Volume inequalities for complex isotropic measures

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Abstract. A new notion of complex isotropic measures is introduced and volume inequalities for their L_p-cosine and their sine transform are established.

1. Introduction

A finite nonnegative Borel measure μ on S^{n-1} is said to be (real) isotropic if

$$\|x\|^2 = \int_{S^{n-1}} \langle x, v\rangle^2 d\mu(v),$$

for $x \in \mathbb{R}^n$, where $\langle x, v\rangle$ denotes the standard scalar product. Isotropic measures have been the focus of recent studies, in particular, in relation with a variety of extremal problems for convex bodies (see, e.g., [17, 18, 20, 27, 29–32, 36, 39]).

One purpose of this article is to introduce the concept of complex isotropic measures: A finite nonnegative Borel measure μ on the sphere $S_c^{n-1} = S^{2n-1}$ of $\mathbb{C}^n$ is said to be complex isotropic if

$$\|x\|^2 = \int_{S^{2n-1}} |\langle x, v\rangle_c|^2 d\mu(v),$$

for $x \in \mathbb{C}^n$, where $\langle x, v\rangle_c$ denotes the complex scalar product. It turns out that the class of complex isotropic measures is larger than the class of (real) isotropic measures (see Theorem 3.1). The idea to find analogs of known results from Euclidean geometry in complex vector spaces is not new. In recent years, the study of convex bodies in $\mathbb{C}^n$ has received considerable attention (see, e.g., [1–5, 13, 15, 19, 22–26, 33, 34, 37, 41, 42]).

In this paper, we establish volume inequalities for the L_p-cosine and the sine transform of complex isotropic measures. Volume inequalities for the L_p-cosine transform of

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Keywords. Complex isotropic measure, The L_p-cosine transform, The sine transform.
2010 Mathematics Subject Classification. 52A20, 52A40.
Supported in part by the National Natural Science Foundation of China (Grant NO.11371239), Shanghai Leading Academic Discipline Project (Project Number: J50101) and the Research Fund for the Doctoral Programs of Higher Education of China (20123108110001).

isotropic measures were obtained by Lutwak, Yang and Zhang [29] and Barthe [11], which generalized results of Ball [7,8] and Barthe [9]. Volume inequalities for the sine transform of isotropic measures were recently obtained by Maresch and Schuster [32].

For $p \geq 1$, the L_p-cosine transform $\mathcal{C}_p\mu$ of a finite Borel measure μ on S^{n-1} is the continuous function defined by

$$(\mathcal{C}_p\mu)(\xi) = \Big(\int_{S^{n-1}} |\langle \xi, v\rangle|^p d\mu(v) \Big)^{1/p}, \quad \xi \in \mathbb{R}^n.$$

The L_p-cosine transform of an isotropic Borel measure μ on S^{n-1} determines a norm on $\mathbb{R}^n$ whose unit ball we denote by $C_p(\mu)^*$.

The (spherical) sine transform $\mathcal{S}\mu$ of a finite Borel measure μ on S^{n-1} is the continuous function defined by

$$(\mathcal{S}\mu)(\xi) = \int_{S^{n-1}} \sqrt{1 - \langle \xi, v\rangle^2} d\mu(v), \quad \xi \in S^{n-1}.$$

The sine transform of an isotropic Borel measure μ on S^{n-1} determines a norm on $\mathbb{R}^n$ (by 1-homogeneous extension) whose unit ball we denote by $S(\mu)^*$.

Volume inequalities for $C_p(\mu)^*,\ S(\mu)^*$ and their polars, $C_p(\mu),\ S(\mu)$ for which suitably normalized Lebesgue measures are extremal, are easily obtained by using inequalities such as the Urysohn and the Hölder inequality.

Optimal reverse inequalities for $C_p(\mu)^*,\ C_p(\mu)$ where equality holds when μ is a cross measure (see Section 4 for definition), are much more difficult to establish. Two approaches have been developed. Ball [7,8] and Barthe [9,11] attacked these problems using the Brascamp-Lieb inequality [14, 28, 38] and the reverse Brascamp-Lieb inequality [9]; Motivated by their results, Lutwak, Yang and Zhang [29] gave self-contained proofs that rely on the Ball-Barthe Lemma and mass transport techniques (see [29] for more details).

Optimal reverse inequalities for $S(\mu)^*$, and its polar, $S(\mu)$ for which equality holds when μ is a cross measure were conjectured by Maresch and Schuster [32]. Using the multidimensional Brascamp-Lieb inequality and its reverse due to Lieb [28] and Barthe [9], Maresch and Schuster also obtained asymptotically optimal bounds for these reverse inequalities.

The main purpose of this paper is to establish complex versions of the existing results for the L_p-cosine and the sine transform. Inspired by the results of Ball-Barthe and Maresch-Schuster, the method we adopted to treat these complex counterparts relies on the multidimensional Brascamp-Lieb inequality and its reverse due to Lieb [28] and Barthe [9].

With our first results we extend the volume inequalities for $C_p(\mu)^*$ and $C_p(\mu)$ to the complex case. In order to state these results, denote by $p^* \in [1,\infty]$ the Hölder conjugate of $p \in [1,\infty]$, that is

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

For $n, p \in (0,\infty)$, denote by $\kappa_{2n}(p)$, the volume of the unit ball of $\ell_p(\mathbb{C}^n)$ (see Proposition 6.1), that is,

$$\kappa_{2n}(p) = \frac{\pi^n(\Gamma(1+\frac{2}{p}))^n}{\Gamma(1+\frac{2n}{p})}.$$

Define $\kappa_{2n}(\infty) = \lim_{p\to\infty} \kappa_{2n}(p) = \pi^n$, and abbreviate $\kappa_{2n}(2)$ by κ_{2n}. Note that for positive integers n, the Euclidean unit ball of $\mathbb{R}^{2n}$ has precisely volume κ_{2n}. For $p \in (0,\infty)$, define $\alpha_{2n,p}$ by

$$\alpha_{2n,p} = \Big[\frac{\Gamma(n+1)\Gamma(\frac{p}{2}+1)}{\Gamma(n+\frac{p}{2})}\Big]^{\frac{2n}{p}},$$

and define $\alpha_{2n,\infty} = \lim_{p\to\infty} \alpha_{2n,p} = 1$.

For each $p \in [1,\infty)$, define the convex body $C_p^c(\mu)$ in $\mathbb{R}^{2n}$ to be the body whose support function, for $\xi \in S^{2n-1}$, is given by

$$h_{C_p^c(\mu)}(\xi) = \Big(\int_{S^{2n-1}} |\langle \xi, v\rangle_c|^p d\mu(v)\Big)^{1/p},$$

and, for $p = \infty$, let

$$h_{C_\infty^c(\mu)}(\xi) = \lim_{p\to\infty} h_{C_p^c(\mu)}(\xi) = \sup_{v\in \mathrm{supp}\mu} |\langle x, v\rangle_c|,$$

where supp μ denotes the support of the measure μ.

Our volume inequalities for the L_p-cosine transform of complex isotropic measures (see [29, p.163] for the real counterpart) can be stated as follows:

Theorem 1 Suppose $p \in [1,\infty]$. If μ is a complex isotropic measure on S^{2n-1}, then

$$\kappa_{2n}(p^*) \le |C_p^c(\mu)| \le \kappa_{2n}\alpha_{2n,p}. \tag{1}$$

There is equality in the left inequality if μ is a complex cross measure. There is equality in the right inequality if μ is suitably normalized Lebesgue measure.

Theorem 2 Suppose $p \in [1,\infty]$. If μ is a complex isotropic measure on S^{2n-1}, then

$$\kappa_{2n}/\alpha_{2n,p} \le |C_p^c(\mu)^*| \le \kappa_{2n}(p). \tag{2}$$

There is equality in the left inequality if μ is suitably normalized Lebesgue measure. There is equality in the right inequality if μ is a complex cross measure.

For the definition of complex cross measures we refer to Section 4. If $p = 2$, then equality in the left hand inequality of (1) and in the right hand inequality of (2) holds for arbitrary complex isotropic measures.

In order to extend the volume inequalities for $S(\mu)^*$ and $S(\mu)$ to the complex case define

$$\beta_{2n} := \frac{(2n)(2n-1)(n-1)^{4n}}{\Gamma(2n-1)^{1/(n-1)}}$$

and let $S^c(\mu)$ be the convex body whose support function, for $\xi \in S^{2n-1}$, is given by

$$h_{S^c(\mu)}(\xi) = \int_{S^{2n-1}} \sqrt{1 - |\langle \xi, v \rangle_c|^2} d\mu(v).$$

Our volume inequalities for the sine transform of complex isotropic measures (see [32, p.4] for the real counterpart) can be stated as follows:

Theorem 3 If μ is a complex isotropic measure on S^{2n-1}, then

$$\kappa_{2n}\beta_{2n}\Big(\frac{2n-1}{2n(n-1)}\Big)^{2n} \le |S^c(\mu)| \le \kappa_{2n}\Big(\frac{2n(n-1)}{2n-1}\Big)^{2n}. \tag{3}$$

There is equality in the right inequality if μ is suitably normalized Lebesgue measure.

Theorem 4 If μ is a complex isotropic measure on S^{2n-1}, then

$$\kappa_{2n}\Big(\frac{2n-1}{2n(n-1)}\Big)^{2n} \le |S^c(\mu)^*| \le \frac{\kappa_{2n}}{\beta_{2n}}\Big(\frac{2n(n-1)}{2n-1}\Big)^{2n}. \tag{4}$$

There is equality in the left inequality if μ is suitably normalized Lebesgue measure.

As in [32, Theorem 4.3], we can also show that the left inequality in (3) and the right inequality in (4) are asymptotically optimal. More precisely, up to a factor tending to one as n goes infinity, $|S^c(\mu)|$ is minimized and $|S^c(\mu)^*|$ is maximized by complex cross measures, respectively. However, the problems whether $|S(\mu)|$ (or $|S^c(\mu)|$) is minimized and $|S(\mu)^*|$ (or $|S^c(\mu)^*|$) is maximized precisely by real (or complex) cross measures remains open.

In Section 2 we collect definitions and basic facts about convex bodies and complex vector spaces. In Section 3 we introduce the notion of complex isotropic measures and explain their difference to real isotropic measures. The L_p-cosine transform and sine transform in complex vector spaces will be introduced in Section 4. In Section 5, we will recall the multidimensional Brascamp-Lieb inequality and its reverse form. The proofs of our main results are contained in Sections 6 and 7.

2. Notations and preliminaries

For general reference about convex geometry, the reader may wish to consult the books of Gardner [16], Koldobsky [21] and Schneider [35].

For $x, y \in \mathbb{R}^n$, we denote their scalar product by $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ and the Euclidean norm of x by $\|x\| = \sqrt{\langle x, x\rangle}$. The unit sphere $\{x \in \mathbb{R}^n : \|x\| = 1\}$ is denoted by S^{n-1}. Similarly, for $x, y \in \mathbb{C}^n$, we denote their complex scalar product by $\langle x, y\rangle_c = \sum_{i=1}^n x_i \overline{y_i}$ and the modulus of x by $\|x\| = \sqrt{\langle x, x\rangle_c}$. The unit sphere $\{x \in \mathbb{C}^n : \|x\| = 1\}$ is denoted by S^{2n-1}. We use $|K|$ for the volume of a compact set K.

If K is a nonempty compact convex subset of $\mathbb{R}^n$, then

$$h_K(x) = h(K, x) := \max\{\langle x, y\rangle : y \in K\}, \tag{5}$$

for $x \in \mathbb{R}^n$, is its support function. A nonempty compact convex set is uniquely determined by its support function. Support functions are homogeneous of degree 1, that is,

$$h_K(rx) = rh_K(x), \tag{6}$$

for all $x \in \mathbb{R}^n$ and $r \geq 0$, and are therefore often regarded as functions on S^{n-1}. They are also subadditive, i.e.,

$$h_K(x + y) \leq h_K(x) + h_K(y), \tag{7}$$

for all $x, y \in \mathbb{R}^n$. Any real-valued function on $\mathbb{R}^n$ that is sublinear, that is, both homogeneous of degree 1 and subadditive, is the support function of a unique compact convex set.

A convex body is a compact convex subset of $\mathbb{R}^n$ containing the origin in its interior. The Minkowski functional $\|\cdot\|_K$ of a convex body K is defined by $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$.

The support function of a convex body K and the Minkowski functional of the polar body K^* is related by

$$h(K, \cdot) = \|\cdot\|_{K^*}, \tag{8}$$

where the polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } y \in K\}. \tag{9}$$

For $\phi \in SL(\mathbb{R}^n)$ and a convex body K, we have

$$(\phi K)^* = \phi^{-t} K^*, \tag{10}$$

where ϕ^{-t} is the inverse of the transpose of ϕ. Using the polar coordinate formula for volume, it is easy to see that the volume of a convex body $K \in \mathbb{R}^n$ is given by

$$|K| = \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} \exp(-\|x\|_K^p) dx, \tag{11}$$

where integration is with respect to Lebesgue measure on $\mathbb{R}^n$. The classical Urysohn inequality [35, p. 318] provides an upper bound for the volume of a convex body in terms of the average value of its support function: If K is a convex body in $\mathbb{R}^n$, then

$$\Big(\frac{|K|}{\kappa_n}\Big)^{1/n} \le \frac{1}{n\kappa_n}\int_{S^{n-1}} h(K,v)dv, \tag{12}$$

with equality if and only if K is a ball. Here the integral is with respect to spherical Lebesgue measure.

In the following we list some basic facts about complex vector spaces.

Origin symmetric complex convex bodies in $\mathbb{C}^n$ are the unit balls of norms on $\mathbb{C}^n$. We denote by $\|\cdot\|_K$ the norm corresponding to the body K, that is,

$$K = \{z \in \mathbb{C}^n : \|z\|_K \le 1\}.$$

In order to define volume, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard mapping

$$\xi = (\xi_1, \cdots, \xi_n) = (\xi_{11} + i\xi_{12}, \cdots, \xi_{n1} + i\xi_{n2}) \overset{\tau}{\mapsto} (\xi_{11}, \xi_{12}, \cdots, \xi_{n1}, \xi_{n2}). \tag{13}$$

Since norms on $\mathbb{C}^n$ satisfy the equality

$$\|\lambda z\| = |\lambda|\|z\|, \forall z \in \mathbb{C}^n, \forall \lambda \in \mathbb{C},$$

origin symmetric complex convex bodies correspond to those origin symmetric bodies K in $\mathbb{R}^{2n}$ that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each $\theta \in [0, 2\pi]$ and each $(\xi_{11}, \xi_{12}, \cdots \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$:

$$\|\xi\|_K = \|R_\theta(\xi_{11}, \xi_{12}), \cdots, R_\theta(\xi_{n1}, \xi_{n2})\|_K, \tag{14}$$

where R_θ stands for the counterclockwise rotation of $\mathbb{R}^2$ by the angle θ with respect to the origin. We shall simply say that K is R_θ-invariant if it satisfies equation (14).

For $\xi \in \mathbb{C}^n$ such that $\|\xi\| = 1$, denote by

$$H_\xi = \{z \in \mathbb{C}^n : \langle z, \xi\rangle_c = \sum_{k=1}^{n} z_k\overline{\xi_k} = 0\}, \tag{15}$$

the complex hyperplane through the origin, perpendicular to ξ. Under the standard mapping from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ the hyperplane H_ξ turns into a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \cdots, \xi_{n1}, \xi_{n2}) \quad \text{and} \quad \xi^\dagger = (-\xi_{12}, \xi_{11}, \cdots, -\xi_{n2}, \xi_{n1}).$$

The orthogonal two-dimensional subspace $H_\xi^\perp$ has orthonormal basis $\xi, \xi^\dagger$.

We identify $\ell_p(\mathbb{C}^n)$ with the real $2n$-dimensional space equipped with the norm

$$\|x\|_{B_p(\mathbb{C}^n)} = [(x_{11}^2 + x_{12}^2)^{p/2} + \cdots + (x_{n1}^2 + x_{n2}^2)^{p/2}]^{1/p}, \tag{16}$$

if $1 \le p < \infty$, and

$$\|x\|_{B_\infty(\mathbb{C}^n)} = \max_{1\le j\le n} (x_{j1}^2 + x_{j2}^2)^{1/2}, \tag{17}$$

where we used $B_p(\mathbb{C}^n)$ for the unit ball of $\ell_p(\mathbb{C}^n)$. If $p \ge 1$, $B_p(\mathbb{C}^n)$ is an R_θ-invariant convex body in $\mathbb{R}^{2n}$. As usual, we denote by $B_p(\mathbb{R}^n)$ the $\ell_p(\mathbb{R}^n)$-balls. Note that $B_2(\mathbb{C}^n) = B_2(\mathbb{R}^{2n})$.

The following proposition establishes a familiar relation for the spaces $\ell_p(\mathbb{C}^n)$.

Proposition 2.1 Suppose $p \in [1, \infty]$. Then

$$(B_p(\mathbb{C}^n))^* = B_{p^*}(\mathbb{C}^n).$$

Proof. We first assume that $1 < p < \infty$. Using Hölder's inequality twice, we have

$$\begin{aligned} \langle x, y\rangle &= \sum_{i=1}^n (x_{i1}y_{i1} + x_{i2}y_{i2}) \le \sum_{i=1}^n (x_{i1}^2 + x_{i1}^2)^{1/2}(y_{i1}^2 + y_{i1}^2)^{1/2} \\ &\le \Big(\sum_{i=1}^n (x_{i1}^2 + x_{i1}^2)^{p^*/2}\Big)^{1/p^*} \Big(\sum_{i=1}^n (y_{i1}^2 + y_{i1}^2)^{p/2}\Big)^{1/p} \\ &= \|x\|_{B_{p^*}(\mathbb{C}^n)} \|y\|_{B_p(\mathbb{C}^n)}, \end{aligned}$$

for $x = (x_1, x_2, \cdots, x_n) = (x_{11}, x_{12}, x_{21}, x_{22}, \cdots, x_{n1}, x_{n2})$ and $y = (y_1, y_2, \cdots, y_n) = (y_{11}, y_{12}, y_{21}, y_{22}, \cdots, y_{n1}, y_{n2})$. In addition, equality holds precisely when

$$y_{i1} = \alpha\|x_i\|^{p^*-2}x_{i1} \quad \text{and} \quad y_{i2} = \alpha\|x_i\|^{p^*-2}x_{i2} \quad \text{for } \alpha \ge 0,\ i = 1, \cdots, n. \tag{18}$$

Hence,

$$\Big\langle \frac{x}{\|x\|_{B_{p^*}(\mathbb{C}^n)}}, \frac{y}{\|y\|_{B_p(\mathbb{C}^n)}} \Big\rangle \le 1. \tag{19}$$

Since $y/\|y\|_{B_p(\mathbb{C}^n)}$ lies on the boundary of the convex body $B_p(\mathbb{C}^n)$, formula (9) immediately implies that

$$\frac{x}{\|x\|_{B_{p^*}(\mathbb{C}^n)}} \in (B_p(\mathbb{C}^n))^*.$$

Therefore,

$$\|x\|_{(B_p(\mathbb{C}^n))^*} \le \|x\|_{B_{p^*}(\mathbb{C}^n)}.$$

Note that, since for given $x \in \mathbb{R}^{2n}$ equality in (19) is attained for suitable y, the last inequality is in fact an equality. Indeed, if $\|x\|_{(B_p(\mathbb{C}^n))^*} < \|x\|_{B_{p^*}(\mathbb{C}^n)}$, by (19), we obtain

$$\Big\langle \frac{x}{\|x\|_{(B_p(\mathbb{C}^n))^*}}, \frac{y}{\|y\|_{B_p(\mathbb{C}^n)}} \Big\rangle > 1$$

when y is chose in (18). However, this contradicts (9). Thus, we obtain

$$(B_p(\mathbb{C}^n))^* = B_{p^*}(\mathbb{C}^n), \quad \text{for } p \in (1, \infty).$$

The cases $p = 1$ and $p = \infty$ are treated in a similar way. For $p = \infty$, it follows that

$$\begin{aligned} \langle x, y \rangle &= \sum_{i=1}^{n} (x_{i1}y_{i1} + x_{i2}y_{i2}) \leq \sum_{i=1}^{n} (x_{i1}^2 + x_{i1}^2)^{1/2} (y_{i1}^2 + y_{i1}^2)^{1/2} \\ &\leq \sum_{i=1}^{n} (x_{i1}^2 + x_{i1}^2)^{1/2} \max_{1 \leq i \leq n} (y_{i1}^2 + y_{i1}^2)^{1/2} \\ &= \|x\|_{B_1(\mathbb{C}^n)} \|y\|_{B_\infty(\mathbb{C}^n)}, \end{aligned}$$

with equality precisely when

$$y_{i1} = \frac{\alpha x_{i1}}{\|x_i\|} \quad \text{and} \quad y_{i2} = \frac{\alpha x_{i2}}{\|x_i\|} \quad \text{for } \alpha \geq 0,\ i = 1, \cdots, n.$$

Now the result follows from a similar argument as for $p \in (1, \infty)$. □

3. Real and complex isotropic measures

We say a measure on S^{2n-1} is R_θ-invariant if it assumes the same value on a set and its R_θ image for each $\theta \in [0, 2\pi]$.

Recall that a Borel measure μ on S^{n-1} is isotropic provided

$$\int_{S^{n-1}} \langle \xi, v \rangle^2 d\mu(v) = \|\xi\|^2, \tag{20}$$

for all $\xi \in \mathbb{R}^n$.

A Borel measure μ on S^{2n-1} is complex isotropic provided

$$\int_{S^{2n-1}} |\langle \xi, v \rangle_c|^2 d\mu(v) = \|\xi\|^2, \tag{21}$$

for all $\xi \in \mathbb{C}^n$. Since we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, the following simple fact is crucial when considering complex isotropic measures:

$$|\langle \xi, v \rangle_c|^2 = \langle \xi, v \rangle^2 + \langle \xi, v^\dagger \rangle^2 = \langle \xi, v \rangle^2 + \langle \xi^\dagger, v \rangle^2. \tag{22}$$

Consequently, (21) becomes

$$\int_{S^{2n-1}} [\langle \xi, v\rangle^2 + \langle \xi, v^\dagger\rangle^2] d\mu(v) = \|\xi\|^2. \tag{23}$$

Geometrically, this means that

$$\int_{S^{2n-1}} \|\xi|\ \mathrm{span}\{v, v^\dagger\}\|^2 d\mu(v) = \|\xi\|^2, \tag{24}$$

whereas (20) can be rewritten as

$$\int_{S^{n-1}} \|\xi|\ \mathrm{span}\{v\}\|^2 d\mu(v) = \|\xi\|^2. \tag{25}$$

Here, $\|\xi|\ \mathrm{span}\{v, v^\dagger\}\|$ is the length of the orthogonal projection of ξ onto the 2-dimensional subspace $\mathrm{span}\{v, v^\dagger\}$ and $\|\xi|\ \mathrm{span}\{v\}\|$ is the length of the orthogonal projection of ξ onto the 1-dimensional subspace $\mathrm{span}\{v\}$. Note that (23) can also be rewritten as

$$\int_{S^{2n-1}} (v \otimes v + v^\dagger \otimes v^\dagger) d\mu(v) = I_{2n}, \tag{26}$$

where I_{2n} denotes the identity operator on $\mathbb{R}^{2n}$, and $v \otimes v$ is the matrix such that $(v \otimes v)_{ij} = v_i v_j$ for $v = (v_1, \cdots, v_{2n}) \in S^{2n-1}$.

From the definition of isotropic measures (20), it is easy to see that an isotropic measure cannot be concentrated on a great subsphere of S^{n-1}. Similarly, since, by (22), (23) can also be written as

$$\int_{S^{2n-1}} [\langle \xi, v\rangle^2 + \langle \xi^\dagger, v\rangle^2] d\mu(v) = \|\xi\|^2, \tag{27}$$

the complex isotropic measure μ is not concentrated on $H_\xi \cap S^{2n-1}$ for any $\xi \in S^{2n-1}$, where the hyperplane H_ξ is defined by (15).

Let $e_1, \cdots, e_{2n}$ denote the canonical basis for $\mathbb{R}^{2n}$. From (27), by taking $\xi = e_{2i-1}$ for $1 \leq i \leq n$, we obtain

$$\int_{S^{2n-1}} (v_{i1}^2 + v_{i2}^2) d\mu(v) = 1.$$

Summing over $1 \leq i \leq n$, it follows that

$$\mu(S^{2n-1}) = n. \tag{28}$$

Theorem 3.1

(i) If μ is an isotropic measure on S^{2n-1}, then $\frac{1}{2}\mu$ is complex isotropic.

(ii) There exists a complex isotropic measure μ on S^{2n-1} such that 2μ is not isotropic.

(iii) If μ is a complex isotropic measure and R_θ-invariant for every $\theta \in [0, 2\pi]$, then 2μ is isotropic.

Proof.

(i). By (20), we need to verify (23). We have

$$\begin{aligned}\frac{1}{2}\int_{S^{2n-1}}[\langle\xi, v\rangle^2 + \langle\xi, v^\dagger\rangle^2]d\mu(v) &= \frac{1}{2}\Big[\int_{S^{2n-1}}\langle\xi, v\rangle^2 d\mu(v) + \int_{S^{2n-1}}\langle\xi^\dagger, v\rangle^2 d\mu(v)\Big] \\ &= \frac{1}{2}[\|\xi\|^2 + \|\xi^\dagger\|^2] = \|\xi\|^2.\end{aligned}$$

(ii). Using (24) and (25), it is easy to construct a measure to prove this fact. For example, taking the discrete measure μ such that supp $\mu \subset \{e_1, e_2, e_3 \ldots, e_{2n}\}$, $\mu(\{e_1\}) = 1$, $\mu(\{e_2\}) = 0$, and $\mu(\{e_i\}) = \frac{1}{2}$ for $3 \le i \le 2n$.

(iii). Since μ is a complex isotropic R_θ-invariant measure we have, by (23),

$$\begin{aligned}\|\xi\|^2 &= \int_{S^{2n-1}}[\langle\xi, v\rangle^2 + \langle\xi, v^\dagger\rangle^2]d\mu(v) \\ &= \int_{S^{2n-1}}\langle\xi, v\rangle^2 d\mu(v) + \int_{S^{2n-1}}\langle\xi, v^\dagger\rangle^2 d\mu(v) \\ &= \int_{S^{2n-1}}\langle\xi, v\rangle^2 d\mu(v) + \int_{S^{2n-1}}\langle\xi, v^\dagger\rangle^2 d\mu(v^\dagger) \\ &= 2\int_{S^{2n-1}}\langle\xi, v\rangle^2 d\mu(v).\end{aligned}$$

□

By Theorem 3.1, the class of complex isotropic measures is larger than the one of real isotropic measures. Moreover, complex isotropic measures which are R_θ-invariant are in one-to-one correspondence with isotropic measures.

In [11], Barthe showed that any isotropic measure can be approximated by a sequence of discrete isotropic measures. Inspired by this result, we prove a similar result for complex isotropic measures.

Since we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we shall consider the following 1-1 map $\iota : L(\mathbb{C}^n) \to L(\mathbb{R}^{2n})$:

$$\begin{pmatrix} c_1^{11} + ic_2^{11} & \cdots & c_1^{1n} + ic_2^{1n} \\ \vdots & & \vdots \\ c_1^{n1} + ic_2^{n1} & \cdots & c_1^{nn} + ic_2^{nn} \end{pmatrix} \stackrel{\iota}{\longmapsto} \begin{pmatrix} c_1^{11} & -c_2^{11} & \cdots & c_1^{1n} & -c_2^{1n} \\ c_2^{11} & c_1^{11} & \cdots & c_2^{1n} & c_1^{1n} \\ \vdots & \vdots & & \vdots & \vdots \\ c_1^{n1} & -c_2^{n1} & \cdots & c_1^{nn} & -c_2^{nn} \\ c_2^{n1} & c_1^{n1} & \cdots & c_2^{nn} & c_1^{nn} \end{pmatrix}.$$

It is easy to verify that $\tau(Tx) = \iota(T)\tau(x)$ for $T \in L(\mathbb{C}^n)$, $x \in \mathbb{C}^n$, where τ is defined by (13), and $\iota(AB) = \iota(A)\iota(B)$ for $A, B \in L(\mathbb{C}^n)$. Moreover, $(\iota(T)x)^\dagger = \iota(T)x^\dagger$ for $T \in$

$L(\mathbb{C}^n),\ x \in \mathbb{R}^{2n}$ and $T \in L(\mathbb{C}^n)$ is a Hermitian matrix if and only if $\iota(T)$ is a symmetric matrix.

Theorem 3.2 Suppose μ is an arbitrary complex isotropic measure on S^{2n-1}. Then there exists a sequence $\mu_k, k \in \mathbb{N}$, of discrete complex isotropic measures such that μ_k converges weakly to μ as $k \to \infty$.

Proof. Given $\epsilon > 0$, we choose a finite maximal ϵ-net N_ϵ in S^{2n-1} such that S^{2n-1} is partitioned into Borel sets $(U_x)_{x\in N}$ with $U_x \subset B(x, \epsilon)$ (the closed ball of radius ϵ around x). Given a finite measure μ on S^{2n-1}, we consider its approximation

$$\mu_\epsilon = \sum_{x\in N_\epsilon} \mu(U_x)\delta[x],$$

where $\delta[x]$ denotes the Dirac measure at x. For any continuous function $f : S^{2n-1} \to \mathbb{R}$,

$$\Big|\int f d\mu - \int f d\mu_\epsilon\Big| = \Big|\sum_{x\in N_\epsilon} \int_{U_x} (f - f(x))d\mu\Big| \le \mu(S^{2n-1})\omega_f(\epsilon), \tag{29}$$

where $\omega_f(\epsilon)$ is the modulus of continuity of f. Thus, μ_ϵ converges weakly to μ as $\epsilon \to 0$.

Since the complex isotropic measure μ is not concentrated on $H_\xi \cap S^{2n-1}$ for any $\xi \in S^{2n-1}$, also μ_ϵ is not concentrated on $H_\xi \cap S^{2n-1}$ for any $\xi \in S^{2n-1}$ when ϵ is small enough. Therefore,

$$\xi^* M_\epsilon \xi = \int_{S^{2n-1}} |\langle \xi, v\rangle_c|^2 d\mu_\epsilon(v) > 0, \quad \text{for all} \quad \xi \in \mathbb{C}^n/\{o\},$$

where ξ^* is the conjugate transpose of ξ and the complex matrix M_ϵ is defined by

$$M_\epsilon = \int_{S^{2n-1}} v \otimes_c v d\mu_\epsilon(v)$$

with $(v \otimes_c v)_{ij} = v_i\overline{v_j}$ for $v = (v_1, \cdots, v_n) \in \mathbb{C}^n$. Consequently, M_ϵ is positive definite when ϵ is small enough. Thus, we can write $M_\epsilon = U_\epsilon^* diag(\lambda_1(\epsilon), \cdots, \lambda_n(\epsilon))U_\epsilon$ such that U_ϵ is a unitary matrix and $\lambda_i(\epsilon) > 0$ for all $1 \le i \le n$, where U_ϵ^* is the conjugate transpose of U_ϵ (see e.g., [40, Theorem 6.1]). It follows that $S_\epsilon = M_\epsilon^{-1/2} = U_\epsilon^* diag(\lambda_1^{-1/2}(\epsilon), \cdots, \lambda_n^{-1/2}(\epsilon))U_\epsilon$. Hence, S_ϵ is a Hermitian matrix, i.e., $\iota(S_\epsilon)$ is a symmetric matrix. Observe that $\iota(v \otimes_c v) = v \otimes v + v^\dagger \otimes v^\dagger$. Therefore,

$$\iota(M_\epsilon) = \int_{S^{2n-1}} (v \otimes v + v^\dagger \otimes v^\dagger)d\mu_\epsilon(v). \tag{30}$$

Using the fact that $(\iota(T)x)^\dagger = \iota(T)x^\dagger$ for $T \in L(\mathbb{C}^n),\ x \in \mathbb{R}^{2n}$ and $\iota(AB) = \iota(A)\iota(B)$ for $A, B \in$

$L(\mathbb{C}^n)$, we have

$$\begin{aligned}
&\int_{S^{2n-1}} \Big[\frac{\iota(S_\epsilon)v}{\|\iota(S_\epsilon)v\|} \otimes \frac{\iota(S_\epsilon)v}{\|\iota(S_\epsilon)v\|} + \Big(\frac{\iota(S_\epsilon)v}{\|\iota(S_\epsilon)v\|}\Big)^\dagger \otimes \Big(\frac{\iota(S_\epsilon)v}{\|\iota(S_\epsilon)v\|}\Big)^\dagger\Big] \|\iota(S_\epsilon)v\|^2 d\mu_\epsilon(v) \\
= &\int_{S^{2n-1}} \Big[\frac{\iota(S_\epsilon)v}{\|\iota(S_\epsilon)v\|} \otimes \frac{\iota(S_\epsilon)v}{\|\iota(S_\epsilon)v\|} + \frac{\iota(S_\epsilon)v^\dagger}{\|\iota(S_\epsilon)v\|} \otimes \frac{\iota(S_\epsilon)v^\dagger}{\|\iota(S_\epsilon)v\|}\Big] \|\iota(S_\epsilon)v\|^2 d\mu_\epsilon(v) \\
= &\int_{S^{2n-1}} \Big(\iota(S_\epsilon)v \otimes \iota(S_\epsilon)v + \iota(S_\epsilon)v^\dagger \otimes \iota(S_\epsilon)v^\dagger\Big) d\mu_\epsilon(v) \\
= &\ \iota(S_\epsilon) \int_{S^{2n-1}} (v \otimes v + v^\dagger \otimes v^\dagger) d\mu_\epsilon(v) \iota(S_\epsilon) \\
= &\ \iota(S_\epsilon)\iota(M_\epsilon)\iota(S_\epsilon) = \iota(S_\epsilon M_\epsilon S_\epsilon) = \iota(I_n) = I_{2n}.
\end{aligned}$$

Let

$$u = \frac{\iota(S_\epsilon)v}{\|\iota(S_\epsilon)v\|} \quad \text{and} \quad \nu_\epsilon = \sum_{x \in N_\epsilon} \mu_\epsilon(x) \|\iota(S_\epsilon)x\|^2 \delta\Big[\frac{\iota(S_\epsilon)x}{\|\iota(S_\epsilon)x\|}\Big].$$

It follows that

$$\int_{S^{2n-1}} (u \otimes u + u^\dagger \otimes u^\dagger) d\nu_\epsilon(u) = I_{2n}.$$

That is, the measure ν_ϵ is complex isotropic.

For any continuous function $f : S^{2n-1} \to \mathbb{R}$,

$$\begin{aligned}
\Big| \int f d\nu_\epsilon - \int f d\mu_\epsilon \Big| &\le \sum_{x \in N_\epsilon} \mu_\epsilon(x) \Big| \|\iota(S_\epsilon)x\|^2 f\Big(\frac{\iota(S_\epsilon)x}{\|\iota(S_\epsilon)x\|}\Big) - f(x) \Big| \\
&\le \mu(S^{2n-1}) \max_{x \in S^{2n-1}} \Big| \|\iota(S_\epsilon)x\|^2 f\Big(\frac{\iota(S_\epsilon)x}{\|\iota(S_\epsilon)x\|}\Big) - f(x) \Big| \\
&\le \mu(S^{2n-1}) \max_{x \in S^{2n-1}} \|\iota(S_\epsilon)x\|^2 \Big| f\Big(\frac{\iota(S_\epsilon)x}{\|\iota(S_\epsilon)x\|}\Big) - f(x) \Big| \\
&\quad + \mu(S^{2n-1}) \max_{x \in S^{2n-1}} \Big| \|\iota(S_\epsilon)x\|^2 - 1 \Big| \max_{x \in S^{2n-1}} |f(x)|.
\end{aligned}$$

From the assumption that μ is complex isotropic, we deduce (26). Since μ_ϵ converges weakly converges to μ as $\epsilon \to 0$ and $\iota(M_\epsilon)$ involves n^2 continuous functions, (30) shows that $\iota(M_\epsilon)$ converges to I_{2n} in the maximum norm, i.e., there exists a function $\omega_1(\epsilon)$ with limit zero at zero such that $|(\iota(M_\epsilon) - I_{2n})_{ij}| \le \omega_1(\epsilon)$. Since $\iota(S_\epsilon) = \iota(M_\epsilon)^{-1/2}$, the above quantity can be bounded from above by a function $\omega_2(\epsilon)$ which is zero at zero, and depending on ω_1, on the modulus of continuity ω_f and on $\max |f|$.

Hence, (29) yields

$$\Big| \int f d\nu_\epsilon - \int f d\mu \Big| \le \Big| \int f d\nu_\epsilon - \int f d\mu_\epsilon \Big| + \Big| \int f d\mu_\epsilon - \int f d\mu \Big| \le \omega_2(\epsilon) + \mu(S^{2n-1})\omega_f(\epsilon).$$

Therefore, there exists a subsequence $\nu_{\epsilon_k}, k \in \mathbb{N}$, of discrete complex isotropic measure such that ν_{ϵ_k} weakly converges to μ. $\square$

4.The L_p-cosine and the sine transform in complex vector spaces

The following lemma is a complex version of a lemma obtained by Lutwak, Yang and Zhang [29, Lemma 4.3].

Lemma 4.1 Suppose μ is a complex isotropic Borel measure on S^{2n-1}. Let $\{v_1, \cdots, v_n\} \subset S^{2n-1}$ such that $v_i \notin \text{span}\{v_j, v_j^\dagger\}$ for $i \neq j$, and

$$\text{supp}\ \mu \subseteq \{\text{span}\{v_1, v_1^\dagger\} \cap S^{2n-1}, \cdots, \text{span}\{v_n, v_n^\dagger\} \cap S^{2n-1}\},$$

then $\mu(\text{span}\{v_i, v_i^\dagger\} \cap S^{2n-1}) = 1$ for $1 \leq i \leq n$ and $\{v_1, v_1^\dagger, \cdots, v_n, v_n^\dagger\}$ is an orthonormal basis of $\mathbb{R}^{2n}$.

Proof. Since μ is complex isotropic, (23) and the fact that $|\langle \xi, v\rangle_c| = |\langle \xi, R_\theta v\rangle_c|$ for all $\xi \in \mathbb{R}^{2n}$, yield

$$\sum_{i=1}^{n} a_i[\langle \xi, v_i\rangle^2 + \langle \xi, v_i^\dagger\rangle^2] = \|\xi\|^2,$$

where $a_i = \mu(\text{span}\{v_i, v_i^\dagger\} \cap S^{2n-1})$. Taking $\xi = v_j$, we obtain $\langle v_j, v_j\rangle^2 + \langle v_j, v_j^\dagger\rangle^2 = 1$. Meanwhile,

$$\sum_{i=1}^{n} a_i[\langle v_j, v_i\rangle^2 + \langle v_j, v_i^\dagger\rangle^2] = 1, \tag{31}$$

which shows that $a_j \leq 1$. However, by (28), we have $\sum_{i=1}^{n} a_i = n$, and hence $a_j = 1$. Hence, from (31), we deduce that $|\langle v_j, v_i\rangle| = |\langle v_j, v_i^\dagger\rangle| = 0$ for $j \neq i$. $\square$

Lemma 4.2 Let $\{v_1, v_1^\dagger, \cdots, v_n, v_n^\dagger\}$ be an orthonormal basis of $\mathbb{R}^{2n}$ and let μ be a Borel measure on S^{2n-1} such that

$$\text{supp}\ \mu = \{\text{span}\{v_1, v_1^\dagger\} \cap S^{2n-1}, \cdots, \text{span}\{v_n, v_n^\dagger\} \cap S^{2n-1}\},$$

and

$$\mu(\text{span}\{v_i, v_i^\dagger\} \cap S^{2n-1}) = 1$$

for $1 \leq i \leq n$. Then, μ is complex isotropic, but not necessarily isotropic.

Proof. Since $\{v_1, v_1^\dagger, \cdots, v_n, v_n^\dagger\}$ is an orthonormal basis of $\mathbb{R}^{2n}$, we have for all $\xi \in \mathbb{R}^{2n}$,

$$\begin{aligned} \xi &= \sum_{i=1}^{n} [\langle \xi, v_i\rangle v_i + \langle \xi, v_i^\dagger\rangle v_i^\dagger] \\ &= \sum_{i=1}^{n} \mu(\text{span}\{v_i, v_i^\dagger\} \cap S^{2n-1})[\langle \xi, v_i\rangle v_i + \langle \xi, v_i^\dagger\rangle v_i^\dagger]. \end{aligned}$$

Thus,

$$\begin{aligned}\|\xi\|^2 &= \langle \xi, \xi\rangle = \sum_{i=1}^{n} \mu(\text{span}\{v_i, v_i^\dagger\} \cap S^{2n-1})[\langle \xi, v_i\rangle^2 + \langle \xi, v_i^\dagger\rangle^2] \\ &= \int_{S^{2n-1}} [\langle \xi, v\rangle^2 + \langle \xi, v^\dagger\rangle^2] d\mu(v).\end{aligned}$$

Thus, μ is complex isotropic by (23). However, μ may not be isotropic as can be seen from the example given in the proof of Theorem 3.1 (ii). □

An important example of (real) isotropic measures on S^{n-1} are the cross measures, i.e., even isotropic measures concentrated on $\{\pm v_1, \cdots, \pm v_n\}$, where $v_1, \cdots, v_n$ is an orthonormal basis of $\mathbb{R}^n$. The basic cross measure is the cross measure such that $v_i = e_i$ for $1 \le i \le n$. In view of Lemma 4.1 and Lemma 4.2, we introduce complex cross measures μ, that is, R_θ-invariant complex isotropic measures such that supp $\mu = \{\text{span}\{v_1, v_1^\dagger\} \cap S^{2n-1}, \cdots, \text{span}\{v_n, v_n^\dagger\} \cap S^{2n-1}\}$, where $v_1, v_1^\dagger \cdots, v_n, v_n^\dagger$ is an orthonormal basis of $\mathbb{R}^{2n}$. The basic complex cross measure is the complex cross measure such that $v_i = e_{2i-1}$ for $1 \le i \le n$. Note that a complex cross measure is just a rotation of the basic complex cross measure, since $\{v_1, v_1^\dagger, \cdots, v_n, v_n^\dagger\}$ is an orthonormal basis of $\mathbb{R}^{2n}$ by Lemma 4.1.

The convex bodies defined by the L_p-cosine and the sine transform in real vector spaces are well understood. Assume that the measure μ is not concentrated on a great subsphere of S^{n-1}. For each $p \in [1, \infty)$, the origin-symmetric convex body $C_p(\mu)$ in $\mathbb{R}^n$ is defined to be the body whose support function, for $\xi \in S^{n-1}$, is given by

$$h_{C_p(\mu)}(\xi)^p = \int_{S^{n-1}} |\langle \xi, v\rangle|^p d\mu(v),$$

and, for $p = \infty$, is given by

$$h_{C_\infty(\mu)}(\xi) = \lim_{p\to\infty} h_{C_p(\mu)}(\xi) = \sup_{v\in \text{supp}\mu} |\langle \xi, v\rangle|.$$

The origin-symmetric convex body $S(\mu)$ in $\mathbb{R}^n$ is defined to be the body whose support function, for $\xi \in S^{n-1}$, is given by

$$h_{S(\mu)}(\xi) = \int_{S^{n-1}} (1 - \langle \xi, v\rangle^2)^{1/2} d\mu(v).$$

Now, we introduce their complex counterparts. Assume that the measure μ is not concentrated on $H_\xi \cap S^{2n-1}$ for any $\xi \in S^{2n-1}$. For each $p \in [1, \infty)$, define the R_θ-invariant convex body $C_p^c(\mu)$ in $\mathbb{R}^{2n}$ to be the body whose support function , for $\xi \in S^{2n-1}$, is given by

$$h_{C_p^c(\mu)}(\xi)^p = \int_{S^{2n-1}} |\langle \xi, v\rangle_c|^p d\mu(v) = \int_{S^{2n-1}} [\langle \xi, v\rangle^2 + \langle \xi, v^\dagger\rangle^2]^{\frac{p}{2}} d\mu(v), \tag{32}$$

and, for $p=\infty$, is given by

$$h_{C^c_\infty(\mu)}(\xi)=\lim_{p\to\infty} h_{C^c_p(\mu)}(\xi)=\sup_{v\in\operatorname{supp}\mu}|\langle \xi,v\rangle_c|=\sup_{v\in\operatorname{supp}\mu}[\langle \xi,v\rangle^2+\langle \xi,v^\dagger\rangle^2]^{\frac{1}{2}}. \tag{33}$$

Define the R_θ-invariant convex body $S^c(\mu)$ in $\mathbb{R}^{2n}$ to be the body whose support function, for $\xi\in S^{2n-1}$, is given by

$$h_{S^c(\mu)}(\xi)=\int_{S^{2n-1}}\sqrt{1-|\langle \xi,v\rangle_c|^2}d\mu(v)=\int_{S^{2n-1}}\sqrt{1-\langle \xi,v\rangle^2-\langle \xi,v^\dagger\rangle^2}d\mu(v). \tag{34}$$

As shown in [29], if μ is a basic cross measure, then $C_p(\mu)=B_{p^*}(\mathbb{R}^n)$. Analogously, if μ is a basic complex cross measure, then, by Lemma 4.1, we have $\mu(\operatorname{span}\{e_{2i-1},e^\dagger_{2i-1}\}\cap S^{2n-1})=1$. Together with (32), (16), (8) and Proposition 2.1, it follows that for $1\le p<\infty$, $\xi\in S^{2n-1}$,

$$\begin{aligned}
h_{C^c_p(\mu)}(\xi)^p &= \int_{S^{2n-1}}[\langle \xi,v\rangle^2+\langle \xi,v^\dagger\rangle^2]^{\frac{p}{2}}d\mu(v)\\
&= \sum_{i=1}^n \mu(\operatorname{span}\{e_{2i-1},e^\dagger_{2i-1}\}\cap S^{2n-1})[\langle \xi,e_{2i-1}\rangle^2+\langle \xi,e^\dagger_{2i-1}\rangle^2]^{\frac{p}{2}}\\
&= \sum_{i=1}^n[\langle \xi,e_{2i-1}\rangle^2+\langle \xi,e^\dagger_{2i-1}\rangle^2]^{\frac{p}{2}}=\sum_{i=1}^n(\xi_{i1}^2+\xi_{i2}^2)^{p/2}\\
&= \|\xi\|^p_{B_p(\mathbb{C}^n)}=h_{(B_p(\mathbb{C}^n))^*}(\xi)^p=h_{B_{p^*}(\mathbb{C}^n)}(\xi)^p,
\end{aligned}$$

i.e., $C^c_p(\mu)=B_{p^*}(\mathbb{C}^n)$. By (33), (17) and (8), we also have $C^c_\infty(\mu)=B_1(\mathbb{C}^n)$.

Theorem 4.3 Assume that the measure μ is not concentrated on $H_\xi\cap S^{2n-1}$ for any $\xi\in S^{2n-1}$. Then, $C^c_p(\mu)$, $1\le p\le\infty$, and $S^c(\mu)$ are R_θ-invariant convex bodies.

Proof. The assumption that the measure μ is not concentrated on $H_\xi\cap S^{2n-1}$ for any $\xi\in S^{2n-1}$ implies that $C^c_p(\mu)$ ($p\in[1,\infty]$) contains the origin in its interior. We may rewrite (32) as follows:

$$h_{C^c_p(\mu)}(x)^p=\int_{S^{2n-1}}\|x|\operatorname{span}\{v,v^\dagger\}\|^p d\mu(v), \tag{35}$$

for $x\in\mathbb{R}^{2n}$. Here, $\|x|\operatorname{span}\{v,v^\dagger\}\|$ is the length of the orthogonal projection of x onto the 2-dimensional subspace $\operatorname{span}\{v,v^\dagger\}$. In order to prove $C^c_p(\mu)$ is convex we need to show that $h_{C^c_p(\mu)}$ satisfies (6) and (7). Clearly, $h_{C^c_p(\mu)}$ is homogeneous of degree 1. Observe that

$$\begin{aligned}
\|(x+y)|\operatorname{span}\{v,v^\dagger\}\| &= \|x|\operatorname{span}\{v,v^\dagger\}+y|\operatorname{span}\{v,v^\dagger\}\|\\
&\le \|x|\operatorname{span}\{v,v^\dagger\}\|+\|y|\operatorname{span}\{v,v^\dagger\}\|.
\end{aligned}$$

The latter inequality, together with the fact that the L_p norm $\|\cdot\|_p$ with respect to μ is increasing for $p \in [1,\infty)$ and Minkowski's inequality, yield

$$\begin{aligned} h_{C_p^c(\mu)}(x+y) &= \Big(\int_{S^{2n-1}} \|(x+y)|\ \mathrm{span}\{v,v^\dagger\}\|^p d\mu(v)\Big)^{\frac{1}{p}} \\ &\leq \Big(\int_{S^{2n-1}} (\|x|\ \mathrm{span}\{v,v^\dagger\}\| + \|y|\ \mathrm{span}\{v,v^\dagger\}\|)^p d\mu(v)\Big)^{\frac{1}{p}} \\ &\leq \Big(\int_{S^{2n-1}} \|x|\ \mathrm{span}\{v,v^\dagger\}\|^p d\mu(v)\Big)^{\frac{1}{p}} + \Big(\int_{S^{2n-1}} \|y|\ \mathrm{span}\{v,v^\dagger\}\|^p d\mu(v)\Big)^{\frac{1}{p}} \\ &= h_{C_p^c(\mu)}(x) + h_{C_p^c(\mu)}(y). \end{aligned}$$

Similarly, one can verify that $C_\infty^c(\mu)$ is convex.

Just as in the real case, we can extend the domain of $h_{S^c(\mu)}$ from S^{2n-1} to $\mathbb{R}^{2n}$ by

$$h_{S^c(\mu)}(x) = \int_{S^{2n-1}} \|x|\ [\mathrm{span}\{v,v^\dagger\}]^\perp\| d\mu(v), \tag{36}$$

for $x \in \mathbb{R}^{2n}$. Here, $\|x|\ [\mathrm{span}\{v,v^\dagger\}]^\perp\|$ is the length of the orthogonal projection of x onto the $(2n-2)$-dimensional subspace $[\mathrm{span}\{v,v^\dagger\}]^\perp$. In order to prove that $S^c(\mu)$ contains the origin in its interior under the assumption that the measure μ is not concentrated on $H_\xi \cap S^{2n-1}$ for any $\xi \in S^{2n-1}$, we need the following simple fact: for each $v \in S^{2n-1}$ there exists $\xi \in S^{2n-1}$ such that

$$\mathrm{span}\{v,v^\dagger\} \subseteq H_\xi.$$

Indeed, there exists $\xi \in S^{2n-1}$ such that $\langle \xi, v\rangle = \langle \xi, v^\dagger\rangle = 0$. Thus, $\xi \perp \mathrm{span}\{v,v^\dagger\}$ and $\xi^\dagger \perp \mathrm{span}\{v,v^\dagger\}$, which, by (15), proves the claim. Next, we show that the set $S^c(\mu)$ is convex. For $x, y \in \mathbb{R}^{2n}$,

$$\begin{aligned} h_{S^c(\mu)}(x+y) &= \int_{S^{2n-1}} \|(x+y)|\ [\mathrm{span}\{v,v^\dagger\}]^\perp\| d\mu(v) \\ &\leq \int_{S^{2n-1}} (\|x|\ [\mathrm{span}\{v,v^\dagger\}]^\perp\| + \|y|\ [\mathrm{span}\{v,v^\dagger\}]^\perp\|) d\mu(v) \\ &= h_{S^c(\mu)}(x) + h_{S^c(\mu)}(y). \end{aligned}$$

Now, since $|\langle R_\theta \xi, v\rangle_c| = |\langle \xi, v\rangle_c|$, (8) and (14), definitions (32), (33), (34), yield that $C_p^c(\mu)^*$ and $S^c(\mu)^*$ are R_θ-invariant. Moreover, since $R_\theta \in SO(\mathbb{R}^{2n})$, (10) implies that also $C_p^c(\mu)$ and $S^c(\mu)$ are R_θ-invariant. □

Remark: Note that given a complex isotropic measure μ, there always exists an R_θ-invariant complex isotropic measure $\tilde{\mu}$ such that

$$\int_{S^{2n-1}} |\langle \xi, v\rangle_c|^2 d\mu(v) = \int_{S^{2n-1}} |\langle \xi, v\rangle_c|^2 d\tilde{\mu}(v).$$

By Theorem 3.1 (iii), $2\tilde{\mu}$ is also isotropic. Thus, one can consider the convex body $C_p^c(\tilde{\mu})$ and $S^c(\tilde{\mu})$ instead. However, we can not directly apply the results for real isotropic measures (except for $p = 2$) due to Barthe [11], Lutwak, Yang and Zhang [29], Maresch and Schuster [32]. The reason is that to determine $C_p^c(\tilde{\mu})$'s extremum we have to consider projections onto 2-dimensional subspaces span$\{v, v^\dagger\}$ rather than 1-dimensional projections as for $C_p(\tilde{\mu})$, and, similarly, for $S^c(\tilde{\mu})$. Thus, we use the multidimensional Brascamp-Lieb inequality and its reverse as motivated by [11, 32].

5. The multidimensional Brascamp-Lieb inequality and its reverse

The tools we use to investigate the complex L_p-cosine and sine transforms are Theorem 5.2 and Theorem 5.3, below. These results can be seen as special cases of the following multidimensional Brascamp-Lieb inequality and its reverse due to Lieb [28] and Barthe [9, Theorem 6].

Theorem 5.1 Let m, n be integers. For $i = 1, \cdots, m$ let E_i be subspaces of $\mathbb{R}^n$ of dimension n_i and let P_i be the orthogonal projections onto E_i. Assume that there exist positive numbers $(c_i)_{i=1}^m$ such that

$$\sum_{i=1}^{m} c_i P_i = I_n.$$

If for $i = 1, \cdots, m$, f_i is a non-negative integrable function on E_i, then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(P_i x)^{c_i} dx \leq \prod_{i=1}^{m} \Big(\int_{E_i} f_i \Big)^{c_i}, \tag{37}$$

and

$$\int_{\mathbb{R}^n}^{*} \sup \Big\{ \prod_{i=1}^{m} f_i(y_i)^{c_i} : x = \sum_{i=1}^{m} c_i y_i, y_i \in E_i \Big\} dx \geq \prod_{i=1}^{m} \Big(\int_{E_i} f_i \Big)^{c_i}. \tag{38}$$

Note that the normalized Brascamp-Lieb inequality which was first discovered by Ball can be easily deduced from (37) by setting $n_i = 1$ for $i = 1, \cdots, m$. Similarly, Barthe's normalized reverse Brascamp-Lieb inequality can be easily deduced from (38) by setting $n_i = 1$ for $i = 1, \cdots, m$. These inequalities have had a profound impact on convex geometric analysis (see, e.g., [6–12, 17, 18]). Using the same notations, we write $P_{\text{span}\{u,u^\dagger\}}$ and $P_{[\text{span}\{u,u^\dagger\}]^\perp}$, $u \in S^{2n-1}$, for the orthogonal projections onto the 2-dimensional subspace span$\{u, u^\dagger\}$ and the $(2n-2)$-dimensional subspace $[\text{span}\{u, u^\dagger\}]^\perp$, respectively. Using Theorem 5.1 with $n_i = 2$ for $i = 1, \cdots, m$, we obtain:

Theorem 5.2 Suppose that $v_1, \cdots, v_m \in S^{2n-1}$ and there exist $c_1, \cdots, c_m > 0$ such that

$$\sum_{i=1}^{m} c_i P_{\mathrm{span}\{v_i, v_i^\dagger\}} = I_{2n}. \tag{39}$$

Then for all integrable functions $f_i : \mathrm{span}\{v_i, v_i^\dagger\} \to [0, \infty)$, $1 \le i \le m$,

$$\int_{\mathbb{R}^{2n}} \prod_{i=1}^{m} f_i^{c_i}(x|\ \mathrm{span}\{v_i, v_i^\dagger\})dx \le \prod_{i=1}^{m} \Big(\int_{\mathrm{span}\{v_i, v_i^\dagger\}} f_i \Big)^{c_i}$$

and

$$\int_{\mathbb{R}^{2n}}^{*} \sup \Big\{ \prod_{i=1}^{m} f_i(y_i)^{c_i} : x = \sum_{i=1}^{m} c_i y_i, y_i \in \mathrm{span}\{v_i, v_i^\dagger\} \Big\} dx \ge \prod_{i=1}^{m} \Big(\int_{\mathrm{span}\{v_i, v_i^\dagger\}} f_i \Big)^{c_i}.$$

The following result is also deduced from Theorem 5.1 by combining (37) and (38), for $n_i = 2n - 2$, $i = 1, \cdots, m$.

Theorem 5.3 Suppose that $v_1, \cdots, v_m \in S^{2n-1}$ and that there exist $c_1, \cdots, c_m > 0$ such that

$$\sum_{i=1}^{m} c_i P_{[\mathrm{span}\{v_i, v_i^\dagger\}]^\perp} = I_{2n}. \tag{40}$$

If $f_i, g_i : [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp \to [0, \infty)$, $1 \le i \le m$, are integrable functions such that

$$\int_{[\mathrm{span}\{v_i, v_i^\dagger\}]^\perp} f_i = \int_{[\mathrm{span}\{v_i, v_i^\dagger\}]^\perp} g_i = 1,$$

then

$$\int_{\mathbb{R}^{2n}} \prod_{i=1}^{m} f_i(x|\ [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp)^{c_i} dx \le \int_{\mathbb{R}^{2n}} \sup \Big\{ \prod_{i=1}^{m} g_i(y_i)^{c_i} : x = \sum_{i=1}^{m} c_i y_i, y_i \in [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp \Big\} dx.$$

6. The L_p-cosine transform of complex isotropic measures

Recall that, for $p \in (0, \infty)$,

$$\kappa_{2n}(p) = \frac{\pi^n (\Gamma(1 + \frac{2}{p}))^n}{\Gamma(1 + \frac{2n}{p})}, \quad \text{and} \quad \kappa_{2n}(\infty) = \pi^n.$$

The following lemma is proved in the same way as its real counterpart (see e.g., [21, p.32]).

Proposition 6.1 Suppose $p \in [1, \infty]$. Then

$$|B_p(\mathbb{C}^n)| = \kappa_{2n}(p).$$

Proof. Clearly, $|B_\infty(\mathbb{C}^n)| = \pi^n$. Thus, we may assume that $1 \leq p < \infty$. Recall that

$$\|x\|_{B_p(\mathbb{C}^n)} = [(x_{11}^2 + x_{12}^2)^{p/2} + \cdots + (x_{n1}^2 + x_{n2}^2)^{p/2}]^{1/p}.$$

Therefore, on the one hand

$$\int_{\mathbb{R}^{2n}} \exp\Big(- \|x\|_{B_p(\mathbb{C}^n)}^p \Big) dx = \prod_{i=1}^{n} \Big(\int_{\mathbb{R}^2} e^{-(x_{i1}^2 + x_{i2}^2)^{p/2}} dx_{i1} dx_{i2} \Big) = \Big(\pi \Gamma\Big(1 + \frac{2}{p}\Big)\Big)^n.$$

On the other hand, by (11), we obtain

$$\int_{\mathbb{R}^{2n}} \exp\Big(- \|x\|_{B_p(\mathbb{C}^n)}^p \Big) dx = \Gamma\Big(1 + \frac{2n}{p}\Big) |B_p(\mathbb{C}^n)|.$$

Comparing these two expressions for the same integral, yields the desired result. □

Corollary 6.2 Suppose $p \in [1, \infty]$ and μ is a complex cross measure. Then

$$|C_p^c(\mu)^*| = \kappa_{2n}(p).$$

Proof. Observe that if μ is a basic complex cross measure, then $C_p^c(\mu)^* = B_p(\mathbb{C}^n)$. If μ is a complex cross measure, then we may assume that supp $\mu = \{\text{span}\{v_1, v_1^\dagger\} \cap S^{2n-1}, \cdots, \text{span}\{v_n, v_n^\dagger\} \cap S^{2n-1}\}$, where $v_1, v_1^\dagger \cdots, v_n, v_n^\dagger$ denotes some orthonormal basis vectors of $\mathbb{R}^{2n}$. From Lemma 4.1, it follows that $\mu(\text{span}\{v_i, v_i^\dagger\} \cap S^{2n-1}) = 1$ for $1 \leq i \leq n$. Since $\{v_1, v_1^\dagger, \cdots, v_n, v_n^\dagger\}$ is an orthonormal basis of $\mathbb{R}^{2n}$, there exists $U \in SO(\mathbb{R}^{2n})$, such that $v_i = Ue_{2i-1}, v_i^\dagger = Ue_{2i}$ for $i = 1, \cdots, m$. By (8) and (32), we have for $x \in \mathbb{R}^{2n}$,

$$\begin{aligned}
\|x\|_{C_p^c(\mu)^*}^p &= h_{C_p^c(\mu)}(x)^p = \int_{S^{2n-1}} [\langle x, v\rangle^2 + \langle x, v^\dagger\rangle^2]^{\frac{p}{2}} d\mu(v) \\
&= \sum_{i=1}^{m} \mu(\text{span}\{v_i, v_i^\dagger\} \cap S^{2n-1})(\langle x, v_i\rangle^2 + \langle x, v_i^\dagger\rangle^2)^{p/2} \\
&= \sum_{i=1}^{m} (\langle x, Ue_{2i-1}\rangle^2 + \langle x, Ue_{2i}\rangle^2)^{p/2} \\
&= \sum_{i=1}^{m} (\langle U^{-1}x, e_{2i-1}\rangle^2 + \langle U^{-1}x, e_{2i}\rangle^2)^{p/2} \\
&= \|U^{-1}x\|_{B_p(\mathbb{C}^n)}^p = \|x\|_{UB_p(\mathbb{C}^n)}^p
\end{aligned}$$

Therefore, the convex body $C_p^c(\mu)^*$ is a rotation of $B_p(\mathbb{C}^n)$, which concludes the proof by Proposition 6.1. □

Corollary 6.3 Suppose $p \in [1, \infty]$ and μ is a complex cross measure. Then

$$|C_p^c(\mu)| = \kappa_{2n}(p^*).$$

The proofs of Theorem 6.4 and Theorem 6.5 are similar to their real counterparts by Ball [8] and Barthe [9, 11].

Theorem 6.4 Suppose $p \in [1, \infty]$. If μ is a complex isotropic measure on S^{2n-1}, then

$$|C_p^c(\mu)^*| \leq \kappa_{2n}(p).$$

There is equality if μ is a complex cross measure.

Proof. Suppose that supp $\mu = \{v_1, \cdots, v_m\}$ and let $\mu(\{v_i\}) = c_i > 0$. Then condition (39) is satisfied since the measure μ is a discrete complex isotropic measure on S^{2n-1}. By (8) and (35), we have for $x \in \mathbb{R}^{2n}$,

$$\|x\|^p_{C_p^c(\mu)^*} = h_{C_p^c(\mu)}(x)^p = \sum_{i=1}^{m} c_i \|x| \operatorname{span}\{v_i, v_i^\dagger\}\|^p, \tag{41}$$

if $1 \leq p < \infty$, and

$$\|x\|_{C_\infty^c(\mu)^*} = h_{C_\infty^c(\mu)}(x) = \sup_{v \in \{v_1, \cdots, v_m\}} \|x| \operatorname{span}\{v, v^\dagger\}\|. \tag{42}$$

Case $p = \infty$**:** From (42), we obtain

$$C_\infty^c(\mu)^* = \Big\{x \in \mathbb{R}^{2n} : \|x| \operatorname{span}\{v_i, v_i^\dagger\}\| \leq 1 \text{ for all } v_i, 1 \leq i \leq m\Big\}. \tag{43}$$

Define functions $f_i : \operatorname{span}\{v_i, v_i^\dagger\} \to [0, \infty), 1 \leq i \leq m$, by

$$f_i(y) = \mathcal{X}_{[0,1]}(\|y\|),$$

where $\mathcal{X}_{[0,1]}$ is the characteristic function of $[0, 1]$.

From (43), Theorem 5.2 and (28), we obtain

$$\begin{aligned} |C_\infty^c(\mu)^*| &= \int_{\mathbb{R}^{2n}} \prod_{i=1}^{m} \mathcal{X}_{[0,1]}(\|x| \operatorname{span}\{v_i, v_i^\dagger\}\|) dx \\ &= \int_{\mathbb{R}^{2n}} \prod_{i=1}^{m} f_i^{c_i}(x| \operatorname{span}\{v_i, v_i^\dagger\}) dx \leq \prod_{i=1}^{m} \Big(\int_{\operatorname{span}\{v_i, v_i^\dagger\}} f_i \Big)^{c_i} \\ &= \prod_{i=1}^{m} \Big(\int_{\operatorname{span}\{v_i, v_i^\dagger\}} \mathcal{X}_{[0,1]}(\|x\|) dx \Big)^{c_i} = \pi^n. \end{aligned}$$

Case $1 \le p < \infty$**:** Define functions $f_i : \mathrm{span}\{v_i, v_i^\dagger\} \to [0, \infty), 1 \le i \le m$, by

$$f_i(y) = \exp(-\|y\|^p).$$

Combining (11), (41), Theorem 5.2 and (28), yields

$$\begin{aligned}\Gamma\Big(1+\frac{2n}{p}\Big)|C_p^c(\mu)^*| &= \int_{\mathbb{R}^{2n}} \exp(-\|x\|^p_{C_p^c(\mu))^*})dx = \int_{\mathbb{R}^{2n}} \exp(-c_i\|x|\ \mathrm{span}\{v_i, v_i^\dagger\}\|^p)dx \\ &= \int_{\mathbb{R}^{2n}} \prod_{i=1}^m f_i^{c_i}(x|\ \mathrm{span}\{v_i, v_i^\dagger\})dx \le \prod_{i=1}^m \Big(\int_{\mathrm{span}\{v_i, v_i^\dagger\}} f_i\Big)^{c_i} \\ &= \prod_{i=1}^m \Big(\int_{\mathrm{span}\{v_i, v_i^\dagger\}} e^{-\|x\|^p} dx\Big)^{c_i} = \Big(\pi\Gamma\Big(1+\frac{2}{p}\Big)\Big)^n.\end{aligned}$$

Therefore, $|C_p^c(\mu)^*| \le \kappa_{2n}(p)$.

Now let μ be an arbitrary complex isotropic measure on S^{2n-1}. Theorem 3.2 implies that there exists a sequence $\mu_k, k \in \mathbb{N}$, of discrete complex isotropic measures such that μ_k converges weakly to μ as $k \to \infty$. Thus,

$$\lim_{k\to\infty} h(C_p^c(\mu_k), v) = h(C_p^c(\mu), v) \text{ for every } v \in S^{2n-1}.$$

Since pointwise convergence of support functions implies the convergence of the respective convex bodies in the Hausdorff metric (see e.g., [35]), the continuity of volume and polarity on convex bodies completes the proof.

The equality case follows from Corollary 6.2. □

Theorem 6.5 Suppose $p \in [1, \infty]$. If μ is a complex isotropic measure on S^{2n-1}, then

$$|C_p^c(\mu)| \ge \kappa_{2n}(p^*).$$

There is equality if μ is a complex cross measure.

Proof. By the concluding arguments of the proof of Theorem 6.4, we only need to consider the case that μ is a discrete complex isotropic measure on S^{2n-1}. Suppose that supp $\mu = \{v_1, \cdots, v_m\}$ and $\mu(\{v_i\}) = c_i > 0$. Recall that for $x \in \mathbb{R}^{2n}$,

$$\|x\|^p_{C_p^c(\mu)^*} = h_{C_p^c(\mu)}(x)^p = \sum_{i=1}^m c_i\|x|\ \mathrm{span}\{v_i, v_i^\dagger\}\|^p, \tag{44}$$

if $1 \le p < \infty$, and

$$\|x\|_{C_\infty^c(\mu)^*} = h_{C_\infty^c(\mu)}(x) = \sup_{v\in\{v_1,\cdots,v_m\}} \|x|\ \mathrm{span}\{v, v^\dagger\}\|. \tag{45}$$

Case $p = 1$**:** From (44) and $\|x| \; [\mathrm{span}\{v_i, v_i^\dagger\}]\| = h(B_2(\mathbb{R}^{2n})| \; [\mathrm{span}\{v_i, v_i^\dagger\}], x)$, we have

$$\begin{aligned} h_{C_1^c(\mu)}(x) &= \sum_{i=1}^{m} c_i \|x| \; \mathrm{span}\{v_i, v_i^\dagger\}\| \\ &= \sum_{i=1}^{m} c_i h(B_2(\mathbb{R}^{2n})| \; [\mathrm{span}\{v_i, v_i^\dagger\}], x) \\ &= h\Big(\sum_{i=1}^{m} c_i B_2(\mathbb{R}^{2n})| \; [\mathrm{span}\{v_i, v_i^\dagger\}], x\Big), \end{aligned}$$

where the addition in last equality is Minkowski addition of convex sets. Hence, it follows that

$$C_1^c(\mu) = \{x \in \mathbb{R}^{2n} : x = \sum_{i=1}^{m} c_i y_i, y_i \in B_2(\mathbb{R}^{2n})| \; \mathrm{span}\{v_i, v_i^\dagger\}\}. \tag{46}$$

Define functions $f_i : \mathrm{span}\{v_i, v_i^\dagger\} \to [0, \infty), 1 \le i \le m$, by

$$f_i(y) = \mathcal{X}_{[0,1]}(\|y\|).$$

From (46), Theorem 5.2 and (28), we obtain

$$\begin{aligned} |C_1^c(\mu)| &= \int_{\mathbb{R}^{2n}} \sup\Big\{\prod_{i=1}^{m} \mathcal{X}_{[0,1]}(\|y_i\|)^{c_i} : x = \sum_{i=1}^{m} c_i y_i, y_i \in \mathrm{span}\{v_i, v_i^\dagger\}\Big\} dx \\ &= \int_{\mathbb{R}^{2n}} \sup\Big\{\prod_{i=1}^{m} f_i(y_i)^{c_i} : x = \sum_{i=1}^{m} c_i y_i, y_i \in \mathrm{span}\{v_i, v_i^\dagger\}\Big\} dx \\ &\ge \prod_{i=1}^{m} \Big(\int_{\mathrm{span}\{v_i, v_i^\dagger\}} f_i\Big)^{c_i} = \prod_{i=1}^{m} \Big(\int_{\mathrm{span}\{v_i, v_i^\dagger\}} \mathcal{X}_{[0,1]}(\|x\|) dx\Big)^{c_i} = \pi^n. \end{aligned}$$

Case $1 < p \le \infty$**:** We claim that

$$\|x\|_{C_p^c(\mu)}^{p^*} \le \inf\Big\{\sum_{i=1}^{m} c_i (r_{i1}^2 + r_{i1}^2)^{p^*/2} : \sum_{i=1}^{m} c_i (r_{i1} v_i + r_{i2} v_i^\dagger) = x\Big\}. \tag{47}$$

For $1 < p < \infty$, and $x = \sum\limits_{i=1}^{m} c_i (r_{i1} v_i + r_{i2} v_i^\dagger)$, Hölder's inequality applied twice and (44) yield

$$\begin{aligned}
\langle x, y\rangle &= \sum_{i=1}^{m} c_i(r_{i1}\langle y, v_i\rangle + r_{i2}\langle y, v_i^\dagger\rangle) \\
&\le \sum_{i=1}^{m} c_i(r_{i1}^2 + r_{i2}^2)^{\frac{1}{2}}(\langle y, v_i\rangle^2 + \langle y, v_i^\dagger\rangle^2)^{\frac{1}{2}} \\
&\le \Big(\sum_{i=1}^{m} c_i(r_{i1}^2 + r_{i2}^2)^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}} \Big(\sum_{i=1}^{m} c_i(\langle y, v_i\rangle^2 + \langle y, v_i^\dagger\rangle^2)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \\
&= \Big(\sum_{i=1}^{m} c_i(r_{i1}^2 + r_{i2}^2)^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}} \|y\|_{C_p^c(\mu)^*},
\end{aligned}$$

and for $p = \infty$, $x = \sum_{i=1}^{m} c_i(r_{i1}v_i + r_{i2}v_i^\dagger)$, we have, by (45),

$$\begin{aligned}
\langle x, y\rangle &= \sum_{i=1}^{m} c_i(r_{i1}\langle y, v_i\rangle + r_{i2}\langle y, v_i^\dagger\rangle) \\
&\le \sum_{i=1}^{m} c_i(r_{i1}^2 + r_{i2}^2)^{\frac{1}{2}}(\langle y, v_i\rangle^2 + \langle y, v_i^\dagger\rangle^2)^{\frac{1}{2}} \\
&\le \Big[\sum_{i=1}^{m} c_i(r_{i1}^2 + r_{i2}^2)^{\frac{1}{2}}\Big] \cdot \sup_{v \in \{v_1, \cdots, v_m\}} (\langle y, v\rangle^2 + \langle y, v^\dagger\rangle^2)^{\frac{1}{2}} \\
&= \Big[\sum_{i=1}^{m} c_i(r_{i1}^2 + r_{i2}^2)^{\frac{1}{2}}\Big] \cdot \|y\|_{C_\infty^c(\mu)^*}.
\end{aligned}$$

Let $m_x = (\sum_{i=1}^{m} c_i(r_{i1}^2 + r_{i2}^2)^{p^*/2})^{1/p^*}$. From the above two inequalities, (9) and the fact that $y/\|y\|_{C_p^c(\mu)^*}$ lies on the boundary of the convex body $C_p^c(\mu)^*$, we have

$$\frac{x}{m_x} \in C_p^c(\mu).$$

Thus,

$$\Big\|\frac{x}{m_x}\Big\|_{C_p^c(\mu)} \le 1.$$

That is,

$$\|x\|_{C_p^c(\mu)} \le (\sum_{i=1}^{m} c_i(r_{i1}^2 + r_{i2}^2)^{p^*/2})^{1/p^*},$$

for all $x = \sum_{i=1}^{m} c_i(r_{i1}v_i + r_{i2}v_i^\dagger)$. Taking the infimum concludes the proof of the claim.

Define functions $f_i : \mathrm{span}(v_i, v_i^\dagger) \to [0, \infty), 1 \le i \le m$, by

$$f_i(y) = \exp(-(\|y\|^{p^*}).$$

Combining (11), (47), Theorem 5.2 and (28), yields

$$
\begin{aligned}
&\Gamma\Big(1+\frac{2n}{p^*}\Big)|C_p^c(\mu)| \\
= & \int_{\mathbb{R}^{2n}} \exp(-\|x\|_{C_p^c(\mu)}^{p^*})dx \\
\geq & \int_{\mathbb{R}^{2n}} \sup\Big\{\prod_{i=1}^m \exp(-c_i(r_{i1}^2+r_{i2}^2)^{\frac{p^*}{2}}) : \sum_{i=1}^m c_i(r_{i1}v_i + r_{i2}v_i^\dagger) = x\Big\}dx \\
= & \int_{\mathbb{R}^{2n}} \sup\Big\{\prod_{i=1}^m f_i(y_i)^{c_i} : x = \sum_{i=1}^m c_i y_i, y_i \in \operatorname{span}\{v_i, v_i^\dagger\}\Big\}dx \geq \prod_{i=1}^m \Big(\int_{\operatorname{span}\{v_i,v_i^\dagger\}} f_i\Big)^{c_i} \\
= & \prod_{i=1}^m \Big(\int_{\operatorname{span}\{v_i,v_i^\dagger\}} e^{-\|x\|^{p^*}} dx\Big)^{c_i} = \Big(\pi\Gamma(1+\frac{2}{p^*})\Big)^n.
\end{aligned}
$$

Therefore, $|C_p^c(\mu)| \geq \kappa_{2n}(p^*)$.

The equality case follows from Corollary 6.3. □

Next, we establish the lower bound for the volume of $C_p^c(\mu)^*$ and the upper bound for the volume of $C_p^c(\mu)$. The following lemma is needed.

Lemma 6.6 Suppose $p \in [1, \infty)$. Then

$$
\int_{S^{2n-1}} (\langle u, v\rangle^2 + \langle u, v^\dagger\rangle^2)^{p/2} du = \frac{2\pi^n \Gamma(\frac{p}{2}+1)}{\Gamma(n+\frac{p}{2})}
$$

for each $v \in S^{2n-1}$.

Proof. Note that $v, v^\dagger$ are mutually orthogonal unit vectors, thus there exists $U \in SO(\mathbb{R}^{2n})$, such that $v = Ue_1, v^\dagger = Ue_2$. Thus, by the rotation invariance of the spherical Lebesgue measure, it suffices to consider $v = e_1$. Using polar coordinates, we obtain

$$
\begin{aligned}
&\int_{S^{2n-1}} (u_1^2+u_2^2)^{\frac{p}{2}} du \\
= & \int_{B_2(\mathbb{R}^2)} \Big(\int_{\sqrt{1-u_1^2-u_2^2}S^{2n-3}} (u_1^2+u_2^2)^{\frac{p}{2}} du_3\cdots du_{2n}\Big)(1-u_1^2-u_2^2)^{-\frac{1}{2}} du_1 du_2 \\
= & (2n-2)\kappa_{2n-2} \int_{B_2(\mathbb{R}^2)} (u_1^2+u_2^2)^{\frac{p}{2}}(1-u_1^2-u_2^2)^{n-2} du_1 du_2 \\
= & (2n-2)\kappa_{2n-2} \cdot 2\pi \int_0^1 r^{p+1}(1-r^2)^{n-2} dr \\
= & \frac{2\pi^n\Gamma(\frac{p}{2}+1)}{\Gamma(n+\frac{p}{2})}.
\end{aligned}
$$

□

Theorem 6.7 Suppose $p \in [1, \infty]$. If μ is a complex isotropic measure on S^{2n-1}, then

$$\kappa_{2n}/\alpha_{2n,p} \le |C_p^c(\mu)^*| \qquad \text{and} \qquad |C_p^c(\mu)| \le \kappa_{2n}\alpha_{2n,p},$$

with equality in either inequality if μ is suitably normalized Lebesgue measure.

Proof. To establish the first inequality, observe that for $p \in [1, \infty)$, the polar coordinate formula for volume and (8), together with the Hölder inequality, definition (32), Fubini's theorem, Lemma 6.6 and (28), implies

$$\begin{aligned}
\Big(\frac{|C_p^c(\mu)^*|}{\kappa_{2n}}\Big)^{-\frac{p}{2n}} &= \Big(\frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}} h_{C_p^c(\mu)}(u)^{-2n}du\Big)^{-\frac{p}{2n}} \\
&\le \frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}} h_{C_p^c(\mu)}(u)^{p}du \\
&= \frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}} \Big(\int_{S^{2n-1}} (\langle u,v\rangle^2 + \langle u, v^\dagger\rangle^2)^{\frac{p}{2}} d\mu(v)\Big)du \\
&= \frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}} \Big(\int_{S^{2n-1}} (\langle u,v\rangle^2 + \langle u, v^\dagger\rangle^2)^{\frac{p}{2}} du\Big)d\mu(v) \\
&= \frac{\Gamma(n)\Gamma(\frac{p}{2}+1)}{\Gamma(n+\frac{p}{2})}\int_{S^{2n-1}} d\mu(v) = \alpha_{2n,p}^{\frac{p}{2n}}
\end{aligned}$$

with equality if and only if $C_p^c(\mu)$ is a ball. Using Lemma 6.6 with $p = 2$, and (27), we have

$$d\mu(u) = \frac{1}{2\kappa_{2n}}du$$

is complex isotropic. Thus, Lemma 6.6 shows that $C_p^c(\mu)$ ($p \in [1, \infty)$) is a ball for this measure μ.

The second inequality is proved by using the classical Urysohn inequality (12) for $p \in [1, \infty)$. By Hölder's inequality, definition (32), Fubini's theorem, Lemma 6.6 and (28), we have

$$\begin{aligned}
\Big(\frac{|C_p^c(\mu)|}{\kappa_{2n}}\Big)^{\frac{1}{2n}} &\le \frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}} h_{C_p^c(\mu)}(u)du \\
&\le \Big[\frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}} h_{C_p^c(\mu)}(u)^{p}du\Big]^{\frac{1}{p}} \\
&= \Big[\frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}} \Big(\int_{S^{2n-1}} (\langle u,v\rangle^2 + \langle u, v^\dagger\rangle^2)^{\frac{p}{2}} d\mu(v)\Big)du\Big]^{\frac{1}{p}} \\
&= \Big[\frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}} \Big(\int_{S^{2n-1}} (\langle u,v\rangle^2 + \langle u, v^\dagger\rangle^2)^{\frac{p}{2}} du\Big)d\mu(v)\Big]^{\frac{1}{p}} \\
&= \Big[\frac{\Gamma(n)\Gamma(\frac{p}{2}+1)}{\Gamma(n+\frac{p}{2})}\int_{S^{2n-1}} d\mu(v)\Big]^{\frac{1}{p}} = \alpha_{2n,p}^{\frac{1}{2n}}
\end{aligned}$$

with equality if and only if $C_p^c(\mu)$ is a ball. As before, there exists a suitably normalized Lebesgue measure such that this measure is complex isotropic and $C_p^c(\mu)$ $(p \in [1,\infty))$ is a ball.

For $p = \infty$, we obtain

$$\begin{aligned} h_{C_\infty^c(\mu)}(\xi) &= \sup_{v\in \mathrm{supp}\mu} \|\xi|\ \mathrm{span}\{v, v^\dagger\}\| \\ &\leq \sup_{v\in S^{2n-1}} \|\xi|\ \mathrm{span}\{v, v^\dagger\}\| \\ &= \sup_{v\in S^{2n-1}} h((B_2(\mathbb{R}^{2n})|\ \mathrm{span}\{v, v^\dagger\}), \xi) \\ &\leq h((B_2(\mathbb{R}^{2n}), \xi), \end{aligned} \tag{48}$$

for each $\xi \in S^{2n-1}$. Note that the last inequality is actually an equality since we can set $v = \xi$. Therefore, we have $C_\infty^c(\mu) \subseteq B_2(\mathbb{R}^{2n})$ and, thus, $B_2(\mathbb{R}^{2n}) \subseteq C_\infty^c(\mu)^*$. When the measure μ is a suitably normalized Lebesgue measure, then supp $\mu = S^{2n-1}$. Equality holds in (48), that is, $C_\infty^c(\mu) = B_2(\mathbb{R}^{2n})$. Together with $\alpha_{2n,\infty} = 1$, this concludes the case $p = \infty$. □

Theorem 6.4, Theorem 6.5, and Theorem 6.7, together yield Theorem 1 and Theorem 2.

7. The sine transform of complex isotropic measures

In the following we adapt ideas of Maresch and Schuster [32].

Theorem 7.1 If μ is a complex isotropic measure on S^{2n-1}, then

$$|S^c(\mu)^*| \leq |S^c(\mu)|/\beta_{2n}.$$

Proof. Suppose that supp $\mu = \{v_1, \cdots, v_m\}$ and let $\mu(\{v_i\}) = \bar{c}_i > 0$. Since μ is complex isotropic, it follows that $\mu(S^{2n-1}) = \sum_{i=1}^m \bar{c}_i = n$ by (28). Therefore, using $P_{[\mathrm{span}\{v_i, v_i^\dagger\}]^\perp} = I_{2n} - v_i \otimes v_i - v_i^\dagger \otimes v_i^\dagger$, we have

$$\frac{1}{n-1}\sum_{i=1}^m \bar{c}_i P_{[\mathrm{span}\{v_i, v_i^\dagger\}]^\perp} = I_{2n}. \tag{49}$$

We define c_i by

$$c_i = \frac{\bar{c}_i}{n-1}, \quad \text{for } i = 1, \cdots, m.$$

From (8), (36) and the fact that $\|x|\ [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp\| = h(B_2(\mathbb{R}^{2n})|\ [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp, x)$, we obtain

$$\begin{aligned}
\|x\|_{(S^c(\mu))^*} &= h_{S^c(\mu)}(x) = \sum_{i=1}^m \bar{c}_i \|x| \ [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp\| \\
&= \sum_{i=1}^m \bar{c}_i h(B_2(\mathbb{R}^{2n})| \ [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp, x) \\
&= h\Big(\sum_{i=1}^m \bar{c}_i B_2(\mathbb{R}^{2n})| \ [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp, x\Big),
\end{aligned}$$

where the addition in the last equality is Minkowski addition of convex sets. Hence, it follows that

$$S^c(\mu) = \{x \in \mathbb{R}^{2n} : x = \sum_{i=1}^m \bar{c}_i y_i, y_i \in B_2(\mathbb{R}^{2n})| \ [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp\}.$$

Consequently,

$$|S^c(\mu)| = \int_{\mathbb{R}^{2n}} \sup\Big\{\prod_{i=1}^m \mathcal{X}_{[0,n-1]}(\|y_i\|)^{c_i} : x = \sum_{i=1}^m c_i y_i, y_i \in [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp\Big\} dx, \qquad (50)$$

where $\mathcal{X}_{[0,n-1]}$ is the characteristic function of $[0, n-1]$.

From (11) with $p = 1$ and (36), we obtain

$$\begin{aligned}
|S^c(\mu)^*| &= \frac{1}{(2n)!} \int_{\mathbb{R}^{2n}} \exp(-\|x\|_{S^c(\mu)^*}) dx \\
&= \frac{1}{(2n)!} \int_{\mathbb{R}^{2n}} \exp(-\sum_{i=1}^m \bar{c}_i \|x| \ [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp\|) dx \\
&= \frac{1}{(2n)!} \int_{\mathbb{R}^{2n}} \prod_{i=1}^m \exp(-(n-1)\|x| \ [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp\|)^{c_i} dx.
\end{aligned} \qquad (51)$$

Define functions $f_i, g_i : [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp \to [0, \infty), 1 \le i \le m$, by

$$f_i(y) = \frac{(n-1)^{2n-2}}{\Gamma(2n-1)\kappa_{2n-2}} \exp(-(n-1)\|y\|) \qquad (52)$$

and

$$g_i(y) = \frac{1}{(n-1)^{2n-2}\kappa_{2n-2}} \mathcal{X}_{[0,n-1]}(\|y\|). \qquad (53)$$

Note that the normalizations are chosen such that

$$\int_{[\mathrm{span}\{v_i, v_i^\dagger\}]^\perp} f_i = \int_{[\mathrm{span}\{v_i, v_i^\dagger\}]^\perp} g_i = 1.$$

Since $\sum_{i=1}^m c_i = n/(n-1)$, by (49)-(53) and Theorem 5.3, it follows that

$$\begin{aligned}
|S^c(\mu)^*| &= \frac{(\Gamma(2n-1)\kappa_{2n-2})^{\frac{n}{n-1}}}{(2n)!(n-1)^{2n}} \int_{\mathbb{R}^{2n}} \prod_{i=1}^m f_i(x|\ [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp)^{c_i} dx \\
&\leq \frac{(\Gamma(2n-1)\kappa_{2n-2})^{\frac{n}{n-1}}}{(2n)!(n-1)^{2n}} \int_{\mathbb{R}^{2n}} \sup\Big\{ \prod_{i=1}^m g_i(y_i)^{c_i} : x = \sum_{i=1}^m c_i y_i, y_i \in [\mathrm{span}\{v_i, v_i^\dagger\}]^\perp \Big\} dx \\
&= \frac{\Gamma(2n-1)^{1/(n-1)}}{(2n)(2n-1)(n-1)^{4n}} |S^c(\mu)| = |S^c(\mu)|/\beta_{2n}.
\end{aligned}$$

Now let μ be an arbitrary complex isotropic measure on S^{2n-1}. Theorem 3.2 implies that there exists a sequence $\mu_k, k \in \mathbb{N}$, of discrete complex isotropic measures such that μ_k converges weakly to μ as $k \to \infty$. Thus,

$$\lim_{k\to\infty} h(S^c(\mu_k), v) = h(S^c(\mu), v) \text{ for every } v \in S^{2n-1}.$$

Since pointwise convergence of support functions implies the convergence of the respective convex bodies in the Hausdorff metric (see e.g., [35]), the continuity of volume and polarity on convex bodies completes the proof. □

Next, we establish the lower bound for the volume of $S^c(\mu)^*$ and the upper bound for the volume of $S^c(\mu)$. The following lemma is needed.

Lemma 7.2 For each $v \in S^{2n-1}$,

$$\int_{S^{2n-1}} \sqrt{1 - \langle u, v\rangle^2 - \langle u, v^\dagger\rangle^2} du = \frac{4\pi^n}{(2n-1)\Gamma(n-1)}.$$

Proof. Note that $v, v^\dagger$ are mutually orthogonal unit vectors, thus there exists $U \in SO(\mathbb{R}^{2n})$, such that $v = Ue_1, v^\dagger = Ue_2$. Thus, by the rotation invariance of the spherical Lebesgue measure, it is sufficient to consider $v = e_1$. Using polar coordinates, we obtain

$$\begin{aligned}
&\int_{S^{2n-1}} \sqrt{1 - u_1^2 - u_2^2} du \\
&= \int_{B_2(\mathbb{R}^2)} \Big(\int_{\sqrt{1-u_1^2-u_2^2} S^{2n-3}} \sqrt{1 - u_1^2 - u_2^2} du_3 \cdots du_{2n} \Big) (1 - u_1^2 - u_2^2)^{-\frac{1}{2}} du_1 du_2 \\
&= (2n-2)\kappa_{2n-2} \int_{B_2(\mathbb{R}^2)} (1 - u_1^2 - u_2^2)^{n-\frac{3}{2}} du_1 du_2 \\
&= (2n-2)\kappa_{2n-2} \cdot 2\pi \int_0^1 r(1-r^2)^{n-\frac{3}{2}} dr \\
&= \frac{4\pi^n}{(2n-1)\Gamma(n-1)}.
\end{aligned}$$

□

Theorem 7.3 If μ is a complex isotropic measure on S^{2n-1}, then

$$\kappa_{2n}\Big(\frac{2n-1}{2n(n-1)}\Big)^{2n} \le |S^c(\mu)^*| \quad\text{and}\quad |S^c(\mu)| \le \kappa_{2n}\Big(\frac{2n(n-1)}{2n-1}\Big)^{2n},$$

with equality in either inequality if μ is suitably normalized Lebesgue measure.

Proof. By the polar coordinate formula for volume, (8), the Hölder inequality, definition (34), Fubini's theorem, Lemma 7.2 and (28), we have

$$\begin{aligned}
\Big(\frac{|S^c(\mu)^*|}{\kappa_{2n}}\Big)^{-1/(2n)} &= \Big(\frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}} h(S^c(\mu),u)^{-2n}du\Big)^{-1/(2n)}\\
&\le \frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}} h(S^c(\mu),u)du\\
&= \frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}}\Big(\int_{S^{2n-1}}\sqrt{1-\langle u,v\rangle^2-\langle u,v^\dagger\rangle^2}d\mu(v)\Big)du\\
&= \frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}}\Big(\int_{S^{2n-1}}\sqrt{1-\langle u,v\rangle^2-\langle u,v^\dagger\rangle^2}du\Big)d\mu(v)\\
&= \frac{2n-2}{2n-1}\int_{S^{2n-1}} d\mu(v) = \frac{2n(n-1)}{2n-1}.
\end{aligned}$$

with equality if and only if $S^c(\mu)$ is a ball. Using Lemma 6.6 with $p=2$, and (27), we see that

$$d\mu(u) = \frac{1}{2\kappa_{2n}}du$$

is complex isotropic. Thus, Lemma 7.2 yields that $S^c(\mu)$ is a ball for this measure μ.

The second inequality is proved by using the classical Urysohn inequality (12). By definition (34), Fubini's theorem, Lemma 7.2 and (28), we have

$$\begin{aligned}
\Big(\frac{|S^c(\mu)|}{\kappa_{2n}}\Big)^{1/(2n)} &\le \frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}} h(S^c(\mu),u)du\\
&= \frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}}\Big(\int_{S^{2n-1}}\sqrt{1-\langle u,v\rangle^2-\langle u,v^\dagger\rangle^2}d\mu(v)\Big)du\\
&= \frac{1}{2n\kappa_{2n}}\int_{S^{2n-1}}\Big(\int_{S^{2n-1}}\sqrt{1-\langle u,v\rangle^2-\langle u,v^\dagger\rangle^2}du\Big)d\mu(v)\\
&= \frac{2n-2}{2n-1}\int_{S^{2n-1}} d\mu(v) = \frac{2n(n-1)}{2n-1}.
\end{aligned}$$

with equality if and only if $S^c(\mu)$ is a ball. As before, a suitably normalized Lebesgue measure is complex isotropic and $S^c(\mu)$ is a ball. □

Theorem 3 and Theorem 4 now follow from Theorem 7.1 and Theorem 7.3.

Finally, we show that the left inequality in (3) and the left inequality in (4) are asymptotically optimal.

Theorem 7.4 If ν_n for $n \geq 3$, are complex cross measures on S^{2n-1} then

$$\lim_{n\to\infty}\frac{1}{\kappa_{2n}\beta_{2n}}\Big(\frac{2n(n-1)}{2n-1}\Big)^{2n}|S^c(\nu_n)| = \lim_{n\to\infty}\frac{\beta_{2n}}{\kappa_{2n}}\Big(\frac{2n-1}{2n(n-1)}\Big)^{2n}|S^c(\nu_n)^*| = 1$$

Proof. As in the proof of Corollary 6.2, we only need to consider the basic complex cross measures. Thus, supp $\nu_n = \{\mathrm{span}\{e_1, e_1^\dagger\} \cap S^{2n-1}, \cdots, \mathrm{span}\{e_{2n-1}, e_{2n-1}^\dagger\} \cap S^{2n-1}\}$ and $\nu_n(\mathrm{span}\{e_{2i-1}, e_{2i-1}^\dagger\} \cap S^{2n-1}) = 1$ for $1 \leq i \leq n$.

By (36), we obtain

$$\begin{aligned} h(S^c(\nu_n),\xi) &= \sum_{i=1}^{n}\mu(\mathrm{span}\{e_{2i-1}, e_{2i-1}^\dagger\} \cap S^{2n-1})\|\xi|\ [\mathrm{span}(e_i, e_i^\dagger)]^\perp\| \\ &= \sum_{i=1}^{n}\|\xi|\ [\mathrm{span}\{e_i, e_i^\dagger\}]^\perp\| = \sum_{i=1}^{n}\sqrt{1-\xi_{i1}^2-\xi_{i2}^2}, \end{aligned}$$

subject to

$$\|\xi\| = \|(\xi_{11},\xi_{12},\cdots,\xi_{n1},\xi_{n2})\| = \sum_{i=1}^{n}(\xi_{i1}^2+\xi_{i2}^2) = 1.$$

Using the Lagrange multiplier rule, it is easy to see that

$$\max_{\xi\in S^{2n-1}} h(S^c(\nu_n),\xi) = n\sqrt{1-\frac{1}{n}},$$

and the maximum is attained at the points $\bigcap_{i=1}^{n}\{\xi|\ \xi_{i1}^2+\xi_{i2}^2 = \frac{1}{n}\}$. Therefore, we have the inclusion

$$S^c(\nu_n) \subseteq n\sqrt{1-\frac{1}{n}}B_2(\mathbb{R}^{2n}).$$

Using Theorem 3 and Theorem 4, we immediately obtain the following volume bounds for $S^c(\nu_n)$ and $(S^c(\nu_n))^*$:

$$\kappa_{2n}\beta_{2n}\Big(\frac{2n-1}{2n(n-1)}\Big)^{2n} \leq |S^c(\nu_n)| \leq \kappa_{2n}n^{2n}(1-\frac{1}{n})^n, \tag{54}$$

and

$$\frac{\kappa_{2n}}{n^{2n}}(1-\frac{1}{n})^{-n} \leq |S^c(\nu_n)^*| \leq \frac{\kappa_{2n}}{\beta_{2n}}\Big(\frac{2n(n-1)}{2n-1}\Big)^{2n}. \tag{55}$$

Using Stirling's formula and the definition of the constant β_{2n}, we have

$$\lim_{n\to\infty}\frac{1}{\beta_{2n}}\Big(\frac{2n(n-1)}{2n-1}\Big)^{2n}n^{2n}(1-\frac{1}{n})^n = 1.$$

Combining (54) and (55), concludes the proof.

Acknowledgement

The authors are indebted to the referee for the valuable suggestions and the very careful reading of the original manuscript.

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