# ON THE LOOMIS-WHITNEY INEQUALITY FOR ISOTROPIC MEASURES

QINGZHONG HUANG AND AI-JUN LI

ABSTRACT. The Ball-Loomis-Whitney inequality for isotropic measures is extended from volume to all intrinsic volumes along with a complete description of equality conditions. The proof is based on a reverse intrinsic volume inequality for zonoids.

## 1. INTRODUCTION

The classical Loomis-Whitney inequality states that the $n$-dimensional volume $\lambda_n$ of a compact set $A$ in $\mathbb{R}^n$ is dominated by the geometric mean of the $(n-1)$-dimensional volumes $\lambda_{n-1}$ of its coordinate projections:

$$\lambda_n(A)^{n-1} \leq \prod_{i=1}^{n} \lambda_{n-1}(\mathrm{P}_{e_i^\perp} A), \tag{1.1}$$

with equality if and only if $A$ is a coordinate box (a rectangular parallelepiped whose facets are parallel to the coordinate hyperplanes) in $\mathbb{R}^n$. Here, $\mathrm{P}_{e_i^\perp} A$ denotes the orthogonal projection of $A$ onto the 1-codimensional subspace $e_i^\perp$ perpendicular to $e_i$ and $\{e_1, \ldots, e_n\}$ is the standard Euclidean basis of $\mathbb{R}^n$. Inequality (1.1) was first proved by Loomis and Whitney [19] in 1949 and has been widely studied in recent years (see e.g., [1, 2, 7–12, 17, 18, 28]).

An important generalization of (1.1) was established by Ball [2], which states that (1.1) still holds when the basis vectors $e_1, \ldots, e_n$ are replaced by a sequence of directions satisfying John's condition [16]. More specifically, let $K$ be a convex body (a compact convex set with non-empty interior) in $\mathbb{R}^n$. If there are unit

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vectors $(u_i)_{i=1}^m$ and positive numbers $(c_i)_{i=1}^m$ satisfying John's condition

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n, \tag{1.2}$$

then

$$\lambda_n(K)^{n-1} \le \prod_{i=1}^{m} \lambda_{n-1}(\mathrm{P}_{u_i^\perp} K)^{c_i}. \tag{1.3}$$

Here, $u_i \otimes u_i$ is the rank-one orthogonal projection onto the space spanned by the unit vector $u_i$ and $I_n$ is the identity map on $\mathbb{R}^n$. Clearly, inequality (1.3) reduces to (1.1) when $u_i = e_i$ and $c_i = 1$ for all $i = 1, \dots, n$.

A finite nonnegative Borel measure $\mu$ on the unit sphere $S^{n-1}$ of $\mathbb{R}^n$ is said to be isotropic if

$$\int_{S^{n-1}} u \otimes u d\mu(u) = I_n. \tag{1.4}$$

The measure $\mu$ is said to be even if it assumes the same value on antipodal sets. Note that condition (1.4) reduces to (1.2) if the isotropic measure $\mu$ in (1.4) is of the form $(1/2)\sum_{i=1}^m (c_i\delta_{u_i} + c_i\delta_{-u_i})$ on $S^{n-1}$ ($\delta_x$ stands for the Dirac mass at $x$). In particular, an isotropic measure of the form $(1/2)\sum_{i=1}^n (\delta_{v_i} + \delta_{-v_i})$, where $(v_i)_1^n$ is an orthonormal basis of $\mathbb{R}^n$, is called a *cross measure*. We shall say that $K$ is a cube formed by the cross measure $\mu$ (concentrated on $\pm v_1, \dots, \pm v_n$), if there is a positive number $\alpha$ such that

$$K = \alpha \sum_{i=1}^{n} [-v_i, v_i]. \tag{1.5}$$

The body $K$ is said to be a box formed by $\mu$, if there are positive numbers $(\alpha_i)_1^n$ such that

$$K = \sum_{i=1}^{n} \alpha_i [-v_i, v_i]. \tag{1.6}$$

Volume inequalities for isotropic measures have been the focus of intensive research in recent years, see, e.g., [3–6, 14, 15, 21–24, 26, 27].

Denote the $j$-th intrinsic volume of a convex body $K$ in $\mathbb{R}^n$ by $V_j(K)$, $0 \le j \le n$. This notion extends the usual concept of volume, that is $V_n(K) = \lambda_n(K)$, and it turns out that intrinsic volumes do not depend on the dimension of the ambient space. Moreover, up to a constant, $V_{n-1}(K)$ and $V_1(K)$ are the surface area and the mean width of $K$, respectively.

The main purpose of this paper is to extend (1.3) from volume to all intrinsic volumes and from discrete to general isotropic measures. Also complete equality conditions will be established.

**Theorem 1.1.** *Let $K$ be a convex body in $\mathbb{R}^n$ and $1 \leq j \leq n$. If $\mu$ is an even isotropic measure on $S^{n-1}$, then*

$$V_j(K)^{j-1} \leq \frac{\binom{n}{j}^{j-1}}{\binom{n-1}{j-1}^j} \exp\Big(\frac{j}{n}\int_{S^{n-1}} \log V_{j-1}(\mathrm{P}_{u^\perp}K)d\mu(u)\Big). \tag{1.7}$$

*For $j = 1$, inequality (1.7) holds with equality. For $2 \leq j \leq n-1$, equality in (1.7) holds if and only if $\mu$ is a cross measure on $S^{n-1}$ and $K$ is a cube formed by $\mu$, up to translations. For $j = n$, equality in (1.7) holds if and only if $\mu$ is a cross measure on $S^{n-1}$ and $K$ is a box formed by $\mu$, up to translations.*

Note that for discrete isotropic measures $\mu$ and $j = n$, inequality (1.7) reduces to (1.3). For more information about the Loomis-Whitney inequality for intrinsic volumes, see [9, 11, 12].

Assume that $\mu$ is an even Borel measure on $S^{n-1}$, whose support, $\operatorname{supp}\mu$, is not contained in a subsphere of $S^{n-1}$. Let $\alpha : S^{n-1} \to (0, +\infty)$ be an even continuous function. The *zonoid* $Z_\alpha$ with generating measure $\alpha d\mu$ is defined as the convex body whose support function, for $x \in \mathbb{R}^n$, is given by

$$h_{Z_\alpha}(x) = \int_{S^{n-1}} |x \cdot u|\alpha(u)d\mu(u), \tag{1.8}$$

where $x \cdot u$ denotes the standard inner product of $x$ and $u$. Zonoids are limits of Minkowski sums of line segments and play an important role in the classical Brunn-Minkowski theory (see e.g., [25, p.191]). The proof of Theorem 1.1 relies on the following reverse inequality for zonoids.

**Theorem 1.2.** *Let $\mu$ be an even isotropic measure on $S^{n-1}$, $\alpha : S^{n-1} \to (0, +\infty)$ an even continuous function and $1 \leq j \leq n$. If $Z_\alpha$ is the zonoid with generating measure $\alpha d\mu$, then*

$$V_j(Z_\alpha) \geq 2^j \binom{n}{j} \exp\Big(\frac{j}{n}\int_{S^{n-1}} \log \alpha(u)d\mu(u)\Big). \tag{1.9}$$

*For $j = 1$, equality in (1.9) holds if and only if the function $\alpha$ is constant on* $\operatorname{supp}\mu$. *For $2 \leq j \leq n-1$, equality in (1.9) holds if and only if $\mu$ is a cross measure on $S^{n-1}$ and $Z_\alpha$ is a cube formed by $\mu$. For $j = n$, equality in (1.9) holds if and only if $\mu$ is a cross measure on $S^{n-1}$ and $Z_\alpha$ is a box formed by $\mu$.*

Inequality (1.9) for $\alpha \equiv 1$ was proved by Hug and Schneider [15], and generalized a previously established inequality for volumes by Lutwak, Yang and Zhang [21]. For discrete measures, but without the equality conditions, inequality (1.9) for volumes was proved by Ball [2].

## 2. Background material

We list some basic facts about convex bodies. As general references we recommend the books of Gardner [13] and Schneider [25].

As usual, $B_2^n$ denotes the Euclidean unit ball in $\mathbb{R}^n$. Its volume is denoted by $\kappa_n$. For $x \in \mathbb{R}^n$, we denote the Euclidean norm of $x$ by $\|x\|$. If $K$ is a convex body in $\mathbb{R}^n$, then its support function $h_K : \mathbb{R}^n \to \mathbb{R}$ is defined for $x \in \mathbb{R}^n$ by

$$h_K(x) = \max\{x \cdot y : y \in K\}.$$

Let $V(K_1, \ldots, K_n)$ denote the *mixed volume* of the compact convex sets $K_1, \ldots, K_n$ in $\mathbb{R}^n$. Mixed volumes arise as coefficients in the expansion of $\lambda_n(t_1K_1 + \cdots + t_mK_m)$ as a homogeneous polynomial of degree $n$ in the parameters $t_1, \ldots, t_m \geq 0$:

$$\lambda_n(t_1K_1 + \cdots + t_mK_m) = \sum_{i_1,\ldots,i_n=1}^{m} t_{i_1} \cdots t_{i_n} V(K_{i_1}, \ldots, K_{i_n}),$$

where $t_1K_1 + \cdots + t_mK_m$ is the Minkowski linear combination of the compact convex sets $K_1, \ldots, K_m$ in $\mathbb{R}^n$. The notation $V(K, j; L, n-j)$ means $K$ appears $j$ times and $L$ appears $n-j$ times. It is well-known that

$$V(L; K, j-1; B_2^n, n-j) = \frac{1}{n} \int_{S^{n-1}} h_L(v) dS_{j-1}(K, v), \tag{2.1}$$

where $S_{j-1}(K, \cdot)$ is the area measure of order $j-1$ of $K$. Note that $S_{n-1}(K, \cdot)$ is the classical surface area measure of $K$. The *$j$-th intrinsic volume*, $0 \leq j \leq n$, of a compact convex set $K$ is defined by

$$V_j(K) = \frac{\binom{n}{j}}{\kappa_{n-j}} V(K, j; B_2^n, n-j).$$

We shall use the following formula (see e.g., [13, p.408]): If $2 \leq j \leq n$, then

$$V_{j-1}(\mathrm{P}_u K) = \frac{\binom{n-1}{j-1}}{2\kappa_{n-j}} \int_{S^{n-1}} |u \cdot v| dS_{j-1}(K, v). \tag{2.2}$$

A fundamental inequality about mixed volumes and intrinsic volumes (see e.g., [13, p.420]) states that, if $1 \leq j \leq n$, then for convex bodies $K_1, K_2$ in $\mathbb{R}^n$,

$$V(K_1; K_2, j-1; B_2^n, n-j)^j \geq \left( \frac{\kappa_{n-j}}{\binom{n}{j}} \right)^j V_j(K_2)^{j-1} V_j(K_1). \tag{2.3}$$

For $1 < j \le n$, equality in (2.3) holds if and only if $K_1$ and $K_2$ are homothetic, that is, they coincide up to a translation and a dilatation. For $j = 1$, inequality (2.3) holds with equality.

It is well-known that the isotropicity assumption (1.4) is equivalent to

$$\|x\|^2 = \int_{S^{n-1}} |x \cdot u|^2 d\mu(u)$$

for all $x \in \mathbb{R}^n$. Moreover, taking the trace in (1.4), we see that

$$\mu(S^{n-1}) = n. \tag{2.4}$$

Next, we need the following formula for the zonoid $Z_\alpha$. The $j$-th intrinsic volume of $Z_\alpha$, $1 \le j \le n$, can be expressed as (see e.g., [25, (5.83)])

$$V_j(Z_\alpha) = \frac{2^j}{j!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} [u_1, \dots, u_j] \alpha(u_1) \cdots \alpha(u_j) d\mu(u_1) \cdots d\mu(u_j), \tag{2.5}$$

where $[u_1, \dots, u_j]$ denotes the $j$-dimensional volume of the parallelepiped spanned by the vectors $u_1, \dots, u_j$.

## 3. Proof of Theorem 1.2

*Proof of Theorem 1.2.* For $u_1, \dots, u_j \in S^{n-1}$, let

$$U_k := \operatorname{span}\{u_1, \dots, u_k\}, \quad \text{for } k = 1, \cdots, j-1,$$

where $\operatorname{span}\{u_1, \dots, u_k\}$ denotes the linear span of the vectors $u_1, \dots, u_k$. If $u_1, \dots, u_k$ are linearly independent, then

$$[u_1, \dots, u_j] = \|u_1\| \|\mathrm{P}_{U_1^\perp} u_2\| \cdots \|\mathrm{P}_{U_{j-2}^\perp} u_{j-1}\| \|\mathrm{P}_{U_{j-1}^\perp} u_j\|. \tag{3.1}$$

By the isotropicity assumption (1.4), we have

$$\mathrm{P}_{U_k^\perp} = \int_{S^{n-1}} \mathrm{P}_{U_k^\perp} u \otimes \mathrm{P}_{U_k^\perp} u d\mu(u). \tag{3.2}$$

If $u_1, \dots, u_k$ are linearly independent, then $\mathrm{P}_{U_k^\perp}$ has rank $n-k$. Taking traces in (3.2), we see that

$$\int_{S^{n-1}} \|\mathrm{P}_{U_k^\perp} u\|^2 d\mu(u) = n - k. \tag{3.3}$$

Define the set $\Omega_k$, $k = 1, \dots, j$, by

$$\Omega_k = \{(u_1, \dots, u_k) \in (S^{n-1})^k : [u_1, u_2, \dots, u_k] \neq 0\}. \tag{3.4}$$

Note that if $u_k \in \operatorname{span}\{u_1, \dots, u_{k-1}\}$, then we have

$$\|\mathrm{P}_{U_{k-1}^\perp} u_k\| = 0. \tag{3.5}$$

Applying Fubini's theorem iteratively, integrating first with respect to $u_j$, then $u_{j-1}$ and so on, and using (3.4), (3.1), (3.5), and (3.3), we obtain

$$\begin{aligned}
&\int_{(S^{n-1})^j}[u_1,u_2,\dots,u_j]^2 d\mu(u_1)\cdots d\mu(u_j)\\
&=\int_{\Omega_j}[u_1,u_2,\dots,u_j]^2 d\mu(u_1)\cdots d\mu(u_j)\\
&=\int_{\Omega_j}(\|\mathrm{P}_{U_1^\perp}u_2\|\cdots\|\mathrm{P}_{U_{j-1}^\perp}u_j\|)^2 d\mu(u_1)\cdots d\mu(u_j)\\
&=\int_{\Omega_{j-1}}\Big(\int_{S^{n-1}}\|\mathrm{P}_{U_{j-1}^\perp}u_j\|^2 d\mu(u_j)\Big)(\|\mathrm{P}_{U_1^\perp}u_2\|\cdots\|\mathrm{P}_{U_{j-2}^\perp}u_{j-1}\|)^2 d\mu(u_1)\cdots d\mu(u_{j-1})\\
&=(n-j+1)\int_{\Omega_{j-1}}(\|\mathrm{P}_{U_1^\perp}u_2\|\cdots\|\mathrm{P}_{U_{j-2}^\perp}u_{j-1}\|)^2 d\mu(u_1)\cdots d\mu(u_{j-1})\\
&\qquad\cdots\\
&=\frac{n!}{(n-j)!}. \qquad (3.6)
\end{aligned}$$

By (3.6), the measure $\frac{(n-j)!}{n!}[u_1,\dots,u_j]^2 d\mu(u_1)\cdots d\mu(u_j)$ is a probability measure. In a similar way, we obtain

$$\int_{(S^{n-1})^j}[u,u_2,\dots,u_j]^2 d\mu(u_2)\cdots d\mu(u_j)=\frac{(n-1)!}{(n-j)!},\quad \text{for all } u\in S^{n-1}. \qquad (3.7)$$

Note that the discrete case of formula (3.6) was proved by Lutwak, Yang, and Zhang [20, Lemma 2.1] in a different way.

It follows from (3.7) that

$$\begin{aligned}
&\int_{(S^{n-1})^j}\log\alpha(u_1)[u_1,\dots,u_j]^2 d\mu(u_1)\cdots d\mu(u_j)\\
&=\int_{S^{n-1}}\log\alpha(u_1)\Big[\int_{(S^{n-1})^{j-1}}[u_1,\dots,u_j]^2 d\mu(u_2)\cdots d\mu(u_j)\Big]d\mu(u_1)\\
&=\frac{(n-1)!}{(n-j)!}\int_{S^{n-1}}\log\alpha(u_1)d\mu(u_1).
\end{aligned}$$

Thus, for each $1\le i\le j$,

$$\int_{(S^{n-1})^j}\log\alpha(u_i)[u_1,\dots,u_j]^2 d\mu(u_1)\cdots d\mu(u_j)=\frac{(n-1)!}{(n-j)!}\int_{S^{n-1}}\log\alpha(u_i)d\mu(u_i). \qquad (3.8)$$

From (2.5), the fact that $[u_1,\dots,u_j]\le 1$, Jensen's inequality and (3.8), we obtain

$$V_j(Z_\alpha)=\frac{2^j}{j!}\int_{(S^{n-1})^j}[u_1,\dots,u_j]\alpha(u_1)\cdots\alpha(u_j)d\mu(u_1)\cdots d\mu(u_j)$$

$$
\begin{aligned}
&\geq \frac{2^j}{j!}\int_{(S^{n-1})^j}[u_1,\dots,u_j]^2\alpha(u_1)\cdots\alpha(u_j)d\mu(u_1)\cdots d\mu(u_j)\\
&\geq \frac{2^j n!}{j!(n-j)!}\exp\Big(\frac{(n-j)!}{n!}\int_{(S^{n-1})^j}\log(\alpha(u_1)\cdots\alpha(u_j))\\
&\qquad\times[u_1,\dots,u_j]^2 d\mu(u_1)\cdots d\mu(u_j)\Big)\\
&= 2^j\binom{n}{j}\exp\Big(\frac{j}{n}\int_{S^{n-1}}\log\alpha(u)d\mu(u)\Big),
\end{aligned}
\tag{3.9}
$$

which is the desired inequality.

Now, we deal with the equality conditions of (3.9). For $j=1$, the first inequality in (3.9) holds with equality. Equality in the second inequality in (3.9) holds if and only if the function $\alpha$ is constant on $\operatorname{supp}\mu$. For $2\leq j\leq n$, equality in the first inequality in (3.9) implies

$$
[u_1,\dots,u_j]=1 \quad\text{or}\quad [u_1,\dots,u_j]=0 \tag{3.10}
$$

for $\mu\otimes\cdots\otimes\mu$-almost all $(u_1,\dots,u_j)\in(S^{n-1})^j$. By continuity, (3.10) holds whenever $u_i\in\operatorname{supp}\mu$, for $i=1,\dots,j$. Thus, for arbitrary linearly independent $u_1,\dots,u_j\in\operatorname{supp}\mu$, the vectors $u_1,\dots,u_j$ must be pairwise orthogonal. In particular, $\operatorname{supp}\mu\subset\{\pm v_1\}\cup v_1^{\perp}$ for given $v_1\in\operatorname{supp}\mu$. If we choose $v_2\in\operatorname{supp}\mu\cap v_1^{\perp}$, then $\operatorname{supp}\mu\subset\{\pm v_2\}\cup v_2^{\perp}$. Therefore, $\operatorname{supp}\mu\subset\{\pm v_1\}\cup\{\pm v_2\}\cup\operatorname{span}\{v_1,v_2\}^{\perp}$. Repeating these steps, we see that $\operatorname{supp}\mu\subset\{\pm v_1,\dots,\pm v_n\}$, where $v_1,\dots,v_n$ form an orthonormal basis of $\mathbb{R}^n$. Since $\mu$ is not concentrated on any great subsphere of $S^{n-1}$ and even, we conclude that $\mu$ is a cross measure on $S^{n-1}$. Equality in the second inequality in (3.9) implies that the function $\alpha(u_1)\cdots\alpha(u_j)$ is constant on the support of the probability measure $\frac{(n-j)!}{n!}[u_1,\dots,u_j]^2d\mu(u_1)\cdots d\mu(u_j)$. As shown in [21, Lemma A.1], this implies that $\alpha(u_1)\cdots\alpha(u_j)$ is constant for linearly independent $u_1,\dots,u_j\in\operatorname{supp}\mu$. We claim that $\alpha$ must be constant on $\operatorname{supp}\mu$ when $j\neq n$. Suppose that $\operatorname{supp}\mu=\{\pm v_1,\dots,\pm v_n\}$ and that

$$
\alpha(v_1)\alpha(v_3)\cdots\alpha(v_j)=\alpha(v_2)\alpha(v_3)\cdots\alpha(v_j).
$$

Then we have $\alpha(v_1)=\alpha(v_2)$, and hence $\alpha$ is constant on $\operatorname{supp}\mu$ for $1\leq j\leq n-1$. By the definition of $Z_\alpha$ (1.8), we get that $Z_\alpha$ is a cube formed by the cross measure $\mu$. If $j=n$, we see that the numbers $\alpha(v_i)$ may be distinct. That is, $Z_\alpha$ is a box formed by the cross measure $\mu$. $\square$

## 4. Proof of Theorem 1.1

We are now in a position to prove the Loomis-Whitney inequality for intrinsic volumes.

*Proof of Theorem 1.1.* For $j = 1$, inequality (1.7) trivially holds with equality.

For $2 \le j \le n$ and $u \in \operatorname{supp} \mu$, let

$$\alpha^{-1}(u) = \int_{S^{n-1}} |u \cdot v| dS_{j-1}(K, v) = \frac{2\kappa_{n-j}}{\binom{n-1}{j-1}} V_{j-1}(\mathrm{P}_{u^\perp} K), \tag{4.1}$$

where the second equality follows from (2.2). From (2.3), (2.1), the definition of $Z_\alpha$ (1.8), Fubini's theorem, (4.1) and (2.4), we obtain

$$\begin{aligned}
V_j(K)^{j-1} &\le \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1} V(Z_\alpha; K, j-1; B_2^n, n-j)^j \\
&= \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1} \Big( \frac{1}{n} \int_{S^{n-1}} h_{Z_\alpha}(v) dS_{j-1}(K, v) \Big)^j \\
&= \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1} \Big( \frac{1}{n} \int_{S^{n-1}} \int_{S^{n-1}} |v \cdot u| \alpha(u) d\mu(u) dS_{j-1}(K, v) \Big)^j \\
&= \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1} \Big( \frac{1}{n} \int_{S^{n-1}} \int_{S^{n-1}} |v \cdot u| dS_{j-1}(K, v) \alpha(u) d\mu(u) \Big)^j \\
&= \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1}.
\end{aligned}$$

Thus, by (1.9) and (4.1), we have

$$\begin{aligned}
V_j(K)^{j-1} &\le \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1} \\
&\le \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j \left( 2^j \binom{n}{j} \exp\Big( \frac{j}{n} \int_{S^{n-1}} \log \alpha(u) d\mu(u) \Big) \right)^{-1} \\
&= \frac{\binom{n}{j}^{j-1}}{2^j \kappa_{n-j}^j} \exp\left( \frac{j}{n} \int_{S^{n-1}} \log\Big( \frac{2\kappa_{n-j}}{\binom{n-1}{j-1}} V_{j-1}(\mathrm{P}_{u^\perp} K) \Big) d\mu(u) \right) \\
&= \frac{\binom{n}{j}^{j-1}}{\binom{n-1}{j-1}^j} \exp\Big( \frac{j}{n} \int_{S^{n-1}} \log V_{j-1}(\mathrm{P}_{u^\perp} K) d\mu(u) \Big),
\end{aligned} \tag{4.2}$$

which is the desired inequality.

By (2.3), equality in the first inequality in (4.2) holds if and only if $K$ and $Z_\alpha$ are homothetic when $2 \le j \le n$. Combining this with the equality conditions of

Theorem 1.2, we immediately see that $K$ is of the form (1.5) and (1.6), respectively, up to translations.

Conversely, we will show that equality in (1.7) holds if $K$ is of the form (1.5) and (1.6), respectively, up to translations. As in the proof of (4.2), we only have to verify that $K$ and $Z_\alpha$ are homothetic when $2 \leq j \leq n$. For $2 \leq j \leq n-1$, let

$$K = \alpha C + v_0,$$

for some vector $v_0 \in \mathbb{R}^n$ and a positive number $\alpha$, where $C = \sum_{i=1}^n [-v_i, v_i]$ and $\operatorname{supp} \mu = \{\pm v_1, \dots, \pm v_n\}$ for some orthonormal basis $(v_i)_1^n$ of $\mathbb{R}^n$. From (4.1), we get

$$\begin{aligned}
\alpha(v_k) &= \Big( \int_{S^{n-1}} |v_k \cdot v| dS_{j-1}(K, v) \Big)^{-1} \\
&= \Big( \int_{S^{n-1}} |v_k \cdot v| dS_{j-1}(\alpha C + v_0, v) \Big)^{-1} \\
&= \alpha^{1-j} \Big( \int_{S^{n-1}} |v_k \cdot v| dS_{j-1}(C, v) \Big)^{-1}
\end{aligned}$$

for every $k = 1, \dots, n$. Thus, $\alpha(v_k) = \alpha^{1-j}\beta^{-1}$, where $\beta = \int_{S^{n-1}} |v_k \cdot v| dS_{j-1}(C, v)$ is a constant for all $k = 1, \dots, n$. Therefore, by the definition of $Z_\alpha$ (1.8),

$$Z_\alpha = \frac{1}{2} \alpha^{1-j} \beta^{-1} C.$$

That is, $K$ and $Z_\alpha$ are homothetic for $2 \leq j \leq n-1$. For $j = n$, let

$$K = \sum_{i=1}^n \alpha_i [-v_i, v_i] + v_0,$$

for some vector $v_0 \in \mathbb{R}^n$ and positive numbers $(\alpha_i)_1^n$, where $\operatorname{supp} \mu = \{\pm v_1, \dots, \pm v_n\}$ and $(v_i)_1^n$ is an orthonormal basis of $\mathbb{R}^n$. Obviously, $S_{n-1}(K, \{v_k\}) = 2^{n-1} \alpha_1 \cdots \alpha_n / \alpha_k$. It follows from (4.1) that $\alpha(v_k) = S_{n-1}^{-1}(K, \{v_k\}) = \alpha_k / 2^{n-1} \alpha_1 \cdots \alpha_n$ for every $k = 1, \dots, n$. Therefore, by the definition of $Z_\alpha$ (1.8),

$$Z_\alpha = \frac{1}{2^n \alpha_1 \cdots \alpha_n} \sum_{i=1}^n \alpha_i [-v_i, v_i].$$

That is, $K$ and $Z_\alpha$ are homothetic for $j = n$.

□

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(Q. Huang) College of Mathematics, Physics and Information Engineering, Jiaxing University, Jiaxing 314001, China

*E-mail address*: hqz376560571@163.com

(A.-J. Li) School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo City 454000, China

*E-mail address*: liaijun72@163.com