ON THE LOOMIS-WHITNEY INEQUALITY FOR ISOTROPIC MEASURES

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Abstract. The Ball-Loomis-Whitney inequality for isotropic measures is extended from volume to all intrinsic volumes along with a complete description of equality conditions. The proof is based on a reverse intrinsic volume inequality for zonoids.

1. Introduction

The classical Loomis-Whitney inequality states that the $n$-dimensional volume $\lambda_n$ of a compact set $A$ in $\mathbb{R}^n$ is dominated by the geometric mean of the $(n - 1)$-dimensional volumes $\lambda_{n-1}$ of its coordinate projections:

$$\lambda_n(A)^{n-1} \leq \prod_{i=1}^{n} \lambda_{n-1}(P_{e_i^\perp} A),$$

with equality if and only if $A$ is a coordinate box (a rectangular parallelepiped whose facets are parallel to the coordinate hyperplanes) in $\mathbb{R}^n$. Here, $P_{e_i^\perp} A$ denotes the orthogonal projection of $A$ onto the 1-codimensional subspace $e_i^\perp$ perpendicular to $e_i$ and $\{e_1, \ldots, e_n\}$ is the standard Euclidean basis of $\mathbb{R}^n$. Inequality (1.1) was first proved by Loomis and Whitney [19] in 1949 and has been widely studied in recent years (see e.g., [1, 2, 7–12, 17, 18, 28]).

An important generalization of (1.1) was established by Ball [2], which states that (1.1) still holds when the basis vectors $e_1, \ldots, e_n$ are replaced by a sequence of directions satisfying John’s condition [16]. More specifically, let $K$ be a convex body (a compact convex set with non-empty interior) in $\mathbb{R}^n$. If there are unit
vectors \((u_i)_{i=1}^m\) and positive numbers \((c_i)_{i=1}^m\) satisfying John’s condition
\[
\sum_{i=1}^m c_i u_i \otimes u_i = I_n,
\tag{1.2}
\]
then
\[
\lambda_n(K)^{n-1} \leq \prod_{i=1}^m \lambda_{n-1}(P_{u_i^\perp} K)^{c_i}.
\tag{1.3}
\]
Here, \(u_i \otimes u_i\) is the rank-one orthogonal projection onto the space spanned by the unit vector \(u_i\) and \(I_n\) is the identity map on \(\mathbb{R}^n\). Clearly, inequality (1.3) reduces to (1.1) when \(u_i = e_i\) and \(c_i = 1\) for all \(i = 1, \ldots, n\).

A finite nonnegative Borel measure \(\mu\) on the unit sphere \(S^{n-1}\) of \(\mathbb{R}^n\) is said to be isotropic if
\[
\int_{S^{n-1}} u \otimes u d\mu(u) = I_n.
\tag{1.4}
\]
The measure \(\mu\) is said to be even if it assumes the same value on antipodal sets. Note that condition (1.4) reduces to (1.2) if the isotropic measure \(\mu\) in (1.4) is of the form \((1/2) \sum_{i=1}^m (c_i \delta_{u_i} + c_i \delta_{-u_i})\) on \(S^{n-1}\) (\(\delta_x\) stands for the Dirac mass at \(x\)). In particular, an isotropic measure of the form \((1/2) \sum_{i=1}^n (\delta_{v_i} + \delta_{-v_i})\), where \((v_i)_{i=1}^n\) is an orthonormal basis of \(\mathbb{R}^n\), is called a cross measure. We shall say that \(K\) is a cube formed by the cross measure \(\mu\) (concentrated on \(\pm v_1, \ldots, \pm v_n\)), if there is a positive number \(\alpha\) such that
\[
K = \alpha \sum_{i=1}^n [-v_i, v_i].
\tag{1.5}
\]
The body \(K\) is said to be a box formed by \(\mu\), if there are positive numbers \((\alpha_i)_{i=1}^n\) such that
\[
K = \sum_{i=1}^n \alpha_i [-v_i, v_i].
\tag{1.6}
\]
Volume inequalities for isotropic measures have been the focus of intensive research in recent years, see, e.g., [3–6, 14, 15, 21–24, 26, 27].

Denote the \(j\)-th intrinsic volume of a convex body \(K\) in \(\mathbb{R}^n\) by \(V_j(K)\), \(0 \leq j \leq n\). This notion extends the usual concept of volume, that is \(V_n(K) = \lambda_n(K)\), and it turns out that intrinsic volumes do not depend on the dimension of the ambient space. Moreover, up to a constant, \(V_{n-1}(K)\) and \(V_1(K)\) are the surface area and the mean width of \(K\), respectively.

The main purpose of this paper is to extend (1.3) from volume to all intrinsic volumes and from discrete to general isotropic measures. Also complete equality conditions will be established.
Theorem 1.1. Let $K$ be a convex body in $\mathbb{R}^n$ and $1 \leq j \leq n$. If $\mu$ is an even isotropic measure on $S^{n-1}$, then

$$V_j(K)^{j-1} \leq \frac{(n)^{j-1}}{(n-1)^j} \exp \left( \frac{j}{n} \int_{S^{n-1}} \log V_{j-1}(P_{u\perp K}) d\mu(u) \right).$$

(1.7)

For $j = 1$, inequality (1.7) holds with equality. For $2 \leq j \leq n-1$, equality in (1.7) holds if and only if $\mu$ is a cross measure on $S^{n-1}$ and $K$ is a cube formed by $\mu$, up to translations. For $j = n$, equality in (1.7) holds if and only if $\mu$ is a cross measure on $S^{n-1}$ and $K$ is a box formed by $\mu$, up to translations.

Note that for discrete isotropic measures $\mu$ and $j = n$, inequality (1.7) reduces to (1.3). For more information about the Loomis-Whitney inequality for intrinsic volumes, see [9, 11, 12].

Assume that $\mu$ is an even Borel measure on $S^{n-1}$, whose support, $\text{supp} \mu$, is not contained in a subsphere of $S^{n-1}$. Let $\alpha : S^{n-1} \rightarrow (0, +\infty)$ be an even continuous function. The zonoid $Z_\alpha$ with generating measure $\alpha d\mu$ is defined as the convex body whose support function, for $x \in \mathbb{R}^n$, is given by

$$h_{Z_\alpha}(x) = \int_{S^{n-1}} |x \cdot u| \alpha(u) d\mu(u),$$

(1.8)

where $x \cdot u$ denotes the standard inner product of $x$ and $u$. Zonoids are limits of Minkowski sums of line segments and play an important role in the classical Brunn-Minkowski theory (see e.g., [25, p.191]). The proof of Theorem 1.1 relies on the following reverse inequality for zonoids.

Theorem 1.2. Let $\mu$ be an even isotropic measure on $S^{n-1}$, $\alpha : S^{n-1} \rightarrow (0, +\infty)$ an even continuous function and $1 \leq j \leq n$. If $Z_\alpha$ is the zonoid with generating measure $\alpha d\mu$, then

$$V_j(Z_\alpha) \geq 2^j \binom{n}{j} \exp \left( \frac{j}{n} \int_{S^{n-1}} \log \alpha(u) d\mu(u) \right).$$

(1.9)

For $j = 1$, equality in (1.9) holds if and only if the function $\alpha$ is constant on $\text{supp} \mu$. For $2 \leq j \leq n-1$, equality in (1.9) holds if and only if $\mu$ is a cross measure on $S^{n-1}$ and $Z_\alpha$ is a cube formed by $\mu$. For $j = n$, equality in (1.9) holds if and only if $\mu$ is a cross measure on $S^{n-1}$ and $Z_\alpha$ is a box formed by $\mu$.

Inequality (1.9) for $\alpha \equiv 1$ was proved by Hug and Schneider [15], and generalized a previously established inequality for volumes by Lutwak, Yang and Zhang [21]. For
discrete measures, but without the equality conditions, inequality \((1.9)\) for volumes was proved by Ball [2].

2. Background material

We list some basic facts about convex bodies. As general references we recommend the books of Gardner [13] and Schneider [25].

As usual, \(B_n^2\) denotes the Euclidean unit ball in \(\mathbb{R}^n\). Its volume is denoted by \(\kappa_n\). For \(x \in \mathbb{R}^n\), we denote the Euclidean norm of \(x\) by \(\|x\|\). If \(K\) is a convex body in \(\mathbb{R}^n\), then its support function \(h_K : \mathbb{R}^n \to \mathbb{R}\) is defined for \(x \in \mathbb{R}^n\) by \(h_K(x) = \max\{x \cdot y : y \in K\}\).

Let \(V(K_1, \ldots, K_n)\) denote the \textit{mixed volume} of the compact convex sets \(K_1, \ldots, K_n\) in \(\mathbb{R}^n\). Mixed volumes arise as coefficients in the expansion of \(\lambda_n(t_1K_1 + \cdots + t_mK_m)\) as a homogeneous polynomial of degree \(n\) in the parameters \(t_1, \ldots, t_m \geq 0\):

\[
\lambda_n(t_1K_1 + \cdots + t_mK_m) = \sum_{i_1, \ldots, i_n=1}^{m} t_{i_1} \cdots t_{i_n} V(K_{i_1}, \ldots, K_{i_n}),
\]

where \(t_1K_1 + \cdots + t_mK_m\) is the Minkowski linear combination of the compact convex sets \(K_1, \ldots, K_m\) in \(\mathbb{R}^n\). The notation \(V(K, j; L, n-j)\) means \(K\) appears \(j\) times and \(L\) appears \(n-j\) times. It is well-known that

\[
V(L; K, j-1; B_n^2, n-j) = \frac{1}{n} \int_{S^{n-1}} h_L(v) dS_{j-1}(K, v), \tag{2.1}
\]

where \(S_{j-1}(K, \cdot)\) is the area measure of order \(j-1\) of \(K\). Note that \(S_{n-1}(K, \cdot)\) is the classical surface area measure of \(K\). The \(j\)-th intrinsic volume, \(0 \leq j \leq n\), of a compact convex set \(K\) is defined by

\[
V_j(K) = \left(\frac{n}{\kappa_{n-j}}\right)^j V(K; j; B_n^2, n-j).
\]

We shall use the following formula (see e.g., [13, p.408]): If \(2 \leq j \leq n\), then

\[
V_{j-1}(P_\nu K) = \left(\frac{n-1}{2\kappa_{n-j}}\right)^{\frac{n-1}{2}} \int_{S^{n-1}} |u \cdot v| dS_{j-1}(K, v). \tag{2.2}
\]

A fundamental inequality about mixed volumes and intrinsic volumes (see e.g., [13, p.420]) states that, if \(1 \leq j \leq n\), then for convex bodies \(K_1, K_2\) in \(\mathbb{R}^n\),

\[
V(K_1; K_2, j-1; B_n^2, n-j)^j \geq \left(\frac{\kappa_{n-j}}{\kappa_n}\right)^j V_j(K_2)^{j-1} V_j(K_1). \tag{2.3}
\]
For $1 < j \leq n$, equality in (2.3) holds if and only if $K_1$ and $K_2$ are homothetic, that is, they coincide up to a translation and a dilatation. For $j = 1$, inequality (2.3) holds with equality.

It is well-known that the isotropicity assumption (1.4) is equivalent to

$$
\|x\|^2 = \int_{S^{n-1}} |x \cdot u|^2 d\mu(u)
$$

for all $x \in \mathbb{R}^n$. Moreover, taking the trace in (1.4), we see that

$$
\mu(S^{n-1}) = n. \quad (2.4)
$$

Next, we need the following formula for the zonoid $Z_{\alpha}$. The $j$-th intrinsic volume of $Z_{\alpha}$, $1 \leq j \leq n$, can be expressed as (see e.g., [25, (5.83)])

$$
V_j(Z_{\alpha}) = \frac{2^j}{j!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} [u_1, \ldots, u_j] \alpha(u_1) \cdots \alpha(u_j) d\mu(u_1) \cdots d\mu(u_j), \quad (2.5)
$$

where $[u_1, \ldots, u_j]$ denotes the $j$-dimensional volume of the parallelepiped spanned by the vectors $u_1, \ldots, u_j$.

3. Proof of Theorem 1.2

Proof of Theorem 1.2. For $u_1, \ldots, u_j \in S^{n-1}$, let

$$
U_k := \text{span}\{u_1, \ldots, u_k\}, \quad \text{for } k = 1, \cdots, j - 1,
$$

where $\text{span}\{u_1, \ldots, u_k\}$ denotes the linear span of the vectors $u_1, \ldots, u_k$. If $u_1, \ldots, u_k$ are linearly independent, then

$$
[u_1, \ldots, u_j] = \|u_1\| \|P_{U_j^\perp} u_2\| \cdots \|P_{U_{j-2}^\perp} u_{j-1}\| \|P_{U_{j-1}^\perp} u_j\|. \quad (3.1)
$$

By the isotropicity assumption (1.4), we have

$$
P_{U_k^\perp} = \int_{S^{n-1}} P_{U_k^\perp} u \otimes P_{U_k^\perp} u d\mu(u). \quad (3.2)
$$

If $u_1, \ldots, u_k$ are linearly independent, then $P_{U_k^\perp}$ has rank $n - k$. Taking traces in (3.2), we see that

$$
\int_{S^{n-1}} \|P_{U_k^\perp} u\|^2 d\mu(u) = n - k. \quad (3.3)
$$

Define the set $\Omega_k$, $k = 1, \ldots, j$, by

$$
\Omega_k = \{(u_1, \ldots, u_k) \in (S^{n-1})^k : [u_1, u_2, \ldots, u_k] \neq 0\}. \quad (3.4)
$$

Note that if $u_k \in \text{span}\{u_1, \ldots, u_{k-1}\}$, then we have

$$
\|P_{U_{k-1}^\perp} u_k\| = 0. \quad (3.5)
$$
Applying Fubini’s theorem iteratively, integrating first with respect to \(u_j\), then \(u_{j-1}\) and so on, and using (3.4), (3.1), (3.5), and (3.3), we obtain

\[
\int_{(S^{n-1})^j} [u_1, u_2, \ldots, u_j]^2 d\mu(u_1) \cdots d\mu(u_j)
\]

\[
= \int_{\Omega_j} [u_1, u_2, \ldots, u_j]^2 d\mu(u_1) \cdots d\mu(u_j)
\]

\[
= \int_{\Omega_j} (\|P_{U_j} u_2\| \cdots \|P_{U_{j-1}} u_j\|)^2 d\mu(u_1) \cdots d\mu(u_j)
\]

\[
= \int_{\Omega_{j-1}} \left( \int_{S^{n-1}} \|P_{U_{j-1}} u_j\|^2 d\mu(u_j) \right) \left( \|P_{U_{j-1}} u_2\| \cdots \|P_{U_{j-2}} u_{j-1}\| \right)^2 d\mu(u_1) \cdots d\mu(u_{j-1})
\]

\[
= (n - j + 1) \int_{\Omega_{j-1}} \left( \|P_{U_{j-1}} u_2\| \cdots \|P_{U_{j-2}} u_{j-1}\| \right)^2 d\mu(u_1) \cdots d\mu(u_{j-1})
\]

\[
\cdots
\]

\[
= \frac{n!}{(n-j)!}. \tag{3.6}
\]

By (3.6), the measure \(\frac{(n-j)!}{n!}[u_1, \ldots, u_j]^2 d\mu(u_1) \cdots d\mu(u_j)\) is a probability measure. In a similar way, we obtain

\[
\int_{(S^{n-1})^j} [u, u_2, \ldots, u_j]^2 d\mu(u_2) \cdots d\mu(u_j) = \frac{(n-1)!}{(n-j)!}, \quad \text{for all } u \in S^{n-1}. \tag{3.7}
\]

Note that the discrete case of formula (3.6) was proved by Lutwak, Yang, and Zhang [20, Lemma 2.1] in a different way.

It follows from (3.7) that

\[
\int_{(S^{n-1})^j} \log \alpha(u_1)[u_1, \ldots, u_j]^2 d\mu(u_1) \cdots d\mu(u_j)
\]

\[
= \int_{S^{n-1}} \log \alpha(u_1) \left[ \int_{(S^{n-1})^{j-1}} [u_1, \ldots, u_j]^2 d\mu(u_2) \cdots d\mu(u_j) \right] d\mu(u_1)
\]

\[
= \frac{(n-1)!}{(n-j)!} \int_{S^{n-1}} \log \alpha(u_1) d\mu(u_1).
\]

Thus, for each \(1 \leq i \leq j\),

\[
\int_{(S^{n-1})^j} \log \alpha(u_i)[u_1, \ldots, u_j]^2 d\mu(u_1) \cdots d\mu(u_j) = \frac{(n-1)!}{(n-j)!} \int_{S^{n-1}} \log \alpha(u_i) d\mu(u_i). \tag{3.8}
\]

From (2.5), the fact that \([u_1, \ldots, u_j] \leq 1\), Jensen’s inequality and (3.8), we obtain

\[
V_j(Z_{\alpha}) = \frac{2^j}{j!} \int_{(S^{n-1})^j} [u_1, \ldots, u_j] \alpha(u_1) \cdots \alpha(u_j) d\mu(u_1) \cdots d\mu(u_j)
\]
By the definition of $\alpha$, this implies that

$$
\geq \frac{2^j n!}{j!(n-j)!} \exp \left( \frac{(n-j)!}{n!} \int_{(S^{n-1})^j} \log(\alpha(u_1) \cdots \alpha(u_j)) \right) 
\times \left[ u_1, \ldots, u_j \right]^2 \left[ d\mu(u_1) \cdots d\mu(u_j) \right] 
= 2^j \binom{n}{j} \exp \left( \frac{j}{n} \int_{S^{n-1}} \log \alpha(u) d\mu(u) \right), \tag{3.9}
$$

which is the desired inequality.

Now, we deal with the equality conditions of (3.9). For $j = 1$, the first inequality in (3.9) holds with equality. Equality in the second inequality in (3.9) holds if and only if the function $\alpha$ is constant on $\text{supp } \mu$. For $2 \leq j \leq n$, equality in the first inequality in (3.9) implies

$$
[u_1, \ldots, u_j] = 1 \quad \text{or} \quad [u_1, \ldots, u_j] = 0 \tag{3.10}
$$

for $\mu \otimes \cdots \otimes \mu$-almost all $(u_1, \ldots, u_j) \in (S^{n-1})^j$. By continuity, (3.10) holds whenever $u_i \in \text{supp } \mu$, for $i = 1, \ldots, j$. Thus, for arbitrary linearly independent $u_1, \ldots, u_j \in \text{supp } \mu$, the vectors $u_1, \ldots, u_j$ must be pairwise orthogonal. In particular, $\text{supp } \mu \subset \{ \pm v_1 \} \cup v_1^\perp$ for given $v_1 \in \text{supp } \mu$. If we choose $v_2 \in \text{supp } \mu \cap v_1^\perp$, then $\text{supp } \mu \subset \{ \pm v_2 \} \cup v_2^\perp$. Therefore, $\text{supp } \mu \subset \{ \pm v_1 \} \cup \{ \pm v_2 \} \cup \text{span} \{ v_1, v_2 \}^\perp$. Repeating these steps, we see that $\text{supp } \mu \subset \{ \pm v_1, \ldots, \pm v_n \}$, where $v_1, \ldots, v_n$ form an orthonormal basis of $\mathbb{R}^n$. Since $\mu$ is not concentrated on any great subsphere of $S^{n-1}$ and even, we conclude that $\mu$ is a cross measure on $S^{n-1}$. Equality in the second inequality in (3.9) implies that the function $\alpha(u_1) \cdots \alpha(u_j)$ is constant on the support of the probability measure $\frac{(n-j)!}{n!} [u_1, \ldots, u_j]^2 d\mu(u_1) \cdots d\mu(u_j)$. As shown in [21, Lemma A.1], this implies that $\alpha(u_1) \cdots \alpha(u_j)$ is constant for linearly independent $u_1, \ldots, u_j \in \text{supp } \mu$. We claim that $\alpha$ must be constant on $\text{supp } \mu$ when $j \neq n$. Suppose that $\text{supp } \mu = \{ \pm v_1, \ldots, \pm v_n \}$ and that

$$
\alpha(v_1)\alpha(v_3) \cdots \alpha(v_j) = \alpha(v_2)\alpha(v_3) \cdots \alpha(v_j).
$$

Then we have $\alpha(v_1) = \alpha(v_2)$, and hence $\alpha$ is constant on $\text{supp } \mu$ for $1 \leq j \leq n - 1$. By the definition of $Z_\alpha$ (1.8), we get that $Z_\alpha$ is a cube formed by the cross measure $\mu$. If $j = n$, we see that the numbers $\alpha(v_i)$ may be distinct. That is, $Z_\alpha$ is a box formed by the cross measure $\mu$. \[\square\]
We are now in a position to prove the Loomis-Whitney inequality for intrinsic volumes.

Proof of Theorem 1.1. For $j = 1$, inequality (1.7) trivially holds with equality.

For $2 \leq j \leq n$ and $u \in \text{supp} \mu$, let

$$\alpha^{-1}(u) = \int_{S^{n-1}} |u \cdot v| dS_{j-1}(K, v) = \frac{2\kappa_{n-j}}{\binom{n}{j-1}} V_{j-1}(P_u \perp K), \quad (4.1)$$

where the second equality follows from (2.2). From (2.3), (2.1), the definition of $Z_\alpha$ (1.8), Fubini’s theorem, (4.1) and (2.4), we obtain

$$V_j(K)^{j-1} \leq \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1} V(Z_\alpha; K, j - 1; B^n_2, n - j)^j$$

$$= \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1} \left( \frac{1}{n} \int_{S^{n-1}} h_{Z_\alpha}(v) dS_{j-1}(K, v) \right)^j$$

$$= \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1} \left( \frac{1}{n} \int_{S^{n-1}} \int_{S^{n-1}} |v \cdot u| \alpha(u) d\mu(u) dS_{j-1}(K, v) \right)^j$$

$$= \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1} \left( \frac{1}{n} \int_{S^{n-1}} |v \cdot u| dS_{j-1}(K, v) \alpha(u) d\mu(u) \right)^j$$

$$= \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1}. \quad (4.2)$$

Thus, by (1.9) and (4.1), we have

$$V_j(K)^{j-1} \leq \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j V_j(Z_\alpha)^{-1}$$

$$\leq \left( \frac{\binom{n}{j}}{\kappa_{n-j}} \right)^j \left( 2^j \binom{n}{j} \exp \left( \frac{j}{n} \int_{S^{n-1}} \log \alpha(u) d\mu(u) \right) \right)^{-1}$$

$$= \left( \frac{\binom{n}{j}}{2^j \kappa_{n-j}} \right)^j \exp \left( \frac{j}{n} \int_{S^{n-1}} \log \left( \frac{2\kappa_{n-j}}{(n-1)\binom{n}{j-1}} V_{j-1}(P_u \perp K) \right) d\mu(u) \right)$$

$$= \left( \frac{\binom{n}{j}}{(n-1)\binom{n}{j-1}} \right)^j \exp \left( \frac{j}{n} \int_{S^{n-1}} \log V_{j-1}(P_u \perp K) d\mu(u) \right), \quad (4.2)$$

which is the desired inequality.

By (2.3), equality in the first inequality in (4.2) holds if and only if $K$ and $Z_\alpha$ are homothetic when $2 \leq j \leq n$. Combining this with the equality conditions of
Theorem 1.2, we immediately see that $K$ is of the form (1.5) and (1.6), respectively, up to translations.

Conversely, we will show that equality in (1.7) holds if $K$ is of the form (1.5) and (1.6), respectively, up to translations. As in the proof of (4.2), we only have to verify that $K$ and $Z_\alpha$ are homothetic when $2 \leq j \leq n$. For $2 \leq j \leq n - 1$, let

$$K = \alpha C + v_0,$$

for some vector $v_0 \in \mathbb{R}^n$ and a positive number $\alpha$, where $C = \sum_{i=1}^{n} [-v_i, v_i]$ and $\text{supp} \mu = \{\pm v_1, \ldots, \pm v_n\}$ for some orthonormal basis $(v_i)_1^n$ of $\mathbb{R}^n$. From (4.1), we get

$$\alpha(v_k) = \left( \int_{S^{n-1}} |v_k \cdot v| dS_{j-1}(K, v) \right)^{-1} = \left( \int_{S^{n-1}} |v_k \cdot v| dS_{j-1}(\alpha C + v_0, v) \right)^{-1} = \alpha^{1-j} \left( \int_{S^{n-1}} |v_k \cdot v| dS_{j-1}(C, v) \right)^{-1}$$

for every $k = 1, \ldots, n$. Thus, $\alpha(v_k) = \alpha^{1-j} \beta^{-1}$, where $\beta = \int_{S^{n-1}} |v_k \cdot v| dS_{j-1}(C, v)$ is a constant for all $k = 1, \ldots, n$. Therefore, by the definition of $Z_\alpha$ (1.8),

$$Z_\alpha = \frac{1}{2} \alpha^{1-j} \beta^{-1} C.$$

That is, $K$ and $Z_\alpha$ are homothetic for $2 \leq j \leq n - 1$. For $j = n$, let

$$K = \sum_{i=1}^{n} \alpha_i [-v_i, v_i] + v_0,$$

for some vector $v_0 \in \mathbb{R}^n$ and positive numbers $(\alpha_i)_1^n$, where $\text{supp} \mu = \{\pm v_1, \ldots, \pm v_n\}$ and $(v_i)_1^n$ is an orthonormal basis of $\mathbb{R}^n$. Obviously, $S_{n-1}(K, \{v_k\}) = 2^{n-1} \alpha_1 \cdots \alpha_n / \alpha_k$. It follows from (4.1) that $\alpha(v_k) = S_{n-1}^{-1}(K, \{v_k\}) = \alpha_k / 2^{n-1} \alpha_1 \cdots \alpha_n$ for every $k = 1, \ldots, n$. Therefore, by the definition of $Z_\alpha$ (1.8),

$$Z_\alpha = \frac{1}{2^n \alpha_1 \cdots \alpha_n} \sum_{i=1}^{n} \alpha_i [-v_i, v_i].$$

That is, $K$ and $Z_\alpha$ are homothetic for $j = n$.

\[\square\]

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