AN ASYMMETRIC ORLICZ CENTROID INEQUALITY FOR PROBABILITY MEASURES

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ABSTRACT. Using M-addition, an asymmetric Orlicz centroid inequality for absolutely continuous probability measures is established corresponding to Paouris and Pivovarov's recent result on the symmetric case. As an application, we extend Haberl and Schuster's asymmetric L_p centroid inequality from star bodies to compact sets.

1. Introduction

Recently, some elements of the Orlicz Brunn-Minkowski theory have been emerged, namely Orlicz projection and centroid bodies as well as the Orlicz Minkowski problem (see, e.g., [2, 7, 11, 12, 14, 15, 17, 18, 21, 26, 27, 28]). This extension is motivated by asymmetric concepts within the L_p Brunn-Minkowski theory developed by Ludwig [13], Haberl and Schuster [8, 9], and Haberl, Schuster and Xiao [10]. In this paper, we show asymmetric elements in the Orlicz centroid inequality for absolutely continuous probability measures.

In order to state the results regarding to the Orlicz centroid bodies, several notations are needed. Let $\mathrm{Conv}(\mathbb{R})$ be the class of convex functions $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(0) = 0$ and such that ϕ is either strictly decreasing on $(-\infty, 0]$ or ϕ is strictly increasing on $[0, \infty)$.

2000 *Mathematics Subject Classification.* 52A20, 52A40.

Key words and phrases. M-addition, Orlicz centroid inequality, Asymmetric Orlicz centroid bodies, Asymmetric L_p centroid bodies.

Partially support by the National Natural Science Foundation of China (11071156), Shanghai Leading Academic Discipline Project (J50101) and the Research Fund for the Doctoral Programs of Higher Education of China (20123108110001).

Let $\mathrm{Conv}[0,\infty)$ be the class of convex, strictly increasing functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(0) = 0$. We denote B_2^n for the Euclidean ball of radius one and D_n for the Euclidean ball of volume one. Lebesgue measure restricted to D_n is λ_{D_n}. Let $\mathcal{P}_{[n]}$ be the class of probability measures on $\mathbb{R}^n$ that are absolutely continuous with respect to Lebesgue measure. For a convex body K (i.e., a compact convex set in $\mathbb{R}^n$ with non-empty interior) denote by $h(K, y) = \max\{x \cdot y : x \in K\}$, for $y \in \mathbb{R}^n$, the support function of K.

In [18], Lutwak, Yang, and Zhang prove the following Orlicz centroid inequality (see also Li and Leng in [12]). Later, Zhu [28] extends this inequality from the body that contains the origin in its interior to the star bodies (see section 2 for precise definition) with respect to the origin.

Orlicz centroid inequality for convex bodies. *If $\phi \in \mathrm{Conv}(\mathbb{R})$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior with $\mathrm{vol}_n(K) = 1$. Define the Orlicz centroid body $\Gamma_\phi K$ to be the convex body whose support function at $y \in \mathbb{R}^n$ is given by*

$$h(\Gamma_\phi K, y) = \inf\Big\{\lambda > 0 : \int_K \phi\Big(\frac{x \cdot y}{\lambda}\Big) dx \le 1\Big\},$$

then

$$\mathrm{vol}_n(\Gamma_\phi(K)) \ge \mathrm{vol}_n(\Gamma_\phi(D_n))$$

with equality holds if and only if K is an ellipsoid centered at the origin.

When $\phi(t) = |t|$, the inequality is the volume-normalized classical centroid inequality. When $\phi(t) = |t|^p$, and $p > 1$, the inequality is the L_p centroid inequality [1, 16]. Haberl and Schuster's asymmetric L_p centroid inequality [9] is the case $\phi(t) = (|t| + \alpha t)^p$, for $-1 \le \alpha \le 1$ of the inequality.

By using probability arguments, Paouris and Pivovarov [21] recently obtain the following result.

Symmetric Orlicz centroid inequality for probability measures *Let $\phi \in \mathrm{Conv}[0,\infty)$ and $\mu \in \mathcal{P}_{[n]}$. Define the symmetric Orlicz centroid* *body $\Gamma_\phi(\mu)$ of μ corresponding to ϕ by its support function*

$$h(\Gamma_\phi(\mu), y) = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \phi\Big(\frac{|x\cdot y|}{\lambda}\Big) d\mu \le 1\Big\}.$$

If f denotes the density of μ and if $\|f\|_\infty \le 1$, then

$$\mathrm{vol}_n(\Gamma_\phi(\mu)) \ge \mathrm{vol}_n(\Gamma_\phi(\lambda_{D_n})).$$

The main purpose of this paper is to exhibit the asymmetric version of the above theorem. Our starting point is that observing Paouris and Pivovarov's recent work [21, 22, 23] are closely related to the concept of M-addition, which is first introduced by Protasov [24] and generalized by Gardner, Hug and Weil [6]. As it has been shown in [6], M is 1-unconditional and $M \subset [0,\infty)^m$ play an important role in M-addition. Indeed, the above theorem had been proved by M is 1-unconditional under some Orlicz assumptions in [21]. By letting $M \subset [0,\infty)^m$, we obtain the following theorem.

Theorem (Asymmetric Orlicz centroid inequality for probability measures) *Let $\phi \in \mathrm{Conv}[0,\infty)$ and $\mu \in \mathcal{P}_{[n]}$. Define the asymmetric Orlicz centroid body $\Gamma_\phi^+(\mu)$ of μ corresponding to ϕ by its support function*

$$h(\Gamma_\phi^+(\mu), y) = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \phi\Big(\frac{(x\cdot y)_+}{\lambda}\Big) d\mu \le 1\Big\},$$

where $(x\cdot y)_+ = \max\{x\cdot y, 0\}$. If f denotes the density of μ and if $\|f\|_\infty \le 1$, then

$$\mathrm{vol}_n(\Gamma_\phi^+(\mu)) \ge \mathrm{vol}_n(\Gamma_\phi^+(\lambda_{D_n})).$$

From this theorem, Haberl and Schuster's asymmetric L_p centroid inequality [9] can be extended from star bodies (about the origin) to compact sets (see Corollary 4.1). The ideas and techniques of Paouris and Pivovarov play a critical role throughout this paper. It would be impossible to overstate our reliance on their work.

The rest of this paper is organized as follows: In Section 2, some basic notations and preliminaries are provided. In Section 3, we complete the proof of the asymmetric Orlicz centroid inequality for probability measures. In Section 4, we establish the asymmetric L_p centroid inequality for compact sets, which extends a result of Haberl and Schuster.

2. **Notations and preliminaries**

In this section we present some terminologies and notations we shall use throughout. For general reference the reader may wish to consult the books of Gardner [5], Schneider [25].

The setting for this article is n-dimensional Euclidean space $\mathbb{R}^n$. We write $\mathbb{R}^n_+ = [0, \infty)^n$. The standard basis in $\mathbb{R}^n$ will be denoted by $e_1, e_2, \cdots, e_n$. We say the vector $x \in \mathbb{R}^n$ is a positive combination of the vectors $x_1, \cdots, x_k \in \mathbb{R}^n$ if

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with} \quad \lambda_i \geq 0 \quad (i = 1, \cdots, k).$$

For $A \subset \mathbb{R}^n$, the set of all positive combinations of any finitely many elements of A is denoted by $\operatorname{pos}\{A\}$.

Let $\mathcal{K}^n$ be the class of compact convex sets in $\mathbb{R}^n$ with non-empty interior, let $\mathcal{K}^n_o$ be the class of members of $\mathcal{K}^n$ containing the origin in their interiors.

A compact set $K \subset \mathbb{R}^n$ is called a star body, if every straight line that passes through the origin crosses the boundary of the set at exactly two points and the boundary of K is continuous in the sense that the Minkowski functional of K, defined by

$$\|x\|_K = \inf\{\lambda \geq 0 : x \in \lambda K\} \tag{2.1}$$

is a continuous function on $\mathbb{R}^n$.

The polar set K^o of $K \in \mathcal{K}^n_o$ is the convex body defined by

$$K^o = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}. \tag{2.2}$$

If $K \in \mathcal{K}^n_o$ is a convex body, then it follows from the definitions of support functions and Minkowski functionals, and the definition of polar body, that

$$h(K^o, \cdot) = \| \cdot \|_K. \tag{2.3}$$

The Hausdorff metric $\delta^H(K, L)$ between sets $K, L \in \mathcal{K}^n$ can be defined by

$$\delta^H(K, L) = \sup_{y \in S^{n-1}} |h(K, y) - h(L, y)|. \tag{2.4}$$

For $p \geq 1$, $K, L \in \mathcal{K}_o^n$, and $\alpha, \beta \geq 0$ (not both zero), the L_p Minkowski combination $\alpha \cdot K +_p \beta \cdot L$ is the convex body defined by

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p. \tag{2.5}$$

Introduced by Firey [4] in the 1960's, this notion is the basis of what has become known as the L_p Brunn-Minkowski theory.

We shall call a convex body K in $\mathbb{R}^n$ 1-unconditional if it is symmetric with respect to each coordinate hyperplane. In other words, if $x = (x_1, \cdots, x_n) \in K$ then the whole rectangle $[-|x_1|, |x_1|] \times \cdots \times [-|x_n|, |x_n|]$ is contained in K.

The concept of M-addition is introduced by Protasov [24] in 1997: Let M be an arbitrary subset of $\mathbb{R}^2$ and define the M-sum $K \oplus_M L$ of arbitrary sets K and L in $\mathbb{R}^n$ by

$$K \oplus_M L = \{ax + by : x \in K, y \in L, (a, b) \in M\}. \tag{2.6}$$

Recently, Gardner, Hug and Weil [6] generalize this concept as follows: Let M be an arbitrary subset of $\mathbb{R}^m$ and define the M-combination $\oplus_M(K_1, K_2, \cdots, K_m)$ of arbitrary sets $K_1, K_2, \cdots, K_m$ in $\mathbb{R}^n$ by

$$\oplus_M(K_1, K_2, \cdots, K_m) = \Big\{ \sum_{i=1}^m a_i x_i : x_i \in K_i, (a_1, a_2, \cdots, a_m) \in M \Big\}.$$

Note that a special set M of (2.6) is treated by Lutwak, Yang and Zhang [19], which aims to extend L_p-addition from convex bodies to compact sets.

Unlike general cases, we consider the case that $K_1, \cdots, K_m \in \mathbb{R}^n$ are replaced by $x_1, \cdots, x_m \in \mathbb{R}^n$; i.e.,

$$\oplus_M (x_1, x_2, \cdots, x_m) = \Big\{ \sum_{i=1}^{m} a_i x_i : (a_1, a_2, \cdots, a_m) \in M \Big\}. \tag{2.7}$$

To make clear that $M \subset \mathbb{R}^m$, we denote M_m for M. Note that for $M_m \in \mathcal{K}^m$, $\oplus_{M_m}(x_1, \cdots, x_m)$ belongs to $\mathcal{K}^n$. Indeed, for points $(a_1, \cdots, a_m)$, $(b_1, \cdots, b_m) \in M_m$, let

$$s = \sum_{i=1}^{m} a_i x_i, \;\; t = \sum_{i=1}^{m} b_i x_i,$$

if $s, t \in \oplus_M(x_1, x_2, \cdots, x_m)$, then the convexity of M_m implies that $(1-\lambda)s + \lambda t \in \oplus_{M_m}(x_1, \cdots, x_m)$ for $0 < \lambda < 1$. Moreover, by the definition of support functions, we have

$$\begin{aligned} h(\oplus_{M_m}(x_1, \cdots, x_m), y) &= \max_{(a_1, \cdots, a_m) \in M_m} \Big(\sum\nolimits_{i=1}^{m} a_i x_i \Big) \cdot y \\ &= \max_{(a_1, \cdots, a_m) \in M_m} \sum\nolimits_{i=1}^{m} a_i (x_i \cdot y) \\ &= h_{M_m}(x_1 \cdot y, \cdots, x_m \cdot y). \end{aligned} \tag{2.8}$$

3. Asymmetric Orlicz centroid inequality

In order to state the following Lemma 3.1, we have to take the probabilistic setting. Assume that $\mu_1, \mu_2, \cdots$ are probability measures in $\mathcal{P}_{[n]}$ and f_i denotes the density of μ_i (for $i = 1, 2, \cdots$). Suppose that we have the following sequences of independent random vectors:

(1) $X_1, X_2, \cdots$ with X_i distributed according to f_i;

(2) $Z_1, Z_2, \cdots$ with Z_i distributed according to $\mathbb{1}_{D_n}$.

If we adopt the common convention that all random vectors are defined on a common underlying probability space $(\Omega, \Sigma, \mathbb{P})$ and $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$, then the random vector X_i with n real components defined on the probability space $(\Omega, \Sigma, \mathbb{P}_{X_i})$ is a vector valued function $X_i : \Omega \to \mathbb{R}^n$ with the property that $\{\omega \in \Omega : X_i(\omega) \leq x_i\} \in \Sigma$ for all $x_i \in \mathbb{R}^n$. Here, the notation $X_i \leq x_i$ is shorthand for $X_{1i} \leq x_{1i}, \cdots, X_{ni} \leq x_{ni}$, where $X_i = (X_{1i}, \cdots, X_{ni})$ and $x_i =$

$(x_{1i}, x_{2i}, \cdots, x_{ni}) \in \mathbb{R}^n$. The probability space $(\Omega, \Sigma, \mathbb{P}_{Z_i})$ is analogous with $(\Omega, \Sigma, \mathbb{P}_{X_i})$.

For given the convex body $M_N \subset \mathbb{R}^N (N \geq n)$, from (2.7), the random vectors $X_1, X_2, \cdots, X_N$ and $Z_1, Z_2, \cdots, Z_N$ produce respectively the random convex bodies in $\mathbb{R}^n$, i.e.,

$$\oplus_{M_N}(X_1, \cdots, X_N),\ \oplus_{M_N}(Z_1, \cdots, Z_N) \in \mathcal{K}^n.$$

They can be regarded as from Ω to $\mathcal{K}^n$ ($\mathcal{K}^n$ is a metric space in the Hausdorff metric δ^H, see e.g., [25]) measurable maps.

We say that an event happens almost surely (a.s.) if it happens with probability one. In addition, we say the sequence of random convex bodies $\{K_N\}_{N=n}^{\infty}$ converge to a convex body K almost surely as $N \to \infty$ if

$$\mathbb{P}(\lim_{N\to\infty} \delta^H(K_N, K) = 0) = 1, \tag{3.1}$$

where δ^H is the Hausdorff metric defined in (2.4).

Lemma 3.1. *Suppose that $(M_N)_{N=n}^{\infty}$ is a sequence of convex bodies with $M_N \subset \mathbb{R}^N$. Suppose $\mathcal{C}_X, \mathcal{C}_Z$ are (random) convex bodies in $\mathbb{R}^n$ defined by the following*

$$\mathcal{C}_X := \lim_{N\to\infty} \oplus_{M_N}(X_1, \cdots, X_N) \qquad (a.s.), \tag{3.2}$$

$$\mathcal{C}_Z := \lim_{N\to\infty} \oplus_{M_N}(Z_1, \cdots, Z_N) \qquad (a.s.), \tag{3.3}$$

where the meaning of almost surely convergence of random convex bodies is in (3.1). If there is constants $R_1, R_2 > 0$ such that for any $N \geq n$

$$\oplus_{M_N}(X_1, \cdots, X_N)) \subseteq R_1 B_2^n \qquad (a.s.), \tag{3.4}$$

$$\oplus_{M_N}(Z_1, \cdots, Z_N)) \subseteq R_2 B_2^n \qquad (a.s.), \tag{3.5}$$

and suppose further that $\|f_i\|_\infty \leq 1$ for each $i = 1, 2, \cdots$. Then

$$\mathbb{E}\mathrm{vol}_n(\mathcal{C}_X) \geq \mathbb{E}\mathrm{vol}_n(\mathcal{C}_Z).$$

Proof. Let $T_N = [X_1 \cdots X_N]$ be an $n \times N$ matrix with columns $X_1 \cdots X_N$, and $M_N \subset \mathbb{R}^N$ a convex body. To prove this lemma, according to Paouris and Pivovarov's result in [21, Corollary 5.1], we only need to prove that

$$T_N M_N = \oplus_{M_N}(X_1, \cdots, X_N).$$

In fact, from the definition of M_N-addition (2.7) and the operation of matrixes, we obtain immediately that

$$T_N M_N = \Big\{ \sum_{i=1}^{N} a_i X_i : (a_1, a_2, \cdots, a_N) \in M_N \Big\} = \oplus_{M_N}(X_1, \cdots, X_N).$$

□

Recall for $\phi \in \mathrm{Conv}[0, \infty)$ the asymmetric Orlicz centroid body $\Gamma_\phi^+(\mu)$ of μ corresponding to ϕ is defined by its support function

$$h(\Gamma_\phi^+(\mu), y) = \inf \Big\{ \lambda > 0 : \int_{\mathbb{R}^n} \phi\Big(\frac{(x \cdot y)_+}{\lambda} \Big) d\mu(x) \le 1 \Big\}. \tag{3.6}$$

Together with that $(\cdot)_+$ is subadditive and that ϕ is a convex, strictly increasing function, similar to the proof of [18, Lemma 2.2], we obtain that $h(\Gamma_\phi^+(\mu), y)$ is the support function of a convex body. We assume that $h(\Gamma_\phi^+(\mu), y)$ is finite for each $y \in S^{n-1}$. (If $h(\Gamma_\phi^+(\mu), y) = \infty$ for some $y \in S^{n-1}$, then $\mathrm{vol}_n(\Gamma_\phi^+(\mu)) = \infty$ and Theorem in the introduction is trivially true.) Set

$$B_{\phi/N} = \Big\{ z = (z_1, \cdots, z_N) \in \mathbb{R}^N : \frac{1}{N} \sum_{i=1}^{N} \phi(|z_i|) \le 1 \Big\}.$$

One can check that $B_{\phi/N}$ is convex, 1-unconditional, bounded and the origin is an interior point. By (2.1), we have

$$\|z\|_{B_{\phi/N}} := \inf\{\lambda > 0 : z \in \lambda B_{\phi/N}\}.$$

Combining with (2.3), we have

$$\begin{aligned} h_{B^o_{\phi/N}}(z_1, \cdots, z_N) &= \|(z_1, \cdots, z_N)\|_{B_{\phi/N}} \\ &= \inf \Big\{ \lambda > 0 : \frac{1}{N} \sum_{i=1}^{N} \phi\Big(\frac{|z_i|}{\lambda} \Big) \le 1 \Big\}. \end{aligned} \tag{3.7}$$

Now, we consider the following M_N^+-addition:

$$M_N^+ = B^+_{\phi/N} = B^o_{\phi/N} \cap \mathbb{R}^N_+,$$

comparing with Paouris and Pivovarov's M_N-addition [21]:

$$M_N = B^o_{\phi/N}.$$

From (2.2), we obtain that if K is 1-unconditional, then K^o is 1-unconditional. As $B_{\phi/N}$ is 1-unconditional, it follows that $B^o_{\phi/N}$ is 1-unconditional.

Lemma 3.2. *With the above notations, we have*

$$h_{B^+_{\phi/N}}(z_1, \cdots, z_N) = \inf\Big\{\lambda > 0 : \frac{1}{N}\sum_{i=1}^{N} \phi\Big(\frac{(z_i)_+}{\lambda}\Big) \le 1\Big\}.$$

Proof. We will consider the following two cases.

(i) For the case that $z_1, \cdots, z_N > 0$. Since $B^o_{\phi/N}$ is 1-unconditional, it follows that $h_{B^+_{\phi/N}}(z_1, \cdots, z_N) = h_{B^o_{\phi/N}}(z_1, \cdots, z_N)$.

(ii) For the case that some components of $(z_1, \cdots, z_N)$ are nonpositive. Without loss of generality, we may assume that $z = (z_1, \cdots, z_N)$ such that $z_1, \cdots, z_k > 0$ and $z_{k+1}, \cdots, z_N \le 0$. Suppose $a(z) = (a_1(z), \cdots, a_N(z)) \in B^+_{\phi/N}$ such that $h_{B^+_{\phi/N}}(z_1, \cdots, z_N) = a(z) \cdot z$. We claim that there exits $a'(z) = (a_1(z), \cdots, a_k(z), 0, \cdots, 0)$ such that $h_{B^+_{\phi/N}}(z_1, \cdots, z_N) = a'(z) \cdot z$. Indeed, observe $B^o_{\phi/N}$ is 1-unconditional and $a(z) \in B^+_{\phi/N}$, we have $a'(z) \in B^+_{\phi/N}$. Note that for $a(z) \in B^+_{\phi/N}$, we get $a_i(z) \cdot z_i \le 0$ for $i = k+1 \cdots, N$. The claim follows from the definition of the support function. So,

$$h_{B^+_{\phi/N}}(z_1, \cdots, z_N) = h_{B^+_{\phi/N}}(z_1, \cdots, z_k, 0, \cdots, 0),$$

where $z_1, \cdots, z_k > 0$ and $z_{k+1}, \cdots, z_N \le 0$.

Together with (3.7) in both cases, the lemma follows. $\square$

From (2.8) and Lemma 3.2, we can get

$$\begin{aligned} h(\oplus_{B^+_{\phi/N}}(x_1,\cdots,x_N),y) &= h_{B^+_{\phi/N}}(x_1\cdot y,\cdots,x_N\cdot y) \\ &= \inf\Big\{\lambda>0:\frac{1}{N}\sum_{i=1}^N \phi\Big(\frac{(x_i\cdot y)_+}{\lambda}\Big)\le 1\Big\}. \end{aligned} \tag{3.8}$$

The following lemma is basically similar to Paouris and Pivovarov's symmetric version [21, Lemma 5.4]. For the sake of completeness, we give a proof of the following lemma.

Lemma 3.3. *Let $\mu\in\mathcal{P}_{[n]}$. Let $x_1,x_2,\cdots$ be a sequence of vectors in $\mathbb{R}^n$ and suppose that*

$$\mathrm{pos}\{x_1,\cdots,x_{n+1}\}=\mathbb{R}^n. \tag{3.9}$$

Let $\phi\in\mathrm{Conv}[0,\infty)$. Assume that for each $y\in S^{n-1}$ and each $\lambda>0$ such that

$$\lim_{N\to\infty}\Big|\frac{1}{N}\sum_{i=1}^N\phi\Big(\frac{(x_i\cdot y)_+}{\lambda}\Big)-\int_{\mathbb{R}^n}\phi\Big(\frac{(x\cdot y)_+}{\lambda}\Big)d\mu(x)\Big|=0. \tag{3.10}$$

Then

$$\Gamma^+_\phi(\mu)=\lim_{N\to\infty}\oplus_{B^+_{\phi/N}}(x_1,\cdots,x_N). \tag{3.11}$$

Proof. Since pointwise convergence of support functions implies uniform convergence (see e.g. [25, Theorem 1.8.12]), it is sufficient to show for each $y\in S^{n-1}$

$$\lim_{N\to\infty}h(\oplus_{B^+_{\phi/N}}(x_1,\cdots,x_N),y)=h(\Gamma^+_\phi(\mu),y). \tag{3.12}$$

Fix $y\in S^{n-1}$. By (3.9), there exists $i\in\{1,\cdots,n+1\}$ such that $x_i\cdot y>0$. Together with ϕ is strictly increasing and $\phi(0)=0$, we have

$$\frac{1}{N}\sum_{i=1}^N\phi\Big(\frac{(x_i\cdot y)_+}{\lambda}\Big)>0$$

for $N \geq n+1$ and $\lambda > 0$. Since ϕ is convex and strictly increasing, it follows that both of the functions $P_N, P : (0,\infty) \to (0,\infty)$ defined by

$$P_N(\lambda) := \frac{1}{N}\sum_{i=1}^{N} \phi\Big(\frac{(x_i \cdot y)_+}{\lambda}\Big)$$

and

$$P(\lambda) := \int_{\mathbb{R}^n} \phi\Big(\frac{(x \cdot y)_+}{\lambda}\Big) d\mu(x)$$

are continuous and strictly decreasing.

Hence, from (3.8) and the definition of $h(\Gamma_\phi^+(\mu), y)$, we have

$$\lambda(N) := h(\oplus_{B^+_{\phi/N}}(x_1, \cdots, x_N), y) = \inf\{\lambda > 0 : P_N(\lambda) \leq 1\} \tag{3.13}$$

and

$$\lambda_0 := h(\Gamma_\phi^+(\mu), y) = \inf\{\lambda > 0 : P(\lambda) \leq 1\}. \tag{3.14}$$

Now, we argue by contradiction, assume that (3.12) is false; i.e., there exists $\varepsilon_0 > 0$ and a subsequence $(N_j)_{j=1}^\infty \subset \mathbb{N}$ such that either

(i) $\lambda(N_j) \geq \lambda_0 + \varepsilon_0$ for $j = 1, 2, \cdots$, or

(ii) $\lambda(N_j) \leq \lambda_0 - \varepsilon_0$ for $j = 1, 2, \cdots$.

For the case (i). Let

$$\lambda_* := \inf_j \lambda(N_j). \tag{3.15}$$

By the assumption (i), we have

$$\lambda_* \geq \lambda_0 + \varepsilon_0. \tag{3.16}$$

Let $\eta > 0$. From (3.13), (3.15) and the fact that P_{N_j} is decreasing, it follows that

$$1 < P_{N_j}(\lambda(N_j) - \eta) \leq P_{N_j}(\lambda_* - \eta).$$

Moreover, by (3.10), we have

$$1 \leq \lim_{j\to\infty} P_{N_j}(\lambda_* - \eta) = P(\lambda_* - \eta).$$

Since $\eta > 0$ is arbitrary, and P is continuous, we have $1 \leq P(\lambda_*)$. On the other hand, from (3.16), (3.14) and P is a strictly decreasing continuous function, we have $P(\lambda_*) \leq P(\lambda_0 + \varepsilon_0) < 1$. This leads to a contradiction.

For the case (ii). Let

$$\lambda^* := \sup_j \lambda(N_j). \tag{3.17}$$

By the assumption (ii), we have

$$\lambda^* \leq \lambda_0 - \varepsilon_0. \tag{3.18}$$

Let $\eta > 0$. From (3.13), (3.17) and the fact that P_{N_j} is decreasing, it follows that

$$P_{N_j}(\lambda^* + \eta) \leq P_{N_j}(\lambda(N_j) + \eta) \leq 1.$$

Moreover, by (3.10), we have

$$P(\lambda^* + \eta) = \lim_{j\to\infty} P_{N_j}(\lambda^* + \eta) \leq 1.$$

Together with (3.14), we get $\lambda_0 \leq \lambda^* + \eta$. Since $\eta > 0$ is arbitrary, we have $\lambda_0 \leq \lambda^*$, contradicting (3.18).

□

Now, using above three lemmas, we prove the asymmetric Orlicz centroid inequality for probability measures.

Theorem. *Let $\phi \in \mathrm{Conv}[0,\infty)$ and $\mu \in \mathcal{P}_{[n]}$. If f denotes the density of μ and if $\|f\|_\infty \leq 1$, then*

$$\mathrm{vol}_n(\Gamma_\phi^+(\mu)) \geq \mathrm{vol}_n(\Gamma_\phi^+(\lambda_{D_n})),$$

where λ_{D_n} is the Lebesgue measure restricted to D_n.

Proof. We can assume that there exist $x_1, \cdots, x_{n+1} \in \mathrm{supp}(\mu)$ such that

$$\mathrm{pos}\{x_1, \cdots, x_{n+1}\} = \mathbb{R}^n. \tag{3.19}$$

Otherwise, it follows that $\mathrm{pos}\{\mathrm{supp}(\mu)\} \neq \mathbb{R}^n$. Then from (3.6) and $\phi(0) = 0$, we can let $h(\Gamma_\phi^+(\mu), y) = \infty$ for $y \notin \mathrm{pos}\{\mathrm{supp}(\mu)\}$. Thus, $\mathrm{vol}_n(\Gamma_\phi^+(\mu)) = \infty$ and the theorem is trivially true.

First, we prove that the assumptions of Theorem satisfy that (3.2) and (3.4) in Lemma 3.1.

Let $X_1, X_2, \cdots$ be independent and identically distributed according to f (the density of μ). By standard approximation arguments, we can assume that μ is compactly supported, namely there exists $R_X > 0$ such that

$$\operatorname{supp}(\mu) \subset R_X B_2^n. \tag{3.20}$$

It follows that

$$(X_i \cdot y)_+ \leq R_X \tag{3.21}$$

for all $i \in \mathbb{N}$ and $y \in S^{n-1}$.

Fixed $y \in S^{n-1}$ and $\lambda > 0$. Let $\widetilde{X}_i = \phi((X_i, y)_+/\lambda)$ for $i \in \mathbb{N}$, together with ϕ is strictly increasing, $\mu(\mathbb{R}^n) = 1$ and (3.21), we have

$$\mathbb{E}|\widetilde{X}_1| = \mathbb{E}\phi\Big(\frac{(X_1, y)_+}{\lambda}\Big) = \int_{\mathbb{R}^n} \phi\Big(\frac{(x \cdot y)_+}{\lambda}\Big) d\mu(x) \leq \phi\Big(\frac{R_X}{\lambda}\Big) < \infty.$$

Thus, by the strong law of large numbers (e.g. [3, Theorem 8.3.5]), we obtain that

$$\frac{\widetilde{X}_1 + \cdots + \widetilde{X}_N}{N} = \frac{1}{N}\sum_{i=1}^{N} \phi\Big(\frac{(X_i \cdot y)_+}{\lambda}\Big) \to \mathbb{E}\widetilde{X}_1 = \int_{\mathbb{R}^n} \phi\Big(\frac{(x \cdot y)_+}{\lambda}\Big) d\mu(x) \ (a.s.),$$

this is, X_i's satisfy (3.10) almost surely. Moreover (3.9) is satisfied by (3.19). Then, by applying Lemma 3.3, in the Hausdorff metric, we obtain that

$$\Gamma_\phi^+(\mu) = \lim_{N\to\infty} \oplus_{B^+_{\phi/N}}(X_1, \cdots, X_N) \qquad (a.s.).$$

This shows that (3.2) in Lemma 3.1 is satisfied (where $\Gamma_\phi^+(\mu)$ and $B^+_{\phi/N}$ correspond to $\mathcal{C}_X$ and M_N in Lemma 3.1, respectively).

For convenience, let

$$\lambda_X := \frac{R_X}{\phi^{-1}(1)}.$$

Note that ϕ is strictly increasing, therefore, for any N and any $y \in S^{n-1}$, by (3.21), we have

$$\frac{1}{N}\sum_{i=1}^{N} \phi\Big(\frac{(X_i \cdot y)_+}{\lambda_X}\Big) \leq \frac{1}{N}\sum_{i=1}^{N} \phi\Big(\frac{R_X}{\lambda_X}\Big) = \frac{1}{N}\sum_{i=1}^{N} \phi(\phi^{-1}(1)) = 1,$$

together with (3.8), it follows that

$$h(\oplus_{B^+_{\phi/N}}(X_1, \cdots, X_N), y) \leq \lambda_X.$$

Hence, for any N,

$$\oplus_{B^+_{\phi/N}}(X_1, \cdots, X_N) \subseteq \lambda_X B^n_2. \tag{3.22}$$

This shows that (3.4) in Lemma 3.1 is satisfied.

Next, we prove that the assumptions of Theorem satisfy that (3.3) and (3.5) in Lemma 3.1.

Let $Z_1, Z_2, \cdots$ be the sequence that Z_i distributed according to $\mathbb{1}_{D_n}$. Note that (3.20), we can set $R_Z = (1/\mathrm{vol}_n(B^n_2))^{1/n}$, it follows that

$$(Z_i \cdot y)_+ \leq R_Z \tag{3.23}$$

for all $i \in \mathbb{N}$ and $y \in S^{n-1}$.

Fixed $y \in S^{n-1}$ and $\lambda > 0$. Let $\widetilde{Z}_i = \phi((Z_i, y)_+/\lambda)$ for $i \in \mathbb{N}$, together with ϕ is strictly increasing, $\lambda_{D_n}(\mathbb{R}^n) = 1$ and (3.23), we have

$$\mathbb{E}|\widetilde{Z}_1| = \mathbb{E}\phi\Big(\frac{(Z_1, y)_+}{\lambda}\Big) = \int_{D_n} \phi\Big(\frac{(z \cdot y)_+}{\lambda}\Big)dz \leq \phi\Big(\frac{R_Z}{\lambda}\Big) < \infty.$$

Thus, by the strong law of large numbers, we obtain that

$$\frac{\widetilde{Z}_1 + \cdots + \widetilde{Z}_N}{N} = \frac{1}{N}\sum_{i=1}^N \phi\Big(\frac{(Z_i \cdot y)_+}{\lambda}\Big) \to \mathbb{E}\widetilde{Z}_1 = \int_{D_n} \phi\Big(\frac{(z \cdot y)_+}{\lambda}\Big)dz \ (a.s.).$$

This is, Z_i's satisfy (3.10) almost surely. Moreover, (3.19) is obviously true when $\mu = \lambda_{D_n}$, thus (3.9) is satisfied. Then, applying Lemma 3.3, in the Hausdorff metric,

$$\Gamma^+_\phi(\lambda_{D_n}) = \lim_{N\to\infty} \oplus_{B^+_{\phi/N}}(Z_1, \cdots, Z_N) \qquad (a.s.).$$

This shows that (3.3) in Lemma 3.1 is satisfied (where $\Gamma^+_\phi(\lambda_{D_n})$ corresponds to $\mathcal{C}_Z$ in Lemma 3.1).

Let

$$\lambda_Z := \frac{R_Z}{\phi^{-1}(1)},$$

recall that ϕ is strictly increasing, therefore, for any N and any $y \in S^{n-1}$, by the inequality (3.23), we obtain that

$$\frac{1}{N}\sum_{i=1}^N \phi\Big(\frac{(Z_i \cdot y)_+}{\lambda_Z}\Big) \leq \frac{1}{N}\sum_{i=1}^N \phi\Big(\frac{R_Z}{\lambda_Z}\Big) = \frac{1}{N}\sum_{i=1}^N \phi(\phi^{-1}(1)) = 1.$$

By (3.8), it follows that

$$h(\oplus_{B^+_{\phi/N}}(Z_1, \cdots, Z_N), y) \leq \lambda_Z,$$

hence, for any N,

$$\oplus_{B^+_{\phi/N}}(Z_1, \cdots, Z_N) \subseteq \lambda_Z B_2^n. \tag{3.24}$$

This shows that (3.5) in Lemma 3.1 is satisfied.

Finally, by Lemma 3.1, we obtain that

$$\mathbb{E}\mathrm{vol}_n(\Gamma^+_\phi(\mu) \geq \mathbb{E}\mathrm{vol}_n(\Gamma^+_\phi(\lambda_{D_n}).$$

Note that for given μ, from the definition (3.6) of $\Gamma^+_\phi(\mu)$, $\Gamma^+_\phi(\mu)$ is an non-random set, and it is obvious that $\Gamma^+_\phi(\lambda_{D_n})$ is an non-random set. Thus,

$\mathbb{E}\mathrm{vol}_n(\Gamma^+_\phi(\mu) = \mathrm{vol}_n(\Gamma^+_\phi(\mu)$, and $\mathbb{E}\mathrm{vol}_n(\Gamma^+_\phi(\lambda_{D_n}) = \mathrm{vol}_n(\Gamma^+_\phi(\lambda_{D_n})$. This completes the proof of theorem. $\square$

4. Asymmetric L_p centroid inequality

Let $K \subset \mathbb{R}^n$ be a compact set with $\mathrm{vol}_n(K) = 1$. For $p \geq 1$, let $\Gamma^+_p(K)$ denote the asymmetric L_p centroid body of K; i.e., the body with support function

$$h(\Gamma^+_p(K), y) = \Big(\int_K (x \cdot y)^p_+ dx\Big)^{1/p}.$$

Using the notation in [9], we define

$$\Gamma^-_p(K) = \Gamma^+_p(-K),$$

and denote the convex body $\Gamma^\alpha_p(K)$, for $-1 \leq \alpha \leq 1$, by

$$\Gamma^\alpha_p(K) = \frac{(1+\alpha)^p}{(1+\alpha)^p + (1-\alpha)^p} \cdot \Gamma^+_p(K) +_p \frac{(1-\alpha)^p}{(1+\alpha)^p + (1-\alpha)^p} \cdot \Gamma^-_p(K).$$

Note that the origin may be on the boundary of $\Gamma^+_p(K)$ or $\Gamma^-_p(K)$. As shown in [4], the L_p Minkowski combination (2.5) also applies to K, L contains the origin as its boundary point.

Now, we apply our theorem by setting $\phi = t^p$ and the density f of μ by

$$f = \frac{(1+\alpha)^p}{(1+\alpha)^p + (1-\alpha)^p}\mathbb{1}_K + \frac{(1-\alpha)^p}{(1+\alpha)^p + (1-\alpha)^p}\mathbb{1}_{-K}$$

with $\mathrm{vol}_n(K) = 1$. Then, we have

Corollary 4.1. *Let $K \subset \mathbb{R}^n$ be a compact set with $\mathrm{vol}_n(K) = 1$. Then*

$$\mathrm{vol}_n(\Gamma_p^\alpha(K)) \geq \mathrm{vol}_n(\Gamma_p^\alpha(D_n)),$$

where D_n is the Euclidean ball of volume one.

For star bodies $K \in \mathbb{R}^n$ this inequality, together with the equality conditions, is previously proved by Haberl and Schuster [9, Theorem 6.4]. Note that their result can also be deduced by Zhu's Orlicz centroid inequality for star bodies [28]. Moreover, let $\alpha = 1$ in Corollary 4.1, we have

Corollary 4.2. *Let $K \subset \mathbb{R}^n$ be a compact set with $\mathrm{vol}_n(K) = 1$. Then*

$$\mathrm{vol}_n(\Gamma_p^+(K)) \geq \mathrm{vol}_n(\Gamma_p^+(D_n)),$$

where D_n is the Euclidean ball of volume one.

The following L_p centroid inequality can be proved by Paouris and Pivovarov's theorem [21] (see also [20]). On the other hand, let $\alpha = 0$ in Corollary 4.1, we also have this inequality.

Corollary 4.3. *Let $K \subset \mathbb{R}^n$ be a compact set with $\mathrm{vol}_n(K) = 1$. Then*

$$\mathrm{vol}_n(\Gamma_p(K)) \geq \mathrm{vol}_n(\Gamma_p(D_n)),$$

where D_n is the Euclidean ball of volume one.

For star bodies $K \in \mathbb{R}^n$ this inequality, together with the equality conditions, is proved in [16]; see [1] for an alternate proof.

Acknowledgements

The authors are indebted to the anonymous referees for the valuable suggestions and the very careful reading of the original manuscript.

References

[1] S. Campi and P. Gronchi, *The L^p-Busemann-Petty centroid inequality*, Adv. Math. **167** (2002), 128-141.

[2] F. Chen, J. Zhou, C. Yang, *On the reverse Orlicz Busemann-Petty centroid inequality*, Adv. in Appl. Math. **47** (2012), 820-828.

[3] R. M. Dudley, *Real Analysis and Probability*, Cambridge Studies in Advanced Mathematics, **74**, Cambridge University Press, Cambridge, 2002.

[4] Wm. J. Firey, *p-means of convex bodies*, Math. Scand. **10** (1962), 17-24.

[5] R. J. Gardner, *Geometric tomography,* second edition, Encyclopedia of Mathematics and its Applications, **58**, Cambridge University Press, Cambridge, 2006.

[6] R. J. Gardner, D. Hug, and W. Weil, *Operations between sets in geometry*, J. Europ. Math. Soc. (to appear).

[7] C. Haberl, E. Lutwak, D. Yang, G. Zhang, *The even Orlicz Minkowski problem*, Adv. Math. **224** (2010), 2485-2510.

[8] C. Haberl, F. Schuster, *Asymmetric Affine L_p Sobolev Inequalities*, J. Funct. Anal. **257** (2009), 641-658.

[9] C. Haberl, F. Schuster, *General L_p affine isoperimetric inequalities*, J. Differ. Geom. **83** (2009), 1-26.

[10] C. Haberl, F. Schuster and J. Xiao *An Asymmetric Affine Pólya-Szegö Principle*, Math. Ann. **352** (2012), 517-542.

[11] Q. Huang and B. He, *On the Orlicz Minkowski Problem for Polytopes*, Discrete Comput. Geom. **83** (2012), 281-297.

[12] A. J. Li and G. Leng, *A new proof of the Orlicz Busemann-Petty centroid inequality*, Proc. Amer. Math. Soc. **139(4)** (2011), 1473-1481.

[13] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), 4191-4213.

[14] M. Ludwig, *General affine surface areas*, Adv. Math. **224** (2010), 2346-2360.

[15] M. Ludwig, M. Reitzner, *A classification of $SL(n)$ invariant valuations*, Ann. Math. **172(2)** (2010), 1219-1267.

[16] E. Lutwak, D. Yang, and G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111-132.

[17] E. Lutwak, D. Yang, G. Zhang, *Orlicz projection bodies*, Adv. Math. **223** (2010), 220-242.

[18] E. Lutwak, D. Yang, G. Zhang, *Orlicz centroid bodies*, J. Differential Geom. **84** (2010), 365-387.

[19] E. Lutwak, D. Yang, and G. Zhang, *The Brunn-Minkowski-Firey inequality for non-convex sets*, Adv. in Appl. Math. **48** (2012), 407-413.

[20] G. Paouris, *On the existence of supergaussian directions on convex bodies*, Mathematika **58** (2012), 389-408.

[21] G. Paouris and P. Pivovarov, *A probabilistic take on isoperimetric-type inequalities,* Adv. Math. **230** (2012), 1402-1422.

[22] G. Paouris and P. Pivovarov, *Intrinsic volumes and linear contractions,* Proc. Amer. Math. Soc. (in press).

[23] G. Paouris and P. Pivovarov, *Small-ball probabilities for the volume of random convex sets*, Preprint.

[24] V. Y. Protasov, *A generalized joint spectral radius. A geometric approach* (in Russian), Izv. Ross. Akad. Nauk Ser. Mat. **61** (1997), 99-136; translation in Izv. Math. **61** (1997), 995-1030.

[25] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, **44**, Cambridge University Press, Cambridge, 1993.

[26] G. Wang, G. Leng and Q. Huang, *Volume inequalities for Orlicz zonotopes*, J. Math. Anal. Appl. **391** (2012), 183-189.

[27] D. Ye, *Inequalities for general mixed affine surface areas*, J. London Math. Soc. **85** (2012), 101-120.

[28] G. Zhu, *The Orlicz centroid inequality for star bodies*, Adv. in Appl. Math. **48** (2012), 432-445.

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