# Rényi Mutual Information in Holographic Warped CFTs 

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#### Abstract

The study of Rényi mutual information (RMI) sheds light on the AdS/CFT correspondence beyond classical order. In this article, we study the Rényi mutual information between two intervals at large distance in two-dimensional holographic warped conformal field theory, which is conjectured to be dual to gravity on $\mathrm{AdS}_{3}$ or warped $\mathrm{AdS}_{3}$ spacetimes under Dirichlet-Neumann boundary conditions. By using the operator product expansion of twist operators up to level 3, we read the leading oder and the next-to-leading order RMI in the large central charge and small cross-ratio limits. The leading order result is furthermore confirmed using the conformal block expansion. Finally, we match the next-to-leading order result by a 1-loop calculation in the bulk.


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## 1 Introduction

Holographic entanglement entropy opens a new window to study the AdS/CFT correspondence [1]. It has been proposed [2]3] that the entanglement entropy of a subregion $A$ in the boundary CFT can be holographically computed by the area of the minimal surface $\Sigma_{A}$ in the bulk,

$$
\begin{equation*}
S_{A}=\frac{\operatorname{Area}\left(\Sigma_{A}\right)}{4 G_{N}} \tag{1.1}
\end{equation*}
$$

where $\Sigma_{A}$ is homologous to the entangling surface $A$. This so-called Ryu-Takayanagi (RT) formula is reminiscent of the Bekenstein-Hawking entropy for the black hole. Actually, it has
been shown in [4] that the holographic entanglement entropy can be taken as a kind of generalized gravitational entropy and it can be computed by the classical action of the corresponding gravitational configuration. The quantum correction to the holographic entanglement entropy would then be related to the semi-classical gravitational effects in the bulk [5, 6].

The semiclassical gravity picture of holographic entanglement entropy is most manifest in the $\mathrm{AdS}_{3} / \mathrm{CFT}_{2}$ correspondence. In $\mathrm{AdS}_{3}$ gravity, under Brown-Henneaux boundary conditions, the asymptotic symmetry group is generated by two copies of the Virasoro algebra with central charge [7]

$$
\begin{equation*}
c=\frac{3 l}{2 G_{N}} \tag{1.2}
\end{equation*}
$$

where $l$ is the AdS radius. This suggests that a consistent quantum gravity on $\mathrm{AdS}_{3}$ with a semiclassical limit should be holographic dual to a 2D CFT, and that such holographic CFTs should have a large central charge [8]. Further studies from modular invariance indicates that holographic CFTs also have a sparse spectrum of light states, which further implies that the vacuum is the dominant contribution to the torus partition at low temperature/energy limit [9]. In this setup, the holographic entanglement entropy has been intensely studied. When the entangling surface in the CFT is just a single interval, the minimal surface in the bulk is just a geodesic, whose length matches with the single-interval entanglement entropy [2]. For the multi-interval case, the holographic entanglement entropy has been proved in [10, 11] to be equal to the one in the CFT1.

Furthermore, other important entanglement quantities such as the Rényi entropy and the mutual information can be studied in the $\mathrm{AdS}_{3} / \mathrm{CFT}_{2}$ correspondence quantitatively as well. The Rényi entropy encodes rich information of the reduced density matrix and the entanglement. In general such quantities are rather difficult to compute, as it requires the partition function of the field theory on a manifold of nontrivial topology and conical singularity. Holographically, one has to take into account the backreaction of a cosmic brane [4, 15]. However, in the semiclassical limit, the computation of the Rényi entropy is feasible on both the field theory and the gravity sides, even for the multi-interval case. On the field theory side, assuming that the conformal block is dominated by the vacuum module at large $c$, one may use the recursive relation of the Virasoro block [10] or the operator product expansion (OPE) of the twist operators [16, 17] to compute the correlation functions of the twist operators. In particular, the OPE of the twist operators allows us to read not only the leading-order(LO) result in the $1 / c$ expansion, which is linear in $c$ and corresponds to the semiclassical action of the gravitational configuration, but also the next-to-leading-order(NLO) result, which corresponds to the 1-loop correction to the holographic Rényi entropy [6, 17, and even the next-next-to-leading-order

[^0](NNLO) result, which corresponds to the 2-loop quantum correction in gravity [17, 18. On the gravity side, though the gravitational configurations corresponding to the higher genus Riemann surfaces resulting from the replica trick are hard to construct explicitly, their semiclassical action can be read via the Zograf-Takhtadzhyan action and the monodromy method [11], and the 1 -loop correction can be read by finding the Schottky uniformization [6]. The computations from the field theory and the bulk gravity agree remarkably well $[17,19,20$. Furthermore, Rényi entropy with higher $n$ also provides examples where the sparseness condition does not necessarily imply that the identity block dominates the conformal block expansion, and new phase could appear 21,22$]^{2}$.

Three-dimensional quantum gravity should be defined with respect to appropriate boundary conditions. Besides the usual Brown-Henneaux boundary conditions, there exist other sets of consistent asymptotic boundary conditions [24, 52]. In particular, under the Compére-Song-Strominger(CSS) boundary conditions, the asymptotic symmetry group of $\mathrm{AdS}_{3}$ gravity is generated by the Virasoro-Kac-Moody algebra [24, which can be realized in a warped conformal field theory (WCFT) [25, 26]. This leads to the $\mathrm{AdS}_{3} /$ WCFT correspondence. In this correspondence, the holographic entanglement entropy presents some novel features due to the nontrivial boundary conditions. The single-interval entropy is not simply captured by the length of a geodesic with homologous condition in the bulk, but needs modification [27, 28]. It is definitely interesting to consider the Rényi entropy in the multi-interval case, which could shed new light on the $\mathrm{AdS}_{3} / \mathrm{WCFT}$ correspondence.

In this note, we study the Rényi mutual information of two disjoint intervals in holographic warped CFT with the assumption that the vacuum module dominates the correlation function in the limit of large central charge. We consider the case that the intervals are far apart so that the cross ratio $x$ is small. For warped CFTs, the vacuum module contains all the Virasoro and $U(1)$ Kac-Moody descendants. We use the OPE of the twist operators to compute the partition function. We propose a warped conformal transformation from $n$-sheeted geometry to the plane such that the OPE coefficients can be read from the one-point functions of the quasi-primary operators. We consider the quasi-primary operators up to level 3 , and find the Rényi mutual information up to $x^{3}$, including both the leading-order(LO) result and the next-to-leading order(NLO) result in the $1 / c$ expansion.

As a consistency check, we reproduce the leading-order result in the Rényi mutual information by studying the conformal blocks of the warped CFT in the large $c$ limit. The warped conformal block factorizes into a Virasoro block and a Kac-Moody block. Consequently, the

[^1]leading order contribution of the Rényi mutual information can be put in the following form
\[

$$
\begin{align*}
I_{n}^{(L O)}(x)= & -\frac{(n-1) c}{12 n} \log (1-x) \\
& + \text { (holomorphic part of RMI in holographic CFTs), } \tag{1.3}
\end{align*}
$$
\]

where the first line comes purely from the Kac-Moody block, and the second line comes from the Virasoro block.

The next-to-leading-order result is expected to match the 1-loop partition function of handlebody configurations, which can be read by summing over images from the 1-loop result of BTZ [48]

$$
\begin{equation*}
Z_{C S S}^{1-\text { loop }}=\prod_{\gamma \in \mathcal{P}^{\prime}} \prod_{l^{\prime}=1}^{\infty} \frac{1}{1-q_{\gamma}^{l^{l}}} \prod_{l=2}^{\infty} \frac{1}{1-q_{\gamma}^{l}} \tag{1.4}
\end{equation*}
$$

where $\mathcal{P}^{\prime}$ is a set of representatives of primitive conjugacy classes of the modified Schottky group $\Gamma^{\prime}$ which should now be compatible with the CSS boundary conditions. Note that the CSS boundary conditions make a selection on both the linearized fluctuations and the allowed images we sum over. As was shown in [48], only the holomorphic vector modes and tensor modes contribute to the partition on a fixed background, consistent with the chiral Virasoro-Kac-Moody symmetry in the dual WCFT. In addition, the saddle points should be obtained from images of given BTZ solutions generated by an $S L(2, \mathbb{R}) \times U(1)$ quotient instead of $S L(2, \mathbb{R}) \times S L(2, \mathbb{R})$. Nevertheless, we provide an argument that the quotients acts the same way in the holomorphic sector, which enables us to perform the 1-loop calculation in the bulk, and find that it indeed agrees with the NLO result from WCFT calculation up to $x^{3}$. This is to be expected as WCFT is chiral, and the difference in the uniformization does not affect the holomorphic part.

The remaining parts of this paper are arranged as follows. In section 2, we set up the notations in warped CFT and discuss the operator product expansion. In section 3, we give the detailed prescriptions to calculate the mutual Rényi information, involving the partition function and twist operator expansion. We introduce the warped conformal transformation relating the $n$-sheeted geometry in the single interval (on the plane) case to the flat warped geometry. In section 4, we study the large distance expansion of the Rényi mutual information in the holographic warped CFT. First, we classify the quasi-primary operators in the orbifold theory, give them explicitly up to level 3 and calculate their coefficients. Then we read the Rényi mutual information and expand it in orders of $1 / c$. In section 5 , we provide a consistency check on the leading order result by computing the 4 -point function of twist operators, assuming that the conformal block is dominated by the vacuum Verma module in the large $c$ limit. This allows us to reach the conclusion (1.3). Furthermore we compare the next-to-leading order
result with the bulk computation (1.4) in section 6 . In section 7 , we end with conclusion and discussions. Some technical details are collected in the appendices.

## 2 The ABCs of WCFTs

### 2.1 Symmetries and spectral flow

In this subsection, we provide a brief review of warped CFTs [26,29], starting with a definition on a Lorentzian cylinder, instead of the plane. The motivation comes from the holographic models where the bulk spacetime has a $S L(2, R) \times U(1)$ local isometry, and a $S^{1} \times R$ boundary topology [24,49]. Let us start with the so-called canonical cylinder parametrized by $(\hat{x}, \hat{y})$, with a canonical spatial circle $(\hat{x}, \hat{y}) \sim(\hat{x}+2 \pi, \hat{y})$. Consider a two-dimensional local field theory on the $(\hat{x}, \hat{y})$ cylinder, with the following local symmetry.

$$
\begin{equation*}
\hat{x} \rightarrow f(\hat{x}), \quad \hat{y} \rightarrow \hat{y}+g(\hat{x}) \tag{2.1}
\end{equation*}
$$

where $f(\hat{x})$ and $g(\hat{x})$ are periodic functions of $\hat{x}$. (2.1) is the defining property of warped CFTs, which are non-relativistic. The two directions are not on the equal footing in the sense that $\hat{x}$ allows reparametrization while $y$ does not. The left moving stress tensor is denoted as $T(\hat{x})$ and the left moving $U(1)$ current is denoted as $P(\hat{x})$. Under the warped transformation (2.1), the stress tensor and the $\mathrm{U}(1)$ current transform as

$$
\begin{align*}
P(f(\hat{x})) & =f(\hat{x})^{\prime-1}\left(P(\hat{x})+\frac{k g^{\prime}(\hat{x})}{2}\right)  \tag{2.2}\\
T(f(\hat{x})) & =f(\hat{x})^{\prime-2}\left(T(\hat{x})-\frac{c}{12} s(f(\hat{x}), \hat{x})-g^{\prime}(\hat{x}) P(\hat{x})-\frac{k g^{\prime}(\hat{x})^{2}}{4}\right) \tag{2.3}
\end{align*}
$$

where $s(f, \hat{x})$ is the Schwarzian derivative,

$$
\begin{equation*}
s(f, \hat{x})=\frac{f^{\prime \prime \prime}}{f^{\prime}}-\frac{3}{2}\left(\frac{f^{\prime \prime}}{f^{\prime}}\right)^{2} \tag{2.4}
\end{equation*}
$$

and the derivative is with respect to $\hat{x}$. The Fourier modes form the canonical warped algebra

$$
\begin{gather*}
{\left[L_{n}, L_{m}\right]=(n-m) L_{n+m}+\frac{c}{12} n\left(n^{2}-1\right) \delta_{n+m, 0}}  \tag{2.5}\\
{\left[L_{n}, P_{m}\right]=-m P_{m+n}}  \tag{2.6}\\
{\left[P_{n}, P_{m}\right]=\frac{k}{2} n \delta_{n+m, 0}} \tag{2.7}
\end{gather*}
$$

We would like to classify the WCFTs by the central charge $c$, level $k$, as well as a vacuum charge $b$ on the canonical cylinder,

$$
\begin{equation*}
P_{0}^{v a c}=i b \tag{2.8}
\end{equation*}
$$

which also determines the expectation value of the Viraosoro zero mode on the vacuum as

$$
\begin{equation*}
L_{0}^{v a c} \equiv a-\frac{c}{24}, \quad a=-\frac{b^{2}}{k} . \tag{2.9}
\end{equation*}
$$

The relation between $L_{0}^{v a c}$ and $P_{0}^{v a c}$ is determined by the unitary bound as discussed in [26].
Now let us consider WCFTs on other geometries. According to the transformation rules (2.1), under a shift

$$
\begin{equation*}
\hat{x} \rightarrow \hat{x}, \quad \hat{y} \rightarrow \hat{y}-i \mu \hat{x}, \tag{2.10}
\end{equation*}
$$

the charges will transformation as

$$
\begin{equation*}
P_{n ; \mu}=P_{n}+\frac{i k \mu}{2} \delta_{n, 0}, \quad L_{n ; \mu}=L_{n}+i \mu P_{n}-\left(\frac{k \mu^{2}}{4}+\frac{c}{24}\right) \delta_{n, 0} \tag{2.11}
\end{equation*}
$$

In particular, the vacuum charge will be shifted to $\left\langle P_{0}\right\rangle=i\left(b+\frac{k \mu}{2}\right)$. By choosing the parameter

$$
\begin{equation*}
b=-\frac{k \mu}{2}, \tag{2.12}
\end{equation*}
$$

we can therefore define a reference cylinder with zero $U(1)$ charge, and a spatial circle $\left(\hat{x}, y^{\prime}\right) \sim$ $\left(\hat{x}+2 \pi, y^{\prime}-2 \pi i \mu\right)$. We can make a smooth cover of the reference cylinder by multiplying the circle by $n$ while keeping the direction invariant. Therefore the spatial circle becomes $\left(\hat{x}, y^{\prime}\right) \sim\left(\hat{x}+2 \pi n, y^{\prime}-2 \pi i n \mu\right)$.

From either the canonical cylinder or the reference cylinder, we can define a "plane" by an exponential map in the Virasoro direction, while keeping the $U(1)$ direction unchanged. To do so, we should view $\hat{x}$ as a null direction with $\hat{x}=t+\phi, \phi \sim \phi+2 \pi$, and take the analytic continuation $t_{E}=i t$. Since the spatial circle does not change the $y$ coordinate, $y$ is to be viewed as proportional to the time direction. Namely, we define the canonical plane $\mathcal{C}$ parameterized by ( $z, \hat{y}$ ), by the transformation

$$
\begin{equation*}
\hat{x} \rightarrow z=e^{i \hat{x}}=e^{-t_{E}+i \phi}, \quad \hat{y} \rightarrow \hat{y} \tag{2.13}
\end{equation*}
$$

with points identified by

$$
\begin{equation*}
(z, y) \sim\left(z e^{2 \pi i}, \hat{y}\right) \tag{2.14}
\end{equation*}
$$

and non-vanishing vacuum charges

$$
\begin{equation*}
<P_{0}>_{\mathcal{C}}=i b, \quad<L_{0}>_{\mathcal{C}}=a=-\frac{b^{2}}{k} . \tag{2.15}
\end{equation*}
$$

The vacuum charges can be interpreted as inserting an operator $V$ at the origin, and the currents will be non-vanishing

$$
\begin{equation*}
\langle T(z)\rangle_{V}=\frac{a}{z^{2}}, \quad\langle P(z)\rangle_{V}=\frac{b}{z} . \tag{2.16}
\end{equation*}
$$

Similarly, the reference plane $\mathcal{C}^{\prime}$ parameterized by $\left(z, y^{\prime}\right)$, is obtained from the the reference cylinder by

$$
\begin{equation*}
\hat{x} \rightarrow z=e^{i \hat{x}}=e^{-t_{E}+i \phi}, \quad y^{\prime} \rightarrow y^{\prime} . \tag{2.17}
\end{equation*}
$$

The reference plane has the nice feature that all vacuum charges are zero, and therefore no operator insertion at the origin. The information of non-trivial vacuum charges is now encoded in the non-trivial boundary conditions

$$
\begin{equation*}
(z, \bar{z}) \sim\left(z e^{2 \pi i}, y^{\prime}-2 \pi i \mu\right) \tag{2.18}
\end{equation*}
$$

The smooth cover of the reference plane $\mathcal{C}^{\prime}$ is made by multiplying the circle by $n$ while keeping the direction invariant, then points will be identified as $\left(z, y^{\prime}\right) \sim\left(z e^{2 \pi n i}, y^{\prime}-2 \pi i n \mu\right)$. This property will help us understand the uniformazation map (3.10). Note that neither the reference plane nor the canonical plane are the complex plane in the usual sense, as they are not parameterized by a pair of holomorphic and anti-holomorphic coordinates which are complex conjugate to each other. Nevertheless, as a mathematical trick, we can still view $z$ as a holomorphic coordinate on the usual complex plane.

Furthermore, in this paper we will formulate physical questions on the physical cylinder with a thermal circle $(x, y) \sim(x+i \beta, y-i \bar{\beta})$, or with a spatial circle $(x, y) \sim(x+L, y-\bar{L})$. The physical cylinder can be mapped to the canonical cylinder with a rescaling and tilting, and furthermore mapped to the canonical plane or the reference plane.

### 2.2 Operator product expansion

In this subsection we discussion the operator product expansion in WCFTs, and determine the coefficients in terms of the three-point coefficients.

Operators in WCFTs can be organized using primary operators and their Virasoro-KacMoody descendants. Primary operators at the origin are labeled by the conformal weight $h$ and the $U(1)$ charge $Q$, with

$$
\begin{equation*}
L_{n}|O\rangle=0, \quad P_{n}|O\rangle=0, \forall n>0 \tag{2.19}
\end{equation*}
$$

and the descendants are linear combinations of $L_{-1}^{N_{1}} L_{-2}^{N_{2}} \cdots P_{-1}^{N_{1}} P_{-2}^{N_{2}} \cdots|O\rangle$. Moving the operators from the origin by $L_{-1}$ and $P_{0}$, one gets the complete set of operator basis of the theory at any point. Using the commutation relations (2.5), all the Virasoro-Kac-Moody generators can be rewritten as the polynomials of $L_{-1}, L_{-2}, P_{-1}$. Therefore it is possible to further organize a warped conformal family by quasi-primary operators and their global descendants, which is the basis we use in this paper. Using this basis, the OPE ansatz can be written as

$$
\begin{equation*}
\phi\left(x_{1}, y_{1}\right) \phi^{\dagger}\left(x_{2}, y_{2}\right)=\sum_{k, n} c_{k} a_{k, n} \frac{e^{i Q y_{12}}}{x_{12}^{2 h-h_{k}-n}} \partial_{x_{2}}^{n} \phi_{k}\left(x_{2}, y_{2}\right) . \tag{2.20}
\end{equation*}
$$

where $\phi_{k}$ are all the quasi-primary operators in the theory with zero $U(1)$ charge. The $x_{12}$ part is fixed by the conformal weight. The ansatz is chosen such that $a_{k, n}$ only depends on the weights while $c_{k}$ contains the dynamical information of the theory. The summation over $k, n$ includes all the quasi-primary operators and their descendants, which form a complete basis.

In the following we will determine the coefficients $c_{k}$ and $a_{k, n}$ using two and three point functions. Two point functions of quasi-primary operators in the $S L(2, R) \times U(1)$ invariant vacuum can be fixed by the global symmetry [29]

$$
\begin{equation*}
\left\langle\phi\left(x_{1}, y_{1}\right) \phi^{\dagger}\left(x_{2}, y_{2}\right)\right\rangle=d e^{i Q y_{12}} \frac{1}{x_{12}^{2 h}} \tag{2.21}
\end{equation*}
$$

where $Q$ is the $\mathrm{U}(1)$ charge of $\phi, \phi^{\dagger}$ has the opposite charge to $\phi$, and $d$ is a normalization factor. Using the OPE (2.20) and taking derivatives on the two-point function of the neutral operator $\phi_{k}$, we obtain the following three-point function

$$
\begin{equation*}
\left\langle\phi(x) \phi^{\dagger}(1) \phi_{k}(0)\right\rangle_{x}=\sum_{n} c_{k} d_{k} a_{k, n} e^{i Q y_{12}} \frac{(-1)^{n}\left(2 h_{k}\right)_{n}}{(x-1)^{2 h-h_{k}-n}} \tag{2.22}
\end{equation*}
$$

where $\left(h_{k}\right)_{n} \equiv\left(h_{k}\right)\left(h_{k}+1\right) \cdots\left(h_{k}+n-1\right)$ is the raising Pochhammer symbol.
On the other hand, three-point function can also be determined by global symmetry and can be rewritten in a general form

$$
\begin{equation*}
\left\langle\phi\left(x_{1}, y_{1}\right) \phi^{\dagger}\left(x_{2}, y_{2}\right) \phi_{k}\left(x_{3}, y_{3}\right)\right\rangle=c_{\phi \phi^{\dagger} \phi^{\prime}} e^{i Q y_{12}} \frac{1}{x_{12}^{2 h-h_{k}}} \frac{1}{x_{31}^{h_{k}}} \frac{1}{x_{23}^{h_{k}}} \tag{2.23}
\end{equation*}
$$

where $c_{\phi \phi^{\dagger} \phi_{k}}$ is the three-point coefficient, characterizing the dynamics of the theory. (2.23) can be expanded in terms of $x-1$,

$$
\begin{equation*}
\left\langle\phi(x) \phi^{\dagger}(1) \phi_{k}(0)\right\rangle=c_{\phi \phi^{\dagger} \phi_{k}} \frac{e^{i Q y_{12}}}{(x-1)^{2 h-h_{k}} x^{h_{k}}}=\sum_{n} c_{\phi \phi^{\dagger} \phi_{k}} e^{i Q y_{12}} \frac{(-1)^{n}\left(h_{k}\right)_{n} / n!}{(x-1)^{2 h-h_{k}-n}} \tag{2.24}
\end{equation*}
$$

Comparing the two expressions (2.22) and (2.24), we find

$$
\begin{equation*}
c_{k}=\frac{c_{\phi \phi^{\dagger} \phi_{k}}}{d_{k}}, \quad a_{k, n}=\frac{\left(h_{k}\right)_{n}}{n!\left(2 h_{k}\right)_{n}} . \tag{2.25}
\end{equation*}
$$

### 2.3 Comments on holographic WCFTs

Similar to their CFT cousins, WCFTs are specified by the operator spectrum labelled by $h_{\phi_{k}}, Q_{\phi_{k}}$ and OPE coefficients $C_{i j k}$. It is interesting to ask what are the necessary conditions for a WCFT to be holographically dual to quantum theories of gravity with a semiclassical limit. From the asymptotic symmetry analysis [24], we learn that holographic WCFTs usually have: i) a large central charge, and ii) a negative level. In addition, in order that the entropy formula [26] to be valid for large central charge and finite temperature, we expect iii) a sparse
spectrum for the light operators, similar to 9. However, a precise statement for the sparseness condition is not yet spelled out 3 . Similar to holographic CFTs, we also expect two closely related but non-equivalent conditions to the sparseness condition. One is that iv) the vacuum block dominates the warped conformal block expansion, and the other is that v) the WCFT is maximally chaotic 39$]^{4}$. Throughout this paper, we explicitly assume condition i) a large central charge and iv) vacuum dominance for holographic WCFTs.

## 3 Rényi mutual information in WCFT

Quantum entanglement plays a central role in many fields of physics. It is interesting to use measures of entanglement to probe WCFTs and their holographic duals. Single interval entanglement entropy and Rényi entropy in WCFT was first calculated in [27] and revisited in [28] by generalizing the Rindler method [23] and the warped Cardy formula [26]. A more general procedure of the generalized Rindler method was proposed in [32]. In [28], an additional parameter was introduced which is essential to match the bulk calculation in $\mathrm{AdS}_{3}$ and warped $\mathrm{AdS}_{3}$ spacetime [28] and in lower spin gravity [33]. Entanglement entropy on excited states has also been discussed in 39].

While the aforementioned results are classical and for single intervals, in this paper we will go one step further by calculating the Rényi mutual information for two disjoint intervals and obtain both the classical result and the leading quantum corrections. In this section, we revisit the calculation of Rényi entropy for single interval in WCFT and describe the strategy to calculate Rényi mutual information by adapting the OPE method developed in the context of $\mathrm{CFT}_{2}$ [17,20].

### 3.1 Twist operator and uniformization

In this subsection, we revisit the calculation of Rényi entropy for single interval in WCFT and convert the previous Rindler transformation to a uniformization map, which facilitates later calculations of Rényi mutual information in Euclidean language.

Entanglement entropy measures the correlation between the subregion and its environment. Consider a spacelike subregion $A$, separated with its environment by an entangling surface, which is a co-dimension 2 submanifold. The entanglement entropy of $A$ is defined to be the von Neumann entropy of the reduced density matrix [30, 31,

$$
\begin{equation*}
S_{A}=-\operatorname{Tr}_{A} \rho_{A} \log \rho_{A} \tag{3.1}
\end{equation*}
$$

[^2]where the reduced density matrix $\rho_{A}$ is obtained by taking the partial trace of the density matrix of the whole system,
\[

$$
\begin{equation*}
\rho_{A}=\operatorname{Tr}_{\bar{A}} \rho . \tag{3.2}
\end{equation*}
$$

\]

In quantum field theories, the computation of the entanglement entropy is quite difficult due to the infinite degrees of freedom and the non-local operator $\log \rho_{A}$. The usual way to compute the entanglement entropy is to apply the replica trick 34 by making $n$ copies of the original theory and glue them together cyclicly along the interval $A$. An orbifold theory is obtained by modding out $Z_{n}$ of the tensor product of $n$ copies of the original theory. Denote the original manifold by $\Sigma$, its $n$-th smooth cover $\Sigma_{n}$, and define Rényi entropy as

$$
\begin{equation*}
S_{n}=\frac{1}{1-n} \log \frac{\mathcal{Z}_{n}}{\mathcal{Z}^{n}} \tag{3.3}
\end{equation*}
$$

where $\mathcal{Z}_{n}$ is the partition function on $\Sigma_{n}$, and $\mathcal{Z}$ is on $\Sigma$. The entanglement entropy can then be read from the $n \rightarrow 1$ limit after proper analytic continuation

$$
\begin{equation*}
S_{A}=\lim _{n \rightarrow 1} S_{n} . \tag{3.4}
\end{equation*}
$$

To proceed, one may introduce twist operators $\sigma$ and $\tilde{\sigma}$ in the vacuum of the orbifold theory. The twist operators are operators of co-dimension two, nonlocal in the spacetime $d>2$. In two dimensions, the twist operators $\sigma$ and $\tilde{\sigma}$ are local operators, located at the endpoints of the subregions. Correlation functions on $\Sigma_{n}$ can be rewritten in the orbifold theory with twist operator insertions,

$$
\begin{align*}
\left\langle O_{i}(x)\right\rangle_{\Sigma_{n}} & =\frac{\left\langle O_{i}(x) \sigma \tilde{\sigma}\right\rangle_{1}}{\langle\sigma \tilde{\sigma}\rangle_{1}}  \tag{3.5}\\
\left\langle\left(O_{i} O_{j}\right)(x)\right\rangle_{\Sigma_{n}} & =\frac{\left\langle O_{i}(x) O_{j}(x) \sigma \tilde{\sigma}\right\rangle_{1}}{\langle\sigma \tilde{\sigma}\rangle_{1}} \tag{3.6}
\end{align*}
$$

where $\langle\ldots\rangle_{1}$ denotes the correlation with the insertion of the twist operators in the orbifold theory, and $x$ denotes collectively the coordinates on $\Sigma_{n}$. For $\mathrm{CFT}_{2}$, the quantum numbers of the twist operator can be found by using a uniformization map from $\Sigma_{n}$ to the plane $\mathcal{C}$, the transformation rule of the stress tensor, and the Ward identity. Then the partition function can be computed by the correlation function of the twist operators [35, 36].

Whereas the aforementioned method using twist operators in $\mathrm{CFT}_{2}$ is more conveniently formulated in Euclidean signature, previous computations of Rényi entropy for single intervals for WCFT [27, 28] have been based on the Rindler method in Lorentzian signature. One goal of this subsection is to translate the Rindler transformation to the uniformization map in the Euclidean signature, which facilitates the calculation of the Rényi mutual information later.

In the calculation of Rényi entropy and entanglement entropy for single intervals in WCFTs [27, 28], the key step is to find the generalized Rindler transformation [32] which maps entanglement entropy to thermal entropy. Consider a subregion $A$ on a manifold $\Sigma$, a generalized

Rindler transformation is a symmetry transformation of the theory which maps the domain of causality $\mathcal{D}$ of the subregion $A$ to a Rindler spacetime $\tilde{\Sigma}$ characterized by a thermal identification. The modular flow generator is the generator of the thermal identification on $\tilde{\Sigma}$ rewritten on $\Sigma$, and is required to annihilate the vacuum, leave $\mathcal{D}$ invariant, and map $\partial \mathcal{D}$ to itself. As the Rindler transformation is a symmetry transformation, the partition function on $\tilde{\Sigma}_{n}$ and the one on $\Sigma_{n}$ are equivalent up to a unitary transformation, and the thermal entropy on $\tilde{\Sigma}_{n}$ is hence the Rényi entropy for the subregion $A$ on the vacuum. In general, thermal entropy is also difficult to calculate directly. However, for theories with nice modular properties, such as WCFT, a Carly-like formula can be derived [26], and the partition function on $n$-copied Rindler space $\tilde{\Sigma}_{n}$, the thermal entropy on $\tilde{\Sigma}_{n}$, and hence the Rényi entropy for $A$ can all be calculated. One can also read the quantum numbers of the twist operators.

More explicitly, consider an interval $A$ bounded by two end points $(0,0)$ and $(l, 0)$ on a cylinder $\Sigma$ parameterized by $(x, y)$, with a thermal circle $(x, y) \sim(x+i \beta, y-i \bar{\beta})$. Results for the spatial circle can be obtained by replacing $i \beta, i \bar{\beta}$ by $L, \bar{L}$, respectively. With a change of convention $\alpha_{\text {there }}=2 \pi \mu_{\text {here }}$ and a shift of the interval, the Rindler transformation in a WCFT [28] can be written as

$$
\begin{align*}
\tanh \frac{\pi \tilde{x}}{\tilde{\beta}} & =\frac{\tanh \frac{\pi\left(x-\frac{l}{2}\right)}{\beta}}{\tanh \frac{\pi l}{2 \beta}},  \tag{3.7}\\
\tilde{y}+\left(\frac{\overline{\tilde{\beta}}}{\tilde{\beta}}-\frac{2 \pi \mu}{\tilde{\beta}}\right) \tilde{x} & =y+\left(\frac{\bar{\beta}}{\beta}-\frac{2 \pi \mu}{\beta}\right) x . \tag{3.8}
\end{align*}
$$

We would like to consider a manifold $\Sigma$ with zero temperature, which can be taken as the plane limit of (3.7) with $\beta \rightarrow \infty, p \equiv \frac{\bar{\beta}-2 \pi \mu}{\beta}$ fixed. Note that the Rindler transformation maps the domain of dependence $\mathcal{D}$ to the Rindler space $\tilde{\Sigma}$, with a thermal circle $(\tilde{x}, \tilde{y}) \sim(\tilde{x}+i \tilde{\beta}, \tilde{y}-i \tilde{\bar{\beta}})$. Making $n$ replicas corresponds to a thermal circle $(\tilde{x}, \tilde{y}) \sim(\tilde{x}+i n \tilde{\beta}, \tilde{y}-i n \tilde{\tilde{\beta}}) .5$ By redefining

$$
\begin{equation*}
z=e^{\frac{2 \pi \tilde{\tilde{x}}}{n \beta}}, \quad \hat{y}=y^{\prime}-\frac{2 b}{k} \log z=\tilde{y}+\frac{\tilde{\tilde{\beta}}}{\tilde{\beta}} \tilde{x} \tag{3.9}
\end{equation*}
$$

the replicated Rindler space will be mapped to the canonical plane or the reference plane, and

[^3]we get a uniformization man $\sqrt{6}$ in the Euclidean signature,
\[

$$
\begin{equation*}
z=\left(\frac{w}{w-l}\right)^{\frac{1}{n}}, \quad \hat{y}-n \mu \log z=y^{\prime}-\left(\frac{2 b}{k}+n \mu\right) \log z=y-p w . \tag{3.10}
\end{equation*}
$$

\]

where we have replaced $x$ by $w$ in the plane limit, and the phase assignment is chosen to map points on the $k$-th sheet to a patch on the plane with $\arg z \in\left[0, \frac{2 \pi}{n}\right)$. The one-point function on the replicated geometry $\Sigma_{n}$ will then become

$$
\begin{aligned}
\langle T(w)\rangle_{\Sigma_{n}} & =\frac{l^{2}}{w^{2}(w-l)^{2}} \frac{h_{n}}{n}+p \frac{l}{w(w-l)} \frac{i Q_{n}}{n}-\frac{k p^{2}}{4}, \\
\langle P(w)\rangle_{\Sigma_{n}} & =\frac{l}{w(w-l)} \frac{i Q_{n}}{n}-\frac{k p}{2} .
\end{aligned}
$$

Here the conformal dimension and the charge of the twist operator in the orbifold theory are given by the vacuum charges and spectral flow parameters

$$
\begin{equation*}
h_{n}=n\left(\frac{c\left(n^{2}-1\right)}{24 n^{2}}+\frac{a}{n^{2}}-\frac{b}{n} \mu-\frac{k}{4} \mu^{2}\right), \quad Q_{n}=-i\left(b+\frac{n k \mu}{2}\right) \tag{3.11}
\end{equation*}
$$

where $k$ is the Kac level, $c$ is the central charge of the Virasoro algebra, $a, b$ are the nonvanishing vacuum charges on the canonical plane as discussed in section 2. The above quantum numbers for the twist operator as well as the one-point functions indeed agree with those of [29] in the plane limit with $\beta \rightarrow \infty, p \equiv \lim _{\beta \rightarrow \infty} \frac{\bar{\beta}-2 \pi \mu}{\beta}$ fixed.

For generality, we have kept three parameters in the transformation. The vacuum charges $a$ or $b$ (determined by only one parameter) on the canonical cylinder/plane is a defining property of the WCFT theory, the twist parameter $\mu$ introduces a shift in $y$ when $w$ goes around a branching point on $\Sigma$, and is responsible for the short distance behavior in the orbifold theory. The parameter $p$ can be viewed as an additional spectral flow parameter on $\Sigma_{n}$ and provides the constant pieces of the one-point functions on $\Sigma_{n}$. From purely WCFT analysis, the freedom in choosing the $U(1)$ directions seems to lead to these three free parameters. For example, [27] set $\mu=0$, [28] kept both $b$ and $\mu$ aribtrary. However, from holography, $b$ and $\mu$ do not correspond to independent quantities in the bulk calculation either in $\mathrm{AdS}_{3}$ [28] or warped AdS black holes [33. They have to be chosen appropriately. Hence one may wonder whether $\mu$ and $b$ are physically independent from a purely WCFT perspective. Indeed, if we further require that the twist operator has zero charges as $n \rightarrow 1$, the parameter $\mu$ should be chosen such that the twist operator has zero charge and conformal dimension. This further requirement sets $b=-\frac{k \mu}{2}$ as was noticed in [39], and leads to two independent parameters. In particular, when $p=0$, the plane limit of $\Sigma$ is just the reference plane. In the following, we will still use all the parameters, while keep in mind that there is only one free parameter in $a, b$, and $\mu$.

[^4]Let us further comment on the plane limit. In this paper we will use the approach of operator product expansion to calculate the Rényi mutual information. Since the operator product expansion captures the short distance behavior, there is no difference in the calculation of the coefficients, no matter whether the interval is on a plane or on the cylinder. We choose to deal with them on the plane for simplicity. We should also note that a further spectral flow parameterized by $p$ has no effect on the two-point functions except the exponential part. In the case of neutral operators, the entire two-point function will not change at all. The final result of Rényi mutual information will be the same for arbitrary $p$, since the exponential factors cancel each other out, which is consistent with the picture that the causal development of the interval is just a strip independent of $y$ in WCFTs which are non-relativistic [27, 28]. In the following discussions, we will set $p=0$ unless otherwise specified.

To summarize, we point out that the uniformization map (3.10) has the following properties,

- The uniformization map (3.10) is a warped conformal transformation.
- (3.10) can be obtained from the Rindler transformation (3.7) by a plane limit on the right hand side, and an exponential map from the left hand side. The conformal weight and charge of the twist operator agree with that found in [29] using the Rindler method.
- When $n=1$, the two end points on $\Sigma$ are mapped to the origin and $\infty$ on the canonical plane respectively, with an additional shift in the $U(1)$ direction. The twist operator has zero quantum numbers $h_{1}=Q_{1}=0$, with the choice $b=-\frac{2 \mu}{k}$. When $p=0$, the physical plane $\Sigma$ is the reference plane.
- For $n \neq 1$, the map is multivalued. In particular, when $p=0$, going around the circle around $(0,0)$ and $(l, 0)$ on $\Sigma_{n}$ becomes $\left(\frac{w}{w-l} e^{2 \pi i n}, y \sim y-2 \pi i n \mu\right)$, which enlarges the circle on the reference plane $\left(z, y^{\prime}\right) \sim\left(z e^{2 \pi i}, y^{\prime}-2 \pi \mu\right)$ while keeping the same direction. This is more obvious from the corresponding cylinders. Points on the $k$-th sheet on $\Sigma_{n}$ is mapped to a slice on the plane with $\arg z \in\left[0, \frac{2 \pi}{n}\right)$.
- The uniformation (3.10) in the Viraosoro direction is exactly the same as that of the holomorphic part of a $\mathrm{CFT}_{2}$.


### 3.2 Rényi mutual information

In this subsection, we give a prescription of calculating Rényi mutual information for WCFTs using the OPE of twist operators for two disjoint subregions.

Rényi mutual information is another important notion of quantum entanglement. For two
disjoint subregions $A$ and $B$, the Rényi mutual information is defined as

$$
\begin{equation*}
I_{n}=S_{n}(A)+S_{n}(B)-S_{n}(A \cup B) \tag{3.12}
\end{equation*}
$$

which can also be expressed by the correlation function of the twist operators

$$
\begin{equation*}
I_{n}=\frac{1}{n-1} \log \frac{\langle\sigma \tilde{\sigma} \sigma \tilde{\sigma}\rangle}{\langle\sigma \tilde{\sigma}\rangle\langle\sigma \tilde{\sigma}\rangle} \tag{3.13}
\end{equation*}
$$

Note that the twist operators are inserted at proper locations to separate the different subregions. $\sigma \tilde{\sigma}$ is just short for

$$
\begin{equation*}
\langle\sigma \tilde{\sigma} \cdots\rangle=\frac{\left\langle\sigma_{n} \tilde{\sigma}_{n} \cdots\right\rangle}{\left\langle\sigma_{1} \tilde{\sigma}_{1} \cdots\right\rangle^{n}} \tag{3.14}
\end{equation*}
$$

which means that the reduced density matrix should be normalized properly. By taking the $n \rightarrow 1$ limit, we have the mutual information

$$
\begin{equation*}
S(A, B)=S(A)+S(B)-S(A \cup B), \tag{3.15}
\end{equation*}
$$

which is always positive due to the subadditivity property of entanglement entropy. The mutual information characterizes the entanglement between two subregions such that even if the two subregions are far apart, the mutual information is non-vanishing due to quantum correlations 37.

In practice, it is hard to compute the Rényi mutual information as the $n$-folded geometry can have not only nontrivial topology, but also singularities. Nevertheless, one may apply the operator product expansion of the twist operators to read the partition functions in the large distance limit. In two dimensional CFTs, the twist operators are local primary operators such that one can use the technology of the OPE of primary operators to read the mutual information between two disjoint intervals [16, 17, 38. In particular, for holographic CFTs, one may focus on the vacuum module such that the quasiprimary operators can be classified level by level and the Rényi mutual information can be read in powers of the cross ratio and $1 / c$. For the 2D warped CFTs, we can apply the same strategy to calculate the four point function for twist operators and the Rényi mutual information. Namely, we will first write down an OPE ansatz for the twist operators, find the dominating terms in holographic WCFTs, and calculate each contributing term. We outline the main steps below.

We consider the Rényi mutual information of two disjoint intervals, with the end points of $A$ at $\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)$, and the end points of $B$ at $\left(x_{3}, y_{3}\right),\left(x_{4}, y_{4}\right)$. The Rényi mutual information (3.13) can be rewritten in terms of correlation functions of the twist operators, we have

$$
\begin{equation*}
I_{n}(x)=\frac{1}{n-1} \log \frac{F_{n}(x)}{F_{1}(x)^{n}} \tag{3.16}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{n}(x)=\frac{\left\langle\sigma_{n}(\infty) \tilde{\sigma}_{n}(1) \sigma_{n}(x) \tilde{\sigma}_{n}(0)\right\rangle}{\left\langle\sigma_{n}(\infty) \tilde{\sigma}_{n}(1)\right\rangle\left\langle\sigma_{n}(x) \tilde{\sigma}_{n}(0)\right\rangle} \tag{3.17}
\end{equation*}
$$

Note that the $y$-dependent pre-factors in the $F_{n}(x)$ cancel each other. Since the exchanged operators are all neutral, there will be no exponential factor in the two-point functions. As a consequence, the Rényi mutual information will be $y$-independent. Also, it depends only on the warped conformal invariant cross ratio $x$

$$
\begin{equation*}
x=\frac{x_{12} x_{34}}{x_{13} x_{24}} \tag{3.18}
\end{equation*}
$$

where $x_{i}$ 's are the endpoints of the two intervals.
Twist operators in WCFTs will also be viewed as quasi-primary operators with conformal weight $h_{n}$ and charge $Q_{n}$ (3.11). Their OPE ansatz follows (2.20),

$$
\begin{equation*}
\sigma_{n}\left(x_{1}, y_{1}\right) \tilde{\sigma}_{n}\left(x_{2}, y_{2}\right)=\sum_{k, m} c_{k} a_{k, m} \frac{e^{i Q y_{12}}}{x_{12}^{2 h_{n}-h_{k}-m}} \partial_{x_{2}}^{m} \phi_{k}\left(x_{2}, y_{2}\right) . \tag{3.19}
\end{equation*}
$$

The $\phi_{k}$ runs over every quasi-primary operators in the orbifold theory, and carries vanishing $\mathrm{U}(1)$ charge. Expanding the twist operators is equivalent to insert a set of complete basis in the four-point function of the twist operators.

Now the task is to determine the OPE coefficients $c_{k}$ for quasi-primary operators. From (2.25), we learn that the coefficient $c_{k}$ of $\phi_{k}$ is determined by the three-point coefficient with the twist operators $c_{k}=\frac{c_{\sigma \tilde{\sigma}} \phi_{k}}{d_{k}}$, which can furthermore be read from its one-point function in the $n$-folded geometry of a single interval using

$$
\begin{equation*}
\left\langle\phi_{k}\left(x_{3}\right)\right\rangle_{\Sigma_{n}}=\frac{\left\langle\sigma\left(x_{1}\right) \tilde{\sigma}\left(x_{2}\right) \phi_{k}\left(x_{3}\right)\right\rangle}{\left\langle\sigma\left(x_{1}\right) \tilde{\sigma}\left(x_{2}\right)\right\rangle} \tag{3.20}
\end{equation*}
$$

where $\Sigma_{n}$ denotes the $n$-folded geometry. Then we get

$$
\begin{equation*}
c_{k}=\frac{1}{d_{k}} x_{12}^{-h_{k}} \lim _{x_{3} \rightarrow \infty} x_{3}^{2 h_{k}}<\phi_{k}\left(x_{3}\right)>_{\Sigma_{n}} \tag{3.21}
\end{equation*}
$$

Finally one-point functions of the quasi-primary operators can be calculated using the transformation laws (2.2) and the uniformization map (3.10) which maps the $n$-folded geometry to the canonical plane.

Using the OPE ansatz (3.19) and the coefficients (2.25) (3.21), the function $F_{n}(x)$ can be expanded in terms of the global conformal blocks,

$$
\begin{align*}
F_{n}(x) & =\left.\sum_{\left\{\phi_{k}\right\}} d_{k} c_{k}^{2} x^{h_{k}} \sum_{m_{1}, m_{2}} a_{k, m_{1}} a_{k, m_{2}} x^{\frac{m_{1}+m_{2}}{2}} \partial_{x_{1}}^{m_{1}} \partial_{x^{2}}^{m_{2}}\left(x_{1}-x_{2}\right)^{2 h_{k}}\right|_{x_{1}=1, x_{2}=0} \\
& =\sum_{\left\{\phi_{k}\right\}} d_{k} c_{k}^{2} x^{h_{k}}{ }_{2} F_{1}\left(h_{k}, h_{k}, 2 h_{k}, x\right) . \tag{3.22}
\end{align*}
$$

where the $n$ dependence is hidden in $c_{k}=\frac{c_{\sigma_{n} \tilde{n} \phi_{k}}}{d_{k}}$, and ${ }_{2} F_{1}\left(h_{k}, h_{k}, 2 h_{k}, x\right)$ is hypergeometric function. The summation over $\left\{\phi_{k}\right\}$ is on the quasi-primary operators in the propagating channels.

For holographic WCFTs, as discussed in the end of section 2.2, we assume that the fourpoint function of the twist operators is dominated by the vacuum module, similar to holographic CFTs [53]. Hereafter the summation (3.19) in the OPE of the twist operators is only over the quasi-primary operators in the vacuum module, which can be constructed in a systematical way. Note that a difference from usual holographic CFTs is that now the vacuum module is generated by the Virasoro and Kac-Moody creation operators acting on the vacuum. As $x$ is the conformal invariant cross ratio, the large distance expansion corresponds to the small cross ratio. From the power $x^{h_{k}}$, we see that the quasi-primary operators of low scaling dimensions give the leading order contributions. Consequently, we can work out the contributions of the quasi-primary operators to the Rényi mutual information level by level.

We end this section with a brief summary of the OPE methods. To to calculate the Rényi mutual information, one needs to deal with the partition function on the $n$-sheeted geometry, which is a formidable task due to the non-trivial topology. One may turn to the correlation functions of the twist operators instead. Using the OPE of the twist operators and only considering the vacuum module, it is feasible to read the Rényi mutual information of two disjoint intervals. The essential point is that the OPE coefficients of the quasi-primaries in the vacuum module can be read by their one-point functions in the replicated geometry. By applying the warped conformal transformation (3.10), the coefficients can be determined analytically. Then the main tasks are finding the operators in the orbifold theory and calculating their one-point functions in the $n$-sheeted geometry.

## 4 Rényi mutual information from OPEs

In this section, we use the OPE method outlined in section 3.2 to study the Rényi mutual information of two disjoint intervals in holographic warped CFT. As the mutual information can be expanded in powers of the cross ratio, the leading contribution is captured by the quasiprimary operators of low scaling dimensions. In this work, we would like to find the contributions up to $x^{3}$. This requires us to find all quasi-primary operators in the orbifold theory up to level 3 and compute their OPE coefficients.

### 4.1 Quasi-primary operators in the orbifold theory

As the first step, we should find the construction of the quasi-primary operators in the vacuum module in the original warped CFT. The states in the vacuum module of the warped CFT
are created by the Virasoro generators $L_{-m}(m \geq 2)$ and Kac-Moody generators $P_{-n}(n \geq 1)$ acting on the $S L(2, R) \times U(1)$ invariant vacuum. The generating function of the quasiprimary operators in the vacuum at level $N$ is

$$
\begin{aligned}
Z(q) & =\frac{1}{1-q} \prod_{m=2}^{\infty} \frac{1}{\left(1-q^{m}\right)^{2}} \\
& =1+q+3 q^{2}+5 q^{3}+10 q^{4}+16 q^{5}+29 q^{6}+45 q^{7}+75 q^{8}+\cdots \\
& =\sum_{N=0}^{\infty} P(N) q^{N} .
\end{aligned}
$$

Here $P(N)$ is the partition function, which gives the number of operators at level $N$. The numbers are as follows,

| Level | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Number of operators | 1 | 1 | 3 | 5 | 10 | 16 | 29 | 45 | 75 |
| Number of quasi-primaries | 1 | 1 | 2 | 2 | 5 | 6 | 13 | 16 | 30 |

We list the quasi-primary operators explicitly up to level 3 .

- Level 0 , there is the identity operator $I$, corresponding to the vacuum state.
- Level 1 , there is one quasi-primary operator, $-i P$, which is hermitian. It corresponds to the state $P_{-1}|0\rangle$.
- Level 2 , there are two quasi-primary operators. One is the stress tensor $T$, which corresponds to the state $L_{-2}|0\rangle$. The other one is $A=T+\frac{c}{k}(P P)$, which corresponds to the state $\left(L_{-2}-\frac{c}{k} P_{-1} P_{-1}\right)|0\rangle$.
- Level 3 , there are two quasi-primary operators. One is $i\left(P^{3}\right)$, which corresponds to the state $P_{-1} P_{-1} P_{-1}|0\rangle$. The other one is $B=i P^{3}+i k(P T)$, which corresponds to the state $\left(P_{-1}^{3}-k P_{-1} L_{-2}\right)|0\rangle$.

We denote the normal ordered product of the operators $A$ and $B$ by $(A B)(w)$. The notation of normal ordered product we use is,

$$
\begin{equation*}
(A B)(w)=\frac{1}{2 \pi i} \oint_{w} \frac{d z}{z-w} A(z) B(w) \tag{4.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(P^{3}\right)=\left(P P^{2}\right)=\left(P^{2} P\right) \tag{4.2}
\end{equation*}
$$

Note that the OPE of $T$ and $P$ has singular terms, so that we cannot separate the contribution from the ones involving stress tensor and the ones involving $U(1)$ current. We use the Schmidt orthogonalization process to make the quasi-primary operators at each level independent of
each other in the following calculation. One should also note that proper number of $\sqrt{-1}$ 's should be included to make sure every quasi-primary operators above are hermitian.

Let us turn to the construction of the quasi-primary operators in the vacuum module in the orbifold CFT. Usually, the spectrum of a CFT includes the normal sector and the twist sector. Here we take a different point of view. Instead of classifying the operators according to their monodromy property, we classify the operators in the orbifold theory by the operators in $n$-replicated theory, taking into account of the replica symmetry [17, 35]. The generating function of $n$-copied theory reads

$$
\begin{aligned}
Z(q) & =\left(\frac{1}{1-q} \prod_{m=2}^{\infty} \frac{1}{\left(1-q^{m}\right)^{2}}\right)^{n} \\
& =1+n q+\frac{5 n+n^{2}}{2} q^{2}+\frac{14 n+15 n^{2}+n^{3}}{6} q^{3}+\cdots \\
& =\sum_{N=0}^{\infty} P(N) q^{N},
\end{aligned}
$$

where the partition functions $P(N)$ give the number of operators at the level $N$. We get the number of quasi-primary operators by subtraction. The numbers are listed in the table below.

| Level | 0 | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Number of quasi-primaries | 1 | $n$ | $\frac{n(n+3)}{2}$ | $\frac{n^{3}+12 n^{2}-n}{6}$ | $\frac{n^{4}+26 n^{2}+71 n^{2}+22 n}{24}$ |

The explicit forms of the quasi-primary operators at each level are listed as follows.

- Level 0 , there is the identity operator $I$.
- Level 1 , there is one kind of quasi-primary operators $-i P_{i}$. The subscript index $i$ labels the $i$-th copy of the theory the operators belong to. There are $n$ such operators in total.
- Level 2 , there are three kinds of quasi-primary operators. The first kind is $T_{i}$, and its number is $n$. The second type is $A_{i}$, and its total number is also $n$. The last type is $P_{i} P_{j}$. We require $0 \leq i<j \leq n-1$ to avoid double counting. The total number of such operators is $\frac{n(n-1)}{2}$. In sum, the total number of the operators at this level is $\frac{n(n+3)}{2}$.
- Level 3, there are six kinds of quasi-primary operators. They are $B_{i}, i P_{i}^{3},-i T_{i} P_{j}$, $-i P_{i} A_{j}, i P_{i} P_{j} P_{k}$, and $-i\left(P_{i} \partial P_{j}-P_{j} \partial P_{i}\right)$ respectively. When the same kind of operators appear in the composite operators, their indices should be arranged increasingly in order to avoid repeatedly counting. The corresponding degeneracies of these operators are $n$, $n, n(n-1), n(n-1), \frac{n(n-1)(n-2)}{6}$, and $\frac{n(n-1)}{2}$ respectively. And the total number of the operators at this level is $\frac{n^{3}+12 n^{2}-n}{6}$.

It is worth noting that not all of the quasi-primary operators can be written as the tensor product of the ones in the single copy theory. Such operators actually belong to the twist sector of the orbifold CFT.

### 4.2 Coefficients in the OPE

To compute the function $F_{n}(x)$, we have to determine the OPE coefficients $c_{k}$ and the normalization factor $d_{k}$ in (3.22). The normalization factors can be determined simply by algebraic relation, while the OPE coefficients can be determined by the the one-point functions of the above quasi-primary operators in the $n$-sheeted geometry with a single interval. As now the quasi-primary operators are constructed by the stress tensor and the operator $P$, their onepoint functions are determined purely by the warped conformal transformation (3.10), similar to the case in usual large $c$ CFT [17]. The twist operator here is actually a composite operator including the standard one and the operator which generate the non-trivial vacuum. Since we has included such effect in the OPE coefficients, now we calculate the correlators of twist operators in the reference plane where the vacuum is trivial. We collect the main results of the OPE coefficients in this subsection, leaving some calculation details to the appendix A

- Level 0 , the coefficients before the identity operator is 1 according to our normalization.
- Level 1 , the quasi-primary operators are $-i P_{i}$, with

$$
\begin{equation*}
c_{-i P}=-i \frac{2 b}{n k}-i \mu . \tag{4.3}
\end{equation*}
$$

- Level 2 , for $T_{i}$,

$$
\begin{equation*}
c_{T}=\frac{n^{2}-1}{12 n^{2}}+\frac{2 a}{c n^{2}}-\frac{2 b \mu}{c n}-\frac{k \mu^{2}}{2 c} . \tag{4.4}
\end{equation*}
$$

For $A_{i}$,

$$
\begin{equation*}
c_{A}=\frac{(2 b+k n \mu)^{2}}{2 c k n^{2}} . \tag{4.5}
\end{equation*}
$$

For $P_{i} P_{j}$,

$$
\begin{equation*}
c_{P_{i} P_{j}}=\left(\frac{4 b^{2}}{k^{2} n^{2}}+\frac{1}{2 k n^{2} s_{i j}^{2}}+\frac{4 b \mu}{k n}+\mu^{2}\right) . \tag{4.6}
\end{equation*}
$$

- Level 3, for $i P_{i}^{3}$,

$$
\begin{equation*}
c_{i P^{3}}=i \frac{(2 b+k n \mu)\left(2(2 b+k n \mu)^{2}-k n^{2}+k\right)}{12 k^{3} n^{3}} . \tag{4.7}
\end{equation*}
$$

For $B_{i}=i P_{i}^{3}+i k(P T)_{i}$,

$$
\begin{equation*}
c_{B}=\frac{i\left(n^{2}-1\right)(2 b+k n \mu)}{12 k^{2} n^{3}} . \tag{4.8}
\end{equation*}
$$

For $-i T_{i} P_{j}$,

$$
\begin{equation*}
c_{-i T_{i} P_{j}}=-i \frac{(2 b+k n \mu)\left(-24 a+6 n \mu(4 b+k n \mu)-c n^{2}+c+6 s_{i j}^{-2}\right)}{12 c k n^{3}} . \tag{4.9}
\end{equation*}
$$

For $-i P_{i} A_{j}$,

$$
\begin{equation*}
c_{-i P_{i} A_{j}}=-i \frac{(2 b+k n \mu)\left((2 b+k n \mu)^{2}+k s_{i j}^{-2}\right)}{2 c k^{2} n^{3}} . \tag{4.10}
\end{equation*}
$$

For $i P_{i} P_{j} P_{k}$,

$$
c_{i P_{i} P_{j} P_{k}}=i \frac{(2 b+k n \mu)^{3}}{k^{3} n^{3}}+i \frac{(k n \mu+2 b)\left(s_{i j}^{-2}+s_{i k}^{-2}+s_{j k}^{-2}\right)}{2 k^{2} n^{3}} .
$$

For $-i\left(P_{i} \partial P_{j}-P_{j} \partial P_{i}\right)$,

$$
\begin{equation*}
c_{-i\left(P_{i} \partial P_{j}-P_{j} \partial P_{i}\right)}=\frac{c_{i j}}{4 k n^{3} s_{i j}^{3}}, \tag{4.11}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{i j} \equiv \cos \frac{\theta_{i}-\theta_{j}}{2 n}=\cos \frac{\pi(i-j)}{n}, \quad s_{i j} \equiv \sin \frac{\pi(i-j)}{n} . \tag{4.12}
\end{equation*}
$$

With the coefficients above and the twist OPE ansatz, we can read the Rényi mutual information. It is worth noting that there is a cancellation between the numerator and denominator.

### 4.3 Rényi entropy on the cylinder

As a consistent check, we first consider the Rényi entropy of single interval with end points $\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)$ on a cylinder $\Sigma$ with a spatial circle $(x, y) \sim(x+L, y-\bar{L})$. The Rényi entropy obtained in [27] and [28] can be put in the same form as

$$
\begin{equation*}
S_{n}=\frac{i}{1-n}\left(Q_{n}-n Q_{1}\right)\left(y_{12}-\frac{\bar{L}}{L} x_{12}\right)+\frac{1}{1-n}\left(n h_{1}-h_{n}\right) \log \left(\frac{L}{\pi \epsilon} \sin \frac{\pi x_{12}}{L}\right) \tag{4.13}
\end{equation*}
$$

where conformal dimension and the charge of the twist operator in the orbifold theory are given by the vacuum charges and spectral flow parameters

$$
\begin{equation*}
h_{n}=n\left(\frac{c\left(n^{2}-1\right)}{24 n^{2}}+\frac{a}{n^{2}}-\frac{b}{n} \mu-\frac{k}{4} \mu^{2}\right), \quad Q_{n}=-i\left(b+\frac{n k \mu}{2}\right) . \tag{4.14}
\end{equation*}
$$

Note that the form of (4.13) depends on the spectral flow parameters only through the conformal weight and charge of the spectral flow parameters. Further requiring $h_{1}=Q_{1}=0, \mu$ and $b$ are related by [39,

$$
\begin{equation*}
b=-\frac{k \mu}{2} \tag{4.15}
\end{equation*}
$$

Then the quantum numbers of the twist operator become

$$
\begin{equation*}
h_{n}^{i n v} \equiv h_{n}-\frac{Q_{n}^{2}}{n k}=\frac{c\left(n^{2}-1\right)}{24 n}, \quad Q_{n}=(n-1) \frac{i k \mu}{2} \tag{4.16}
\end{equation*}
$$

Now we show that the result (4.13) can be reproduced using the OPE method outlined in section 3. For simplicity, let us first consider $\bar{L}=0$. In terms of the twist operators, the Rényi entropy is read from

$$
\begin{equation*}
S_{n}=\frac{1}{1-n} \log \frac{\left\langle\sigma_{n}(x) \tilde{\sigma}_{n}(0)\right\rangle}{\left\langle\sigma_{1}(x) \tilde{\sigma}_{1}(0)\right\rangle^{n}} . \tag{4.17}
\end{equation*}
$$

After substituting the twist OPE ansatz into this formula and separating the contributions from the weight and the charge, it becomes

$$
\begin{equation*}
S_{n}=\frac{i}{1-n}\left(Q_{n}-n Q_{1}\right) y_{12}+\frac{1}{1-n}\left(n h_{1}-h_{n}\right) \log \frac{x_{12}}{\epsilon}+\frac{1}{1-n} \log \frac{G(n)}{G(1)^{n}} \tag{4.18}
\end{equation*}
$$

where

$$
\begin{equation*}
G(n)=\sum_{\left\{\phi_{k}\right\}} c_{k} x^{h_{k}} \sum_{m} x^{m} a_{k, m} \partial_{x^{\prime}}^{m}<\phi_{k}\left(x^{\prime}\right)>\left.\right|_{x^{\prime}=0} . \tag{4.19}
\end{equation*}
$$

comes from nonvanishing one-point functions on the cylinder. We use the following transformation to map the plane to the cylinder,

$$
\begin{equation*}
z=e^{\frac{2 \pi i}{L} x}, \quad \hat{y}=y+\mu \log z \tag{4.20}
\end{equation*}
$$

where $(z, \hat{y})$ are coordinates on the canonical plane, $(x, y)$ are coordinates on the cylinder $\Sigma$. As was argued in section 3.2, the parameter $\mu$ should be chosen as (4.15) to make the twist operator chargeless at $n=1$. The expectation values of the quasi-primary operators up to level 2 on the cylinder are respectively

$$
\begin{gather*}
\langle P(x)\rangle=0, \quad\langle T(x)\rangle=\frac{\pi^{2} c}{6 L^{2}}, \quad\left\langle P^{2}(x)\right\rangle=\frac{\pi^{2} k}{6 L^{2}},  \tag{4.21}\\
\langle A(x)\rangle=0, \quad\left\langle P_{i} P_{j}(x)\right\rangle=0 . \tag{4.22}
\end{gather*}
$$

Note that all the above one-point functions on $\Sigma$ are independent of the vacuum charge $b$, provided that the parameter $\mu$ is chosen as (4.15). After doing the summation in (4.19), we have

$$
\begin{equation*}
G(n)=1+\frac{\pi^{2} c}{6 L^{2}} c_{T} n x_{12}^{2}+O\left(x^{3}\right) \tag{4.23}
\end{equation*}
$$

with

$$
\begin{equation*}
c_{T}=\frac{2}{n c} h_{n} \tag{4.24}
\end{equation*}
$$

which can be read from the general formula (4.4) at $n=1$. Therefore

$$
\begin{equation*}
\frac{1}{1-n} \log \frac{G(n)}{G(1)^{n}}=-\frac{1}{1-n}\left(n h_{1}-h_{n}\right) \frac{\pi^{2}}{6 L^{2}}\left(x_{12}\right)^{2}+O\left(x_{12}^{3}\right) \tag{4.25}
\end{equation*}
$$

Plugging (4.25) into the expansion (4.18), it is easy to check it indeed matches the direct calculation (4.13) up to $O\left(x_{12}^{3}\right)$, when $\bar{L}=0$. To obtain the result (4.13) on an arbitrary spatial circle, we need to perform one further spectral flow transformation with $p=\frac{\bar{L}}{L}$, and the final result is to simply replace $y$ with $y_{12}-p x_{12}=y-\frac{\bar{L}}{L} x_{12}$ as expected from (4.13).

### 4.4 The small $x$ expansion of Rényi mutual information

The normalization coefficients of the relevant quasi-primary operators can be found in the appendix A After using the summation formulae in the appendix B , we find the Rényi mutual information of two disjoint intervals given by (3.16) and (3.22) can be expanded up to order $x^{3}$

$$
\begin{align*}
I_{n}= & \left(\frac{2 b^{2}(n+1)}{k n}+2 b \mu\right) x \\
& +\left(\frac{b^{2}(n+1)}{k n}+b \mu+\frac{c(n-1)(n+1)^{2}}{288 n^{3}}+\frac{(n+1)\left(n^{2}+11\right)}{1440 n^{3}}\right) x^{2} \\
& +\left(\frac{2 b^{2}(n+1)}{3 k n}+\frac{2 b \mu}{3}+\frac{c(n-1)(n+1)^{2}}{288 n^{3}}+\frac{(n+1)\left(23 n^{4}+233 n^{2}-40\right)}{30240 n^{5}}\right) x^{3} \\
& +O\left(x^{4}\right) \tag{4.26}
\end{align*}
$$

where we have kept the spectral flow parameter $\mu$ arbitrary, but used the relation between the vacuum charges between $b$ and $a$. Further using the vanishing charge condition at $n=1$, $b=-\frac{k \mu}{2}$, we find that the RMI only depends on the central charge $c$ and the combination $a=-\frac{b^{2}}{k}$, which will be specified by the theory itself. Using the above relations, we can always rewrite the RMI as follows,

$$
\begin{equation*}
I_{n}=I_{n}^{(L O)}+I_{n}^{(N L O)}+O\left(\frac{1}{c}\right) \tag{4.27}
\end{equation*}
$$

where the leading order result depends on both the central charge and the vacuum charge

$$
\begin{align*}
I_{n}^{(L O)}= & \frac{2 b^{2}(n-1)}{k n}\left(x+\frac{x^{2}}{2}+\frac{x^{3}}{3}\right) \\
& +\frac{c(n-1)(n+1)^{2}}{288 n^{3}}\left(x^{2}+x^{3}\right)+O\left(x^{4}\right) \tag{4.28}
\end{align*}
$$

whereas the next-to-leading-order result is of order 1 and does not depend on either the central charge or the vacuum charges.

$$
\begin{equation*}
I_{n}^{(N L O)}=\frac{(n+1)\left(n^{2}+11\right)}{1440 n^{3}} x^{2}+\frac{(n+1)\left(23 n^{4}+233 n^{2}-40\right)}{30240 n^{5}} x^{3}+O\left(x^{4}\right) \tag{4.29}
\end{equation*}
$$

The RMI for two disjoint intervals at large separation are hence given by (4.27)-(4.29), which are the main results of this paper. We end this subsection with the following comments

- Up to level 3, there are no higher order terms at $o(1 / c)$. This is reminiscent of the observation in holographic CFTs [17] that two loop effects only can be seen from level four. It will be interesting to find a simple explanation, or check it from a bulk calculation.
- The first line in (4.28) comes from the contribution of the operators $P_{i}$, while the remaining parts in (4.28) are the same as that of the holomorphic part of a $\mathrm{CFT}_{2}$. (4.28)
suggests that the contributions from Virasoro and Kac-Moody sectors can factorize. This is a priori not obvious in our explicit constructions of the quasi-primary operators level by level, as we were using a basis where the Virasoro and Kac-Moody sectors do not decouple. As will be shown in next subsection, factorization will be explicit using the method of conformal block expansion.
- The reason for grouping the first line of (4.28) as the leading order result is that these terms come from the vacuum charges, instead of the Virasoro-Kac-Moody descendants. In the bulk we expect these terms to correspond to the on-shell action of a classical geometry. The vacuum charge $b$ depends on the theory, and will take different values in different examples of holography. For example, when the bulk dual is Einstein gravity on $\mathrm{AdS}_{3}$ with the CSS boundary condition, there is a further relation between the parameters 28] from $L_{0}^{v a c} \equiv a-\frac{c}{24}=-\frac{b^{2}}{k}-\frac{c}{24}=0$. Then it becomes obvious that (4.28) is indeed of order $c$ and corresponds to the classical contribution.
- The next-to-leading order result (4.29) up to $O\left(x^{4}\right)$ comes only from the Kac-Moody contribution. As will be seen later, this property can be reproduced in a bulk calculation. This is different from the holomorphic part of holographic CFTs which is zero at this order [17].


## 5 Rényi mutual information from conformal blocks

In this section we provide another way to read the Rényi mutual information at classical level by computing the four-point functions of the twist operators directly. In the orbifold WCFT, the central charge and Kac level is $n c$ and $n k$. The striped four-point function of twist operators $F_{n}(x)$ can be written as a sum over the warped conformal blocks, and each block further factorizes as [41,42] Kac-Moody block and Virasoro block,

$$
\begin{equation*}
F_{n}(x)=\frac{1}{x^{2 h_{n}}} \sum_{p} C_{\sigma \tilde{\sigma} p}^{2} V_{T}\left(n c-1, h_{n}^{i n v}, h_{p} ; x\right) V_{P}\left(n k, Q_{n}, x\right) \tag{5.1}
\end{equation*}
$$

where $C_{\sigma \tilde{\sigma} p}$ is the OPE coefficient and the twist operator has the spectral flow invariant conformal weights $h_{n}^{i n v}$ and $\mathrm{U}(1)$ charge given in (4.16). The sum is over all primary operators labelled by $p$ with non-vanishing three point functions with $\sigma \tilde{\sigma}$. By charge conservation, we learn that $q_{p}=0$. The Kac-Moody block is independent of the summation, and has a closed form [41,

$$
\begin{equation*}
V_{P}\left(k, q_{i}, x\right)=x^{\frac{2 q_{1}^{2}}{k}}(1-x)^{\frac{-2 q_{1} q_{2}}{k}} . \tag{5.2}
\end{equation*}
$$

$F_{n}(x)$ can be written as a product of two parts

$$
\begin{equation*}
F_{n}(x)=(1-x)^{\frac{-2 Q_{n}^{2}}{n k}}\left\{\frac{1}{x^{2 h_{n}^{i n v}}} \sum_{p} C_{\sigma \tilde{\sigma} p}^{2} V_{T}\left(n c-1, h_{n}^{i n v}, h_{p} ; x\right)\right\} \tag{5.3}
\end{equation*}
$$

The part in the braces is the same as the holomorphic part in a CFT [10 with the shift of the central charge and conformal weights, while the remaining part comes entirely from the Kac-Moody blocks. For holographic WCFTs with a large $c$ and assuming that the vacuum block dominates the sum in (5.3), the leading order contribution to the RMI (3.16) can be written as

$$
\begin{align*}
I_{n}^{(L O)}(x)= & \frac{2 b^{2}(n-1)}{k n} \log (1-x) \\
& + \text { (holomorphic part of RMI in holographic CFTs). } \tag{5.4}
\end{align*}
$$

where the second line comes from the Virasoro block for the identity operator, and is exactly the same as the holomorphic part of RMI in holographic CFTs.

Now we check that (5.4) agrees with our previous result (4.28), which requires the an explicit expression of the Virasoro block, which is in general not know in closed form. However, when the theory has a large $c$ expansion, the Virasoro block exponentiates as,

$$
\begin{equation*}
V_{T}\left(c, h_{i}, h_{p} ; x\right) \sim \exp \left\{-\frac{c}{6} f\left(h_{i}, h_{p}, x\right)+O(1)+O\left(\frac{1}{c}\right)+\cdots\right\} . \tag{5.5}
\end{equation*}
$$

Let us first consider the mutual information with $n=1, F_{1}(x)$ as defined in (3.17) can be read from a heavy-heavy-light-light (HHLL) four point function, which is dominated from vacuum block specified by $f_{0}\left(h_{i}, x\right) \equiv f\left(h_{i}, 0, x\right)$, since the twist operators are light operators as $n \rightarrow 1$. $f_{0}\left(h_{i}, x\right)$ can be calculated by the monodromy method in the Liouville field theory. The expression of $f_{0}\left(h_{i}, x\right)$ is an expansion in terms of the weights of light operators, but in closed form in terms of the weights of heavy operators and the cross ratio $x$ [10. When the weights of the external operators are the same, the expansion of $f_{0}(h, x)$ reads [54],

$$
\begin{equation*}
f_{0}(h, x)=\frac{12 h}{c} \log x+o\left(\frac{h}{c}\right) . \tag{5.6}
\end{equation*}
$$

This gives the mutual information $I_{1}$ when the two intervals are far away from each other,

$$
\begin{equation*}
I_{1}(x)=\frac{2 Q_{1}^{2}}{k} \log (1-x)=0, \quad \text { if } x<\frac{1}{2} \tag{5.7}
\end{equation*}
$$

where $Q_{1}=0$ is the charge of the twist operator at $n=1$. This obviously agrees with (4.28) for $n=1$, and is also consistent with the intuition that the mutual information for two disjoint intervals should vanish before reaching the point of phase transition.

More generally, for the Rényi mutual information, it amounts to calculating the correlation function of four heavy operators, since the weights of twist operators grow with $c$ when $n \neq 1$.

The Virasoro block at large $c$ can be expanded in terms of cross ratio $x$ by the recursion relations, no matter what the weights are [10, 16]

$$
\begin{equation*}
f_{0}(h, x)=12 \delta \log x-3072 \delta^{2} q(x)^{2}+\frac{24576}{5} \delta^{2}\left(1-64 \delta+704 \delta^{2}\right) q(x)^{4}+\cdots \tag{5.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\delta=\frac{h}{c}, \quad q(x)=e^{-\pi K(1-x) / K(x)}=\frac{1}{16} x+\frac{1}{32} x^{2}+\cdots \tag{5.9}
\end{equation*}
$$

and $K(x)$ is the complete elliptic integral of the first kind. When the two intervals are far away from each other, the Rényi mutual information can be expanded in terms of $x$,

$$
\begin{align*}
I_{n}^{(L O)}(x)= & \frac{2 b^{2}(n-1)}{k n} \log (1-x) \\
& +\frac{c(n-1)(n+1)^{2}}{288 n^{3}}\left(x^{2}+x^{3}\right) \\
& +\frac{c(n-1)(n+1)^{2}\left(1309 n^{4}-2 n^{2}-11\right) x^{4}}{414720 n^{7}}+O\left(x^{5}\right) \tag{5.10}
\end{align*}
$$

Note that the first line comes from the Kac-Moody block and is valid to all orders of $x$, while the second and third lines requires vacuum dominance and is valid at large $c$, and is a power series of $x$. (5.10) matches with the leading $c$ result in (4.28) up to $x^{3}$, which is a consistent check of the two methods.

## 6 Holographic calculations from the bulk

In this section we consider the holographic calculation of the Rényi mutual information on AdS with the CSS boundary conditions in Einstein gravity.

Let us first review the story in the context of $\mathrm{AdS}_{3} / \mathrm{CFT}$. The bulk calculation of Rényi entropy is to evaluate the partition function on a geometry whose asymptotic boundary, in the sense of Brown-Henneuax, is a given Riemann surface $\Sigma_{n}$. All the classical configurations under Brown-Henneuax boundary conditions are the quotient of the global $\mathrm{AdS}_{3}$ by a discrete subgroup $\Gamma$ of $S L(2, R)_{L} \times S L(2, R)_{R}$. For technical reasons, it is usually more convenient to first consider the Euclidean theory and perform analytic continuation [43]. It is conjectured that the dominant contributions to the partition function are handlebody solutions which can be constructed using Schottky uniformation [11. After proper regularizations, the on-shell action described by the Zograf-Takhtadzhyan(ZT) action [11,44] agrees with the large $c$ result in the field theory calculation [10, 16, 17]. The 1-loop determinant [6] agrees with the $O(c)$ result [16, 17].

For AdS/WCFT, we expect a similar story, namely we expect the large $c$ result in the field theory calculation to match certain semiclassical calculation in the bulk, while the $O(c)$ is to
match some 1-loop calculation. We would borrow heavily techniques from AdS/CFT which are commonly formulated in Euclidean signature. On the other hand, we need to switch to Lorentzian signature temporarily as it will be easy to see the difference between CFT and WCFT. In section 3.1, we show how to relate the Rinlder transformation (3.7) to the uniformization map (3.10). We proceed with the assumption that similar procedure can be taken in more general cases with appropriate analytic continuation so that our results finds an explanation in Lorentzian signature. In the following we first comment on the leading order contribution (4.28) and then match the next-to-leading order result (4.29) from a the one-loop partition function in the bulk.

### 6.1 Comments on the classical part

Holographically, the large $c$ result in the WCFT calculation is expected to match certain semiclassical calculation in the bulk dual. For single intervals, a holographic derivation of the entanglement entropy and Rényi entropy has been performed in [28] in the context of WAdS/WCFT and AdS/WCFT. The bulk extension of the Rindler transformation (3.7) in Lorentzian signature can be explicitly constructed from a quotient by elements of $S L(2, R) \times$ $U(1)$. The novel feature for holographic entanglement entropy in the context of AdS/WCFT is that the spacelike geodesic in the bulk is attached to the boundary end points through null geodesics, which are along the bulk modular flow.

Similarly, we expect the above leading order result (5.10) for RMI to correspond to the semiclassical action of the gravitational configuration. In Lorentzian signature, one may try to construct the bulk configurations whose asymptotic boundary is $\Sigma_{n}$ in the sense of CSS boundary conditions, and evaluate the on-shell action taking careful account of regularization. Schematically, the bulk construction of the $n$-smooth cover can be constructed from quotients of of $S L(2, R) \times U(1)$ and the Chiral Liouville action 45 might be useful to evaluate the on-shell action. One observation is that the contribution from the Virasoro block, i.e. the second and third line of (5.10), indeed agrees with a half of the result in ordinary holographic CFTs [17], which indeed agrees with the bulk calculation of the ZT action using the monodromy method [11]. While it would be interesting to explicitly perform the complete bulk computation, we will leave it for future work.

### 6.2 One-loop partition function in gravity

In this subsection, we focus on the next-to-leading order result (4.29), and show that it can be reproduced by the 1-loop quantum correction to the gravitational configuration in $\mathrm{AdS}_{3}$ gravity with CSS boundary conditions.

Let us first review the story in the context of $\mathrm{AdS}_{3} / \mathrm{CFT}$. All the classical configurations under Brown-Henneuax boundary conditions are quotients of the global $\mathrm{AdS}_{3}$ by a discrete subgroup $\Gamma$ of $S L(2, R)_{L} \times S L(2, R)_{R}$. For handlebody solutions, the subgroup $\Gamma$ is a Schottky group, a finitely generated free group, such that all nontrivial elements are loxodromic. The 1-loop partition functions of the handle-body solutions with genus $g$ are studied in [46, 47] by using the heat-kernel and the method of images

$$
\begin{align*}
Z_{g ; B H}^{(1)} & =\prod_{\gamma \in \mathcal{P}} Z_{B H}^{(1)}\left(q_{\gamma}, \bar{q}_{\gamma}\right)  \tag{6.1}\\
& =\prod_{\gamma \in \mathcal{P}} Z_{B H}^{(1)}\left(q_{\gamma}\right) \bar{Z}_{B H}^{(1)}\left(\bar{q}_{\gamma}\right) . \tag{6.2}
\end{align*}
$$

where $\mathcal{P}$ is a set of representatives of primitive conjugacy classes of the Schottky group $\Gamma$. $q_{\gamma}$ is defined by writing the two eigenvalues of $\gamma \in \Gamma$ as $q_{\gamma}^{ \pm 1 / 2}$ with $\left|q_{\gamma}\right|<1$, and $Z_{B H}^{(1)}(q, \bar{q})$ is the one loop determinant on a fixed saddle point solution with genus 1 such as the BTZ black hole. In the second line of the above equation, we have used the fact that the one loop contribution under the Brown-Hennaux boundary conditions factorizes into holomorphic part and antiholomorphic part $Z_{B H}^{(1)}(q, \bar{q})=Z_{B H}^{(1)}(q) \bar{Z}_{B H}^{(1)}(\bar{q})$. In general, $\quad Z_{B H}^{(1)}(q)=\prod_{s} \prod_{m=s}^{\infty} \frac{1}{\left(1-q^{m}\right)}$, where the product over $s$ is with respect to the spins of massless fluctuations. For pure $\mathrm{AdS}_{3}$ gravity with Brown-Henneaux boundary conditions, there is only massless graviton and $s=2$.

For $\mathrm{AdS}_{3}$ gravity with the CSS boundary conditions, a similar story is expected to hold, namely we still expect that the generic 1-loop result on a fixed saddle point can be obtained from the image of BTZ result under the Schottky group. Namely, we still expect the general result (6.1) to hold. However, two crucial differences are expected due to the boundary conditions: i) the 1-loop determinant around on BTZ black hole $Z_{C S S}^{(1)}(q, \bar{q})$ and ii) the action of the Schottky group.

The 1-loop determinant for the BTZ black hole $Z_{C S S}^{(1)}(q, \bar{q})$ was studied in [48] using the quasi-normal mode method,

$$
\begin{equation*}
Z_{C S S}^{(1)}(q, \bar{q})=\prod_{l^{\prime}=1}^{\infty} \frac{1}{1-q^{l^{\prime}}} \prod_{l=2}^{\infty} \frac{1}{1-q^{l}} . \tag{6.3}
\end{equation*}
$$

where $q$ is determined by the BTZ temperatures. There are two remarkable points on this determinant. Firstly the determinant is the product of two factors, one for spin two and the other for spin one. Secondly, the fact that only $q$ appears in the final result indicates that only the left-mover survives the CSS boundary conditions. Therefore we will drop the dependence on $\bar{q}$ henceforth. In fact, (6.3) is just the contribution to the Virasoro-Kac-Moody character from the descendants [24,42, which is consistent with the fact that the asymptotic symmetry is now generated by a Virasoro algebra and a Kac-Moody algebra, and that the holographic dual should be a WCFT.

For more general gravitational configurations compatible with the CSS boundary conditions, we expect that the 1-loop partition functions can be built from the BTZ result similar to (6.1),

$$
\begin{equation*}
Z_{g ; C S S}^{(1)}=\prod_{\gamma \in \mathcal{P}^{\prime}} Z_{C S S}^{(1)}\left(q_{\gamma}\right)=\prod_{\gamma \in \mathcal{P}} \prod_{l^{\prime}=1}^{\infty} \frac{1}{1-q_{\gamma}^{l^{\prime}}} \prod_{l=2}^{\infty} \frac{1}{1-q_{\gamma}^{l}} \tag{6.4}
\end{equation*}
$$

where $\mathcal{P}^{\prime}$ is a set of representatives of primitive conjugacy classes of the modified Schottky group $\Gamma^{\prime}$ which now should be compatible with the CSS boundary conditions. Using one loop partition function given by (6.4), the quantum correction to the holographic Rényi mutual information at one-loop level reads,

$$
\begin{align*}
\left.I_{n}\right|_{1-\text { loop }} & =-\left.S_{n}(A \cup B)\right|_{1-\text { loop }}=\frac{-1}{1-n}\left(\left.\log Z_{n}\right|_{1-\text { loop }}-\left.n \log Z_{1}\right|_{1-\text { loop }}\right) \\
& =\frac{1}{n-1} \sum_{\gamma \in \mathcal{P}^{\prime}}\left\{\sum_{l=1}^{\infty} \log \left(1-q_{\gamma}^{l}\right)+\sum_{l=2}^{\infty} \log \left(1-q_{\gamma}^{l}\right)\right\} \tag{6.5}
\end{align*}
$$

where we have used the fact that the holographic Rényi entropy of single interval receives no quantum correction.

Now the key question is to determine the Schottky group $\Gamma^{\prime}$ that is compatible with the boundary conditions. First note that the phase space of Einstein gravity under the CSS boundary conditions have the same fixed right moving energy. In particular, the zero mode solutions are BTZ black holes with fixed right moving energy. All other configurations can be constructed from such BTZ black holes by first uncompactifying the non-contractable circle, and then identify points by elements of $S L(2, R)_{L} \times U(1)_{R}$, instead of the full fledged $S L(2, R)_{L} \times S L(2, R)_{R}$. Therefore we expect differences from the CFT results in the right moving sector. Nevertheless, the symmetry algebra, the single interval uniformization map (3.10) and the bulk Rindler transformation [28] strongly suggest that coordinate transformations in the left moving sector remain the same. Now we provide a heuristic argument. Schematically, to calculate the RMI (6.5) in the bulk, we need to find a $n$-th smooth cover $M_{n}$ in the bulk which is asymptotic, in the sense of CSS boundary conditions, to the $n$-sheeted geometry $\Sigma_{n} . M_{n}$ is a quotient of $\mathrm{AdS}_{3}$, and can be obtained from a reference geometry by a local coordinate transformation. The reference geometry can be taken as the BTZ black hole with zero left-moving temperature, and fixed right-moving temperature. For example, with the conventions of [28], the reference geometry is $T_{u}=0$ and a fixed value of $T_{v}$. We assume that the dominant contribution to the partition function is still given by the handle-body solutions which can be obtained from modified version of Schottky uniformization which is compatible with the boundary conditions. Using the holographic dictionary, the coordinate transformation from the reference geometry to $M_{n}$ will reduce to a warped conformal transformation in the form $z=f(w), \quad \hat{y}=y+g(z)$,
where $\left(z, y^{\prime}\right)$ are coordinates on the plane, and $(w, y)$ are coordinates on $\Sigma_{n}$. Note that subtleties might appear due to the state dependence of the bulk to boundary map. Furthermore, the boundary warped conformal map is determined by local properties near the branching points, as well as the global properties. Near each branching point $w_{i}$, the transformation can be expanded as $z=\left(w-w_{i}\right)^{\frac{1}{n}}, \hat{y}=y+n \mu \log z$. Namely, they agree with the uniformization (3.10) for a single interval near all branching points. The global properties will be determined by some monodromy conditions. For the $w$ direction, it will be the same monodromy conditions as in $\mathrm{CFT}_{2}$ [6, 11]. As all the above requirements in the Virasoro direction $z$ is exactly the same as the holomorphic coordinate in a $\mathrm{CFT}_{2}$, the action of the Schottky group also acts the same way in the holomorphic sector. Therefore we expect to use group elements in (6.4) whose action on the right moving sector is different, while on the left moving sector is the same as in the Brown-Henneaux boundary conditions. Since the BTZ 1-loop result is purely left moving, we expect no difference from the action of the usual Schottky group on the holomorphic sector.

With the above argument, in the following we will apply directly the 1-loop results of the holomorphic sector in AdS/CFT to AdS/WCFT, and show that (6.5) indeed agrees with our next-to-leading result (4.29) up to order $x^{3}$. In particular we will use $q_{\gamma}$ 's from the previous study for double interval in the context of AdS/CFT, see [6] by Barrella et. al. In the following we give a brief review on the computation.

The Schottky uniformization in the usual AdS/CFT case can be determined by the monodromy method. The single-valued coordinate $z$ on the $n$-sheeted geometry in the case with two disjoint intervals is the ratio of two independent solutions of the following differential equation,

$$
\begin{equation*}
\phi^{\prime \prime}(w)+\frac{1}{2} \sum_{i=1}^{4}\left(\frac{\Delta}{\left(w-w_{i}\right)}+\frac{\gamma_{i}}{w-w_{i}}\right) \phi(w)=0, \quad z=\frac{\phi_{1}(w)}{\phi_{2}(w)} \tag{6.6}
\end{equation*}
$$

The map $z$ is single-valued with $\Delta=\frac{1}{2}\left(1-\frac{1}{n^{2}}\right)$, since it is actually the series solution in the powers of $\left(w-w_{i}\right)^{1 / n}$ at each point $w_{i}$, which is consistent with our uniformization formula (3.10) for the left moving coordinate. When the solutions go around a closed loop enclosing branch points, they experience potential monodromy. The trivial monodromy at infinity fixes three of the accessory parameters $\gamma_{i}$, while the remaining one can be fixed by the monodromy around the trivial circles defining the Schottky group. The Schottky generators are respectively

$$
\begin{equation*}
L_{1}=M_{2} M_{1}, \quad L_{k}=M_{2}^{k-1} L_{1} M_{2}^{-k+1}=M_{2}^{k} T^{-1} M_{2} T M_{2}^{-k+1}, \quad k=2, \cdots, n-1 \tag{6.7}
\end{equation*}
$$

where $M_{1}, M_{2}$ are the monodromies obtained by encircling the branch points delimiting the nontrivial cycle and $T$ is the transformation matrix of the two basis of the power series solutions at these two points. All possible primitive words are made up of the Schottky generators and their inverses, up to conjugation in the Schottky group. There are infinite primitive classes for
the cases with genus greater than one. Fortunately we are looking for the result in terms of the power series of the cross ratio $x$. Consequently there are finite primitive classes contributing to a fixed power of $x$ in the partition function. Each pair of $T^{-1}$ and $T$ in the words gives the leading contribution of order $x^{2 l}$ in the single term $\log \left(1-q^{l}\right)$. So at the leading order in $x$ expansion, the contribution comes from the so-called one-consecutively-decreasing words (1-CDW) with only one pair of $T^{-1}$ and $T$,

$$
\begin{equation*}
\gamma_{k, m}=L_{k+m} L_{k+m-1} \cdots L_{m+1}=M_{2}^{k+m} T^{-1} M_{2}^{k} T M_{2}^{m} . \tag{6.8}
\end{equation*}
$$

For any given word length $k$ with $k \in[1, n-1]$, there are $n-k$ independent 1-CDWs, and the eigenvalues of $\gamma_{k, m}$ are independent of $m$. One of the eigenvalues of the 1-CDW above is vanishing, while the other one is

$$
\begin{equation*}
q_{\gamma_{k}}^{-1 / 2} \equiv q_{\gamma_{k, m}}^{-1 / 2}=-\frac{4 n^{2} \sin ^{2} \frac{k \pi}{n}}{x}+2+2\left(n^{2}-1\right) \sin ^{2} \frac{k \pi}{n}+O(x) \tag{6.9}
\end{equation*}
$$

Up to $x^{3}$, contributions to the 1-loop correction to the holographic Rényi mutual information only come from the afarmentioned 1-CDW $\gamma_{k, m}$,

$$
\begin{align*}
\left.I_{n}\right|_{1-\text { loop }} & =\frac{1}{n-1} \sum_{k=1}^{n-1} \sum_{m=1}^{n-k}\left\{\sum_{l=1}^{\infty} \log \left(1-q_{\gamma_{k}}^{l}\right)+\sum_{l=2}^{\infty} \log \left(1-q_{\gamma_{k}}^{l}\right)\right\}+\sum_{\text {other } \gamma} \cdots \\
& =-\frac{1}{n-1} \sum_{k=1}^{n-1}(n-k) q_{\gamma_{k}}+O\left(x^{4}\right) \tag{6.10}
\end{align*}
$$

In the first line, $\cdots$ denotes contributions from all other loxodromic element $\gamma$. From the above computation, we can directly see that there is no contributions from spin 2 up to $x^{3}$. Combining with (6.9),

$$
\begin{align*}
\left.I_{n}\right|_{1-l o o p} & =-\frac{n}{2(n-1)} \sum_{k=1}^{n-1}\left\{\frac{s_{k}^{-4}}{16 n^{4}} x^{2}+\frac{n^{2} s_{k}^{-4}+s_{k}^{-6}-s_{k}^{-4}}{16 n^{6}} x^{3}+O\left(x^{4}\right)\right\} \\
& =\frac{(n+1)\left(n^{2}+11\right)}{1440 n^{3}} x^{2}+\frac{(n+1)\left(23 n^{4}+233 n^{2}-40\right)}{30240 n^{5}} x^{3}+O\left(x^{4}\right), \tag{6.11}
\end{align*}
$$

where in the first line $s_{k}=\sin \frac{k \pi}{n}$. This is indeed (4.29).
There are a few remarkable points:

1. Due to the CSS boundary conditions, the graviton in the bulk is "chiral", so only $q_{\gamma}$ 's appear in the relation (6.4).
2. From the sum in (6.10), we can see that the contribution up to $x^{3}$ originates from the massless vector field. The contribution from spin 2 field will appear starting from $x^{4}$. This property also agrees with the WCFT result in section 4.4.
3. The perfect matching between the WCFT result (4.29) and the bulk result (6.11) is very suggestive. To obtain the result (6.11), the above calculations only used the action of the Schottky group on the left moving sector, and is insensitive to the right movers. It supports that the Schottky group indeed acts the same way on the left-moving sector despite that we are using the CSS boundary conditions.
4. As mentioned before, we expect modifications to quotients in the $y$ direction due to the CSS boundary conditions. This is to be expected from the uniformization map (3.10), as well as the Rindler transformation in the bulk [28]. It would be interesting to find the action in the $U(1)$ direction, and show explicitly that it does not contribute to the 1-loop result.

Another way to reproduce the 1-loop determinant (6.4) from field theory is along the lines suggested in 50]. The essential point is that in the large $c$ limit, the conformal field theory becomes effectively free. After redefining the generators of the Virasoro-Kac-Moody algebra, we have

$$
\begin{equation*}
\left[\hat{L}_{n}, \hat{L}_{m}\right]=\delta_{n+m, 0}, \quad\left[\hat{L}_{n}, \hat{P}_{m}\right]=0, \quad\left[\hat{P}_{n}, \hat{P}_{m}\right]=\delta_{n+m, 0} \tag{6.12}
\end{equation*}
$$

Therefore, in the large $c$ limit, the states are generated not only by the Virasoro generators, but also by the Kac-Moody generators. This is very similar to the case of CFT with $\mathcal{W}_{3}$ symmetry, but now there is a vector symmetry. Another subtle point is that now the symmetry has only holomorphic sector such that the partition function (6.4) depends only on $q_{\gamma}$.

## 7 Conclusion and Discussion

In this work, we studied the Rényi mutual information of two disjoint intervals in the holographic warped CFT. In the semiclassical limit, we are allowed to focus on the vacuum module of the theory. We applied the OPE of the twist operators to collect the contributions of the quasi-primary operators up to level 3 . In the large $c$ limit and the large distance limit, the result can be organized in powers of $1 / c$ and the cross ratio $x$. The LO result is linear in $c$ and is expected to correspond to the classical action of the gravitational configuration. We performed a consistency check for the LO result by using the large $c$ expansion of the warped conformal block and found agreement up to $x^{3}$. More interestingly, we found that the LO Rényi mutual information can be closely related to the one in the usual holographic CFT by the relation (5.4): besides the contribution from the Kac-Moody block, which is of a logarithmic term, the remaining part is just half of the one in holographic CFT. It would certainly be important to understand this relation from the bulk gravity point of view.

The NLO result, which is independent of $c$, should correspond to the 1-loop partition function of the gravitational configuration according to the holographic dictionary. For a handlebody configuration, its 1-loop partition function can be rewritten in the form of (6.4). As argued in section 6.2, assuming the CSS boundary conditions do not affect the action of the quotient in the Virasoro direction, we calculated 1-loop corrections to the Rényi mutual information and found agreement with the WCFT NLO result up to order $x^{3}$. This not only validates our treatment, but also provides nontrivial support to the $\mathrm{AdS}_{3} /$ WCFT correspondence at 1-loop.

In the bulk calculation for the 1-loop correction, the CSS boundary conditions suggest that the gravitational configurations are quotients of $\mathrm{AdS}_{3}$ by elements from $S L(2, R) \times U(1)$. We provided an argument that the quotient acts the same way in the $x$ direction as in the usual AdS/CFT story. This allows us to borrow the results from previous calculations in AdS/CFT. The agreement with the field theory result is indeed compatible with this argument. However, it would be nice to construct the gravitational configurations directly from quotients of $\mathrm{AdS}_{3}$ by the group $S L(2, R) \times U(1)$, which respect the CSS boundary conditions in an obvious way. Such a construction has been done in [28] in order to calculate the entanglement entropy for a single interval. This is feasible by applying the coordinate transformation between arbitrary solutions in the CSS phase space metric similar to [51], and carefully matching to the boundary uniformization.

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## A Coefficients and Normalization

In this appendix, we show how to calculate the one-point functions on the $n$-sheeted geometry in the single interval case. First, the one-point functions on the plane with non-trivial vacuum can be got by using the Ward identity and the normal order product. Then, we determine the transformation law of the composite operators and read out one-point functions on the $n$-sheeted geometry.

As an example, we calculate the coefficient of $A=T+\frac{c}{k}\left(P^{2}\right)$. The one-point function of
$T$ in the non-trivial vacuum is,

$$
\begin{equation*}
\langle T(z)\rangle_{V}=\lim _{y \rightarrow \infty, x \rightarrow 0} \frac{\langle V(y) T(z) V(x)\rangle}{\langle V(y) V(x)\rangle}=\frac{a}{z^{2}} \tag{A.1}
\end{equation*}
$$

where $V$ is the operator related with the non-trivial vacuum. The transformation law of $T$ is,

$$
\begin{equation*}
T(w)=w^{\prime-2}\left(T(z)-\frac{c}{12} s(w, z)-g^{\prime}(z) P(z)-\frac{k g^{\prime}(z)^{2}}{4}\right) \tag{A.2}
\end{equation*}
$$

where $s(w, z)$ is the Schwarzian derivative, and $g(z)=n \mu \log z$. So the one-point function of $T$ on the $n$-sheeted geometry is,

$$
\begin{equation*}
\langle T(w)\rangle_{\Sigma_{n}}=\frac{l^{2}}{w^{2}(w-l)^{2}}\left(\frac{c\left(n^{2}-1\right)+24 a}{24 n^{2}}-\frac{b \mu}{n}-\frac{k \mu^{2}}{4}\right) . \tag{A.3}
\end{equation*}
$$

Now, let us consider the composite operator $\left(P^{2}\right)$. The one-point function of $\left(P^{2}\right)$ on the plane with non-trivial vacuum is,

$$
\begin{equation*}
\left\langle\left(P^{2}\right)\left(z_{2}\right)\right\rangle_{V}=\frac{1}{2 \pi i} \oint \frac{1}{z_{1}-z_{2}}\left\langle P\left(z_{1}\right) P\left(z_{2}\right)\right\rangle_{V} d z_{1} \tag{A.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle P\left(z_{1}\right) P\left(z_{2}\right)\right\rangle_{V}=\frac{b^{2}}{z_{1} z_{2}}-\frac{k}{2 z_{12}^{2}} \tag{A.5}
\end{equation*}
$$

so

$$
\begin{equation*}
\left\langle\left(P^{2}\right)(z)\right\rangle_{V}=\frac{b^{2}}{z^{2}} \tag{A.6}
\end{equation*}
$$

The transformation law of $\left(P^{2}\right)$ should be calculated by applying the normal order product procedure,

$$
\begin{align*}
\left(P^{2}\right)(w)= & \frac{1}{2 \pi i} \oint \frac{1}{w_{1}-w} P\left(w_{1}\right) P(w) d w_{1} \\
= & \frac{1}{2 \pi i} \oint d z_{1} \frac{1}{w\left(z_{1}\right)-w(z)} w^{\prime}(z)^{-1}\left(P\left(z_{1}\right)+\frac{k g^{\prime}\left(z_{1}\right)}{2}\right)\left(P(z)+\frac{k g^{\prime}(z)}{2}\right) \\
= & \frac{1}{2 \pi i} \oint d z_{1} \frac{1}{w\left(z_{1}\right)-w(z)} w^{\prime}(z)^{-1}\left(\frac{k}{2\left(z_{1}-z\right)^{2}}+\left(P^{2}\right)\right. \\
& \left.+\frac{k}{2} P\left(z_{1}\right) g^{\prime}(z)+\frac{k}{2} P(z) g^{\prime}\left(z_{1}\right)+\frac{k^{2}}{4} g^{\prime}\left(z_{1}\right) g^{\prime}(z)\right) \\
= & \frac{k\left(3 w^{\prime \prime}(z)^{2}-2 w^{(3)}(z) w^{\prime}(z)\right)}{24 w^{\prime}(z)^{4}}+w^{\prime}(z)^{-2}\left(P^{2}\right)(z) \\
& +\frac{b k g^{\prime}(z)}{z w^{\prime}(z)^{2}}+\frac{k^{2} g^{\prime}(z)^{2}}{4 w^{\prime}(z)^{2}} \tag{A.7}
\end{align*}
$$

Using the explicit expression of the warped conformal transformation (3.10), we get

$$
\begin{equation*}
\left\langle P^{2}(w)\right\rangle_{\Sigma_{n}}=\frac{l^{2}}{w^{2}(w-l)^{2}}\left(\frac{24 b^{2}+k\left(1-n^{2}\right)}{24 n^{2}}+\frac{b k \mu}{n}+\frac{k^{2} \mu^{2}}{4}\right) . \tag{A.8}
\end{equation*}
$$

Combining the results above, we find

$$
\begin{equation*}
\langle A(w)\rangle_{\Sigma_{n}}=\frac{l^{2}}{w^{2}(w-l)^{2}}\left(\frac{a k+b^{2} c}{k n^{2}}+\frac{b(c-1) \mu}{n}+\frac{1}{4}(c-1) k \mu^{2}\right) . \tag{A.9}
\end{equation*}
$$

After taking the limit, the coefficient reads,

$$
\begin{equation*}
c_{A}=\frac{2\left(\frac{a k+b^{2} c}{k n^{2}}+\frac{b(c-1) \mu}{n}+\frac{1}{4}(c-1) k \mu^{2}\right)}{(c-1) c}=\frac{(2 b+k n \mu)^{2}}{2 c k n^{2}} . \tag{A.10}
\end{equation*}
$$

The above operator is defined in the same replica. For the composite operators with the operators on different replicas, their one-point functions are more tricky to compute. This is because that the warped transformation (3.10) is not single-valued, nor does its derivatives. For example, consider the operator $P_{i} P_{j}$ with $i \neq j$. It transforms as

$$
\begin{equation*}
P_{i} P_{j}(w)=w_{i}^{\prime-1} w_{j}^{\prime-1}\left(P\left(z_{i}\right)+\frac{k g^{\prime}\left(z_{i}\right)}{2}\right)\left(P\left(z_{j}\right)+\frac{k g^{\prime}\left(z_{j}\right)}{2}\right) \tag{A.11}
\end{equation*}
$$

where $w_{j}$ is not single-valued. As $w \rightarrow \infty$,

$$
\begin{equation*}
w_{j}^{\prime}(x) \rightarrow \frac{l}{n w^{2}} e^{2 \pi i j / n} . \tag{A.12}
\end{equation*}
$$

Combining with the two-point function of $P\left(z_{1}\right) P\left(z_{2}\right)$ in the non-trivial vacuum, we get

$$
\begin{equation*}
c_{P_{i} P_{j}}=\left(\frac{4 b^{2}}{k^{2} n^{2}}+\frac{1}{2 k n^{2} s_{i j}^{2}}+\frac{4 b \mu}{k n}+\mu^{2}\right) . \tag{A.13}
\end{equation*}
$$

The other operators involved in our study can be treated in the similar manner.
The normalization factors of the quasi-primary operators can be computed straightforwardly by using the algebra. For the quasi-primary operators appearing in our study, their normalization factors are listed as following

$$
\begin{align*}
& d_{-i P}=\frac{k}{2}, \quad d_{T}=\frac{c}{2}, \quad d_{P_{i} P_{j}}=\frac{k^{2}}{4}, \quad d_{A}=\frac{c(c-1)}{2},  \tag{A.14}\\
& d_{-i T_{i} P_{j}}=\frac{c k}{4}, \quad d_{i P_{i} P_{j} P_{k}}=\frac{k^{3}}{8}, \quad d_{-i P_{i} A_{j}}=\frac{k c(c-1)}{4},  \tag{A.15}\\
& d_{P^{3}}=\frac{3 k^{3}}{4}, \quad d_{B}=\frac{1}{4} k^{3}(c-1), \quad d_{-i P_{i} \partial P_{j}+P_{j} \partial P_{i}}=k^{2} . \tag{A.16}
\end{align*}
$$

## B Summation

In our computation, we run into the following type of summations

$$
\begin{equation*}
\sum_{0 \leq i \leq j<n} s_{i j}^{-m}=\frac{n}{2} f_{m}(n)=\frac{n}{2} \sum_{j=1}^{n-1} \frac{1}{\left(\sin \frac{j \pi}{n}\right)^{m}} . \tag{B.1}
\end{equation*}
$$

Actually the function $f_{m}(n)$ can be calculated by using the inverse Mellin transformation,

$$
\begin{equation*}
\frac{1}{(\sin a \pi)^{m}}=\int_{0}^{\infty} x^{a-1} g_{\Delta}(x) d x \tag{B.2}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{\Delta}(x)=\frac{2^{\Delta-2}}{\pi^{2} \Gamma[\Delta] \sqrt{x}} \Gamma\left[\frac{\Delta}{2}+i \frac{\log x}{2 \pi}\right] \Gamma\left[\frac{\Delta}{2}-i \frac{\log x}{2 \pi}\right] . \tag{B.3}
\end{equation*}
$$

We can do the summation first and then calculate the integral to get the desired results. There are two other kinds of summations involved in the calculation. They are respectively

$$
\begin{gather*}
\sum_{0 \leq i<j<k \leq n-1} s_{i j}^{-2}+s_{i k}^{-2}+s_{j k}^{-2}=\frac{1}{6}(n-2)(n-1) n(n+1),  \tag{B.4}\\
\sum_{0 \leq i<j<k \leq n-1}\left(s_{i j}^{-2}+s_{i k}^{-2}+s_{j k}^{-2}\right)^{2}=\frac{1}{90}(n-2)(n-1) n(n+1)\left(n^{2}+8 n+27\right) . \tag{B.5}
\end{gather*}
$$

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[^0]:    ${ }^{1}$ For other aspects on the holographic entanglement entropy, see $[3,12,13$ and the references in 14$]$.

[^1]:    ${ }^{2}$ We thank A.Belin for pointing this out to us.

[^2]:    ${ }^{3}$ A detailed analysis is in process in 55.
    ${ }^{4}$ See [40] for a discussion between the last three conditions for holographic CFTs. We thank L. Apolo for discussions on this point.

[^3]:    ${ }^{5}$ In this paper we use the exponential map and Wick rotation to obtain the Euclidean version in a heuristic way. A more proper analysis in Lorentzian signature is to use the Schwinger-Keldysh construction, similar to the discussions for CFT in Lorentzian signature 56. As was discussed in [32, the Rindler time is given by $\tau_{A}=\pi\left(\frac{\tilde{x}}{\tilde{\beta}}-\frac{\tilde{y}}{\tilde{\beta}}\right)$. The Rindler space is only a wedge of flat spacetime, and different patches can be obtained by defining a single-valued Rindler time $\tau=\tau_{A}+\frac{m \pi}{2} i, m \in \mathbb{Z}$, which can furthermore be obtained from $\tilde{x}=\tilde{x}_{A}+\frac{i m \tilde{\beta}}{4}, \tilde{y}=\tilde{y}_{A}-\frac{i m \overline{\bar{\beta}}}{4}$. In particular, all points in the complement $\bar{A}$ will be mapped to a Rindler space with $m=2$. The thermal circle corresponds to gluing the patch with $m=4$ to the patch with $m=0$ along $A$. Making $n$ replicas corresponds to interpolating the different copies from $0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow m=4 n \rightarrow 0$. See also 57 for related work. We leave all possible subtleties related to analytic continuations to future work.

[^4]:    ${ }^{6}$ The uniformization $\operatorname{map}(3.10$ with $p=0$ was also found in 39, where entanglement entropy on excited states and quantum chaos have also been discussed.

