MIRROR SYMMETRIC SOLUTIONS TO THE CENTRO-AFFINE MINKOWSKI PROBLEM

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ABSTRACT. The centro-affine Minkowski problem, a critical case of the L_p-Minkowski problem in the $n+1$ dimensional Euclidean space is considered. By applying methods of calculus of variations and blow-up analyses, two sufficient conditions for the existence of solutions to the centro-affine Minkowski problem are established.

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1. Introduction

In this paper, we study the existence of solutions to the following Monge-Ampère type equation

$$\det(\nabla^2 H + HI) = \frac{f}{H^{n+2}} \quad \text{on } S^n, \tag{1.1}$$

where f is a given positive function, H is the support function of a bounded convex body K in $\mathbb{R}^{n+1}$, I is the unit matrix and $\nabla^2 H = (\nabla_{ij} H)$ is the Hessian matrix of covariant derivatives of H with respect to an orthonormal frame on S^n.

Let K be a convex body whose boundary is smooth and has positive Gauss curvature. If K contains the origin in its interior, then the quantity

$$\frac{1}{H^{n+2}\det(\nabla^2 H + HI)}$$

is called centro-affine Gauss curvature of ∂K. Notations related to the centro-affine Gauss curvature, such as affine support function and affine distance, appeared in the subject Affine Differential Geometry (see, e.g., [45], pp. 62-63). The question describing the centro-affine Gauss curvature is called the centro-affine Minkowski problem, which was posed by Chou and Wang [12].

Obviously, the centro-affine Minkowski problem is equivalent to solving equation (1.1). Equation (1.1) is also a special case of the L_p-Minkowski problem posed by Lutwak [39].

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Let K be a convex body in $\mathbb{R}^{n+1}$ that contains the origin in its interior, then the L_p surface area measure, $S_p(K,\cdot)$, of K is a Borel measure on S^n defined for a Borel $\omega \subset S^n$, by

$$S_p(K,\omega) = \int_{x\in\nu_K^{-1}(\omega)} (x\cdot\nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial' K \to S^n$ is the Gauss map of K, defined on $\partial' K$, the set of boundary points of K that have a unique outer unit normal, and $\mathcal{H}^n$ is (n)-dimensional Hausdorff measure.

The L_p surface area measure was introduced by Lutwak [39], which is a core notation in modern convex geometric analysis. In recent years the L_p surface area measure appeared in, e.g., [8, 24, 25, 26, 36, 37, 39, 46, 49, 50, 51, 58, 60]. As special cases, $S_1(K,\cdot)$ (is also denoted by S_K in this paper) is the classical surface area measure of K, $S_0(K,\cdot)$ is the cone-volume measure of K, and $S_{-n-1}(K,\cdot)$ is the centro-affine surface area measure of K.

In [39], Lutwak posed the following question:

L_p **Minkowski problem:** *Find necessary and sufficient conditions on a finite Borel measure, μ, on the unit sphere S^n so that μ is the L_p surface area measure of a convex body in $\mathbb{R}^{n+1}$.*

The L_p-Minkowski problem is a central problem in modern convex geometric analysis. It has attracted great attention over the last two decades, and was studied by, e.g., Lutwak [39], Lutwak & Oliker [40], Lutwak, Yang & Zhang [41], Chou & Wang [12], Guan & Lin [21], Hug, Lutwak, Yang & Zhang [29], Böröczky, Lutwak, Yang & Zhang [6, 7], Stancu [52, 53], Huang, Liu, & Xu [28], Jian, Lu, & Wang [30], Jian & Wang [31], Lu & Jian [34], Lu & Wang [35], Dou & Zhu [16], Böröczky, Hegedűs and Zhu [5], Zhu [65, 63, 64] and Sun and Long [54]. Analogues of the Minkowski problems were studied in, e.g., [3, 4, 11, 15, 17, 18, 19, 20, 22, 23, 27, 33, 59]. Applications of solutions to the L_p-Minkowski problem can be found in the work, e.g., Cianchi, Lutwak, Yang & Zhang [14], Lutwak, Yang & Zhang [42], and Zhang [61].

Obviously, for the case where μ has a density function, the L_p-Minkowski problem is equivalent to solving the following Monge-Ampère type equation

$$\det(\nabla^2 H + HI) = fH^{p-1} \quad \text{on } S^n. \tag{1.2}$$

Clearly, equation (1.1) is the special case of equation (1.2) for $p = -n-1$. It is known that equation (1.1) also arises in anisotropic Gauss curvature flows [13, 57] and image processing [2]. Besides, equation (1.1) can be reduced to a singular Monge-Ampère equation in the half Euclidean space $\mathbb{R}^{n+1}_+$, the regularity of which was extensively studied in [31, 32].

The PDE (1.1) remains invariant under projective transforms on S^n. When f is a constant function, equation (1.1) only has constant solution up to a projective transformation. This result has been known for a long time, see [9] for example, which implies that there is no a priori estimates on solutions for general f. Besides, equation (1.1) corresponds to the critical case of the Blaschke-Santaló inequality in convex geometry [44]:

$$\mathrm{vol}(K) \inf_{\xi \in K} \int_{S^n} \frac{dS(x)}{(H(x) - \xi \cdot x)^{n+1}} \leq (n+1)\omega_{n+1}^2, \tag{1.3}$$

where K is any convex body, $\mathrm{vol}(K)$ is the volume of the convex body K, and ω_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$. Chou and Wang [12] found an obstruction for solutions to equation (1.1). This situation is similar, in some aspects, to the prescribed scalar curvature problem on S^n, which involves critical exponents of Sobolev inequalities and the Kazdan-Warner obstruction for solutions [10, 48]. Because of these features, the existence of solutions to equation (1.1) is a rather complicated problem. For $n = 1$, the existence of solutions of equation (1.1) was investigated in [1, 3, 11, 13, 16, 33, 56]. Especially, in [33] Jiang, Wang and Wei proved an existence result under some nondegenerate and topological degree conditions imposed on general f. However, there are few existence results for general f in higher dimensions.

Recently, by applying a blow-up analysis method, Lu and Wang [35] obtained a priori estimates for equation (1.1) and found a sufficient condition for the existence when f is a rotationally symmetric function for all $n \geq 1$. In [34], the results of [35] were generalized slightly by a topological degree method. In [63, 65], Zhu considered the centro-affine Minkowski problem for measures, and proved the existences of solutions to the centro-affine Minkowski problem for discrete measures.

A function f defined on S^n is said to be $(n+1)$-*mirror symmetric*, if for any

$$x = (x_1, \cdots, x_{n+1}) \in S^n,$$

there is

$$f(x_1, \cdots, x_i, \cdots, x_{n+1}) = f(x_1, \cdots, \pm x_i, \cdots, x_{n+1}), \quad \forall\, 1 \leq i \leq n+1.$$

Obviously, a support function H is $(n+1)$-mirror symmetric if and only if the convex body K in $\mathbb{R}^{n+1}$ determined by H is $(n+1)$-mirror symmetric, namely K is symmetric with respect to all coordinate hyperplanes.

We denote σ_n to be the area of the unit n-sphere, and ω_{n+1} to be the volume of the unit ball in $\mathbb{R}^{n+1}$. Note that,

$$\sigma_n = (n+1)\omega_{n+1}.$$

Hence, the mean value of a continuous function, f, on S^n is

$$\frac{1}{(n+1)\omega_{n+1}} \int_{S^n} f(u) dS(u).$$

It is the first aim of this paper to establish:

Theorem 1.1. *If f is an $(n+1)$-mirror symmetric positive continuous function on the unit sphere S^n, such that the value of f restricted to the $n+1$ coordinate hyperplanes is less than the mean value of f on the unit sphere, then equation* (1.1) *admits an $(n+1)$-mirror symmetric solution.*

Obviously, the condition given in our first theorem involves only values of f itself, which is completely different from previous results that involve derivatives of f, see, e.g., [1, 33, 34, 35].

Before stating our second theorem, we need to extend the definitions of $ni(f)$ and $pi(f)$ from rotationally symmetric function (defined in [35]) to $(n+1)$-mirror symmetric function.

Let $e_1 = (1, 0, ..., 0), e_2 = (0, 1, ..., 0), e_{n+1} = (0, 0, ..., 0, 1)$ and let

$$S_j^{n-1} = \{x \in S^n : x_j = 0\},$$

for $j = 1, \cdots, n+1$. We use *coordinate* 2*-plane* to denote any 2-dimensional plane in $\mathbb{R}^{n+1}$ spanned by $\{e_i, e_j\}$ with $i \neq j$. For $j = 1, 2, \cdots, n+1$, we define functions NI_j and PI_j on S_j^{n-1} as follows:

$$NI_j(x) = \begin{cases} -\tilde{f}''(\frac{\pi}{2}), & n \geq 2, \\ \int_0^\pi (\tilde{f}'(\theta) - \tilde{f}'(\frac{\pi}{2})) \tan\theta \, d\theta, & n = 1, \end{cases}$$

and

$$PI_j(x) = \int_0^\pi \tilde{f}'(\theta) \cot\theta \, d\theta,$$

where $\tilde{f}$ is the restriction of f on the half great circle on S^n through the three points x and $\pm e_j$, parameterized by an arc parameter $\theta \in [0, \pi]$. One can easily see that NI_j and PI_j are well defined for smooth $(n+1)$-mirror symmetric functions, and are also $(n+1)$-mirror symmetric in their domain S_j^{n-1}. Note that for given x, $NI_j(x)$ and $PI_j(x)$ are just $ni(f)$ and $pi(f)$ defined in [35]. When $n = 1$, the domain of NI_j or PI_j is a set of two points. Hence, NI_j and PI_j are numbers for $n = 1$.

It is the second aim of this paper to establish:

Theorem 1.2. *Suppose f is a positive $(n+1)$-mirror symmetric function on S^n that satisfies one of the following*

(1) *When $n = 1$, $f \in C^2(S^n)$, $PI_1 > 0$ and $PI_2 > 0$;*
(2) *When $n \geq 2$, $f \in C^6(S^n)$, for each $j = 1, 2, \cdots, n+1$, $NI_j \leq 0$ on S_j^{n-1} with at least one negative value on each coordinate 2-plane, and that $PI_j \geq 0$ on S_j^{n-1} with at least one positive value on S_j^{n-1}.*

Then equation (1.1) *admits an $(n+1)$-mirror symmetric solution.*

Remark 1.3. *Here, we list some useful comments on Theorem 1.2:*

(a) *When $n = 1$, by the definitions, one can easily see that*

$$NI_1 = -PI_2, \quad NI_2 = -PI_1.$$

So our Theorem 1.2 is the same as the existence result in [35] *for $n = 1$.*

(b) *When $n = 2$, the assumptions about NI_j can be relaxed to $\int_{S_j^{n-1}} NI_j < 0$.*
(c) *Besides the definitions above, we can compute $ni(f)$ and $pi(f)$ by* (3.28) *and* (3.29).

We next to show that the function in the following example satisfies the conditions in Theorem 1.2.

Example 1.4. *Let*

$$f(x) = 2 - \sum_{i=1}^{n+1} x_i^4, \quad \forall\, x \in S^n.$$

Obviously, f is an $(n+1)$-mirror symmetric smooth positive function. For assumptions on NI_j and PI_j in Theorem 1.2, by the symmetry of f with respect to $x_1, x_2, \cdots, x_{n+1}$, we only need to verify them for NI_1 and PI_1.

For each $x = (0, x_2, \cdots, x_{n+1})$ in the equator S_1^{n-1}, the corresponding half great circle connecting $(\pm 1, 0, \cdots, 0)$ is given by

$$\gamma(\theta) = (\cos\theta, x_2 \sin\theta, \cdots, x_{n+1}\sin\theta).$$

Hence,

$$f(\theta) = f(\gamma(\theta)) = 2 - \cos^4\theta - \sin^4\theta \sum_{i=2}^{n+1} x_i^4.$$

By definition, when $n \geq 2$

$$NI_1(0, x_2, \cdots, x_{n+1}) = -f''(\frac{\pi}{2}) = -4\sum_{i=2}^{n+1} x_i^4 \leq -\frac{4}{n},$$

and

$$\begin{aligned} PI_1(0, x_2, \cdots, x_{n+1}) &= \int_0^\pi f'(\theta)\cot\theta\, d\theta \\ &= \int_0^\pi 4\cos^2\theta \Big(\cos^2\theta - \sum_{i=2}^{n+1} x_i^4 \sin^2\theta \Big)\, d\theta \\ &\geq \int_0^\pi 4\cos^2\theta \left(\cos^2\theta - \sin^2\theta\right)\, d\theta \\ &= \pi. \end{aligned}$$

Note that any generalized solution to equation (1.1) must be positive on S^n, see Corollary 2.4 in [35]. Therefore, the regularity of solutions obtained in our theorems follows the standard regularity theory about Monge-Ampère equation, see [12] for example.

The proofs of our main theorems are based on an analogous variational argument on the L_p-Minkowski problem given in Section 5 of [12]. It was proved in [12] that, for $1 < q < n+2$, a nonnegative solution H to the equation

$$\det(\nabla^2 H + HI) = \frac{f}{H^q} \quad \text{on } S^n, \tag{1.4}$$

can be obtained by considering the maximizing problem

$$\sup_{|K|=1} \inf_{\xi \in K} J[H - \xi \cdot x], \tag{1.5}$$

where the supremum is taken among all convex bodies K with volume 1, the infimum is taken among all points $\xi \in K$, H is the support function of K, and the functional J is given by

$$J[H] = \frac{1}{q-1} \int_{S^n} \frac{f}{H^{q-1}}.$$

For positive f and $1 < q < n+2$, Chou and Wang [12] proved that the maximizing sequence for (1.5) is uniformly bounded, and hence converges uniformly to a maximizer h. Further more, the maximizer h is proved to be C^1 and strictly convex. Computing the variation at h, and rescaling h by a proper constant, Chou and Wang obtained a generalized solution to (1.4).

The argument fails for $q = n+2$, because a maximizing sequence for (1.5) may fail to be uniformly bounded, and then a maximizer may not exist. However, if we can find a maximizer via a uniformly bounded maximizing sequence, the above argument still works, providing a solution to equation (1.4) with $q = n+2$, namely equation (1.1). The key step of proofs for our theorems is to show that any sequence of convex bodies that maximizing problem (1.5) and satisfying our conditions is uniformly bounded.

2. Proof of theorem 1.1

We first standardize some notation and list some basic facts about convex bodies for this section. For general reference regarding convex bodies see Schneider [47].

For $x, y \in \mathbb{R}^{n+1}$, we will write $x \cdot y$ for the standard inner product of x and y, and write $|x|$ for the Euclidean norm of x. We write $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ for the boundary of the Euclidean unit ball B^{n+1} in $\mathbb{R}^{n+1}$.

The set of $(n+1)$-mirror symmetric continuous functions on S^n is denoted by $C_{un}(S^n)$, and the set of positive continuous $(n+1)$-mirror symmetric functions on S^n is denoted by $C^+_{un}(S^n)$. An $(n+1)$-*mirror symmetric convex body* in this paper is a convex body that symmetric with respect to the $n+1$ coordinate hyperplanes. The set of $(n+1)$-mirror symmetric convex bodies is denoted by $\mathcal{K}^{n+1}_{un}$. Obviously, if $K \in \mathcal{K}^{n+1}_{un}$ then K is origin symmetric.

The *support function* $H_K : \mathbb{R}^{n+1} \to \mathbb{R}$ of a convex body K is defined, for $x \in \mathbb{R}^{n+1}$, by

$$H_K(x) = \max\{x \cdot y : y \in K\}.$$

Obviously, for $c \geq 0$ and $x \in \mathbb{R}^{n+1}$, we have

$$H_{cK}(x) = H_K(cx) = cH_K(x).$$

Let K be a convex body that contains the origin in its interior. Then the radial function $\rho_K : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}^+$ is defined by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}.$$

Let K be a convex body that contains the origin in its interior. Then the polar body, K^*, of K is defined by

$$K^* = \left\{x \in \mathbb{R}^{n+1} : x \cdot y \leq 1, \text{ for all } y \in K\right\}.$$

Obviously, for $\lambda > 0$

$$(\lambda K)^* = \frac{1}{\lambda} K^*,$$

and for $x \in \mathbb{R}^{n+1} \setminus \{0\}$,

$$\rho_{K^*}(x) = \frac{1}{H_K(x)}.$$

The following symmetric version of Blaschke-Santalo inequality will be needed: Let K be an origin symmetric convex body in $\mathbb{R}^{n+1}$, then

$$V(K)V(K^*) \leq \omega_{n+1}^2,$$

with equality if and only if K is an origin symmetric ellipsoid.

The following lemma gives a lower bound of the volumes product of a convex body and its polar.

Lemma 2.1. *Let K be an origin symmetric convex body in $\mathbb{R}^{n+1}$, then*

$$V(K)V(K^*) \geq \frac{\omega_{n+1}^2}{(n+1)^{\frac{n+1}{2}}}.$$

Proof. By John's theorem (see, e.g., [43]), there exists a unique origin symmetric ellipsoid E with maximum volume such that

$$E \subset K \subset \sqrt{n+1}E.$$

Hence,

$$(\sqrt{n+1}E)^* \subset K^* \subset E^*.$$

By this and the fact that $\left(\sqrt{n+1}E\right)^* = \frac{1}{\sqrt{n+1}}E^*$,

$$\begin{aligned} V(K)V(K^*) &\geq V(E)V(\frac{1}{\sqrt{n+1}}E^*) \\ &= \left(\frac{1}{n+1}\right)^{\frac{n+1}{2}} V(E)V(E^*) \\ &= \frac{\omega_{n+1}^2}{(n+1)^{\frac{n+1}{2}}}. \end{aligned}$$

□

The following lemma will be needed.

Lemma 2.2. *Let $\{K_i\}_{i=1}^{\infty}$ be a sequence of $(n+1)$-mirror symmetric convex bodies such that the diameter, $d(K_i)$, of K_i converges to $+\infty$, and $V(K_i) \leq c$ for some positive c and all $i \in \mathbb{N}$. Then, there exist $1 \leq i_0 \leq n$ and a subsequence, i_j, of i such that*

$$\lim_{j\to\infty} H_{K_{i_j}}(e_{i_0}) = 0.$$

Proof. Since K_i is an $(n+1)$-mirror symmetric convex body, K is origin symmetric. By John's theorem, there exists an unique ellipsoid E_i with maximum volume such that

$$E_i \subset K_i \subset \sqrt{n+1}E_i.$$

Obviously, E_i is an $(n+1)$-mirror symmetric ellipsoid (otherwise E_i is not unique). Thus,

$$H_{E_i}(e_1)\cdots H_{E_i}(e_{n+1}) = \frac{V(E_i)}{\omega_{n+1}} \leq \frac{V(K_i)}{\omega_{n+1}} \leq \frac{c}{\omega_{n+1}}. \tag{2.1}$$

By (2.1) and the fact that

$$d(E_i) = 2\max\{H_{E_i}(e_1), ..., H_{E_i}(e_{n+1})\}$$

goes to ∞, there exist $1 \leq i_0 \leq n+1$ and a subsequence, i_j, of i such that

$$\lim_{j\to\infty} H_{E_{i_j}}(e_{i_0}) = 0.$$

Since

$$E_{i_j} \subset K_{i_j} \subset \sqrt{n+1}E_{i_j},$$
$$\lim_{j\to\infty} H_{K_{i_j}}(e_{i_0}) = 0.$$

□

For $(n+1)$-mirror symmetric case, the following lemma solves problem (1.5).

Lemma 2.3. *If f is an $(n+1)$-mirror symmetric positive continuous function on S^n satisfies*

$$\max\left\{f(u) : u \in S^n \cap \{e_1^{\perp} \cup ... \cup e_{n+1}^{\perp}\}\right\} < \frac{1}{(n+1)\omega_{n+1}} \int_{S^n} f(u)dS(u),$$

then there exists a convex body $K \in \mathcal{K}_{un}^{n+1}$ with $V(K) = 1$ such that

$$\int_{S^n} H_K(u)^{-n-1} f(u)dS(u) = \sup\left\{\int_{S^n} H_L(u)^{-n-1} f(u)dS(u) : L \in \mathcal{K}_{un}^{n+1}, V(L) = 1\right\}.$$

Proof. By conditions, we can take a sequence $K_i \in \mathcal{K}_{un}^{n+1}$ such that $V(K_i) = 1$ and

$$\lim_{i\to\infty} \int_{S^n} H_{K_i}(u)^{-n-1} f(u)dS(u) = \sup\left\{\int_{S^n} H_L(u)^{-n-1} f(u)dS(u) : L \in \mathcal{K}_{un}^{n+1}, V(L) = 1\right\}.$$

Since $B_1 = \omega_{n+1}^{-1/(n+1)} B^{n+1} \in \mathcal{K}_{un}^{n+1}$ and $V(B_1) = 1$,

$$\lim_{i\to\infty} \int_{S^n} H_{K_i}(u)^{-n-1} f(u)dS(u) \geq \int_{S^n} H_{B_1}(u)^{-n-1} f(u)dS(u). \tag{2.2}$$

We next prove that K_i is bounded. We only need to show that if $\lim_{i\to\infty} d(K_i)=\infty$, then (2.2) can not hold.

Since f is continuous on S^n and

$$\max\left\{f(u): u\in S^n\cap\{e_1^\perp\cup...\cup e_{n+1}^\perp\}\right\}<\frac{1}{(n+1)\omega_{n+1}}\int_{S^n} f(u)dS(u),$$

there exists a t_0 $(0<t_0<1)$ such that

$$\max\left\{f(u): u\in\{S_1\cup\cdots\cup S_{n+1}\}\right\}<\frac{1}{(n+1)\omega_{n+1}}\int_{S^n} f(u)dS(u),$$

where

$$S_i=\{u: u\in S^n, |u\cdot e_i|\le t_0\}$$

is a spherical stripe near the great subsphere $S^n\cap e_i^\perp$.

Since $S_1\cup\cdots\cup S_{n+1}$ is a closed set and f is continuous, there exists a $\delta_0>0$ such that

$$\max\left\{f(u): u\in\{S_1\cup\cdots\cup S_{n+1}\}\right\}<\frac{1}{(n+1)\omega_{n+1}}\int_{S^n} f(u)dS(u)-\delta_0. \tag{2.3}$$

Since K_i is an $(n+1)$-mirror symmetric convex body with $V(K)=1$ and K_i is not bounded, K^* is an $(n+1)$-mirror symmetric convex body with $V(K^*)\le\omega_{n+1}^2$ (from the Blaschke-Santalo inequality) and K_i^* is not bounded. By Lemma 2.2, without loss of generality, we can suppose

$$\lim_{i\to\infty} H_{K_i^*}(e_1)=0. \tag{2.4}$$

Since K_i is between the two hyperplanes $x_1=H_{K_i^*}(e_1)$ and $x_1=-H_{K_i^*}(e_1)$,

$$\rho_{K_i^*}(u)\le\frac{H_{K_i^*}(e_1)}{t_0}$$

for all $u\in(S^n\backslash S_1)$. By this and (2.4), for $\varepsilon>$ there exists a $N_0\in\mathbb{N}$ such that

$$\rho_{K_i^*}(u)<\varepsilon \tag{2.5}$$

for all $i>N_0$ and $u\in(S^n\backslash S_1)$.

Let

$$b=\max_{u\in S^n} f(u).$$

When $n>N_0$, from the fact that $H_{K_i}=1/\rho_{K_i^*}(u)$; inequality (2.3) and the definition of b; the fact that $V(K_i^*)\le\omega_{n+1}^2$, Lemma 2.1, inequality (2.5) and the fact that

$$\int_{S^n\backslash S_1} dS(u)\le(n+1)\omega_{n+1},$$

we have

$$\begin{aligned}\int_{S^n} H_{K_i}(u)^{-n-1}f(u)dS(u) &= \int_{S_1} \rho_{K_i^*}(u)^{n+1}f(u)dS(u) + \int_{S^n\setminus S_1} \rho_{K_i^*}(u)^{n+1}f(u)dS(u)\\ &< \left(\frac{1}{(n+1)\omega_{n+1}}\int_{S^n} f(u)dS(u) - \delta_0\right)(n+1)V(K_i^*)\\ &\quad + b\int_{S^n\setminus S_1} \rho_{K_i^*}(u)^{n+1}dS(u)\\ &= \left(\frac{1}{(n+1)\omega_{n+1}}\int_{S^n} f(u)dS(u)\right)(n+1)V(K_i^*)\\ &\quad - (n+1)\delta_0 V(K_i^*) + b\int_{S^n\setminus S_1} \rho_{K_i^*}(u)^{n+1}dS(u)\\ &\le \int_{S^n} \omega_{n+1}f(u)dS(u) - \delta_0\frac{\omega_{n+1}^2}{(n+1)^{\frac{n+1}{2}-1}} + (n+1)b\omega_{n+1}\varepsilon^{n+1}.\end{aligned}$$

Since ε is arbitrary,

$$\lim_{i\to\infty}\int_{S^n} H_{K_i}(u)^{-n-1}f(u)dS(u) < \int_{S^n}\omega_{n+1}f(u)dS(u) = \int_{S^n} H_{B_1}(u)^{-n-1}f(u)dS(u),$$

where $B_1 = \omega_{n+1}^{-\frac{1}{n+1}}B^{n+1}$. This contradicts with (2.2).

Therefore, K_i is bounded. By the Blaschke selection theorem, there exists a $K \in \mathcal{K}_{un}^{n+1}$ with $V(K) = 1$ such that

$$\int_{S^n} H_K(u)^{-n-1}f(u)dS(u) = \sup\left\{\int_{S^n} H_L(u)^{-n-1}f(u)dS(u) : L \in \mathcal{K}_{un}^{n+1}, V(L) = 1\right\}.$$

□

Let $H \in C^+(S^n)$, then the Aleksandrov body associated with H is defined by

$$K = \bigcap_{u\in S^n}\left\{x \in \mathbb{R}^{n+1} : x\cdot u \le H(u)\right\}.$$

The volume $V(H)$ of a function $H \in C^+(S^n)$ is defined as the volume of the Aleksandrov body associated with H. Let $I \subset \mathbb{R}$ be an interval containing 0 and $H_t(u) = H(t,u) : I\times S^n \to (0,\infty)$ be continuous. For $t \in I$, let K_t be the Aleksandrov body associated with H_t.

The following lemma proved by Haberl et al. [23] will be needed.

Lemma 2.4. *Suppose $I \subset \mathbb{R}$ is an open interval containing 0 and that the function $H_t = H(t,u) : I \times S^n \to (0,\infty)$ is continuous. If, as $t \to 0$, then convergence in*

$$\frac{H_t - H_0}{t} \to f = \frac{\partial H_t}{\partial t}\bigg|_{t=0}$$

is uniform on S^n, and if K_t denotes the Aleksandov body associated with H_t, then

$$\lim_{t\to 0}\frac{V(K_t)-V(K_0)}{t}=\int_{S^n} f dS_{K_0}.$$

Let B be a Borel subset of S^n, and B_i $(1\le i\le n+1)$ be the set such that B_i and B are symmetric with respect to $e_i^\perp$. In this section, an $(n+1)$*-mirror symmetric Borel measure*, μ, on S^n is a measure that

$$\mu(B)=\mu(B_i)$$

for all $1\le i\le n+1$ and all Borel subset, B, of S^n. The set of $(n+1)$-mirror symmetric Borel measures on S^n is denoted by $\mathcal{M}_{un}(S^n)$.

The following lemma will be needed.

Lemma 2.5. *Let μ_1 and μ_2 be two $(n+1)$-mirror symmetric Borel measures on S^n. If for all $f\in C_{un}(S^n)$*

$$\int_{S^n} f d\mu_1=\int_{S^n} f d\mu_2,$$

then

$$\mu_1=\mu_2.$$

Proof. We only need to prove that

$$\mu_1(B)=\mu_2(B)$$

for any Borel $B\subset S^n$. Without loss of generality, we can assume that B is symmetric with respect to $e_1^\perp,...,e_{n+1}^\perp$.

For $S\subset S^n$ and $u\in S$, define

$$d(u,S)=\inf\big\{|u-v|:v\in S\big\}.$$

Let F be a closed subset of S^n that is symmetric with respect to $e_1^\perp,...,e_{n+1}^\perp$, and for $m\in\mathbb{N}$ let

$$A_m=\big\{u\in S^n, d(u,F)<\frac{1}{m}\big\}.$$

Then, A_m is open and

$$f_m(u)=\begin{cases}1-\frac{d(u,A_m)}{d(u,F)}, & u\notin F,\\ 1, & u\in F\end{cases}$$

is an $(n+1)$-mirror symmetric continuous function on S^n. By conditions of the lemma,

$$\int_{S^n} f_m d\mu_1=\int_{S^n} f_m d\mu_2.$$

On the other hand, f_m converges pointwise and monotonically to the characteristic function, $\mathbf{1}_F$, of F. By monotone convergence,

$$\mu_1(F)=\mu_2(F)$$

for all closed set F. Thus,

$$\mu_1(O) = \mu_2(O)$$

for all open set O.

For $\varepsilon > 0$, we can find a closed set F (symmetric with respect to $e_1^\perp, ..., e_{n+1}^\perp$) and an open set O (symmetric with respect to $e_1^\perp, ..., e_{n+1}^\perp$) such that

$$F \subset B \subset O,$$

and

$$\mu_i(O \setminus F) < \varepsilon$$

for $i = 1, 2$.

Since $\mu_1(O) = \mu_2(O)$,

$$\begin{aligned}|\mu_1(B) - \mu_2(B)| &\leq |\mu_1(B) - \mu_1(O)| + |\mu_2(B) - \mu_1(O)| \\ &= |\mu_1(B) - \mu_1(O)| + |\mu_2(B) - \mu_2(O)| \\ &< 2\varepsilon.\end{aligned}$$

Since ε is arbitrary,

$$\mu_1(B) = \mu_2(B).$$

□

Now, we have prepared enough to prove the main theorem. We only need to prove the following theorem.

Theorem 2.6. *If f is an $(n+1)$-mirror symmetric positive continuous function on S^n that satisfies*

$$\max\left\{f(u) : u \in S^n \cap \{e_1^\perp \cup ... \cup e_{n+1}^\perp\}\right\} < \frac{1}{(n+1)\omega_{n+1}} \int_{S^n} f(u)dS(u),$$

then there exists a convex body K_1 such that the centro-affine surface area measure of K_1 has density f.

Proof. For $q \in C_{un}^+(S^n)$, define the continuous functional, Φ, by

$$\Phi(q) = V(q) \int_{S^n} q(u)^{-n-1} f(u)dS(u).$$

We are searching for a function at which Φ attains a maximum. The search can be restricted to support functions from $\mathcal{K}_{un}^{n+1}$. In fact for a $q \in C_{un}^+(S^n)$, let K be the Alksandrov body associated with q, we have $0 < H_K \leq q$ and $V(q) = V(H_K)$. Since $f > 0$, it follows that $\Phi(H_K) \geq \Phi(q)$. Since Φ is a 0-homogeneous function, the search can be restricted to support functions of $(n+1)$-mirror symmetric convex bodies with volume 1.

By the above discussion and Lemma 2.3, there exists a convex body $K \in \mathcal{K}_{un}^{n+1}$ with $V(K) = 1$ such that

$$\Phi(H_K) = \sup_{q \in C_{un}^+(S^n)} V(q) \int_{S^n} q^{-n-1} f dS(u). \tag{2.6}$$

Suppose $g \in C_{un}(S^n)$, choose $|t|$ small enough so that $H_K^{-n-1} + tg \in C_{un}^+(S^n)$. Let

$$H_t = \left(H_K^{-n-1} + tg\right)^{-\frac{1}{n+1}}.$$

Then $K_0 = K$, $K_t \in \mathcal{K}_{un}^{n+1}$, and

$$\frac{H_t - H_0}{t} \to -\frac{1}{n+1} H_{K_0}^{2+n} g,$$

uniformly on S^n, as $t \to 0$. By Lemma 2.4,

$$\frac{d}{dt} V(K_t)\Big|_{t=0} = -\frac{1}{n+1} \int_{S^n} g H_{K_0}^{2+n} dS_{K_0}.$$

By (2.6), the fact $K_t \in \mathcal{K}_{un}^{n+1}$, and the fact that $V(K_0) = 1$,

$$\begin{aligned} 0 &= \frac{d}{dt} \Phi(H_t)\Big|_{t=0} \\ &= \left(-\frac{1}{n+1} \int_{S^n} g H_{K_0}^{2+n} dS_{K_0}\right) \left(\int_{S^n} H_{K_0}(u)^{-n-1} f(u) dS(u)\right) + \int_{S^n} g f dS(u). \end{aligned}$$

Let

$$K_1 = \left(\frac{\int_{S^n} H_{K_0}(u)^{-n-1} f(u) dS(u)}{n+1}\right)^{\frac{1}{2(n+1)}} K_0,$$

we have

$$\int_{S_{n-1}} g H_{K_1}^{2+n} dS_{K_1} = \int_{S^n} g f dS(u).$$

From Lemma 2.5, $g \in C_{un}(S^n)$ is arbitrary, and $K_1 \in \mathcal{K}_{un}^{n+1}$, we have, the centro-affine surface area measure of K_1 has a density f. $\square$

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. First, recall John's Lemma (for non-symmetric case) in convex geometry, see, e.g., [43, 55]. It says that for any convex body K in $\mathbb{R}^{n+1}$, there is a minimum ellipsoid of K, denoted by E, such that

$$\frac{1}{n+1} E \subset K \subset E,$$

where $\lambda E = \{x_0 + \lambda(x - x_0) : x \in E\}$ and x_0 is the center of E. We say K is *normalized* if the E is a ball.

It is known that for any convex body K in $\mathbb{R}^{n+1}$, one can choose a unimodular linear transformation $A^T \in SL(n+1)$, which transforms K into a normalized convex body K_A. In the following, we use H_A to denote the support function of K_A. Then

$$H_A(x) = |Ax| \cdot H\left(\frac{Ax}{|Ax|}\right), \quad x \in S^n, \tag{3.1}$$

where H is the support function of K. See [35] for more details about this type of transformation. Associating with the linear transformation, we have the following integral formula.

Lemma 3.1. *For any given integral function g on S^n, and any matrix $A \in GL(n+1)$, we have the variable substitution formula*

$$\int_{S^n} g(y)\, dS(y) = \int_{S^n} g\left(\frac{Ax}{|Ax|}\right) \cdot \frac{|\det A|}{|Ax|^{n+1}}\, dS(x). \tag{3.2}$$

Proof. This lemma easily follows from the formula for integral variable substitution in $\mathbb{R}^{n+1}$. □

As a direct result of (3.1) and (3.2), we have that

$$\int_{S^n} \frac{f}{H^{n+1}} = \int_{S^n} \frac{f_A}{H_A^{n+1}}, \quad f_A(x) = f\left(\frac{Ax}{|Ax|}\right) \tag{3.3}$$

for any unimodular linear transformation $A \in SL(n+1)$.

Now we start the proof. We will use a variational method and blow-up analyses to prove Theorem 1.2. To obtain one solution to equation (1.1), following [12] we consider the maximizing problem

$$\sup_{|K|=1} \inf_{\xi \in K} J[H(x) - \xi \cdot x], \tag{3.4}$$

where the supremum is taken among all convex bodies K with volume 1, the infimum is taken among all points $\xi \in K$, H is the support function of K, and the functional J is given by

$$J[H] = \int_{S^n} \frac{f}{H^{n+1}}. \tag{3.5}$$

By virtue of the Blaschke-Santaló inequality (1.3), we see the maximizing problem (3.4) has a finite upper bound. According to the variational argument in [12], if (3.4) has a convergent maximizing sequence, then equation (1.1) has a solution. Therefore our main work is to find a uniformly bounded maximizing sequence. Unfortunately, a maximizing sequence need not to be uniformly bounded in general. For example, when f is constant, any ellipsoid centered at the origin is a maximizing point. So our aim is to impose proper restrictions on convex bodies K and function f, such that any maximizing sequence is uniformly bounded.

Since we assume that f is $(n+1)$-mirror symmetric, we restrict ourselves to $(n+1)$-mirror symmetric convex bodies K when considering the maximizing problem (3.4). Now the variational argument above still works, and solutions coming from the maximizing problem are also $(n+1)$-mirror symmetric. We observe in this situation that

$$\inf_{\xi \in K} J[H(x) - \xi \cdot x] = J[H].$$

Also for any $(n+1)$-mirror symmetric convex body K in $\mathbb{R}^{n+1}$, the normalizing unimodular linear transformation A can be required to be

$$A = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1}) \in SL(n+1). \tag{3.6}$$

In the following, we make use of blow-up analysis method to prove that under the assumptions on f in Theorem 1.2, problem (3.4) has a uniformly bounded maximizing sequence, which then completes the proof.

For this purpose, let $\{K_k\}$ be a maximizing sequence of (3.4), and assume that the corresponding support functions H_k satisfy $\sup_{S^n} H_k \to \infty$ as $k \to \infty$. We will deduce a contradiction by this assumption. Hence, the sequence $\{H_k\}$ will be uniformly bounded, and the theorem will be proved.

For each k choose a unimodular linear transformation A_k as (3.6) which normalizes the K_k. Let K_{A_k} be the normalized convex body of K_k, and H_{A_k} be the support function of K_{A_k}. Then K_{A_k} is uniformly bounded from both below and above, by their volumes 1 and centers origin. On account of Blaschke's selection theorem, we assume without loss of generality that K_{A_k} converges to some normalized convex body $\hat{K}$, namely H_{A_k} converges to $\hat{H}$, the support function of $\hat{K}$, uniformly on S^n. Applying (3.3) and the bounded convergence theorem, one gets

$$
\begin{aligned}
J_{\sup} &:= \lim_{k\to\infty} J[H_k] \\
&= \lim_{k\to\infty} \int_{S^n} \frac{f_{A_k}}{H_{A_k}^{n+1}} \\
&= \int_{S^n} \frac{\hat{f}}{\hat{H}^{n+1}},
\end{aligned}
\tag{3.7}
$$

where $\hat{f}$ is the limit function of f_{A_k}. We will find some $(n+1)$-mirror symmetric support function H such that the volume of the convex body determined by H is 1 and

$$J[H] > J_{\sup},$$

which is a contradiction. Hence it will assert the maximizing sequence $\{H_k\}$ is uniformly bounded.

To do this, we need to explore $\hat{f}$ carefully. Write the unimodular linear transformation A_k as

$$A_k = \mathrm{diag}(\lambda_{1,k}, \lambda_{2,k}, \cdots, \lambda_{n,k}, \lambda_{n+1,k}),$$

and assume without loss of generality that

$$\lambda_{1,k} \geq \lambda_{2,k} \geq \cdots \geq \lambda_{n,k} \geq \lambda_{n+1,k}.$$

From (3.3), we have

$$f_{A_k}(x_1, x_2, \cdots, x_n, x_{n+1}) = f\left(\frac{\lambda_{1,k}x_1, \lambda_{2,k}x_2, \cdots, \lambda_{n,k}x_n, \lambda_{n+1,k}x_{n+1}}{\sqrt{\lambda_{1,k}^2x_1^2 + \lambda_{2,k}^2x_2^2 + \cdots + \lambda_{n,k}^2x_n^2 + \lambda_{n+1,k}^2x_{n+1}^2}}\right).$$

Denote

$$\delta_m := \lim_k \frac{\lambda_{m,k}}{\lambda_{1,k}}, \quad m = 1, \cdots, n+1,$$

then $\delta_m \in [0,1]$, $\delta_1 = 1$, $\delta_{n+1} = 0$ by the blow-up assumption that $\sup_{S^n} H_k \to \infty$ as $k \to \infty$. Hence

$$\hat{f}(x_1, x_2, \cdots, x_n, x_{n+1}) = f\left(\frac{x_1, \delta_2 x_2, \cdots, \delta_n x_n, 0}{\sqrt{x_1^2 + \delta_2^2 x_2^2 + \cdots + \delta_n^2 x_n^2}}\right). \tag{3.8}$$

We first consider the situation where $\delta_m \in \{0,1\}$. Then there are only n cases, namely

$$\begin{cases} \delta_1, \cdots, \delta_m = 1, \\ \delta_{m+1}, \cdots, \delta_{n+1} = 0, \end{cases}$$

for $m = 1, \cdots, n$. Now (3.8) becomes into

$$\hat{f}(x_1, \cdots, x_{n+1}) = f\left(\frac{x_1, \cdots, x_m, 0, \cdots, 0}{\sqrt{x_1^2 + \cdots + x_m^2}}\right). \tag{3.9}$$

To find some support function H such that $J[H] > J_{\sup}$, we consider the family of convex bodies $\hat{K}_{A(a)}$ with $A(a) \in SL(n+1)$ given by

$$A(a) = \begin{pmatrix} a^{\frac{n+1-m}{n+1}} I_m & 0 \\ 0 & a^{-\frac{m}{n+1}} I_{n+1-m} \end{pmatrix}, \quad a > 0,$$

where I_m and I_{n+1-m} are unit matrixes of order m and $n+1-m$ respectively. Obviously, the volume of $\hat{K}_{A(a)}$ are all 1 since that of $\hat{K}$ is 1. We compute by (3.3) that

$$\begin{aligned} J(a) := J[\hat{H}_{A(a)}] &= \int_{S^n} \frac{f}{\hat{H}_{A(a)}^{n+1}} \\ &= \int_{S^n} \frac{f_{A(a)^{-1}}}{\hat{H}^{n+1}} := \int_{S^n} \frac{f_a}{\hat{H}^{n+1}}, \end{aligned}$$

where the function f_a is defined as

$$f_a(x_1, \cdots, x_{n+1}) = f\left(\frac{x_1, \cdots, x_m, a x_{m+1}, \cdots, a x_{n+1}}{\sqrt{x_1^2 + \cdots + x_m^2 + a^2(x_{m+1}^2 + \cdots + x_{n+1}^2)}}\right), \quad a \geq 0. \tag{3.10}$$

Observing (3.7) and (3.9), one gets that

$$J(0) = J_{\sup}. \tag{3.11}$$

Now we only need to find some $a > 0$ such that $J(a) > J(0)$, namely

$$\int_{S^n} \frac{f_a - f_0}{\hat{H}^{n+1}} > 0. \tag{3.12}$$

This can be achieved by analysing the asymptotic behavior of the above integral when $a \to 0^+$. As a preparation, we introduce some notations for convenience. For the function f defined on S^n, one can extend it to $\mathbb{R}^{n+1}$ such that it is homogeneous of degree zero. For a point $x \in \mathbb{R}^{n+1}$, we write $x = (y, z)$ where

$$y = (x_1, \cdots, x_m), \quad z = (x_{m+1}, \cdots, x_{n+1}).$$

Then we can use the standard notations in Euclidean space such as f'_z for the gradient and f''_{zz} for the Hessian of f with respect to z. From now on, we always use these conventions unless explicitly stated otherwise.

To prove (3.12), we need the following lemma.

Lemma 3.2. *Let $\varphi \in C^\alpha(S^n)$ be an $(n+1)$-mirror symmetric positive function where $\alpha \in (0, 1)$. Given f_a as (3.10), there exists some sufficiently small $a > 0$ such that*

$$\int_{S^n} \varphi(x) \left(f_a(x) - f_0(x)\right) dS(x) > 0 \tag{3.13}$$

if $f \in C^2(S^n)$ for $n = 1$ and $f \in C^6(S^n)$ for $n \geq 2$, satisfying

either (a) *when $m \geq 2$, $f''_{zz}(y, 0)$ is positive semi-definite for any $|y| = 1$, and*

$$\int_{|y|=1} \operatorname{tr} f''_{zz}(y, 0)\, dS(y) > 0;$$

or (b) *when $m = 1$,*

$$\int_{\{x=(y,z)\in S^n : z/|z|=\xi\}} \frac{f'_y(x) \cdot y}{1 - |y|^2} dS(x) \leq 0, \quad \forall\, \xi \in S^{n-1},$$

and

$$\int_{S^{n-1}} dS(\xi) \int_{\{x=(y,z)\in S^n : z/|z|=\xi\}} \frac{f'_y(x) \cdot y}{1 - |y|^2} dS(x) < 0.$$

Proof. Let Λ_a denote the integral on the left hand side of (3.13). To prove (3.13), we consider three cases: $m \geq 3$, $m = 2$, $m = 1$, respectively.

First, assume $m \geq 3$. By virtue of the Taylor's expansion, for almost any $x = (y, z) \in S^n$, there exists a $t(x) \in (0, a)$ such that

$$\Lambda_a = \frac{1}{2} \int_{S^n} \varphi(x) z f''_{zz}(y, tz)\, z^T dS(x) \cdot a^2,$$

where the first order item vanishes since $f'_z(y, 0) = 0$ by its mirror-symmetry. Using the coarea formula twice, we have

$$\begin{aligned} 2a^{-2}\Lambda_a &= \int_{|y|\leq 1} \frac{dy}{\sqrt{1 - |y|^2}} \int_{|z|=\sqrt{1-|y|^2}} \varphi(y, z) z f''_{zz}(y, tz)\, z^T dS(z) \\ &= \int_0^1 dr \int_{|y|=r} \frac{dS(y)}{\sqrt{1 - r^2}} \int_{|z|=\sqrt{1-r^2}} \varphi(y, z) z f''_{zz}(y, tz)\, z^T dS(z). \end{aligned} \tag{3.14}$$

Writing

$$\rho(r) = \sqrt{1 - r^2}, \tag{3.15}$$

and applying the area formula for the scaling transformations, one gets that

(3.16)
$$\begin{aligned}2a^{-2}\Lambda_a &= \int_0^1 dr \int_{|y|=r} dS(y) \int_{|z|=1} \varphi(y,\rho z) z f''_{zz}(y,t\rho z)\, z^T \rho^{n+1-m} dS(z) \\ &= \int_0^1 dr \int_{|y|=1} dS(y) \int_{|z|=1} \varphi(ry,\rho z) z f''_{zz}(ry,t\rho z)\, z^T \rho^{n+1-m} r^{m-1} dS(z).\end{aligned}$$

Observing that f''_{zz} is homogeneous of degree -2, we find

$$\begin{aligned}\left|z f''_{zz}(ry,t\rho z)\, z^T r^{m-1}\right| &= \left| z f''_{zz}\left(\frac{ry,t\rho z}{\sqrt{r^2+t^2\rho^2}}\right) z^T \cdot \frac{r^{m-1}}{r^2+t^2\rho^2}\right| \\ &\leq \|f''_{zz}\|_{S^n} \cdot r^{m-3},\end{aligned}$$

which is bounded since $m \geq 3$. Applying the bounded convergence theorem to (3.16), one gets that

$$\lim_{a\to 0^+} 2a^{-2}\Lambda_a = \int_0^1 dr \int_{|y|=1} dS(y) \int_{|z|=1} \varphi(ry,\rho z) z f''_{zz}(y,0)\, z^T \rho^{n+1-m} r^{m-3} dS(z).$$

Observing the assumption (a), we see (3.13) holds for sufficiently small $a > 0$.

Next, assume $m = 2$. For almost any $x = (y,z) \in S^n$, we construct the following function

$$g(t;x) := f\left(\sqrt{1-t^2|z|^2}\cdot \frac{y}{|y|}, tz\right), \quad t \in [0,1].$$

By the assumption that $f \in C^6(S^n)$ is $(n+1)$-mirror symmetric, one can verify that $g'''(t;x) := \frac{\partial^3 g}{\partial t^3}(t;x)$ is bounded by $\|f\|_{C^6(S^n)}$ up to a positive constant depending only on n, for every $t \in [0,1]$ and almost every $x \in S^n$. Since g is an even function of t, by virtue of the Taylor's expansion, we have

(3.17)
$$\begin{aligned}g(t;x) - g(0;x) &= \frac{1}{2}\left[z f''_{zz}\left(\frac{y}{|y|},0\right) z^T - f'_y\left(\frac{y}{|y|},0\right)\cdot \frac{y}{|y|}|z|^2\right] t^2 + O(1)t^3 \\ &= \frac{1}{2} z f''_{zz}\left(\frac{y}{|y|},0\right) z^T t^2 + O(1)t^3,\end{aligned}$$

where the second equality holds because of $\nabla f(x)\cdot x = 0$ by its homogeneity. Note that

$$g\left(\frac{a}{\sqrt{|y|^2+a^2|z|^2}};x\right) = f_a(x), \quad \forall\, a \in [0,1].$$

It follows from (3.17) that

$$\begin{aligned}\Lambda_a &= \frac{1}{2}\int_{S^n} \varphi(x) z f''_{zz}\left(\frac{y}{|y|}, 0\right) z^T \left(\frac{a}{\sqrt{|y|^2 + a^2|z|^2}}\right)^2 dS(x) \\ &\quad + O(1)\int_{S^n} \left(\frac{a}{\sqrt{|y|^2 + a^2|z|^2}}\right)^3 dS(x) \\ &:= \frac{1}{2} I_a + O(1) II_a.\end{aligned} \tag{3.18}$$

Similar to (3.14), we compute

$$I_a = \int_0^1 dr \int_{|y|=r} \frac{dS(y)}{\rho(r)} \int_{|z|=\rho(r)} \varphi(y,z) z f''_{zz}\left(\frac{y}{|y|}, 0\right) z^T \cdot \frac{a^2}{r^2 + a^2\rho^2} dS(z),$$

where $\rho(r)$ is the same as (3.15). Using the area formula for the scaling transformations, we obtain that

$$I_a = \int_0^1 dr \int_{|y|=1} dS(y) \int_{|z|=1} \varphi(ry, \rho z) z f''_{zz}(y,0)\, z^T \rho^{n+1-m} \cdot \frac{a^2 r^{m-1}}{r^2 + a^2\rho^2} dS(z).$$

Note that $m = 2$ and $\varphi \in C^\alpha(S^n)$, one can see that as $a \to 0^+$,

$$\begin{aligned}I_a &= \int_0^1 \frac{a^2 r dr}{r^2 + a^2\rho^2} \int_{|y|=1} dS(y) \int_{|z|=1} \varphi(0,z) z f''_{zz}(y,0)\, z^T dS(z) + O(a^2) \\ &= -a^2 \log a \cdot \left(\int_{|y|=1} dS(y) \int_{|z|=1} \varphi(0,z) z f''_{zz}(y,0)\, z^T dS(z) + o(1)\right).\end{aligned} \tag{3.19}$$

Following the same steps to compute the I_a, we find that

$$II_a = O(a^2), \quad \text{as } a \to 0^+. \tag{3.20}$$

Substituting (3.19) and (3.20) in (3.18), one has

$$\Lambda_a = -a^2 \log a \cdot \left(\frac{1}{2}\int_{|y|=1} dS(y) \int_{|z|=1} \varphi(0,z) z f''_{zz}(y,0)\, z^T dS(z) + o(1)\right), \text{ as } a \to 0^+.$$

Now our assumption (a) tells that (3.13) is true when a is positive and becomes very close to 0.

Finally, assume $m = 1$. We start from the following equality,

$$f_a(x) - f_0(x) = \int_0^a f'_z(y, tz) \cdot z dt$$

for almost every $x = (y,z) \in S^n$. Then

$$\begin{aligned}\Lambda_a &= \int_{S^n} \varphi(x) dS(x) \int_0^a f'_z(y, tz) \cdot z dt \\ &= \int_0^a dt \int_{S^n} \varphi(x) f'_z(y, tz) \cdot z dS(x).\end{aligned} \tag{3.21}$$

To deal with the inner integral above, for $0 < t < a$, we use the variable substitution (3.2) with

$$A = \begin{pmatrix} t^{\frac{n}{n+1}} & 0 \\ 0 & t^{-\frac{1}{n+1}} I_n \end{pmatrix}$$

to obtain that

$$\begin{aligned} \int_{S^n} \varphi(x) f_z'(y, tz) \cdot z dS(x) &= t \int_{S^n} \varphi\left(\frac{ty, z}{|ty, z|}\right) f_z'\left(\frac{tx}{|ty, z|}\right) \cdot \frac{z}{|ty, z|^{n+2}} dS(x) \\ &= \int_{S^n} \varphi\left(\frac{ty, z}{|ty, z|}\right) \frac{f_z'(x) \cdot z}{|ty, z|^{n+1}} dS(x) \\ &:= G(t), \end{aligned} \tag{3.22}$$

where we have used the notation $|ty, z| = |(ty, z)| = \sqrt{t^2|y|^2 + |z|^2}$ and the fact that f_z' is homogeneous of degree -1 for the second equality. Hence (3.21) is turned to be the following

$$\Lambda_a = \int_0^a G(t) dt. \tag{3.23}$$

One need to analyse the asymptotic behavior of $G(t)$ as $t \to 0^+$. For this, we claim that *for every* $x = (y, z) \in S^n$,

$$|f_z'(x)| \le 2 \|f\|_{C^2(S^n)} |z|. \tag{3.24}$$

Indeed, if $|z| \ge 1/\sqrt{2}$, then

$$|f_z'(x)| \le \|f\|_{C^2(S^n)} \le 2 \|f\|_{C^2(S^n)} |z|.$$

If $|z| \le 1/\sqrt{2}$, then $|y| \ge 1/\sqrt{2}$. Noting $f_z'(y, 0) = 0$, we estimate $f_z'(x)$ as follows:

$$\begin{aligned} |f_z'(x)| &= |f_z'(y, z) - f_z'(y, 0)| \\ &\le \sup_{\lambda \in (0,1)} |f_{zz}''(y, \lambda z)| \cdot |z| \\ &= \sup_{\lambda \in (0,1)} \left| f_{zz}''\left(\frac{y, \lambda z}{|y, \lambda z|}\right) \frac{1}{|y, \lambda z|^2} \right| \cdot |z| \\ &\le 2 \|f\|_{C^2(S^n)} |z|. \end{aligned}$$

Hence the inequality (3.24) holds. Therefore, we have

$$\left| \varphi\left(\frac{ty, z}{|ty, z|}\right) \frac{f_z'(x) \cdot z}{|ty, z|^{n+1}} \right| \le \frac{C}{|z|^{n-1}}, \quad \forall\, x \in S^n \tag{3.25}$$

with $C = 2 \|\varphi\|_{C(S^n)} \|f\|_{C^2(S^n)}$. Since $m = 1$, simple calculations show that

$$\int_{S^n} \frac{1}{|z|^{n-1}} dS(x) = C(n) < +\infty.$$

Noting the definition of $G(t)$ given in (3.22), and using the estimate (3.25) and the dominated convergence theorem, one can see that

$$\begin{aligned}\lim_{t\to 0^+} G(t) &= \int_{S^n} \varphi\left(0, \frac{z}{|z|}\right) \frac{f_z'(x)\cdot z}{|z|^{n+1}} dS(x) \\ &= -\int_{S^n} \varphi\left(0, \frac{z}{|z|}\right) \frac{f_y'(x)\cdot y}{|z|^{n+1}} dS(x),\end{aligned} \tag{3.26}$$

where the fact $\nabla f(x)\cdot x = 0$ has been used for the second equality. Hence, by (3.23) we get that

$$\lim_{a\to 0^+} a^{-1}\Lambda_a = \lim_{t\to 0^+} G(t).$$

This, together with the coarea formula, implies that

$$\lim_{a\to 0^+} a^{-1}\Lambda_a = -\int_{S^{n-1}} \varphi(0,\xi)\, dS(\xi) \int_{\{x=(y,z)\in S^n : z/|z|=\xi\}} \frac{f_y'(x)\cdot y}{1-|y|^2} dS(x). \tag{3.27}$$

Applying the assumption (b) to (3.27) we see that (3.13) holds for sufficiently small $a > 0$. In this way, we have completed the proof of the lemma. □

We continue to prove Theorem 1.2. As was said before, it is sufficient to prove (3.12). By virtue of Lemma 3.2, it is enough to verify the assumptions on f given in Theorem 1.2 imply the ones (a) and (b) in Lemma 3.2. In fact, any half great circle γ connecting some pair of axial antipodal points, say $(\pm 1, 0, \cdots, 0)$ for example, can be parameterized as

$$\gamma(\theta) = (\cos\theta, \sin\theta\cdot\xi), \quad \theta\in[0,\pi],$$

where $\xi \in S^{n-1}$. Then the restriction of f on the half great circle γ is written as

$$f(\theta) = f(\gamma(\theta)) = f(\cos\theta, \sin\theta\cdot\xi).$$

We compute

$$\begin{aligned} f'(\theta) &= -\sin\theta\, f_1' + \cos\theta\, f_\xi'\cdot\xi \\ &= -\sin\theta\, f_1' - \cos\theta\, f_1'\cot\theta \\ &= -\frac{f_1'}{\sin\theta}, \end{aligned}$$

where that $\nabla f(x)\cdot x = 0$ was used for the second equality. Therefore one immediately gets that

$$-ni(f) = f''(\pi/2) = f_{11}''(0,\xi), \quad \text{for } n \geq 2, \tag{3.28}$$

and

$$\int_\gamma \frac{f_1'(x)\cdot x_1}{1-|x_1|^2} dS(x) = -\int_0^\pi f'(\theta)\cot\theta\, d\theta = -pi(f), \quad \text{for } n\geq 1. \tag{3.29}$$

Similar computations as (3.28) and (3.29) hold for f along any other half great circles. Hence we can apply Lemma 3.2 to the f in Theorem 1.2. Therefore one finds some $a > 0$ such that the inequality (3.12) is true.

For general situation where $\delta_m \in [0,1]$, we can reduce it to the special case that we have just considered above. Indeed, we observe the fact that $\hat{H}$ is normalized is not necessary in our above proof. Let

$$A_\delta = \mathrm{diag}(a_1, a_2, \cdots, a_n, a_{n+1}),$$

where for $m = 1, \cdots, n$,

$$a_m = \begin{cases} 1, & \text{if } \delta_m = 0, \\ \delta_m^{-1}, & \text{if } 0 < \delta_m \le 1, \end{cases}$$

and a_{n+1} is chosen such that $A_\delta \in SL(n+1)$. Then by (3.3) and (3.7), we have

$$\begin{aligned} J_{\sup} &= \int_{S^n} \frac{\hat{f}}{\hat{H}^{n+1}} \\ &= \int_{S^n} \frac{\hat{f}_{A_\delta}}{\hat{H}_{A_\delta}^{n+1}} := \int_{S^n} \frac{\tilde{f}}{\tilde{H}^{n+1}}. \end{aligned}$$

Simple computation shows that if $\tilde{f}$ is written as the form of the right hand side of (3.8), then δ_m is equal to 0 or 1. Replacing $\hat{f}$ by $\tilde{f}$ and $\hat{H}$ by $\tilde{H}$ in the previous arguments, we still get (3.12).

To summarize, we have got a contradiction for any possible blow-up sequences under the assumptions of Theorem 1.2, according to our previous discussion, which completes the proof of the theorem.

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