THE L_p MINKOWSKI PROBLEM FOR POLYTOPES FOR $p < 0$

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Abstract. Existence of solutions to the L_p Minkowski problem is proved for all $p < 0$. For the critical case of $p = -n$, which is known as the centro-affine Minkowski problem, this paper contains the main result in [71] as a special case.

1. Introduction

A *convex body* in n-dimensional Euclidean space, $\mathbb{R}^n$, is a compact convex set that has non-empty interior. If $p \in \mathbb{R}$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the L_p surface area measure, $S_p(K, \cdot)$, of K is a Borel measure on the unit sphere, S^{n-1}, defined for each Borel $\omega \subset S^{n-1}$ by

$$S_p(K,\omega) = \int_{x\in\nu_K^{-1}(\omega)} (x \cdot \nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of boundary points of K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.

The L_p surface area measure was introduced by Lutwak [40]. The L_p surface area measure contains three important measures as special cases: the L_1 surface area measure is the classic surface area measure; the L_0 surface area measure is the cone-volume measure; the L_{-n} surface area measure is the centro-affine surface area measure. Today, the L_p surface area measure is a central notation in convex geometry analysis, and appeared in, e.g., [3, 8, 21–28, 36–51, 53, 55–59, 64–66].

The following L_p Minkowski problem that posed by Lutwak [40] is considered as one of the most important problems in modern convex geometry analysis.

L_p **Minkowski problem:** *Find necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} so that μ is the L_p surface area measure of a convex body in $\mathbb{R}^n$.*

The associated partial differential equation for the L_p Minkowski problem is the following Mong-Ampère type equation: For a given positive function f on the unit sphere, solve

$$h^{1-p} \det(h_{ij} + h\delta_{ij}) = f, \tag{1.1}$$

where h_{ij} is the covariant derivative of h with respect to an orthonormal frame on S^{n-1} and δ_{ij} is the Kronecker delta.

The solutions of the L_p Minkowski problem have important applications to affine isoperimetric inequalities, see, e.g., Zhang [69], Lutwak, Yang and Zhang [45], Ciachi, Lutwak,

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Yang and Zhang [12], Haberl and Schuster [25–27]. The solutions to the L_p Minkowski problem are also related with some important flows (see, e.g., [1, 2, 60, 61]).

When $p = 1$, the L_p Minkowski problem is the classical Minkowski problem. The existence and uniqueness for the solution of this problem was solved by Minkowski, Aleksandrov, and Fenchel and Jessen (see Schneider [56] for references). Regularity of the Minkowski problem was studied by e.g., Caffarelli [7], Cheng and Yau [10], Nirenberg [52] and Pogorelov [54].

For $p \neq 1$, the L_p Minkowski problem was studied by, e.g., Lutwak [40], Lutwak and Oliker [41], Lutwak, Yang and Zhang [46], Chou and Wang [11], Guan and Lin [19], Hug, Lutwak, Yang and Zhang [22], Böröczky, Hegedűs and Zhu [4], Böröczky, Lutwak, Yang and Zhang [5,6], Chen [9], Dou and Zhu [14], Haberl, Lutwak, Yang and Zhang [22], Huang, Liu and Xu [30], Jian, Lu and Wang [32], Jian and Wang [33], Jiang, Wang and Wei [34], Lu and Wang [35], Stancu [60, 61], Sun and Long [62] and Zhu [70–72]. Analogues of the Minkowski problems were studied in, e.g., [13, 15, 16, 18, 20, 29, 67].

The uniqueness of solutions to the L_p Minkowski for $p > 1$ can be shown by applying the L_p Minkowski inequality established by Lutwak [40]. However, little is know about the L_p Minkowski inequality for the case where $p < 1$. This is one of the main reasons that most of the previous work on the L_p Minkowski problem was limited to the case where $p > 1$.

The critical case where $p = -n$ of the L_p Minkowski problem is called the centro-affine Minkowski problem, which describes the centro-affine surface area measure. This problem is especially important due to the affine invariant of the partial differential equation (1.1). It is known that the centro-affine Minkowski problem has connections with several important geometric problems (see, e.g., Jian and Wang [33] for reference). The centro-affine Minkowski problem was explicitly posed by Chow and Wang [11]. Recently, the centro-affine Minkowski problem was studied by Lu and Wang [35] for rotationally symmetric case and was studied by Zhu [71] for discrete measures.

When $p < -n$, very few results are known for the L_p Minkowski problem. So far as the author knows, in $\mathbb{R}^2$, the L_p Minkowski problem for all $p < 0$ was studied by Dou and Zhu [14], Sun and Long [62]. It is the aim of this paper to study the L_p Minkowski problem for all $p < 0$ and $n \geq 2$.

It is know that the Minkowski problem and the L_p Minkowski problem (for $p > 1$) for arbitrary measures can be solved by an approximation argument by first solving the polytopal case (see, e.g., [31] or [56] pp. 392-393). This is one of the reasons why the Minkowski problem and the L_p Minkowski problem for polytopes are of great importance.

A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ provided that it has positive n-dimensional volume. The convex hull of a subset of these points is called a *facet* of the polytope if it lies entirely on the boundary of the polytope and has positive $(n-1)$-dimensional volume. Let P be a polytope which contains the origin in its interior with N facets whose outer unit normals are $u_1, ..., u_N$, and such that the facet with outer unit normal u_k has area a_k and distance h_k from the origin for all $k \in \{1, ..., N\}$. Then,

$$S_p(P, \cdot) = \sum_{k=1}^{N} h_k^{1-p} a_k \delta_{u_k}(\cdot).$$

where δ_{u_k} denotes the delta measure that is concentrated at the point u_k.

A finite subset U of S^{n-1} is said to be *in general position* if any k elements of U, $1 \le k \le n$, are linearly independent.

In [71], the author solved the centro-affine Minkowski problem for polytopes whose outer unit normals are in general position:

Theorem A. *Let μ be a discrete measure on the unit sphere S^{n-1}. Then μ is the centro-affine surface area measure of a polytope whose outer unit normals are in general position if and only if the support of μ is in general position and not concentrated on a closed hemisphere.*

A linear subspace X $(0 < \dim X < n)$ of $\mathbb{R}^n$ is said to be *essential* with respect to a Borel measure μ on S^{n-1} if $X \cap \operatorname{supp}(\mu)$ is not concentrated on any closed hemisphere of $X \cap S^{n-1}$.

Obviously, if the support of a discrete measure μ is in general position, then the set of essential subspaces of μ is empty. On the other hand, in $\mathbb{R}^n$ $(n \ge 3)$, one can easily construct a discrete measure μ such that μ does not have essential subspace but the support of μ is not in general position. Therefore, the set of discrete measures whose supports are in general position is a subset of the set of discrete measures that do not have essential subspaces.

It is the aim of this paper to solve the L_p Minkowski problem for discrete measures that do not have essential subspaces. Obviously, the following main theorem of this paper contains Theorem A as a special case.

Theorem 1.1. *Let $p < 0$ and μ be a discrete measure on the unit sphere S^{n-1}. Then μ is the L_p surface area measure of a polytope whose L_p surface area measure does not have essential subspace if and only if μ does not have essential subspace and not concentrated on a closed hemisphere.*

2. Preliminaries

In this section, we standardize some notations and list some basic facts about convex bodies. For general references regarding convex bodies, see, e.g., [17, 56, 63].

The sets in this paper are subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we write $x \cdot y$ for the standard inner product of x and y, $|x|$ for the Euclidean norm of x, and S^{n-1} for the unit sphere of $\mathbb{R}^n$.

Suppose S is a subset of $\mathbb{R}^n$, then the positive hull, $\operatorname{pos}(S)$, of S is the set of all positive combinations of any finitely many elements of S. Let $\operatorname{lin}(S)$ be the smallest linear subspace of $\mathbb{R}^n$ containing S. The diameter of a subset, S, of $\mathbb{R}^n$ is defined by

$$d(S) = \max\{|x - y| : x, y \in S\}.$$

The convex hull of a subset, S, of $\mathbb{R}^n$ is defined by

$$\mathrm{Conv}\ (S) = \{\lambda x + (1-\lambda) y : 0 \le \lambda \le 1 \text{ and } x, y \in S\}.$$

For convex bodies K_1, K_2 in $\mathbb{R}^n$ and $s_1, s_2 \ge 0$, the Minkowski combination is defined by

$$s_1 K_1 + s_2 K_2 = \{s_1 x_1 + s_2 x_2 : x_1 \in K_1, x_2 \in K_2\}.$$

The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ of a convex body K is defined, for $x \in \mathbb{R}^n$, by

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$

Obviously, for $s \geq 0$ and $x \in \mathbb{R}^n$,

$$h(sK, x) = h(K, sx) = sh(K, x).$$

If K is a convex body in $\mathbb{R}^n$ and $u \in S^{n-1}$, then the *support set* $F(K, u)$ of K in direction u is defined by

$$F(K, u) = K \cap \{x \in \mathbb{R}^n : x \cdot u = h(K, u)\}.$$

The *Hausdorff distance* of two convex bodies K_1, K_2 in $\mathbb{R}^n$ is defined by

$$\delta(K_1, K_2) = \inf\{t \geq 0 : K_1 \subset K_2 + tB^n, K_2 \subset K_1 + tB^n\},$$

where B^n is the unit ball.

Let $\mathcal{P}$ be the set of polytopes in $\mathbb{R}^n$. If the unit vectors $u_1, ..., u_N$ are not concentrated on a closed hemisphere, let $\mathcal{P}(u_1, ..., u_N)$ be the subset of $\mathcal{P}$ such that a polytope $P \in \mathcal{P}(u_1, ..., u_N)$ if the the set of the outer unit normals of P is a subset of $\{u_1, ..., u_N\}$. Let $\mathcal{P}_N(u_1, ..., u_N)$ be the subset of $\mathcal{P}(u_1, ..., u_N)$ such that a polytope $P \in \mathcal{P}_N(u_1, ..., u_N)$ if, $P \in \mathcal{P}(u_1, ..., u_N)$, and P has exactly N facets.

3. An extremal problem related to the L_p Minkowski problem

Suppose $p < 0$, $\alpha_1, ..., \alpha_N > 0$, the unit vectors $u_1, ..., u_N$ are not concentrated on a closed hemisphere, and $P \in \mathcal{P}(u_1, ..., u_N)$. Define the function, $\Phi_P : \text{Int } (P) \to \mathbb{R}$, by

$$\Phi_P(\xi) = \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi \cdot u_k)^p.$$

In this section, we study the extremal problem

$$\sup\{\inf_{\xi \in \text{Int } (Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1\}. \tag{3.1}$$

The main purpose of this section is to prove that a dilation of the solution to problem (3.1) solves the corresponding L_p Minkowski problem.

Lemma 3.1. *If $p < 0$, $\alpha_1, ..., \alpha_N > 0$, the unit vectors $u_1, ..., u_N$ are not concentrated on a closed hemisphere and $P \in \mathcal{P}(u_1, ..., u_N)$, then there exists a unique $\xi(P) \in$ Int (P) such that*

$$\Phi_P(\xi(P)) = \inf_{\xi \in \mathit{Int}\ (P)} \Phi_P(\xi).$$

Proof. Since $p < 0$, the function $f(t) = t^p$ is strictly convex on $(0, +\infty)$. Hence, for $0 < \lambda < 1$ and $\xi_1, \xi_2 \in \text{Int } (P)$,

$$\begin{aligned}
\lambda \Phi_P(\xi_1) + (1-\lambda)\Phi_P(\xi_2) &= \lambda \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi_1 \cdot u_k)^p + (1-\lambda) \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi_2 \cdot u_k)^p \\
&= \sum_{k=1}^{N} \alpha_k \left[\lambda (h(P, u_k) - \xi_1 \cdot u_k)^p + (1-\lambda)(h(P, u_k) - \xi_2 \cdot u_k)^p\right] \\
&\geq \sum_{k=1}^{N} \alpha_k \left[h(P, u_k) - (\lambda \xi_1 + (1-\lambda)\xi_2) \cdot u_k\right]^p \\
&= \Phi_P(\lambda \xi_1 + (1-\lambda)\xi_2).
\end{aligned}$$

Equality hold if and only if $\xi_1 \cdot u_k = \xi_2 \cdot u_k$ for all $k = 1, ..., N$. Since $u_1, ..., u_N$ are not concentrated on a closed hemisphere, $\mathbb{R}^n = \text{lin}\{u_1, ..., u_N\}$. Thus, $\xi_1 = \xi_2$. Hence, Φ_P is strictly convex on Int (P).

From the fact that $P \in \mathcal{P}(u_1, ..., u_N)$, we have, for any $x \in \partial P$, there exists a $u_{i_0} \in \{u_1, ..., u_N\}$ such that

$$h(P, u_{i_0}) = x \cdot u_{i_0}.$$

Thus, $\Phi_P(\xi) \to \infty$ whenever $\xi \in$ Int (P) and $\xi \to x$. Therefore, there exists a unique interior point $\xi(P)$ of P such that

$$\Phi_P(\xi(P)) = \inf_{\xi \in \text{Int }(P)} \Phi_P(\xi).$$

□

Obviously, for $\lambda > 0$ and $P \in \mathcal{P}(u_1, ..., u_N)$,

$$\xi(\lambda P) = \lambda \xi(P), \tag{3.2}$$

and if $P_i \in \mathcal{P}(u_1, ..., u_N)$ and P_i converges to a polytope P, then $P \in \mathcal{P}(u_1, ..., u_N)$.

Lemma 3.2. *If $p < 0$, $\alpha_1, ..., \alpha_N > 0$, the unit vectors $u_1, ..., u_N$ are not contained in a closed hemisphere, $P_i \in \mathcal{P}(u_1, ..., u_N)$, and P_i converges to a polytope P, then $\lim_{i\to\infty} \xi(P_i) = \xi(P)$ and*

$$\lim_{i\to\infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

Proof. Since P_i converges to P and $\xi(P_i) \in$ Int (P_i), $\xi(P_i)$ is bounded. Let ξ_0 be the limit point of a subsequence, $\xi(P_{i_j})$, of $\xi(P_i)$. We claim that $\xi_0 \in$ Int (P). Otherwise, ξ_0 is a boundary point of P with $\lim_{j\to\infty} \Phi_{P_{i_j}}(\xi_{P_{i_j}}) = \infty$, which contradicts the fact that

$$\overline{\lim_{j\to\infty}} \Phi_{P_{i_j}}(\xi(P_{i_j})) \leq \overline{\lim_{j\to\infty}} \Phi_{P_{i_j}}(\xi(P)) = \Phi(\xi(P)) < \infty. \tag{3.3}$$

We claim that $\xi_0 = \xi(P)$. Otherwise,

$$\begin{aligned}\lim_{j\to\infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) &= \Phi_P(\xi_0)\\ &> \Phi_P(\xi(P))\\ &= \lim_{j\to\infty} \Phi_{P_{i_j}}(\xi(P)).\end{aligned}$$

This contradicts the fact that

$$\Phi_{P_{i_j}}(\xi(P_{i_j})) \leq \Phi_{P_{i_j}}(\xi(P)).$$

Hence, $\lim_{i\to\infty} \xi(P_i) = \xi(P)$ and

$$\lim_{i\to\infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

□

Lemma 3.3. *If $p < 0$, $\alpha_1, ..., \alpha_N > 0$, the unit vectors $u_1, ..., u_N$ are not concentrated on a closed hemisphere and $P \in \mathcal{P}(u_1, ..., u_N)$, then*

$$\sum_{k=1}^{N} \alpha_k \frac{u_k}{[h(P, u_k) - \xi(P) \cdot u_k]^{1-p}} = 0.$$

Proof. Define f :Int$(P) \to \mathbb{R}^n$ by

$$f(x)=\sum_{k=1}^{N}\alpha_k(h(P,u_k)-x\cdot u_k)^p.$$

By conditions,

$$f(\xi(P))=\inf_{x\in Int(P)} f(x).$$

Thus,

$$\sum_{k=1}^{N}\alpha_k\frac{u_{k,i}}{[h(P,u_k)-\xi(P)\cdot u_k]^{1-p}}=0,$$

for all $i=1,...,n$, where $u_k=(u_{k,1},...,u_{k,n})^T$. Therefore,

$$\sum_{k=1}^{N}\alpha_k\frac{u_k}{[h(P,u_k)-\xi(P)\cdot u_k]^{1-p}}=0.$$

□

Lemma 3.4. *Suppose $p<0$, $\alpha_1,..,\alpha_N>0$, the unit vectors $u_1,...,u_N$ are not concentrated on a closed hemisphere, and there exists a $P\in\mathcal{P}_N(u_1,...,u_N)$ with $\xi(P)=o$, $V(P)=1$ such that*

$$\Phi_P(o)=\sup\left\{\inf_{\xi\in Int\ (Q)}\Phi_Q(\xi):Q\in\mathcal{P}(u_1,...,u_N)\ \textit{and}\ V(Q)=1\right\}.$$

Then,

$$S_p(P_0,\cdot)=\sum_{k=1}^{N}\alpha_k\delta_{u_k}(\cdot),$$

where $P_0=\left(\sum_{j=1}^{N}\alpha_jh(P,u_j)^p/n\right)^{\frac{1}{n-p}}P$.

Proof. By conditions, there exists a polytope $P\in\mathcal{P}_N(u_1,...,u_N)$ with $\xi(P)=o$ and $V(P)=1$ such that

$$\Phi_P(o)=\sup\{\inf_{\xi\in\mathrm{Int}\ (Q)}\Phi_Q(\xi):Q\in\mathcal{P}(u_1,...,u_N)\text{ and }V(Q)=1\},$$

where $\Phi_Q(\xi)=\sum_{k=1}^{N}\alpha_k(h(Q,u_k)-\xi\cdot u_k)^p$.

For $\tau_1,...,\tau_N\in\mathbb{R}$, choose $|t|$ small enough so that the polytope P_t defined by

$$P_t=\bigcap_{i=1}^{N}\{x:x\cdot u_i\le h(P,u_i)+t\tau_i\}$$

has exactly N facets. By [56] (Lemma 7.5.3),

$$\frac{\partial V(P_t)}{\partial t}=\sum_{i=1}^{N}\tau_ia_i,$$

where a_i is the area of $F(P,u_i)$. Let $\lambda(t)=V(P_t)^{-\frac{1}{n}}$, then $\lambda(t)P_t\in\mathcal{P}^n_N(u_1,...,u_N)$, $V(\lambda(t)P_t)=1$ and

$$\lambda'(0)=-\frac{1}{n}\sum_{i=1}^N\tau_iS_i. \tag{3.4}$$

Define $\xi(t):=\xi(\lambda(t)P_t)$, and

$$\begin{aligned}\Phi(t)&:=\min_{\xi\in\lambda(t)P_t}\sum_{k=1}^N\alpha_k(\lambda(t)h(P_t,u_k)-\xi\cdot u_k)^p\\&=\sum_{k=1}^N\alpha_k(\lambda(t)h(P_t,u_k)-\xi(t)\cdot u_k)^p.\end{aligned} \tag{3.5}$$

It follows from Lemma 3.3 that

$$\sum_{k=1}^N\alpha_k\frac{u_{k,i}}{[\lambda(t)h(P_t,u_k)-\xi(t)\cdot u_k]^{1-p}}=0,$$

for $i=1,...,n$, where $u_k=(u_{k,1},...,u_{k,n})^T$. In addition, since $\xi(P)$ is the origin,

$$\sum_{k=1}^N\alpha_k\frac{u_k}{h(P,u_k)^{1-p}}=0. \tag{3.6}$$

Let $F=(F_1,...,F_n)$ be a function from an open neighbourhood of the origin in $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ such that

$$F_i(t,\xi_1,...,\xi_n)=\sum_{k=1}^N\alpha_k\frac{u_{k,i}}{[\lambda(t)h(P_t,u_k)-(\xi_1u_{k,1}+...+\xi_nu_{k,n})]^{1-p}}$$

for $i=1,...,n$. Then,

$$\left.\frac{\partial F_i}{\partial t}\right|_{(t,\xi_1,...,\xi_n)}=\sum_{k=1}^N\frac{(p-1)\alpha_ku_{k,i}\left[\lambda'(t)h(P_t,u_k)+\lambda(t)\tau_k\right]}{\left[\lambda(t)h(P_t,u_k)-(\xi_1u_{k,1}+...+\xi_nu_{k,n})\right]^{2-p}},$$

$$\left.\frac{\partial F_i}{\partial \xi_j}\right|_{(t,\xi_1,...,\xi_n)}=\sum_{k=1}^N\frac{(1-p)\alpha_ku_{k,i}u_{k,j}}{\left[\lambda(t)h(P_t,u_k)-(\xi_1u_{k,1}+...+\xi_nu_{k,n})\right]^{2-p}}$$

are continuous on a small neighbourhood of $(0,0,...,0)$ with

$$\left(\left.\frac{\partial F}{\partial \xi}\right|_{(0,...,0)}\right)_{n\times n}=\sum_{k=1}^N\frac{(1-p)\alpha_k}{h(P,u_k)^{2-p}}u_k\cdot u_k^T,$$

where $u_ku_k^T$ is an $n\times n$ matrix.

Since $u_1, ..., u_N$ are not contained in a closed hemisphere, $\mathbb{R}^n = \text{lin}\{u_1, ..., u_N\}$. Thus, for any $x \in \mathbb{R}^n$ with $x \neq 0$, there exists a $u_{i_0} \in \{u_1, ..., u_N\}$ such that $u_{i_0} \cdot x \neq 0$. Then,

$$\begin{aligned} x^T \cdot \left(\sum_{k=1}^N \frac{(1-p)\alpha_k}{h(P,u_k)^{2-p}} u_k \cdot u_k^T \right) \cdot x &= \sum_{k=1}^N \frac{(1-p)\alpha_k}{h(P,u_k)^{2-p}} (x \cdot u_k)^2 \\ &\geq \frac{(1-p)\alpha_{i_0}}{h(P,u_{i_0})^{2-p}} (x \cdot u_{i_0})^2 > 0. \end{aligned}$$

Therefore, $(\frac{\partial F}{\partial \xi}\big|_{(0,...,0)})$ is positive defined. By this, the fact that $F_i(0,...,0) = 0$ for all $i = 1, ..., n$, the fact that $\frac{\partial F_i}{\partial \xi_j}$ is continuous on a neighbourhood of $(0, 0, ..., 0)$ for all $0 \leq i, j \leq n$ and the implicit function theorem, we have

$$\xi'(0) = (\xi_1'(0), ..., \xi_n'(0))$$

exists.

From the fact that $\Phi(0)$ is an extreme value of $\Phi(t)$ (in Equation (3.5)), Equation (3.4) and Equation (3.6), we have

$$\begin{aligned} 0 &= \Phi'(0)/p \\ &= \sum_{k=1}^N \alpha_k h(P,u_k)^{p-1} \left(\lambda'(0) h(P,u_k) + \tau_k - \xi'(0) \cdot u_k \right) \\ &= \sum_{k=1}^N \alpha_k h(P,u_k)^{p-1} \left[-\frac{1}{n} \left(\sum_{i=1}^N a_i \tau_i \right) h(P,u_k) + \tau_k \right] - \xi'(0) \cdot \left[\sum_{k=1}^N \alpha_k \frac{u_k}{h(P,u_k)^{1-p}} \right] \\ &= \sum_{k=1}^N \alpha_k h(P,u_k)^{p-1} \tau_k - \left(\sum_{i=1}^N a_i \tau_i \right) \frac{\sum_{k=1}^N \alpha_k h(P,u_k)^p}{n} \\ &= \sum_{k=1}^N \left(\alpha_k h(P,u_k)^{p-1} - \frac{\sum_{j=1}^N \alpha_j h(P,u_j)^p}{n} a_k \right) \tau_k. \end{aligned}$$

Since $\tau_1, ..., \tau_N$ are arbitrary,

$$\frac{\sum_{j=1}^N \alpha_j h(P,u_j)^p}{n} h(P,u_k)^{1-p} a_k = \alpha_k,$$

for all $k = 1, ..., N$. By letting

$$P_0 = \left(\frac{\sum_{j=1}^N \alpha_j h(P,u_j)^p}{n} \right)^{\frac{1}{n-p}} P,$$

we have

$$S_p(P_0, \cdot) = \sum_{k=1}^N \alpha_k \delta_{u_k}(\cdot).$$

$\square$

4. The proof of the main theorem

In this section, we prove the main theorem of this paper.

The following lemmas will be needed.

Lemma 4.1. *Let* $\{h_{1j}\}_{j=1}^{\infty}, ..., \{h_{Nj}\}_{j=1}^{\infty}$ *be* N $(N \geq 2)$ *sequences of real numbers. Then, there exists a subsequence,* $\{j_n\}_{n=1}^{\infty}$*, of* $\mathbb{N}$ *and a rearrangement,* $i_1, ..., i_N$*, of* $1, ..., N$ *such that*

$$h_{i_1 j_n} \leq h_{i_2 j_n} \leq ... \leq h_{i_N j_n},$$

for all $n \in \mathbb{N}$.

Proof. For each fixed j, the number of the possible order (from small to big) of $h_{1j}, ..., h_{Nj}$ is $N!$. Therefore, there exists a subsequence, $\{j_n\}_{n=1}^{\infty}$, of $\mathbb{N}$ and a rearrangement, $i_1, ..., i_N$, of $1, ..., N$ such that

$$h_{i_1 j_n} \leq h_{i_2 j_n} \leq ... \leq h_{i_N j_n},$$

for all $n \in \mathbb{N}$. □

Lemma 4.2. *Suppose the unit vectors* $u_1, ..., u_N$ *are not concentrated on a closed hemisphere, and for any subspace,* X*, of* $\mathbb{R}^n$ *with* $1 \leq \dim X \leq n-1$*,* $\{u_1, ..., u_N\} \cap X$ *is concentrated on a closed hemisphere of* $S^{n-1} \cap X$*. If* P_m *is a sequence of polytopes with* $V(P_m) = 1$*,* $o \in Int\,(P_m)$ *and* $P_m \in \mathcal{P}(u_1, ..., u_N)$*, then* P_m *is bounded.*

Proof. We only need to prove that if the diameter, $d(P_i)$, of P_i is not bounded, then there exists a subspace, X, of $\mathbb{R}^n$ with $1 \leq \dim(X) \leq n-1$ and $\{u_1, ..., u_N\} \cap X$ is not concentrated on a closed hemisphere of $S^{n-1} \cap X$.

Let μ be a discrete measure on the unit sphere such that $\text{supp}(\mu) = \{u_1, ..., u_N\}$, $\mu(u_i) = \alpha_i > 0$ for $1 \leq i \leq N$. Obviously, we only need to prove the lemma under the condition that $\xi(P_m) = o$ for all $m \in \mathbb{N}$.

By Lemma 4.1, we may assume that

$$h(P_m, u_1) \leq ... \leq h(P_m, u_N). \tag{4.0}$$

By this and the condition that $V(P_m) = 1$ and $\lim_{m\to\infty} d(P_m) = \infty$,

$$\lim_{m\to\infty} h(P_m, u_1) = 0 \text{ and } \lim_{m\to\infty} h(P_m, u_N) = \infty.$$

By this and (4.0), there exists an i_0 $(1 \leq i_0 \leq N)$ such that

$$\overline{\lim}_{m\to\infty} \frac{h(P_m, u_{i_0})}{h(P_m, u_1)} = \infty, \tag{4.1}$$

and for $1 \leq i \leq i_0 - 1$

$$\overline{\lim}_{m\to\infty} \frac{h(P_m, u_i)}{h(P_m, u_1)} \tag{4.2}$$

exists and equals to a positive number.

Let

$$\Sigma = \text{pos}\{u_1, ..., u_{i_0-1}\}$$

and

$$\Sigma^* = \{x \in \mathbb{R}^n : x \cdot u_i \leq 0 \text{ for all } 1 \leq i \leq i_0 - 1\}.$$

Let $1 \leq j \leq i_0 - 1$ and $x \in \Sigma^* \cap S^{n-1}$. From the condition that $\xi(P_m)$ is the origin and Lemma 3.3, we have

$$\sum_{i=0}^{N} \frac{\alpha_i (x \cdot u_i)}{[h(P_m, u_i)]^{1-p}} = 0.$$

By this and the fact that $x \in \Sigma^* \cap S^{n-1}$,

$$\begin{aligned} 0 &\geq \alpha_j (x \cdot u_j) \\ &= -\sum_{i \neq j} \left[\frac{h(P_m, u_j)}{h(P_m, u_i)}\right]^{1-p} \alpha_i (x \cdot u_i) \\ &\geq \sum_{i \geq i_0} \left[\frac{h(P_m, u_j)}{h(P_m, u_i)}\right]^{1-p} \alpha_i (x \cdot u_i) \\ &\geq -\sum_{i \geq i_0} \left[\frac{h(P_m, u_j)}{h(P_m, u_i)}\right]^{1-p} \alpha_i \end{aligned}$$

By this, (4.0), (4.1) and (4.2), $\alpha_j (x \cdot u_j)$ is no bigger than 0 and no less than any negative number. Hence,

$$x \cdot u_j = 0$$

for all $j = 1, ..., i_0 - 1$ and $x \in \Sigma^* \cap S^{n-1}$. Thus,

$$\Sigma^* \cap \mathrm{lin}\{u_1, ..., u_{i_0 - 1}\} = \{0\}. \tag{4.3}$$

Obviously, $\{u_1, ..., u_{i_0-1}\}$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \mathrm{lin}\{u_1, ..., u_{i_0-1}\}$. Otherwise, there exists an $x_0 \in \mathrm{lin}\{u_1, ..., u_{i_0-1}\}$ with $x_0 \neq 0$ such that $x_0 \cdot u_i \leq 0$ for all $1 \leq i \leq i_0 - 1$. This contradicts with (4.3).

We next prove that

$$\mathrm{lin}\{u_1, ..., u_{i_0-1}\} \neq \mathbb{R}^n.$$

Otherwise, from the fact that $u_1, ..., u_{i_0-1}$ are not concentrated on a closed hemisphere of

$$\mathrm{lin}\{u_1, ..., u_{i_0-1}\} \cap S^{n-1},$$

we have, the convex hull of $\{u_1, ..., u_{i_0-1}\}$ (denoted by Q) is a polytope in $\mathbb{R}^n$ and contains the origin as an interior. Let F be a facet of Q such that $\{s u_{i_0} : s > 0\} \cap F \neq \emptyset$. Since F is the union of finite $(n-1)-$dimensional simplexes and the vertexes of these simplexes are subsets of $\{u_1, ..., u_{i_0-1}\}$, there exists a subset, $\{u_{i_1}, ..., u_{i_n}\}$, of $\{u_1, ..., u_{i_0-1}\}$ such that

$$u_{i_0} \in \mathrm{pos}\{u_{i_1}, ..., u_{i_n}\}.$$

Since $o \in \mathrm{Int}\ (Q)$, there exists $r > 0$ such that $rB^n \subset Q$. Choose $t > 0$ such that $tu \in F \cap \mathrm{pos}\{u_{i_1}, ..., u_{i_n}\}$. Then,

$$tu = \beta_{i_1} u_{i_1} + ... + \beta_{i_n} u_{i_n},$$

where $\beta_{i_1}, ..., \beta_{i_n} \geq 0$ with $\beta_{i_1} + ... + \beta_{i_n} = 1$. If we let $a_{i_j} = \beta_{i_j}/t$ for $j = 1, ..., n$, we have

$$u = a_{i_1} u_{i_1} + ... + a_{i_n} u_{i_n}.$$

Obviously, $a_{i_j} \geq 0$ with

$$a_{i_j} = \beta_{i_j}/t \leq 1/r$$

for all $j = 1, ..., n$. Hence,

$$\begin{aligned} h(P_m, u_{i_0}) &= h(P_m, a_{i_1}u_{i_1} + ... + a_{i_n}u_{i_n}) \\ &\leq a_{i_1}h(P_m, u_{i_1}) + ... + a_{i_n}h(P_m, u_{i_n}) \\ &\leq \frac{1}{r}\left[h(P_m, u_{i_1}) + ... + h(P_m, u_{i_n})\right], \end{aligned}$$

for all $m \in \mathbb{N}$. This contradicts (4.1) and (4.2). Therefore,

$$\text{lin}\{u_1, ..., u_{i_0-1}\} \neq \mathbb{R}^n.$$

Let $X = \text{lin}\{u_1, ..., u_{i_0-1}\}$. Then, $1 \leq \dim X \leq n-1$ but $\{u_1, ..., u_N\} \cap X = \{u_1, ..., u_{i_0-1}\}$ is not concentrated on a closed hemisphere of $S^{n-1} \cap X$, which contradicts the conditions of this lemma. Therefore, $d(P_m)$ is bounded. $\square$

The following lemmas will be needed (see, e.g., [72]).

Lemma 4.3. *If P is a polytope in $\mathbb{R}^n$ and $v_0 \in S^{n-1}$ with $V_{n-1}(F(P, v_0)) = 0$, then there exists a $\delta_0 > 0$ such that for $0 \leq \delta < \delta_0$*

$$V(P \cap \{x : x \cdot v_0 \geq h(P, v_0) - \delta\}) = c_n\delta^n + ... + c_2\delta^2,$$

where $c_n, ..., c_2$ are constants that depend on P and v_0.

Lemma 4.4. *Suppose $p < 0$, $\alpha_1, ..., \alpha_N > 0$, and the unit vectors $u_1, ..., u_N$ are not concentrated on a hemisphere. If for any subspace X with $1 \leq \dim X \leq n-1$, $\{u_1, ..., u_N\} \cap X$ is always concentrated on a closed hemisphere of $S^{n-1} \cap X$, then there exists a $P \in \mathcal{P}_N(u_1, ..., u_N)$ such that $\xi(P) = o$, $V(P) = 1$, and*

$$\Phi_P(o) = \sup\{\inf_{\xi \in Int\ (Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1\},$$

where $\Phi_Q(\xi) = \sum_{k=1}^N \alpha_k(h(Q, u_k) - \xi \cdot u_k)^p$.

Proof. Obviously, for $P, Q \in \mathcal{P}(u_1, ..., u_N)$, if there exists a $x \in \mathbb{R}^n$ such that $P = Q + x$, then

$$\Phi_P(\xi(P)) = \Phi_Q(\xi(Q)).$$

Thus, we can choose a sequence of polytopes $P_i \in \mathcal{P}(u_1, ..., u_N)$ with $\xi(P_i) = o$ and $V(P_i) = 1$ such that $\Phi_{P_i}(o)$ converges to

$$\sup\{\inf_{\xi \in \text{Int } (Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1\}.$$

By the conditions of this lemma and Lemma 4.2, P_i is bounded. From the Blaschke selection theorem, there exists a subsequence of P_i that converges to a polytope P such that $P \in \mathcal{P}(u_1, ..., u_N)$, $V(P) = 1$, $\xi(P) = o$ and

$$\Phi_P(o) = \sup\{\inf_{\xi \in \text{Int } (Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1\}. \tag{4.4}$$

We claim that $F(P, u_i)$ are facets for all $i = 1, ..., N$. Otherwise, there exists an $i_0 \in \{1, ..., N\}$ such that

$$F(P, u_{i_0})$$

is not a facet of P.

Choose $\delta > 0$ small enough so that the polytope

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta\} \in \mathcal{P}(u_1, ..., u_N),$$

and (by Lemma 4.3)

$$V(P_\delta) = 1 - (c_n\delta^n + ... + c_2\delta^2),$$

where $c_n, ..., c_2$ are constants that depend on P and direction u_{i_0}.

From Lemma 3.2, for any $\delta_i \to 0$ it always true that $\xi(P_{\delta_i}) \to o$. We have,

$$\lim_{\delta\to 0} \xi(P_\delta) = o.$$

Let δ be small enough so that $h(P, u_k) > \xi(P_\delta)\cdot u_k + \delta$ for all $k \in \{1, ..., N\}$, and let

$$\lambda = V(P_\delta)^{-\frac{1}{n}} = (1 - (c_n\delta^n + ... + c_2\delta^2))^{-\frac{1}{n}}.$$

From this and Equation (3.2), we have

(4.5)

$$\begin{aligned}
\Phi_{\lambda P_\delta}(\xi(\lambda P_\delta)) &= \sum_{k=1}^{N} \alpha_k \big(h(\lambda P_\delta, u_k) - \xi(\lambda P_\delta)\cdot u_k\big)^p \\
&= \lambda^p \sum_{k=1}^{N} \alpha_k \big(h(P_\delta, u_k) - \xi(P_\delta)\cdot u_k\big)^p \\
&= \lambda^p \sum_{k=1}^{N} \alpha_k \big(h(P, u_k) - \xi(P_\delta)\cdot u_k\big)^p - \alpha_{i_0}\lambda^p\big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0}\big)^p \\
&\quad + \alpha_{i_0}\lambda^p\big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta\big)^p \\
&= \sum_{k=1}^{N} \alpha_k \big(h(P, u_k) - \xi(P_\delta)\cdot u_k\big)^p + (\lambda^p - 1)\sum_{k=1}^{N} \alpha_k \big(h(P, u_k) - \xi(P_\delta)\cdot u_k\big)^p \\
&\quad + \alpha_{i_0}\lambda^p\Big[\big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta\big)^p - \big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0}\big)^p\Big] \\
&= \Phi_P(\xi(P_\delta)) + B(\delta),
\end{aligned}$$

where

$$\begin{aligned}
B(\delta) &= (\lambda^p - 1)\left(\sum_{k=1}^{N} \alpha_k \big(h(P, u_k) - \xi(P_\delta)\cdot u_k\big)^p\right) \\
&\quad + \alpha_{i_0}\lambda^p\Big[\big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta\big)^p - \big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0}\big)^p\Big] \\
&= \Big[(1 - (c_n\delta^n + ... + c_2\delta^2))^{-\frac{p}{n}} - 1\Big]\left(\sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi(P_\delta)\cdot u_k)^p\right) \\
&\quad + \alpha_{i_0}\lambda^p\Big[\big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta\big)^p - \big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0}\big)^p\Big].
\end{aligned}$$

From the facts that $d_0 = d(P) > h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} > h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta > 0$, $p < 0$ and the fact that $f(t) = t^p$ is convex on $(0, \infty)$, we have

$$\big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta\big)^p - \big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0}\big)^p > (d_0 - \delta)^p - d_0^p > 0.$$

Hence,

$$\begin{aligned}
B(\delta) &= (\lambda^p - 1)\left(\sum_{k=1}^{N} \alpha_k \big(h(P, u_k) - \xi(P_\delta)\cdot u_k\big)^p\right) \\
&\quad + \alpha_{i_0}\lambda^p\Big[\big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta\big)^p - \big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0}\big)^p\Big] \\
&> \Big[(1-(c_n\delta^n + ... + c_2\delta^2))^{-\frac{p}{n}} - 1\Big]\left(\sum_{k=1}^{N}\alpha_k(h(P,u_k) - \xi(P_\delta)\cdot u_k)^p\right) \\
&\quad + \alpha_{i_0}\lambda^p\big[(d_0-\delta)^p - d_0^p\big].
\end{aligned} \tag{4.6}$$

On the other hand,

$$\lim_{\delta\to 0}\sum_{k=1}^{N}\alpha_k\big(h(P,u_k) - \xi(P_\delta)\cdot u_k\big)^p = \sum_{k=1}^{N}\alpha_k h(P,u_k)^p, \tag{4.7}$$

$$(d_0 - \delta)^p - d_0^p > 0, \tag{4.8}$$

and

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{(1-(c_n\delta^n + ... + c_2\delta^2))^{-\frac{p}{n}} - 1}{(d_0-\delta)^p - d_0^p} \\
&= \lim_{\delta\to 0}\frac{(-\frac{p}{n})(1-(c_n\delta^n + ... + c_2\delta^2))^{-\frac{p}{n}-1}(-nc_n\delta^{n-1} - ... - 2c_2\delta)}{p(d_0-\delta)^{p-1}(-1)} = 0.
\end{aligned} \tag{4.9}$$

From Equations (4.6), (4.7), (4.8), (4.9), and the fact that $p < 0$, we have $B(\delta) > 0$ for small enough $\delta > 0$. From this and Equation (4.5), there exists a $\delta_0 > 0$ such that $P_{\delta_0} \in \mathcal{P}(u_1, ..., u_N)$ and

$$\Phi_{\lambda_0 P_{\delta_0}}(\xi(\lambda_0 P_{\delta_0})) > \Phi_P(\xi(P_{\delta_0})) \geq \Phi_P(\xi(P)) = \Phi_P(o),$$

where $\lambda_0 = V(P_{\delta_0})^{-\frac{1}{n}}$. Let $P_0 = \lambda_0 P_{\delta_0} - \xi(\lambda_0 P_{\delta_0})$, then $P_0 \in \mathcal{P}^n(u_1, ..., u_N)$, $V(P_0) = 1$, $\xi(P_0) = o$ and

$$\Phi_{P_0}(o) < \Phi_P(o). \tag{4.10}$$

This contradicts Equation (4.4). Therefore, $P \in \mathcal{P}_N(u_1, ..., u_N)$. □

Now we have prepared enough to prove the main theorem of this paper. We only need to prove the following:

Theorem 4.5. *Suppose $p < 0$, $\alpha_1, ..., \alpha_N > 0$, and the unit vectors $u_1, ..., u_N$ are not concentrated on a hemisphere. If for any subspace X with $1 \leq \dim X \leq n-1$, $\{u_1, ..., u_N\} \cap X$ is always concentrated on a closed hemisphere of $S^{n-1} \cap X$, then there exists a polytope $P_0 \in \mathcal{P}_N(u_1, ..., u_N)$ such that*

$$S_p(P_0, \cdot) = \sum_{k=1}^{N}\alpha_k\delta_{u_k}(\cdot).$$

Proof. Theorem 4.5 can be directly got by Lemma 3.4 and Lemma 4.4. □

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