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Convex bodies with minimal volume product in $\mathbb{R}^2$ — a new proof

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ABSTRACT

A new proof of the Mahler conjecture in $\mathbb{R}^2$ is given. In order to prove the result, we introduce a new method — the *vertex removal method*; i.e., for any origin-symmetric polygon P, there exists a linear image ϕP contained in the unit disk B^2, and there exist three contiguous vertices of ϕP lying on the boundary of B^2. We can show that the volume-product of P decreases when we remove the middle vertex of the three vertices.

1. Introduction

If $K \subset \mathbb{R}^n$ is an origin-symmetric convex body, let K^* denote its polar body which is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1, \forall y \in K\}.$$

Define the volume-product $\mathcal{P}(K)$ of K as

$$\mathcal{P}(K) = \mathrm{Vol}(K)\mathrm{Vol}(K^*).$$

The famous Mahler conjecture [6] is to find a lower bound of $\mathcal{P}(K)$. Is it true that we always have

$$\mathcal{P}(K) \ge \mathcal{P}(B^n_\infty), \tag{1.1}$$

where $B^n_\infty = \{x \in \mathbb{R}^n : |x_i| \le 1, 1 \le i \le n\}$?

For $n = 2$, Mahler [5] proved that the answer is affirmative, and in 1986 Reisner [9] showed that equality holds only for parallelograms. For $n = 2$, a new proof of inequality (1.1) was obtained by Campi and Gronchi [2]. In the n-dimensional case, the conjecture has been verified for some special classes of bodies, namely, for 1-unconditional bodies, [7,10,11], and for zonoids, [3,9].

In 1987, Bourgain and Milman [1] proved that there exists a universal constant $c > 0$ such that $\mathcal{P}(K) \ge c^n\mathcal{P}(B^n_\infty)$, which is now known as the Bourgain–Milman inequality. Very recently, Kuperberg [4] found a beautiful new approach to the Bourgain–Milman inequality. What's especially remarkable about Kuperberg's inequality is that it provides an explicit value for c.

For $n = 2$, Mahler [5] proved that the volume-product of a polygon is concave down as a pair of opposite sides are pivoted and the volume-product can be maximized as all pairs of sides are repeatedly pivoted, at the same time the polygon converges to a regular polygon. In this paper, to prove the Mahler conjecture for $n = 2$, we provide a new method — the *vertex removal method*, which is illustrated as follows: Firstly, we prove that any origin-symmetric polygon P can be transformed

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into a new polygon contained in the unit disk B^2, in which there are three contiguous vertices lying on the boundary of B^2 (see Theorem 3.1). Next, we prove that the volume-product of the new polygon decreases as the middle vertex of the three contiguous vertices moves toward its adjacent vertices on the boundary of B^2 (see Theorem 3.3). Therefore, we obtain that the volume-product of P decreases when we remove the middle vertex of the three vertices. Compared with Mahler' proof, the *vertex removal method* provides a specific downward path to a square from any origin-symmetric polygon. For $n = 3$, the conjecture could probably be solved following the same idea.

It is worth mentioning that Meyer [8] proved the Mahler conjecture for the general convex bodies in $\mathbb{R}^2$. He showed that the volume-product of a convex body is always bigger than that of a triangle and established the case of equality.

2. Definition, notation and preliminaries

As usual, S^{n-1} denotes the unit sphere, B^n the unit ball centered at the origin, o the origin and $\|\cdot\|$ the norm in Euclidean n-space $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$, then $x \cdot y$ is the inner product of x and y.

If K is a set, ∂K is its boundary, int K is its interior, and conv K denotes its convex hull. Let $\mathbb{R}^n \setminus K$ denote the complement of K; i.e.,

$$\mathbb{R}^n \setminus K = \{x \in \mathbb{R}^n : x \notin K\}. \tag{2.1}$$

If K is an n-dimensional convex subset of $\mathbb{R}^n$, then $V(K)$ denotes $\mathrm{Vol}_n(K)$.

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. Let $\mathcal{K}^n_o$ denote the subset of $\mathcal{K}^n$ that contains the origin in its interior. Let $h(K, \cdot) : S^{n-1} \to \mathbb{R}$, denote the support function of $K \in \mathcal{K}^n_o$; i.e.,

$$h(K, u) = \max\{u \cdot x \mid x \in K\}, \quad u \in S^{n-1}. \tag{2.2}$$

Let $\rho(K, \cdot) : S^{n-1} \to \mathbb{R}$, denote the radial function of $K \in \mathcal{K}^n_o$; i.e.,

$$\rho(K, u) = \max\{\lambda \geq 0 \mid \lambda u \in K\}, \quad u \in S^{n-1}. \tag{2.3}$$

A linear transformation (or affine transformation) of $\mathbb{R}^n$ is a map ϕ from $\mathbb{R}^n$ to itself such that $\phi x = Ax$ (or $\phi x = Ax + t$, respectively), where A is an $n \times n$ matrix and $t \in \mathbb{R}^n$. For a convex body P and a linear transformation ϕ, we define ϕP as the linear image of P about ϕ.

Geometrically, an affine transformation in Euclidean space is one that preserves:

(i) The collinearity relation between points; i.e., three points which lie on a line continue to be collinear after the transformation.
(ii) Ratios of distances along a line; i.e., for distinct collinear points P_1, P_2, P_3, the ratio $|P_2 - P_1|/|P_3 - P_2|$ is preserved.

If $K \in \mathcal{K}^n_o$, it is easy to verify that (see p. 44 in [12])

$$h(K^*, u) = \frac{1}{\rho(K, u)} \quad \text{and} \quad \rho(K^*, u) = \frac{1}{h(K, u)}. \tag{2.4}$$

If P is a polygon; i.e., $P = \mathrm{conv}\{A_1, \ldots, A_m\}$, where A_i $(i = 1, \ldots, m)$ are vertices of P. Let a_i denote the vector of A_i. By the definition of the polar body, we have

$$P^* = \bigcap_{i=1}^{m} \{x \in \mathbb{R}^2 : x \cdot a_i \leq 1\}, \tag{2.5}$$

which implies that P^* is the intersection of m closed half-planes with exterior normal vector a_i. The distance of the straight line $\{x \in \mathbb{R}^2 : x \cdot a_i = 1\}$ from the origin is $1/\|a_i\|$. Thus, if P is an inscribed polygon in a unit circle, then P^* is a polygon circumscribed around the unit circle. In the proof of Theorem 3.3, we shall use these properties.

For $K, L \in \mathcal{K}^n$, the Hausdorff distance is defined by

$$d(K, L) = \min\{\lambda \geq 0 : K \subset L + \lambda B^n, L \subset K + \lambda B^n\}, \tag{2.6}$$

which can be conveniently defined by (see p. 53 in [12])

$$d(K, L) = \max_{u \in S^{n-1}} |h(K, u) - h(L, u)|. \tag{2.7}$$

Therefore, a sequence of convex bodies K_i converges to K if and only if the sequence of support functions $h(K_i, \cdot)$ converges uniformly to $h(K, \cdot)$.

The following lemmas are well-known and important for our proof:

Lemma 2.1. *For any origin-symmetric convex body $K \subset \mathbb{R}^n$, $\mathcal{P}(K)$ is linear invariant, that is, for every linear transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$, we have $\mathcal{P}(\phi K) = \mathcal{P}(K)$.*

Lemma 2.2. *The volume-product $\mathcal{P}(K)$ is continuous under the Hausdorff metric.*

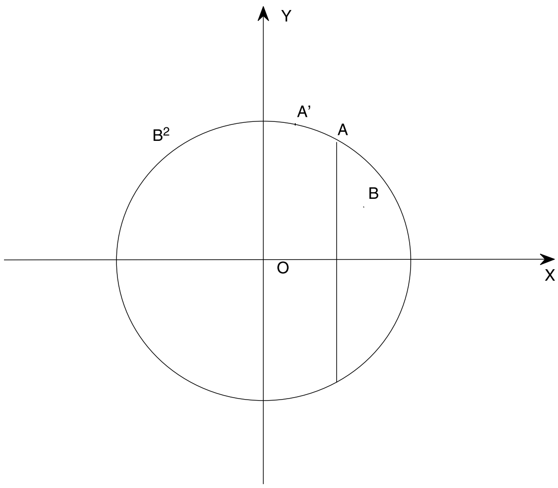

Fig. 3.1. Linear transformation ϕ such that $A' = \phi A$.

3. Main result and its proof

Theorem 3.1. *In $\mathbb{R}^2$, for any origin-symmetric polygon P, there exists a linear image $P' = \phi P$ which satisfies that $P' \subset B^2$ and there exist three contiguous vertices of P' contained in ∂B^2.*

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *In Fig. 3.1, $A(x_1, y_1)$ and $A'(x_2, y_2)$ are on ∂B^2. ϕ is a linear transformation from $\mathbb{R}^2$ to itself such that $A' = \phi A$. If $B(x, y) \in B^2$ and $0 < x_2 < x_1 < x$, then $\phi B \in B^2$.*

Proof. Let $B' = \phi B$ and $B' = (x', y')$. Since ϕ is a linear transformation, we get

$$\frac{x}{x_1} = \frac{x'}{x_2} \quad \text{and} \quad \frac{y}{y_1} = \frac{y'}{y_2}.$$

Therefore,

$$x' = \frac{xx_2}{x_1} \quad \text{and} \quad y' = \frac{yy_2}{y_1}.$$

Noting that $x_1^2 + y_1^2 = 1$, $x_2^2 + y_2^2 = 1$ and $x^2 + y^2 \le 1$, we have

$$\begin{aligned} x'^2 + y'^2 &= \frac{x^2x_2^2}{x_1^2} + \frac{y^2y_2^2}{y_1^2} \\ &\le \frac{x^2x_2^2}{x_1^2} + \frac{(1-x^2)(1-x_2^2)}{1-x_1^2} \\ &= 1 - \frac{(x_1^2 - x_2^2)(x^2 - x_1^2)}{x_1^2(1-x_1^2)} \\ &\le 1. \end{aligned} \tag{3.1}$$

Hence, $\phi B \in B^2$. □

Now we prove Theorem 3.1.

Proof. Since P is an origin-symmetric polygon, the number of sides of P is even and the opposite sides of P are parallel. Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ denote all vertices of P, where B_i is the symmetric point of A_i about the origin. Our proof is in three steps.

The first step, transform the parallelogram $A_1A_2B_1B_2$ into the rectangular $A_1'A_2'B_1'B_2'$ inscribed in B^2. Now P is transformed into P_1 (see (2) or (2)′ in Figs. 3.2.1 and 3.2.2).

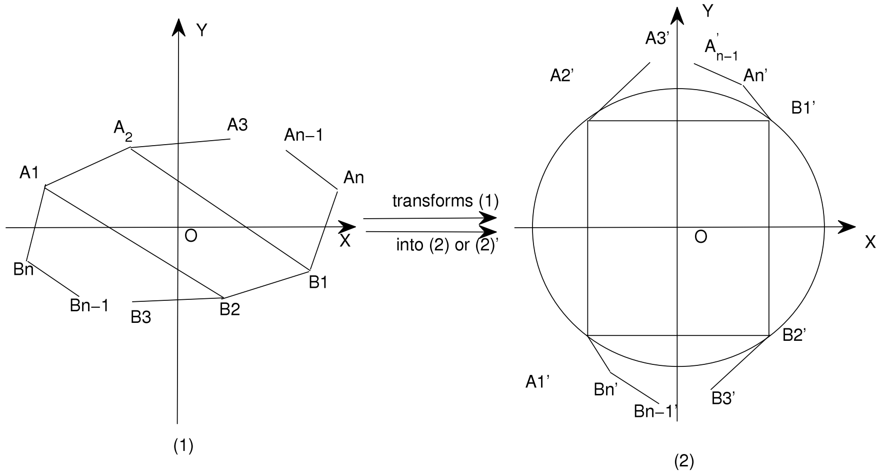

Fig. 3.2.1. Transforms (1) into (2) or (2)′.

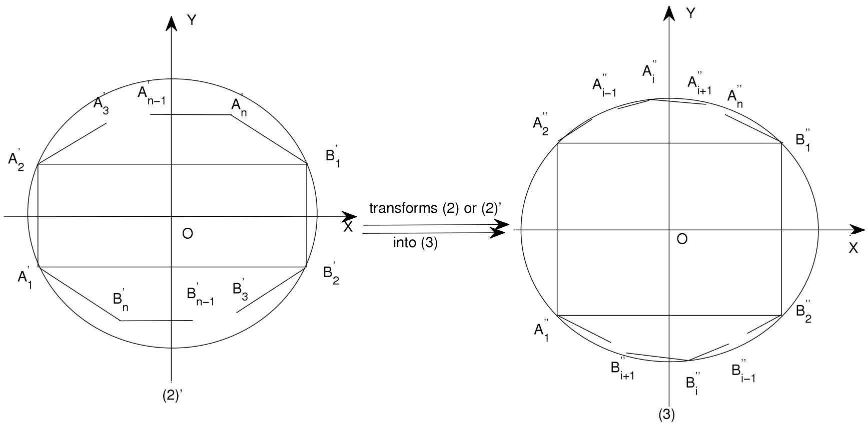

Fig. 3.2.2. Transforms (2) or (2)′ into (3).

The second step, transform P_1 into P_2 (see (3) in Fig. 3.2.2). Consider the following two cases for the polygon P_1,

(i) If there exist some vertices of P_1 satisfying

$$\{A_i' : i \in I \subset \{3, \dots, n\}\} \subset \mathbb{R}^2 \setminus B^2,$$

then we transform P_1 into $P_2 \subset B^2$. The transformation shortens the segment $A_1'A_2'$ into $A_1''A_2''$ and satisfies that some vertices $\{A_i'' : i \in I_1 \subset \{3, \dots, n\}\}$ lie on ∂B^2.

(ii) If vertices A_i' and B_i' $(i = 3, \dots, n)$ of P_1 satisfy

$$\{A_3', \dots, A_n', B_3', \dots, B_n'\} \subset \operatorname{int} B^2,$$

then we transform P_1 into $P_2 \subset B^2$. The transformation lengthens the segment $A_1'A_2'$ into $A_1''A_2''$ and satisfies that some vertices $\{A_i'' : i \in I_1 \subset \{3, \dots, n\}\}$ lie on ∂B^2.

The third step, transform P_2 into P' (see Fig. 3.3). If $A_1'', A_2'', A_i''(i \in I_1)$ are three contiguous vertices on ∂B^2, then this theorem has already been proved; otherwise we rotate P_2 about the origin, we can get a new polygon P_3' such that $A_2''A_i''$ parallels the X-axis (see (4) in Fig. 3.3). Then lengthen segments $A_2''B_i''$ and $A_i''B_2''$ into $A_2^{(3)}B_i^{(3)}$ and $A_i^{(3)}B_2^{(3)}$, respectively, satisfying that some vertices $\{A_j^{(3)} : j \in I_2 \subset \{3, \dots, i-1\}\}$ lie on ∂B^2 and $\{A_j^{(3)} : j \in \{3, \dots, i-1\} \setminus I_2\} \subset B^2$. By Lemma 3.2, vertices $\{A_{i+1}^{(3)}, \dots, A_n^{(3)}, B_1^{(3)}\}$ are still in the internal of B^2 (see (5) in Fig. 3.3). Let P_3 denote the new polygon. We get $P_3 \subset B^2$.

There are $i-3$ vertices between the vertex $A_2^{(3)}$ and vertex $A_i^{(3)}$. If $i-3 = 1$, then P_3 is the polygon satisfying the theorem. If $i-3 \geq 2$, consider the following two cases:

(i) If $j-2 \geq i-j$ (where $j \in I_2$), rotate P_3 about the origin such that $A_2^{(3)}A_j^{(3)}$ parallels the X-axis.

(ii) If $j-2 \leq i-j$ (where $j \in I_2$), rotate P_3 about the origin such that $A_j^{(3)}A_i^{(3)}$ parallels the X-axis.

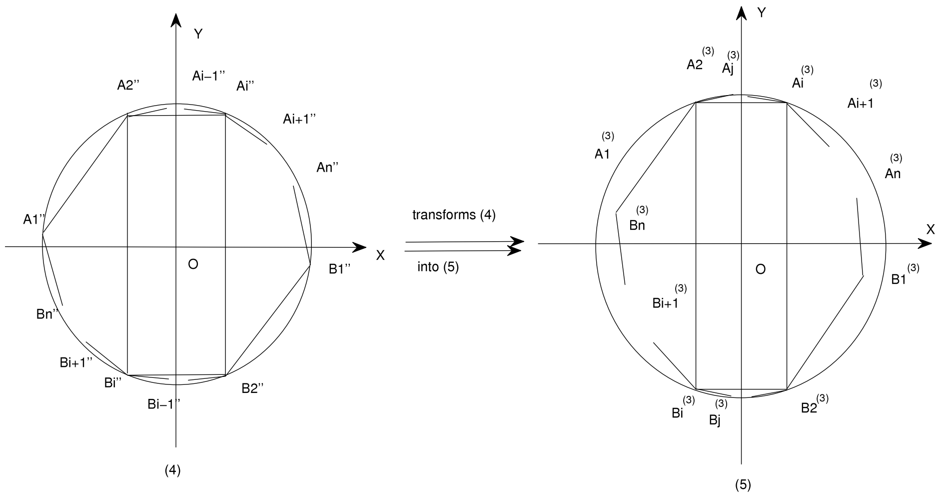

Fig. 3.3. Transforms (4) into (5).

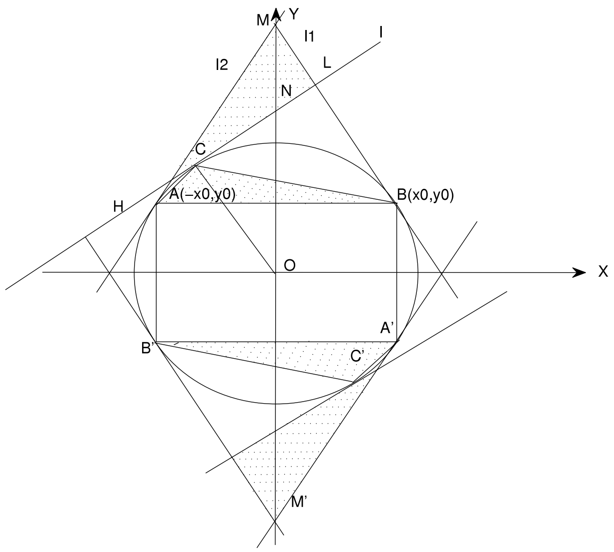

Fig. 3.4. Polygon P' and its polar body.

We denote the new polygon as P_4. It is clear that there are less than $i-3$ vertices between $A_2^{(3)}$ and $A_j^{(3)}$ (or between $A_j^{(3)}$ and $A_i^{(3)}$). By induction, we can get a new polygon $P' \subset B^2$ with three contiguous vertices contained in ∂B^2 after finite transformations. $\square$

By the above theorem, we consider the volume-product of the polygon with three contiguous vertices contained in ∂B^2.

Theorem 3.3. *Suppose that $P' \subset B^2$ is an origin-symmetric polygon and A, C, B are three contiguous vertices of P' contained in ∂B^2, then $\mathcal{P}(P'') \le \mathcal{P}(P')$, where P'' is a new polygon obtained from P' by removing vertices C and C'.*

Proof. Suppose that the side AB parallels the X-axis (see Fig. 3.4.) and straight lines l, l_1 and l_2 are three tangent lines to the unit circle B^2 passing through points C, A and B, respectively. Let $A=(-x_0, y_0)$, then $B=(x_0, y_0)$, where ${x_0}^2+{y_0}^2=1$. Let $\theta = \angle XOC$. It is clear that $\pi/2 \le \theta \le \pi - \arctan(y_0/x_0)$ when the point C is in the second quadrant. We have the following equations of straight lines:

$$l_1 : y - y_0 = \frac{x_0}{y_0}(x + x_0),$$

$$l_2 : y - y_0 = -\frac{x_0}{y_0}(x - x_0),$$

$$l : y - \sin\theta = -\frac{\cos\theta}{\sin\theta}(x - \cos\theta).$$

Let the point N denote the intersection of l and the Y-axis and the point M denote the intersection of l_1 and the Y-axis. We can easily get $N(0, 1/\sin\theta)$ and $M(0, 1/y_0)$. Let H and L denote the intersections of l and l_1, l_2, respectively. Solve the following systems of equations:

$$\begin{cases} y - \sin\theta = -\dfrac{\cos\theta}{\sin\theta}(x - \cos\theta) \\ y - y_0 = \dfrac{x_0}{y_0}(x + x_0) \end{cases} \tag{3.2}$$

and

$$\begin{cases} y - \sin\theta = -\dfrac{\cos\theta}{\sin\theta}(x - \cos\theta) \\ y - y_0 = -\dfrac{x_0}{y_0}(x - x_0). \end{cases} \tag{3.3}$$

We get the abscissas of points H and L:

$$x_1 = \frac{y_0 - \sin\theta}{y_0\cos\theta + x_0\sin\theta}$$

and

$$x_2 = \frac{y_0 - \sin\theta}{y_0\cos\theta - x_0\sin\theta}.$$

Let $S_{\triangle MHL}$ denote the area of $\triangle MHL$, it follows that

$$S_{\triangle MHL} = \frac{x_0}{y_0} \cdot \frac{\sin\theta - y_0}{\sin\theta + y_0}.$$

Let $V = V(P'')$ and $V^0 = V(P''^*)$, where P'' denotes the new polygon obtained from P' by removing vertices C and C', then $\mathcal{P}(P')$ is a function $f(\theta)$, where

$$f(\theta) = (V + 2x_0(\sin\theta - y_0))\left(V^0 - \frac{2x_0}{y_0} \cdot \frac{\sin\theta - y_0}{\sin\theta + y_0}\right) \tag{3.4}$$

and

$$\frac{\pi}{2} \le \theta \le \pi - \arctan\left(\frac{y_0}{x_0}\right).$$

We have

$$f'(\theta) = 2x_0\cos\theta \cdot \frac{(V^0y_0 - 2x_0)(\sin\theta + y_0)^2 + 2y_0(4x_0y_0 - V)}{y_0(\sin\theta + y_0)^2}. \tag{3.5}$$

In (3.4), since $\cos\theta \le 0$ and $y_0(\sin\theta + y_0)^2 \ge 0$, to prove $f'(\theta) \le 0$, let $t = \sin\theta$, we need only to show that $g(t) \ge 0$, where

$$g(t) = (V^0y_0 - 2x_0)(t + y_0)^2 + 2y_0(4x_0y_0 - V), \quad t \in [y_0, 1]. \tag{3.6}$$

Let $S_{\square ABA'B'}$ and $S_{\triangle ABM}$ denote the area of a rectangle $ABA'B'$ and triangle ABM, respectively. Since

$$V(P''^*) \ge V(\mathrm{conv}\{A, M, B, A', M', B'\}),$$

we get

$$V^0 \ge S_{\square ABA'B'} + 2S_{\triangle ABM} = 4x_0y_0 + 2x_0\left(\frac{1}{y_0} - y_0\right),$$

therefore,

$$\begin{aligned} V^0y_0 - 2x_0 &\ge \left(4x_0y_0 + 2x_0\left(\frac{1}{y_0} - y_0\right)\right)y_0 - 2x_0 \\ &= 2x_0y_0^2 \\ &> 0. \end{aligned} \tag{3.7}$$

Thus, the graph of $g(t)$ is a parabola opening upward. Since the axis of symmetry of the parabola is $t = -y_0$, $g(t)$ is increasing for $t \in [y_0, 1]$. To prove $g(t) \geq 0$, it suffices to prove

$$g(y_0) = 2y_0(2V^0 y_0^2 - V) \geq 0. \tag{3.8}$$

Let $\mathcal{D}$ denote the area of a circular segment enclosed by the arc BA' and the chord BA', then

$$\begin{aligned} V^0 &\geq S_{\square ABA'B'} + 2S_{\triangle ABM} + 2\mathcal{D} \\ &= 4x_0y_0 + 2x_0\left(\frac{1}{y_0} - y_0\right) + 2\mathcal{D} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} V &\leq S_{\square ABA'B'} + 2\mathcal{D} \\ &= 4x_0y_0 + 2\mathcal{D}. \end{aligned} \tag{3.10}$$

To prove (3.8), it suffices to prove

$$2\left(4x_0y_0 + 2x_0\left(\frac{1}{y_0} - y_0\right) + 2\mathcal{D}\right)y_0^2 \geq 4x_0y_0 + 2\mathcal{D}, \tag{3.11}$$

which is equivalent to

$$2x_0y_0^3 \geq \mathcal{D}(1 - 2y_0^2). \tag{3.12}$$

And since

$$\mathcal{D} \leq (1 - x_0) \cdot 2y_0, \tag{3.13}$$

hence, it suffices to prove

$$x_0y_0^3 \geq y_0(1 - x_0)(1 - 2y_0^2), \tag{3.14}$$

which is equivalent to

$$x_0^3 - 2x_0^2 + 1 \geq 0, \tag{3.15}$$

which is clearly correct when $0 < x_0 < 1$.

We get that $f(\theta)$ is decreasing when $\theta \in [\pi/2, \pi - \arctan(y_0/x_0)]$, hence the function $f(\theta)$ has a minimum value at $\theta = \pi - \arctan(y_0/x_0)$. The point C coincides with the point A when $\theta = \pi - \arctan(y_0/x_0)$. Therefore $\mathcal{P}(P'') \leq \mathcal{P}(P')$. □

By Theorem 3.3, we can get the following Corollaries 3.4 and 3.5.

Corollary 3.4. *If $P \subset \mathbb{R}^2$ is an origin-symmetric polygon, then $\mathcal{P}(P) \geq \mathcal{P}(S)$, where S is a square.*

Proof. By Theorems 3.1 and 3.3 and the linear invariance of $\mathcal{P}(P)$, if the number of sides of the polygon P is $2n$, there exists a polygon P_1 with $2(n-1)$ sides satisfying $\mathcal{P}(P_1) \leq \mathcal{P}(P)$. Repeating this process $n-2$ times, we can get a square S satisfying $\mathcal{P}(P) \geq \mathcal{P}(S)$. □

Corollary 3.5. *If $K \subset \mathbb{R}^2$ is an origin-symmetric convex body and $S \subset \mathbb{R}^2$ is an origin-symmetric square, then $\mathcal{P}(K) \geq \mathcal{P}(S)$.*

Proof. For any origin-symmetric convex body $K \subset \mathbb{R}^2$, there exists a sequence of origin-symmetric polytopes $\{P_i\}$ converging to K under the Hausdorff metric (see p. 54 in [12]). By Corollary 3.4 and Lemma 2.2., we have

$$\mathcal{P}(K) = \lim_{n\to\infty} \mathcal{P}(P_i) \geq \mathcal{P}(S). \quad \square \tag{3.16}$$

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