A One Loop Problem of the Matrix Big Bang Model

Miao Li[1,2,3] and Wei Song[1,3]

[1] *Institute of Theoretical Physics*
Academia Sinica, P. O. Box 2735
Beijing 100080
and

[2] *Interdisciplinary Center of Theoretical Studies*
Academia Sinica, Beijing 100080, China
and

[3] *Interdisciplinary Center for Theoretical Study*
University of Science and Technology of China, Hefei, Anhui 230026, China

mli@itp.ac.cn
wsong@itp.ac.cn

We compute the one-loop effective action of two D0-branes in the matrix model for a cosmological background, and find vanishing static potential. However, there is a non-vanishing v^2 term not predicted in a supergravity calculation. This term is complex and signals an instability of the two D0-brane system, it may also indicate that the matrix model is incorrect.

Dec. 2005

1. Introduction

Formulating string/M theory in a time-dependent background remains an elusive problem. The only observables in string theory are S-matrix elements, this is certainly true in the perturbative formulation of string theory where conformal symmetry on the world-sheet plays the role of the guiding principle in constructing consistent asymptotically flat background, it is also true in a nonperturbative formulation of M theory, the matrix theory, where scattering amplitudes among D0-branes and their bound states are assumed to exist. However, S-matrix does not exist for most of interesting cosmological backgrounds, it certainly does not exist for our universe. Perhaps, a reformulation of observables is the key to extending string/M theory to include time-dependent backgrounds.

The matrix model proposed by Craps et al. is an attempt to formulate string theory in a time-dependent background [1], the metric in this model depends on a null coordinate and in the Einstein frame it exhibits a null singularity at the "big bang" point. This model was subsequently generalized to a class of more general backgrounds in [2], and to a class of even more general backgrounds in [3], [4], and [5] (a concrete model in this class was previously studied in detail in [6]). For related work on time-dependent backgrounds, see [7], [8], [9], and [10].

So far, except for the decoupling argument presented in [1], there has been no independent check on the correctness of the matrix proposal. The effective action of a D0-brane in the background generated by another D0-brane was derived in [11], where it is noticed that the usual double expansion in the relative velocity v and the inverse of the relative separation b fails when time is sufficiently close to the big bang point. Although there is no definition of scattering amplitude between two D0-branes too, we believed that it makes sense to talk about the effective action at later times. In the present work we shall make the usual one loop calculation to see whether we can obtain the small velocity expansion of [11]. To our surprise, we shall see that the v^2 term in the one-loop calculation does not vanish and is complex. This is a rather astonishing result.

We are faced with two possibilities, our result may indicate that the matrix proposal is incorrect, or it may signal an instability of the two D0-brane system at later times, since the v^2 term in the effective action is complex. However, we can not locate a physical reason for this instability at present.

The layout of this paper is as follows. We use the background field method of [13] to write down a gauge-fixed action and expand it to the second order. We compute the one-loop contribution of the off-diagonal fluctuations to the effective action of two D0-branes in sect.3 when the relative velocity vanishes, and find it equal to zero. The v^2 term in the one-loop contribution is calculated in sect.4, and we find a non-vanishing complex term. We show that the small velocity expansion makes sense in the flat matrix theory and the v^2 terms cancel in appendix A. Appendixes B is devoted to discussions on the propagators.

2. Basic setup

In [1], the authors consider a flat type IIA background, with a null linear dilaton, $\phi = -Qx^+$. The Einstein metric has a curvature singularity at $x^+ = -\infty$. A matrix string action is proposed in [1] to describe the theory nonperturbatively. The type IIA background can be obtained by compactifying M theory on a circle, along the ninth direction. In [2], the background is lifted to M theory, and the corresponding matrix theory is BFSS like [12]. D0 brane interaction is found out by considering the shock wave solution in [11]. To get the shock wave solution, the authors have compactified the ninth direction and averaged the source over that direction. The Routhian of a graviton in the presence of another is $\frac{1}{2}p_- \sum_{n=1}^{\infty} c_n v^2 [\kappa_{11}^2 e^{-2Qx^+} p'_- v^2 r^{-6}]^{n-1}$, where p_- is the null momentum of the test particle, $(2\pi)^2 RR'p'_-$ is that of the source particle, and c_n is some fixed numerical coefficient, especially, $c_1 = 1, c_2 = \frac{1}{8\pi^2}$. R is the radius of the M-theory and R' is that of 9th direction. From the form of the Routhian, one can see that there is no static potential between two gravitons, and there is no v^2 correction, either. In this approach, we are expanding the effective action in terms of $\kappa_{11}^2 e^{-2Qx^+} p'_- v^2 r^{-6}$. Note that $\kappa^2 = \frac{\kappa_{11}^2 e^{-2Qx^+}}{2\pi R'} = \frac{\kappa_{11}^2 g_s^2}{2\pi R'}$ is just the physical gravitational constant in IIA string theory. Therefore it is clear that the expansion is a supergravity perturbation.

In the present paper, we will just consider the case of two D0 branes, and hence $p_- = \frac{1}{R}$, and $p'_- = \frac{1}{R 2\pi R 2\pi R'}$. We shall in this paper work in the scheme of [2], and use the matrix model action instead of the matrix string action. We compute the effective potential of two D0 branes with separation both in the ninth direction and in the transverse directions. To compare our matrix model calculation with the supergravity result in [11], the separation in the ninth direction should be integrated out in the end.

The matrix theory action includes the bosonic part S_B and fermionic part S_F. Set the Planck scale l_p to 1, the two parts can be written as

$$S_B = \int dt Tr\{\frac{1}{2R}(D_t X^i)^2 + \frac{1}{2R} e^{-2Qt} (D_\tau X^9)^2 + \frac{R}{4} e^{2Qt} [X^i, X^j]^2 + \frac{R}{2} [X^9, X^j]^2\},$$
$$S_F = \int dt Tr\{i\theta^T D_t \theta - Re^{Qt} \theta^T \gamma_i [X^i, \theta] - R\theta^T \gamma_9 [X^9, \theta]\}, \tag{2.1}$$

where $i,j = 1,\cdots 8$, runs over the eight transverse directions, and $D_t = \partial_t + i[A,$ is the covariant derivative. Rescale $t \to (2^{1/3}R)^{-1}t, Q \to 2^{1/3}RQ$ and $X^\mu \to 2^{1/3}X^\mu$ to absorb the R in the action, we have

$$\begin{aligned} S_B &= \int dtTr\{(D_tX^i)^2 + e^{-2Qt}(D_tX^9)^2 + \frac{1}{2}e^{2Qt}[X^i,X^j]^2 + [X^9,X^j]^2\}, \\ S_F &= \int dtTr\{i\theta^T D_t\theta - e^{Qt}\theta^T\gamma_i[X^i,\theta] - \theta^T\gamma_9[X^9,\theta]\}. \end{aligned} \tag{2.2}$$

To calculate the effective potential, we use the background field method [13]. Expand the action (2.2) around the classical background field B^μ by setting $X^\mu = B^\mu + Y^\mu$, $\mu = 1,2\cdots 9$. The fluctuation part of the action is a sum of five terms

$$S = S_i + S_9 + S_A + S_{fermi} + S_{ghost}. \tag{2.3}$$

In the following, we will determine the explicit form of each term. It is convenient to choose the gauge

$$G \equiv \partial_t A - ie^{2Qt}[B^i,X^i] - i[B^9,X^9] = 0. \tag{2.4}$$

In the standard gauge fixing procedure, we need to insert

$$1 = \Delta_{fp}\int [d\xi]\delta(G - f(t)g(t)) \tag{2.5}$$

into the path integral, where ξ is a gauge parameter, $f(t)$ is chosen to be $f(t) = e^{Qt}$ for later convenience, $g(t)$ is any function. The path integral is independent of the choice of $g(t)$, so we can multiply the path integral by $\int[dg(t)]e^{-ig(t)^2}$. Δ_{fp} is given by the variation of G under gauge transformation, independent of $g(t)$. Thus by changing the order of integration, we can integrate out $g(t)$, and get a gauge fixing term

$$S_{gf} = -e^{-2Qt}G^2. \tag{2.6}$$

So the bosonic action of the fluctuation is

$$\begin{aligned} S_{Y^i} &= \int dtTr\{(\partial_t Y^i)^2 + e^{2Qt}([B^i,Y^j]^2 + e^{-2Qt}[B^9,Y^j]^2 + [B^i,Y^i]^2 \\ &\quad + 2[B^i,Y^j][Y^i,Y^j] + \frac{1}{2}[Y^i,Y^j]^2)\}, \\ S_{Y^9} &= \int dte^{-2Qt}Tr\{(\partial_t Y^9)^2 + e^{2Qt}([B^i,Y^9]^2 + e^{-2Qt}[B^9,Y^9]^2 \\ &\quad + 2[B^i,Y^9][Y^i,Y^9] + 2[Y^i,B^9][Y^i,Y^9] + [Y^i,Y^9]^2)\}, \\ S_A &= \int dtTr\{-e^{-2Qt}(\partial_t A)^2 - e^{-2Qt}[A,B^9]^2 - [A,B^i]^2 \\ &\quad + 4i\partial_t B^i[A,Y^i] + 4ie^{-2Qt}\partial_t B^9[A,Y^9] - 4Qie^{-2Qt}B^9[A,Y^9] \\ &\quad + 2i\partial_t Y^i[A,Y^i] - 2[A,B^i][A,Y^i] - [A,Y^i]^2 \\ &\quad + 2ie^{-2Qt}\partial_t Y^9[A,Y^9] - 2e^{-2Qt}[A,B^9][A,Y^9] - e^{-2Qt}[A,Y^9]^2\}. \end{aligned} \tag{2.7}$$

Since we are considering two D0-branes, the Yang-Mills fields are just 2×2 matrix. We will choose the background to be diagonal,

$$B^1 = \frac{vt}{2}\sigma_3, B^2 = \frac{b}{2}\sigma_3, B^9 = \frac{c}{2}\sigma_3. \tag{2.8}$$

The background for A and other transverse directions are chosen to be zero. This corresponds to, in comoving coordinate, two zero-branes moving towards each other with relative velocity v in the x^1 direction, and transverse separation b in the x^2 direction and c in the x^9 direction. Write the matrix in terms of $U(2)$ generators,

$$\begin{aligned} Y^i &= \frac{1}{2}({Y_0}^i \mathbf{1}_2 + {Y_a}^i \sigma^a), \quad Y^9 = \frac{1}{2} e^{Qt}({Y_0}^9 \mathbf{1}_2 + {Y_a}^9 \sigma^a), \\ A &= \frac{1}{2} e^{Qt}(A_0 \mathbf{1}_2 + A_a \sigma^a), \quad \theta = \frac{1}{2}(\theta_0 \mathbf{1}_2 + \theta_a \sigma^a). \end{aligned} \tag{2.9}$$

where $a = 1, 2, 3$. The 0 components in this decomposition describe the free motion of the center of mass and will not be written explicitly in the following. Then up to quadratic terms, the actions for the fluctuations are

$$\begin{aligned} S_i = \frac{1}{2} \int dt \{ & {Y_1}^i(-{\partial_t}^2 - b^2 e^{2Qt} - v^2 t^2 e^{2Qt} - c^2){Y_1}^i \\ & + {Y_2}^i(-{\partial_t}^2 - b^2 e^{2Qt} - v^2 t^2 e^{2Qt} - c^2){Y_2}^i \\ & + {Y_3}^i(-{\partial_t}^2){Y_3}^i \}, \\ S_9 = \frac{1}{2} \int dt \{ & {Y_1}^9(-{\partial_t}^2 + Q^2 - b^2 e^{2Qt} - v^2 t^2 e^{2Qt} - c^2){Y_1}^9 \\ & + {Y_2}^9(-{\partial_t}^2 + Q^2 - b^2 e^{2Qt} - v^2 t^2 e^{2Qt} - c^2){Y_2}^9 \\ & + {Y_3}^9(-{\partial_t}^2 + Q^2){Y_3}^9 \}, \\ S_A = -\frac{1}{2} \int dt \{ & A_1(-{\partial_t}^2 + Q^2 - b^2 e^{2Qt} - v^2 t^2 e^{2Qt} - c^2)A_1 \\ & + A_2(-{\partial_t}^2 + Q^2 - b^2 e^{2Qt} - v^2 t^2 e^{2Qt} - c^2)A_2 \\ & + A_3(-{\partial_t}^2 + Q^2)A_3 \\ & + 4ve^{Qt}(A_1 {Y_2}^1 - A_2 {Y_1}^1) - 4Qc(A_1 {Y_2}^9 - A_2 {Y_1}^9) \}. \end{aligned} \tag{2.10}$$

Define

$$\begin{aligned} Y_2^9 &= \frac{1}{\sqrt{2}}(A_1^+ + A_1^-), \quad A_1 = \frac{1}{i\sqrt{2}}(A_1^+ - A_1^-), \\ Y_1^9 &= \frac{1}{\sqrt{2}}(A_2^+ + A_2^-), \quad A_2 = \frac{1}{i\sqrt{2}}(A_2^- - A_2^+). \end{aligned} \tag{2.11}$$

Then the actions for A and X_9 become

$$S_{A^+} = \frac{1}{2}\int dt\{A^+{}_1(-\partial_t{}^2 + Q^2 - b^2e^{2Qt} - v^2t^2e^{2Qt} - c^2 - i2Qc)A^+{}_1$$
$$+A^+{}_2(-\partial_t{}^2 + Q^2 - b^2e^{2Qt} - v^2t^2e^{2Qt} - c^2 - i2Qc)A^+{}_2$$
$$+A_3(-\partial_t{}^2 + Q^2)A_3 + i2\sqrt{2}ve^{Qt}[(A^+{}_1 - A^-{}_1)Y_2{}^1 + (A^+{}_2 - A^-{}_2)Y_1{}^1]\},$$
$$S_{A^-} = \frac{1}{2}\int dt\{A^-{}_1(-\partial_t{}^2 + Q^2 - b^2e^{2Qt} - v^2t^2e^{2Qt} - c^2 + i2Qc)A^-{}_1$$
$$+A^-{}_2(-\partial_t{}^2 + Q^2 - b^2e^{2Qt} - v^2t^2e^{2Qt} - c^2 + i2Qc)A^-{}_2$$
$$+Y_3{}^9(-\partial_t{}^2 + Q^2)Y_3{}^9\}. \tag{2.12}$$

Define new fermionic fields,

$$\theta_+ = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2), \quad \theta_- = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2). \tag{2.13}$$

Then the action is

$$S_f = \int dt\theta_-^T(i\partial_t + vte^{Qt}\gamma_1 + be^{Qt}\gamma_2 + c\gamma_9)\theta_+ + \frac{1}{2}\theta_3{}^T(i\partial_t)\theta_3. \tag{2.14}$$

The ghost action is determined by the infinitesimal gauge transformation of G,

$$S_g = \int dtC_1{}^*(-\partial_t^2 - b^2e^{2Qt} - v^2t^2e^{2Qt} - c^2)C_1 + C_2{}^*(-\partial_t^2 - b^2e^{2Qt} - v^2t^2e^{2Qt} - c^2)C_2$$
$$+ C_3{}^*(-\partial_t^2)C_3. \tag{2.15}$$

Before doing any calculation, we can see that the fluctuation action for X_3^μ is independent of the separation, and hence has nothing to do with the interaction of the two zero branes. We will not consider them in the following.

3. Static case

First we will analyze the situation when $v = 0$. This corresponds to two zero-branes static in the comoving coordinates. To calculate the one loop interaction, we need to integrate out the quadratic fluctuation, which can be written in the form of determinants,

$$\begin{aligned}
&\det{}^{-\frac{1}{2}}(-\partial_t^2 - b^2e^{2Qt} - c^2), && \text{for} \quad Y_{1,2}{}^i, i = 1, \cdots 8,\\
&\det{}^{-\frac{1}{2}}(-\partial_t^2 + Q^2 - b^2e^{2Qt} - c^2 - i2Qc), && \text{for} \quad A^+{}_1, A^+{}_2,\\
&\det{}^{-\frac{1}{2}}(-\partial_t^2 + Q^2 - b^2e^{2Qt} - c^2 + i2Qc), && \text{for} \quad A^-{}_1, A^-{}_2,\\
&\det(-\partial_t^2 - b^2e^{2Qt} - c^2), && \text{for} \quad C_{1,2},\\
&\det(i\partial_t + be^{Qt}\gamma_2 + c\gamma_9), && \text{for} \quad \theta_+.
\end{aligned} \tag{3.1}$$

We use Schwinger proper time formalism to calculate the determinants. For any Hermitian operator Δ, the determinant is represented by

$$\delta \equiv \ln(\det \Delta) = -\int_0^\infty \frac{ds}{s} Tr e^{-i\Delta s}. \tag{3.2}$$

Thus we need to calculate the heat kernel, $K(t', t; s) \equiv < t'|e^{-i\Delta s}|t >$. $K(t, t'; s)$ satisfies the differential equation and the boundary condition,

$$\begin{aligned} i\partial_s K(t', t; s) &= \Delta K(t', t; s), \\ K(t', t; 0) &= \delta(t - t'). \end{aligned} \tag{3.3}$$

For the first determinant in (3.1), $\Delta = -\partial_t^2 - b^2 e^{2Qt} - c^2$. To solve (3.3), we first solve the static shrödinger equation

$$\lambda y_\lambda(t) = (-\partial_t^2 - b^2 e^{2Qt} - c^2) y_\lambda(t). \tag{3.4}$$

The two linearly dependent solutions of (3.4) are Bessel functions

$$J_{\pm\kappa}(x), \quad \text{for} \quad \kappa \notin Z, \quad \text{or} \quad J_\kappa(x), \quad Y_\kappa(x), \quad \text{for} \quad \kappa \in Z, \tag{3.5}$$

where $x = \frac{b}{Q} e^{Qt}$, $\quad -(Q\kappa)^2 = \lambda + c^2$. $Y_n(x)$ has singlarity at $x = 0$, and are not in consideration. Since the operator Δ is hermitian, λ is real, and κ is either real or pure imaginary.

Using an integral of Bessel function (eq. 6.574.2 of [15])

$$\int_0^\infty \frac{dx}{x} J_\nu(x) J_\mu(x) = \frac{2 \sin \pi(\frac{\mu-\nu}{2})}{\pi(\mu+\nu)(\mu-\nu)}, \tag{3.6}$$

an orthonormal basis can be constructed,

$$\begin{aligned} y_\omega(t) &= \sqrt{\frac{Q\omega}{2\sinh(\pi\omega)}} [J_{i\omega}(x) + J_{-i\omega}(x)], \quad \omega > 0, \\ f_n(t) &\equiv \sqrt{4Qn} J_{2n}(x), \quad n = 1, 2, \cdots \end{aligned} \tag{3.7}$$

To check the orthogonality,

$$\begin{aligned} &\int_{-\infty}^\infty dt y_\omega(t) y^*_{\omega'}(t) = \frac{Q\omega}{2\sinh(\pi\omega)} \int_0^\infty \frac{dx}{Qx} [J_{i\omega+\epsilon}(x) + J_{-i\omega+\epsilon}(x)][J_{-i\omega'+\epsilon}(x) + J_{i\omega'+\epsilon}(x)] \\ &= \frac{\omega}{\sinh(\pi\omega)} \{ \frac{\sinh[\frac{\pi}{2}(\omega+\omega')]}{(\omega+\omega')} \frac{\epsilon}{\pi[(\frac{\omega-\omega'}{2})^2 + \epsilon^2]} + \frac{\sinh[\frac{\pi}{2}(\omega-\omega')]}{(\omega-\omega')} \frac{\epsilon}{\pi[(\frac{\omega+\omega'}{2})^2 + \epsilon^2]} \} \\ &= \delta(\omega - \omega') + \delta(\omega + \omega') \\ &= \delta(\omega - \omega'), \\ &\int_{-\infty}^\infty dt f_n(t) f_m(t) = \delta_{n,m}, \\ &\int_{-\infty}^\infty dt f_\omega(t) f_n(t) \propto \sin(n\pi) = 0. \end{aligned} \tag{3.8}$$

We have deformed $\pm i\omega$ by a small real part, $\pm i\omega \to \pm i\omega + \epsilon$ and used the identity $\delta(z) = \lim_{\epsilon\to 0}\frac{\epsilon}{\pi(z^2+\epsilon^2)}$. In the fourth line $\omega > 0$ is taken into account.

To check the completeness, we will need to prove that all $J_\kappa(x), \Re\kappa > 0, \kappa \neq 2n, n \in Z_+$ can be expanded in the basis. Define

$$\begin{aligned} \tilde{J}_\kappa(\omega) &= \int_0^\infty dt y_\omega^*(t) J_\kappa(x), \quad \tilde{J}_\kappa^n = \int_0^\infty dt f_n(t) J_\kappa(x), \\ \tilde{\tilde{J}}_\kappa(x) &= \int_0^\infty d\omega \tilde{J}_\kappa(\omega) y_\omega(t) + \sum_{n=1}^\infty \tilde{J}_\kappa^n f_n(t). \end{aligned} \tag{3.9}$$

Using (3.6), one finds that

$$\tilde{J}_\kappa(\omega) = \sqrt{\frac{\omega}{2Q\sinh(\pi\omega)}}\frac{4\cosh(\frac{\pi\omega}{2})\sin(\frac{\pi\kappa}{2})}{\pi(\kappa^2+\omega^2)}, \quad \tilde{J}_\kappa^n = 4\sqrt{\frac{n}{Q}}\frac{(-1)^n\sin(\pi\frac{\kappa}{2})}{\pi(\kappa^2-4n^2)}. \tag{3.10}$$

Hence,

$$\int_0^\infty d\omega \tilde{J}_\kappa(\omega) y_\omega(t) = \int_{-\infty}^\infty d\omega \frac{\sin(\frac{\pi\kappa}{2})\omega J_{i\omega+\epsilon}(x)}{\pi\sinh(\frac{\pi\omega}{2})(\kappa^2+\omega^2)}. \tag{3.11}$$

The large order behavior of the Bessel function is

$$J_\mu(x) \sim e^{\mu+\mu\ln\frac{x}{2}-(\mu+\frac{1}{2})\ln\mu}. \tag{3.12}$$

Then the integral (3.11) can be evaluated by closing the contour in the lower half plane. Simple poles are at $\omega = -2ni, -i\kappa$.

$$\begin{aligned} \int_0^\infty d\omega \tilde{J}_\kappa(\omega) y_\omega(t) &= J_\kappa(x) - \frac{8}{\pi}\sin(\frac{\pi\kappa}{2})\sum_{n=1}^\infty (-1)^n \frac{n}{\kappa^2-4n^2} J_{2n}(x) \\ &= J_\kappa(x) - \sum_{n=1}^\infty \tilde{J}_\kappa^n f_n(t), \end{aligned} \tag{3.13}$$

Therefore,

$$\tilde{\tilde{J}}_\kappa(x) = J_\kappa(x). \tag{3.14}$$

When $\kappa = \pm i\omega + \epsilon$, the above equations still hold. Then the other linear combination of $J_{\pm\omega}(x), \quad J_{i\omega}(x) - J_{-i\omega}(x)$ can be also expanded in terms of the basis (3.7), and so are not included in the basis. In fact, we have shown that any normalizable eigenfunction can be expanded in terms of this basis, which is enough to guarantee that the basis is complete. (The completeness of this set of the eigenfunctions was discussed previously in [16].)

Then the heat kernel can be expanded in terms the orthonormal basis,

$$K(t',t;s)=\int_0^\infty d\omega y_\omega(t')^* y_\omega(t) e^{-i[(Q\omega)^2-c^2]s}+\sum_{n=1}^\infty f_n(t')f_n(t)e^{i[(2Qn)^2+c^2]s}. \tag{3.15}$$

3.1. The bosonic effective potential

Having found out the heat kernel, it is straight forward to write down the determinant explicitly, by $\delta_i=-\int_0^\infty \frac{ds}{s}\int_{-\infty}^\infty dt K(t,t;s)$. The trace in (3.2) is now an integral over t. In order to compare with the result obtained on the supergravity side [11], we need to compactify the 9-direction and smear the result over the circle. This is equivalent to sum the images in the covering space and then average over the compactified circle. On the matrix theory side, we need to calculate the one loop effective potential of two D0-branes separated also by c in the x^9 direction, integrate over c and then divide by $2\pi R'$. This procedure is expected to give us the result that is to be compared with our earlier result in [11]. Here R' is the radius of X^9. Now there are altogether four integrals in our calculation of determinant, the integral over t, s, ω, and c. We can first do the the integral over c. Then the smeared determinant becomes

$$\begin{aligned}\delta_i=&\frac{-1}{2\sqrt{-\pi i}R'}\int_0^\infty\frac{ds}{s^{\frac{3}{2}}}\int_{-\infty}^\infty dt\{\int_{-\infty}^\infty d\omega\frac{Q\omega}{2\sinh(\pi\omega)}J_{i\omega}(x)[J_{i\omega}(x)+J_{-i\omega}(x)]e^{-i(Q\omega)^2 s}\\ &+\sum_{n=1}^\infty 4QnJ_{2n}^2(x)e^{i(2Qn)^2 s}\}\\ =&\frac{-1}{2\sqrt{-\pi i}R'}\int_0^\infty\frac{ds}{s^{\frac{3}{2}}}\int_{-\infty}^\infty dt\{\int_{-\infty}^\infty d\omega\frac{Q\omega}{2\sinh(\pi\omega)}J_{i\omega}(x)[J_{i\omega}(x)+J_{-i\omega}(x)]\\ &\times[1-i(Q\omega)^2 s+\cdots]+\sum_{n=1}^\infty 4QnJ_{2n}^2(x)[1+i(2Qn)^2 s+\cdots]\}.\end{aligned} \tag{3.16}$$

Here we have extended the integral range of ω from $(0,\infty)$ to $(-\infty,\infty)$. We use the notation $\pm i=e^{\frac{\pm\pi i}{2}}$, $\sqrt{\pm i}=e^{\pm\frac{\pi}{4}i}$, and $\ln(i)=\frac{\pi i}{2}$. We have rewritten the exponential in the form of power series. Using the large order behavior of Bessel function, we have

$$\begin{aligned}\frac{\omega^{2n+1}}{\sinh(\pi\omega)}J_{i\omega}(x)J_{-i\omega}(x)&\sim\frac{\omega^{2n+1}}{\sinh(\pi\omega)}\exp[-2i\omega\ln i-\ln\omega]=\frac{\omega^{2n}e^{\pi\omega}}{\sinh(\pi\omega)},\\ \frac{\omega^{2n+1}}{\sinh(\pi\omega)}J_{i\omega}^2(x)&\sim\frac{\omega^{2n+1}}{\sinh(\pi\omega)}\exp[2i\omega+2i\omega\ln\frac{x}{2}-2i\omega\ln\omega-2i\omega\ln i-\ln\omega]\\ &=\frac{\omega^{2n}\exp[2i\omega+2i\omega\ln\frac{x}{2}-2i\omega\ln\omega+\pi\omega]}{\sinh(\pi\omega)}.\end{aligned} \tag{3.17}$$

Close the contour in the lower half plane, we can see that the integral of each term proportional to $J^2_{i\omega}(x)$ at the infinity is zero. Because of the third line of (3.17), we will meet a divergence at infinity in each term proportional to $J_{i\omega}(x)J_{-i\omega}(x)$. Note that this divergence is independent of the x, and therefore can be subtracted. Then

$$\begin{aligned}&\int_{-\infty}^{\infty} d\omega \frac{Q\omega}{2\sinh(\pi\omega)} J_{i\omega}(x)[J_{i\omega}(x)+J_{-i\omega}(x)][1-i(Q\omega)^2 s+\cdots],\\ &= -4QnJ^2_{2n}(x)[1+i(2Qn)^2 s+\cdots].\end{aligned} \tag{3.18}$$

The above just cancels with the summation in (3.16) term by term. Although we are not sure about the convergence of the expansion, the exact cancelation of each term between the integral (3.18) and the summation in (3.16) has show that $\delta_i = 0$ up to a physical irrelevant constant. So the bosons coming from the i directions give no contribution to the effective potential.

For the second and the third determinants in (3.1), The heat kernel becomes

$$\begin{aligned}K_+(t',t;s) &= \int_0^\infty d\omega y^*_\omega(x')y_\omega(x)e^{-i[(Q\omega)^2-(c+iQ)^2]s} + \sum_{n=1}^{\infty} f_n(t')f_n(t)e^{i[(2Qn)^2+(c+iQ)^2]s},\\ K_-(t',t;s) &= \int_0^\infty d\omega y^*_\omega(x')y_\omega(x)e^{-i[(Q\omega)^2-(c-iQ)^2]s} + \sum_{n=1}^{\infty} f_n(t')f_n(t)e^{i[(2Qn)^2+(c-iQ)^2]s}.\end{aligned} \tag{3.19}$$

Then take the same procedure as in eqs. (3.16), (3.17), and (3.18), we will find that the bosons coming from the gauge field and X^9 give no contribution to the effective potential, either.

The ghost determinant is the same with that of X^i, and hence give the same result except for a minus sign.

In a word, we find that there is no static potential coming from the bosons.

3.2. The fermionic effective potential

In the fermionic sector, there are 16 degrees of freedom for each $SU(2)$ index. Since there are only three gamma matrix relevant here, we can choose a basis to make the gamma matrix and the field block diagonal,

$$\gamma_1 = \sigma_2 \otimes \mathbf{1}_8, \quad \gamma_2 = \sigma_3 \otimes \mathbf{1}_8, \quad \gamma_9 = \sigma_1 \otimes \mathbf{1}_8. \tag{3.20}$$

Define

$$K_{\alpha\beta}(t',t;s) =< t'|\exp(-i\Delta_f(\hat{t})s)|t>_{\alpha\beta},$$

where $\Delta_f = i\partial_t + be^{Qt}\gamma_2 + c\gamma_9$, $\alpha, \beta = 1, 2$ label the two 8×8 block matrix. Then $K_{\alpha\beta}(t', t; s)$ satisfies the following differential equation and initial condition

$$i\partial_s K_{\alpha\beta}(t', t; s) = (\Delta_f(\hat{t}))_{\alpha\rho} K_{\rho\beta}(t', t; s), \quad K_{\alpha\beta}(t', t; 0) = \delta_{\alpha\beta}\delta(t' - t). \tag{3.21}$$

To find the solution, we write

$$\begin{pmatrix} K_{1\beta}(t', t; s) \\ K_{2\beta}(t', t; s) \end{pmatrix} = \int_{-\infty}^{\infty} d\lambda K_\beta(t', t; \lambda) e^{-i\lambda s}.$$

In the following, we just write K_β for short. Then from (3.21), K_β satisfy the following differential equations

$$\begin{aligned} &(i\partial_t + be^{Qt} - \lambda)K_{1\beta} + cK_{2\beta} = 0, \\ &(i\partial_t - be^{Qt} - \lambda)K_{2\beta} + cK_{1\beta} = 0. \end{aligned} \tag{3.22}$$

These equations are equivalent to

$$\begin{aligned} &cK_{2\beta} = -(i\partial_t + be^{Qt} - \lambda)K_{1\beta}, \\ &(\partial_t^2 + b^2 e^{2Qt} + c^2 - \lambda^2 + 2i\lambda\partial_t - iQbe^{Qt})K_{1\beta} = 0. \end{aligned} \tag{3.23}$$

Denote $\lambda/(Q)$ by ω, and c/Q by c'. Take the ansatz $K_{1\beta} = f(w, c', t')\psi(\omega, c', t)$. Then $f(\omega, c', t')$ factorize and the equation for $\psi(\omega, c', t)$ has two linearly independent solutions, $x^{(-1/2-i\omega)} M_{1/2,ic'}(-2ix)$ and $x^{(-1/2-i\omega)} M_{1/2,-ic'}(-2ix)$. Where $M_{\lambda,\mu}(x)$ is a Whittaker function. A general solution for (3.23) is

$$\begin{aligned} K_\beta =& f(\omega, c', t') x^{(-1/2-i\omega)} \begin{pmatrix} M_{1/2,ic'}(-2ix) \\ M_{-1/2,ic'}(-2ix) \end{pmatrix} \\ &+ g(\omega, c', t') x^{(-1/2-i\omega)} \begin{pmatrix} M_{1/2,-ic'}(-2ix) \\ -M_{-1/2,-ic'}(-2ix) \end{pmatrix}, \end{aligned} \tag{3.24}$$

where $f(\omega, c', t')$ and $g(\omega, c', t')$ are chosen to satisfy the initial condition. Then

$$\begin{aligned} K_{\alpha 1}(t', t; s) =& \int_{-\infty}^{\infty} d\omega Q (\frac{x}{x'})^{-i\omega} \frac{\exp(-iQ\omega s)}{-4i\sqrt{xx'}} [M_{-1/2,-ic'}(-2ix') \begin{pmatrix} M_{1/2,ic'}(-2ix) \\ M_{-1/2,ic'}(-2ix) \end{pmatrix} \\ &+ M_{-1/2,ic'}(-2ix') \begin{pmatrix} M_{1/2,-ic'}(-2ix) \\ -M_{-1/2,-ic'}(-2ix) \end{pmatrix}], \\ K_{\alpha 2}(t', t; s) =& \int_{-\infty}^{\infty} d\omega Q (\frac{x}{x'})^{-i\omega} \frac{\exp(-iQ\omega s)}{-4i\sqrt{xx'}} [M_{1/2,-ic'}(-2ix') \begin{pmatrix} M_{1/2,ic'}(-2ix) \\ M_{-1/2,ic'}(-2ix) \end{pmatrix} \\ &- M_{1/2,ic'}(-2ix') \begin{pmatrix} M_{1/2,-ic'}(-2ix) \\ -M_{-1/2,-ic'}(-2ix) \end{pmatrix}]. \end{aligned} \tag{3.25}$$

When $s=0$, ω can be integrated out and gives $\delta(t-t')$. Then

$$K_{\alpha\beta}(t',t;0)=\delta_{\alpha\beta}\delta(t-t')\frac{D(x,x',c')}{-4i\sqrt{xx'}},$$

where

$$D(x,x',c')\equiv M_{1/2,ic'}(-2ix)M_{-1/2,-ic'}(-2ix')+M_{1/2,-ic'}(-2ix)M_{-1/2,ic'}(-2ix'). \quad (3.26)$$

In appendix B, we will prove that $D(x,x,c')=-4ix$. So (3.25) satisfies the initial condition in (3.21).

Finally,

$$\begin{aligned}\delta_f &= -\int_{-\infty}^{\infty}\frac{dc}{2\pi R'}\int_0^{\infty}\frac{ds}{s}trK_{\alpha,\beta}(t,t';s)\\ &= -\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}\frac{dc}{2\pi R'}\int_0^{\infty}\frac{ds}{s}\int_{-\infty}^{\infty}d\omega Q\exp[-iQ\omega s]\\ &\times D^{-1}[M_{1/2,ic'}(-2ix)M_{-1/2,-ic'}(-2ix)+M_{1/2,-ic'}(-2ix)M_{-1/2,ic'}(-2ix)]\\ &= -\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}\frac{dc}{2\pi R'}\int_0^{\infty}\frac{ds}{s}\int_{-\infty}^{\infty}d\omega Q\exp[-iQ\omega s]\end{aligned} \quad (3.27)$$

The tr in the first line means a trace of the 2×2 matrix and an integral over t. The integral is divergent but is independent of b and α. In fact, this is just the phase shift generated by a free operator $i\frac{d}{dt}$. Regularize the phase shift by subtracting $-\int_0^{\infty}\frac{ds}{s}tre^{-iH_0 s}$, H_0 is the Lagrange for free fermions. We can see that the fermions give no contribution to the effective potential.

Then we can draw the conclusion that there is no static effective potential.

4. Effective interaction at the order v^2

Now we are going to investigate the case when there is a small relative velocity between the zero branes. Since it is difficult to compute the determinants directly, we will perturbatively expand around $v=0$.

From (2.10), (2.12), (2.14)and (2.15), we can see that the only possible terms term odd in v in the perturbation series come from the the fermionic action (2.14). These terms vanish because the trace of odd number of Gamma matrix is zero.We shall in this section calculate various v^2 terms.

Denote the v^2 terms coming from the first order bosonic contribution by b_1, the second order bosonic contribution by b_2, the first order ghost contribution by g_1, and the second order fermionic contribution by f_2. Then,

$$
\begin{aligned}
b_1 =& - i\int_{-\infty}^{\infty}\frac{dc}{2\pi R'}\int_{-\infty}^{\infty}dt\frac{1}{2}v^2t^2e^{2Qt}[\langle Y_1^i(t)Y_1^i(t)\rangle + \langle Y_2^i(t)Y_2^i(t)\rangle,\\
&+ \langle A_1^+(t)A_1^+(t)\rangle + \langle A_2^+(t)A_2^+(t)\rangle + \langle A_1^-(t)A_1^-(t)\rangle + \langle A_2^-(t)A_2^-(t)\rangle],\\
g_1 =& i\int_{-\infty}^{\infty}\frac{dc}{2\pi R'}\int_{-\infty}^{\infty}dtv^2t^2e^{2Qt}[\langle C_1(t)C_1^*(t)\rangle + \langle C_2(t)C_2^*(t)\rangle],\\
b_2 =& \frac{1}{2}\int_{-\infty}^{\infty}\frac{dc}{2\pi R'}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2(v\sqrt{2})^2e^{Qt_1}e^{Qt_2}\\
&\times \{[< A^+{}_1(t_1)A^+{}_1(t_2) > + < A^-{}_2(t_1)A^-{}_2(t_2) >] < Y_2{}^1(t_1)Y_2{}^1(t_2) >\\
&+ [< A^-{}_1(t_1)A^-{}_1(t_2) > + < A^+{}_2(t_1)A^+{}_2(t_2) >] < Y_1{}^1(t_1)Y_1{}^1(t_2) >\},\\
f_2 =& \frac{1}{2}\int_{-\infty}^{\infty}\frac{dc}{2\pi R'}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2v^2t_1t_2e^{Qt_1}e^{Qt_2}\\
&\times \mathrm{Tr}[\gamma_1G_f(t_1,t_2)\gamma_1G_f(t_2,t_1)].
\end{aligned}
\tag{4.1}
$$

Since we are only interested in the effective potential, we do not have to do all the integrals in the second order contributions. Define $t_1 = t + \frac{1}{2}\tau, \quad t_2 = t - \frac{1}{2}\tau$, integrate out τ and c, we are left with an integral of t, which combined with the first order perturbation, will give the effective potential to v^2 order. In Appendix A, we will show how this procedure is carried out when the background is flat. We hope that this procedure also goes through here, as we shall see, there is a problem arising at this order.

The propagator $< Y_a^i(t_2)Y_a^i(t_1) > \equiv G_i(t_2,t_1)$, satisfies the differential equation

$$
(-\partial_{t_1}^2 - b^2e^{2Qt_1} - c^2)G_i(t_2,t_1) = i\delta(t_1 - t_2),
\tag{4.2}
$$

and is related to the heat kernel by $G_i(t_2,t_1) = -\int_0^\infty dsK(t_2,t_1;s)$.

$$
\begin{aligned}
G_i(t_2,t_1) =& -\int_0^\infty dsK(t_2,t_1;s)\\
=& i\int_{-\infty}^{\infty}d\omega\frac{Q\omega}{2\sinh(\pi\omega)}J_{i\omega}(x_1)[J_{-iw}(x_2) + J_{i\omega}(x_2)]/[(Q\omega)^2 - c^2 - i\epsilon]\\
&+ i\sum_{n=1}^{\infty}4QnJ_{2n}(x_1)J_{2n}(x_2)/[-(2Qn)^2 - c^2 - i\epsilon]\\
=& -\theta(t_1 - t_2)\frac{\pi}{2Q\sinh(\pi c')}J_{-ic'}(x_2)[J_{ic'}(x_1) + J_{-ic'}(x_1)]\\
&-\theta(t_2 - t_1)\frac{\pi}{2Q\sinh(\pi c')}J_{-ic'}(x_1)[J_{ic'}(x_2) + J_{-ic'}(x_2)].
\end{aligned}
\tag{4.3}
$$

From the second line of (4.3) to the forth line, we have integrated ω by contour integral, and assumed $c' > 0$. When $c' < 0$, just replace c' with $-c'$. Using the asymptotical behavior of Bessel function at large order (3.12), we see that $J_{i\omega}(x_1)J_{-i\omega}(x_2) \sim e^{iQ\omega(t_1-t_2)+\pi\omega-\ln\omega}$. When $t_1 - t_2 > 0$, we should close the contour in the upper half plane, and otherwise the lower half plane. The poles at $\pm 2ni$ cancels with the sum in the third line of (4.3). So only the poles at $\pm(c+i\epsilon)$ contribute to the propagator. It is difficult to obtain a compact result. We will take a $b \to \infty$ limit to obtain the asymptotic behavior. Or equivalently, we let $b/(Q)$ to be of order 1, and let $t \to \infty$.

$$G_i(t_2,t_1) \sim -\theta(t_1-t_2)\frac{\cos(x_2+\frac{1}{2}i\pi c'-\frac{\pi}{4})\cos(x_1-\frac{\pi}{4})}{Q\sqrt{x_1x_2}\sinh(\frac{\pi c'}{2})} + (t_1 \leftrightarrow t_2). \tag{4.4}$$

We use $G_i(t,t)$ to calculate b_1, which becomes

$$G_i(t,t) \sim -\frac{[\sin(2x)+1]\coth(\frac{\pi c'}{2})}{2Qx} - \frac{i\cos(2x)}{2Qx}. \tag{4.5}$$

Define

$$\begin{aligned} G_+(1,2) &=< A^+{}_1(t_1)A^+{}_1(t_2) >=< A^+{}_2(t_1)A^+{}_2(t_2) >, \\ G_-(1,2) &=< A^-{}_1(t_1)A^-{}_1(t_2) >=< A^-{}_2(t_1)A^-{}_2(t_2) > . \end{aligned} \tag{4.6}$$

They satisfy the following differential equations

$$\begin{aligned} (-\partial_{t_1}^2 - b^2e^{2Qt_1} - (c+iQ)^2)G_+(t_2,t_1) &= i\delta(t_1-t_2), \\ (-\partial_{t_1}^2 - b^2e^{2Qt_1} - (c-iQ)^2)G_-(t_2,t_1) &= i\delta(t_1-t_2). \end{aligned} \tag{4.7}$$

The solution G_+ can also be obtained from the heat kernel.

$$\begin{aligned} G_+(t_2,t_1) =& -\int_0^\infty dsK_+(t_2,t_1;s) \\ =& \int_0^\infty d\omega \frac{iQ\omega[J_{i\omega}(x_1)+J_{-i\omega}(x_1)][J_{i\omega}(x_2)+J_{-i\omega}(x_2)]}{2\sinh(\pi\omega)[(Q\omega)^2-(c+iQ)^2]} \\ &+ i\sum_{n=1}^\infty \frac{4QnJ_{2n}(x_1)J_{2n}(x_2)}{-(2Qn)^2-(c+iQ)^2} \\ =& -\frac{\theta(t_1-t_2)\pi}{2Q\sinh[\pi(c'+i)]}J_{-i(c'+i)}(x_2)[J_{i(c'+i)}(x_1)+J_{-i(c'+i)}(x_1)] \\ &-\frac{\theta(t_2-t_1)\pi}{2Q\sinh[\pi(c'+i)]}J_{-i(c'+i)}(x_1)[J_{i(c'+i)}(x_2)+J_{-i(c'+i)}(x_2)] \\ \sim& i\theta(t_1-t_2)\frac{\cos[x_2+\frac{1}{2}i\pi(c'+i)-\frac{\pi}{4}]\cos(x_1-\frac{\pi}{4})}{Q\sqrt{x_1x_2}\cosh(\frac{\pi c'}{2})} + (t_1 \leftrightarrow t_2), \end{aligned} \tag{4.8}$$

and

$$G_+(t,t) \sim -\frac{[\sin(2x)+1]\tanh(\frac{\pi c'}{2})}{2Qx} - \frac{i\cos(2x)}{2Qx}. \tag{4.9}$$

To get G_-, just replace c' by $-c'$ in (4.8).

Then the first order contribution adds up to

$$\begin{aligned} -iV_1 &= -i\int_{-\infty}^{\infty} \frac{dc}{2\pi R'} v^2 t^2 e^{2Qt}[6G_i(t,t) + G_+(t,t) + G_-(t,t)] \\ &\sim \frac{i3v^2t^2e^{Qt}[1+\sin(2x)]}{b}\int_0^{\infty}\frac{Qdc'}{\pi R'}\coth(\frac{\pi c'}{2}) - \frac{4v^2t^2e^{Qt}\cos(2x)}{b}\int_0^{\infty}\frac{Qdc'}{\pi R'} \\ &\sim \frac{i3v^2t^2e^{Qt}}{b}\int_0^{\infty}\frac{Qdc'}{\pi R'}\coth(\frac{\pi c'}{2}). \end{aligned} \tag{4.10}$$

In the last step, we have omitted the trigonometric functions because they are periodic functions and fluctuate violently at large argument. The integral of c' seems to give a divergence, but this is just caused by our using the asymptotic expansion of the bessel function. That step hides the depression of the large c'. In fact, from the large order behavior (3.17), we can see that there is indeed no divergence in the c'. Hereafter, we can just put the this divergence aside.

To compute b_2, we again need to take the limit $|x_1| \gg 1$ and $|x_2| \gg 1$. This is equivalent to $t_1 \gg 1$ and $t_2 \gg 1$. In this case, we use the asymptotic expansion of Bessel function when x_1 and x_2 are large in (4.3) and (4.8). Multiply (4.3) and (4.8), and integrate out $\tau \equiv t_1 - t_2$, we will get the dependence on $x = \frac{be^{Qt}}{Q}$, $\quad t = \frac{1}{2}(t_1+t_2)$. The relevant integral is

$$\begin{aligned} -iV_{b2} =& \int_0^{\infty}\frac{dc}{\pi R'}2v^2e^{2Qt}\int_{-\infty}^{\infty} d\tau G_i(t_1,t_2)[G_+(t_2,t_1)+G_-(t_2,t_1)] \\ \sim& \int_0^{\infty}\frac{dc}{\pi R'}\int_0^{\infty} d\tau \frac{\theta(t_1-t_2)v^2}{b^2}\{i\coth(\frac{\pi c'}{2})[\cos(2x_2)+\frac{1}{2}\sin(2x_1+2x_2) \\ &+\frac{1}{2}\sin(2x_1-2x_2)] - [1-\sin(2x_2)+\sin(2x_1) \\ &-\frac{1}{2}\cos(2x_1-2x_2)+\frac{1}{2}\cos(2x_1+2x_2)]\} + (t_1 \leftrightarrow t_2) \\ \sim& \int_0^{\infty}\frac{dc'}{\pi R'}\frac{v^2}{b^2}\{i\pi\coth(\frac{\pi c'}{2})[I_0(4x)-\mathbf{L}_0(4x)] - \int_{-\infty}^{\infty}Qd\tau + 2K_0(4x)\}. \end{aligned} \tag{4.11}$$

Considering $x_1 \gg 1$, and $x_2 \gg 1$, The trigonometric functions with argument x_1, x_2, and $2x_1+2x_2$ fluctuate quickly in the $x \gg 1$ limit, they average to zero and hence can be omitted. The term $\int_{-\infty}^{\infty} Qd\tau$ seems to be divergent. However, remember the limit we are taking here, $t_1, t_2 \gg 1$, and t fixed, so the range of both t_1 and t_2 is proportional to t. Then the range of $\tau = t_1 - t_2$ is also proportional to t, and the term proportional to $\int_{-\infty}^{\infty} Q d\tau$ is finite and increases with t.

To calculate f_2, we will need the fermionic propagator, defined by $G_{\alpha,\beta}(t_1, t_2) \equiv \langle T\theta_+(t_1)\theta_-^T(t_2)\rangle_{\alpha\beta}$. It satisfies the following differential equation,

$$[i\partial_{t_1} + be^{Qt_1}\gamma_2 + c\gamma_9]G_{\alpha,\beta}(t_1, t_2) = -i\delta_{\alpha,\beta}\delta(t_1 - t_2). \tag{4.12}$$

The propagator is related to the heat kernel $K_{\alpha\beta}(t_2, t_1; s)$ roughly by $G_{\alpha,\beta}(t_1, t_2) = \int_0^\infty ds K_{\alpha\beta}(t_2, t_1; s)$. But there is some subtly in determining the time ordering in each term. This is related to the boundary conditions. We are not going to solve the problem in this way. Instead, we take the $b \to 0$ limit. The limiting case will be the propagator for massive fermions, which will be analyzed in Appendix A. When $b \to 0$, the argument of the Whittaker function is small. We will have

$$M_{\lambda,\mu}(z) \sim z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}}. \tag{4.13}$$

We determine the time ordered propagator by comparing its small b limit with the propagator of a massive fermion in Appendix A , consequently

$$\begin{aligned}
G_{11}(t_1, t_2) &= (4i\sqrt{x_1 x_2})^{-1}[\theta(t_1 - t_2)M_{-1/2,-ic'}(-2ix_2)M_{1/2,ic'}(-2ix_1) \\
&\qquad -\theta(t_2 - t_1)M_{-1/2,ic'}(-2ix_2)M_{1/2,-ic'}(-2ix_1)] \\
G_{22}(t_1, t_2) &= (4i\sqrt{x_1 x_2})^{-1}[\theta(t_1 - t_2)M_{1/2,-ic'}(-2ix_2)M_{-1/2,ic'}(-2ix_1) \\
&\qquad -\theta(t_2 - t_1)M_{1/2,ic'}(-2ix_2)M_{-1/2,-ic'}(-2ix_1)] \\
G_{12}(t_1, t_2) &= (4i\sqrt{x_1 x_2})^{-1}[\theta(t_1 - t_2)M_{1/2,-ic'}(-2ix_2)M_{1/2,ic'}(-2ix_1) \\
&\qquad +\theta(t_2 - t_1)M_{1/2,ic'}(-2ix_2)M_{1/2,-ic'}(-2ix_1)] \\
G_{21}(t_1, t_2) &= (4i\sqrt{x_1 x_2})^{-1}[\theta(t_1 - t_2)M_{-1/2,-ic'}(-2ix_2)M_{-1/2,ic'}(-2ix_1) \\
&\qquad +\theta(t_2 - t_1)M_{-1/2,ic'}(-2ix_2)M_{-1/2,-ic'}(-2ix_1)].
\end{aligned} \tag{4.14}$$

Again let $b/(Q)$ to be of order 1, and take the $t_{1,2} \gg 1$ limit also, we will get a finite integral with respect to both τ. In the following, we will need to use the asymptotical expansion of Whittaker function at large argument.

$$\begin{aligned}
M_{\frac{1}{2},\pm ic'}(-2ix) &\sim \frac{\Gamma(\pm 2ic' + 1)}{\Gamma(\pm ic' + 1)} \exp(\mp\pi c' + ix)\sqrt{-2ix} \\
M_{-\frac{1}{2},\pm ic'}(-2ix) &\sim \frac{\Gamma(\pm 2ic' + 1)}{\Gamma(\pm ic' + 1)} \exp(-ix)\sqrt{-2ix}.
\end{aligned} \tag{4.15}$$

Then

$$
\begin{aligned}
\mathrm{Tr}[\gamma_1 G_f(t_1,t_2)\gamma_1 G_f(t_2,t_1)] = &[G_{11}(t_2,t_1)G_{22}(t_1,t_2) + G_{22}(t_2,t_1)G_{11}(t_1,t_2) \\
&-G_{12}(t_1,t_2)G_{12}(t_2,t_1) - G_{21}(t_1,t_2)G_{21}(t_2,t_1)] \\
=&(8x_1x_2)^{-1}\theta(t_1-t_2)[M^2_{-1/2,-ic'}(-2ix_2)M^2_{1/2,ic'}(-2ix_1) \\
&+ M^2_{1/2,-ic'}(-2ix_2)M^2_{-1/2,ic'}(-2ix_1)] + (t_1 \leftrightarrow t_2) \\
\sim& -\frac{\cosh(2ix_2 - 2ix_1 + 2\pi c')}{\cosh^2(\pi c')}
\end{aligned}
\tag{4.16}
$$

The fermionic contribution to the effective potential is thus

$$
\begin{aligned}
-iV_f \equiv& \int_0^\infty \frac{dc}{\pi R'} \int_{-\infty}^\infty d\tau 8v^2 e^{2Qt}(t^2 - \frac{\tau^2}{4})\mathrm{Tr}[\gamma_1 G_f(t_1,t_2)\gamma_1 G_f(t_2,t_1)] \\
\sim& \int_0^\infty \frac{8dc'}{\pi R'}\{-\frac{v^2 e^{2Qt}\cosh(2\pi c')}{\cosh^2(\pi c')}K_0(x)[4t^2 - \frac{\pi^2}{Q^2}] \\
&-\frac{i2\pi v^2 t^2 e^{2Qt}\sinh(2\pi c')}{\cosh^2(\pi c')}[I_0(4x) - \mathbf{L}_0(4x)]\}
\end{aligned}
\tag{4.17}
$$

From (4.10), (4.11), and (4.17) we see that the effective potential proportional to v^2 does not vanish. The late time potential contains both a real part and an imaginary part. In the following, when we use "proportional to", we mean that we ignore some numerical coefficient, including the the integral of c'. The leading real part comes from the bosons, (4.10), proportional to $\frac{-Qv^2t^2e^{Qt}}{b}$. It increases with t. The leading imaginary part also comes from the bosonic part, (4.11), proportional to $-i\frac{v^2}{b^2}\int_{-\infty}^{\infty} Qd\tau$. This term is finite and increases as t as we have explained following eq.(4.11). We may also pay attention to the subleading terms, which are finite, and may have some physical significance. The subleading imaginary part comes from the fermionic contribution, which is proportional to $-iv^2t^2e^{2Qt}K_0(4x) \sim \frac{-iv^2t^2e^{2Qt}}{\sqrt{\frac{be^{Qt}}{Q}}}e^{\frac{-4e^{Qt}}{Q}}$. The subleading real contribution comes also from the fermionic contribution, which is proportional to $v^2t^2e^{2Qt}[I_0(4x) - \mathbf{L}_0(4x)] \sim \frac{v^2t^2e^{2Qt}}{\sqrt{\frac{be^{Qt}}{Q}}}e^{\frac{-4e^{Qt}}{Q}}$. Both the real part and the imaginary part of the effective potential are proportional to positive power of Q, so when $Q \to 0$, both vanish. The subleading effective potential also vanish as $t \to \infty$.

5. Conclusion and Discussions

We study the effective potential between two D0-branes in a time-dependent matrix theory at the one loop level. When the two D0-branes have no relative motion in the comoving coordinates, we find that there is no effective potential. This result is expected if there is supersymmetry, thanks to the cancelation between bosons and fermions. What is surprising is that there is no supersymmetry in our case. The bosonic and fermionic phase shifts are both divergent but do not dependent on the physical parameter, the seperation b. So upon suitable regularization, they are both zero.

When we consider the case when $v \neq 0$, the exact form of the effective potential is not calculated because of the integrand is too complicated. Too see that the potential is non-trivial, we examined the behavior of the potential in later times. The v^2 corrections do not cancel in one loop calculation. Moreover, there exists an imaginary part in addition to a real part. This result seems to contradict with our supergravity calculation. When we compactify the X^9 direction, we get a type IIA string theory with string coupling constant $g_s = e^{-Qt}$, and the effective 10 dimensional gravitational constant is $\kappa^2 \propto g_s^2 = e^{-2Qt}$. Supergravity loop expansion is in terms of gravitational constant. But we see no sign of this expansion in matrix calculation. Furthermore, the imaginary part of the effective potential may imply an instability of the 2 D0-brane system. As the two D0-branes move apart in the comoving coordinates, certain modes in the two D0-brane system become tachyonic, and the imaginary part just signals creation of these modes.

Acknowledgments

We are grateful to Cheng Guo, Qing Guo Huang, Tao Wang and Peng Zhang for useful discussions. This work was supported by a grant from CNSF. We are also grateful to the referee for pointing out the problem of completeness in our previous version.

Appendix A. Perturbation in flat background

When the background is flat, BFSS matrix model has been tested to two loops. Here we will use our perturbation method to repeat the result to one loop order. Set $b = 0$ and $Q = 0$, we just return to the situation investigated by [14]. The $v = 0$ case is similar. The determinants we are going to compute becomes

$$\begin{aligned} &\det{}^{10}(-\partial_t^2 - c^2) \quad \text{for} \quad Y^\mu_{1,2}, \mu = 1, \cdots 9 \quad \text{and} \quad A_{1,2}, \\ &\det{}^{-2}(-\partial_t^2 - c^2) \quad \text{for} \quad C_{1,2}, \\ &\det{}^{-8}(i\partial_t + c\gamma_9) \quad \text{for} \quad \theta_+. \end{aligned} \tag{A.1}$$

The propagators are $G_b(t,t') = -\frac{1}{2c}e^{ic|t-t'|}$ for all the bosons and the ghosts. For the fermions,

$$G_{11}(t,t') = G_{22}(t,t') = -\epsilon(t-t')e^{ic|t-t'|},$$
$$G_{12}(t,t') = G_{21}(t,t') = -\frac{1}{2}e^{ic|t-t'|}, \tag{A.2}$$

where $\epsilon(t-t') = \frac{1}{2}[\theta(t-t') - \theta(t'-t)]$. For the $v \neq 0$ case, we need

$$\begin{aligned}
b_1 &= -i\int_{-\infty}^{\infty} dt\frac{1}{2}v^2t^2[\langle Y_1^\mu(t)Y_1^\mu(t)\rangle + \langle Y_2^\mu(t)Y_2^\mu(t)\rangle \\
&+ \langle A_1(t)A_1(t)\rangle + \langle A_2(t)A_2(t)\rangle], \\
g_1 &= i\int_{-\infty}^{\infty} dtv^2t^2[\langle C_1(t)C_1^*(t)\rangle + \langle C_2(t)C_2^*(t)\rangle], \\
b_2 &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1dt_2v^2\{\langle A_1(t_1)A_1(t_2)\rangle\langle {Y_2}^1(t_1){Y_2}^1(t_2)\rangle \\
&+ \langle A_2(t_1)A_2(t_2)\rangle\langle {Y_1}^1(t_1){Y_1}^1(t_2)\rangle\}, \\
f_2 &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1dt_2v^2t_1t_2 \times 8Tr[\gamma_1G_f(t_1,t_2)\gamma_1G_f(t_2,t_1)].
\end{aligned} \tag{A.3}$$

In order to get the effective action, we do not need to perform all the integrals. Define $t = \frac{t_1+t_2}{2}$ $\tau = t_1 - t_2$, integrate out τ, and sum over all terms above, we will get the effective potential before the smearing:

$$\begin{aligned}
b_1 &= i\int_{-\infty}^{\infty} dtv^2t^2\frac{5}{c}, \\
g_1 &= -i\int_{-\infty}^{\infty} dtv^2t^2\frac{1}{c}, \\
b_2 &= -i\int_{-\infty}^{\infty} dtv^2\frac{1}{2c^3}, \\
f_2 &= -i\int_{-\infty}^{\infty} dtv^2(\frac{4t^2}{c} - \frac{1}{2c^3}).
\end{aligned} \tag{A.4}$$

The various factors comes from the counting of degree of freedom. They sum up to zero. So there is no v^2 term in the effective action.

Appendix B. The proof of an identity

Here we will give the proof of the following identity

$$D(c,z) \equiv M_{1/2,ic}(z)M_{-1/2,-ic}(z) + M_{1/2,-ic}(z)M_{-1/2,ic}(z) = 2z, \tag{B.1}$$

where z is pure imaginary. In the following, we will treat $D(c,z)$ as a function of c, and view z as a parameter. Using the steepest descendent method, we can get the large $|c|$ behavior of Whittaker function, $M_{\frac{1}{2},\pm ic}(z) \sim z^{\frac{1}{2}\pm ic}, M_{-\frac{1}{2},\pm ic}(z) \sim z^{\frac{1}{2}\pm ic}$, when $|c| \to \infty$. So $\lim_{|c|\to\infty} D(c,z) \sim 2z$. Using $M_{\lambda,\mu}(z) = e^{-z/2} z^{\mu+\frac{1}{2}} \Phi(\mu - \lambda + \frac{1}{2}, 2\mu + 1; z)$, and the relation $\Phi(\alpha,\gamma;z) = e^z \Phi(\alpha - \gamma, \gamma; -z)$, we can write $D(c,z)$ in terms of Φ as

$$D(c,z) = z[\Phi(ic, 2ic+1; z)\Phi(-ic, -2ic+1; -z) + \Phi(ic, 2ic+1; -z)\Phi(-ic, -2ic+1; z)]. \tag{B.2}$$

$\Phi(\alpha,\gamma;z)$ as a function of γ has single poles at $\gamma = -n$, and analytic elsewhere. Near the pole, $\lim_{2ic+1\to -n} \Phi(ic, 2ic+1; z) \sim \frac{(-1)^n}{n!(2ic+1+n)} \begin{pmatrix} \frac{1}{2}(n-1) \\ n+1 \end{pmatrix} z^{n+1} \Phi(\frac{n+1}{2}, n+2; z)$, where

$$\begin{pmatrix} \frac{1}{2}(n-1) \\ n+1 \end{pmatrix} \equiv \frac{1}{2}(n-1)[\frac{1}{2}(n-1) - 1] \cdots [\frac{1}{2}(n-1) - n]/[(n+1)!].$$

If n is odd, the above is zero, so the potential poles in the upper half plane are at $2ic+1 = -2n$. However,

$$\begin{aligned} &\lim_{2ic+1\to -2n} D(c,z) \\ =&\frac{z}{(2n)!(2ic+1+2n)} \begin{pmatrix} \frac{1}{2}(2n-1) \\ 2n+1 \end{pmatrix} \Phi(\frac{2n+1}{2}, 2n+2; z)\Phi(\frac{2n+1}{2}, 2n+2; -z) \\ &\times[z^{2n+1} + (-z)^{2n+1}] = 0. \end{aligned} \tag{B.3}$$

The same phenomenon happens when $2ic+1 \to 2n$. Thus $D(c,z)$ is analytic in the complex plane as a function of c. Since $D(c,z)$ approaches to $2z$ as the $|c| \to \infty$, $D(c,z) = 2z$ by Cauthy integral formula.

References

[1] B. Craps, S. Sethi and E. Verlinde,"A Matrix Big Bang," hep-th/0506180.

[2] M. Li, "A class of cosmological matrix models," hep-th/0506260, Phys.Lett. B626 (2005) 202.

[3] B. Chen, "The Time-dependent Supersymmetric Configurations in M-theory and Matrix Models," hep-th/0508191, B. Chen.

[4] Y-l He and P. Zhang, "Exactly Solvable Model of Superstring in Plane-wave Background with Linear Null Dilaton," hep-th/0509113.

[5] T. Ishino, H. Kodama and N. Ohta, "Time-dependent Solutions with Null Killing Spinor in M-theory and Superstrings", hep-th/0509173,Phys. Lett. B 631 (2005) 68.

[6] S. R. Das and J. Michelson, "pp Wave Big Bangs: Matrix Strings and Shrinking Fuzzy Spheres," hep-th/0508068, Phys.Rev. D72 (2005) 086005.

[7] M. Berkooz, Z. Komargodski, D. Reichmann and V. Shpitalnik, "Flow of Geometries and Instantons on the Null Orbifold," hep-th/0507067.

[8] J.-H. She, "A Matrix Model for Misner Universe and Closed String Tachyons," hep-th/0509067 ;"Winding String Condensation and Noncommutative Deformation of Spacelike Singularity," hep-th/0512299.

[9] Yasuaki Hikida, Rashmi R. Nayak, Kamal L. Panigrahi, "D-branes in a Big Bang/Big Crunch Universe: Misner space," hep-th/0508003,JHEP 0509 (2005) 023.

[10] Yasuaki Hikida and Ta-Sheng Tai,"D-instantons and Closed String Tachyon in Misner Space," hep-th/0510129.

[11] M. Li, W. Song,"Shock waves and cosmological matrix models," hep-th/0507185, JHEP 0510 (2005) 073.

[12] T. Banks, W. Fischler, S. Shenker and L. Susskind, M theory as A Matrix Model: A Conjecture, hep-th/9610043, Phys. Rev. D55 (1997) 5112.

[13] L. F. Abbott, "The Background Field Theory Method Beyond One Loop," *Nucl. Phys.* **B185**(1981) 189.

[14] K.Becker and M.Becker,"A Two-Loop Test of M(atrix) Theory",hep-th/9705091.

[15] I.S.Gradshteyn and L.M.Ryzhik, "Table of integrals, series, and products".

[16] S. Fredenhagen, and V. Schomerus, "On Minisuperspace Models of S-branes", hep-th/0308205, JHEP 0312 (2003) 003.