THE ORLICZ CENTROID INEQUALITY FOR STAR BODIES

GUANGXIAN ZHU

Abstract. Lutwak, Yang and Zhang established the Orlicz centroid inequality for convex bodies and conjectured that their inequality can be extended to star bodies. In this paper, we confirm this conjecture.

1. Introduction

The centroid body operator is one of the central notions in convex geometry. Blaschke conjectured that the ratio between the volume of an origin-symmetric convex body and that of the volume of its centroid body attains its maximum precisely when the body is an origin symmetric ellipsoid (see e.g., [15, 26, 37, 59]). By applying Busemann's random simplex inequality (see [4]), Petty proved Blaschke's conjecture, extended the definition of centroid bodies, and gave centroid bodies their name [57]. Petty's theorem is known as the Busemann-Petty centroid inequality (see e.g., [15, 35–37, 59]).

With the development of the L_p Brunn-Minkowski theory and its dual (see e.g., [15, 33, 34, 59]), and the applications of this theory (see e.g., [1–3,5–11,13,18–22,24,27–39,41–44,46–55,58,60–62,64–68]), the L_p analogues of centroid inequality became a central focus. The fundamental inequality for L_p centroid bodies was established by Lutwak, Yang and Zhang [39] with an independent approach presented by Campi and Gronchi [5]. After that, Haberl and Schuster proved a general asymmetric L_p centroid inequality [22]. For additional references regarding centroid body inequalities and L_p centroid body inequalities and their applications see e.g., [14, 16, 17, 25, 53–55, 69].

In [40] and [45] Lutwak, Yang and Zhang extended the L_p Brunn-Minkowski theory to an Orlicz Brunn-Minkowski theory. In [40] they established the Orlicz centroid body inequality for convex bodies. In this paper their inequality, along with its equality conditions, will be extended from convex to star bodies.

Throughout let $\phi : \mathbb{R} \to [0, \infty)$ be convex and let $\phi(0) = 0$. Thus ϕ is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. We require that either one is happening strictly, that is ϕ is either strictly decreasing on $(-\infty, 0]$ or strictly increasing on $[0, \infty)$. The class of such ϕ is denoted by $\mathcal{C}$, and the subset of $\mathcal{C}$ that contains strictly convex functions is denoted by $\mathcal{C}_s$.

Let K is a star body (see Section 2 for precise definition) with respect to the origin in $\mathbb{R}^n$ with volume $|K|$, and $\phi \in \mathcal{C}$. The *Orlicz centroid body* $\Gamma_\phi K$ of K is the convex body whose support function at $x \in \mathbb{R}^n$ is given by

$$h(\Gamma_\phi K; x) = \inf \left\{ \lambda > 0 : \frac{1}{|K|} \int_K \phi\left(\frac{x \cdot y}{\lambda}\right) dy \le 1 \right\}, \tag{1.1}$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$ and the integration is with respect to Lebesgue measure on $\mathbb{R}^n$. Obviously, when $\phi(t) = |t|^p$, with $p \ge 1$, the *Orlicz centroid body* becomes the L_p *centroid body.*

In [40], Lutwak, Yang and Zhang proved the following theorem,

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Theorem A. *If $\phi \in \mathcal{C}$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the volume ratio*

$$|\Gamma_\phi K|/|K|$$

is minimized if and only if K is an ellipsoid centered at the origin.

By using the class reduction technique (introduced in [35]), Lutwak, Yang and Zhang showed that once the L_p Busemann-Petty centroid inequality has been established for convex bodies, then the inequality can be extended to all star bodies (see [39]). However, it is unclear whether there exists a similar class reduction technique that is applicable for the Orlicz centroid inequality. They also posted the following open problem:

Conjecture. *If $\phi \in \mathcal{C}$ and K is a star body with respect to the origin, then the volume ratio $|\Gamma_\phi K|/|K|$ is minimized if and only if K is an ellipsoid centered at the origin.*

In this paper, we extend the methods (used in [40]) for convex bodies to star bodies. As a result, we can confirm the above conjecture,

Theorem. *If $\phi \in \mathcal{C}$ and K is a star body with respect to the origin, then the volume ratio*

$$|\Gamma_\phi K|/|K|$$

is minimized when K is an ellipsoid centered at the origin. If $\phi \in \mathcal{C}_s$, then ellipsoids centered at the origin are the only minimizers.

This paper is organized as follows. In section 2, we recall some basic facts about convex bodies, star bodies and compact sets. In section 3, basic properties for the Steiner symmetrization of star bodies are developed. In section 4, we prove two auxiliary inequalities. In section 5, we extend two inequalities proved for convex bodies in [40] to the class of star bodies. In section 6, we complete the proof of the Orlicz centroid inequality for star bodies.

2. Some basics facts about convex bodies, star bodies and compact sets

All the subsets of $\mathbb{R}^n$ appearing in this paper are compact sets unless otherwise stated. If K is a Borel subset of $\mathbb{R}^n$ and K is contained in an $i-$dimensional affine subspace of $\mathbb{R}^n$ but not in any affine subspace of lower dimension, then $|K|$ denotes the $i-$dimensional Lebesgue measure of K. For $x \in \mathbb{R}^n$, we will write $|x|$ for the Euclidean norm of x. For $A \in$GL(n) we write A^t for the transpose of A, A^{-t} for the inverse of the transpose of A, and $|A|$ for the absolute value of the determinant of A. We write $e_1, ..., e_n$ for the standard orthonormal basis of $\mathbb{R}^n$ and when we write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ we always assume that e_n is associated with the last factor.

Let $\mathcal{K}^n$ denote the set of convex bodies (compact convex sets with nonempty interiors), $\mathcal{K}^n_o$ denote those convex bodies that contain the origin in their interiors. A compact set $K \subset \mathbb{R}^n$ is a star-shaped set (with respect to the origin) if the intersection of every straight line through the origin with K is a line segment. Let $K \subset \mathbb{R}^n$ be a compact star shaped set (with respect to the origin), the radial function $\rho(K, \cdot) : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}$ is defined by $\rho(K, x) = \rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}$. If ρ_K is strictly positive and continuous, then we call K a star body (with respect to the origin), denotes the class of star bodies (respect to the origin o) in $\mathbb{R}^n$ by $\mathcal{S}^n_o$.

If K, L are two compact sets in $\mathbb{R}^n$ and $\lambda \in \mathbb{R}$, their *Minkowski sum* $K + L$ is defined by,

$$K + L = \{x + y : x \in K, y \in L\},$$

and for $\lambda > 0$, the *scalar multiplication* λK is given by

$$\lambda K = \{\lambda x : x \in K\}.$$

For two compact sets K, L in $\mathbb{R}^n$, the *Hausdorff distance* between them is defined by

$$d(K,L) = \min\left\{t \geq 0 : K \subset L + tB^n, L \subset K + tB^n\right\}.$$

Let $h(K;\cdot) = h_K : \mathbb{R}^n \to \mathbb{R}$ denote the *support function* of the convex body $K \in \mathcal{K}^n$; i.e., $h(K;x) = \max\{x \cdot y : y \in K\}$. It is known that the *Hausdorff distance* between two convex bodies K and L is given by

$$d(K,L) = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

Obviously for $K, L \in \mathcal{K}^n$, we have $K \subset L$ if and only if $h_K \leq h_L$. For $c > 0$ and $x \in \mathbb{R}^n$, we have $h_{cK}(x) = ch_K(x)$ and $h_K(cx) = ch_K(x)$. More generally for $A \in \mathrm{GL}\ (n)$ we have

$$h_{AK}(x) = h_K(A^t x),$$

and

$$h_{K+L}(u) = h_K(u) + h_L(u).$$

For a direction $e_n = u \in S^{n-1}$, a convex body $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$ and $(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we will usually write $h(K; x', t)$ rather than $h(K;(x',t))$. Let K_u denote the image of the orthogonal projection of K onto $u^\perp$, and let

$$K = \{(y', z) : -\underline{l}_u(K, y') \leq z \leq \bar{l}_u(K, y'), y' \in K_u\},$$

where $\underline{l}_u(K;\cdot) : K_u \to \mathbb{R}$ and $\bar{l}_u(K;\cdot) : K_u \to \mathbb{R}$ are the *lowergraph* and *uppergraph functions* of K in the direction u. The following lemma will be needed (see e.g., [40]).

Lemma 2.1. *Suppose $K \in \mathcal{K}_o^n$ and $u \in S^{n-1}$. For $y' \in$ relint K_u, the uppergraph function and lowergraph function of K in the direction u are given by*

$$\bar{l}_u(K; y') = \min_{x' \in u^\perp}\left\{h(K; x', 1) - x' \cdot y'\right\},$$

and

$$\underline{l}_u(K; y') = \min_{x' \in u^\perp}\left\{h(K; x', -1) - x' \cdot y'\right\}.$$

3. Steiner symmetrization of star bodies

In this section we discuss properties of the Steiner symmetrization of star bodies. For a compact set K with nonzero measure, the intersection of K with any straight line is a compact set on the line (so the intersection is a one dimensional Lebesgue measurable set). The Steiner symmetrized body $S_u K$ of K with respect to the hyperplane $u^\perp$ is characterized by the following properties: First, $S_u K$ is symmetric with respect to the hyperplane $u^\perp$. Second, any straight line that is parallel to u and intersects K or $S_u K$, intersects also the other and both intersections have the same one-dimensional measure. Third, the intersection of a straight line parallel to u with $S_u K$ is a segment or a point in $u^\perp$. A further property of Steiner symmetrization is that, if K is a compact set, then $S_u K$ is also compact for any $u \in S^{n-1}$ (see e.g., [12]).

Let $\{u_i^\perp\}_{1 \leq i \leq k}$ be a finite set of hyperplanes. A multiple symmetrization is a composite of the form

$$S^* = S_{u_k^\perp} \circ S_{u_{k-1}^\perp} \circ \cdots \circ S_{u_1^\perp}.$$

For a nonempty compact set K, let $\mathfrak{S}(K)$ denote the set of all $S^*(K)$ multiple symmetrizations of K. The following well known lemma proved by Lusterink and Gross will be needed (see e.g., [23] p.170-173).

Lemma 3.1. *Let K be a nonempty compact set, then there is a sequence $\{K_i\} \subset \mathfrak{S}(K)$ and a closed ball $r\bar{B}^n$ centered at the origin of radius r such that $|r\bar{B}^n| = |K|$ and $K_i \to r\bar{B}^n$ with respect to the Hausdorff distance.*

Lemma 3.2. *Let K be a star-shaped set with respect to o, then K is a star body with respect to o if and only if for any $u \in S^{n-1}$, all the points of $\{tu : 0 \le t < \rho_K(u)\}$ are interior points of K.*

Proof. Assume K is a star body (respect to o) but there exist a $u_0 \in S^{n-1}$ and a $t_0 \ge 0$, such that $t_0u_0 \in \{tu_0 : 0 \le t < \rho_K(u_0)\}$ is not an interior point of K. Let $\delta = \frac{1}{2}(\rho_K(u_0) - t_0)$, since t_0u_0 is not an interior point of K, there exist an open ball $(t_0 + \delta)B^n$ centered at the origin of radius $(t_0 + \delta)$ and a sequence of points P_i such that $P_i \in ((t_0 + \delta)B^n \cap (\mathbb{R}^n \setminus K))$ and $P_i \to t_0u_0$. Let $u_i = (oP_i)/|oP_i| \in S^{n-1}$, then $u_i \to u_0$. Since P_i are not from K, $\rho_K(u_i) < |oP_i|$ for all $i \in \mathbb{N}$ and $|oP_i| \to t_0 < \rho_K(u_0)$. We have that $\rho_K(u)$ is not continuous at u_0, which is a contradiction. So for any $u \in S^{n-1}$ all the points of $\{tu : 0 \le t < \rho_K(u)\}$ are interior points of K.

If for any $u \in S^{n-1}$ all the points of $\{tu : 0 \le t < \rho_K(u)\}$ are interior points of K, but K is not a star body. Which means $\rho_K(u)$ is not continuous on S^{n-1}, then there exist a $\delta > 0$, a $u_0 \in S^{n-1}$, and a sequence of $u_i \in S^{n-1}$ such that $u_i \to u_0$ but $|\rho_K(u_i) - \rho_K(u_0)| > \delta$ for all i. Thus, we can either find an infinite subsequence of the u_i (without loss of generality we can suppose it is u_i) such that $\rho_K(u_i) - \rho_K(u_0) > \delta$, or we can find an infinite subsequence of the u_i (without loss of generality we can suppose it is the u_i) such that $\rho_K(u_0) - \rho_K(u_i) > \delta$. For the first case, since $u_i \to u_0$ and $\rho_K(u_i)u_i$ is bounded, the sequence $\rho_K(u_i)u_i$ has at least one limit point P. Since K is compact, $P \in K$ and obviously $P \in \{tu_0 : t \ge \rho_K(u_0) + \delta\}$, which is a contradiction. For the second case, since $u_i \to u_0$ and $\rho_K(u_i)u_i$ is bounded, the sequence $\rho_K(u_i)u_i$ has at least one limit point P, obviously $P \in K$ and $P \in \{tu_0 : 0 \le t \le \rho_K(u_0) - \delta\}$. Since K is a star-shaped set, the sequence of points $Q_i = (\rho_K(u_i) + 1/i)u_i$ is not in K. Obviously P is a limit point of Q_i and P is not an interior point of K, which is a contradiction. Therefore K is a star body. □

Theorem 3.3. *If $K \in \mathcal{S}_o^n$ and $u \in S^{n-1}$, then $S_uK \in \mathcal{S}_o^n$.*

Proof. Since K is a compact set, S_uK is compact (see e.g., [12]). For any $v \in S^{n-1}$ let $a_0 = a_0(v) = \sup\{a : a > 0, av \in S_uK\}$. Since S_uK is compact, $a_0v \in S_uK$. Furthermore we claim that for any s $(0 \le s < a_0)$ we have, the point $P = sv$ is an interior point of S_uK. Then the intersection of S_uK with any straight line through o is a segment, and except the two end points, all the points of this segment are interior points of S_uK. Thus by Lemma 3.2, S_uK is a star body.

For any point $P = sv$ $(0 \le s < a_0)$, write $(sv)_u$ and $(a_0v)_u$ for the projections of sv and a_0v onto $u^\perp$. Since K is a star body, for any point $Q \in K \cap \{(a_0v)_u + tu : t \in \mathbb{R}\}$, we have

$$(s/a_0)Q \in K \cap \{(sv)_u + tu : t \in \mathbb{R}\}.$$

So the set $(s/a_0)(K \cap \{(a_0v)_u + tu : t \in \mathbb{R}\})$ is a subset of $K \cap \{(sv)_u + tu : t \in \mathbb{R}\}$, and it is compact. By Lemma 3.2, for any point

$$Q \in (s/a_0)(K \cap \{(a_0v)_u + tu : t \in \mathbb{R}\}),$$

Q is an interior point of K, so we can find an open cube Δ_Q such that $Q \in \Delta_Q$, $\Delta_Q \subset K$ and the edges of Δ_Q are parallel to the axes. Thus we have an open cover of the compact set

$$(s/a_0)(K \cap \{(a_0v)_u + tu : t \in \mathbb{R}\}),$$

so we can choose a finite open cover of $(s/a_0)(K \cap \{(a_0v)_u + tu : t \in \mathbb{R}\})$, and denote this cover by $\Delta_{Q_1}, \Delta_{Q_2}, ..., \Delta_{Q_m}$. Obviously in $u^\perp$, the point $(Q_1)_u = (Q_2)_u = \cdots = (Q_m)_u = (sv)_u$ is an interior point of $\Delta' = \cap_{i=1}^m (\Delta_{Q_i})_u$ (where $(\Delta_{Q_i})_u$ is the projection of Δ_{Q_i} onto $u^\perp$). Let

$$t_M = |(\cup_{i=1}^m \Delta_{Q_i}) \cap \{(sv)_u + tu : t \in \mathbb{R}\}|$$

and

$$\delta_0 = |K \cap \{(a_0 v)_u + tu : t \in \mathbb{R}\}|,$$

then

$$(s/a_0)\{(a_0 v)_u \times [-\delta_0/2, \delta_0/2]\} \subset \Delta' \times (-t_M/2, t_M/2) \subset S_u K.$$

Since $\Delta' \times (-t_M/2, t_M/2)$ is an open set and

$$P = sv \in (s/a_0)\{(a_0 v)_u \times [-\delta_0/2, \delta_0/2]\},$$

P is an interior point of $S_u K$. Thus, by Lemma 3.2, $S_u K$ is a star body. $\square$

4. Two auxiliary inequalities

In this section we prove two basic inequalities that will be needed in the following sections.

Lemma 4.1. *Let $\phi \in \mathcal{C}$, $a_1 a_2 < 0, b_1, b_2 \in \mathbb{R}$, and $c_1, c_2 > 0$, then*

$$f(t) = c_1 \phi(a_1 t + b_1) + c_2 \phi(a_2 t + b_2)$$

is a convex function and there exists a t_0 such that $f(t)$ is decreasing on $(-\infty, t_0]$, increasing on $[t_0, +\infty)$, and $\lim_{t\to-\infty} f(t) = \lim_{t\to+\infty} f(t) = +\infty$. If $\phi \in \mathcal{C}_s$, then $f(t)$ is a strictly convex function and there exists a unique t_0 such that $f(t)$ is strictly decreasing on $(-\infty, t_0]$, strictly increasing on $[t_0, +\infty)$ and $\lim_{t\to-\infty} f(t) = \lim_{t\to+\infty} f(t) = +\infty$.

Proof. Let $f_i(t) = c_i \phi(a_i t + b_i)$. Since $\phi(t)$ is convex on $\mathbb{R}$ and $c_1, c_2 > 0$, for any $0 \le \lambda_1 \le 1$, $\lambda_2 = 1 - \lambda_1$ and $t_1, t_2 \in \mathbb{R}$, we have

$$\begin{aligned} f_i(\lambda_1 t_1 + \lambda_2 t_2) &= c_i \phi[a_i(\lambda_1 t_1 + \lambda_2 t_2) + b_i] \\ &= c_i \phi[\lambda_1 (a_i t_1 + b_i) + \lambda_2 (a_i t_2 + b_i)] \\ &\le c_i [\lambda_1 \phi(a_i t_1 + b_i) + \lambda_2 \phi(a_i t_2 + b_i)] \\ &= \lambda_1 f_i(t_1) + \lambda_2 f_i(t_2). \end{aligned} \tag{4.1}$$

So $f(t) = f_1(t) + f_2(t)$ is convex on $\mathbb{R}$. Obviously when $\phi \in \mathcal{C}_s$, the functions $f_1(t), f_2(t), f(t)$ are strictly convex.

Let $t_m = \min\{-\frac{b_1}{a_1}, -\frac{b_2}{a_2}\}$, $t_M = \max\{-\frac{b_1}{a_1}, -\frac{b_2}{a_2}\}$. If $t_m = t_M$ (denoted also by t_0), then, since $\phi \in \mathcal{C}$ and $a_1 a_2 < 0$, $c_1, c_2 > 0$, both f_1 and f_2 are increasing on $[t_0, +\infty)$ and decreasing on $(-\infty, t_0]$, so is $f = f_1 + f_2$. If $t_m < t_M$, then $f(t)$ is increasing on $[t_M, +\infty)$ and decreasing on $(-\infty, t_m]$. Let $f(t_0) = \min_{t_m \le t \le t_M} f(t)$, if $t_0 = t_M$ then $f(t)$ is increasing on $[t_0, +\infty)$; if $t_0 < t_M$ then choose any $t_0 < t_1 < t_2 \le t_M$ and let $\lambda = (t_2 - t_1)/(t_2 - t_0)$, then $0 \le \lambda \le 1$ and $\lambda t_0 + (1 - \lambda) t_2 = t_1$, so

$$f(t_1) = f[\lambda t_0 + (1-\lambda) t_2] \le \lambda f(t_0) + (1-\lambda) f(t_2) \le f(t_2),$$

therefore $f(t)$ is increasing on $[t_0, t_M]$. Since $f(t)$ is continuous on $\mathbb{R}$, $f(t)$ is increasing on $[t_0, +\infty)$. Similarly we can prove that $f(t)$ is decreasing on $(-\infty, t_0]$. Since $a_1 a_2 < 0$, $c_1, c_2 > 0$ and $\phi \in \mathcal{C}$, $\lim_{t\to+\infty} f(t) = \lim_{t\to-\infty} f(t) = +\infty$. Obviously when $\phi \in \mathcal{C}_s$, $f(t)$ is strictly decreasing on $(-\infty, t_0]$ and strictly increasing on $[t_0, +\infty)$ (otherwise $f(t)$ will not be strictly convex), and $\lim_{t\to+\infty} f(t) = \lim_{t\to-\infty} f(t) = +\infty$. $\square$

Lemma 4.2. *Let $f(t) \ge 0$ be a continuous function, decreasing on $(-\infty, t_0]$ and increasing on $[t_0, +\infty)$. If E is a compact subset of $\mathbb{R}$, then*

$$\int_E f(t)dt \ge \int_{t_0 - \delta_-}^{t_0 + \delta_+} f(t)dt, \tag{4.2}$$

where $\delta_- = |E \cap (-\infty, t_0]|, \delta_+ = |E \cap [t_0, +\infty)|$.

If f is strictly decreasing on $(-\infty, t_0]$, strictly increasing on $[t_0, +\infty)$ and there exists a t_0' not in E and $|E \cap (-\infty, t_0']| > 0, |E \cap [t_0', +\infty)| > 0$, then

$$\int_E f(t)dt > \int_{t_0-\delta_-}^{t_0+\delta_+} f(t)dt. \tag{4.3}$$

Proof. We will prove

$$\int_{E\cap[t_0,+\infty)} f(t)dt \geq \int_{t_0}^{t_0+\delta_+} f(t)dt \tag{4.4}$$

and

$$\int_{E\cap(-\infty,t_0]} f(t)dt \geq \int_{t_0-\delta_-}^{t_0} f(t)dt. \tag{4.5}$$

Since E is a compact set, we have $t_+ = \sup E < +\infty$. Let $t_i = t_0 + \frac{i}{n}(t_+ - t_0)$, (where $0 \leq i \leq n$). When $t_i \leq t < t_{i+1}$ $(0 \leq i < n-1)$, define $f_n(t) = f(t_i)$ and $f_n(t_n) = f(t_+)$. Obviously $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence of simple functions on $E \cap [t_0, +\infty)$ and $f_n(t) \to f(t)$. By the monotone convergence theorem (see e.g., [63]) we have

$$\begin{aligned} \int_{E\cap[t_0,+\infty)} f(t)dt &= \lim_{n\to+\infty} \int_{E\cap[t_0,+\infty)} f_n(t)dt \\ &= \lim_{n\to+\infty} \sum_{i=0}^{n-1} f_n(t_i) \left|E \cap [t_i, t_{i+1}]\right|. \end{aligned} \tag{4.6}$$

Let $t_i' = t_0 + |E \cap [t_0, t_i]|$ $(0 \leq i \leq n)$, define $f_n'(t) = f(t_i')$ when $t \in [t_i', t_{i+1}'], (0 \leq i \leq n-1)$, and $f_n'(t_n') = f(t_0 + \delta_+)$. Then $\{f_n'\}_{n=1}^{+\infty}$ is an increasing sequence of simple functions on $[t_0, t_0 + \delta_+]$ and $f_n'(t) \to f(t)$. By the monotone convergence theorem we have

$$\begin{aligned} \int_{t_0}^{t_0+\delta_+} f(t)dt &= \lim_{n\to+\infty} \int_{t_0}^{t_0+\delta_+} f_n'(t)dt \\ &= \lim_{n\to+\infty} \sum_{i=0}^{n-1} f_n'(t_i') \left|t_{i+1}' - t_i'\right| \\ &= \lim_{n\to+\infty} \sum_{i=1}^{n-1} f_n'(t_i') \left|E \cap [t_i, t_{i+1}]\right|. \end{aligned} \tag{4.7}$$

Since $f(t)$ is increasing on $[t_0, +\infty)$, $f_n'(t_i') \leq f_n(t_i)$, by (4.6) and (4.7), we obtain (4.4). By a similar argument one can prove (4.5), so

$$\int_E f(t)dt \geq \int_{t_0-\delta_-}^{t_0+\delta_+} f(t)dt.$$

Assume now that f is strictly decreasing on $(-\infty, t_0]$, strictly increasing on $[t_0, +\infty)$ and there exist a t_0' such that t_0' is not in E and $|E \cap [t_0', +\infty)| > 0$, $|E \cap (-\infty, t_0']| > 0$. Without loss of generality we can assume that $t_0 \leq t_0' < t_+$. Since E is a compact set, there exists a $\delta_0 > 0$ such that $[t_0', t_0'+\delta_0] \cap E$ is empty. And since f is continuous and strictly increasing on $[t_0, +\infty)$, there exists a δ_0' such that $f(t) - f(t-\delta_0) > \delta_0'$ on $[t_0', t_+]$, and when $t_i > t_i' + \delta_0$, $f(t_i) - f(t_i') > f(t_i) - f(t_i - \delta_0) > \delta_0'$. By (4.6) and (4.7), this yields

$$\int_{E\cap[t_0,+\infty)} f(t)dt - \int_{t_0}^{t_0+\delta_+} f(t)dt \geq |E \cap [t_0', +\infty)|\delta_0' > 0.$$

Together with (4.5), we obtain

$$\int_E f(t)dt > \int_{t_0-\delta_-}^{t_0+\delta_+} f(t)dt.$$

□

5. Steiner symmetrization of Orlicz centroid bodies

In this section, we prove two inequalities for star bodies, both of them were proved by Lutwak, Yang and Zhang for the case of convex bodies in [40].

Lemma 5.1. *If $\phi \in \mathcal{C}$ and $K \in \mathcal{S}_o^n$, then for any $u \in S^{n-1}$, and $x_1', x_2' \in u^\perp$,*

$$h(\Gamma_\phi(S_u K); \frac{1}{2}x_1' + \frac{1}{2}x_2', 1) \leq \frac{1}{2}h(\Gamma_\phi K; x_1', 1) + \frac{1}{2}h(\Gamma_\phi K; x_2', -1),$$

and

$$h(\Gamma_\phi(S_u K); \frac{1}{2}x_1' + \frac{1}{2}x_2', -1) \leq \frac{1}{2}h(\Gamma_\phi K; x_1', 1) + \frac{1}{2}h(\Gamma_\phi K; x_2', -1).$$

If $\phi \in \mathcal{C}_s$, P_1, P_2 are two interior points of K, and the segment $\overline{P_1P_2}$ does not completely lie in K, then, for $u = (P_1 - P_2)/|P_1 - P_2|$, equality can not hold in either of the inequalities.

Proof. According to the affine properties of Orlicz centroid bodies (see [40]), for $A \in$GL(n) and $K \in \mathcal{S}_o^n$, we have $\Gamma_\phi(AK) = A\Gamma_\phi K$. Without loss of generality we can assume that $|K| = |S_u K| = 1$.

Denote by $K' = K_u$ the image of the projection of K onto $u^\perp$. For $y' \in K'$, denote by $\sigma_{y'}(u) = \sigma_{y'} = |K \cap (y' + \mathbb{R}u)|$ the one dimensional measure of $K \cap (y' + \mathbb{R}u)$.

For fixed $x_1', x_2', x_0' = \frac{1}{2}x_1' + \frac{1}{2}x_2' \in K'$ and any $y' \in K', s \in \mathbb{R}$ and $\lambda_1, \lambda_2, \lambda_0 = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 \in \mathbb{R}^+$, by Lemma 4.1 the function

$$g(s) = \frac{\lambda_1}{\lambda_0}\phi\left(\frac{x_1' \cdot y' + s}{\lambda_1}\right) + \frac{\lambda_2}{\lambda_0}\phi\left(\frac{x_2' \cdot y' - s}{\lambda_2}\right)$$

is convex, and there exists a $y_u(y') = y(y')$ such that $g(s)$ is decreasing on $(-\infty, y(y')]$ and increasing on $[y(y'), \infty)$. Let $\sigma_{y'}^+ = |K \cap (y(y') + \mathbb{R}^+ u)|$ and $\sigma_{y'}^- = |K \cap (y(y') + \mathbb{R}^- u)|$.

By Lemma 4.2 we have

$$\int_{K\cap(y'+\mathbb{R}u)} g(s)ds \geq \int_{y(y')-\sigma_{y'}^-}^{y(y')+\sigma_{y'}^+} g(s)ds. \tag{5.1}$$

Let $m_{y'} = m_{y'}(u)$ be the midpoint of $y(y') - \sigma_{y'}^- \leq t \leq y(y') + \sigma_{y'}^+$. By the convexity of $\phi(t)$ we have

$$\frac{\lambda_1}{\lambda_0}\phi\left(\frac{x_1' \cdot y' + t + m_{y'}(u)}{\lambda_1}\right) + \frac{\lambda_2}{\lambda_0}\phi\left(\frac{x_2' \cdot y' + t - m_{y'}(u)}{\lambda_2}\right) \geq 2\phi\left(\frac{x_0' \cdot y' + t}{\lambda_0}\right). \tag{5.2}$$

Let

$$A = \frac{\lambda_1}{\lambda_0}\int_K \phi\left(\frac{(x_1', 1) \cdot y}{\lambda_1}\right)dy + \frac{\lambda_2}{\lambda_0}\int_K \phi\left(\frac{(x_2', -1) \cdot y}{\lambda_2}\right)dy.$$

By Fubini's theorem and (5.1), we have

$$\begin{aligned}
A &= \int_{K'}\int_{K\cap(y'+\mathbb{R}u)} \left[\frac{\lambda_1}{\lambda_0}\phi\left(\frac{x_1'\cdot y'+s}{\lambda_1}\right)+\frac{\lambda_2}{\lambda_0}\phi\left(\frac{x_2'\cdot y'-s}{\lambda_2}\right)\right]dy'ds \\
&\geq \int_{K'}\int_{y(y')-\sigma^-_{y'}}^{y(y')+\sigma^+_{y'}} \left[\frac{\lambda_1}{\lambda_0}\phi\left(\frac{x_1'\cdot y'+s}{\lambda_1}\right)+\frac{\lambda_2}{\lambda_0}\phi\left(\frac{x_2'\cdot y'-s}{\lambda_2}\right)\right]dy'ds \qquad (5.3a)\\
&= \int_{K'}\int_{y(y')-\sigma^-_{y'}}^{y(y')+\sigma^+_{y'}} \frac{\lambda_1}{\lambda_0}\phi\left(\frac{x_1'\cdot y'+s}{\lambda_1}\right)dy'ds+\int_{K'}\int_{y(y')-\sigma^-_{y'}}^{y(y')+\sigma^+_{y'}} \frac{\lambda_2}{\lambda_0}\phi\left(\frac{x_2'\cdot y'-s}{\lambda_2}\right)dy'ds.
\end{aligned}$$

Since $y(y')-\sigma^-_{y'}=m_{y'}-\frac{1}{2}\sigma_{y'}$, $y(y')+\sigma^+_{y'}=m_{y'}+\frac{1}{2}\sigma_{y'}$, by making the change of variables $s=m_{y'}+t$ for the first integral of the last equation in (5.3a), and making the change of variables $s=m_{y'}-t$ for the second integral of the last equation in (5.3a). Together with Fubini's theorem and (5.2) we obtain

$$\begin{aligned}
A &\geq \int_{K'}\int_{-\sigma_{y'}/2}^{\sigma_{y'}/2} \frac{\lambda_1}{\lambda_0}\phi\left(\frac{x_1'\cdot y'+t+m_{y'}}{\lambda_1}\right)dy'dt+\int_{K'}\int_{-\sigma_{y'}/2}^{\sigma_{y'}/2} \frac{\lambda_2}{\lambda_0}\phi\left(\frac{x_2'\cdot y'+t-m_{y'}}{\lambda_2}\right)dy'dt \\
&= \int_{S_uK}\left[\frac{\lambda_1}{\lambda_0}\phi\left(\frac{x_1'\cdot y'+t+m_{y'}}{\lambda_1}\right)+\frac{\lambda_2}{\lambda_0}\phi\left(\frac{x_2'\cdot y'+t-m_{y'}}{\lambda_2}\right)\right]dy'dt \qquad (5.3b)\\
&\geq 2\int_{S_uK}\phi\left(\frac{\left(\frac{1}{2}x_1'+\frac{1}{2}x_2'\right)\cdot y'+t}{\frac{1}{2}\lambda_1+\frac{1}{2}\lambda_2}\right)dy'dt.
\end{aligned}$$

Consequently

$$\frac{\lambda_1}{\lambda_0}\int_K\phi\left(\frac{(x_1',1)\cdot y}{\lambda_1}\right)dy+\frac{\lambda_2}{\lambda_0}\int_K\phi\left(\frac{(x_2',-1)\cdot y}{\lambda_2}\right)dy\geq 2\int_{S_uK}\phi\left(\frac{(x_0',1)\cdot y}{\lambda_0}\right)dy. \qquad (5.3c)$$

Choose any numbers $\lambda_1>h(\Gamma_\phi K;x_1',1)\geq 0$, $\lambda_2>h(\Gamma_\phi K;x_2',-1)\geq 0$. Then, since $|K|=|S_uK|=1$, and by (1.1), we have $\int_K\phi((x_1',1)\cdot y/\lambda_1)dy\leq 1$, $\int_K\phi((x_2',-1)\cdot y/\lambda_2)dy\leq 1$. From this and (5.3c) we obtain

$$1\geq\frac{1}{|S_uK|}\int_{S_uK}\phi\left(\frac{(x_0',1)\cdot y}{\lambda_0}\right)dy.$$

Since λ_0 can be any positive number bigger than $\frac{1}{2}h(\Gamma_\phi K;x_1',1)+\frac{1}{2}h(\Gamma_\phi K,x_2',-1)$, by (1.1) we conclude

$$h(\Gamma_\phi(S_uK);\frac{1}{2}x_1'+\frac{1}{2}x_2',1)\leq\frac{1}{2}h(\Gamma_\phi K;x_1',1)+\frac{1}{2}h(\Gamma_\phi K;x_2',-1). \qquad (5.4)$$

Note, if we making the change of variables $s=m_{y'}-t$ for the first integral of the last equation in (5.3a), and making the change of variable $s=m_{y'}+t$ for the second integral of the second equation in (5.3a) then by similar argument one obtains,

$$h(\Gamma_\phi(S_uK);\frac{1}{2}x_1'+\frac{1}{2}x_2',-1)\leq\frac{1}{2}h(\Gamma_\phi K;x_1',1)+\frac{1}{2}h(\Gamma_\phi K;x_2',-1). \qquad (5.5)$$

Assume now that $\phi\in\mathcal{C}_s$ and there exist two interior points P_1,P_2 of K and a point P such that $P\in\overline{P_1P_2}$ but not from K. Let $u=(P_1-P_2)/|P_1-P_2|$, and choose two open balls $r_iB^n(P_i)$ (where $i=1,2$) centered at P_i of radius r_i. Since K is compact and P is not from K, there exists an open ball $rB^n(P)$ centered at P of radius r such that $rB^n(P)\cap K$ is empty and $(rB^n(P))_u\subset(r_iB^n(P_i))_u$ for $i=1,2$. Thus, for any point $Q\in rB^n(P)$ we have $|K\cap(Q+\mathbb{R}^+u)|>0,|K\cap(Q+\mathbb{R}^-u)|>0$. From the condition ϕ is strictly convex, strictly decreasing on $(-\infty,0]$ and strictly increasing on $[0,+\infty)$. By Lemma 4.1, $g(s)$ is also strictly convex and there exists a unique $y_u(y')$ such that $g(s)$ is strictly decreasing on $(-\infty,y_u(y')]$ and strictly increasing on $[y_u(y'),+\infty)$. Moreover, by

Lemma 4.2 when $y' \in (rB^n(P))_u$ the equality in (5.1) can not hold, and since $|(rB^n(P))_u| > 0$, the equality of the first inequality in (5.3a) can not hold either. Thus, equality in both (5.4) and (5.5) can not hold. $\square$

We note that for the *Steiner symmetral* S_uK of $K \in \mathcal{K}^n$ in the direction u, the lowergraph and uppergraph functions are given by

$$\underline{l}_u(S_uK; y') = \frac{1}{2}\left[\underline{l}_u(K; y') + \bar{l}_u(K; y')\right], \tag{5.6a}$$

and

$$\bar{l}_u(S_uK; y') = \frac{1}{2}\left[\underline{l}_u(K; y') + \bar{l}_u(K, y')\right]. \tag{5.6b}$$

Lemma 5.2. *If $\phi \in \mathcal{C}$ and $K \in \mathcal{S}_o^n$, then for $u \in S^{n-1}$,*

$$\Gamma_\phi(S_uK) \subset S_u(\Gamma_\phi K)$$

If $\phi \in \mathcal{C}_s$ and there exist two interior points P_1, P_2 of K such that the segment $\overline{P_1P_2}$ does not completely lie in K, then, for $u = (P_1 - P_2)/|P_1 - P_2| \in S^{n-1}$, we have

$$\Gamma_\phi(S_uK) \neq S_u(\Gamma_\phi K).$$

Proof. For $y' \in \operatorname{relint}(\Gamma_\phi K)_u$, by Lemma 2.1, there exist $x_1' = x_1'(y'), x_2' = x_2'(y') \in u^\perp$ such that

$$\bar{l}_u(\Gamma_\phi K, y') = h_{\Gamma_\phi K}(x_1', 1) - x_1' \cdot y', \tag{5.7a}$$

and

$$\underline{l}_u(\Gamma_\phi K, y') = h_{\Gamma_\phi K}(x_2', -1) - x_2' \cdot y'. \tag{5.7b}$$

Now by (5.6a), (5.6b), (5.7a), (5.7b) followed by Lemma 2.1, and Lemma 5.1 we have

$$\begin{aligned}
\bar{l}_u(S_u(\Gamma_\phi K); y') &= \frac{1}{2}\bar{l}_u(\Gamma_\phi K; y') + \frac{1}{2}\underline{l}_u(\Gamma_\phi K; y') \\
&= \frac{1}{2}(h_{\Gamma_\phi K}(x_1', 1) - x_1' \cdot y') + \frac{1}{2}(h_{\Gamma_\phi K}(x_2', -1) - x_2' \cdot y') \\
&\geq h_{\Gamma_\phi(S_uK)}(\frac{1}{2}x_1' + \frac{1}{2}x_2', 1) - (\frac{1}{2}x_1' + \frac{1}{2}x_2') \cdot y' \\
&\geq \min_{x' \in u^\perp}\{h_{\Gamma_\phi(S_uK)}(x', 1) - x' \cdot y'\} \\
&= \bar{l}_u(\Gamma_\phi(S_uK); y'),
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
\underline{l}_u(S_u(\Gamma_\phi K); y') &= \frac{1}{2}\bar{l}_u(\Gamma_\phi K; y') + \frac{1}{2}\underline{l}_u(\Gamma_\phi K; y') \\
&= \frac{1}{2}(h_{\Gamma_\phi K}(x_1', 1) - x_1' \cdot y') + \frac{1}{2}(h_{\Gamma_\phi K}(x_2', -1) - x_2' \cdot y') \\
&\geq h_{\Gamma_\phi(S_uK)}(\frac{1}{2}x_1' + \frac{1}{2}x_2', -1) - (\frac{1}{2}x_1' + \frac{1}{2}x_2') \cdot y' \\
&\geq \min_{x' \in u^\perp}\{h_{\Gamma_\phi(S_uK)}(x', -1) - x' \cdot y'\} \\
&= \underline{l}_u(\Gamma_\phi(S_uK); y').
\end{aligned} \tag{5.9}$$

So

$$\Gamma_\phi(S_uK) \subset S_u(\Gamma_\phi K).$$

If $\phi \in \mathcal{C}_s$, P_1, P_2 are two interior points of K such that the segment $\overline{P_1P_2}$ does not completely lie in K, and $u = (P_1 - P_2)/|P_1 - P_2| \in S^{n-1}$. Then by Lemma 5.1 equality in inequality (5.8) and (5.9) can not hold, thus $\Gamma_\phi(S_u K) \neq S_u(\Gamma_\phi K)$. □

6. Orlicz centroid inequality For star bodies

In this section, we prove the Orlicz Busemann-Petty centroid inequality for star bodies.

Theorem. *If $\phi \in \mathcal{C}$ and $K \in \mathcal{S}_o^n$, then the volume ratio*

$$|\Gamma_\phi K|/|K|$$

is minimized when K is an ellipsoid centered at the origin. If $\phi \in \mathcal{C}_s$, then ellipsoids centered at the origin are the only minimizers.

Proof. We prove this theorem in two steps. First we prove the inequality and in the second step we prove the uniqueness of minimizers.

For $K_0 = K \in \mathcal{S}_o^n$ and a sequence of positive number $\varepsilon_m \to 0$, by Lemma 3.1 and Theorem 3.3, there exist a closed ball $r\bar{B}^n$ centered at the origin of radius r and a sequence $u_{11}, u_{12}, ..., u_{1i_1} \in S^{n-1}$, such that

$$d(K_1, r\bar{B}^n) < \varepsilon_1,$$

where $K_1 = S_{u_{1,i_1}^\perp} \circ S_{u_{1,i_1-1}^\perp} \circ \cdots \circ S_{u_{1,1}^\perp} K$ is a star body obtained from K by multiple symmetrization. In particular $|r\bar{B}^n| = |K|$.

From Lemma 5.2 we have

$$|\Gamma_\phi(S_{u_{1,i_1}^\perp} \circ S_{u_{1,i_1-1}^\perp} \circ \cdots \circ S_{u_{1,1}^\perp} K)| \leq |\Gamma_\phi(S_{u_{1,i_1-1}^\perp} \circ \cdots \circ S_{u_{1,1}^\perp} K)|. \tag{6.1}$$

Therefore, in particular,

$$|\Gamma_\phi K_1| \leq |\Gamma_\phi K_0| = |\Gamma_\phi K|. \tag{6.2}$$

If $i = m-1$ then we can find a sequence $u_{m,1}^\perp$, $u_{m,2}^\perp$,...,$u_{m,i_m}^\perp$ such that $d(K_m, r\bar{B}^n) < \varepsilon_m$ and $|\Gamma_\phi K_m| \leq |\Gamma_\phi K_{m-1}|$, where $K_m = S_{u_{m,i_m}^\perp} \circ S_{u_{m,i_m-1}^\perp} \circ \cdots \circ S_{u_{m,1}^\perp} K_{m-1}$ is a star body. When $i = m$, since K_m is a star body, by Lemma 3.1 and Theorem 3.3 we can find a sequence of $u_{m+1,1}^\perp$, $u_{m+1,2}^\perp$,...,$u_{m+1,i_{m+1}}^\perp$, such that $d(K_{m+1}, r\bar{B}^n) < \varepsilon_{m+1}$ and $|\Gamma_\phi K_{m+1}| \leq |\Gamma_\phi K_m|$, where $K_{m+1} = S_{u_{m+1,i_{m+1}}^\perp} \circ S_{u_{m+1,i_{m+1}-1}^\perp} \circ \cdots \circ S_{u_{m+1,1}^\perp} K_m$ is a star body and $|K_{m+1}| = |r\bar{B}^n|$. By induction we obtain a sequence of $\{K_m\}$ with $|K_m| = |K|$ such that

$$d(K_m, r\bar{B}^n) < \varepsilon_m \tag{6.3}$$

for all $m \in \mathbb{N}$, and

$$|\Gamma_\phi K_m| \leq |\Gamma_\phi K_{m-1}| \tag{6.4}$$

for all $m \in \mathbb{N}$.

By (6.3), $K_m \to r\bar{B}^n$ with respect to Hausdorff distance, so $\lim_{m\to+\infty} |\Gamma_\phi K_m| = |\Gamma_\phi(r\bar{B}^n)|$, (see [40]). By (6.2) and (6.4) we obtain that

$$|\Gamma_\phi K| \geq |\Gamma_\phi(r\bar{B}^n)|. \tag{6.5}$$

For $A \in \mathrm{GL}(n)$, we have $\Gamma_\phi(AK) = A\Gamma_\phi K$, (see [40]), thus

$$\frac{|\Gamma_\phi K|}{|K|} \geq \frac{|\Gamma_\phi \bar{B}^n|}{|\bar{B}^n|}.$$

Consequently, the volume ratio $|\Gamma_\phi K|/|K|$ is minimized when K is an ellipsoid centered at the origin.

We turn now to the equality conditions. For this assume that $\phi \in \mathcal{C}_s$ and $K \in \mathcal{S}_o^n$. If $K \in \mathcal{S}_o^n$ is not convex, then for any point $P \in \partial K$, by Lemma 3.2 all the points of the segment $\overline{oP}$ except P are interior points of K. Since K is not convex, we can choose $P_3, P_4 \in \partial K$ such that there exists a point $Q \in \overline{P_3P_4}$, but not in K. Since K is compact, we can choose an open ball $r'B^n(Q)$ centered at Q of radius r', such that $r'B^n(Q) \cap K$ is empty. Also we can choose $P_1 \in \overline{oP_3}, (P_3 \neq P_1)$ and $P_2 \in \overline{oP_4}, (P_4 \neq P_2)$, such that $\overline{P_1P_2} \cap (r'B^n(Q))$ is not empty and from Lemma 3.2 P_1, P_2 are interior points of K. Let $u_1 = (P_1 - P_2)/|P_1 - P_2| \in S^{n-1}$, by Lemma 5.2 we have

$$\Gamma_\phi(S_{u_1}K) \subset S_{u_1}(\Gamma_\phi K), \tag{6.6}$$

and the inclusion is not an identity. If we use $S_{u_1}K$ to replace $K = K_0$ in the first step, by (6.3), (6.4), (6.5), (6.6) and the affine property of Orlicz centroid body, we have

$$\frac{|\Gamma_\phi K|}{|K|} > \frac{|\Gamma_\phi \bar{B}^n|}{|\bar{B}^n|}.$$

If $\phi \in \mathcal{C}_s$ and $K \in \mathcal{S}_o^n$ is a convex body, then, by Theorem A, ellipsoids centered at the origin are the only minimizers of $|\Gamma_\phi K|/|K|$. So when $\phi \in \mathcal{C}_s$ and $K \in \mathcal{S}_o^n$, ellipsoids centered at the origin are the only minimizers. $\square$

After work on this project was completed, the author learned of the work of Paouris [56]. While there is some overlap of results, the methods employed to achieve them are quite different.

This work can be extended to compact sets and will be done in a future paper.

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POLYTECHNIC INSTITUTE OF NEW YORK UNIVERSITY, SIX METROTECH, BROOKLYN, NY 11201.
E-mail address: guangxian.zhu@gmail.com