THE LOGARITHMIC MINKOWSKI PROBLEM FOR POLYTOPES

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Abstract. The logarithmic Minkowski problem asks for necessary and sufficient conditions for a finite Borel measure on the unit sphere so that it is the cone-volume measure of a convex body. This problem was solved recently by Böröczky, Lutwak, Yang and Zhang for even measures (JAMS 2013). This paper solves the case of discrete measures whose supports are in general position.

1. Introduction

A *convex body* in n-dimensional Euclidean space, $\mathbb{R}^n$, is a compact convex set that has non-empty interior. The classical *surface area measure*, S_K, of a convex body K is a Borel measure on the unit sphere, S^{n-1}, defined for a Borel $\omega \subset S^{n-1}$, by

$$S_K(\omega) = \int_{x\in\nu_K^{-1}(\omega)} d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of boundary points of K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.

One of the cornerstones of the Brunn-Minkowski theory is the Minkowski problem. It asks: what are the necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} so that μ is the surface area measure of a convex body in $\mathbb{R}^n$? The answer is: if μ is not concentrated on a great subsphere, then μ is the surface area measure of a convex body if and only if

$$\int_{S^{n-1}} u d\mu(u) = 0;$$

i.e., if μ is considered as a mass distribution on the unit sphere, then its centroid is the origin.

The surface area measure of a convex body has clear geometric significance. Another important measure (defined on the unit sphere) that is associated with a convex body and that has clear geometric importance is the cone-volume measure. If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the *cone-volume measure*, V_K, of K is a Borel measure on S^{n-1} defined for each Borel $\omega \subset S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n}\int_{x\in\nu_K^{-1}(\omega)} x\cdot\nu_K(x)\, d\mathcal{H}^{n-1}(x).$$

The cone-volume measure has many important applications. For instance, by using a probabilistic representation of the normalized volume measure on B_p^n $(:= \{x \in \mathbb{R}^n : \|x\|_p \le 1\})$, Barthe, Guédon, Mendelson and Naor [5] computed moments of linear functionals on B_p^n, which give sharp constants in Khinchine's inequalities on B_p^n and determine the ψ_2-constant of all directions on B_p^n. For additional references regarding cone-volume measure see, e.g., [6, 7, 18, 44, 45, 57, 58, 61, 67].

The Minkowski problem deals with the question of prescribing the surface area measure. An important, natural problem is prescribing the cone-volume measure.

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Logarithmic Minkowski problem: What are the necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} so that μ is the cone-volume measure of a convex body in $\mathbb{R}^n$?

In a special case, this question goes back to to Firey [15]. When $n\mu$ has a density f, with respect to spherical Lebesgue measure, the associated partial differential equation for the logarithmic Minkowski problem is the following Monge-Ampère type equation on S^{n-1}

$$h \det(h_{ij} + h\delta_{ij}) = f, \tag{1.1}$$

where h_{ij} is the covariant derivative of h with respect to an orthonormal frame on S^{n-1} and δ_{ij} is the Kronecker delta.

In [46], Lutwak introduced the notion of the L_p surface area measure and posed the associated L_p Minkowski problem which has the classical Minkowski problem and the logarithmic Minkowski problem as two importance cases.

If $p \in \mathbb{R}$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the L_p surface area measure, $S_p(K, \cdot)$, of K is a Borel measure on S^{n-1} defined for each Borel $\omega \subset S^{n-1}$ by

$$S_p(K, \omega) = \int_{x \in \nu_K^{-1}(\omega)} (x \cdot \nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x).$$

Obviously, $S_1(K, \cdot)$ is the classical surface area measure of K, and $\frac{1}{n}S_0(K, \cdot)$ is the cone-volume measure of K. The notion of the L_p surface area measure has been rapidly attracting much attention; see, e.g., [9, 24, 26, 27, 43, 48–50, 53, 55, 60].

Today, the L_p Minkowski problem is one of the central problems in convex geometric analysis. The problem asks: what are the necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} so that μ is the L_p surface area measure of a convex body in $\mathbb{R}^n$?

When μ has a density f, with respect to spherical Lebesgue measure, the associated partial differential equation for the L_p Minkowski problem is the following Monge-Ampère type equation on S^{n-1}

$$h^{1-p} \det(h_{ij} + h\delta_{ij}) = f. \tag{1.2}$$

Obviously, the L_1 Minkowski problem is the classical Minkowski problem, while the L_0 Minkowski problem is the logarithmic Minkowski problem.

More than a century ago, Minkowski himself completely solved the classical Minkowski problem for the case where the given measure is discrete [56]. The complete solution to this problem for arbitrary measures was given by Aleksandrov, and Fenchel and Jessen (see, e.g., [64] or [1]). Landmark contributions to establishing regularity for the classical Minkowski problem are due to (among others) Lewy [41], Nirenbeng [59], Cheng and Yau [11], Pogorelov [62], and Caffarelli [8].

The even L_p Minkowski problem for $p > 1$ (but with $p \neq n$) was solved by Lutwak [46]. The regular, even L_p Minkowski problem for $p > 1$ (but also with $p \neq n$) was studied by Lutwak and Oliker [47]. In [52], Lutwak, Yang and Zhang showed that for $p \neq n$, the L_p Minkowski problem is equivalent to a volume-normalized L_p Minkowski problem, and in [52] they solved the even volume-normalized L_p Minkowski problem for all $p > 1$. Without the assumption that the measure is even, the L_p Minkowski problem was treated by Guan and Lin [22], and by Chou and Wang [13]. Later, Hug, Lutwak, Yang and Zhang [34] gave an alternate proof of some of the results of Chou and Wang [13], and solved the corresponding volume-normalized L_p Minkowski problem. By now, the L_p Minkowski problem and its extensions, have generated a considerable literature; see, e.g., [6, 10, 13, 21–25, 32–34, 37, 38, 40, 42, 46, 47, 52, 54, 65, 66, 69].

Theory of curvature flows has roots in convexity (see, e.g., [2, 3]) and plays an important role in solving geometric problems. It is known (see, e.g., [2, 3, 12, 16]) that in $\mathbb{R}^n$, homothetic solutions to curvature flows are solutions of the L_p Minkowski problem; and when μ is proportional to Lebesgue measure on S^1, homothetic solutions to isotropic flows (classified by Andrews [3]) are solutions of the L_p Minkowski problem. Remarkable work has been done for the mean curvature flow (see Huisken [35,36]) and the Gauss curvature flow (see, e.g., [3,4,12]) that are related to convex bodies.

The solutions to the Minkowski problem and the L_p Minkowski problem have a number of important applications. For instance, by using the convexification, a notion that depends on the solution of the Minkowski problem, Zhang [72] extended the Petty projection inequality from convex bodies to compact domains, and extended the classical Sobolev inequality to a much stronger one: the affine Sobolev-Zhang inequality. By using the solution of the even L_p Minkowski problem, Lutwak, Yang and Zhang [51] extended the affine Zhang-Sobolev inequality to the L_p affine Zhang-Sobolev inequality, an affine inequality far stronger than the classical L_p Sobolev inequality. Later, Ciachi, Lutwak, Yang and Zhang [14] used the solution of the even L_p Minkowski problem to establish the affine Moser-Trudinger and the affine Morrey-Sobolev inequalities, both inequalities stronger than their classical euclidean counterparts. These affine inequalities were then further strengthened by Haberl and Schuster [28–30], and Wang [70].

Most past work on the L_p Minkowski problem and the Monge-Ampère type PDE (1.2) is limited to the case $p > 1$. When $p < 1$, the L_p Minkowski problem becomes much harder. Even in the case where μ has a non-negative density that equals 0 on a subset of S^{n-1} with positive spherical Lebesgue measure, the Monge-Ampère type PDE (1.2) becomes challenging.

The cone-volume measure is the only one among all the L_p surface area measure that is $\mathrm{SL}(n)$ invariant (see, e.g., [6]). The logarithmic Minkowski problem is clearly the most important of the L_p Minkowski problems, with clear geometric significance because it is the singular case. The polygonal case, in $\mathbb{R}^2$, of the logarithmic Minkowski problem was studied by Stancu [65,66]. In [13], Chou and Wang treated the logarithmic Minkowski problem for the case where the measure has a positive density.

A finite Borel measure μ on S^{n-1} is said to satisfy the *subspace concentration condition* if, for every subspace ξ of $\mathbb{R}^n$, such that $0 < \dim \xi < n$,

$$\mu(\xi \cap S^{n-1}) \leq \frac{\dim \xi}{n} \mu(S^{n-1}), \tag{1.3}$$

and if equality holds in (1.3) for some subspace ξ, then there exists a subspace ξ', that is complementary to ξ in $\mathbb{R}^n$, so that also

$$\mu(\xi' \cap S^{n-1}) = \frac{\dim \xi'}{n} \mu(S^{n-1}).$$

In [6], Böröczky, Lutwak, Yang and Zhang gave the following necessary and sufficient conditions for the existence of solutions to the even logarithmic Minkowski problem.

Theorem A. *A non-zero finite even Borel measure on the unit sphere S^{n-1} is the cone-volume measure of an origin-symmetric convex body in $\mathbb{R}^n$ if and only if it satisfies the subspace concentration condition.*

It is important to point out that the logarithmic Minkowski problem for general measures is much harder than the special case where the measure is a function (the Monge-Ampère type PDE (1.1)). For instance, the subspace concentration condition, which is satisfied by all cone-volume measures of origin-symmetric convex bodies, is also the critical and only condition that is needed for existence. For functions the subspace concentration condition is trivially satisfied but for measure it is precisely what is necessary.

The Minkowski problem and the L_p Minkowski problem for polytopes are of great importance. One reason that the problem for polytopes is so important is that the Minkowski problem and the L_p Minkowski problem (for $p > 1$) for arbitrary measures can be solved by an approximation argument by first solving the polytopal case (see, e.g., [34] or [64], pp.392-393).

It is the aim of this paper to solve the existence question for the logarithmic Minkowski problem for polytopes without the assumption that the measure is even — an assumption that Böröczky, Lutwak, Yang and Zhang were forced to make.

In this paper, a *polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ providing that it has positive n-dimensional volume. The convex hull of a subset of these points is called a *j-dimensional face* (with $1 \leq j \leq n-1$) of the polytope provided it lies entirely on the boundary of the polytope and that it has positive j-dimensional volume. When $j = n-1$, the convex hull, is called a *facet* of the polytope.

If a polytope P contains the origin in its interior, then the *cone-volume* associated with a facet of P is the volume of the convex hull of the facet and the origin. Obviously, if P is a polytope with cone-volumes $\gamma_1, ..., \gamma_N$ and corresponding outer unit normals $u_1, ..., u_N$, then the cone-volume measure of P is given by

$$V_P = \sum_{k=1}^{N} \gamma_k \delta_{u_k},$$

where δ_{u_k} denotes the delta measure that is concentrated at the point u_k.

Definition. A finite set U of unit vectors in $\mathbb{R}^n$ is said to be in *general position* if U is not contained in a closed hemisphere of S^{n-1} and any n elements of U are linearly independent.

Note that, if U has no more than n elements, then U must be contained in a closed hemisphere. Therefore, $N \geq n+1$ is always true if U is in general position.

Polytopes whose facet normals are in general position consist an important class of polytopes. Károlyi and Lovász [39] were among the first to study this class of polytopes and established beautiful decomposition theorem. For convenience, the notion of *general position* in this paper is slightly different from what is defined in [39].

The origin-symmetric convex bodies is an important class of convex bodies. Polytopes whose facet normals are in general position is another important class of convex bodies. Theorem A solved the logarithmic Minkowski problem for even measures. The main theorem in this paper solves the case of discrete measures whose support are in general position:

Theorem. *Let μ be a discrete measure on the unit sphere S^{n-1}. Then μ is the cone-volume measure of a polytope whose outer unit normals are in general position if and only if the supports of μ is in general position.*

In [6], Böröczky, Lutwak, Yang and Zhang (for polytopes see He, Leng and Li [31], and Xiong [71]) showed that if ξ is a subspace of $\mathbb{R}^n$ with $0 < \dim \xi < n$ and μ is the cone-volume measure of an origin-symmetric convex body, then

$$\mu(\xi \cap S^{n-1}) \leq \frac{\dim \xi}{n} \mu(S^{n-1}).$$

This imposes a strict condition on all even measures that may arise as the cone-volume measures of origin-symmetric convex bodies. Surprisingly, without the assumption that the measure be even, the data may be arbitrary. This may provide a possible way to solve the logarithmic Minkowski problem for arbitrary measures, by applying the main theorem in our paper to an approximation argument.

Uniqueness for the logarithmic Minkowski problem was completely settled for even measures in $\mathbb{R}^2$ in [7].

2. Preliminaries

In this section, we standardize some notation and list some basic facts about convex bodies. For general reference regarding convex bodies see, e.g., [17, 19, 20, 64, 68].

The vectors of this paper are column vectors. For $x, y \in \mathbb{R}^n$, we will write $x \cdot y$ for the standard inner product of x and y, and write $|x|$ for the Euclidean norm of x. We write $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ for the boundary of the Euclidean unit ball B^n in $\mathbb{R}^n$. If $x_1, ..., x_{n-1} \in \mathbb{R}^n$ are linearly independent, we write $x_1 \wedge ... \wedge x_{n-1}$ for the unique vector for which

$$\det\big(x_1, ..., x_{n-1}, x_1 \wedge ... \wedge x_{n-1}\big) > 0$$

and $|x_1 \wedge ... \wedge x_{n-1}|$ is equal to the $(n-1)$-dimensional volume of the parallelotope

$$\{\lambda_1 x_1 + ... + \lambda_{n-1} x_{n-1} : 0 \leq \lambda_1, ..., \lambda_{n-1} \leq 1\}.$$

If $1 \leq i \leq n$, K is a Borel subset of $\mathbb{R}^n$, and K is contained in an i-dimensional affine subspace of $\mathbb{R}^n$ but not in any affine subspace of lower dimension, then let $|K|$ be the i-dimensional Lebesgue measure of K.

If K, L are two compact sets in $\mathbb{R}^n$, their *Minkowski sum* $K + L$ is defined by

$$K + L = \{x + y : x \in K, y \in L\},$$

and for $c > 0$, the *scalar multiplication* cK is defined by

$$cK = \{cx : x \in K\}.$$

The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ of a compact convex set K is defined, for $x \in \mathbb{R}^n$, by

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$

Obviously, for $c > 0$ and $x \in \mathbb{R}^n$,

$$h(cK, x) = h(K, cx) = ch(K, x).$$

The *Hausdorff distance* of two compact sets K, L in $\mathbb{R}^n$ is defined by

$$\delta(K, L) = \inf\{t \geq 0 : K \subset L + tB^n, L \subset K + tB^n\}.$$

It is known that the Hausdorff distance between two convex bodies, K and L, is

$$\delta(K, L) = \max_{u \in S^{n-1}} |h(K, u) - h(L, u)|.$$

For a convex body K in $\mathbb{R}^n$, and $u \in S^{n-1}$, the *support hyperplane* $H(K, u)$ in direction u is defined by

$$H(K, u) = \{x \in \mathbb{R}^n : x \cdot u = h(K, u)\},$$

the *half-space* $H^-(K, u)$ in direction u is defined by

$$H^-(K, u) = \{x \in \mathbb{R}^n : x \cdot u \leq h(K, u)\},$$

and the *support set* $F(K, u)$ in direction u is defined by

$$F(K, u) = K \cap H(K, u).$$

For a compact $K \in \mathbb{R}^n$, the diameter of it is defined by

$$d(K) = \max\{|x - y| : x, y \in K\}.$$

If K_1, K_2 are subset of $\mathbb{R}^n$, define

$$d(K_1, K_2) = \inf\{|x_1 - x_2| : x_1 \in K_1, x_2 \in K_2\}.$$

Let $\mathcal{P}$ be the set of polytopes in $\mathbb{R}^n$. If $u_1, ..., u_N \in S^{n-1}$ in general position, let $\mathcal{P}(u_1, ..., u_N)$ be the subset of $\mathcal{P}$ such that a polytope $P \in \mathcal{P}(u_1, ..., u_N)$ if

$$P = \bigcap_{k=1}^{N} H^-(P, u_k).$$

Obviously, if $P \in \mathcal{P}(u_1, ..., u_N)$, then P has at most N facets, and the outer unit normals of P are a subset of $\{u_1, ..., u_N\}$. Let $\mathcal{P}_N(u_1, ..., u_N)$ be the subset of $\mathcal{P}(u_1, ..., u_N)$ such that a polytope $P \in \mathcal{P}_N(u_1, ..., u_N)$ if, $P \in \mathcal{P}(u_1, ..., u_N)$, and P has exactly N facets.

The following lemma will be needed.

Lemma 2.1. *If $1 \leq j \leq n-2$, $u_1, ..., u_N \in S^{n-1}$ are in general position and $P \in \mathcal{P}(u_1, ..., u_N)$, then each j-dimensional face of P is the intersection of $n-j$ facets of P.*

Proof. Since $u_1, ..., u_N$ are in general position and the outer unit normals of P are a subset of $\{u_1, ..., u_N\}$, the dimensions of the intersection of any m $(m > n-j)$ facets of P is less than j. On the other hand, each j-dimensional face of P is the intersection of the family (containing at least $n-j$ members) of facets of P (see, e.g., [20], pp.35, Theorem 7). Therefore, each j-dimensional face of P is the intersection of $n-j$ facets of P. $\square$

3. An extreme problem

In this section, we study an extreme problem. Its solution also solves the logarithmic Minkowski problem. The analogous problem was studied by Chou and Wang [13].

Let $\gamma_1, ..., \gamma_N \in \mathbb{R}^+$, $u_1, ..., u_N \in S^{n-1}$ are in general position and $P \in \mathcal{P}(u_1, ..., u_N)$. Define $\Phi_P :$ Int $(P) \to \mathbb{R}$ by

$$\Phi_P(\xi) = \sum_{k=1}^{N} \gamma_k \log(h(P, u_k) - \xi \cdot u_k).$$

Lemma 3.1. *If $\gamma_1, ..., \gamma_N \in \mathbb{R}^+$, the unit vectors $u_1, ..., u_N$ are in general position and $P \in \mathcal{P}(u_1, ..., u_N)$, then there exists a unique point $\xi(P) \in$ Int (P) such that*

$$\Phi_P(\xi(P)) = \max_{\xi \in \mathit{Int}\ (P)} \Phi_P(\xi).$$

Proof. Since $\log t$ is strictly concave on $(0, \infty)$, for $0 < \lambda < 1$ and $\xi_1, \xi_2 \in$ Int (P),

$$\begin{aligned}
\lambda\Phi_P(\xi_1) + (1-\lambda)\Phi_P(\xi_2) &= \lambda \sum_{k=1}^{N} \gamma_k \log(h(P, u_k) - \xi_1 \cdot u_k) + (1-\lambda) \sum_{k=1}^{N} \gamma_k \log(h(P, u_k) - \xi_2 \cdot u_k) \\
&= \sum_{k=1}^{N} \gamma_k \left[\lambda \log(h(P, u_k) - \xi_1 \cdot u_k) + (1-\lambda) \log(h(P, u_k) - \xi_2 \cdot u_k)\right] \\
&\leq \sum_{k=1}^{N} \gamma_k \log\left[h(P, u_k) - (\lambda\xi_1 + (1-\lambda)\xi_2) \cdot u_k\right] \\
&= \Phi_P(\lambda\xi_1 + (1-\lambda)\xi_2),
\end{aligned}$$

with equality if and only if $\xi_1 \cdot u_k = \xi_2 \cdot u_k$ for all $k = 1, ..., N$. Since $u_1, ..., u_N$ are in general position, $\xi_1 = \xi_2$. Thus, Φ_P is strictly concave on Int (P).

Since $P \in \mathcal{P}(u_1, ..., u_N)$, for any $x \in \partial P$, there exists a $u_{i_0} \in \{u_1, ..., u_N\}$ such that

$$h(P, u_{i_0}) = x \cdot u_{i_0}.$$

Thus, $\Phi_P(\xi) \to -\infty$ whenever $\xi \in \text{Int }(P)$ and $\xi \to \partial P$. Then, there exists a unique interior point $\xi(P)$ of P such that

$$\Phi_P(\xi(P)) = \max_{\xi \in \text{Int }(P)} \Phi_P(\xi).$$

□

By definition, for $\lambda > 0$ and $P \in \mathcal{P}(u_1, ..., u_N)$,

$$\xi(\lambda P) = \lambda \xi(P). \tag{3.1}$$

Obviously, if $P_i \in \mathcal{P}(u_1, ..., u_N)$ and P_i converges to a polytope P, then $P \in \mathcal{P}(u_1, ..., u_N)$.

Lemma 3.2. *If the unit vectors $u_1, ..., u_N$ are in general position, $P_i \in \mathcal{P}(u_1, ..., u_N)$ and P_i converges to a polytope P, then $\lim_{i\to\infty} \xi(P_i) = \xi(P)$ and*

$$\lim_{i\to\infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

Proof. Let $a_0 = \min_{u\in S^{n-1}}\{h(P,u) - \xi(P)\cdot u\} > 0$. Since P_i converges to P and $\xi(P) \in \text{Int }(P)$, there exists a $N_0 > 0$ such that

$$h(P_i, u_k) - \xi(P) \cdot u_k > \frac{a_0}{2},$$

for all $k = 1, ..., N$, whenever $i > N_0$. Thus,

$$\Phi_{P_i}(\xi(P_i)) \geq \Phi_{P_i}(\xi(P)) > (\sum_{k=1}^N \gamma_k) \log \frac{a_0}{2}, \tag{3.2}$$

whenever $i > N_0$.

From the conditions, $\xi(P_i)$ is bounded. Suppose that $\xi(P_i)$ does not converge to $\xi(P)$, then there exists a subsequence P_{i_j} of P_i such that $P_{i_j} \to P$, $\xi(P_{i_j}) \to \xi_0$, but $\xi_0 \neq \xi(P)$. Obviously, $\xi_0 \in P$. We claim that ξ_0 is not a boundary point of P, otherwise $\lim_{j\to\infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) = -\infty$; this is a contradiction with Equation (3.2). If ξ_0 is an interior point of P with $\xi_0 \neq \xi(P)$, then

$$\begin{aligned}\lim_{j\to\infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) &= \Phi_P(\xi_0) \\ &< \Phi_P(\xi(P)) \\ &= \lim_{j\to\infty} \Phi_{P_{i_j}}(\xi(P)).\end{aligned}$$

This is a contradiction with the fact that

$$\Phi_{P_{i_j}}(\xi(P_{i_j})) \geq \Phi_{P_{i_j}}(\xi(P)).$$

Therefore, $\lim_{i\to\infty} \xi(P_i) = \xi(P)$ and thus,

$$\lim_{i\to\infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

□

Consider the extreme problem,

$$\inf\left\{\max_{\xi\in\text{Int }(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, ..., u_N) \text{ and } V(Q) = \sum_{k=1}^N \gamma_k\right\}.$$

Lemma 3.3. *If $\gamma_1, ..., \gamma_N \in \mathbb{R}^+$, the unit vectors $u_1, ..., u_N$ are in general position and there exists a $P \in \mathcal{P}_N(u_1, ..., u_N)$ with $\xi(P) = o$, $V(P) = \sum_{k=1}^N \gamma_k$ such that*

$$\Phi_P(o) = \inf\left\{\max_{\xi\in Int\ (Q)} \Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, ..., u_N) \textit{ and } V(Q) = \sum_{k=1}^N \gamma_k\right\}.$$

Then,

$$V_P = \sum_{k=1}^{N} \gamma_k \delta_{u_k}.$$

Proof. When noticing Equation (3.1), it is sufficient to establish the lemma under the assumption that $\sum_{k=1}^{N} \gamma_k = 1$.

From the conditions, there exists a polytope $P \in \mathcal{P}_N(u_1, ..., u_N)$ with $\xi(P) = o$ and $V(P) = 1$ such that

$$\Phi_P(o) = \inf \left\{ \max_{\xi \in \text{Int }(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, ..., u_N) \text{ and } V(Q) = 1 \right\},$$

where $\Phi_Q(\xi) = \sum_{k=1}^{N} \gamma_k \log(h(Q, u_k) - \xi \cdot u_k)$.

For $\delta_1, ..., \delta_N \in \mathbb{R}$, choose $|t|$ small enough so that the polytope P_t defined by

$$P_t = \bigcap_{i=1}^{N} \{x : x \cdot u_i \leq h(P, u_i) + t\delta_i\}$$

has exactly N facets. Then,

$$V(P_t) = V(P) + t \left(\sum_{i=1}^{N} \delta_i S_i \right) + o(t),$$

where $S_i = |F(P, u_i)|$. Thus,

$$\lim_{t \to 0} \frac{V(P_t) - V(P)}{t} = \sum_{i=1}^{N} \delta_i S_i.$$

Let $\lambda(t) = V(P_t)^{-\frac{1}{n}}$, then $\lambda(t)P_t \in \mathcal{P}_N(u_1, ..., u_N)$, $V(\lambda(t)P_t) = 1$ and

$$\lambda'(0) = -\frac{1}{n} \sum_{i=1}^{N} \delta_i S_i. \tag{3.3}$$

Let $\xi(t) = \xi(\lambda(t)P_t)$, and

$$\begin{aligned} \Phi(t) &= \max_{\xi \in \lambda(t)P_t} \sum_{k=1}^{N} \gamma_k \log(h(\lambda(t)P_t, u_k) - \xi \cdot u_k) \\ &= \sum_{k=1}^{N} \gamma_k \log(\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k). \end{aligned} \tag{3.4}$$

From the definition of $\xi(t)$, Equation (3.4) and the fact that $\xi(t)$ is an interior point of $\lambda(t)P_t$, we have

$$\sum_{k=1}^{N} \gamma_k \frac{u_{k,i}}{\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k} = 0, \tag{3.5}$$

for $i = 1, ..., n$, where $u_k = (u_{k,1}, ..., u_{k,n})^T$. As a special case when $t = 0$, we have

$$\sum_{k=1}^{N} \gamma_k \frac{u_{k,i}}{h(P, u_k)} = 0,$$

for $i = 1, ..., n$. Therefore,

$$\sum_{k=1}^{N} \gamma_k \frac{u_k}{h(P, u_k)} = 0. \tag{3.6}$$

Let

$$F_i(t,\xi_1,...,\xi_n)=\sum_{k=1}^{N}\gamma_k\frac{u_{k,i}}{\lambda(t)h(P_t,u_k)-(\xi_1u_{k,1}+...+\xi_nu_{k,n})}$$

for $i=1,...,n$. Then,

$$\left.\frac{\partial F_i}{\partial \xi_j}\right|_{(0,...,0)}=\sum_{k=1}^{N}\frac{\gamma_k}{h(P,u_k)^2}u_{k,i}u_{k,j}.$$

Thus,

$$\left(\left.\frac{\partial F}{\partial \xi}\right|_{(0,...,0)}\right)_{n\times n}=\sum_{k=1}^{N}\frac{\gamma_k}{h(P,u_k)^2}u_ku_k^T,$$

where $u_ku_k^T$ is a $n\times n$ matrix. For any $x\in\mathbb{R}^n$ with $x\neq 0$, from the fact that $u_1,...,u_N$ are in general position, there exists a $u_{i_0}\in\{u_1,...,u_N\}$ such that $u_{i_0}\cdot x\neq 0$. Then,

$$\begin{aligned}x^T\left(\sum_{k=1}^{N}\frac{\gamma_k}{h(P,u_k)^2}u_ku_k^T\right)x&=\sum_{k=1}^{N}\frac{\gamma_k}{h(P,u_k)^2}(x\cdot u_k)^2\\&\geq\frac{\gamma_{i_0}}{h(P,u_{i_0})^2}(x\cdot u_{i_0})^2>0.\end{aligned}$$

Thus, $(\frac{\partial F}{\partial \xi}\big|_{(0,...,0)})$ is positive defined. From this, the fact that $\xi(0)=0$, Equation (3.5), and the inverse function theorem,

$$\xi'(0)=(\xi_1'(0),...,\xi_n'(0))$$

exists.

From the fact that $\Phi(0)$ is a minimizer of $\Phi(t)$ (in Equation (3.4)), Equation (3.3), the fact $\sum_{k=1}^{N}\gamma_k=1$ and Equation (3.6), we have

$$\begin{aligned}0&=\Phi'(0)\\&=\sum_{k=1}^{N}\gamma_k\frac{\lambda'(0)h(P,u_k)+\lambda(0)\frac{dh(P_t,u_k)}{dt}\big|_{t=0}-\xi'(0)\cdot u_k}{h(P,u_k)}\\&=\sum_{k=1}^{N}\gamma_k\frac{-\frac{1}{n}(\sum_{i=1}^{N}\delta_iS_i)h(P,u_k)+\delta_k-\xi'(0)\cdot u_k}{h(P,u_k)}\\&=-\sum_{i=1}^{N}\frac{S_i\delta_i}{n}+\sum_{k=1}^{N}\frac{\gamma_k\delta_k}{h(P,u_k)}-\xi'(0)\cdot\left[\sum_{k=1}^{N}\gamma_k\frac{u_k}{h(P,u_k)}\right]\\&=\sum_{k=1}^{N}\left(\frac{\gamma_k}{h(P,u_k)}-\frac{S_k}{n}\right)\delta_k.\end{aligned}$$

Since $\delta_1,...,\delta_N$ are arbitrary, $\gamma_k=\frac{1}{n}h(P,u_k)S_k$ for $k=1,...,N$. $\square$

4. The compactness of the extreme problem

In this section, we prove that if a sequence of polytopes from $\mathcal{P}(u_1,...,u_N)$ has a bounded volume it also has a bounded diameter.

Lemma 4.1. *If the unit vectors $u_1,...,u_N$ are in general position and $P\in\mathcal{P}(u_1,...,u_N)$, then $F(P,u_i)$ is either a point or a facet for all $1\leq i\leq N$. Moreover, if $n\geq 3$ and $F(P,u_i)$ is a facet, then the outer unit normals of $F(P,u_i)$ (in $H(P,u_i)$) are in general position.*

Proof. Since

$$\begin{aligned} F(P,u_i) &= H(P,u_i) \cap P \\ &= H(P,u_i) \cap \left(\cap_{k=1}^N H^-(P,u_k) \right) \\ &= \cap_{k=1}^N \left(H(P,u_i) \cap H^-(P,u_k) \right), \end{aligned}$$

$F(P,u_i)$ is a lower dimensional polytope or a point.

For the first part, if $F(P,u_i)$ is a m-dimensional polytope with $1 \leq m \leq n-2$ for some i ($1 \leq i \leq N$), then from Lemma 2.1, there exist $u_{i_1},...,u_{i_{n-m}} \in \{u_1,...,u_N\} \backslash \{u_i\}$ with $F(P,u_{i_1}),...,F(P,u_{i_{n-m}})$ are facets of P such that

$$\begin{aligned} F(P,u_i) &= F(P,u_{i_1}) \cap ... \cap F(P,u_{i_{n-m}}) \\ &= P \cap H(P,u_{i_1}) \cap ... \cap H(P,u_{i_{n-m}}). \end{aligned}$$

Thus,

$$\begin{aligned} F(P,u_i) &= F(P,u_i) \cap H(P,u_i) \\ &= P \cap H(P,u_i) \cap H(P,u_{i_1}) \cap ... \cap H(P,u_{i_{n-m}}). \end{aligned}$$

On the other hand, $u_i, u_{i_1},...,u_{i_{n-1}}$ are linearly independent. Thus, the dimensions of

$$P \cap H(P,u_i) \cap H(P,u_{i_1}) \cap ... \cap H(P,u_{i_{n-m}})$$

are no more than $m-1$. This is a contradiction. Therefore, $F(P,u_i)$ is either a point or a facet of P.

We now turn to the second part. We only need to prove that every $n-1$ distinct vectors chosen from the outer unit normals of $F(P,u_i)$ (in $H(P,u_i)$) are linearly independent. If it is not correct, then from Lemma 2.1 there exist $1 \leq i_1 < ... < i_{n-1} \leq N$ with $i_j \neq i$ for all $1 \leq j \leq n-1$ and $v_1,...,v_{n-1} \in S^{n-1} \cap u_i^\perp$ such that

$$H(P,u_{i_1}) \cap F(P,u_i), ..., H(P,u_{i_{n-1}}) \cap F(P,u_i)$$

are $(n-2)$-dimensional facets of $F(P,u_i)$, $v_1,...,v_{n-1}$ are the corresponding outer unit normals in $H(P,u_i)$, and $v_1,...,v_{n-1}$ are linearly dependent. Then, there exists a vector $v \in S^{n-1} \cap u_i^\perp$ such that

$$v \cdot v_1 = ... = v \cdot v_{n-1} = 0.$$

Thus, v is parallel to $H(P,u_{i_j}) \cap H(P,u_i)$ for all $1 \leq j \leq n-1$. We have,

$$v \cdot u_i = v \cdot u_{i_1} = ... = v \cdot u_{i_{n-1}} = 0.$$

This is a contradiction with the fact that $u_i, u_{i_1},...,u_{i_{n-1}}$ are linearly independent. $\square$

Lemma 4.2. *If the unit vectors $u_1,...,u_N$ are in general position, $1 \leq i_1 < ... < i_{n-1} \leq N$, and $P \in \mathcal{P}(u_1,...,u_N)$, then*

$$L_{i_1,...,i_{n-1}} = \bigcap_{j=1}^{n-1} F(P,u_{i_j})$$

is either empty, a point, or a 1-dimensional face of P. Moreover, if $n \geq 3$ and $L_{i_1,...,i_{n-1}}$ is a 1-dimensional face of P, then $F(P,u_{i_1}),...,F(P,u_{i_{n-1}})$ are facets of P and $L_{i_1,...,i_{n-1}}$ is parallel to

$$u_{i_1} \wedge ... \wedge u_{i_{n-1}}.$$

Proof. Since $u_1,...,u_N$ are in general position,

$$\bigcap_{j=1}^{n-1} H(P,u_{i_j})$$

is a line. Thus,

$$\begin{aligned} L_{i_1,...,i_{n-1}} &= \cap_{j=1}^{n-1} F(P, u_{i_j}) \\ &= P \cap (\cap_{j=1}^{n-1} H(P, u_{i_j})) \end{aligned}$$

is either empty, a point, or a 1-dimensional face of P. On the other hand, from Lemma 4.1, $F(P, u_j)$ is either a point or a facet of P for all $1 \leq j \leq N$. Thus, if $L_{i_1,...,i_{n-1}}$ is a 1-dimensional face of P, then $F(P, u_{i_1}), ..., F(P, u_{i_{n-1}})$ are facets of P and $L_{i_1,...,i_{n-1}}$ is parallel to $u_{i_1} \wedge ... \wedge u_{i_{n-1}}$. □

Theorem 4.3. *If the unit vectors $u_1, ..., u_N$ are in general position, $P_i \in \mathcal{P}(u_1, ..., u_N)$ with $o \in P_i$ and $V(P_i) = 1$, then P_i is bounded.*

Proof. We only need to prove that if the unit vectors $u_1, ..., u_N$ are in general position, $P_i \in \mathcal{P}(u_1, ..., u_N)$ and $d(P_i)$ is not bounded, then $V(P_i)$ is not bounded.

We proceed by induction on the dimensions of the ambient space, $\mathbb{R}^n$. When $n = 2$, let

$$c_2 = \min \left\{ \sqrt{1 - |u_i \cdot u_j|^2} : 1 \leq i < j \leq N \right\}.$$

Since $u_1, ..., u_N$ are in general position, $c_2 > 0$.

Obviously, all the facets (line segments) of P_i are from the finite set

$$\{F(P_i, u_k) : k = 1, ..., N\}.$$

We note that $F(P_i, u_k)$ may be a point, in this case, $|F(P_i, u_k)| = 0$. Then,

$$d(P_i) < \sum_{k=1}^{N} |F(P_i, u_k)|.$$

Since $d(P_i)$ is not bounded, there exists an i_0 $(1 \leq i_0 \leq N)$ such that $|F(P_i, u_{i_0})|$ is not bounded. Since

$$\sum_{k \neq i_0} |F(P_i, u_k)| > |F(P_i, u_{i_0})|,$$

there exists an i_1 $(1 \leq i_1 \leq N)$ different from i_0 such that

$$|F(P_i, u_{i_1})| > \frac{1}{N-1} |F(P_i, u_{i_0})|$$

is not bounded. Thus, for any $M > 0$, there exists an i such that

$$|F(P_i, u_{i_0})| > M$$

and

$$|F(P_i, u_{i_1})| > M.$$

Let $x_0 \in F(P_i, u_{i_1})$ such that

$$d(x_0, H(P_i, u_{i_0})) = \max\{d(x, H(P_i, u_{i_0})) : x \in F(P_i, u_{i_1})\}.$$

Then,

$$\text{Conv } (F(P_i, u_{i_0}) \cup \{x_0\}) \subset \text{Conv } (F(P_i, u_{i_0}) \cup F(P_i, u_{i_1})).$$

Let

$$\text{Proj }_{u_{i_0}}(F(P_i, u_{i_1}))$$

be the projection of $F(P_i, u_{i_1})$ on the line

$$\{tu_{i_0} : t \in \mathbb{R}\}.$$

Since $H(P_i, u_{i_0})$ is a support line of P_i, $F(P_i, u_{i_1})$ lies on one side of $H(P_i, u_{i_0})$. Then,

$$\begin{aligned} d(x_0, H(P_i, u_{i_0})) &\geq \left| \text{Proj }_{u_{i_0}} (F(P_i, u_{i_1})) \right| \\ &= |F(P_i, u_{i_1})| \cdot \sqrt{1 - |u_{i_0} \cdot u_{i_1}|^2} \\ &\geq c_2 M. \end{aligned}$$

Thus,

$$\begin{aligned} S(P_i) &\geq S\left(\text{Conv } (F(P_i, u_{i_0}) \cup F(P_i, u_{i_1}))\right) \\ &\geq S\left(\text{Conv } (F(P_i, u_{i_0}) \cup \{x_0\})\right) \\ &= \frac{1}{2} |F(P_i, u_{i_0})| \cdot d(x_0, H(P_i, u_{i_0})) \\ &\geq \frac{c_2}{2} M^2. \end{aligned}$$

Therefore, $S(P_i)$ is not bounded.

Suppose the Lemma is true for dimensions $n-1$; we next prove that the theorem is true for dimensions n. For dimensions n ($n \geq 3$), let

$$c_n = \min \left\{ \left| u_i \cdot \frac{u_{i_1} \wedge ... \wedge u_{i_{n-1}}}{|u_{i_1} \wedge ... \wedge u_{i_{n-1}}|} \right| : 1 \leq i_1 < ... < i_{n-1} \leq N, i \neq i_j \right\}.$$

Since $u_1, ..., u_N$ are in general position, $c_n > 0$.

From Lemma 2.1 and Lemma 4.2, the set of the 1-dimensional faces of P_i is a subset of

$$\left\{ \bigcap_{j=1}^{n-1} F(P_i, u_{i_j}) : 1 \leq i_1 < ... < i_{n-1} \leq N \right\}.$$

We have

$$d(P_i) \leq \sum_{1 \leq j_1 < ... < j_{n-1} \leq N} |L_{i;j_1,...,j_{n-1}}|,$$

where $|L_{i;j_1,...,j_{n-1}}|$ is the length of $\cap_{k=1}^{n-1} F(P_i, u_{j_k})$ and equals 0 whenever the intersection is a point or empty. Since $d(P_i)$ is not bounded, there exist $i_1, ..., i_{n-1}$ (with $1 \leq i_1 < ... < i_{n-1} \leq N$) such that the length of

$$L_{i;i_1,...,i_{n-1}} = \bigcap_{j=1}^{n-1} F(P_i, u_{i_j})$$

is not bounded. On the other hand

$$\left(\sum_{1 \leq j_1 < ... < j_{n-1} \leq N} |L_{i;j_1,...,j_{n-1}}| \right) - |L_{i;i_1,...,i_{n-1}}| > |L_{i;i_1,...,i_{n-1}}|.$$

Thus, there exist $1 \leq i'_1 < ... < i'_{n-1} \leq N$ such that $L_{i;i_1,...,i_{n-1}} \neq L_{i;i'_1,...,i'_{n-1}}$ and

$$|L_{i;i'_1,...,i'_{n-1}}| > \frac{1}{\binom{N}{n-1} - 1} |L_{i;i_1,...,i_{n-1}}|$$

is not bounded. Without loss of generality, we can suppose

$$i'_1 \neq i_j$$

for all $1 \leq j \leq n-1$.

Therefore, there exists a subsequence i_k of i such that

$$\lim_{k \to \infty} |L_{i_k;i_1,...,i_{n-1}}| = \lim_{k \to \infty} |L_{i_k;i'_1,...,i'_{n-1}}| = \infty.$$

From this and Lemma 4.1, when i_k is big enough, $F(P_{i_k}, u_{i'_1})$ is a facet of P_{i_k}, and the outer unit normals of $F(P_{i_k}, u_{i'_1})$ (in $H(P_{i_k}, u_{i'_1})$, and is a subset of the unit normals of $H(P_{i_k}, u_j) \cap H(P_{i_k}, u_{i'_1})$ for $j \neq i'_1$) are in general position (in $H(P_{i_k}, u_{i'_1})$). By the inductive hypothesis, $|F(P_{i_k}, u_{i'_1})|$ is not bounded. Thus, for any $M > 0$ there exists an i such that $L_{i;i_1,\dots,i_{n-1}}$ and $L_{i;i'_1,\dots,i'_{n-1}}$ are 1-dimensional faces of P_i, $F(P_i, u_{i'_1})$ is a facet of P_i with

$$|L_{i;i_1,\dots,i_{n-1}}| > M,$$

and

$$|F(P_i, u_{i'_1})| > M.$$

Let $x_0 \in L_{i;i_1,i_2,\dots,i_{n-1}}$ such that

$$d(x_0, H(P_i, u_{i'_1})) = \max\{d(x, H(P_i, u_{i'_1})) : x \in L_{i;i_1,i_2,\dots,i_{n-1}}\}.$$

Then,

$$\text{Conv }\left(F(P_i, u_{i'_1}) \cup \{x_0\}\right) \subset \text{Conv }\left(F(P_i, u_{i'_1}) \cup L_{i;i_1,i_2,\dots,i_{n-1}}\right).$$

Let

$$\text{Proj }_{u_{i'_1}}\left(L_{i;i_1,i_2,\dots,i_{n-1}}\right)$$

be the projection of $L_{i;i_1,i_2,\dots,i_{n-1}}$ on the line

$$\{tu_{i'_1} : t \in \mathbb{R}\}.$$

Since $H(P_i, u_{i'_1})$ is a support hyperplane of P_i, $L_{i;i_1,i_2,\dots,i_{n-1}}$ lies on one side of $H(P_i, u_{i'_1})$. From this and Lemma 4.2, we have

$$\begin{aligned} d(x_0, H(P_i, u_{i'_1})) &\geq \left|\text{Proj }_{u_{i'_1}} L_{i;i_1,i_2,\dots,i_{n-1}}\right| \\ &= |L_{i;i_1,i_2,\dots,i_{n-1}}| \cdot \left|u_{i'_1} \cdot \frac{u_{i_1} \wedge u_{i_2} \wedge \dots \wedge u_{i_{n-1}}}{|u_{i_1} \wedge u_{i_2} \wedge \dots \wedge u_{i_{n-1}}|}\right| \\ &\geq c_n M. \end{aligned}$$

Thus,

$$\begin{aligned} V(P_i) &\geq V\left(\text{Conv }(F(P_i, u_{i'_1}) \cup L_{i;i_1,i_2,\dots,i_{n-1}})\right) \\ &\geq V\left(\text{Conv }(F(P_i, u_{i'_1}) \cup \{x_0\})\right) \\ &= \frac{1}{n}|F(P_i, u_{i'_1})| \cdot d(x_0, H(P_i, u_{i'_1})) \\ &\geq \frac{c_n}{n} M^2. \end{aligned}$$

Therefore, $V(P_i)$ is not bounded. □

5. The logarithmic Minkowski problem for polytopes

In this section, we prove the main theorem.

Lemma 5.1. *If $\gamma_1, \dots, \gamma_N$ are positive and the unit vectors $u_1, \dots, u_N$ are in general position, then there exists a $P \in \mathcal{P}_N(u_1, \dots, u_N)$ such that $\xi(P) = o$, $V(P) = \sum_{k=1}^N \gamma_k$ and*

$$\Phi_P(o) = \inf\left\{\max_{\xi \in Int\ (Q)} \Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, \dots, u_N) \text{ and } V(Q) = \sum_{k=1}^N \gamma_k\right\},$$

where $\Phi_Q(\xi) = \sum_{k=1}^N \gamma_k \log(h(Q, u_k) - \xi \cdot u_k)$.

Proof. It is easily seen that it is sufficient to establish the lemma under the assumption that $\sum_{k=1}^{N}\gamma_k = 1$.

Obviously, for $P, Q \in \mathcal{P}_N(u_1, ..., u_N)$, if there exists a $x \in \mathbb{R}^n$ such that $P = Q + x$, then

$$\Phi_P(\xi(P)) = \Phi_Q(\xi(Q)).$$

Thus, we can choose a sequence $P_i \in \mathcal{P}_N(u_1, ..., u_N)$ with $\xi(P_i) = o$ and $V(P_i) = 1$ such that $\Phi_{P_i}(o)$ converges to

$$\inf\left\{\max_{\xi\in \text{Int }(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, ..., u_N) \text{ and } V(Q) = 1\right\}.$$

From Theorem 4.3, P_i is bounded. Thus, from Lemma 3.2 and the Blaschke selection theorem, there exists a subsequence of P_i that converges to a polytope P such that $P \in \mathcal{P}(u_1, ..., u_N)$, $V(P) = 1$, $\xi(P) = o$ and

$$\Phi_P(o) = \inf\left\{\max_{\xi\in \text{Int }(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, ..., u_N) \text{ and } V(Q) = 1\right\}. \tag{5.1}$$

We next prove that $F(P, u_i)$ are facets for all $i = 1, ..., N$. Otherwise, from Lemma 4.1 and the fact that $N \geq n + 1$, there exist $1 \leq i_0 < ... < i_m \leq N$ with $m \geq 0$ such that

$$F(P, u_i)$$

is a point for $i \in \{i_0, ..., i_m\}$ and is a facet of P for $i \in \{1, ..., N\}\backslash\{i_0, ..., i_m\}$.

Choose $\delta > 0$ small enough so that the polytope

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta\}$$

has exactly $(N - m)$ facets and

$$P \cap \{x : x \cdot u_{i_0} \geq h(P, u_{i_0}) - \delta\}$$

is a cone. Then,

$$V(P_\delta) = 1 - c_0\delta^n,$$

where $c_0 > 0$ is a constant that depends on P and direction u_{i_0}.

From Lemma 3.1, for any $\delta_i \to 0$ it is always true that $\xi(P_{\delta_i}) \to o$. We have $\xi(P_\delta) \to o$ as $\delta \to 0$. Let δ be small enough so that $h(P, u_k) > \xi(P_\delta) \cdot u_k + \delta$ for all $k \in \{1, ..., N\}$, let $d_0 = d(P)$, and let $\lambda = V(P_\delta)^{-\frac{1}{n}} = (\frac{1}{1-c_0\delta^n})^{\frac{1}{n}}$. From this, Equation (3.1), the fact that $\sum_{k=1}^{N}\gamma_k = 1$, and $d_0 > h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} > 0$, we have

$$\begin{aligned}
\prod_{k=1}^{N}(h(\lambda P_\delta, u_k) - \xi(\lambda P_\delta)\cdot u_k)^{\gamma_k} &= \lambda\prod_{k=1}^{N}(h(P_\delta, u_k) - \xi(P_\delta)\cdot u_k)^{\gamma_k} \\
&= \lambda\left[\prod_{k=1}^{N}(h(P, u_k) - \xi(P_\delta)\cdot u_k)^{\gamma_k}\right]\left[\frac{h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta}{h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0}}\right]^{\gamma_{i_0}} \\
&= \left[\prod_{k=1}^{N}(h(P, u_k) - \xi(P_\delta)\cdot u_k)^{\gamma_k}\right]\frac{(1 - \frac{\delta}{h(P,u_{i_0})-\xi(P_\delta)\cdot u_{i_0}})^{\gamma_{i_0}}}{(1 - c_0\delta^n)^{\frac{1}{n}}} \\
&\leq \left[\prod_{k=1}^{N}(h(P, u_k) - \xi(P_\delta)\cdot u_k)^{\gamma_k}\right]\frac{(1 - \frac{\delta}{d_0})^{\gamma_{i_0}}}{(1 - c_0\delta^n)^{\frac{1}{n}}}.
\end{aligned}$$

Let $g(\delta) = (1 - c_0\delta^n)^{\frac{1}{n\gamma_{i_0}}} + \frac{1}{d_0}\delta - 1$, then $g(0) = 1$ and

$$g'(\delta) = \frac{1}{d_0} - \frac{c_0}{\gamma_{i_0}}\delta^{n-1}(1 - c_0\delta^n)^{\frac{1}{n\gamma_{i_0}}-1}$$

is bigger than 0 for small positive δ. Thus, there exists a $\delta_0 > 0$ such that P_{δ_0} has exactly $(N-m)$ facets and

$$\frac{\left(1-\frac{\delta_0}{d_0}\right)^{\gamma_{i_0}}}{(1-c_0\delta_0^n)^{\frac{1}{n}}} < 1.$$

Then

$$\prod_{k=1}^{N}\left(h(\lambda_0 P_{\delta_0}, u_k) - \xi(\lambda_0 P_{\delta_0})\cdot u_k\right)^{\gamma_k} < \prod_{k=1}^{N}\left(h(P, u_k) - \xi(P_{\delta_0})\cdot u_k\right)^{\gamma_k},$$

where $\lambda_0 = V(P_{\delta_0})^{-\frac{1}{n}}$. Thus,

$$\Phi_{\lambda_0 P_{\delta_0}}(\xi(\lambda_0 P_{\delta_0})) < \Phi_P(\xi(P_{\delta_0})) \le \Phi_P(\xi(P)) = \Phi_P(o).$$

Let $P_0 = \lambda_0 P_{\delta_0} - \xi(\lambda_0 P_{\delta_0})$, then $P_0 \in \mathcal{P}(u_1, ..., u_N)$, $V(P_0) = 1$, $\xi(P_0) = o$ and

$$\Phi_{P_0}(o) < \Phi_P(o). \tag{5.2}$$

If $m = 0$, then Equation (5.2) is a contradiction with Equation (5.1). If $m \ge 1$, choose positive δ_i so that $\delta_i \to 0$ as $i \to \infty$,

$$P_{\delta_i} = P_0 \cap \left(\cap_{j=1}^m \{x : x\cdot u_{i_j} \le h(P_0, u_{i_j}) - \delta_i\}\right),$$

and $\lambda_i P_{\delta_i} \in \mathcal{P}_N(u_1, ..., u_N)$, where $\lambda_i = V(P_{\delta_i})^{-\frac{1}{n}}$. Obviously, $\lambda_i P_{\delta_i}$ converges to P_0. From Lemma 3.1 and Equation (5.2), we have

$$\begin{aligned}\lim_{i\to\infty}\Phi_{\lambda_i P_{\delta_i}}(\xi(\lambda_i P_{\delta_i})) &= \Phi_{P_0}(o)\\ &< \Phi_P(o)\\ &= \inf\left\{\max_{\xi\in \text{Int }(Q)}\Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, ..., u_N) \text{ and } V(Q) = 1\right\}.\end{aligned}$$

This is a contradiction with Equation (5.1). Therefore, $P \in \mathcal{P}_N(u_1, ..., u_N)$. $\square$

Now, we have prepared enough to prove the main theorem.

Theorem 5.2. *If $\gamma_1, ..., \gamma_N \in \mathbb{R}^+$ and the unit vectors $u_1, ..., u_N$ are in general position, then there exists a polytope P (containing the origin in its interior) such that*

$$V_P = \sum_{k=1}^{N}\gamma_k\delta_{u_k}.$$

Proof. From Lemma 5.1, there exists a $P \in \mathcal{P}_N(u_1, ..., u_N)$ with $\xi(P) = o$ and $V(P) = \sum_{k=1}^N \gamma_k$ such that

$$\Phi_P(o) = \inf\left\{\max_{\xi\in\text{Int }(Q)}\Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, ..., u_N) \text{ and } V(Q) = \sum_{k=1}^{N}\gamma_k\right\}.$$

From this and Lemma 3.3, we have

$$V_P = \sum_{k=1}^{N}\gamma_k\delta_{u_k}.$$

$\square$

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