mathrm{J}K)} \leq \frac{2^n}{\omega_n},$$

with equality if and only if K is a parallelotope. The fact that equality holds in Ball's inequality only for parallelotopes was established by Barthe [3]. He also established the outer volume-ratio inequality: if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$\frac{V(K)}{V(\mathrm{L}K)} \geq \frac{2^n}{n!\omega_n},$$

with equality if and only if K is a cross-polytope.

Recall that when K is an origin-symmetric convex body, $\mathrm{L}_\infty K$ is just the Löwner ellipsoid $\mathrm{L}K$. Thus, combining Theorem 7.1 with Barthe's outer volume ratio inequality, we immediately obtain

Theorem 7.3 *Suppose $K \in \mathcal{K}_o^n$ is origin-symmetric and $\varphi \in \Phi$. Then*

$$\frac{V(K)}{V(\mathrm{L}_\varphi K)} \geq \frac{2^n}{n!\omega_n}.$$

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