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Orlicz mixed quermassintegrals

Dedicated to Professor Ren De-lin on the Occasion of his 80th Birthday

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Abstract The notion of mixed quermassintegrals in the classical Brunn-Minkowski theory is extended to that of Orlicz mixed quermassintegrals in the Orlicz Brunn-Minkowski theory. The analogs of the classical Cauchy-Kubota formula, the Minkowski isoperimetric inequality and the Brunn-Minkowski inequality are established for this new Orlicz mixed quermassintegrals.

Keywords Orlicz Brunn-Minkowski theory, quermassintegral, Minkowski's isoperimetric inequality, integral geometry

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1 Introduction

Beginning with the groundbreaking articles [11, 20, 21] and the very recent work [9], the classical Brunn-Minkowski theory of convex bodies (see, e.g., [8, 10, 28, 30]) was extended to the Orlicz stage, which is known as the Orlicz Brunn-Minkowski theory. Analogous to the way that Orlicz spaces generalize L_p spaces (see [23]), it represents a generalization of the L_p Brunn-Minkowski theory, which emerged in the early 1960s (see [6]), began largely with the initial works [14, 15] in the mid 1990s, and expanded rapidly thereafter (see, e.g., [2–4, 7, 12, 13, 16–19, 22, 25, 27, 29, 31–34]).

A convex body in $\mathbb{R}^n$, the standard Euclidean n-space, is a compact convex set with non-empty interior. Associated with a convex body Q in $\mathbb{R}^n$ are the *quermassintegrals* $W_0(Q), W_1(Q), \dots, W_n(Q)$, which are defined by letting $W_0(Q) = \mathrm{vol}_n(Q)$, the volume of Q; $W_n(Q) = \omega_n = \pi^{n/2}/\Gamma(1 + n/2)$, the volume of the unit ball B in $\mathbb{R}^n$; and for $j = 1, 2, \dots, n-1$ by

$$W_{n-j}(Q) = \frac{\omega_n}{\omega_j} \int_{G_{n,j}} \mathrm{vol}_j(Q \mid \xi) d\sigma_{n,j}(\xi), \tag{1.1}$$

where $G_{n,j}$, $\sigma_{n,j}$, vol_j and $Q \mid \xi$ denote the Grassmannian manifold of j-dimensional linear subspaces of $\mathbb{R}^n$, the normalized Haar measure on $G_{n,j}$, the j-dimensional volume and the orthogonal projection of Q onto ξ, respectively. Equation (1.1) is the well-known Cauchy-Kubota formula.

That studying the first variation of quermassintegrals is an effective approach to find new geometric quantities or measures induced by convex bodies. In the late 1930s, Aleksandrov [1] and Fenchel and

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Jessen [5] independently proved that for a convex body Q in $\mathbb{R}^n$ and $j = 0, 1, \dots, n-1$, there exists a regular Borel measure $S_{n-j-1}(Q, \cdot)$ on S^{n-1}, the unit sphere in $\mathbb{R}^n$, such that for any convex body P,

$$W_j(Q, P) := \frac{1}{n-j} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} W_j(Q + \varepsilon P) = \frac{1}{n} \int_{S^{n-1}} h_P(u) dS_{n-1-j}(Q, u),$$

where h_P is the support function of P, and $Q + \varepsilon P = \{x + \varepsilon y : x \in Q, y \in P\}$. The quantity $W_j(Q, P)$ is called the j-th *mixed quermassintegral* of Q and P.

Let K, L be convex bodies in $\mathbb{R}^n$ with the origin in their interiors, and $\varepsilon \geqslant 0$. Recall that $K +_p \varepsilon \cdot_p L$, the L_p combination of K and L, is defined by

$$h_{K +_p \varepsilon \cdot_p L} = ({h_K}^p + \varepsilon {h_L}^p)^{\frac{1}{p}}.$$

In [14], for $j = 0, 1, \dots, n-1$, Lutwak considered the L_p first variation of the j-th quermassintegral

$$W_{p,j}(K, L) := \frac{p}{n-j} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} W_j(K +_p \varepsilon \cdot_p L).$$

The quantity $W_{p,j}(K, L)$ is called the L_p *mixed quermassintegral* of K and L. In particular, $W_{p,0}(K, L)$ is also denoted by $V_p(K, L)$, the L_p *mixed volume* of K and L.

The aim of this paper is to extend the notion of mixed quermassintegrals to the Orlicz setting. The above cited works [11, 20, 21], and especially the work [9], make it apparent that the time is ripe.

Throughout this paper, we consider a Young function $\varphi : [0, \infty) \to [0, \infty)$, i.e., φ is convex, strictly increasing, with $\varphi(0) = 0$. Let $\alpha, \beta > 0$. It was shown by Gardner, Weil and Hug [9] that the function $h : \mathbb{R}^n \to [0, \infty)$ given by

$$h(x) = \inf \left\{ \lambda > 0 : \alpha \varphi\left(\frac{h_K(x)}{\lambda} \right) + \beta \varphi\left(\frac{h_L(x)}{\lambda} \right) \leqslant \varphi(1) \right\}, \quad \text{for } x \in \mathbb{R}^n$$

is positive definite, homogeneous of order 1 and subadditive, and therefore is precisely the support function of a unique convex body, $\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L$, with the origin in its interior. The convex body $\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L$ is called the *Orlicz combination* of K and L. If $\varphi(t) = t^p$, $p \geqslant 1$, then the Orlicz combination reduces to the L_p combination and particularly the classical Minkowski combination for $p = 1$. For brevity, when $\alpha = 1$ and $\alpha = \beta = 1$, we write $\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L$ as $K +_\varphi \beta \cdot_\varphi L$ and $K +_\varphi L$, respectively.

In Section 3, we compute the Orlicz first variations of quermassintegrals

$$W_{\varphi,j}(K, L) := \frac{\varphi'_-(1)}{n-j} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} W_j(K +_\varphi \varepsilon \cdot_\varphi L), \quad j = 0, 1, \dots, n-1,$$

and obtain their integral representation.

Theorem 1. *Suppose K and L are convex bodies in $\mathbb{R}^n$ containing the origin in their interiors, and φ is a Young function. Then for each $j = 0, 1, \dots, n-1$,*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} W_j(K +_\varphi \varepsilon \cdot_\varphi L) = \frac{n-j}{n\varphi'_-(1)} \int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)} \right) h_K(u) dS_{n-j-1}(K, u).$$

Naturally, the quantity $W_{\varphi,j}(K, L)$ is called the j-th *Orlicz mixed quermassintegral* of K and L regarding φ. In particular, $W_{\varphi,0}(K, L)$ is the *Orlicz mixed volume* $V_\varphi(K, L)$ (see [9]) . Obviously, if $\varphi(t) = t^p$, $p \geqslant 1$, then $W_{\varphi,j}(K, L) = W_{p,j}(K, L)$.

In Section 4, from the viewpoint of integral geometry (see, e.g., [24, 26]), we show the probabilistic essence of Orlicz mixed quermassintegrals. The classical Cauchy-Kubota formula has the following natural Orlicz extension:

Theorem 2. *Suppose K and L are convex bodies in $\mathbb{R}^n$ containing the origin in their interiors, and φ is a Young function. Then for $1 \leqslant q \leqslant j < n$,*

$$W_{\varphi,j}(K, L) = \frac{\omega_n}{\omega_{n-q}} \int_{G_{n,n-q}} W^{(n-q)}_{\varphi,j-q}(K \mid \xi, L \mid \xi) d\sigma_{n,n-q}(\xi),$$

where $W^{(n-q)}_{\varphi,j-q}(K \mid \xi, L \mid \xi)$ *is the Orlicz mixed quermassintegral of the* $(n-q)$*-dimensional convex bodies* $K \mid \xi$ *and* $L \mid \xi$ *in the subspace* ξ*.*

In Section 5, the Minkowski isoperimetric inequality and the Brunn-Minkowski inequality for quermassintegrals are generalized to the Orlicz setting.

Theorem 3. *Suppose* K *and* L *are convex bodies in* $\mathbb{R}^n$ *containing the origin in their interiors, and* φ *is a Young function. Then for each* $j = 0, 1, \ldots, n-1$,

$$\frac{W_{\varphi,j}(K,L)}{W_j(K)} \geqslant \varphi\bigg(\bigg(\frac{W_j(L)}{W_j(K)}\bigg)^{\frac{1}{n-j}}\bigg),$$

and

$$\varphi(1) \geqslant \varphi\bigg(\bigg(\frac{W_j(K)}{W_j(K +_\varphi L)}\bigg)^{\frac{1}{n-j}}\bigg) + \varphi\bigg(\bigg(\frac{W_j(L)}{W_j(K +_\varphi L)}\bigg)^{\frac{1}{n-j}}\bigg).$$

If φ *is strictly convex, each equality holds if and only if* K *and* L *are dilates.*

If $\varphi(t) = t^p$, $p \geqslant 1$, then these inequalities reduce to the L_p Minkowski isoperimetric inequality

$$W_{p,j}(K,L) \geqslant W_j(K)^{\frac{n-j-p}{n-j}} W_j(L)^{\frac{p}{n-j}}$$

and the L_p Brunn-Minkowski inequality

$$W_j(K +_p L)^{\frac{p}{n-j}} \geqslant W_j(K)^{\frac{p}{n-j}} + W_j(L)^{\frac{p}{n-j}},$$

respectively.

2 Preliminaries

In order to keep the paper self-contained, we list here some basic facts about convex bodies that are needed in our investigations. The setting is the standard Euclidean n-space, $\mathbb{R}^n$. As usual, $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$.

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of a convex body K in $\mathbb{R}^n$ is defined by $h_K(x) = \max\{x \cdot y : y \in K\}$ for $x \in \mathbb{R}^n$. The notation $\mathcal{K}^n_o$ denotes the class of convex bodies in $\mathbb{R}^n$ which contain the origin in their interiors. The set $\mathcal{K}^n_o$ is often equipped with the Hausdorff metric δ_H, which is defined by

$$\delta_H(K_1, K_2) = \max_{S^{n-1}} |h_{K_1} - h_{K_2}|,$$

for $K_1, K_2 \in \mathcal{K}^n_o$.

The classical Aleksandrov-Fenchel-Jessen *surface area measure*, $S_{n-1}(Q, \cdot)$, of a convex body Q, is defined as the unique Borel measure on S^{n-1} such that

$$\int_{S^{n-1}} f(u) dS_{n-1}(Q, u) = \int_{\partial Q} f(\nu_Q(y)) d\mathcal{H}^{n-1}(y),$$

for each continuous $f : S^{n-1} \to \mathbb{R}$, where ν_Q is the unit outer normal of ∂Q at $x \in \partial Q$. It is noted that $\nu_Q(x)$ exists for $\mathcal{H}^{n-1}$-almost all $x \in \partial Q$.

The n measures $S_0(Q, \cdot), S_1(Q, \cdot), \ldots, S_{n-1}(Q, \cdot)$ appeared in Section 1 are the measures, which are such that for each $\varepsilon > 0$ and for each Borel subset $\omega \subseteq S^{n-1}$,

$$S_{n-1}(Q + \varepsilon B, \omega) = \sum_{j=0}^{n-1} \binom{n-1}{j} S_{n-1-j}(Q, \omega) \varepsilon^j.$$

The measure $S_j(Q, \cdot)$ is called the *j-th area measure* of Q. The $(n-1)$-th area measure is just the surface area measure. The measure $S_0(Q, \cdot)$ is independent of the body Q, and is just the spherical Lebesgue measure on S^{n-1}.

Throughout this paper, the notation φ and Φ denote a Young function and the class of all Young functions, respectively.

For $K, L \in \mathcal{K}_o^n$ and $\alpha, \beta > 0$, the *Orlicz combination*, $\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L$, is the convex body with support function

$$h_{\alpha\cdot_\varphi K+_\varphi\beta\cdot_\varphi L}(x) = \inf\left\{\lambda > 0 : \alpha\varphi\left(\frac{h_K(x)}{\lambda}\right) + \beta\varphi\left(\frac{h_L(x)}{\lambda}\right) \leqslant \varphi(1)\right\}.$$

It is noted that $h_{\alpha\cdot_\varphi K+_\varphi\beta\cdot_\varphi L}$ can be defined for all nonzero $x \in \mathbb{R}^n$ by the equation

$$\alpha\varphi\left(\frac{h_K(x)}{h_{\alpha\cdot_\varphi K+_\varphi\beta\cdot_\varphi L}(x)}\right) + \beta\varphi\left(\frac{h_L(x)}{h_{\alpha\cdot_\varphi K+_\varphi\beta\cdot_\varphi L}(x)}\right) = \varphi(1),$$

and by $h_{\alpha\cdot_\varphi K+_\varphi\beta\cdot_\varphi L}(0) = 0$. If $\varphi(t) = t^p$, $1 \leqslant p < \infty$, then

$$\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L = \alpha \cdot_p K +_p \beta \cdot_p L.$$

Associated with each positive continuous function f on S^{n-1} is its Aleksandrov body, $A(f)$, defined by

$$A(f) = \bigcap_{u\in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leqslant f(u)\}.$$

Obviously, $A(h_Q) = Q$ for any convex body Q in $\mathbb{R}^n$.

To obtain the Orlicz first variation of volume, it is crucial to use Aleksandrov's variational principle (see [3, Lemma 3.1], [9, Lemma 8.3] or [11, Lemma 1]).

Lemma 2.1. *Let $I \subset \mathbb{R}$ be an interval containing* 0 *and some positive number and the function*

$$h_t(u) = h(t, u) : I \times S^{n-1} \to (0, \infty)$$

be continuous and such that the convergence in

$$h'_+(0, u) = \lim_{t\to 0^+} \frac{h(t, u) - h(0, u)}{t}$$

is uniform on S^{n-1}. Then

$$\left.\frac{d}{dt}\right|_{t=0^+} \mathrm{vol}_n(A(h_t)) = \int_{S^{n-1}} h'_+(0, u)dS_{n-1}(A(h_0), u).$$

According to [9, Lemmas 8.2 and 8.4], it gives that

$$K +_\varphi \varepsilon \cdot_\varphi L \to K \quad \text{as} \quad \varepsilon \to 0^+, \tag{2.1}$$

and

$$\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0^+} h_{K+_\varphi\varepsilon\cdot_\varphi L}(u) = \frac{1}{\varphi'_-(1)}\varphi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u), \tag{2.2}$$

uniformly for $u \in S^{n-1}$. Note that

$$h_\varepsilon(u) = h_{K+_\varphi\varepsilon\cdot_\varphi L}(u) : [0, \infty) \times S^{n-1} \to (0, \infty)$$

is continuous. Hence, from Lemma 2.1 as well as the fact that $A(h_\varepsilon) = K +_\varphi \varepsilon \cdot_\varphi L$ for $\varepsilon \geqslant 0$, it follows that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \mathrm{vol}_n(K +_\varphi \varepsilon \cdot_\varphi L) = \frac{1}{\varphi'_-(1)} \int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)dS_{n-1}(K, u). \tag{2.3}$$

The above formula allows us to define the *Orlicz mixed volume*, $V_\varphi(K, L)$, of $K, L \in \mathcal{K}_o^n$ regarding φ, by

$$V_\varphi(K, L) = \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)dS_{n-1}(K, u).$$

If $\varphi(t) = t^p$, $1 \leqslant p < \infty$, then $V_\varphi(K, L) = V_p(K, L)$, the L_p mixed volume of K and L.

We conclude this section with the next lemma, which will be used in the proof of Theorem 3.1.

Lemma 2.2. *Let $K,L\in\mathcal{K}_o^n$ and $\varphi\in\Phi$. Then for $0\leqslant\varepsilon_1<\varepsilon_2<\infty$, there is the inclusion*

$$K+_\varphi\varepsilon_1\cdot_\varphi L\subset K+_\varphi\varepsilon_2\cdot_\varphi L.$$

Proof. Let $u\in S^{n-1}$ be arbitrary but fixed. For $\varepsilon\geqslant 0$, define the function $\psi_\varepsilon:(0,\infty)\to(0,\infty)$ by

$$\psi_\varepsilon(\lambda)=\varphi(\lambda^{-1}h_K(u))+\varepsilon\varphi(\lambda^{-1}h_L(u)).$$

It is easy to check that $\psi_\varepsilon(\lambda)$ is continuous and strictly decreasing in $\lambda\in(0,\infty)$,

$$\lim_{\lambda\to 0^+}\psi_\varepsilon(\lambda)=\infty,\quad\text{and}\quad\lim_{\lambda\to\infty}\psi_\varepsilon(\lambda)=0.$$

These facts allow us to take the function $\psi_\varepsilon^{-1}:(0,\infty)\to(0,\infty)$, the inverse function of ψ_ε. Note that for $0\leqslant\varepsilon_1<\varepsilon_2<\infty$,

$$\varphi(1)=\psi_{\varepsilon_1}(h_{K+_\varphi\varepsilon_1\cdot_\varphi L}(u))<\psi_{\varepsilon_2}(h_{K+_\varphi\varepsilon_1\cdot_\varphi L}(u)).$$

Hence,

$$h_{K+_\varphi\varepsilon_2\cdot_\varphi L}(u)=\psi_{\varepsilon_2}^{-1}(\varphi(1))>\psi_{\varepsilon_2}^{-1}(\psi_{\varepsilon_2}(h_{K+_\varphi\varepsilon_1\cdot_\varphi L}(u)))=h_{K+_\varphi\varepsilon_1\cdot_\varphi L}(u).$$

Since u is arbitrary, we complete the proof. □

3 Orlicz mixed quermassintegrals

Formula (2.3) expresses the Orlicz first variation of the n-dimensional volume. For the completeness of the study, we consider

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}W_j(K+_\varphi\varepsilon\cdot_\varphi L),\quad j=0,1,\ldots,n-1.$$

The following theorem gives their integral representations.

Theorem 3.1. *Suppose $K,L\in\mathcal{K}_o^n$ and $\varphi\in\Phi$. Then for each $j=0,1,\ldots,n-1$,*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}W_j(K+_\varphi\varepsilon\cdot_\varphi L)=\frac{n-j}{n\varphi'_-(1)}\int_{S^{n-1}}\varphi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)dS_{n-j-1}(K,u).\tag{3.1}$$

Letting $\varphi(t)=t$ and $\varphi(t)=t^p$ with $p\geqslant 1$ in (3.1), it gives the integral representations of $W_j(K,L)$ and $W_{p,j}(K,L)$, respectively.

To prove the theorem, Minkowski's isoperimetric inequality for $W_j(K,L)$ is indispensable: For convex bodies K,L in $\mathbb{R}^n$ and $j=0,1,\ldots,n-1$,

$$W_j(K,L)\geqslant W_j(K)^{\frac{n-j-1}{n-j}}W_j(L)^{\frac{1}{n-j}},\tag{3.2}$$

with equality if and only if K and L are homothetic.

Proof. For brevity, we write $K+_\varphi\varepsilon\cdot_\varphi L$ as K_ε, and define $g:[0,\infty)\to(0,\infty)$ by

$$g(\varepsilon)=W_j(K_\varepsilon)^{\frac{1}{n-j}}.$$

The continuity of $W_j:\mathcal{K}_o^n\to\mathbb{R}$ as well as (2.1) implies that g is continuous at 0. By Lemma 2.2 and the definition of quermassintegrals, we have $W_j(K_{\varepsilon_1})<W_j(K_{\varepsilon_2})$, for $0\leqslant\varepsilon_1<\varepsilon_2<\infty$. Thus, $g(\varepsilon)^{n-j}$ and $g(\varepsilon)$ are both increasing in ε. Hence, we get

$$\begin{aligned}g(0)^{n-j-1}\liminf_{\varepsilon\to 0^+}\frac{g(\varepsilon)-g(0)}{\varepsilon}&=W_j(K)^{\frac{n-j-1}{n-j}}\liminf_{\varepsilon\to 0^+}\frac{W_j(K_\varepsilon)^{\frac{1}{n-j}}-W_j(K)^{\frac{1}{n-j}}}{\varepsilon}\\&=\liminf_{\varepsilon\to 0^+}W_j(K_\varepsilon)^{\frac{n-j-1}{n-j}}\frac{W_j(K_\varepsilon)^{\frac{1}{n-j}}-W_j(K)^{\frac{1}{n-j}}}{\varepsilon}.\end{aligned}$$

According to (3.2), it immediately yields

$$\liminf_{\varepsilon\to 0^+}\frac{g(\varepsilon)-g(0)}{\varepsilon}\geqslant g(0)^{-(n-j-1)}\liminf_{\varepsilon\to 0^+}\frac{W_j(K_\varepsilon)-W_j(K_\varepsilon,K)}{\varepsilon}. \tag{3.3}$$

Similarly, it has

$$\limsup_{\varepsilon\to 0^+}\frac{g(\varepsilon)-g(0)}{\varepsilon}\leqslant g(0)^{-(n-j-1)}\limsup_{\varepsilon\to 0^+}\frac{W_j(K,K_\varepsilon)-W_j(K)}{\varepsilon}. \tag{3.4}$$

The weak continuity of surface area measures as well as (2.1) implies that $S_{n-j-1}(K_\varepsilon,\cdot)$ weakly converges to $S_{n-j-1}(K,\cdot)$ on S^{n-1}, as $\varepsilon\to 0^+$. This and (2.2) yield that

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{W_j(K_\varepsilon)-W_j(K_\varepsilon,K)}{\varepsilon}&=\lim_{\varepsilon\to 0^+}\frac{1}{n}\int_{S^{n-1}}\frac{h_{K_\varepsilon}(u)-h_K(u)}{\varepsilon}dS_{n-j-1}(K_\varepsilon,u)\\&=\frac{1}{n}\int_{S^{n-1}}\lim_{\varepsilon\to 0^+}\frac{h_{K_\varepsilon}(u)-h_K(u)}{\varepsilon}dS_{n-j-1}(K,u)\\&=\frac{1}{n\varphi'_-(1)}\int_{S^{n-1}}\varphi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)dS_{n-j-1}(K,u).\end{aligned}$$

That is,

$$\lim_{\varepsilon\to 0^+}\frac{W_j(K_\varepsilon)-W_j(K_\varepsilon,K)}{\varepsilon}=\frac{1}{n\varphi'_-(1)}\int_{S^{n-1}}\varphi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)dS_{n-j-1}(K,u). \tag{3.5}$$

Similarly, we have

$$\lim_{\varepsilon\to 0^+}\frac{W_j(K,K_\varepsilon)-W_j(K)}{\varepsilon}=\frac{1}{n\varphi'_-(1)}\int_{S^{n-1}}\varphi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)dS_{n-j-1}(K,u). \tag{3.6}$$

Now, combining (3.3), (3.4), (3.5) and (3.6), it implies that $g(\varepsilon)$, and therefore $g(\varepsilon)^{n-j}$, are differentiable at 0. Moreover,

$$\begin{aligned}\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}g(\varepsilon)^{n-j}&=(n-j)g(0)^{n-j-1}\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}g(\varepsilon)\\&=\frac{n-j}{n\varphi'_-(1)}\int_{S^{n-1}}\varphi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)dS_{n-j-1}(K,u).\end{aligned}$$

This completes the proof. □

Theorem 3.1 allows us to introduce the following:

Definition 3.2. For convex bodies $K,L\in\mathcal{K}^n_o$, $\varphi\in\Phi$ and $j=0,1,\ldots,n-1$, define the j-th Orlicz mixed quermassintegral $W_{\varphi,j}(K,L)$ by

$$W_{\varphi,j}(K,L)=\frac{1}{n}\int_{S^{n-1}}\varphi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)dS_{n-j-1}(K,u).$$

Some basic facts are observed:

(1) $W_{\varphi,j}(K,K)=\varphi(1)W_j(K)$.
(2) $W_{\varphi,0}(K,L)=V_\varphi(K,L)$.
(3) If $\varphi(t)=t^p$, $p\geqslant 1$, then $W_{\varphi,j}(K,L)=W_{p,j}(K,L)$.
(4) $W_{\varphi,j}(gK,gL)=W_{\varphi,j}(K,L)$, for all rotations g.
(5) If $L_1\subset L_2$, then $W_{\varphi,j}(K,L_1)\leqslant W_{\varphi,j}(K,L_2)$.
(6) For all $\varepsilon>0$,

$$W_j(K+_\varphi\varepsilon\cdot_\varphi L)/\varphi(1)=W_{\varphi,j}(K+_\varphi\varepsilon\cdot_\varphi L,K)+\varepsilon W_{\varphi,j}(K+_\varphi\varepsilon\cdot_\varphi L,L).$$

Let $\varphi_i,\varphi\in\Phi$ and $i\in\mathbb{N}$. We say that $\varphi_i\to\varphi$, provided

$$\lim_{i\to\infty}\max_{t\in I}|\varphi_i(t)-\varphi(t)|=0,$$

for each compact interval $I\subset[0,\infty)$.

The next lemma shows the continuity of Orlicz mixed quermassintegrals.

Lemma 3.3. *Let $K_i, L_i, K, L \in \mathcal{K}_o^n$ and $i \in \mathbb{N}$.*
(1) *If $K_i \to K$, then $W_{\varphi,j}(K_i, L) \to W_{\varphi,j}(K, L)$.*
(2) *If $L_i \to L$, then $W_{\varphi,j}(K, L_i) \to W_{\varphi,j}(K, L)$.*
(3) *Let $\varphi_i, \varphi \in \Phi$. If $\varphi_i \to \varphi$, then $W_{\varphi_i,j}(K, L) \to W_{\varphi,j}(K, L)$.*

Proof. Note that $K_i \to K$ implies

$$S_{n-j-1}(K_i, \cdot) \to S_{n-j-1}(K, \cdot)$$

weakly on S^{n-1}. In addition, the convergence in

$$\varphi(h_L/h_{K_i})h_{K_i} \to \varphi(h_L/h_K)h_K$$

is uniform on S^{n-1}. Thus, we derive (1) immediately.

Since the convergence in

$$\varphi(h_{L_i}/h_K)h_K \to \varphi(h_L/h_K)h_K$$

is uniform on S^{n-1}, (2) is obtained directly.

Take

$$r_m = \min_{u\in S^{n-1}} h_L(u)/h_K(u) \quad \text{and} \quad r_M = \max_{u\in S^{n-1}} h_L(u)/h_K(u).$$

Note that $\varphi_i \to \varphi$ implies $\varphi_i|_{[r_m,r_M]} \to \varphi|_{[r_m,r_M]}$ uniformly. Therefore, $\varphi_i(h_L/h_K) \to \varphi(h_L/h_K)$ uniformly on S^{n-1}, which implies (3). □

4 The generalized Cauchy-Kubota formula

Quermassintegrals are also fundamental and useful in integral geometry. In this section, we show the probabilistic essence of Orlicz mixed quermassintegrals. The starting point is the Cauchy-Kubota formula.

Recall that for a convex body K in $\mathbb{R}^n$,

$$W_j(K) = \frac{\omega_n}{\omega_{n-j}} \int_{G_{n,n-j}} \mathrm{vol}_{n-j}(K \mid \xi) d\sigma_{n,n-j}(\xi), \quad j = 1, \dots n-1.$$

We generalize this formula to the Orlicz setting. For this aim, the next lemma is needed, which was previously proved in [9]. Here we give a direct proof.

Lemma 4.1. *Let $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$, and $j = 1, \dots n-1$. Then for each $\xi \in G_{n,j}$ and $\varepsilon > 0$,*

$$(K +_\varphi \varepsilon \cdot_\varphi L) \mid \xi = K \mid \xi +_\varphi \varepsilon \cdot_\varphi L \mid \xi.$$

Proof. Let $\xi \in G_{n,j}$ be arbitrary but fixed, and let

$$S^{j-1} = S^{n-1} \cap \xi.$$

For any $u \in S^{j-1}$ and $Q \in \mathcal{K}_o^n$, it has

$$h_Q(u) = h_{Q\,|\,\xi}(u).$$

Thus, applying the definition of $K +_\varphi \varepsilon \cdot_\varphi L$ to u, it gives

$$\varphi\left(\frac{h_{K\,|\,\xi}(u)}{h_{(K+_\varphi\varepsilon\cdot_\varphi L)\,|\,\xi}(u)}\right) + \varepsilon\varphi\left(\frac{h_{L\,|\,\xi}(u)}{h_{(K+_\varphi\varepsilon\cdot_\varphi L)\,|\,\xi}(u)}\right) = \varphi(1).$$

On the other hand, from the definition of $K \mid \xi +_{\varphi,\varepsilon} L \mid \xi$ defined in ξ, it gives

$$\varphi\left(\frac{h_{K\,|\,\xi}(u)}{h_{K\,|\,\xi+_\varphi\varepsilon\cdot_\varphi L\,|\,\xi}(u)}\right) + \varepsilon\varphi\left(\frac{h_{L\,|\,\xi}(u)}{h_{K\,|\,\xi+_\varphi\varepsilon\cdot_\varphi L\,|\,\xi}(u)}\right) = \varphi(1).$$

Hence, $(K +_\varphi \varepsilon \cdot_\varphi L) \mid \xi$ and $K \mid \xi +_\varphi \varepsilon \cdot_\varphi L \mid \xi$ is a same convex body in ξ. □

Theorem 4.2 provides a probabilistic approach to define Orlicz mixed quermassintegrals.

Theorem 4.2. *Suppose $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then for each $j = 1, \ldots, n-1$,*

$$W_{\varphi,j}(K,L) = \frac{\omega_n}{\omega_{n-j}} \int_{G_{n,n-j}} V_\varphi^{(n-j)}(K \mid \xi, L \mid \xi) d\sigma_{n,n-j}(\xi),$$

where $V_\varphi^{(n-j)}(K \mid \xi, L \mid \xi)$ denotes the Orlicz mixed volume of the $(n-j)$-dimensional convex bodies $K \mid \xi$ and $L \mid \xi$ in the subspace ξ.

Proof. From Definition 3.2, the Cauchy-Kubota formula and Lemma 4.1, it follows that

$$W_{\varphi,j}(K,L) = \frac{\varphi'_-(1)}{n-j} \frac{\omega_n}{\omega_{n-j}} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \int_{G_{n,n-j}} \mathrm{vol}_{n-j}(K \mid \xi +_\varphi \varepsilon \cdot_\varphi L \mid \xi) d\sigma_{n,n-j}(\xi).$$

By Theorem 3.1, the above integrand depends smoothly on ε (for small ε). Hence, it gives

$$\begin{aligned} W_{\varphi,j}(K,L) &= \frac{\varphi'_-(1)}{n-j} \frac{\omega_n}{\omega_{n-j}} \int_{G_{n,n-j}} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \mathrm{vol}_{n-j}(K \mid \xi +_\varphi \varepsilon \cdot_\varphi L \mid \xi) d\sigma_{n,n-j}(\xi) \\ &= \frac{\varphi'_-(1)}{n-j} \frac{\omega_n}{\omega_{n-j}} \int_{G_{n,n-j}} \frac{(n-j)}{\varphi'_-(1)} V_\varphi^{(n-j)}(K \mid \xi, L \mid \xi) d\sigma_{n,n-j}(\xi) \\ &= \frac{\omega_n}{\omega_{n-j}} \int_{G_{n,n-j}} V_\varphi^{(n-j)}(K \mid \xi, L \mid \xi) d\sigma_{n,n-j}(\xi), \end{aligned}$$

as desired. □

Up to a constant, the quantity $W_{\varphi,j}(K,L)$ is the expectation of the random variable

$$V_\varphi^{(n-j)}(K \mid \cdot, L \mid \cdot) : G_{n,n-j} \to (0,\infty), \xi \mapsto V_\varphi^{(n-j)}(K \mid \xi, L \mid \xi),$$

which is defined on the probability space $(G_{n,n-j}, \mathcal{B}, \sigma_{n,n-j})$ (where $\mathcal{B}$ is the Borel sigma-algebra on $G_{n,n-j}$).

Letting $\varphi(t) = t^p$ with $p \geqslant 1$ in Theorem 4.2, it yields the formula

$$W_{p,j}(K,L) = \frac{\omega_n}{\omega_{n-j}} \int_{G_{n,n-j}} V_p^{(n-j)}(K \mid \xi, L \mid \xi) d\sigma_{n,n-j}(\xi).$$

A general representation [24,26] of the Cauchy-Kubota formula states that for a convex body K in $\mathbb{R}^n$ and $1 \leqslant q \leqslant j < n$,

$$W_j(K) = \frac{\omega_n}{\omega_{n-q}} \int_{G_{n,n-q}} W_{j-q}^{(n-q)}(K \mid \xi) d\sigma_{n,n-q}(\xi),$$

where $W_{j-q}^{(n-q)}$ denotes the $(j-q)$-th quermassintegral in the subspace ξ. Using an argument similar to that in Theorem 4.2, we can obtain the following theorem.

Theorem 4.3. *Suppose $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then for $1 \leqslant q \leqslant j < n$,*

$$W_{\varphi,j}(K,L) = \frac{\omega_n}{\omega_{n-q}} \int_{G_{n,n-q}} W_{\varphi,j-q}^{(n-q)}(K \mid \xi, L \mid \xi) d\sigma_{n,n-q}(\xi),$$

where $W_{\varphi,j-q}^{(n-q)}(K \mid \xi, L \mid \xi)$ denotes the Orlicz mixed quermassintegral of the $(n-q)$-dimensional convex bodies $K \mid \xi$ and $L \mid \xi$ in the subspace ξ.

5 Inequalities for Orlicz mixed quermassintegrals

Recall that for convex bodies K, L in $\mathbb{R}^n$ and $j = 0, 1, \ldots, n-1$, Minkowski's isoperimetric inequality (3.2) can be rewritten as

$$\frac{W_j(K,L)}{W_j(K)} \geqslant \left(\frac{W_j(L)}{W_j(K)}\right)^{\frac{1}{n-j}}, \tag{5.1}$$

with equality if and only if K and L are homothetic. The next result extends this inequality to the Orlicz setting.

Theorem 5.1. *Suppose $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then for each $j = 0, 1, \ldots, n-1$,*

$$\frac{W_{\varphi,j}(K,L)}{W_j(K)} \geqslant \varphi\bigg(\bigg(\frac{W_j(L)}{W_j(K)}\bigg)^{\frac{1}{n-j}}\bigg).$$

If φ is strictly convex, the equality holds if and only if K and L are dilates.

Proof. The condition on K guarantees that $W_j(K) > 0$. Thus, the measure

$$\frac{h_K}{nW_j(K)} dS_{n-j-1}(K, \cdot)$$

is a probability measure on S^{n-1}. From the convexity of φ combined with Jensen's inequality, the strict monotonicity of φ, together with (5.1), it follows that

$$\begin{aligned}\frac{W_{\varphi,j}(K,L)}{W_j(K)} &= \frac{1}{nW_j(K)} \int_{S^{n-1}} \varphi\bigg(\frac{h_L(u)}{h_K(u)}\bigg) h_K(u) dS_{n-j-1}(K,u) \\ &\geqslant \varphi\bigg(\frac{1}{nW_j(K)} \int_{S^{n-1}} h_L(u) dS_{n-j-1}(K,u)\bigg) \\ &= \varphi\bigg(\frac{W_j(K,L)}{W_j(K)}\bigg) \\ &\geqslant \varphi\bigg(\bigg(\frac{W_j(L)}{W_j(K)}\bigg)^{\frac{1}{n-j}}\bigg).\end{aligned}$$

Now, we verify the equality conditions. The sufficiency is easy to prove. We prove the necessity.

Suppose the equality holds. From the injectivity of φ, we have the equality in (5.1). So, there exist $x \in \mathbb{R}^n$ and $r > 0$ such that $L = rK + x$. Hence,

$$h_L(u) = rh_K(u) + x \cdot u,$$

for all $u \in S^{n-1}$. Since φ is strictly convex, by the equality condition for Jensen's inequality, we have

$$\frac{1}{nW_j(K)} \int_{S^{n-1}} \frac{h_L(u)}{h_K(u)} h_K(u) dS_{n-j-1}(K,u) = \frac{h_L(v)}{h_K(v)},$$

for $S_{n-j-1}(K,\cdot)$-almost all $v \in S^{n-1}$. Therefore, for $S_{n-j-1}(K,\cdot)$-almost all $v \in S^{n-1}$,

$$\frac{1}{nW_j(K)} \int_{S^{n-1}} \bigg(r + \frac{x \cdot u}{h_K(u)}\bigg) h_K(u) dS_{n-j-1}(K,u) = r + \frac{x \cdot v}{h_K(v)}.$$

Note that

$$W_j(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) dS_{n-j-1}(K,u),$$

and the centroid of $S_{n-j-1}(K,\cdot)$ is at the origin. Thus, we have

$$\begin{aligned}0 &= x \cdot \bigg(\frac{1}{nW_j(K)} \int_{S^{n-1}} u dS_{n-j-1}(K,u)\bigg) \\ &= \frac{1}{nW_j(K)} \int_{S^{n-1}} x \cdot u dS_{n-j-1}(K,u) \\ &= \frac{x \cdot v}{h_K(v)},\end{aligned}$$

for $S_{n-j-1}(K,\cdot)$-almost all $v \in S^{n-1}$. By combining the above with the fact that $S_{n-j-1}(K,\cdot)$ is not concentrated on any great subsphere of S^{n-1}, we conclude that x is just the origin, and therefore K and L are dilates. □

Now, we turn to the applications of Theorem 5.1.

Lemma 5.2. *Suppose φ is a strictly convex Young function, and $K, L \in \mathcal{K}_o^n$.*

(1) *If K and L are dilates, then for each $\alpha, \beta > 0$, K and $\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L$ are dilates.*

(2) *Suppose $\alpha, \beta > 0$. If K and $\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L$ are dilates, then K and L are dilates.*

Proof. To prove (1), assume $L = \varepsilon K$ for some constant $\varepsilon > 0$. Let C_S denote the class

$$\{h_K|_{S^{n-1}} : K \in \mathcal{K}_o^n\}.$$

The definition of Orlicz combinations implies that the function $h_{\alpha\cdot_\varphi K +_\varphi \beta\cdot_\varphi L}$ is the unique solution to the equation

$$\alpha\varphi\bigg(\frac{h_K}{f}\bigg) + \beta\varphi\bigg(\frac{\varepsilon h_K}{f}\bigg) = \varphi(1), \quad f \in C_S.$$

On the other hand, it is obvious to prove that there exists a unique $\delta > 0$ such that

$$\alpha\varphi\bigg(\frac{1}{\delta}\bigg) + \beta\varphi\bigg(\frac{\varepsilon}{\delta}\bigg) = \varphi(1),$$

which immediately implies

$$\alpha\varphi\bigg(\frac{h_K}{h_{\delta K}}\bigg) + \beta\varphi\bigg(\frac{\varepsilon h_K}{h_{\delta K}}\bigg) = \varphi(1).$$

Hence, $\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L = \delta K$, which concludes (1).

To prove (2), assume $\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L = \lambda K$ for some constant $\lambda > 0$. Then for arbitrary $u \in S^{n-1}$,

$$\alpha\varphi\bigg(\frac{1}{\lambda}\bigg) + \beta\varphi\bigg(\frac{h_L(u)}{h_{\alpha\cdot_\varphi K +_\varphi \beta\cdot_\varphi L}(u)}\bigg) = \varphi(1),$$

which implies that

$$\varphi\bigg(\frac{h_L(u)}{h_{\alpha\cdot_\varphi K +_\varphi \beta\cdot_\varphi L}(u)}\bigg)$$

is constant for all $u \in S^{n-1}$. This and the injectivity of φ show that $\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L$ and L are dilates. □

Theorem 5.1 yields an Orlicz extension of the Brunn-Minkowski inequality.

Theorem 5.3. *Suppose $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$, and $\alpha, \beta > 0$. Then for each $j = 0, 1, \ldots, n-1$,*

$$\varphi(1) \geqslant \alpha\varphi\bigg(\bigg(\frac{W_j(K)}{W_j(\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L)}\bigg)^{\frac{1}{n-j}}\bigg) + \beta\varphi\bigg(\bigg(\frac{W_j(L)}{W_j(\alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L)}\bigg)^{\frac{1}{n-j}}\bigg).$$

If φ is strictly convex, the equality holds if and only if K and L are dilates.

Proof. For brevity, let

$$\widetilde{K} = \alpha \cdot_\varphi K +_\varphi \beta \cdot_\varphi L.$$

The definition of $\widetilde{K}$ necessarily implies for all $u \in S^{n-1}$ that

$$\varphi(1) = \alpha\varphi\bigg(\frac{h_K(u)}{h_{\widetilde{K}}(u)}\bigg) + \beta\varphi\bigg(\frac{h_L(u)}{h_{\widetilde{K}}(u)}\bigg).$$

Hence, by the definitions of W_j and $W_{\varphi,j}$ and Theorem 5.1, we have

$$\begin{aligned}\varphi(1)W_j(\widetilde{K}) &= \alpha W_{\varphi,j}(\widetilde{K}, K) + \beta W_{\varphi,j}(\widetilde{K}, L)\\ &\geqslant W_j(\widetilde{K})\bigg(\alpha\varphi\bigg(\bigg(\frac{W_j(K)}{W_j(\widetilde{K})}\bigg)^{\frac{1}{n-j}}\bigg) + \beta\varphi\bigg(\bigg(\frac{W_j(L)}{W_j(\widetilde{K})}\bigg)^{\frac{1}{n-j}}\bigg)\bigg).\end{aligned}$$

From Theorem 5.1 and Lemma 5.2, the equality conditions can be obtained immediately. □

When $\varphi(t) = t^p$, $p \geqslant 1$, and $\alpha, \beta > 0$, the above inequality reduces to the L_p Brunn-Minkowski inequality

$$W_j(\alpha \cdot_p K +_p \beta \cdot_p L)^{\frac{p}{n-j}} \geqslant \alpha W_j(K)^{\frac{p}{n-j}} + \beta W_j(L)^{\frac{p}{n-j}}.$$

The next corollary is a weaker version of Theorem 5.3.

Corollary 5.4. *Suppose $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$, and $j = 0, 1, \dots, n-1$. Then for $0 < \alpha < 1$,*

$$W_j(\alpha \cdot_\varphi K +_\varphi (1-\alpha) \cdot_\varphi L) \geqslant W_j(K)^\alpha W_j(L)^{1-\alpha}.$$

Proof. For brevity, let

$$K_\alpha = \alpha \cdot_\varphi K +_\varphi (1-\alpha) \cdot_\varphi L.$$

Since φ is strictly increasing and convex, by Theorem 5.3 and the arithmetic mean-geometric mean (AM-GM) inequality, we have

$$\begin{aligned} \varphi(1) &\geqslant \alpha \varphi\bigg(\bigg(\frac{W_j(K)}{W_j(K_\alpha)}\bigg)^{\frac{1}{n-j}}\bigg) + (1-\alpha)\varphi\bigg(\bigg(\frac{W_j(L)}{W_j(K_\alpha)}\bigg)^{\frac{1}{n-j}}\bigg) \\ &\geqslant \varphi\bigg(\alpha\bigg(\frac{W_j(K)}{W_j(K_\alpha)}\bigg)^{\frac{1}{n-j}} + (1-\alpha)\bigg(\frac{W_j(L)}{W_j(K_\alpha)}\bigg)^{\frac{1}{n-j}}\bigg) \\ &\geqslant \varphi\bigg(\frac{W_j(K)^{\frac{\alpha}{n-j}} W_j(L)^{\frac{1-\alpha}{n-j}}}{W_j(K_\alpha)^{\frac{1}{n-j}}}\bigg). \end{aligned}$$

Therefore,

$$W_j(K_\alpha) \geqslant W_j(K)^\alpha W_j(L)^{1-\alpha}.$$

□

If $\varphi(t) = t$, then the above corollary gives that for each $0 < \alpha < 1$,

$$W_j(\alpha K + (1-\alpha)L) \geqslant W_j(K)^\alpha W_j(L)^{1-\alpha}.$$

The classical difference body, DK, of a convex body K, is defined by $h_{DK} = h_K + h_{-K}$. The next corollary considers the Orlicz version of difference bodies.

Corollary 5.5. *Suppose $K \in \mathcal{K}_o^n$, $\varphi \in \Phi$, and $j = 0, 1, \dots, n-1$. For the convex bodies*

$$\Delta_\varphi K = \frac{1}{2} \cdot_\varphi K +_\varphi \frac{1}{2} \cdot_\varphi (-K) \quad \textit{and} \quad D_\varphi K = K +_\varphi (-K),$$

there exist the inequalities

$$W_j(\Delta_\varphi K) \geqslant W_j(K)$$

and

$$W_j(D_\varphi K) \geqslant \big(\varphi^{-1}(\varphi(1)/2)\big)^{n-j} W_j(K).$$

If φ is strictly convex, each equality holds if and only if K is origin-symmetric.

Along with Orlicz mixed quermassintegrals, we introduce the following quantity.

Definition 5.6. For convex bodies $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$, define

$$\widehat{W}_{\varphi,j}(K, L) = \inf\bigg\{\lambda > 0 : \frac{1}{nW_j(K)} \int_{S^{n-1}} \varphi\bigg(\frac{h_L(u)}{\lambda h_K(u)}\bigg) h_K(u) dS_{n-j-1}(K, u) \leqslant \varphi(1)\bigg\}.$$

It can be checked that if $\varphi(t) = t^p$, $p \geqslant 1$, then

$$\widehat{W}_{\varphi,j}(K, L) = (W_{p,j}(K, L)/W_j(K))^{1/p}.$$

In fact, $\widehat{W}_{\varphi,j}(K, L)$ is the *Orlicz norm* (see [23]) of h_L/h_K with respect to the Borel probability measure

$$\frac{h_K}{nW_j(K)} dS_{n-j-1}(K, \cdot).$$

In light of [11, Lemma 5], we have

$$W_{\varphi,j}\left(K, \frac{L}{\widehat{W}_{\varphi,j}(K,L)}\right) = \varphi(1)W_j(K). \tag{5.2}$$

The quantity $\widehat{W}_{\varphi,j}(K,L)$ provides an approach to extend Minkowski's isoperimetric inequality to the Orlicz setting.

Theorem 5.7. *Suppose $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then for each $j = 0, 1, \ldots, n-1$,*

$$\widehat{W}_{\varphi,j}(K,L) \geqslant \left(\frac{W_j(L)}{W_j(K)}\right)^{\frac{1}{n-j}}.$$

If φ is strictly convex, the equality holds if and only if K and L are dilates.

Proof. From (5.2), Theorem 5.1 and the fact that

$$W_j(\alpha L) = \alpha^{n-j} W_j(L), \quad \alpha > 0,$$

it follows that

$$\begin{aligned}\varphi(1) &\geqslant \varphi\left(\left(W_j\left(\frac{L}{\widehat{W}_{\varphi,j}(K,L)}\right)/W_j(K)\right)^{\frac{1}{n-j}}\right)\\ &= \varphi\left(\frac{(W_j(L)/W_j(K))^{\frac{1}{n-j}}}{\widehat{W}_{\varphi,j}(K,L)}\right).\end{aligned}$$

Note that φ is strictly increasing. Hence, the desired inequality is obtained.

If φ is strictly convex, by Theorem 5.1 again, the equality holds if and only if K and $\widehat{W}_{\varphi,j}(K,L)^{-1}L$ are dilates. □

At the end of this paper, we modify the approach to deal with Orlicz mixed quermassintegrals. Let $K, L \in \mathcal{K}_o^n$ and $j = 0, 1, \ldots, n-1$. Define the Borel probability measure $\overline{V}_{n-j}(K,\cdot)$ on S^{n-1} by

$$d\overline{V}_{n-j}(K,\cdot) = \frac{h_K}{nW_j(K)} dS_{n-j-1}(K,\cdot).$$

It is noted that the measure $\overline{V}_n(K,\cdot)$ is just the normalized cone-volume measure of K. See [2,3,12,22,33].

Following the idea of normalization [19], we introduce the quantity $\overline{W}_{\varphi,j}(K,L)$ by

$$\overline{W}_{\varphi,j}(K,L) = \varphi^{-1}\left(\frac{W_{\varphi,j}(K,L)}{W_j(K)}\right) = \varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) d\overline{V}_{n-j}(K,u)\right),$$

where φ^{-1} is the inverse function of $\varphi \in \Phi$. An advantage of this notion is that, in terms of φ, the quantity $\overline{W}_{\varphi,j}(K,L)$ produces an average of h_L/h_K with respect to the measure $\overline{V}_{n-j}(K,\cdot)$. The inequality in Theorem 5.1 can be rewritten as

$$\overline{W}_{\varphi,j}(K,L) \geqslant \left(\frac{W_j(L)}{W_j(K)}\right)^{\frac{1}{n-j}}.$$

In particular, $\overline{W}_{\varphi,0}(K,L)$ is $\overline{V}_\varphi(K,L)$, the normalized Orlicz mixed volume [35] of K and L. That is,

$$\overline{V}_\varphi(K,L) = \varphi^{-1}\left(\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) d\overline{V}_n(K,u)\right).$$

Correspondingly, there is the inequality

$$\overline{V}_\varphi(K,L) \geqslant \left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right)^{\frac{1}{n}}.$$

If $\varphi(t) = t^p$, $t \geqslant 1$, then $\overline{V}_\varphi(K, L)$ is just the normalized L_p-mixed volume $\overline{V}_p(K, L)$ (see [19]).

Applying Theorem 5.1 to the strictly convex function $\varphi(t) = \mathrm{e}^t - 1$, it yields

$$\log \int_{S^{n-1}} \exp\left(\frac{h_L(u)}{h_K(u)}\right) d\overline{V}_{n-j}(K, u) \geqslant \left(\frac{W_j(L)}{W_j(K)}\right)^{\frac{1}{n-j}},$$

with equality if and only if K and L are dilates. Along with the conjectured log-Minkowski inequality in [2], the following question begs to be asked:

Question. *If K and L are both origin-symmetric convex bodies in $\mathbb{R}^n$ and $j = 0, 1, \ldots, n-1$, then is it the case that*

$$\exp \int_{S^{n-1}} \log\left(\frac{h_L(u)}{h_K(u)}\right) d\overline{V}_{n-j}(K, u) \geqslant \left(\frac{W_j(L)}{W_j(K)}\right)^{\frac{1}{n-j}},$$

with equality if and only if K and L are dilates?

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