Iwasawa Main Conjecture for Supersingular Elliptic curves

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Abstract

In this paper we prove the $\pm$-main conjecture formulated by Kobayashi for elliptic curves with supersingular reduction at p such that $a_p = 0$, using a completely new idea of reducing it to another Iwasawa-Greenberg main conjecture which is more accessible. We also prove as a corollary the p-part of the BSD formula at supersingular primes when the analytic rank is 0. The argument uses in an essential way the recent study on explicit reciprocity law for Beilinson-Flach elements by Kings-Loeffler-Zerbes.

1 Introduction

Let p be an odd prime. Iwasawa theory studies relations between special values of L-functions and arithmetic objects such as class numbers of number fields or more generally p-adic Selmer groups. The central problem for this study is the Iwasawa main conjecture, which roughly speaking, says that the size (or more precisely the characteristic ideal) of certain module parameterizing the p-adic families of Selmer groups is controlled by the so called p-adic L-function, which interpolates p-adic families of the algebraic parts of the corresponding special L-values. Iwasawa main conjecture is also a useful tool in proving the refined Birch-Swinnerton-Dyer (BSD) formula for elliptic curves.

Earlier work on Iwasawa main conjecture includes the work of Mazur-Wiles [42], Wiles [71] for p-adic families of Hecke characters of totally real fields using the Eisenstein congruence on GL_2, Rubin [52] for characters for quadratic imaginary fields using Euler systems of elliptic units, the work of Hida-Tilouine for anticyclotomic characters of general CM fields [18], the work of E.Urban [65] on symmetric square $\mathcal{L}$ functions, the work of Bertolini-Darmon [3] for anticyclotomic main conjecture for modular forms, and the recent work of Kato [25] and Skinner-Urban [61] which proves the Iwasawa main conjecture for ordinary elliptic curves $E/\mathbb{Q}$ (and this list is not complete). We briefly recall the formulation of [61]. Let $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$ with the Galois group denoted as $\Gamma_\mathbb{Q}$. Write $\Lambda_\mathbb{Q} := \mathbb{Z}_p[[\Gamma_\mathbb{Q}]]$. There is a p-adic L-function $\mathcal{L}_E$ interpolating the central critical values of the L-function for E. We define the Selmer group by

$$\mathrm{Sel}_E : \varinjlim_n \ker\{H^1(\mathbb{Q}_n, T_E \otimes \mathbb{Q}_p/\mathbb{Z}_p) \to \prod_{v\nmid p} H^1(I_v, T_E \otimes \mathbb{Q}_p/\mathbb{Z}_p) \times \prod_{v|p} H^1(\mathbb{Q}_v, T_E/T_E^+ \otimes \mathbb{Q}_p/\mathbb{Z}_p)\}$$

where T_E^+ is a rank one submodule of T_E stable under G_v such that the G_v-action is unramified on T_E/T_E^+. The dual Selmer group X_E being the Pontryagin dual of Sel_E. The Iwasawa main conjecture states that X_E is a torsion $\Lambda_\mathbb{Q}$-module and the characteristic ideal of X_E as a module over $\Lambda_\mathbb{Q}$ is generated by $\mathcal{L}_E$. In fact Kato proves one divisibility by constructing an Euler system while Skinner-Urban ([61]) proves the other divisibility using Eisenstein congruences on the larger unitary group $\mathrm{U}(2,2)$.

Now let us turn to the supersingular elliptic curve case. By Taniyama-Shimura conjecture proved by Wiles [72] and Breuil-Conrad-Diamond-Taylor [2] we know there is a normalized cuspidal eigenform $f = \sum_{n=1}^{\infty} a_n q^n$ associated to E. Suppose $a_p = 0$. (This is automatically true if $p \geq 5$). For a supersingular elliptic curve $E/\mathbb{Q}$, Kobayashi ([33]) reformulated Kato's result in terms of the $\pm$ p-adic L-functions $\mathcal{L}_E^{\pm}$ and $\pm$-Selmer groups (we recall in the text). The Iwasawa main conjecture for supersingular elliptic curves is

Conjecture 1.1. *The $\pm$-dual Selmer group $X_E^{\pm}$ is torsion over $\Lambda_{\mathbb{Q}}$ and the characteristic ideal of $X_E^{\pm}$ is generated by $\mathcal{L}_E^{\pm}$ as ideals of $\Lambda_{\mathbb{Q}}$.*

Here we define the characteristic ideal as follows

Definition 1.2. *Let A be a Noetherian normal domain and M a finitely generated A-module. Then the characteristic ideal* $\mathrm{char}_A(M)$ *of M is defined to be*

$$\{x \in A | \mathrm{ord}_P(x) \geq \mathrm{Length}_{A_P} M_P, \textit{for any height one prime } P \textit{ of } A\}.$$

If M is not A-torsion then we define it to be zero.

Before this work the only result for $\pm$-main conjecture is due to Pollack-Rubin ([49]) for CM elliptic curves. However the proof in *loc.cit* does not generalize to all supersingular elliptic curves. It is also a natural attempt to adapt the argument of [61] to the supersingular case. To make this work we similarly take an auxiliary quadratic imaginary field $\mathcal{K}$ such that p splits as $v_0\bar{v}_0$. However this turns out to be quite hard since we do not see any ways to pick up the $\pm$-part of the Selmer groups from Skinner-Urban's construction. Another possibility would be trying to prove the formuation of the main conjecture of Pottharst [47], which, instead of using the $\pm$ theory, studies one (unbounded) p-adic L-function for an U_p-eigenvector and the corresponding Selmer group. Then it is reasonable to believe that one needs to construct families of triangulations for the family of Galois representations on the eigenvariety of $\mathrm{U}(2,2)$. However it is not clear whether there are such families of triangulations at all points we need to study (see [37]). Moreover it seems hard to get the main conjecture before inverting p by this method. Finally the construction of the Eisenstein families and the study of the Fourier-Jacobi expansion in this case both require completely new ideas.

After some unsuccessful tries, a different idea came into our consideration. We first give some backgrounds on Greenberg's work on Iwasawa theory. At the moment suppose T is a geometric (i.e. potentially semistable) $\mathbb{Z}_p$-Galois representation of $G_{\mathbb{Q}}$ and $V := T \otimes \mathbb{Q}_p$. Then we have the Hodge-Tate decomposition

$$V \otimes \mathbb{C}_p = \oplus_i \mathbb{C}_p(i)^{h_i}$$

where $\mathbb{C}_p(i)$ is the i-th Tate twist and h_i is the multiplicity. Let d be the dimension of T and let $d^{\pm}$ be the dimensions of the subspaces whose eigenvalues of the complex conjugation c is ± 1. We assume

- $d^+ = \sum_{i>0} h_i$.

This is put by Greenberg as a p-adic version of the assumption that $L(T, 0)$ (in favorable situations when this makes sense) is critical in the sense of Deligne. Assume moreover the following Panchishkin condition

- There is a d^+-dimensional $\mathbb{Q}_p$-subspace V^+ of V which is stable under the action of the decomposition group G_p at p such that $V^+ \otimes \mathbb{C}_p = \oplus_{i>0}\mathbb{C}_p^{h_i}$.

Write $T^+ := V^+ \cap T$. Under this Panchishkin condition Greenberg defined the following local Selmer condition

$$H^1_f(\mathbb{Q}_p, V/T) = \mathrm{Ker}\{H^1(\mathbb{Q}_p, V/T) \to H^1(\mathbb{Q}_p, \frac{V/T}{V^+/T^+})\}.$$

In other words under the Panchishkin condition the local Selmer condition above is very analogous to the ordinary case, thus making the corresponding Iwasawa main conjecture (when an appropriate p-adic L-function is available) accessible to proof (especially the "lattice construction" discussed in [61, Chapter 4]). The following example is crucial for this paper.

Example 1.3. *Let f be a cuspidal eigenform of weight k and g be a CM form of weight k' with respect to a quadratic imaginary field $\mathcal{K}$ such that p splits. Then g is ordinary at p by definition. Assume $k + k'$ is an odd number. We consider critical values for Rankin-Selberg products $L(f, g, i)$ (which means $L(\rho_f \otimes \rho_g(-i), 0)$ if we write ρ_f and ρ_g for the corresponding Galois representations). We consider two possibilities:*
1. If $k > k'$, then the Panchishkin's condition is true if f is ordinary;
2. If $k' > k$, then the Panchishkin's condition is always true, regardless of whether f is ordinary or not. This can be seen as follows: we have $d^\pm = 2$, ρ_f and ρ_g have Hodge-Tate weights $(0, k-1)$ and $(0, k'-1)$ respectively. The L-values are critical when $k - 1 \leq i \leq k' - 1$. So for those i above $\rho_f \otimes \rho_g(-i)$ has two positive Hodge-Tate weights. On the other hand ρ_g as a $G_{\mathbb{Q}_p}$-representation is the direct sum of two characters. Thus the Panchishkin condition is easily seen.

In the case when f is nearly ordinary the result is proved in [67]. The first thing we do in this paper is prove this Greenberg main conjecture when f corresponds to the supersingular elliptic curve E (this is proved in Theorem 5.3). This theorem in itself has independent interest and has other arithmetic applications. As in [67], the p-adic L-function here appears as the constant of certain Klingen Eisenstein series on the group $\mathrm{U}(3, 1)$ and we make use of the Eisenstein congruences of them with cusp forms. The following new ingredients are important in our argument

- The construction in [9] of families of Klingen Eisenstein series from f and a CM character. This family is semi-ordinary in the sense that some (not all) U_p operators have p-adic units as eigenvalues.

- The above family sits in a two dimensional subspace of the three dimensional weight space for $\mathrm{U}(3, 1)$. The theory of families of semi-ordinary forms that we develop in Section 3.3 on this two dimensional space is essentially a "Hida theory" which are over the two dimensional Iwasawa algebra (instead of over a small affinoid disc as the Colman-Mazur theory. This observation is crucial since the Iwasawa main conjectures are formulated over the Iwasawa algebra.

Now let us go back to the proof of the $\pm$ main conjecture. We call the $\pm$-main conjecture (as extended by B.D. Kim to a two variable one) case one and the "Greenberg type" main conjectures case two. A surprising fact is, these ostensibly different main conjectures are actually equivalent (note that conjecture two does not involve any $\pm$ theory at all)! The Beilinson-Flach elements can be used to build a bridge between case one and case two. In fact the explicit reciprocity law (studied by Kings-Loeffler-Zerbes and Bertolini-Darmon-Rotger) enables us to reformulate the main conjectures in both cases in terms of Beilinson-Flach elements and in fact the new formulations for the two cases are the same. This means we can reduce the proof of one case to the other one. We note here that unlike Kato's zeta elements which are by definition in the bounded Iwasawa cohomology group, the Beilinson-Flach elements form an unbounded family in the non-ordinary case. Therefore we need to construct a bounded "+" Beilinson-Flach element from the unbounded Beilinson-Flach classes constructed by Lei-Loeffler-Zerbes, in the similar flavor as Pollack's work on constructing the $\pm$ p-adic L-functions. This is the very reason why a $\pm$-type main conjecture can be equivalent to a Greenberg type one. This finishes the proof of the lower bound for Selmer group in B.D. Kim's main conjecture. The conjecture of Kobayashi (cyclotomic main conjecture) follows from B.D. Kim's via an easy control theorem of Selmer groups. Our main result is

Theorem 1.4. *Suppose E has square-free conductor N, supersingular reduction at p and $a_p = 0$. Then Conjecture 1.1 is true.*

The square-free conductor assumption is put in [67] (can be removed if we would like to do some technical triple product computations). The assumption for $a_p = 0$ is primarily made for simplicity and we expect the same idea to work to prove the conjecture by F. Sprung [60] when $a_p \neq 0$. We also remark that although we work with supersingular case, however, even in the ordinary case, with the same idea we can deduce new cases of the two variable main conjectures considered in [61] (there the global sign is assumed to be $+1$ while we no longer need this assumption). Finally in the two variable case the upper bound for Selmer group is still missing since there are some technical obstacles (about level raising) to construct the Beilinson-Flach element Euler system in our context. Luckily such upper bound in one variable case is already provided by the work of Kato and Kobayashi.

In the text we will also prove the p-part of the refined BSD formula in the analytic rank 0 case as a corollary (Corollary 8.8). Therefore our result combined with the results in [61] and [59] gives the full refined BSD formula up to powers of 2, for a large class of semi-stable elliptic curves (when analytic rank is 0). The two-variable main conjecture we prove can also be used to deduce the anti-cyclotomic main conjecture of Darmon-Iovita ([6]). We leave this to industrious reader.

In the argument we prove the result for the $+$-main conjecture since the $-$-main conjecture is equivalent (by [33, Theorem 7.4]. This paper is organized as follows: in Section 2 we recall some backgrounds for automorphic forms and p-adic automorphic forms. In Section 3 we develop the theory of semi-ordinary forms and families, following ideas of [64] and arguments in [19, Section 4]. In Section 4 we construct the families of Klingen Eisenstein series using the calculations in [9] with some modifications. In Section 5 we make use of the calculations in [67], and then deduce the main conjecture for Rankin-Selberg products. In Section 6 we develop some local theory and recall the precise formulation of the B.D. Kim's two variable main conjecture. In Section 7 we recall the work of D. Loeffler et al on Beilinson-Flach elements, especially the explicit reciprocity law. We reinterpret these reciprocity laws in terms of the local theory in Section 6. In Secion 8 we put everything together and prove the main result using Poitou-Tate exact sequence. To treat powers of p we use a trick which appeals to Rubin's work on main conjecture for CM fields, and Hida-Tilouine's idea of constructing anti-cyclotomic Selmer group from congruence modules.

Notations:
We let E be an elliptic curve over $\mathbb{Q}$ and let f be the weight two cuspidal normalized eigenform associated to it by the Shimura-Taniyama conjecture of conductor N. Write

$$f=\sum_{n=1}^{\infty} a_n q^n$$

with $a_p=0$. Let T be the Tate module of E and $V=T\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Let $\alpha=\sqrt{-p}$. Then there are two eigenforms $f_\alpha, f_{-\alpha}$ of level Np for U_p-operator in the automorphic representation of f with eigenvalues $\alpha,-\alpha$. Let $\mathcal{K}$ be a quadratic imaginary field in which p splits as $v_0\bar{v}_0$. Let $\mathfrak{d}_{\mathcal{K}}$ be the absolute different of $\mathcal{K}/\mathbb{Q}$. We fix once for all an isomorphism $\iota_p:\mathbb{C}\simeq\mathbb{C}_p$ and suppose v_0 is induced by ι_p.

Let $\mathcal{K}_\infty$ be the unique $\mathbb{Z}_p^2$-extension of $\mathcal{K}$ with $\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$ denoted as $\Gamma_{\mathcal{K}}$. Let $\Lambda=\Lambda_{\mathcal{K}}=\mathbb{Z}_p[[\Gamma_{\mathcal{K}}]]$. We assume v_0 splits into p^t different primes in $\mathcal{K}_\infty$. Let $\mathcal{K}_{cyc}$ be the cyclotomic $\mathbb{Z}_p$ extension of $\mathcal{K}$. We write Γ for the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$ and U the Galois group of the unramified $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$. We fix topological generators γ and u of them with u being the arithmetic Frobenius. Let $\Gamma_n=\Gamma/p^n\Gamma$ and $U_m=U/p^mU$. Let $\Gamma_p\subseteq\Gamma_{\mathcal{K}}$ be the decomposition group of v_0 in $\Gamma_{\mathcal{K}}$. Then $[\Gamma_{\mathcal{K}}:\Gamma_p]=p^t$. We also define the maximal sub-extension $\mathcal{K}^{v_0}$ of $\mathcal{K}_\infty$ such that $\bar{v}_0$ is unramified and define $\mathcal{K}^{\bar{v}_0}$ similarly but switching the roles played by v_0 and $\bar{v}_0$. We define Γ_{v_0} as $\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}^{\bar{v}_0})$ and $\Gamma_{\bar{v}_0}$ as $\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}^{v_0})$. Let γ_{v_0} and $\gamma_{\bar{v}_0}$ be their topological generator. We also identify $U=U_{v_0}=U_{\bar{v}_0}=\mathrm{Gal}(\mathcal{K}_{\infty,v_0}/\mathcal{K}_{cyc,v_0})$. Let $\Gamma^-=\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}_{cyc})$ and γ^- be a topological generator. Let Ψ be the character $G_{\mathcal{K}}\to\Gamma_{\mathcal{K}}\to\Lambda_{\mathcal{K}}^\times$ and let $\mathcal{E}$ be Ψ composed with the reciprocity map in class field theory (normalized by the geometric Frobenius). Define $\Lambda_{\mathcal{K}}^*$ as the Pontryagin dual of $\Lambda_{\mathcal{K}}$. Define $\hat{\mathbb{Z}}_p^{ur}$ as the completion of the $\mathbb{Z}_p$-unramified extension of $\mathbb{Z}_p$. (In the literature it usually means the completion of the maximal unramified extension of $\mathbb{Z}_p$. But the $\mathbb{Z}_p$ is enough for our purposes).

We write $\Phi_m(X)=\sum_{i=1}^{p-1}X^{p^{m-1}i}$ for the p^m-th cyclotomic polynomial. Our α,β will be denoting any elements in the set $\{\pm\sqrt{-p}\}$. Sometimes we will precisely indicate that $\alpha=\sqrt{-p}$, $\beta=-\sqrt{-p}$. Fix a compatible system of roots of unity ζ_{p^n} such that $\zeta_{p^n}^p=\zeta_{p^{n-1}}$. For a character ω of $\mathbb{Q}_p^\times$ we define a ε factor of it as in [38, Page 8]: we define $\varepsilon(\omega)=1$ if it is unramified and

$$\varepsilon(\omega)=\int_{\mathbb{Q}_p^\times}\omega(x^{-1})\lambda(x)dx$$

otherwise. Here λ is an additive character of $\mathbb{Q}_p$ such that the kernel is $\mathbb{Z}_p$ and $\lambda(\frac{1}{p^n})=\zeta_{p^n}$. We can also define the ε factors for Galois characters via class field theory (p is mapped to the geometric Frobenius). For a primitive character of Γ/Γ_n we also define the Gauss sum $\mathfrak{g}(\omega):=\sum_{\gamma\in\Gamma/\Gamma_n}\omega(\gamma)\zeta_{p^n}^\gamma$.

We often write Σ for a finite set of primes containing all bad primes. If D is a quaternion algebra, we will sometimes write $[D^\times]$ for $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_{\mathbb{Q}})$. We similarly write $[\mathrm{U}(2)]$, $[\mathrm{GU}(2,0)]$, etc. We also define $S_n(R)$ to be the set of $n\times n$ Hermitian matrices with entries in $\mathcal{O}_{\mathcal{K}}\otimes_{\mathbb{Z}}R$. Finally we define $G_n=\mathrm{GU}(n,n)$ for the unitary similitude group for the skew-Hermitian matrix $\begin{pmatrix} & 1_n\\ -1_n & \end{pmatrix}$ and $\mathrm{U}(n,n)$ for the corresponding unitary groups. We write $e_{\mathbb{A}}=\prod_v e_v$ where for each place v of $\mathbb{Q}$ and e_v is the usual exponential map at v. We refer to [19] for the discussion of the CM period Ω_∞ and the p-adic period Ω_p. For two automorphic forms f_1, f_2 on U(2) we write $\langle f_1, f_2\rangle = \int_{[\mathrm{U}(2)]} f_1(g)f_2(g)dg$ (we use Shimura's convention for the Haar measures).

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2 Backgrounds

2.1 Greenberg's Main Conjecture

As remarked in the introduction our first step is to prove a Greenberg type main conjecture, which we formulate here. (This will be proved in Section 5). We will take a holomorphic cuspidal automorphic representation π of $\mathrm{GL}_2/\mathbb{Q}$ with even weight and a CM character ξ of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ with infinite type $(\kappa/2, -\kappa/2)$ for some even number $\kappa \geq 6$. Let $f \in \pi$ be the normalized newform and ρ_f the Galois representation of $G_\mathbb{Q}$ associated to it. (We will not assume π has weight two until Section 5). We first define the characteristic ideals and the Fitting ideals. We let A be a Noetherian ring. We write $\mathrm{Fitt}_A(X)$ for the Fitting ideal in A of a finitely generated A-module X. This is the ideal generated by the determinant of the $r \times r$ minors of the matrix giving the first arrow in a given presentation of X:

$$A^s \to A^r \to X \to 0.$$

If X is not a torsion A-module then $\mathrm{Fitt}_A(X) = 0$.

Fitting ideals behave well with respect to base change. For $I \subset A$ an ideal, then:

$$\mathrm{Fitt}_{A/I}(X/IX) = \mathrm{Fitt}_A(X) \bmod I$$

Now suppose A is a Krull domain (a domain which is Noetherian and normal), then the characteristic ideal is defined by:

$$\mathrm{char}_A(X) := \{x \in A : \mathrm{ord}_Q(x) \geq \mathrm{length}_Q(X) \text{ for any } Q \text{ a height one prime of } A\},$$

Again if X is not torsion then we define $\mathrm{char}_A(X) = 0$.

We consider the Galois representation:

$$V_{f,\mathcal{K},\xi} := \rho_f \sigma_{\bar{\xi}^c} \epsilon^{\frac{4-\kappa}{2}} \otimes \Lambda_\mathcal{K}(\Psi_\mathcal{K}^{-c}).$$

Define the Selmer group to be:

$$\mathrm{Sel}_{f,\mathcal{K},\xi} := \ker\{H^1(\mathcal{K}, T_{f,\mathcal{K},\xi}\otimes\mathcal{O}_L[[\Gamma_\mathcal{K}]]^*) \to H^1(I_{\bar{v}_0}, T_{f,\mathcal{K},\xi}\otimes\mathcal{O}_L[[\Gamma_\mathcal{K}]]^*)\times\prod_{v\nmid p} H^1(I_v, T_{f,\mathcal{K},\xi}\otimes\mathcal{O}_L[[\Gamma_\mathcal{K}]]^*)\}$$

where $*$ means Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(-, \mathbb{Q}_p/\mathbb{Z}_p)$ and the Σ-primitive Selmer groups:

$$\mathrm{Sel}^\Sigma_{f,\mathcal{K},\xi} := \ker\{H^1(\mathcal{K}, T_{f,\mathcal{K},\xi}\otimes\mathcal{O}_L[[\Gamma_\mathcal{K}]]^*) \to H^1(I_{\bar{v}_0}, T_{f,\mathcal{K},\xi}\otimes\mathcal{O}_L[[\Gamma_\mathcal{K}]]^*)\times\prod_{v\notin\Sigma} H^1(I_v, T_{f,\mathcal{K},\xi}\otimes\mathcal{O}_L[[\Gamma_\mathcal{K}]]^*)\}$$

and

$$X^{\Sigma}_{f,\mathcal{K},\xi} := (\mathrm{Sel}^{\Sigma}_{f,\mathcal{K},\xi})^*.$$

We are going to define the p-adic L-functions $\mathcal{L}_{f,\mathcal{K},\xi}$ and $\mathcal{L}^{\Sigma}_{f,\mathcal{K},\xi}$ (which are elements in $\mathrm{Frac}(\mathcal{O}_L^{ur}[[\Gamma_{\mathcal{K}}]])$) in section 4. The two-variable Iwasawa main conjecture and its Σ-imprimitive version is the following (see [11]).

Conjecture 2.1.

$$\mathrm{char}_{\mathcal{O}_L^{ur}[[\Gamma_{\mathcal{K}}]]} X_{f,\mathcal{K},\xi} = (\mathcal{L}_{f,\mathcal{K},\xi}),$$
$$\mathrm{char}_{\mathcal{O}_L^{ur}[[\Gamma_{\mathcal{K}}]]} X^{\Sigma}_{f,\mathcal{K},\xi} = (\mathcal{L}^{\Sigma}_{f,\mathcal{K},\xi}).$$

2.2 Groups

Let $\delta \in \mathcal{K}$ be a totally imaginary element such that $-i\delta$ is positive. Let $d = \mathrm{Nm}(\delta)$ which we assume to be a p-adic unit. Let $\mathrm{U}(2) = \mathrm{U}(2,0)$ (resp. $\mathrm{GU}(2) = \mathrm{GU}(2,0)$) be the unitary group (resp. unitary similitude group) associated to the skew-Hermitian matrix $\zeta = \begin{pmatrix} \mathfrak{s}\delta & \\ & \delta \end{pmatrix}$ for some $\mathfrak{s} \in \mathbb{Z}_+$ prime to p. More precisely $\mathrm{GU}(2)$ is the group scheme over $\mathbb{Z}$ defined by: for any $\mathbb{Z}$ algebra A,

$$\mathrm{GU}(2)(A) = \{g \in \mathrm{GL}_2(A \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{K}}) | {}^t\bar{g}\zeta g = \lambda(g)\zeta,\ \lambda(g) \in A^{\times}.\}$$

The map $\mu : \mathrm{GU}(2) \to \mathbb{G}_m,\ g \mapsto \lambda(g)$ is called the similitude character and $\mathrm{U}(2) \subseteq \mathrm{GU}(2)$ is the kernel of μ. Let W be the corresponding Hermitian space over $\mathcal{K}$ and fix a lattice $L \subset W$ over $\mathcal{O}_{\mathcal{K}}$ such that $\mathrm{Tr}_{\mathcal{K}/\mathbb{Q}}\langle L, L\rangle \subset \mathbb{Z}$. Let $G = \mathrm{GU}(3,1)$ (resp. $\mathrm{U}(3,1)$) be the similarly defined unitary similitude group (resp. unitary group) over $\mathbb{Z}$ associated to the skew-Hermitian matrix $\begin{pmatrix} & & 1 \\ & \zeta & \\ -1 & & \end{pmatrix}$. We write its corresponding Hermitian space as $V = X_{\mathcal{K}} \oplus W \oplus Y_{\mathcal{K}}$ where W is the Hermitian space for $\mathrm{GU}(2)$ and $X_{\mathcal{K}}$ and $Y_{\mathcal{K}}$ are one dimensional $\mathcal{K}$-spaces with standard basis x^1 and y^1. Let $X^{\vee} = \mathfrak{d}_{\mathcal{K}}^{-1}x^1$ and $Y = \mathcal{O}_{\mathcal{K}}y^1$ and we call $X^{\vee} \oplus L \oplus Y$ the standard lattice of V. Let $P \subseteq G$ be the parabolic subgroup of $\mathrm{GU}(3,1)$ consisting of those matrices in G of the form $\begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & & & \times \end{pmatrix}$. Let N_P be the unipotent radical of P. Then

$$M_P := \mathrm{GL}(X_{\mathcal{K}}) \times \mathrm{GU}(2) \hookrightarrow \mathrm{GU}(V),\ (a, g_1) \mapsto \mathrm{diag}(a, g_1, \mu(g_1)\bar{a}^{-1})$$

is the Levi subgroup. Let $G_P := \mathrm{GU}(2)(\subseteq M_P) \mapsto \mathrm{diag}(1, g_1, \mu(g))$. Let δ_P be the modulus character for P. We usually use a more convenient character δ such that $\delta^3 = \delta_P$.

Since p splits as $v_0\bar{v}_0$ in $\mathcal{K}$, $\mathrm{GL}_4(\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{v_0}}) \times \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{\bar{v}_0}})$. Here $\mathrm{U}(3,1)(\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{v_0}}) = \mathrm{GL}_4(\mathbb{Z}_p)$ with the projection onto the first factor. Let B and N be the upper triangular Borel subgroup of G and its unipotent radical, respectively. Let $K_p = \mathrm{GU}(3,1)(\mathbb{Z}_p) \simeq \mathrm{GL}_4(\mathbb{Z}_p)$, and for any $n \geq 1$ let K_0^n be the subgroup of K consisting of matrices upper-triangular modulo p^n. Let $K_1^n \subset K_0^n$ be the subgroup of matrices whose diagonal elements are 1 modulo p^n.

The group GU(2) is closely related to a division algebra. Put

$$D = \{g \in M_2(\mathcal{K}) | g^t \zeta \bar{g} = \det(g)\zeta\},$$

then D is a definite quaternion algebra over $\mathbb{Q}$ with local invariants $\mathrm{inv}_v(D) = (-\mathfrak{s}, -D_{\mathcal{K}/\mathbb{Q}})_v$ (the Hilbert symbol). The relation between GU(2) and D is explained by

$$\mathrm{GU}(2) = D^\times \times_{\mathbb{G}_m} \mathrm{Res}_{\mathcal{K}/\mathbb{Q}} \mathbb{G}_m.$$

For each finite place v we write D_v^1 for the set of elements $g_v \in D_v^\times$ such that $|\mathrm{Nm}(g_v)|_v = 1$, where Nm is the reduced norm.

Let Σ be a finite set of primes containing all the primes at which $\mathcal{K}/\mathbb{Q}$ or π or ξ is ramified, the primes dividing $\mathfrak{s}$, the primes such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is compact and the prime 2. Let Σ^1 and Σ^2, respectively be the set of non-split primes in Σ such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is non-compact, and compact.

We define $G_n = \mathrm{GU}(n, n)$ for the unitary similitude group for the skew-Hermitian matrix $\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$ and $\mathrm{U}(n, n)$ for the corresponding unitary groups.

2.3 Hermitian Spaces and Automorphic Forms

Let $(r, s) = (3, 3)$ or $(3, 1)$ or $(2, 0)$. Then the unbounded Hermitian symmetric domain for $\mathrm{GU}(r, s)$ is

$$X^+ = X_{r,s} = \{\tau = \begin{pmatrix} x \\ y \end{pmatrix} | x \in M_s(\mathbb{C}), y \in M_{(r-s)\times s}(\mathbb{C}), i(x^* - x) > iy^*\zeta^{-1}y\}.$$

We use x_0 to denote the Hermitian symmetric domain for GU(2), which is just a point. We have the following embedding of Hermitian symmetric domains:

$$\iota : X_{3,1} \times X_{2,0} \hookrightarrow X_{3,3}$$

$$(\tau, x_0) \hookrightarrow Z_\tau,$$

where $Z_\tau = \begin{pmatrix} x & 0 \\ y & \frac{\zeta}{2} \end{pmatrix}$ for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $G(r, s) = \mathrm{GU}(r, s)$ and $H = H_{r,s} = \mathrm{GL}_r \times \mathrm{GL}_s$. Let $G_{r,s}(\mathbb{R})^+$ be the subgroup of elements of $G_{r,s}(\mathbb{R})$ whose similitude factors are positive. If $s \neq 0$ we define a cocycle:

$$J : G_{r,s}(\mathbb{R})^+ \times X^+ \to H_{r,s}(\mathbb{C})$$

by $J(\alpha, \tau) = (\kappa(\alpha, \tau), \mu(\alpha, \tau))$, where for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\alpha = \begin{pmatrix} a & b & c \\ g & e & f \\ h & l & d \end{pmatrix}$ (blocks matrix with respect to the partition $(s + (r - s) + s)$),

$$\kappa(\alpha, \tau) = \begin{pmatrix} \bar{h}^t x + \bar{d} & \bar{h}^t y + l\bar{\zeta} \\ -\bar{\zeta}^{-1}(\bar{g}^t x + \bar{f}) & -\bar{\zeta}^{-1}\bar{g}^t y + \bar{\zeta}^{-1}\bar{e}\bar{\zeta} \end{pmatrix}, \ \mu(\alpha, \tau) = hx + ly + d$$

in the $\mathrm{GU}(3,1)$ case and

$$\kappa(\alpha,\tau)=\bar{h}^t x+\bar{d},\ \ \mu(\alpha,\tau)=hx+d$$

in the $\mathrm{GU}(3,3)$ case. Let $i\in X^+$ be the point $\begin{pmatrix} i1_s \\ 0 \end{pmatrix}$. Let K_∞^+ be the compact subgroup of $\mathrm{U}(r,s)(\mathbb{R})$ stabilizing i and let K_∞ be the groups generated by K_∞^+ and $\mathrm{diag}(1_{r+s},-1_s)$. Then

$$K_\infty^+\to H(\mathbb{C}),\ \ k_\infty\mapsto J(k_\infty,i)$$

defines an algebraic representation of K_∞^+.

Definition 2.2. *A weight $\underline{k}$ is defined to be an $(r+s)$-tuple*

$$\underline{k}=(a_1,\cdots,a_r;b_1,\cdots,b_s)\in\mathbb{Z}^{r+s}$$

with $a_1\geq\cdots\geq a_r\geq -b_1\geq\cdots-b_s$.

We refer to [19, Section 3.1] for the definition of the algebraic representation $L_{\underline{k}}(\mathbb{C})$ of H with the action denoted by $\rho_{\underline{k}}$ (note the different index for weight) and define a model $L^{\underline{k}}(\mathbb{C})$ of the representation $H(\mathbb{C})$ with the highest weight $\underline{k}$ as follows. The underlying space of $L^{\underline{k}}(\mathbb{C})$ is $L_{\underline{k}}(\mathbb{C})$ and the group action is defined by

$$\rho^{\underline{k}}(h)=\rho_{\underline{k}}({}^t h^{-1}),h\in H(\mathbb{C}).$$

We also note that if each $\underline{k}=(0,...,0;\kappa,...,\kappa)$ then $L^{\underline{k}}(\mathbb{C})$ is one dimensional.

For a weight $\underline{k}$, define $\|\underline{k}\|=\|\underline{k}\|$ by:

$$\|\underline{k}\|:=a_1+\cdots+a_r+b_1+\cdots+b_s$$

and $|\underline{k}|$ by:

$$|\underline{k}|=(b_1+\cdots+b_s)\cdot\sigma+(a_1+\cdots+a_r)\cdot\sigma c\in\mathbb{Z}^I.$$

Here I is the set of embeddings $\mathcal{K}\hookrightarrow\mathbb{C}$ and σ is the Archimedean place of $\mathcal{K}$ determined by our fixed embedding $\mathcal{K}\hookrightarrow\mathbb{C}$. Let χ be a Hecke character of $\mathcal{K}$ with infinite type $|\underline{k}|$, i.e. the Archimedean part of χ is given by:

$$\chi_\infty(z)=(z^{(b_1+\cdots+b_s)}\cdot\bar{z}^{+(a_1+\cdots+a_r)}).$$

Definition 2.3. *Let U be an open compact subgroup in $G(\mathbb{A}_f)$. We denote by $M_{\underline{k}}(U,\mathbb{C})$ the space of holomorphic $L^{\underline{k}}(\mathbb{C})$-valued functions f on $X^+\times G(\mathbb{A}_f)$ such that for $\tau\in X^+$, $\alpha\in G(\mathbb{Q})^+$ and $u\in U$ we have:*

$$f(\alpha\tau,\alpha gu)=\mu(\alpha)^{-\|\underline{k}\|}\rho^{\underline{k}}(J(\alpha,\tau))f(\tau,g).$$

Now we consider automorphic forms on unitary groups in the adelic language. The space of automorphic forms of weight $\underline{k}$ and level U with central character χ consists of smooth and slowly increasing functions $F:G(\mathbb{A})\to L_{\underline{k}}(\mathbb{C})$ such that for every $(\alpha,k_\infty,u,z)\in G(\mathbb{Q})\times K_\infty^+\times U\times Z(\mathbb{A})$,

$$F(z\alpha gk_\infty u)=\rho^{\underline{k}}(J(k_\infty,\boldsymbol{i})^{-1})F(g)\chi^{-1}(z).$$

We can associate a $L_{\underline{k}}$-valued function on $X^+\times G(\mathbb{A}_f)/U$ by

$$f(\tau,g):=\chi_f(\mu(g))\rho^{\underline{k}}(J(g_\infty,\mathbf{i}))F((f_\infty,g))$$

where $g_\infty\in G(\mathbb{R})$ such that $g_\infty(\mathbf{i})=\tau$. If this function is holomorphic then we say that the automorphic form F is holomorphic.

2.4 Galois representations Associated to Cuspidal Representations

In this section we follow [58] to state the result of associating Galois representations to cuspidal automorphic representations on $\mathrm{GU}(r,1)(\mathbb{A}_F)$. Let $n=r+1$. First of all let us fix the notations. Let $\bar{\mathcal{K}}$ be the algebraic closure of $\mathcal{K}$ and let $G_{\mathcal{K}}:=\mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$. For each finite place v of $\mathcal{K}$ let $\bar{\mathcal{K}}_v$ be an algebraic closure of $\mathcal{K}_v$ and fix an embedding $\bar{\mathcal{K}}\hookrightarrow\bar{\mathcal{K}}_v$. The latter identifies $G_{\mathcal{K}_v}:=\mathrm{Gal}(\bar{\mathcal{K}}_v/\mathcal{K}_v)$ with a decomposition group for v in $G_{\mathcal{K}}$ and hence the Weil group $W_{\mathcal{K}_v}\subset G_{\mathcal{K}_v}$ with a subgroup of $G_{\mathcal{K}}$. Let π be a holomorphic cuspidal irreducible representation of $\mathrm{GU}(r,1)(\mathbb{A}_F)$ with weight $\underline{k}=(a_1,\cdots,a_r;b_1,\cdots,b_s)$ and central character χ_π. Let $\Sigma(\pi)$ be a finite set of primes of F containing all the primes at which π is ramified and all the primes dividing p. Then for some L finite over $\mathbb{Q}_p$, there is a Galois representation (by [56], [40] and [58]):

$$R_p(\pi):G_{\mathcal{K}}\to\mathrm{GL}_n(L)$$

such that:
(a)$R_p(\pi)^c\simeq R_p(\pi)^\vee\otimes\rho_{p,\chi_\pi^{1+c}}\epsilon^{1-n}$, $\rho_{p,\chi_\pi^{1+c}}$ denotes the associated Galois character by class field theory and ϵ is the cyclotomic character.
(b)$R_p(\pi)$ is unramified at all finite places not above primes in $\Sigma(\pi)\cup\{$ primes dividing p), and for such a place w:

$$\det(1-R_p(\pi)(\mathrm{frob}_w q_w^{-s}))=L(BC(\pi)_w\otimes\chi^c_{\pi,w},s+\frac{1-n}{2})^{-1}$$

Here the frob_w is the geometric Frobenius and BC means the base change from $\mathrm{U}(r,1)$ to GL_{r+1}. We write V for the representation space and it is possible to take a Galois stable $\mathcal{O}_L$ lattice which we denote as T. One subtle point here is that Skinner only proved the result for automorphic forms of regular weight. We use a simple trick here to deduce it for all cohomological weights. As explained in [58], there is a "very weak base change" of π to $\mathrm{GL}_n/\mathcal{K}$ in the sense that outside a finite set of primes S containing all bad primes, its local component is the local base change of π. The terminology "very weak" means that the S might be strictly larger than the set of bad primes (i.e primes where $\mathcal{K}$ or π is ramified). It suffices to show that the very weak base change is actually locally the base change at all good primes. We use the method of eigenvarieties to deduce this. For any good prime ℓ which is in S, we take an auxiliary split prime q outside S and deform π in an r-dimensional finite slope q-adic family (i.e. over the whole weight space) of cuspidal eigenforms F over some rigid analytic affinoid X. This can be achieved by applying the result in [45], and the construction for unitary group is done in [46]. The family F interpolates a Zariski dense set $\mathcal{Z}$ of cuspidal eigenforms on $\mathrm{GU}(r,1)$ of regular weight (the classicality at sufficiently regular weight is proved in [46]). Moreover by [1, Lemma 7.8.11], by passing to a finite cover of X followed by a blow up, the rigid space carries a rigid analytic family $\mathcal{M}$ of q-adic Galois representations of $G_{\mathcal{K}}$ which interpolates the Galois representations associated to the forms corresponding to points in $\mathcal{Z}$. The Galois representation ρ_π associated to π is the one associated to the very weak base change, and is also the one obtained from specializing $\mathcal{M}$ to the point corresponding to π. From the latter interpretation we see ρ_π restricting to $G_{\mathcal{K}_\ell}$ is unramified and corresponds to the base change of π_ℓ under the local Langlands correspondence (this is seen by using the local-global compatibility at regular weights, and the Zariski density of $\mathcal{Z}$). But this also corresponds to the very weak base change of π at ℓ under the local Langlands correspondence. These imply what we need.

3 Hida Theory for Semi-Ordinary Forms

3.1 Shimura varieties for Unitary Similitude Groups

We will be brief in the following and refer the details to [19, Section 2, 3] (see also [9, Section 2]). Now we consider the group $\mathrm{GU}(3,1)$. For any open compact subgroup $K = K_pK^p$ of $\mathrm{GU}(3,1)(\mathbb{A}_f)$ whose p-component is $K_p = \mathrm{GU}(3,1)(\mathbb{Z}_p)$, we refer to [19, Section 2.1] for the definition and arithmetic models of the associated Shimura variety, which we denote as $S_G(K)_{/\mathcal{O}_{\mathcal{K},(v_0)}}$. The scheme $S_G(K)$ represents the following functor: for any $\mathcal{O}_{\mathcal{K},(v_0)}$-algebra R, $\underline{A}(R) = \{(A, \bar{\lambda}, \iota, \bar{\eta}^p)\}$ where A is an abelian scheme over R with CM by $\mathcal{O}_{\mathcal{K}}$ given by ι, $\bar{\lambda}$ is an orbit of prime-to-p polarizations and $\bar{\eta}^p$ is an orbit of prime-to-p level structures. There is also a theory of compactifications of $S_G(K)$ developed in [35]. We denote $\bar{S}_G(K)$ the toroidal compactification and $S^*_G(K)$ the minimal compactification. We refer to [19, Section 2.7] for details. The boundary components of $S^*_G(K)$ is in one-to-one correspondence with the set of cusp labels defined below. For $K = K_pK^p$ as above we define the set of cusp labels to be:

$$C(K) := (\mathrm{GL}(X_{\mathcal{K}}) \times G_P(\mathbb{A}_f))N_P(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K.$$

This is a finite set. We denote by $[g]$ the class represented by $g \in G(\mathbb{A}_f)$. For each such g whose p-component is 1 we define $K_P^g = G_P(\mathbb{A}_f) \cap gKg^{-1}$ and denote $S_{[g]} := S_{G_P}(K_P^g)$ the corresponding Shimura variety for the group G_P with level group K_P^g. By strong approximation we can choose a set $\underline{C}(K)$ of representatives of $C(K)$ consisting of elements $g = pk^0$ for $p \in P(\mathbb{A}_f^{(\Sigma)})$ and $k^0 \in K^0$ for K^0 the maximal compact subgroup of $G(\mathbb{A}_f)$ defined in [19, Section 1.10].

3.2 Igusa varieties and p-adic automorphic forms

Now we recall briefly the notion of Igusa varieties in [19, Section 2.3]. Let M be the standard lattice of V and $M_p = M \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let $\mathrm{Pol}_p = \{N^{-1}, N^0\}$ be a polarization of M_p. Recall that this means that if N^{-1} and N^0 are maximal isotropic $\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_p$-submodules in M_p, that they are dual to each other with respect to the Hermitian metric on V, and also that:

$$\mathrm{rank}_{\mathbb{Z}_p} N^{-1}_{v_0} = \mathrm{rank}_{\mathbb{Z}_p} N^0_{\bar{v}_o} = 3, \mathrm{rank}_{\mathbb{Z}_p} N^{-1}_{\bar{v}_0} = \mathrm{rank}_{\mathbb{Z}_p} N^0_{v_0} = 1.$$

We mainly follow [19, Section 2.3] in this subsection. The Igusa variety of level p^n is the scheme over $\mathcal{O}_{\mathcal{K},(v_0)}$ representing the quadruple $\underline{A}(R) = \{(A, \bar{\lambda}, \iota, \bar{\eta}^p)\}$ for Shimura variety of $\mathrm{GU}(3,1)$ as above, together with an injection of group schemes

$$j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$$

over R which is compatible with the $\mathcal{O}_{\mathcal{K}}$-action on both hand sides. Note that the existence of j implies that A must be ordinary along the special fiber. There is also a theory of Igusa varieties over $\bar{S}_G(K)$. As in *loc.cit* let $\bar{H}_{p-1} \in H^0(S_G(K)_{/\bar{\mathbb{F}}}, \det(\underline{\omega})^{p-1})$ be the Hasse invariant. Over the minimal compactification some power (say the tth) of the Hasse invariant can be lifted to $\mathcal{O}_{v_0}$. We denote such a lift by E. By the Koecher principle we can regard E as in $H^0(\bar{S}_G(K), \det(\underline{\omega}^{t(p-1)}))$. Let $\mathcal{O}_m := \mathcal{O}_{\mathcal{K},v_0}/p^m\mathcal{O}_{\mathcal{K},v_0}$. Set $T_{0,m} := \bar{S}_G(K)[1/E]_{/\mathcal{O}_m}$. For any positive integer n define $T_{n,m} := I_G(K^n)_{/\mathcal{O}_m}$ and $T_{\infty,m} = \varprojlim_n T_{n,m}$. Then $T_{\infty,m}$ is a Galois cover over $T_{0,m}$ with Galois group $\mathbf{H} \simeq \mathrm{GL}_3(\mathbb{Z}_p) \times \mathrm{GL}_1(\mathbb{Z}_p)$. Let $\mathbf{N} \subset \mathbf{H}$ be the upper triangular unipotent radical. Define:

$$V_{n,m} = H^0(T_{n,m}, \mathcal{O}_{T_{n,m}}).$$

Let $V_{\infty,m} = \varinjlim_n V_{n,m}$ and $V_{\infty,\infty} = \varprojlim_m V_{\infty,m}$ be the space of p-adic automorphic forms on $\mathrm{GU}(3,1)$ with level K. We also define $W_{n,m} = V_{n,m}^{\mathbf{N}}$, $W_{\infty,m} = V_{\infty,m}^{\mathbf{N}}$ and $\mathcal{W} = \varinjlim_n \varinjlim_m W_{n,m}$. We define $V_{n,m}^0$, etc, to be the cuspidal part of the corresponding spaces.

We can make similar definitions for the definite unitary similitude groups G_P as well and define $V_{n,m,P}$,$V_{\infty,m,P}$, $V_{\infty,\infty,P}$, $V_{n,m,P}^{\mathbf{N}}$, $\mathcal{W}_P$, etc.

Let K_0^n and K_1^n be the subgroup of $\mathbf{H}$ consisting of matrices which are in $B_3 \times {}^tB_1$ or $N_3 \times {}^tN_1$ modulo p^n. (These notations are already used for level groups of automorphic forms. The reason for using the same notation here is that automorphic forms with level group $K_\bullet^n$ are p-adic automorphic forms of level group $K_\bullet^n$). We sometimes denote $I_G(K_1^n) = I_G(K^n)^{K_1^n}$ and $I_G(K_0^n) = I_G(K^n)^{K_0^n}$.

We can define the Igusa varieties for G_P as well. For $\bullet = 0, 1$ we let $K_{P,\bullet}^{g,n} := gK_\bullet^n g^{-1} \cap G_P(\mathbb{A}_f)$ and let $I_{[g]}(K_\bullet^n) := I_{G_P}(K_{P,\bullet}^{g,n})$ be the corresponding Igusa variety over $S_{[g]}$. We denote $A_{[g]}^n$ the coordinate ring of $I_{[g]}(K_1^n)$. Let $A_{[g]}^\infty = \varinjlim_n A_{[g]}^n$ and let $\hat{A}_{[g]}^\infty$ be the p-adic completion of $A_{[g]}^\infty$. This is the space of p-adic automorphic forms for the group $\mathrm{GU}(2,0)$ of level group $gKg^{-1} \cap G_P(\mathbb{A}_f)$.

For Unitary Groups

Assume the tame level group K is neat. For any c an element in $\mathbb{Q}_+ \backslash \mathbb{A}_{\mathbb{Q},f}^\times / \mu(K)$, we refer to [19, 2.5] for the notion of c-Igusa schemes $I_{\mathrm{U}(2)}^0(K, c)$ for the unitary groups $\mathrm{U}(2,0)$ (not the similitude group). It parameterizes quintuples $(A, \lambda, \iota, \bar{\eta}^{(p)}, j)_{/S}$ similar to the Igusa schemes for unitary similitude groups but requires λ to be a prime to p c-polarization of A such that $(A, \bar{\lambda}, \iota, \bar{\eta}^{(p)}, j)$ is a quintuple as in the definition of Shimura varieties for $\mathrm{GU}(2)$. Let g_c be such that $\mu(g_c) \in \mathbb{A}_{\mathbb{Q}}^\times$ is in the class of c. Let ${}^cK = g_c K g_c^{-1} \cap U(2)(\mathbb{A}_{\mathbb{Q},f})$. Then the space $I_{\mathrm{U}(2)}^0(K, c)$ is isomorphic to the space of forms on $I_{\mathrm{U}(2)}^0({}^cK, 1)$ (see *loc.cit*).

Embedding of Igusa Schemes

In order to use the pullback formula algebraically we need a map from the Igusa scheme of $\mathrm{U}(3,1) \times \mathrm{U}(0,2)$ to that of $\mathrm{U}(3,3)$ (or from the Igusa scheme of $\mathrm{U}(2,0) \times \mathrm{U}(0,2)$ to that of $\mathrm{U}(2,2)$) given by:

$$i([(A_1, \lambda_1, \iota_1, \eta_1^p K_1, j_1)], [(A_2, \lambda_2, \iota_2, \eta_2^p K_2, j_2)]) = [(A_1 \times A_2, \lambda_1 \times \lambda_2, \iota_1, \iota_2, (\eta_1^p \times \eta_2^p) K_3, j_1 \times j_2)].$$

We define an element $\Upsilon \in \mathrm{U}(3,3)(\mathbb{Q}_p)$ such that $\Upsilon_{v_0} = S_{v_0}^{-1}$ and $\Upsilon'_{v_0} = S_{v_0}^{-1,'}$. Similar to [19], we know that under the complex uniformization, taking the change of polarization into consideration the above map is given by

$$i([\tau, g], [x_0, h]) = [Z_\tau, (g, h)\Upsilon]$$

(see [19, Section 2.6].)

Fourier-Jacobi Expansions

Define $N_H^1 := \{\begin{pmatrix} 1 & 0 \\ * & 1_2 \end{pmatrix}\} \times \{1\} \subset H$. For an automorphic form or p-adic automorphic form F on $\mathrm{GU}(3,1)$ we refer to [9, Section 2.8] for the notion of analytic Fourier-Jacobi expansions

$$FJ_P(g, f) = a_0(g, f) + \sum_\beta a_\beta(y, g, f) q^\beta$$

at $g\in \mathrm{GU}(3,1)(\mathbb{A}_{\mathbb{Q}})$ for $a_\beta(-,g,f):\mathbb{C}^2\to L_{\underline{k}}(\mathbb{C})$ being theta functions with complex multiplication, and algebraic Fourier-Jacobi expansion

$$FJ^h_{[g]}(f)_{N^1_H}=\sum_\beta a^h_{[g]}(\beta,f)q^\beta,$$

at a p-adic cusp $([g],h)$, and $a^h_{[g]}(\beta,f)\in L_{\underline{k}}(A^\infty_{[g]})_{N^1_H}\otimes_{A_{[g]}}H^0(\mathcal{Z}^\circ_{[g]},\mathcal{L}(\beta))$. We define the Siegel operator to be taking the 0-th Fourier-Jacobi coefficient as in *loc.cit.* Over $\mathbb{C}$ the analytic Fourier-Jacobi expansion for a holomorphic automorphic form f is given by:

$$FJ_\beta(f,g)=\int_{\mathbb{Q}\backslash\mathbb{A}} f(\begin{pmatrix}1 & & n\\ & 1_2 & \\ & & 1\end{pmatrix}g)e_{\mathbb{A}}(-\beta n)dn.$$

3.3 Semi-Ordinary Forms

3.3.1 Definitions

In this subsection we develop a theory for families of "semi-ordinary" forms over a two dimensional weight space (the whole weight space for $\mathrm{U}(3,1)$ is three dimensional). The idea goes back to the work of Hida [16] (also [64]) where they defined the concept of being ordinary with respect to different parabolic subgroups (the usual definition of ordinary is with respect to the Borel subgroup), except that we are working with coherent cohomology while Hida and Tilouine-Urban used group cohomology. In our case it means being ordinary with respect to the parabolic subgroup of GL_4 consisting of matrices of the form $\begin{pmatrix}* & * & * & *\\ * & * & * & *\\ & & * & *\\ & & & *\end{pmatrix}$. The crucial point is, our families are over the two dimensional Iwasawa algebra, which is similar to Hida theory for ordinary forms (instead of Coleman-Mazur theory for finite slope forms). Our argument here will mostly be an adaption of the argument in the ordinary case in [19] and we will sometimes be brief and refer to *loc.cit* for some computations so as not to introduce too many notations.

We always use the identification $\mathrm{U}(3,1)(\mathbb{Q}_v)\simeq \mathrm{GL}_4(\mathbb{Q}_p)$. We define $\alpha_i=\mathrm{diag}(1_{4-i},p\cdot 1_i)$. We let $\alpha=\begin{pmatrix}1 & & & \\ & 1 & & \\ & & p & \\ & & & p^2\end{pmatrix}$ and refer to [19, 3.7, 3.8] for the notion of Hida's U_α and U_{α_i} operators associated to α or α_i. We define $e_\alpha=\lim_{n\to\infty}U_\alpha^{n!}$. We are going to study forms and families invariant under e_α and call them "semi-ordinary" forms. Suppose π is an irreducible automorphic representation on $\mathrm{U}(3,1)$ with weight $\underline{k}$ and suppose that π_p is an unramified principal series representation. If we write $\kappa_1=b_1$ and $\kappa_i=-a_{5-i}+5-i$ for $2\le i\le 4$, then there is a semi-ordinary vector in π if and only if we can re-order the Satake parameters as $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ such that

$$\mathrm{val}_p(\lambda_3)=\kappa_3-\frac{3}{2},\mathrm{val}_p(\lambda_4)=\kappa_4-\frac{3}{2}.$$

Galois Representations

The Galois representations associated to cuspidal automorphic representation π in subsection 2.4 which is unramified and semi-ordinary at p for e_α has the following description when restricting to G_{v_0}:

$$R_p(\pi)|_{G_{v_0}} \simeq \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & \xi_{2,v}\epsilon^{-\kappa_2} & * \\ & & & \xi_{1,v}\epsilon^{-\kappa_1} \end{pmatrix} \quad (1)$$

where $\xi_{1,v}$ and $\xi_{2,v}$ are unramified characters and also

$$R_p(\pi)|_{G_{\bar{v}_0}} \simeq \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}.$$

This can be proved by noting that the Newton Polygon and the Hodge Polygon have four out of five vertices coincide (see [64, Proposition7.1]).

3.3.2 Control Theorems

We define $K_0(p,p^n)$ to be the level group with the same components at primes outside p as K and, at p, consists of matrices which are of the form $\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$ modulo p and are of the form $\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$ modulo p^n. We are going to prove some control theorems for the level group $K_0(p,p^n)$. These will be enough to show that the Eisenstein series constructed in [9] do give families in the sense here. (See Section 4.) We refer the definition of the automorphic sheaves $\omega_{\underline{k}}$ of weight $\underline{k}$ and the subsheaf to [19, section 3.2]. There also defined a $\omega^\flat_{\underline{k}}$ in Section 4.1 of *loc.cit* as follows. Let $\mathcal{D} = \bar{S}_G(K) - S_G(K)$ be the boundary of the toroidal compactification and $\underline{\omega}$ the pullback to identity of the relative differential of the Raynaud extension of the universal Abelian variety. Let $\underline{k}'' = (a_1-a_3, a_2-a_3)$. Let $\mathcal{B}$ be the abelian part of the Mumford family of the boundary. Its relative differential is identified with a subsheaf of $\underline{\omega}|_{\mathcal{D}}$. The $\omega^\flat_{\underline{k}} \subset \omega_{\underline{k}}$ is defined to be $\{s \in \omega_{\underline{k}}, s|_{\mathcal{D}} \in \mathscr{F}_{\mathcal{D}}\}$ for $\mathscr{F}_{\mathcal{D}} := \det(\underline{\omega}|_{\mathcal{D}})^{a_3} \otimes \underline{\omega}_{\mathcal{B}}^{\underline{k}''}$, where the last term means the automorphic sheaf of weight $\underline{k}''$ for $\mathrm{GU}(2,0)$.

Weight Space

Let $H = \mathrm{GL}_3 \times \mathrm{GL}_1$ and T be the diagonal torus. Then $\mathbf{H} = H(\mathbb{Z}_p)$. We let $\Lambda_{3,1} = \Lambda$ be the completed group algebra $\mathbb{Z}_p[[T(1+\mathbb{Z}_p)]]$. This is a formal power series ring with four variables. There is an action of $T(\mathbb{Z}_p)$ given by the action on the $j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$. (see [19, 3.4]) This gives the space of p-adic modular forms a structure of Λ-algebra. A $\bar{\mathbb{Q}}_p$-point ϕ of SpecΛ is call arithmetic if it is determined by a character $[\underline{k}].[\zeta]$ of $T(1+p\mathbb{Z}_p)$ where $\underline{k}$ is a weight and $\zeta = (\zeta_1, \zeta_2, \zeta_3; \zeta_4)$ for $\zeta_i \in \mu_{p^\infty}$. Here $[\underline{k}]$ is the character by regarding $\underline{k}$ as a character of $T(1+\mathbb{Z}_p)$ by

$[\underline{k}](t_1,t_2,t_3,t_4) = (t_1^{a_1}t_2^{a_2}t_3^{a_3}t_4^{-b_1})$ and $[\zeta]$ is the finite order character given by mapping $(1+p\mathbb{Z}_p)$ to ζ_i at the corresponding entry t_i of $T(\mathbb{Z}_p)$. We often write this point $\underline{k}_\zeta$. We also define $\omega^{[\underline{k}]}$ a character of the torsion part of $T(\mathbb{Z}_p)$ (isomorphic to $(\mathbb{F}_p^\times)^4$) given by $\omega^{[\underline{k}]}(t_1,t_2,t_3,t_4) = \omega(t_1^{a_1}t_2^{a_2}t_3^{a_3}t_4^{-b_1})$.

Definition 3.1. *We fix $\underline{k}' = (a_1, a_2)$ and $\rho = L_{\underline{k}'}$. Let $\mathcal{X}_\rho$ be the set of arithmetic points $\phi \in \mathrm{Spec}\Lambda_{3,1}$ corresponding to weight $(a_1, a_2, a_3; b_1)$ such that $a_1 \geq a_2 \geq a_3 \geq -b_1 + 4$. (The ζ-part being trivial). Let $\mathrm{Spec}\tilde{\Lambda} = \mathrm{Spec}\tilde{\Lambda}_{(a_1,a_2)}$ be the Zariski closure of $\mathcal{X}_\rho$.*

We define for $q = 0, \flat$

$$V^q_{\underline{k}}(K_0(p,p^n), \mathcal{O}_m) := \{f \in H^0(T_{n,m}, \omega^q_{\underline{k}}), g\cdot f = [\underline{k}]\omega^{[\underline{k}]}\}.$$

(Note the "ω"-part of the nebentypus).
As in [19, 3.3] we have a canonical isomorphism given by taking the "p-adic avartar"

$$H^0(T_{n,m}, \omega_{\underline{k}}) \simeq V_{n,m} \otimes L_{\underline{k}}, f \mapsto \hat{f}$$

and $\beta_{\underline{k}} : V_{\underline{k}}(K_1^n, \mathcal{O}_m) \to V^{\mathbf{N}}_{n,m}$ by $f \mapsto \beta_{\underline{k}}(f) := l_{\underline{k}}(\hat{f})$. The following lemma is [19, lemma 4.2].

Lemma 3.2. *Let $q \in \{0, \flat\}$ and let $V^q_{\underline{k}}(K_0(p,p^n), \mathcal{O}_m) := H^0(T_{n,m}, \omega^q_{\underline{k}})^{K_0(p,p^n)}$. Then we have*

$$H^0(I_G(K_1^n)[1/E], \omega^q_{\underline{k}}) \otimes \mathcal{O}_m = V^q_{\underline{k}}(K_0(p,p^n), \mathcal{O}_m).$$

In fact in our case for $\mathrm{U}(3,1)$ over $\mathbb{Q}$, such base change property is true even for the sheaf $\omega_{\underline{k}}$ in place of $\omega^\flat_{\underline{k}}$. However it is crucial to use $\omega^\flat_{\underline{k}}$ if working with general totally real fields (see the proof of [19, Lemma 4.1]), or with unitary groups other than $\mathrm{U}(r,1)$ (see the notion $\bar{R}$ before [61, Lemma 6.8] for the unitary group $\mathrm{U}(2,2)$). We choose to use $\omega^\flat_{\underline{k}}$ here so as to cite results in [19] directly. We record a contraction property for the operator U_α.

Lemma 3.3. *If $n > 1$, then we have*

$$U_\alpha \cdot V_{\underline{k}}(K_0(p,p^n), \mathcal{O}_m) \subset V_{\underline{k}}(K_0(p,p^{n-1}), \mathcal{O}_m).$$

The proof is the same as [19, Proposition 4.4]. The following proposition follows from the contraction property for e_α:

Proposition 3.4.

$$e_\alpha V^q_{\underline{k}}(K_0(p,p^n), \mathcal{O}_m) = e_\alpha V_{\underline{k}}(K_0(p), \mathcal{O}_m).$$

The following lemma tells us that to study semi-ordinary forms one only needs to look at the sheaf $\omega^\flat_{\underline{k}}$.

Lemma 3.5. *Let $n \geq m > 0$, then*

$$e_\alpha . V^\flat_{\underline{k}}(K_0(p,p^n), \mathcal{O}_m) = e_\alpha \cdot V^q_{\underline{k}}(K_0(p,p^n), \mathcal{O}_m).$$

Proof. Same as [19, lemma 4.10]. □

Similar to the $\beta_{\underline{k}}$ we define a more general $\beta_{\underline{k},\rho}$ as follows: Let ρ be the algebraic representation $L_\rho = L_{\underline{k}'}$ of GL_2 with lowest weight $-\underline{k}' = -(a_1, a_2)$. We identify $L_{\underline{k}}$ with the algebraically induced representation $\mathrm{Ind}^{\mathrm{GL}_3\times\mathrm{GL}_1}_{\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1} \rho \otimes \chi_{a_3} \otimes \chi_{b_1}$ (χ_a means the algebraic character defined by taking the $(-a)$-th power). We define the functional $l_{\underline{k},\rho}$ taking values in $L_{\underline{k}'}$ by evaluating at identity (similar to the definition of $l_{\underline{k}}$). We define $\beta_{\underline{k},\rho}$ similar to $\beta_{\underline{k}}$ but replacing $l_{\underline{k}}$ by $l_{\underline{k},\rho}$.

Proposition 3.6. *If $n \geq m > 0$, then the morphism*

$$\beta_{\underline{k},\rho} : V_{\underline{k}}(K_0(p,p^n), \mathcal{O}_m) \to (V_{n,m} \otimes L_\rho)^{K_0(p,p^n)}$$

is U_α-equivariant, and there is a Hecke-equivariant homomorphism $s_{\underline{k},\rho} : (V_{n,m} \otimes L_\rho)^{K_0(p,p^n)} \to V_{\underline{k}}(K_0(p,p^n), \mathcal{O}_m)$ such that $\beta_{\underline{k},\rho} \circ s_{\underline{k},\rho} = U_\alpha^m$ and $s_{\underline{k},\rho} \circ \beta_{\underline{k},\rho} = U_\alpha^m$. So the kernel and the cokernel of $\beta_{\underline{k},\rho}$ are annihilated by U_α^m.

Proof. Similar to [19, Proposition 4.7]. Our $s_{\underline{k},\rho}$ is defined as follows: for $(\underline{A}, \bar{j})$ over a $\mathcal{O}_m$-algebra R,

$$s_{\underline{k},\rho}(\alpha^m)(\underline{A}, \bar{j}) := \sum_{v_{\chi'} \in \rho \otimes \chi_{a_3} \otimes \chi_{b_1}} \sum_u \frac{1}{\chi_{r,1}(\alpha^m)} \cdot \mathrm{Tr}_{R_0^{\alpha^m u}/R}(f(\underline{A}_{\alpha^m u} \cdot j_{\alpha^m u}))\rho_{\underline{k}}(u) v_{\chi'}.$$

Here the character $\chi_{r,1}$ is defined by

$$\chi_{r,1}(\mathrm{diag}(a_1, a_2, a_3; d)) := (a_1 a_2 a_3)^{-1} d.$$

The $v_{\chi'}$'s form a basis of the representation $\rho \otimes \chi_{a_3} \otimes \chi_{b_1}$ which are eigenvectors for the diagonal torus action with eigenvalues χ''s (the eigenvalues appear with multiplicity one so we use the subscript χ' to denote the corresponding vector). The u runs over a set of representatives of

$$\alpha^{-m} N_H(\mathbb{Z}_p) \alpha^m \cap N_H(\mathbb{Z}_p) \backslash N_H(\mathbb{Z}_p).$$

The $(\underline{A}_{\alpha u}, j_{\alpha u})$ is a certain pair with $\underline{A}_{\alpha u}$ an abelian variety admitting an isogeny to $\underline{A}$ of type α (see [19, 3.7.1] for details) and $R_0^{\alpha u}/R$ being the coordinate ring for $(\underline{A}_{\alpha u}, j_{\alpha u})$ (see 3.8.1 of *loc.cit*). Note that the twisted action of

$$\tilde{\rho}_{\underline{k}}(\alpha^{-1}) v_{\chi'} := p^{-\langle \mu, \underline{k} + \chi' \rangle} v_{\chi'}$$

satisfies $\tilde{\rho}_{\underline{k}}(\alpha^{-1}) v_{\chi'} = 1$ for all the χ' above. Write χ for $\chi_{a_3} \boxtimes \chi_{b_1}$. Note also that for any eigenvector $v_{\chi'} \in \mathrm{Ind}_{\mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1}^{\mathrm{GL}_3 \times \mathrm{GL}_1} \rho \otimes \chi$ for the torus action such that $v_{\chi'} \notin \rho \otimes \chi$, and $\mu \in X_*(T)$ (the co-character group) with $\mu(p) = \alpha$, we have $\langle \mu, \underline{k} + \chi' \rangle < 0$. By the definition of $U_\alpha^m = U_{\alpha^m}$, if $f = \sum_\chi g_\chi \otimes v_\chi$, then

$$U_{\alpha^m} \cdot f(\underline{A}, j) = \sum_{v_{\chi'} \in \rho \otimes \chi} s_{\underline{k},\rho}(\alpha^m) g_{\chi'}(\underline{A}, j) + \sum_{v_{\chi'} \notin \rho \otimes \chi} p^{-\langle m\mu, \underline{k} + \chi' \rangle} \frac{1}{\chi_{r,1}(\alpha^m)} \mathrm{Tr}_{R_0^{\alpha^m u}/R}(g_{\chi'}(\underline{A}_{\alpha^m u}, j)) \otimes \rho_{\underline{k}}(u) v_{\chi'}.$$

For the notation $R_0^{\alpha^m u}$ see [19, 3.8.1] for an explanation. So $\beta_{\underline{k},\rho} \circ s_{\underline{k},\rho}(\alpha^m) = U_{\alpha^m}$ and $s_{\underline{k},\rho}(\alpha^m) \circ \beta_{\underline{k},\rho} = U_{\alpha^m}$. Taking $s_{\underline{k},\rho} := s_{\underline{k},\rho}(\alpha^m)$, then we proved the proposition. □

The following proposition follows from the above one as [19, Proposition 4.9]. Let $\underline{k}$ and ρ be as before.

Proposition 3.7. *If $n \geq m > 0$, then*

$$\beta_{\underline{k},\rho} : e_\alpha \cdot V_{\underline{k}}(K_0(p,p^n), \mathcal{O}_m) \simeq e_\alpha (V_{n,m} \otimes L_\rho)^{K_0(p,p^n)}[\underline{k}].$$

We are going to prove some control theorems and fundamental exact sequence for semi-ordinary forms along this smaller two-dimensional weight space Spec$\tilde{\Lambda}$. The following proposition follows from Lemma 3.2 and Proposition 3.4 in the same way as [19, Lemma 4.10, Proposition 4.11], noting that by the contraction property the level group is actually in $K_0(p)$.

Proposition 3.8. *Let* $e_\alpha.\mathcal{V}_{\underline{k}}(K_0(p,p^n)) := \varinjlim_m e_\alpha \cdot V_{\underline{k}}(K_0(p,p^n),\mathcal{O}_m)$. *Then* $e_\alpha.\mathcal{V}(K_0(p,p^n))$ *is* p*-divisible and*

$$e_\alpha \cdot \mathcal{V}_{\underline{k}}(K_0(p,p^n))[p^m] = e \cdot V_{\underline{k}}(K_0(p,p^n),\mathcal{O}_m) = e_\alpha \cdot H^0(\mathcal{I}_S,\omega_{\underline{k}}) \otimes \mathcal{O}_m.$$

The following proposition is crucial to prove control theorems for semi-ordinary forms along the weight space $\mathrm{Spec}\tilde{\Lambda}$.

Proposition 3.9. *The dimension of* $e_\alpha M_{\underline{k}}(K_0(p,p^n),\mathbb{C})$*'s are uniformly bounded for all* $\underline{k} \in \mathcal{X}_\rho$.

Proof. The uniform bound for group cohomology is proved in [16, Theorem 5.1]. Note that if the control theorem in *loc.cit* is true then the uniform boundedness is an easy consequence. However in *loc.cit* one assumption ([16, Theorem 5.2 (iii)]) is missing, which we do not know if it is true in our case. But an argument using commutative algebra similar to the proof of [16, Lemma 5.1], considering the cohomologies for H^1, H^2 and H^3 altogether still gives the uniform boundedness without knowing the control theorem. (e.g. one considers the exact sequences

$$0 \to \frac{E_0}{T_1} \to E_1 \to N_0[T_1] \to 0;$$

$$0 \to \frac{E_1}{T_2} \to E_2 \to N_1[T_2] \to 0;$$

$$0 \to \frac{N_0}{T_1} \to N_1 \to H_0[T_1] \to 0,$$

where E_i, N_i and T_i are as in [16, Lemma 5.1] with $q = 3$ and H_i's are the corresponding modules for H^1's. These modules are finitely generated modules over Iwasawa algebras over $\mathcal{O}_L$ with $2-i$-variables. Write Λ for the Iwasawa algebra over $\mathcal{O}_L$ of two variables. Note that if the subscheme of $\mathrm{Spec}\Lambda$ defined by $T_1 = 0$ is not contained in the support of the torsion submodule of N_0, then $N_0[T_1]$ is contained in the submodule of N_0 consisting of elements whose stalks are 0 at all points of codimension at most one. Note also that if $\Lambda/T_1\Lambda$ is an Iwasawa algebra over $\mathcal{O}_L$ of one variable, then the $\mathcal{O}_L$-rank of $\frac{N_0}{T_1}[T_2]$ is bounded by the number of generators of the Λ-module N_0, say, using the structure theorem of finitely generated modules over the one-variable Iwasawa algebra. We do not know if this argument can be generalized in other settings.) The bound for coherent cohomology follows by the Eichler-Shimura isomorphism. See [19, Theorem 4.18]. □

The following theorem says that all semi-ordinary forms of sufficiently regular weights are classical, and can be proved in the same way as [19, Theorem 4.19] using Proposition 3.9.

Theorem 3.10. *For each weight* $\underline{k} = (a_1,a_2,a_3;b_1) \in \mathcal{X}_\rho$*, there is a positive integer* $A(\underline{a})$ *depending on* $\underline{a} = (a_1,a_2,a_3)$ *such that if* $b_1 > A(\underline{a},n)$ *then the natural restriction map*

$$e_\alpha M_{\underline{k}}(K_0(p),\mathcal{O}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \simeq e_\alpha \cdot \mathcal{V}_{\underline{k}}(K_0(p))$$

is an isomorphism.

For $q = 0,\phi$ define

$$V^q_{\mathrm{so}} := \mathrm{Hom}(e_\alpha.\mathcal{W}^q,\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\Lambda_{3,1}} \tilde{\Lambda}$$

$$\mathcal{M}^q_{\mathrm{so}}(K,\tilde{\Lambda}) := \mathrm{Hom}_{\tilde{\Lambda}}(V^q_{\mathrm{so}},\tilde{\Lambda}).$$

Thus from the finiteness results and the p-divisibility of the space of semi-ordinary p-adic modular forms, we get the Hida's control theorem

Theorem 3.11. *Let $q=0$ or ϕ. Then*

(1) V^q_{so} is a free $\tilde{\Lambda}$-module of finite rank.

(2) For any $k \in \mathcal{X}_\rho$ we have $\mathcal{M}^q_{\mathrm{so}}(K,\tilde{\Lambda})\otimes\tilde{\Lambda}/P_{\underline{k}} \simeq e_\alpha \cdot M^q_{\underline{k}}(K,\mathcal{O})$.

The proof is same as [19, Theorem 4.21] using Proposition 3.4, 3.7, Theorem 3.10 and Proposition 3.8.

Descent to Prime to p-Level

The following proposition will be used in the proof of Theorem 5.3.

Proposition 3.12. *Suppose $\underline{k}$ is such that $a_1=a_2=0$, $a_3\equiv b_1\equiv 0 (\mathrm{mod}\ p-1)$, $a_2-a_3>>0, a_3+b_1>>0$, . Suppose $F\in e_\alpha M^0_{\underline{k}}(K_0(p),\mathbb{C})$ is an eigenform with trivial nebentypus at p whose mod p Galois representation (semi-simple) is the same as our Klingen Eisenstein series constructed in section 4. Let π_F be the associated automorphic representation. Then $\pi_{F,p}$ is unramified principal series representation.*

Proof. Similar to [19, proposition 4.17]. Let f be the GL_2 cusp form having good supersingular reduction at p in the introduction. Note that $\pi_{F,p}$ has a fixed vector for $K_0(p)$ and $\bar{\rho}_{\pi_f}|_{G_{\mathbb{Q}_p}}$ is irreducible by [7]. By the classification of admissible representations with $K_0(p)$-fixed vector (see e.g. [5, Theorem 3.7]) we know $\pi_{F,p}$ has to be a subquotient of $\mathrm{Ind}_B^{\mathrm{GL}_4}\chi$ for χ an unramified character of $T_n(\mathbb{Q}_p)$. If this induced representation is irreducible then we are done. If not, when $a_2-a_3>>0, a_3+b_1>>0$, since F is semi-ordinary, we must have $\chi=\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4$ such that (with possibly renumbering) $\chi_1=\chi_2|\cdot|$ and χ_3,χ_4 having p-adic weight $\kappa_1=b_1$ and $\kappa_2=3-a_3$. This implies F is in fact ordinary. But we have $\bar{\rho}_F^{ss}$ is the direct sum of $\bar{\rho}_f$ with two characters. This contradicts that $\bar{\rho}_{\pi_f}|_{G_{\mathbb{Q}_p}}$ is irreducible. Thus $\pi_{F,p}$ must by unramified. □

A Definition Using Fourier-Jacobi Expansion

We can define a $\tilde{\Lambda}$-adic Fourier-Jacobi expansion map for families of semi-ordinary families as in [19, 4.6.1] by taking the $\tilde{\Lambda}$-dual of the Pontryagin dual of the usual Fourier-Jacobi expansion map (replacing the e's in *loc.cit* by e_α's). We also define the Siegel operators $\Phi^h_{[g]}$'s by taking the 0-th Fourier-Jacobi coefficient.

Definition 3.13. *Let A be a finite torsion free Λ-algebra. Let $\mathcal{N}_{\mathrm{so}}(K,A)$ be the set of formal Fourier-Jacobi expansions:*

$$F=\{\sum_{\beta\in\mathscr{S}_{[g]}} a(\beta,F)q^\beta, a(\beta,F)\in A\hat{\otimes}\hat{A}^\infty_{[g]}\otimes H^0(\mathcal{Z}^\circ_{[g]},\mathcal{L}(\beta))\}_{g\in X(K)}$$

such that for a Zariski dense set $\mathcal{X}_F\subseteq\mathcal{X}_\rho$ of points $\phi\in\mathrm{Spec}A$ such that the induced point in $\mathrm{Spec}\Lambda$ is some arithmetic weight $\underline{k}_\zeta$, the specialization F_ϕ of F is the highest weight vector of the Fourier-Jacobi expansion of a semi-ordinary modular form with tame level $K^{(p)}$, weight $\underline{k}$ and nebentype at p given by $[\underline{k}][\underline{\zeta}]\omega^{-[\underline{k}]}$ as a character of $K_0(p)$.

Then we have the following

Theorem 3.14.

$$\mathcal{M}_{\mathrm{so}}(K,A)=\mathcal{N}_{\mathrm{so}}(K,A).$$

The proof is the same as [19, Theorem 4.25]. This theorem is used to show that the construction in [9] recalled later does give a semi-ordinary family in the sense of this section.

Fundamental Exact Sequence

Now we prove a fundamental exact sequence for semi-ordinary forms. Let $w_3' = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix}$.

Lemma 3.15. *Let $\underline{k} \in \mathcal{X}_\rho$ and $F \in e_\alpha M_{\underline{k}}(K_0(p,p^n), R)$ and $R \subset \mathbb{C}$. Let $W_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \cup \mathrm{Id}$ be the Weyl group for $G_P(\mathbb{Q}_p)$. There is a constant A such that for any $\underline{k} \in \mathcal{X}_\rho$ such that $a_2 - a_3 > A, a_3 + b_1 > A$, for each $g \in G(\mathbb{A}_f^{(p)})$, $\Phi_{P,wg}(F) = 0$ for any $w \notin W_2 w_3'$.*

The lemma can be proved using the computations in the proof of [19, lemma 4.14]. Note that by semi-ordinarity and the contraction property the level group at p for F is actually $K_0(p)$.

The following is a semi-ordinary version of [19, Theorem 4.16], noting that e_α induces identity after the Siegel operator $\hat{\Phi}^{w_3'}$. The proof is also similar (even easier since the level group at p is in fact in $K_0(p)$ by the contraction property).

Theorem 3.16. *For $\underline{k} \in \mathcal{X}_\rho$, we have*

$$0 \to e_\alpha \mathcal{M}^0_{\underline{k}}(K,A) \to e_\alpha \mathcal{M}_{\underline{k}}(K,A) \xrightarrow{\hat{\Phi}^{w_3'} = \oplus \hat{\Phi}^{w_3'}_{[g]}} \oplus_{g \in C(K)} \mathcal{M}_{\underline{k}'}(K^g_{P,0}(p), A)$$

is exact.

The family version of the fundamental exact sequence can be deduced from Theorem 3.10, 3.11, 3.16, as well as the affine-ness of $S^*_G(K)(1/E)$ (See [19, Theorem 4.16]).

Theorem 3.17.

$$0 \to e_\alpha \mathcal{M}^0(K,A) \to e_\alpha \mathcal{M}(K,A) \xrightarrow{\hat{\Phi}^{w_3'} = \oplus \hat{\Phi}^{w_3'}_{[g]}} \oplus_{g \in C(K)} \mathcal{M}(K^g_{P,0}(p), A) \to 0.$$

4 Eisenstein Series and Families

4.1 Klingen Einstein Series

Archimedean Places

Let (π_∞, V_∞) be a finite dimensional representation of $D_\infty^\times$. Let ψ_∞ and τ_∞ be characters of $\mathbb{C}^\times$ such that $\psi_\infty|_{\mathbb{R}^\times}$ is the central character of π_∞. Then there is a unique representation π_ψ of $\mathrm{GU}(2)(\mathbb{R})$ determined by π_∞ and ψ_∞ such that the central character is ψ_∞. These determine a representation $\pi_\psi \times \tau$ of $M_P(\mathbb{R}) \simeq \mathrm{GU}(2)(\mathbb{R}) \times \mathbb{C}^\times$. We extend this to a representation ρ_∞ of $P(\mathbb{R})$ by requiring $N_P(\mathbb{R})$ acts trivially. Let $I(V_\infty) = \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} \rho_\infty$ (smooth induction) and $I(\rho_\infty) \subset I(V_\infty)$ be the subspace of K_∞ -finite vectors. Note that elements of $I(V_\infty)$ can be realized as functions on K_∞. For any $f \in I(V)$ and $z \in \mathbb{C}^\times$ we define a function f_z on $G(\mathbb{R})$ by

$$f_z(g) := \delta(m)^{\frac{3}{2}+z}\rho(m)f(k), g = mnk \in P(\mathbb{R})K_\infty.$$

There is an action $\sigma(\rho, z)$ on $I(V_\infty)$ by

$$(\sigma(\rho, z)(g))(k) = f_z(kg).$$

Non-Archimedean Places
Let (π_ℓ, V_ℓ) be an irreducible admissible representation of $D^\times(\mathbb{Q}_\ell)$ and π_ℓ is unitary and tempered if D is split at ℓ. Let ψ and τ be characters of $\mathcal{K}_\ell^\times$ such that $\psi|_{\mathbb{Q}_\ell^\times}$ is the central character of π_ℓ. Then there is a unique irreducible admissible representation π_ψ of $\mathrm{GU}(2)(\mathbb{Q}_\ell)$ determined by π_ℓ and ψ_ℓ. As before we have a representation $\pi_\psi \times \tau$ of $M_P(\mathbb{Q}_\ell)$ and extend it to a representation ρ_ℓ of $P(\mathbb{Q}_\ell)$ by requiring $N_P(\mathbb{Q}_\ell)$ acts trivially. Let $I(\rho_\ell) = \mathrm{Ind}_{P(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}\rho_\ell$ be the admissible induction. We similarly define f_z for $f \in I(\rho_\ell)$ and $\rho_\ell^\vee, I(\rho_\ell^\vee), A(\rho_\ell, z, f)$, etc. For $v \notin \Sigma$ we have $D^\times(\mathbb{Q}_\ell) \simeq \mathrm{GL}_2(\mathbb{Q}_\ell)$.
Global Picture
Let $(\pi = \otimes_v \pi_v, V)$ be an irreducible unitary cuspidal automorphic representation of $D^\times(\mathbb{A}_\mathbb{Q})$ we define $I(\rho)$ to be the restricted tensor product of $\otimes_v I(\rho_v)$ with respect to the unramified vectors $f^0_{\varphi_\ell}$ for some $\varphi = \otimes_v \phi_v \in \pi$. We can define f_z, $I(\rho^\vee)$ and $A(\rho, z, f)$ similar to the local case. f_z takes values in V which can be realized as automorphic forms on $D^\times(\mathbb{A}_\mathbb{Q})$. We also write f_z for the scalar-valued functions $f_z(g) := f_z(g)(1)$ and define the Klingen Eisenstein series:

$$E(f, z, g) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f_z(\gamma g).$$

This is absolutely convergent if $\mathrm{Re} z >> 0$ and has meromorphic continuation to all $z \in \mathbb{C}$.

4.2 Siegel Eisenstein Series

Local Picture:
Our discussion in this section follows [61, 11.1-11.3] closely. Let $Q = Q_n$ be the Siegel parabolic subgroup of GU_n consisting of matrices $\begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$. It consists of matrices whose lower-left $n \times n$ block is zero. For a place v of $\mathbb{Q}$ and a character τ of $\mathcal{K}_v^\times$ we let $I_n(\tau_v)$ be the space of smooth $K_{n,v}$-finite functions (here $K_{n,v}$ means the maximal compact subgroup $G_n(\mathbb{Z}_v)$) $f : K_{n,v} \to \mathbb{C}$ such that $f(qk) = \tau_v(\det D_q)f(k)$ for all $q \in Q_n(\mathbb{Q}_v) \cap K_{n,v}$ (we write q as block matrix $q = \begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$). For $z \in \mathbb{C}$ and $f \in I(\tau)$ we also define a function $f(z, -) : G_n(\mathbb{Q}_v) \to \mathbb{C}$ by $f(z, qk) := \chi(\det D_q))|\det A_q D_q^{-1}|_v^{z+n/2} f(k), q \in Q_n(\mathbb{Q}_v)$ and $k \in K_{n,v}$.

For $f \in I_n(\tau_v), z \in \mathbb{C}$, and $k \in K_{n,v}$, the intertwining integral is defined by:

$$M(z, f)(k) := \bar{\tau}_v^n(\mu_n(k)) \int_{N_{Q_n}(F_v)} f(z, w_n r k) dr.$$

For z in compact subsets of $\{\mathrm{Re}(z) > n/2\}$ this integral converges absolutely and uniformly, with the convergence being uniform in k. In this case it is easy to see that $M(z, f) \in I_n(\bar{\tau}_v^c)$. A standard

fact from the theory of Eisenstein series says that this has a continuation to a meromorphic section on all of $\mathbb{C}$.
Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set. By a meromorphic section of $I_n(\tau_v)$ on $\mathcal{U}$ we mean a function $\varphi : \mathcal{U} \mapsto I_n(\tau_v)$ taking values in a finite dimensional subspace $V \subset I_n(\tau_v)$ and such that $\varphi : \mathcal{U} \to V$ is meromorphic.

Global Picture
For an idele class character $\tau = \otimes \tau_v$ of $\mathbb{A}_{\mathcal{K}}^{\times}$ we define a space $I_n(\tau)$ to be the restricted tensor product defined using the spherical vectors $f_v^{sph} \in I_n(\tau_v)$ (invariant under $K_{n,v}$) such that $f_v^{sph}(K_{n,v}) = 1$, at the finite places v where τ_v is unramified.

For $f \in I_n(\tau)$ we consider the Eisenstein series

$$E(f; z, g) := \sum_{\gamma \in Q_n(\mathbb{Q}) \backslash G_n(\mathbb{Q})} f(z, \gamma g).$$

This series converges absolutely and uniformly for (z, g) in compact subsets of $\{\mathrm{Re}(z) > n/2\} \times G_n(\mathbb{A}_{\mathbb{Q}})$. The defined automorphic form is called Siegel Eisenstein series.

The Eisenstein series $E(f; z, g)$ has a meromorphic continuation in z to all of $\mathbb{C}$ in the following sense. If $\varphi : \mathcal{U} \to I_n(\tau)$ is a meromorphic section, then we put $E(\varphi; z, g) = E(\varphi(z); z, g)$. This is defined at least on the region of absolute convergence and it is well known that it can be meromorphically continued to all $z \in \mathbb{C}$.

4.3 Pullback Formula

We define some embeddings of a subgroup of $\mathrm{GU}(3,1) \times \mathrm{GU}(0,2)$ into $\mathrm{GU}(3,3)$. This will be used in the doubling method. First we define $G(3,3)'$ to be the unitary similitude group associated to:

$$\begin{pmatrix} & & 1 & \\ & \zeta & & \\ -1 & & & \\ & & & -\zeta \end{pmatrix}$$

and $G(2,2)'$ to be associated to

$$\begin{pmatrix} \zeta & \\ & -\zeta \end{pmatrix}.$$

We define an embedding

$$\alpha : \{g_1 \times g_2 \in \mathrm{GU}(3,1) \times \mathrm{GU}(0,2), \mu(g_1) = \mu(g_2)\} \to GU(3,3)'$$

and

$$\alpha' : \{g_1 \times g_2 \in \mathrm{GU}(2,0) \times \mathrm{GU}(0,2), \mu(g_1) = \mu(g_2)\} \to GU(2,2)'$$

as $\alpha(g_1, g_2) = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$ and $\alpha'(g_1, g_2) = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$. We also define isomorphisms:

$$\beta : \mathrm{GU}(3,3)' \xrightarrow{\sim} \mathrm{GU}(3,3), (\beta' : \mathrm{GU}(2,2)' \xrightarrow{\sim} \mathrm{GU}(2,2))$$

by:

$$g \mapsto S^{-1}gS, (g \mapsto S'^{-1}gS')$$

where

$$S = \begin{pmatrix} 1 & & & \\ & 1 & & -\frac{\zeta}{2} \\ & & 1 & \\ & -1 & & -\frac{\zeta}{2} \end{pmatrix}, S' = \begin{pmatrix} 1 & -\frac{\zeta}{2} \\ -1 & -\frac{\zeta}{2} \end{pmatrix}.$$

We define

$$i(g_1, g_2) = S^{-1}\alpha(g_1, g_2)S, i'(g_1, g_2) = S'^{-1}\alpha(g_1, g_2)S'.$$

We recall the pullback formula of Shimura (see [61, Proposition 11.1]. The proof there works in our situation as well). Let τ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$. Given a cuspform φ on GU(2) we consider

$$F_{\varphi}(f; z, g) := \int_{\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})} f(z, S^{-1}\alpha(g, g_1h)S)\bar{\tau}(\det g_1 g)\varphi(g_1h)dg_1,$$

$$f \in I_3(\tau), g \in \mathrm{GU}(3,1)(\mathbb{A}_{\mathbb{Q}}), h \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}}), \mu(g) = \mu(h)$$

or

$$F'_{\varphi}(f'; z, g) = \int_{\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})} f'(z, S'^{-1}\alpha'(g, g_1h)S')\bar{\tau}(\det g_1 g)\varphi(g_1h)dg_1$$

$$f' \in I_2(\tau), g \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}}), h \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}}), \mu(g) = \mu(h)$$

This is independent of h. The pullback formulas are the identities in the following proposition.

Proposition 4.1. *Let τ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$.*
(i) If $f' \in I_2(\tau)$, then $F'_{\varphi}(f'; z, g)$ converges absolutely and uniformly for (z, g) in compact sets of $\{\mathrm{Re}(z) > 1\} \times \mathrm{GU}(2,0)(\mathbb{A}_{\mathbb{Q}})$, and for any $h \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}})$ such that $\mu(h) = \mu(g)$

$$\int_{\mathrm{U}(2)(\mathbb{Q})\backslash \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})} E(f'; z, S'^{-1}\alpha'(g, g_1h)S')\bar{\tau}(\det g_1 h)\varphi(g_1h)dg_1 = F'_{\varphi}(f'; z, g).$$

(ii) If $f \in I_3(\tau)$, then $F_{\varphi}(f; z, g)$ converges absolutely and uniformly for (z, g) in compact sets of $\{Re(z) > 3/2\} \times \mathrm{GU}(3,1)(\mathbb{A}_{\mathbb{Q}})$ such that $\mu(h) = \mu(g)$

$$\int_{\mathrm{U}(2)(\mathbb{Q})\backslash \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})} E(f; z, S^{-1}\alpha(g, g_1h)S)\bar{\tau}(\det g_1 h)\varphi(g_1h)dg_1$$
$$= \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{GU}(3,1)(\mathbb{Q})} F_{\varphi}(f; z, \gamma g),$$

with the series converging absolutely and uniformly for (z, g) in compact subsets of $\{\mathrm{Re}(z) > 3/2\} \times \mathrm{GU}(3,1)(\mathbb{A}_{\mathbb{Q}})$.

4.4 p-adic Interpolation

We recall our notations in [9, Section 5.1] and correct some errors in the formulas for parameterization in *loc.cit.* We define an "Eisenstein datum" $\mathcal{D}$ to be a pair (φ, ξ_0) consisting of a cuspidal eigenform φ of prime to p level, trivial character and weight $\underline{k} = (a_1, a_2), a_1 \geq a_2 \geq 0$ on $\mathrm{GU}(r, 0)$ and a Hecke character ξ_0 of $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times$ such that $\xi_0 | \cdot |^{\frac{1}{2}}$ is a finite order character. Let σ be the reciprocity map of class field theory $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times \to G_\mathcal{K}^{ab}$ normalized by the geometric Frobenius. Note $\Gamma_\mathcal{K} = \Gamma^+ \oplus \Gamma_{\bar{v}_0}$. Let $\Psi_1 : G_\mathcal{K} \twoheadrightarrow \Gamma_\mathcal{K} \to \Gamma^+ \hookrightarrow \mathbb{Z}_p[[\Gamma_\mathcal{K}^+]]^\times$ and $\Psi_2 : G_\mathcal{K} \twoheadrightarrow \Gamma_\mathcal{K} \to \Gamma^{\bar{v}_0} \hookrightarrow \mathbb{Z}_p[[\Gamma_{\bar{v}_0}]]^\times$ where the middle arrows are projections with respect to the above direct sum. Then $\Psi_\mathcal{K} = \Psi_1 \cdot \Psi_2$. We define

$$\begin{aligned}
\tau_0 &:= \overline{(\xi_0 | \cdot |^{\frac{1}{2}})}^c, \\
\boldsymbol{\xi} &:= \xi_0 \cdot (\Psi \circ \sigma), \\
\boldsymbol{\tau} &:= \tau_0 \cdot (\Psi_1^{-c} \circ \sigma), \\
\psi_\mathcal{K} &:= \Psi_2.
\end{aligned}$$

We define $\mathcal{X}^{pb}$ ("*pb*" stands for pullback) to be the set of $\bar{\mathbb{Q}}_p$-points $\phi \in \mathrm{Spec}\Lambda_{\mathcal{K}, \mathcal{O}_L}$ such that $\phi \circ \boldsymbol{\tau}((1+p, 1)) = \tau_0((1+p, 1))$,

$$\phi \circ \boldsymbol{\tau}((1, 1+p)) = (1+p)^{\kappa_\phi} \tau_0((1, 1+p))$$

for some integer $\kappa_\phi > 6$, $\kappa_\phi \equiv 0(\mathrm{mod}(p-2))$ and such that the weight $(a_1, a_2, 0; \kappa_\phi)$ is in the absolutely convergent range for P in the sense of Harris [14], and such that

$$\phi \circ \psi_\mathcal{K}(\gamma^-) = (1+p)^{\frac{m_\phi}{2}}$$

for some non-negative integer m_ϕ, and such that the τ_ϕ (to be defined in a moment) is such that, under the identification $\tau_\phi = (\tau_1, \tau_2)$ for $\mathcal{K}_p^\times \simeq \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$, we have τ_1, τ_2, $\tau_1\tau_2$ all have conductor (p).

We denote by $\mathcal{X}$ the set of $\bar{\mathbb{Q}}_p$-points ϕ in $\mathrm{Spec}\Lambda_{\mathcal{K}, \mathcal{O}_L}$ such that

$$\phi \circ \boldsymbol{\tau}((1, 1+p)) = (1+p)^{\kappa_\phi} \zeta_1 \tau_0((1, 1+p)), \phi \circ \boldsymbol{\tau}((p+1, 1)) = \tau_0((p+1, 1))$$

and $\phi \circ \psi_\mathcal{K}(\gamma^-) = \zeta_2$ with ζ_1 and ζ_2 being p-power roots of unity. Let $\mathcal{X}^{gen}$ be the subset of points such that the ζ_1 and ζ_2 above are all primitive p^t roots of unity for some $t \geq 2$.

Remark 4.2. *We will use the points in $\mathcal{X}^{pb}$ for p-adic interpolation of special L-values and Klingen Eisenstein series, and we will use the points in $\mathcal{X}$ to construct a Siegel Eisenstein measure.*

For each $\phi \in \mathcal{X}^{pb}$, we define Hecke characters ψ_ϕ and τ_ϕ of $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times$ by

$$\begin{aligned}
\bar{\tau}_\phi^c(x) &:= \bar{x}_\infty^{\kappa_\phi} (\phi \circ \boldsymbol{\tau})(x) x_{\bar{v}}^{-\kappa_\phi} \cdot | \cdot |^{-\frac{\kappa_\phi}{2}}, \\
\psi_\phi(x) &:= x_\infty^{\frac{m_\phi}{2}} \bar{x}_\infty^{-\frac{m_\phi}{2}} (\phi \circ \psi_\mathcal{K} \circ \sigma) x_v^{-\frac{m_\phi}{2}} x_{\bar{v}}^{\frac{m_\phi}{2}}.
\end{aligned}$$

Let

$$\begin{aligned}
\xi_\phi &= | \cdot |^{\frac{\kappa_\phi - 1}{2}} \bar{\tau}_\phi^c \psi_\phi, \\
\varphi_\phi &= \varphi \otimes \psi_\phi^{-1}.
\end{aligned}$$

The weight $\underline{k}_\phi$ for φ_ϕ at the arithmetic point ϕ is $(a_1 + m_\phi, a_2 + m_\phi)$.

4.5 Explicit Sections

Now we make explicit sections for the Siegel and Klingen Eisenstein series. We choose $g_1, g_3 \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}}), g_2', g_4' \in \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})$ in the same way as [67, subsection 7.4]. Recall their p-components are 1. We use a slight modification of the sections constructed in [9]. For the Siegel section we use the construction $f_{sieg} = \prod_v f_v$ in [9, Section 5.1]. Recall that the f_∞ is a vector valued section. In *loc.cit* we pullback this section under the embedding γ^{-1} and take the corresponding component for the representation $L^{(\underline{k}_\phi,0)} \boxtimes L^{(\kappa)} \boxtimes (L^{(\underline{k}_\phi)} \otimes \det \kappa)$ (notations as in *loc.cit* Section 4). Recall that in [67, section 7] we constructed a character ϑ of $\mathbb{A}_{\mathbb{Q}}^\times$ and elements $g_1 \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. Recall we start with a eigenform $f \in \pi$ new outside p and is an eigenvector for the U_p-operator with eigenvalue α_1. We extend it to a form on $\mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}})$ using the central character ψ and as in [67, 5.10] define

$$f_\Sigma = (\prod_{v\in\Sigma, v\nmid N} \pi(\begin{pmatrix} & 1 \\ \varpi_v & \end{pmatrix}) - \chi_{1,v}(\varpi_v) q_v^{\frac{1}{2}}) f,$$

$$f_\vartheta(g) = \prod_{v \text{ split } \in\Sigma, v\nmid p} \sum_{\{a_v \in \frac{\varpi_v \mathbb{Z}_v^\times}{\varpi_v^{1+s_v}\mathbb{Z}_v}\}_v} \vartheta(\frac{-a_v}{\varpi_v}) f_\Sigma(g \prod_v \begin{pmatrix} 1 & \\ a & 1 \end{pmatrix}_v \begin{pmatrix} \varpi_v^{-s_v} & \\ & 1 \end{pmatrix}_v)$$

where $\varpi_v^{s_v}$ is the conductor of ϑ at v, $\pi_{f,v} = \pi(\chi_{1,v}, \chi_{2,v})$ (choose any order).

Definition 4.3. *Define our φ in Subsection 4.3 to be $\pi(g_1) f_\vartheta$.*

4.6 Construction of A Measure

We first recall the notion of p-adic L-functions for Dirichlet characters which is needed in the proposition below. There is an element $\mathcal{L}_{\bar{\tau}'}$ in $\Lambda_{\mathcal{K},\mathcal{O}_L}$ such that at each arithmetic point $\phi \in \mathcal{X}^{pb}$, $\phi(\mathcal{L}_{\bar{\tau}'}) = L(\bar{\tau}'_\phi, \kappa_\phi - 2).\tau'_\phi(p^{-1}) p^{\kappa_\phi - 2} \mathfrak{g}(\bar{\tau}'_\phi)^{-1}$. For more details see [61, 3.4.3].

Constructing Families

The following theorem is proved in [9, Theorem 1.2].

Proposition 4.4. *Suppose the unitary automorphic representation $\pi = \pi_f$ generated by the weight k form f is such that π_p is an unramified principal series representation with distinct Satake parameters. Let $\tilde{\pi}$ be the dual representation of π.*

(i) There is an element $\mathcal{L}_{f,\mathcal{K}} \in \Lambda_{\mathcal{K},\mathcal{O}_L^{ur}}$ such that for any character ξ_ϕ of $\Gamma_{\mathcal{K}}$, which is the avatar of a Hecke character of conductor p, infinite type $(\frac{\kappa_\phi}{2} + m_\phi, -\frac{\kappa_\phi}{2} - m_\phi)$ with κ_ϕ an even integer which is at least 6, $m_\phi \geq \frac{k-2}{2}$, we have

$$\phi(\mathcal{L}_{f,\mathcal{K}}) = \frac{L(\tilde{\pi}, \xi_\phi, \frac{\kappa_\phi - 1}{2}) \Omega_p^{4m_\phi + 2\kappa_\phi}}{\Omega_\infty^{4m_\phi + 2\kappa_\phi}} c'_\phi . p^{\kappa_\phi - 3} \mathfrak{g}(\xi_{\phi,2})^2 \prod_{i=1}^{2} (\chi_i^{-1} \xi_{\phi,2}^{-1})(p)$$

c'_ϕ is a constant coming from an Archimedean integral.

(ii) There is a set of formal q-expansions $\mathbf{E}_{f,\xi_0} := \{\sum_\beta a_{[g]}^t(\beta) q^\beta\}_{([g],t)}$ for $\sum_\beta a_{[g]}^t(\beta) q^\beta \in \Lambda_{\mathcal{K},\mathcal{O}_L^{ur}} \otimes_{\mathbb{Z}_p} \mathcal{R}_{[g],\infty}$ where $\mathcal{R}_{[g],\infty}$ is some ring to be defined later, $([g], t)$ are p-adic cusp labels, such that

for a Zariski dense set of arithmetic points $\phi \in \mathrm{Spec}_{\mathcal{K},\mathcal{O}_L}$*,* $\phi(\mathbf{E}_{f,\xi_0})$ *is the Fourier-Jacobi expansion of the highest weight vector of the holomorphic Klingen Eisenstein series constructed by pullback formula which is an eigenvector for* U_{t^+} *with non-zero eigenvalue. The weight for* $\phi(\mathbf{E}_{f,\xi_0})$ *is* $(m_\phi - \frac{k-2}{2}, m_\phi + \frac{k-2}{2}, 0; \kappa_\phi)$*.*

(iii) The $a^t_{[g]}(0)$*'s are divisible by* $\mathcal{L}^\Sigma_{f,\mathcal{K},\xi_0}.\mathcal{L}^\Sigma_{\bar{\tau}'}$ *where* $\mathcal{L}^\Sigma_{\bar{\tau}'}$ *is the* p*-adic* L*-function of a Dirichlet character above.*

This is simply a translation of the main theorem in [9] to the situation here.

Definition 4.5. *We will write* $\mathbf{E}_{Kling}$ *later on for this Klingen Eisenstein measure. We also constructed a Siegel Eisenstein measure in [9] which we write as* $\mathcal{E}_{sieg}$*.*

Here at ϕ the weight of the Klingen Eisenstein series constructed is $(a_1+m_\phi, a_2+m_\phi, 0; \kappa)$. We also remark that the need to extend the scalar from $\mathcal{O}_L$ to $\mathcal{O}_L^{ur}$ is due to the fact that in the construction we need to specify points in the Igusa variety for GU(2) when applying equation (7), which can only be defined over $\mathcal{O}_L^{ur}$. To adapt to the situation of section 3, we multiply the family constructed in (ii) above by $\boldsymbol{\psi}(\det -)$ (so that we fix the weight a_1, a_2 and allow a_3, b_1 to vary). According to the control theorems proved in section 3 and Theorem 3.14 the family constructed thereby comes from a semi-ordinary family defined there. By an appropriate weight map $\tilde{\Lambda} \to \mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$ (we omit the precise formula) this gives a $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$-coefficients family in the sense of section 3.3.

The interpolation formula for the p-adic L-function considered above is not satisfying since it involves non-explicit Archimedean constants. But in fact it also has the following interpolation property if $a_1 = a_2 = 0$. For a Zariski dense set of arithmetic points $\phi \in \mathrm{Spec}\Lambda_\mathcal{K}$ such that $\phi \circ \boldsymbol{\xi}$ is the p-adic avatar of a Hecke character ξ_ϕ of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ of infinite type $(-\kappa - \frac{1}{2}, -\frac{1}{2})$ for some $\kappa \geq 6$, of conductor (p^t, p^t) $(t > 0)$ at p, then:

$$\phi(\mathcal{L}^\Sigma_{f,\mathcal{K}}) = \frac{p^{(\kappa-3)t}\xi^2_{1,p}(p^{-t})\mathfrak{g}(\xi_{1,p}\chi^{-1}_{1,p})\mathfrak{g}(\xi_{1,p}\chi^{-1}_{2,p})L^\Sigma(\tilde{\pi}, \xi_\phi, 0)(\kappa-1)!(\kappa-2)!\Omega_p^{2\kappa}}{(2\pi i)^{2\kappa-1}\Omega_\infty^{2\kappa}}. \tag{2}$$

Here $\mathfrak{g}$ is the Gauss sum and $\chi_{1,p}, \chi_{2,p}$ are characters such that $\pi(\chi_{1,p}, \chi_{2,p}) \simeq \pi_{f,p}$. Note that the weight $a_1 = a_2 = 0$ is nothing but the weight considered in [69] and the computations carry out in the same way. Note also the restrictions in [69] on conductors of π and ξ are put to prove the pullback formulas for Klingen Eisenstein series and has nothing to do with interpolation formula for p-adic L-functions. This computation is also done in the forthcoming work [8].) We also remark that in our situation it is possible to determine the constants $c'_{\underline{k}_\phi,0,\kappa_\phi}$ by taking an auxiliary eigenform ordinary at p and comparing our construction with Hida's (although we do not need it in this paper).

We can also construct the complete p-adic L-function $\mathcal{L}_{f,\mathcal{K},\xi}$ by putting back all the local Euler factors at primes in Σ. By doing this we only get elements in $\mathrm{Frac}\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$. In some cases we can study the integrality of it by comparing with other constructions. There is another way of constructing this p-adic L-function using Rankin-Selberg method by adapting the construction in [15]. We let $\mathbf{g}$ be the Hida family of normalized ordinary CM forms corresponding to the family of characters of $\Gamma_\mathcal{K}$ (thus the specialization of $\mathbf{g}$ to weight one is the Eisenstein series corresponding to $1 \oplus \chi_{\mathcal{K}/\mathbb{Q}}$. We apply Hida's construction to the Rankin-Selberg product of f and specializations of $\mathbf{g}$ of weight higher than 2. Note that although Hida's construction assumes both forms are nearly ordinary, however, it works out in the same way in our situation since in the Rankin-Selberg product the form with higher weight is the CM form which is ordinary by our assumption that p splits in $\mathcal{K}$. The p-adic L-functions of Hida are not integral since he used Petersson inner product as the period. The ratio of this Petersson Inner product over the CM period is a Katz p-adic L-function $\mathcal{L}^{Katz}\cdot h_{\mathcal{K}}$ by [22](this interpolates the algebraic part of $L(\chi_\phi\chi_\phi^{-c},1)$ where χ_ϕ is the CM character corresponding to the CM form g_ϕ. Here $h_{\mathcal{K}}$ is the class number for $\mathcal{K}$). Under assumption (1) of Theorem 5.3, we know the local Hecke algebra corresponding to the CM form $\mathbf{g}$ is Gorenstein, and [22] shows that the congruence module for $\mathbf{g}$ is generated by $\mathcal{L}^{Katz}\cdot h_{\mathcal{K}}$. Comparing the interpolation formula 2 with [15, Theorem I] we see that if we multiply Hida's p-adic L-function by $\mathcal{L}^{Katz}\cdot h_{\mathcal{K}}$ then we recover our p-adic L-function in Proposition 4.4. So under assumption (1) of Theorem 5.3 the $\mathcal{L}_{f,\mathcal{K},\xi}$ is in $\mathcal{O}_L^{ur}[[\Gamma_{\mathcal{K}}]]$. By our discussion in [67, Section 6.4] we know that under the assumption (1) of Theorem 5.3 $\mathcal{L}_{f,\mathcal{K},\xi}$ is co-prime to any height one prime of $\mathcal{O}_L^{ur}[[\Gamma_{\mathcal{K}}]]$ which is not a pullback of a height one prime of $\mathcal{O}_L^{ur}[[\Gamma^+]]$. Under assumption (2) of Theorem 5.3 we only know $\mathcal{L}_{f,\mathcal{K},\xi}$ is in $\mathrm{Frac}\mathcal{O}_L^{ur}[[\Gamma_{\mathcal{K}}]]$ and we call the fractional ideal generated by $\mathcal{L}_{f,\mathcal{K},\xi}$ to be $\mathcal{O}_L^{ur}[[\Gamma_{\mathcal{K}}]]\cdot\mathcal{L}_{f,\mathcal{K},\xi}\subset\mathrm{Frac}\mathcal{O}_L^{ur}[[\Gamma_{\mathcal{K}}]]$.

4.7 Galois Representations for Klingen Eisenstein Series

We can also associate a reducible Galois representation to the holomorphic Klingen Eisenstein series constructed with the same recipe as in subsection 2.4. The resulting Galois representation is:

$$\sigma_{\tau'}\sigma_{\psi^c}\epsilon^{-\kappa}\oplus\sigma_{\psi^c}\epsilon^{-3}\oplus\rho_f.\sigma_{\tau^c}\epsilon^{-\frac{\kappa+2}{2}}.$$

5 Proof of Greenberg's Main Results

In this section we assume the π we start with has weight two so that the Jacquet-Langlands correspondence is trivial representation at ∞. This is because we can do the computations at arithemtic points $\phi\in\mathcal{X}^{gen}$ and in this case they are largely carried out in [67].

5.1 p-adic Properties of Fourier-Jacobi Coefficients

Our goal here is to prove Proposition 5.2 which, roughly speaking says that certain Fourier-Jacobi coefficient of $\mathbf{E}_{Kling}$ which is a unit.

Interpolating Petersson Inner Products

Recall that in [67, section 6] we made a construction for interpolating Petersson inner products of forms on definite unitary groups, one invariant under $B(\mathbb{Z}_p)$ and one invariant under ${}^tB(\mathbb{Z}_p)$ (we use B to denote the upper triangular Borel subgroup of GL_2, noting $\mathrm{U}(2)(\mathbb{Z}_p)\simeq\mathrm{GL}_2(\mathbb{Z}_p)$): For a compact open subgroup $K=\prod_v K_v$ of $\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})$ which is $\mathrm{U}(2)(\mathbb{Z}_p)$ at p we take $\{g_i^\triangle\}_i$ a set of representatives for $\mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})/K_0(p)$ where we write $K_0(p)$ also for the open compact group $\prod_{v\nmid p}K_v\times K_0(p)$. Suppose K is sufficiently small so that for all i we have $\mathrm{U}(2)(\mathbb{Q})\cap g_i^\triangle K g_i^{\triangle-1}=1$. For an ordinary Hida family $\mathbf{h}$ of eigenforms with some coefficient ring $\mathbb{I}$ (whose p-part of level group is in ${}^tB(\mathbb{Z}_p)$ modulo powers of p) we construct a set of bounded $\mathbb{I}$-valued measure μ_i on $N^-(p\mathbb{Z}_p)$ as follows. We only need to specify the measure for sets of the form $t^-N^-(\mathbb{Z}_p)(t^-)^{-1}n$ where $n\in N^-(\mathbb{Z}_p)$ and t^- a matrix of the form $\begin{pmatrix}p^{t_1}&\\&p^{t_2}\end{pmatrix}$ with $t_2>t_1$. We assign $\mathbf{h}(g_int^-)\lambda(t^-)^{-1}$ as its measure where $\lambda(t^-)$ is the Hecke eigenvalue of $\mathbf{h}$ for U_{t^-} (which is a unit since $\mathbf{h}$ is ordinary).

This measure is well defined by the expression for Hecke operators U_{t^-}. The above set $\{\mu_i\}_i$ can be viewed as a measure on $\mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})/K^{(p)}$ by requiring it to be invariant under the right action of $B(\mathbb{Z}_p)$, which we denote as $\mu_\mathbf{h}$. For an $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$-valued family of forms $\mathbf{g}$ on $\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})$, we can regard it as a continuous function on $\mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})/K^{(p)}$ (giving $\mathrm{U}(2)(\mathbb{Z}_p)$ the topology as a p-adic Lie group). Thus we can talk about integral of $\mathbf{g}$ against the measure $\mu_\mathbf{h}$, which we write as $\int_{[\mathrm{U}(2)]}\mathbf{g}d_{\mu_\mathbf{h}}$.

We refer to [67, Section 7.5] for the definition of the theta function θ_1 and a functional l_{θ_1} on the space of p-adic automorphic forms on $\mathrm{U}(3,1)$ essentially by taking Fourier-Jacobi coefficients (viewed as a form on $P(\mathbb{A}_\mathbb{Q})$) and pair with the theta function θ_1. It maps an $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$-adic family of forms on $\mathrm{U}(3,1)$ to an $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$-adic family of forms on $\mathrm{U}(2,0)$.

In [67, Section 7.3] we constructed three-dimensional families of CM forms $\mathbf{h}$ and $\boldsymbol{\theta}$ on $\mathrm{U}(2)$ (both invariant under $B(\mathbb{Z}_p)$) associated to families of CM characters $\chi_\mathbf{h}$ and $\chi_{\boldsymbol{\theta}}$ and we write their restrictions to the two dimensional $\mathrm{Spec}\tilde{\Lambda}$ still using the same symbols. The $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$-linear functional

$$F\mapsto\int l_{\theta_1}(F)d_{\mu_{\pi(g_2'\begin{pmatrix} & 1\\ 1 & \end{pmatrix}_p)\mathbf{h}}}$$

is defined on the space of $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$-adic families on $\mathrm{U}(3,1)$. As in [67] we have to show that the image of $\mathbf{E}_{Kling}$ under this functional is coprime to all height one primes of $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$ except (p). So we want to study $\int l_{\theta_1}(\mathbf{E}_{Kling})d\mu_{(\pi(g_2'\begin{pmatrix} & 1\\ 1 & \end{pmatrix}_p)\mathbf{h})}$. Since $\mathbf{E}_{Kling}$ is realized as $\langle\int i^{-1}(\mathcal{E}_{sieg}),\varphi\rangle_{low}$ ($i:\mathrm{U}(3,1)\times\mathrm{U}(0,2)\hookrightarrow\mathrm{U}(3,3)$ and $\langle,\rangle_{low}$ means taking inner product with respect to the $\mathrm{U}(0,2)$-factor) by Proposition 4.1, we need first to study

$$A_1:=\int l_{\theta_1}^{up}i^{-1}(\mathcal{E}_{sieg})d^{up}_{\mu(\pi(g_2'\begin{pmatrix} & 1\\ 1 & \end{pmatrix}_p)\mathbf{h})}$$

regarded as a family of p-adic automorphic forms on $\mathrm{U}(2)$. Here $i^{-1}(\mathcal{E}_{sieg})$ is a measure of forms on $\mathrm{U}(3,1)\times\mathrm{U}(2)$ and the $l_{\theta_1}^{up},d^{up}$ means the functional and integration on the $\mathrm{U}(3,1)$ factor in $\mathrm{U}(3,1)\times\mathrm{U}(0,2)$. Then

$$A:=\langle A_1,\varphi\rangle_{\mathrm{U}(2)}=\int l_{\theta_1}(\mathbf{E}_{Kling})d\mu_{(\pi(g_2'\begin{pmatrix} & 1\\ 1 & \end{pmatrix}_p)\mathbf{h})}$$

We remark that A_1 is invariant under ${}^tK_0(p)$.

We do the Fourier-Jacobi coefficients calculations as in [67], in particular Proposition 5.28 and Corollary 5.29 there at arithmetic points in $\mathcal{X}^{gen}$ whose corresponding characters have conductors p^t. This shows that up to multiplying by an element in $\mathbb{Q}_p^\times$, the A is interpolating

$$p^t\phi(\mathcal{L}_5^\Sigma\mathcal{L}_6^\Sigma)\int_{[\mathrm{U}(2)]}(\pi(g_2')\mathbf{h}_\phi)(g)\boldsymbol{\theta}_\phi(g\begin{pmatrix} & 1\\ 1 & \end{pmatrix}_p)(\pi(g_1)f_\vartheta)(g)dg.$$

Here $\mathbf{h}_\phi$ and $\boldsymbol{\theta}_\phi$ are specializations of $\mathbf{h}$ and $\boldsymbol{\theta}$ at ϕ, $\mathcal{L}_5^\Sigma$ and $\mathcal{L}_6^\Sigma$ are defined in [67, subsection 7.5] which are Σ-primitive p-adic L-functions for certain CM characters. They come from the pullback integral for $\mathbf{h}$ under $\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$. By our choices of characters they are some $\bar{\mathbb{Q}}_p^\times$ multiples of a unit in $\mathcal{O}_L[[\Gamma_\mathcal{K}]]$. In [67] we also constructed families $\tilde{\mathbf{h}}_3,\tilde{\boldsymbol{\theta}}_3$ in the dual automorphic representations for $\mathbf{h}$ and $\boldsymbol{\theta}$. Let $\tilde{f}_{\tilde{\vartheta}}\in\tilde{\pi}$ be chosen the same as as in [67, Section 7.5] at primes outside p. But at p we take it as the stabilization with U_p-eigenvalue α_1^{-1} (recall α_1 is the eigenvalue for the U_p action on f_ϑ). We consider the expression at arithmetic point ϕ

$$\tilde{A}_\phi := p^t\int_{[\mathrm{U}(2)]}\pi(g_4')\tilde{\mathbf{h}}_{3,\phi}(g)\tilde{\boldsymbol{\theta}}_{3,\phi}(g\begin{pmatrix} & 1\\ 1 & \end{pmatrix}_p)\pi(g_3)\tilde{f}_{\tilde{\vartheta}}(g)dg.$$

From our previous discussions they are interpolated by an element $\tilde{A}\in\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. We are going to calculate $A\cdot\tilde{A}$ using Ichino's triple product formula. We do this by calculating it at arithmetic points in $\mathcal{X}^{gen}$. This is enough since these points are Zariski dense. We refer to [67, subsection 7.4] for a summary of the backgrounds of Ichino's formula. The local calculations are the same as *loc.cit* except at the p-adic places where we have different assumptions for ramification. (In [67] the central character for f_ϕ has conductor p^t at p while our π_f here is unramified at p.) We give a lemma for our situation.

Lemma 5.1. *Let $\chi_{h,1},\chi_{h,2},\chi_{\theta,1},\chi_{\theta,2},\chi_{f,1},\chi_{f,2}$ be character of $\mathbb{Q}_p^\times$ whose product is the trivial character and such that $\chi_{h,1},\chi_{\theta,1},\chi_{f,1},\chi_{f,2}$ are unramifed and $\chi_{h,2}\cdot\chi_{\theta,2}$ is unramified. Let $f_p\in\pi(\chi_{f,2},\chi_{f,1})$ and by using the induced representation model f is the characteristic function of K_1wK_1. Similarly we define $\tilde{f}_p\in\pi(\chi_{f,2}^{-1},\chi_{f,1}^{-1})$. So f is a Hecke eigenvector for T_p with eigenvalue $\chi_{f,1}(p)$. Let $h_p\pi(\chi_{h,1},\chi_{h,2}),\theta_p\in\pi(\chi_{\theta,1},\chi_{\theta,2}),\tilde{h}_p\in\pi(\chi_{h,1}^{-1},\chi_{h,2}^{-1}),\tilde{\theta}_p(\chi_{\theta,1}^{-1},\chi_{\theta,2}^{-1})$ be the f_{χ_h}, f_{χ_θ}, $\tilde{f}_{\tilde{\chi}_h}$, $\tilde{f}_{\tilde{\chi}_\theta}$ defined in [67, lemma 7.4]. Then the local triple product integral (defined at the beginning of [67, subsection 7.4])*

$$\frac{I_p(h_p\otimes\theta_p\otimes f_p,\tilde{h}_p\otimes\tilde{\theta}_p\otimes\tilde{f}_p)}{\langle h_p,\tilde{h}_p\rangle\langle\theta_p,\tilde{\theta}_p\rangle\langle f_p,\tilde{f}_p\rangle}$$

is

$$\frac{p^{-t}(1-p)}{1+p}\cdot\frac{1}{1-\chi_{h,1}(p)\chi_{\theta,1}(p)\chi_{f,1}(p)p^{-\frac{1}{2}}}\cdot\frac{1}{1-\chi_{h,1}(p)\chi_{\theta,2}(p)\chi_{f,1}(p)p^{-\frac{1}{2}}}.$$

Proof. This is an easy consequence of [67, lemma 7.4] and [73, Proposition 3.2]. □

Now as in [67, Section 7.5] by computing at arithmetic points $\phi\in\mathcal{X}^{gen}$ and applying Ichino's formula, the local integrals at finite primes are non-zero constants in $\bar{\mathbb{Q}}_p^\times$ (fixed throughout the family). We conclude that up to multiplying by an element in $\bar{\mathbb{Q}}_p^\times$ the $A\cdot\tilde{A}$ equals $\mathcal{L}_5^\Sigma\mathcal{L}_6^\Sigma\mathcal{L}_1\mathcal{L}_2$ where $\mathcal{L}_1$ is the p-adic L-function interpolating the algebraic part of $L(\lambda^2(\chi_{\boldsymbol{\theta}}\chi_{\mathbf{h}})_\phi,\frac{1}{2})$ (λ is the splitting character of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ we use to define theta functions, see [67, Section 3]) which we can choose the Hecke characters properly so that it is a unit in $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$. (Note that since the CM character λ^2 has weight higher than f the result cited in [67, subsection 7.2] of M. Hsieh does not assume that f is ordinary). The $\mathcal{L}_2$ is the algebraic part of $L(f,\chi_\theta^c\chi_h,\frac{1}{2})\in\bar{\mathbb{Q}}_p$ (fixed throughout the family) which we can choose to be non-zero. (See the calculations in [67, subsection 7.5].) The $\mathcal{L}_5^\Sigma$ and $\mathcal{L}_6^\Sigma$ are also units in $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$ up to multiplying by an element in $\bar{\mathbb{Q}}_p^\times$ by our choices of the characters $\chi_{\boldsymbol{\theta}}$ and $\chi_{\mathbf{h}}$.

To sum up we get the following proposition.

Proposition 5.2. *Any height one prime of* $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$ *containing* $\int l_{\theta_1}(\mathbf{E}_{Kling})d\mu_{\pi(g_2'\begin{pmatrix} & 1\\ 1 & \end{pmatrix}_p)\mathbf{h}}$ *must be* (p).

Proof. The above discussion implies that $A\cdot\tilde{A}=\tilde{A}\cdot\int l_{\theta_1}(\mathbf{E}_{Kling})d\mu_{\pi(g_2'\begin{pmatrix} & 1\\ 1 & \end{pmatrix}_p)\mathbf{h}}$ is a unit in $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$ times an element in $\bar{\mathbb{Q}}_p^\times$. Thus the proposition follows. □

5.2 Proof of Greenberg's Main Conjecture

To state our result we need one more definition: suppose g is a cuspidal eigenform on $\mathrm{GL}_2/\mathbb{Q}$ which is nearly ordinary at p. We have a p-adic Galois representation $\rho_g: G_\mathbb{Q}\to\mathrm{GL}_2(\mathcal{O}_L)$ for some $L/\mathbb{Q}_p$ finite. We say g satisfies:
(irred) If the residual representation $\bar{\rho}_g$ is absolutely irreducible.
Also it is known that $\rho_g|_{G_p}$ is isomorphic to an upper triangular one. We say it satisfies:
(dist) If the Galois characters of G_p giving the diagonal actions are distinct modulo the maximal ideal of $\mathcal{O}_L$. Now we prove the following theorem which is one divisibility of Conjecture 2.1.

Theorem 5.3. *Let* π *be an irreducible cuspidal automorphic representation of* $\mathrm{GL}_2/\mathbb{Q}$ *of weight* 2*, square free level* N *and trivial character. Let* ρ_π *be the associated Galois representation. Assume* π_p *is good supersingular with distinct Satake paramters. Suppose also for some odd non-split* q*,* $q||N$*. Let* ξ *be a Hecke character of* $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ *with infinite type* $(-\frac{1}{2},-\frac{1}{2})$*. Suppose* $(\xi|.|^{\frac{1}{2}})|_{\mathbb{A}_\mathbb{Q}^\times}=\omega\circ\mathrm{Nm}$ *(*ω *is the Techimuller character).*
(1) Suppose the CM form g_ξ *associated to the character* ξ *satisfies* (dist) *and* (irred) *defined above and that for each inert or ramified prime* v *we have the conductor of* ξ_v *is not* (ϖ_v) *where* ϖ_v *is a uniformizer for* $\mathcal{K}_v$ *and that:*

$$\epsilon(\pi_v,\xi_v,\frac{1}{2})=\chi_{\mathcal{K}/\mathbb{Q},v}(-1).$$

Then we have $\mathcal{L}_{f,\xi,\mathcal{K}}\in\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$ *and* $(\mathcal{L}_{f,\mathcal{K},\xi})\supseteq\mathrm{char}_{\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]}(X_{f,\mathcal{K},\xi})$ *as ideals of* $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$*.*
(2) If we drop the conditions (irred) *and* (dist) *and the conditions on the local signs in (1), but assume that the p-adic avatar of* $\xi|.|^{\frac{1}{2}}(\omega^{-1}\circ\mathrm{Nm})$ *factors through* $\Gamma_\mathcal{K}$*, then*

$$(\mathcal{L}_{f,\mathcal{K},\xi})\supseteq\mathrm{char}_{\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]\otimes_{\mathcal{O}_L}L}(X_{f,\mathcal{K},\xi})$$

is true as fractional ideals of $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]\otimes_{\mathcal{O}_L}L$*.*

We note that the assumption on the existence of q is to make sure that we can choose the unitary group Subsection 2.2 properly so that the Jacquet-Langlands correspondence of f to the corresponding quaternion algebra D exists.

Proof. We refer to [67, section 8.1] for the definitions for Hecke operators for $\mathrm{U}(3,1)$ at unramified primes. Let $K_\mathcal{D}$ be an open compact subgroup of $\mathrm{U}(3,1)(\mathbb{A}_\mathbb{Q})$ maximal at p and all primes outside Σ such that the Klingen Eisenstein series we construct is invariant under $K_\mathcal{D}$. We let $\mathbb{T}_\mathcal{D}$ be the reduced Hecke algebra generated by the Hecke operators at unramified primes space of the two variable family of semi-ordinary cusp forms with level group $K_\mathcal{D}$, the U_i operator at p, and then take the reduced quotient. Let the Eisenstein ideal $I_\mathcal{D}$ of $\mathbb{T}_\mathcal{D}$ to be generated by $\{t-\lambda(t)\}_t$ for t in the abstract Hecke algebra and $\lambda(t)$ is the Hecke eigenvalue of t acting on $\mathbf{E}_{Kling}$ and let $\mathcal{E}_{\mathcal{D}}$ be the inverse image of $I_{\mathcal{D}}$ in $\mathcal{O}_L[[\Gamma_{\mathcal{K}}]] \subset \mathbb{T}_{\mathcal{D}}$.

Now the main theorem can be proven in almost the same way as [67, Section 8], using Proposition 5.2 and 4.4. One uses the fundamental exact sequence Theorem 3.17 to show that $(\mathcal{L}^\Sigma) \supseteq \mathcal{E}_{\mathcal{D}}$ as in Lemma 8.4 of *loc.cit.* Then use the lattice construction (Proposition 8.2 there) to show that $\mathcal{E}_{\mathcal{D}}$ contains the characteristic ideal of the dual Selmer group. Note also that to prove part (2) of the main theorem we need to use Lemma 8.3 of *loc.cit.* The only difference is to check the condition (9) in Section 8.3 of *loc.cit*: We suppose our pseudo-character $R = R_1 + R_2 + R_3$ where R_1 and R_2 are 1-dimensional and R_3 is 2-dimensional. Then by residual irreducibility we can associate a 2-dimensional $\mathbb{T}_{\mathcal{D}}$-coefficient Galois representation. Take an arithmetic point x in the absolute convergence region for Eisenstein series such that $a_2 - a_3 >> 0$ and $a_3 + b_1 >> 0$ and consider the specialization of the Galois representation to x. First of all as in [61, Theorem 7.3.1] a twist of this descends to a Galois representation of $G_{\mathbb{Q}}$ which we denote as $R_{3,x}$. By our description for the local Galois representations for semi-ordinary forms at p we know that $R_{3,x}$ has Hodge-Tate weight $0, 1$ and is crystalline (by the corresponding property for $R_x = R_1 + R_2 + R_3$, note that R_x corresponds to a Galois representation for a classical form unramified at p by Theorem 3.10, 3.11 and Proposition 3.12). If p is at least 5 then $R_{3,x}$ is modular over a solvable totally real field $F/\mathbb{Q}$ by [63, Theorem B]. If p is 3 then by [28] it must be modular unless the residual representation were induced from a Galois character for $\mathbb{Q}(\sqrt{(-1)^{\frac{p-1}{2}}p})$. As we noted before $\bar{\rho}_f|_{G_p}$ is irreducible by [7]. So the restriction of it to I_p has semi-simplification as $\mathrm{diag}(\omega_2^i, \omega_2^{pi})$ where ω_2 is the fundamental character of level 2 and i is some integer. Since ρ_f is crystalline of weight $(0, 1)$ the i has to be congruent to 1 modulo $(p-1)$. But if $\bar{\rho}_f$ is induced from the ramified quadratic field extension the i has to be a multiple of $\frac{(p+1)}{2}$, a contradiction if $p = 3$. To sum up in any case $R_{3,x}$ is modular over a solvable totally real field. These implies some solvable base change of R_x to a totally real field is CAP, contradicting the result of [14, Theorem 2.5.6].

Once we get one divisibility for $\mathcal{L}^\Sigma_{f,\mathcal{K},\xi}$, up to height one primes which are pullbacks of height one primes of $\mathcal{O}_L^{ur}[[\Gamma_{\mathcal{K}}^+]]$ (coming from local Euler factors at non-split primes in Σ, by our discussion in [67, Section 6.4] on μ-invariants), the corresponding result for $\mathcal{L}_{f,\mathcal{K},\xi}$ also follows by using [13, Proposition 2.4] as in [67, End of 8.3] (note that $\mathcal{K}_\infty$ contains the cyclotomic $\mathbb{Z}_p$-extension). □

6 The Two Variable ± Main Conjectures

6.1 Local Theory and Two-Variable Main Conjecture

In this subsection we develop some local theory. The main goal is to construct two-variable regulator maps Col^+ and LOG^+ which are important for our argument. The Col^+ is essentially constructed by Kim [26] and the LOG^+ is not in literature.

We note that for any prime v above p the field $\mathcal{K}_v$ is the composition of the maximal unramified $\mathbb{Z}_p$ extension of $\mathbb{Q}_p$ and the cyclotomic $\mathbb{Z}_p$-extension. So it is necessary to study the Galois cohomology of this composed extension. We have an isomorphism $\mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[X]]$ sending γ to $(1 + X)$. Define $\omega_n^+(X) := X \prod_{2 \le m \le n, 2|m} \Phi_m(X)$ and $\omega_n^-(X) := \prod_{1 \le m \le n, 2\nmid m} \Phi_m(X)$ (our definition is slightly different from [33]). We recall some notions from [26] with some modifications. For $k/\mathbb{Q}_p$ an unramified extension of degree d let $\mathcal{O}_k$ be its integer ring, consider the field $k(\zeta_{p^{n+1}})$ and let $\mathfrak{m}_{k(\zeta_{p^{n+1}})}$ be the maximal ideal of its valuation ring $\mathcal{O}_{k(\zeta_{p^{n+1}})}$. Let k_n be the $\mathbb{Z}/p^n\mathbb{Z}$ sub-extension of $k(\zeta_{p^{n+1}})$ with $\mathfrak{m}_{k,n}$ the maximal ideal of its integer ring. We define

$$E^+[k(\mu_{p^{n+1}})] = \{x \in E(k(\mu_{p^{n+1}}))|\mathrm{tr}_{k(\mu_{p^{n+1}})/k(\mu_{p^{\ell+2}})}(x) \in E(k(\mu_{p^{\ell+1}})), 0 \le \ell < n, 2|\ell\}.$$

We also define the $+$-norm subgroup

$$\hat{E}^+[\mathfrak{m}_{k(\mu_{p^{n+1}})}] = \{x \in \hat{E}(\mathfrak{m}_{k(\mu_{p^{n+1}})})|\mathrm{tr}_{k(\mu_{p^{n+1}})/k(\mu_{p^{\ell+2}})}(x) \in \hat{E}(\mathfrak{m}_{k(\mu_{p^{\ell+1}})}), 0 \le \ell < n, 2|\ell\}.$$

Let

$$\log_f(X) = \sum_{n=1}^{\infty}(-1)^n \frac{f^{(2n)}(X)}{p^n}$$

for $f^{(n)} = f^{\varphi^{n-1}} \circ f^{\varphi^{n-2}} \circ \cdots f(X)$. As in [26], for $z \in \mathcal{O}_k^\times$ we define a point $c_{n,z} \in \hat{E}[\mathfrak{m}_{k(\zeta_{p^n})}]$ such that

$$\log_{\hat{E}}(c_{n,z}) = [\sum_{i=1}^{\infty}(-1)^{i-1} z^{\varphi^{-(n+2i)}} \cdot p^i] + \log_{f_z^{\varphi^{-n}}}(z^{\varphi^{-n}} \cdot (\zeta_{p^n} - 1))$$

where φ is the Frobenius on k and $f_z(x) := (x+z)^p - z^p$. Then the following lemma is proved in [26, Page 5].

Lemma 6.1.

$$\mathrm{tr}_{k(\zeta_{p^{n+2}})/k(\zeta_{p^{n+1}})} c_{n+2,z} = -c_{n,z}.$$

We also use the same notation $c_{n,z}$ for $\mathrm{tr}_{k(\zeta_{p^n})/k_{n-1}} c_{n,z} \in \mathfrak{m}_{k,n-1}$ as well. Let $k = k^m$ be unramified $\mathbb{Z}/p^m\mathbb{Z}$-extension of $\mathbb{Q}_p$. We sometimes write $k_{n,m}$ for the above defined k_n with this $k = k^m$. Let $\Lambda_{n,m} = \mathbb{Z}_p[\mathrm{Gal}(k_{n,m}/\mathbb{Q}_p)]$.

Lemma 6.2. *For even n's one can choose a system $\{c_{n,m}\}_{n,m}$ for $c_{n,m} \in \hat{E}^+[\mathfrak{m}_{k_{n,m}}]$ such that*

$$\mathrm{tr}_{k_{n,m+1}/k_{n,m}} c_{n,m+1} = c_{n,m},$$

$$\mathrm{tr}_{k_{n,m}/k_{n-2,m}} c_{n,m} = -c_{n-2,m}.$$

Proof. This can be done in the following way: choose $d := \{d_m\}_m \in \varprojlim_m \mathcal{O}_{k^m}$ where the transition is given by the trace map such that d generates this inverse limit over $\mathbb{Z}_p[[U]]$ (existence is guaranteed by the normal basis theorem). If $d_m = \sum_j a_{m,j}\zeta_j$ where ζ_j are roots of unity and $a_{m,j} \in \mathbb{Z}_p$. Define $c_{n,m} = \sum a_{m,j} c_{n,\zeta_j}$. We prove the first identity and the second one is a consequence of the above lemma.

For any $z = \zeta_j$ a root of unity whose conductor is prime to p, we have

$$\begin{aligned} f_z^{\varphi^{-n}}(z^{\varphi^{-n}}(\zeta_{p^n}-1)) &= f_z^{\varphi^{2k-n-1}} \circ f_z^{\varphi^{2k-n-2}} \circ \cdots \circ f_z^{\varphi^{-n}}(z^{\varphi^{-n}}(\zeta_{p^n}-1)) \\ &= f_z^{\varphi^{2k-n-1}} \circ f_z^{\varphi^{2k-n-2}} \circ \cdots \circ f_z^{\varphi^{1-n}}(z^{\varphi^{-n}}(z^{\varphi^{-n}}(\zeta_{p^{n-1}}-1)) \\ &= \cdots \\ &= z^{\varphi^{2k-n}}(\zeta_{p^{n-2k}}-1) \end{aligned}$$

if $2m < n$ and equals 0 otherwise. So

$$\begin{aligned}\log_{\hat{E}} c_{n,m} &= \sum_{i,j}(-1)^{i-1}\cdot a_{m,j}\zeta_j^{\varphi^{-(n+2i)}}\cdot p^i + \sum_j\sum_{2k<n}(-1)^k a_{m,j}\frac{\zeta_j^{\varphi^{2k-n}}(\zeta_{p^{n-2k}}-1)}{p^k}\\ &= \sum_i(-1)^{i-1}p^i(d_m)^{\varphi^{-(n+2i)}} + \sum_{2k<n}\frac{(-1)^k(\zeta_{p^{n-2k}}-1)}{p^k}(d_m)^{\varphi^{2k-n}}.\end{aligned}$$

Thus

$$\begin{aligned}\log_{\hat{E}} \mathrm{tr}_{m/m-1}c_{n,m} &= \sum_i(-1)^{i-1}p^i(\mathrm{tr}d_m)^{\varphi^{-(n+2i)}} + \sum_{2k<n}\frac{(-1)^k(\zeta_{p^{n-2k}}-1)}{p^k}(\mathrm{tr}d_m)^{\varphi^{2k-n}}\\ &= \sum_i(-1)^{i-1}p^i d_{m-1}^{\varphi^{-(n+2i)}} + \sum_{2k<n}\frac{(-1)^k(\zeta_{p^{n-2k}}-1)}{p^k}d_{m-1}^{\varphi^{2k-n}}\\ &= \log_{\hat{E}} c_{n,m-1}.\end{aligned}$$

□

Definition 6.3. *Let n be an even number. Define*

$$\Lambda^+_{n,m} = \Lambda_{n,m}/\omega^+_n(X),$$
$$\Lambda^-_{n,m} = \Lambda_{n,m}/X\omega^-_n(X).$$

Lemma 6.4. *We have the following exact sequence*

$$0 \to \hat{E}(p\mathcal{O}_{k^m}) \to \Lambda^+_{n,m}c_{n,m} \oplus \Lambda^-_{n,m}c_{n-1,m} \to \hat{E}^+(\mathfrak{m}_{k_{n,m}}) \to 0.$$

The middle term is isomorphic to $\Lambda^+_{n,m} \oplus \Lambda^-_{n,m}$. The $c_{n,m}$ generates $\hat{E}[\mathfrak{m}_{k_{n,m}}]$ as a $\Lambda^+_{n,m}$-module.

Proof. The surjectivity to $\hat{E}^+(\mathfrak{m}_{k_{n,m}})$ is essentially proved in [26, Proposition 2.6] (compare also to the computations in the previous lemma). The other parts are easily proven (compare also with [33, Proposition 8.12]). □

Now we define the two-variable +-Coleman maps

$$H^1(k_{n,m},T)/H^1_+(k_{n,m},T) \simeq \Lambda^+_{m,n}$$

where $H^1_+(k_{n,m},T)$ is the exact annihilator of $E^+(k_{n,m})\otimes\mathbb{Q}_p/\mathbb{Z}_p$ under the Tate pairing. We define $P^+_{c_{n,m}}$ by

$$z \mapsto \sum_{\sigma\in\mathrm{Gal}(k_{n,m}/\mathbb{Q}_p)}(c^\sigma_{n,m},z)_{m,n}\sigma.$$

As is seen in [33, Proposition 8.19] the image of $P^+_{c_{n,m}}$ is contained in $\omega^-_n(X)\Lambda_{m,n}$ if we identify $\mathbb{Z}_p[\Gamma_n]$ with $\mathbb{Z}_p[X]/\omega_n(X)$ by sending γ to $1+X$. We define $\Lambda^+_{m,n} := \Lambda_{m,n}/\omega^+_n(X) \simeq \omega^-_n(X)\Lambda_{m,n}$. The + Coleman map $\mathrm{Col}^+_{n,m}$ is defined to make the following diagram commutative.

$$\begin{array}{ccc} H^1(k_{n,m},T) & \xrightarrow{\mathrm{Col}^+_{n,m}} & \Lambda^+_{n,m} \\ \downarrow & & \downarrow{\times\omega^-_n} \\ H^1(k_{n,m},T)/H^1_+(k_{n,m},T) & \xrightarrow{P^+_{c_{n,m}}} & \Lambda_{n,m} \end{array}$$

As is seen in the proof of [26, Theorem 2.7, 2.8] the $\mathrm{Col}^+_{m,n}$ is an isomorphism and they group together to define the following isomorphism.

Definition 6.5.

$$\mathrm{Col}^+ : \varprojlim_n \varprojlim_m \frac{H^1(k_{n,m}, T)}{H^1_+(k_{n,m}, T)} \simeq \Lambda.$$

The ++-Selmer group is defined by

$$\mathrm{Sel}^{++}(E/\mathcal{K}_\infty) := \ker\{\varinjlim_{K'} H^1(K', E[p^\infty]) \to \prod_{v|p} \frac{\varinjlim H^1(k_{n,m}, E[p^\infty])}{\varinjlim E^+(k_{n,m}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \times \prod_{v\nmid p} \varinjlim H^1(I_v, E[p^\infty])\}$$

and X^{++} its Pontryagin dual.

As noted at the end of [26] there are p-adic L-functions constructed by Loeffler, which are elements in $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

$$L^{++}_{f,p} := \frac{L_{p,\alpha,\alpha} - L_{p,\alpha,\beta} - L_{p,\beta,\alpha} + L_{p,\beta,\beta}}{4\alpha^2 \log^-_{v_0} \log^-_{\bar{v}_0}}$$

$$L^{+-}_{f,p} := \frac{L_{p,\alpha,\alpha} + L_{p,\alpha,\beta} - L_{p,\beta,\alpha} - L_{p,\beta,\beta}}{4\alpha \log^-_{v_0} \log^+_{\bar{v}_0}}$$

$$L^{-+}_{f,p} := \frac{L_{p,\alpha,\alpha} - L_{p,\alpha,\beta} + L_{p,\beta,\alpha} - L_{p,\beta,\beta}}{4\alpha \log^+_{v_0} \log^-_{\bar{v}_0}}$$

$$L^{--}_{f,p} := \frac{L_{p,\alpha,\alpha} + L_{p,\alpha,\beta} + L_{p,\beta,\alpha} + L_{p,\beta,\beta}}{4 \log^+_{v_0} \log^+_{\bar{v}_0}}$$

for $L_{p,\alpha,\alpha}$ interpolating

$$\alpha^{-\mathrm{ord}_{v_0}\mathfrak{f}_\chi}\alpha^{-\mathrm{ord}_{\bar{v}_0}\mathfrak{f}_\chi}\frac{L(E,\chi,1)}{\mathfrak{g}(\chi)\cdot|\mathfrak{f}_\chi|\cdot\Omega^+_E\Omega^-_E}$$

for χ a character of $\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$ and $\mathfrak{f}_\chi$ its conductor and similarly for the other three. (Here the roles played by $\pm$ are switched from [26] and is compatible with [33]). The $\log^-_{v_0}$ and $\log^-_{\bar{v}_0}$ will be defined at the beginning of subsection 7.3. The $\Omega^\pm_E$ are the $\pm$-periods of the newform f associated to the elliptic curve E multiplied by $(2\pi i)$, respectively (we refer to [62, 9.2, 9.3] for details). In fact Loeffler used another period factor which he called Ω_Π instead of $\Omega^+_E \cdot \Omega^-_E$ and proved that his double signed p-adic L-functions are in Λ. A priory we only know our $L^{\pm\pm}_{f,p}$ are in $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ because of different periods. (It is possible to prove they are in Λ but this is not needed for our argument). There is another period Ω^{can} called the canonical period defined in *loc.cit* using congruence numbers. We have the following

Lemma 6.6. *Up to multiplying by a p-adic unit we have*

$$\Omega^{\mathrm{can}} = \Omega^+_E \cdot \Omega^-_E.$$

Proof. This is just [62, Lemma 9.5]. □

Now we are ready to formulate the two-variable "++" main conjecture.

Conjecture 6.7. *The two variable ++- main conjecture states that X^{++} is a torsion $\Lambda_{\mathcal{K}}$-module and the characteristic ideal of X^{++} is generated by $L_{f,p}^{++}$ as an ideal of $\Lambda_{\mathcal{K}}$.*

We also refer to the *weak version* of the above conjecture by requiring that for any height one prime P of $\Lambda = \mathbb{Z}_p[[\Gamma \times \Gamma^-]]$ which is not a pullback of a height one prime of $\mathbb{Z}_p[[\Gamma^-]]$, the length of X_P^{++} over Λ_P is equal to $\mathrm{ord}_P L_p^{++}$.

Now we record a useful lemma.

Lemma 6.8. *The $\varprojlim_n \varprojlim_m H^1(k_{n,m}, T)$ is a free of rank two module over Λ and $H^1(k_{n,m}, T)$ is a free rank two module over $\Lambda_{n,m}$.*

Proof. We first note that T/pT is an irreducible module over $G_{\mathbb{Q}_p}$ [7]. Then it follows from the Euler characteristic formula that $H^1(\mathbb{Q}_p, T/pT)$ is a rank two $\mathbb{F}_p$ vector space. On the other hand one can prove that the inverse limit in the lemma has generic rank two over Λ (see e.g. in [44, appendix A]). Thus the first statement is true. The other statement is seen by noting that $H^1(k_{n,m}, T) = \varprojlim_n \varprojlim_m H^1(k_{n,m}, T)/(\gamma^n - 1, u^m - 1) \varprojlim_n \varprojlim_m H^1(k_{n,m}, T)$, which again follows from the irreducibility of T/pT as a $G_{\mathbb{Q}_p}$-module and the Galois cohomology long exact sequence. □

For the purpose of later argument we need one more regulator map LOG^+. We construct it in an explicit way. By the freeness of $H^1(k_{n,m}, T)$ over $\Lambda_{n,m}$ and that $\omega^+(X)c_{n,m} = 0$, we see that for any even n there is $b_{n,m} \in H^1(k_{n,m}, T)$ such that $\omega_n^-(X) \cdot b_{n,m} = (-1)^{\frac{n+2}{2}} c_{n,m}$. It is easily seen that one can choose the $b_{n,m}$'s such that $\mathrm{tr}_{k_{n,m}/k_{n-1,m}} b_{n,m} = b_{n-1,m}$ and $\mathrm{tr}_{k_{n,m}/k_{n,m-1}} b_{n,m} = b_{n,m-1}$.

Lemma 6.9. *$H_+^1(k_{n,m}, T)$ is a free $\Lambda_{n,m}$-module of rank one generated by $b_{n,m}$.*

Proof. For $y \in E(k_{n,m}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ and $x \in H_+^1(k_{n,m}, T)$ we can show that

$$\langle \omega_n^-(X) \cdot x, y \rangle = 0$$

from [26, Proposition 2.6] and [33, (8.29)]. Moreover we have for any $n' > 0$,

$$E(k_{n,m})/p^{n'} E(k_{n,m}) \hookrightarrow H^1(k_{n,m}, T/p^{n'} T)$$

has $\mathbb{Z}_p/p^{n'}\mathbb{Z}_p$-torsion-free cokernel since each term is free $\mathbb{Z}_p/p^{n'}\mathbb{Z}_p$-module of finite rank. So $E(k_{n,m})$ and $E(k_{n,m}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ are orthogonal complements of each other under local Tate pairing. So

$$\omega_n^-(X) \cdot x \in \omega_n^-(X) H^1(k_{n,m}, T) \cap \mathrm{Im}(E(k_{n,m}) \to H^1(k_{n,m}, T)).$$

By Lemma 6.4 we have $\omega_n^-(X) x \in \Lambda_{n,m} c_{n,m}$. This proves the lemma. □

Let $x = \varprojlim_n \varprojlim_m x_{n,m} \in \varprojlim_n \varprojlim_m H_+^1(k_{n,m}, T)$. If $x_{n,m} = f_{n,m} \cdot b_{n,m}$ for $f \in \Lambda_{n,m}$ then $\sum_{\tau \in \Gamma_n \times U_m} x_{n,m}^\tau \cdot \tau = f_{n,m} \cdot \sum_\tau b_{n,m}^\tau \cdot \tau$.

Definition 6.10. *We define* $\mathrm{LOG}^+ : \varprojlim_n \varprojlim_m H_+^1(k_{n,m}, T) \simeq \Lambda$ *by* $x \to \varprojlim_n \varprojlim_m f_{n,m}$.

Now recall that v_0 splits into p^t primes in $\mathcal{K}_\infty/\mathcal{K}$. We take a set of representatives $\{\gamma_1, \cdots, \gamma_{p^t}\}$ of $\Gamma_{\mathcal{K}}/\Gamma_p$. Write

$$H^1(\mathcal{K}_{v_0}, T \otimes \Lambda_{\mathcal{K}}) = \oplus_i H^1(\mathcal{K}_{v_0}, T \otimes \mathbb{Z}_p[[\Gamma_p]]) \cdot \gamma_i.$$

We define

$$\mathrm{Col}^+ x = \sum_i \gamma_i \cdot (\mathrm{Col}^+ x_i) \in \Lambda_{\mathcal{K}}.$$

We define LOG^+ similarly on $H_+^1(\mathcal{K}_{v_0}, T \otimes \Lambda_{\mathcal{K}})$. The following proposition will be useful.

Proposition 6.11. *Let ϕ be a finite order character of $\Gamma\times U$ such that $\phi(\gamma)$ and $\phi(u)$ are primitive p^n, p^m-th roots of unity. Then for integer m and even n,*

$$\sum_{\sigma\in\Gamma_n\times U_m}\log_{\hat{E}} x^{\sigma}_{n,m}\cdot\phi(\sigma)=(-1)^{\frac{n+2}{2}}\cdot\frac{\phi^{-1}(f_{n,m})\sum\log_{\hat{E}} c^{\sigma}_{n,m}\phi(\sigma)}{\omega^-_n(\phi^{-1})}. \tag{3}$$

$$\sum_{\sigma\in\Gamma_n\times U_m}\log_{\hat{E}}(c_{n,m})^{\sigma}\phi(\sigma)=\mathfrak{g}(\phi|_{\Gamma})\cdot\phi(u)^n\cdot\sum_{u'\in U_m}\phi(u')d^{u'}_m. \tag{4}$$

$$P^+_{c_{n,m}}(z)=(\sum_{\sigma}\log_{\hat{E}}(c^{\sigma}_{n,m})\cdot\sigma)(\sum_{\sigma}\exp^*(z^{\sigma})\cdot\sigma^{-1}). \tag{5}$$

Proof. Straightforward computation. The third identity used the description of the Tate pairing in [38, Page 5]. ☐

6.2 The One Variable Main Conjecture of Kobayashi

Now we briefly recall Kobayashi's one variable (cyclotomic) main conjecture. On the analytic side there is a + p-adic L-function $\mathcal{L}^+_{E,\mathbb{Q}}$ such that

$$\mathcal{L}^+_{E,\mathbb{Q}}(\zeta-1)=(-1)^{\frac{n+2}{2}}\frac{p^n\cdot L(E,\chi,1)}{\omega^-_n(\zeta)\mathfrak{g}(\chi)\cdot|\mathfrak{f}_\chi|\Omega_E}$$

if χ is a character of Γ with conductor p^n, $2|n>0$ and $\chi(\gamma)=\zeta$. On the other hand we define the +-Selmer group

$$\mathrm{Sel}^+_{E,\mathbb{Q},n}:=\ker\{H^1(\mathbb{Q}_{p,n},E[p^\infty])\to\prod_{v|p}\frac{H^1(\mathbb{Q}_{p,n},E[p^\infty])}{E^+(\mathbb{Q}_{p,n})\otimes\mathbb{Q}_p/\mathbb{Z}_p}\times\prod_{v\nmid p}H^1(I_v,E[p^\infty])\}.$$

Define $X^+_{E,\mathbb{Q}}:=(\varinjlim_n\mathrm{Sel}^+_{E,\mathbb{Q},n})^*$. This is a module over $\Lambda_{\mathbb{Q}}$.

Conjecture 6.12. *Kobayashi's main conjecture states that $X^+_{E,\mathbb{Q}}$ is a torsion $\Lambda_{\mathbb{Q}}$-module and the characteristic ideal of $X^+_{E,\mathbb{Q}}$ is generated by $\mathcal{L}^+_{E,\mathbb{Q}}$ as ideals of $\Lambda_{\mathbb{Q}}$.*

Kobayashi proved one containment $(\mathcal{L}^+_{E,\mathbb{Q}})\subseteq\mathrm{char}_{\Lambda_{\mathbb{Q}}}(X^+_{E,\mathbb{Q}})$ in [33], using results of Kato [25].

6.3 Special Case of Greenberg's Main Conjecture

We apply Theorem 5.3 to a special case that we will use to deduce the ±-main conjecture. We change the notations a little. On the arithmetic side we defined

$$\mathrm{Sel}^2_{\mathcal{K},f}=\ker\{H^1(\mathcal{K},T\otimes\Lambda^*(\Psi))\to\prod_{v\nmid p}H^1(\mathcal{K}_v,T\otimes\Lambda^*(\Psi))\times H^1(\mathcal{K}_{\bar{v}_0},T\otimes\Lambda^*(\Psi)).$$

$$X^2_{\mathcal{K},f}:=(\mathrm{Sel}^2_{\mathcal{K},f})^*.$$

On the analytic side there is a corresponding p-adic L-function $\mathcal{L}^2_{f,\mathcal{K}}\in\mathrm{Frac}(W(\bar{\mathbb{F}}_p))[[\Gamma_{\mathcal{K}}]])$ (taking the character ξ to be trivial character. The $W(R)$ means the Witt vector for R), which is the $\mathcal{L}_{f,\mathcal{K},1}$ we constructed in Section 4, with the following interpolation property. For a Zariski dense set of arithmetic points $\phi \in \mathrm{Spec}\Lambda$ such that $\xi_\phi := \phi \circ \Psi$ is the avatar of a Hecke character of infinite type $(\frac{\kappa}{2}, -\frac{\kappa}{2})$ with $\kappa \geq 6$ we have

$$\phi(\mathcal{L}^2_{f,\mathcal{K}}) = C\frac{p^{(\kappa-3)t}\xi^{-2}_{2,p}\chi^{-1}_{1,p}\chi^{-1}_{2,p}(p^{-t})\mathfrak{g}(\xi^{-1}_{2,p}\chi^{-1}_{1,p})\mathfrak{g}(\xi^{-1}_{2,p}\chi^{-1}_{2,p})L(\mathcal{K}, \pi_f, \bar{\xi}^c_\phi, \frac{\kappa}{2} - \frac{1}{2})(\kappa-1)!(\kappa-2)!\Omega^{2\kappa}_p}{(2\pi i)^{2\kappa-1}\Omega^{2\kappa}_\infty}.$$

Here Ω_∞ and Ω_p are the CM periods and p-adic periods for $\mathcal{K}$. The C is a constant in $\bar{\mathbb{Q}}^\times_p$, $\chi_{1,p}, \chi_{2,p}$ is such that the unitary representation $\pi_f \simeq \pi(\chi_{1,p}, \chi_{2,p})$ with $\mathrm{val}_p(\chi_{1,p}(p)) = -\frac{1}{2}$, $\mathrm{val}(\chi_{2,p}(p)) = \frac{1}{2}$. This case corresponds to part (2) of Theorem 5.3. This p-adic L-function can also be constructed by Rankin-Selberg method as in [15]. See [67, Remark 7.2] for a detailed discussion. In fact Hida's construction gives an element in $\mathrm{Frac}(\Lambda_\mathcal{K})$ and the above $\mathcal{L}^2_{f,\mathcal{K}}$ is obtained by multiplying Hida's by a Katz p-adic L-function $\mathcal{L}^{Katz}_\mathcal{K} \in \hat{\mathbb{Z}}^{ur}_p[[\Gamma_\mathcal{K}]]$ and the class number $h_\mathcal{K}$ of $\mathcal{K}$. The $\mathcal{L}^{Katz}_\mathcal{K}$ interpolates algebraic part of special L-values $L(0, \chi_\phi\chi^{-c}_\phi)$ where χ_ϕ are CM characters of $\Gamma_\mathcal{K}$ (see [22]). The denominator of Hida's p-adic L-function is related to certain congruence modules, which we are going to study in Section 8 using Rubin's work on CM main conjecture. (In fact one can show that this $\mathcal{L}^2_{f,\mathcal{K}}$ is in $W(\bar{\mathbb{F}}_p)[[\Gamma_\mathcal{K}]]$.)

Recall we have chosen $d = \varprojlim_m d_m \in \varprojlim \mathcal{O}^\times_{k^m}$ where the transition map is the trace map. We define $F_{d,2} \in \hat{\mathbb{Z}}^{ur}_p[[U]]$ as

$$\varprojlim_m \sum_{u \in U_v/p^m U_v} d^u_m \cdot u^2.$$

Then the discussion in [38, Section 6.4] on Katz p-adic L-functions (see also the discussion in Section 3.2 of *loc.cit*) implies that $\mathcal{L}^{Katz}_\mathcal{K}/F_{d,2}$ is actually an element in $\mathbb{Z}_p[[\Gamma_\mathcal{K}]]\backslash\{0\}$ (Note the coefficients). This can be seen as follows: as remarked at the end of [38, Section 6.4] the Katz p-adic L-function is obtained by applying the two-variable regulator map there to the image of the elliptic units in the Iwasawa cohomology. On the other hand from the construction of this regulator map in [38, Definition 4.6], noting that since $\chi \mapsto \chi \mapsto \chi\chi^{-c}$ induces square map on anticyclotomic characters, $F_{d,2}$ is a generator of the Yager module S_∞ there (this is the $S_{\mathbb{Q}^{ur}_p/\mathbb{Q}_p}$ in Section 7.2) as a free rank one $\mathbb{Z}_p[[U]]$-module. Thus $\mathcal{L}^{Katz}_\mathcal{K}/F_{d,2}$ is a $\mathbb{Z}_p$-coefficient power series. So there is an $\mathcal{L}'_{f,\mathcal{K}} \in \mathrm{Frac}\Lambda_\mathcal{K}$ such that

$$\mathcal{L}'_{f,\mathcal{K}} \cdot F_{d,2} = \mathcal{L}^2_{f,\mathcal{K}}.$$

We have the following Straightforward consequence of part (2) of Theorem 5.3.

Theorem 6.13. *Assume E has square-free conductor N and there is at least one prime $\ell | N$ where $\mathcal{K}$ is non-split. Suppose moreover that $E[p]|_{G_\mathcal{K}}$ is absolutely irreducible. Then the characteristic ideal of $X^2_{\mathcal{K},f}$ is contained in the fractional ideal generated by $\mathcal{L}'_{f,\mathcal{K}}$ as ideals of $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

We similarly have a *weak version* of this theorem by requiring the inequality for any height one prime P of $\Lambda_\mathcal{K}$ which is not a pullback of a height one prime of $\mathbb{Z}_p[[\Gamma^-]]$ instead of for all height one primes.

7 Beilinson-Flach Elements

7.1 Some Preliminaries

We write $\mathbb{Z}_p[[\Gamma^-]] = \mathbb{Z}_p[[T]], \gamma^- \mapsto 1 + T$. Recall $\mathbf{g}$ be the Hida family of normalized CM forms attached to characters of $\Gamma_\mathcal{K}$ with the coefficient ring $\Lambda_\mathbf{g} := \mathbb{Z}_p[[T]]$ (the trivial character of $\Gamma_\mathcal{K}$ is a specialization of this family). We write $\mathcal{L}_{\mathbf{g}}$ for the fraction ring of $\Lambda_{\mathbf{g}}$. As in [30] let $M(f)^*$ ($M(\mathbf{g})^*$) be the part of the cohomology of the modular curves which is the Galois representation associated to f ($\mathbf{g}$). The corresponding coefficients for $M(f)^*$ and $M(\mathbf{g})^*$ is $\mathbb{Q}_p$ and $\mathcal{L}_{\mathbf{g}}$. (Note that the Hida family $\mathbf{g}$ is not quite a Hida family considered in *loc.cit.* It plays the role of a branch **a** there). Note also that $\mathbf{g}$ is cuspidal (which is called "generically non-Eisenstein" in an earlier version) in the sense of [32]. We have $M(\mathbf{g})^*$ is a rank two $\mathcal{L}_{\mathbf{g}}$ vector space and, there is a short exact sequence of $\mathcal{L}_{\mathbf{g}}$ vector spaces with $G_{\mathbb{Q}_p}$ action:

$$0 \to \mathscr{F}_{\mathbf{g}}^+ \to M(\mathbf{g})^* \to \mathscr{F}_{\mathbf{g}}^- \to 0$$

with $\mathscr{F}_{\mathbf{g}}^{\pm}$ being rank one $\mathcal{L}_{\mathbf{g}}$ vector spaces such that the Galois action on $\mathscr{F}_{\mathbf{g}}^-$ is unramified. Since $\mathbf{g}$ is a CM family with p splits in $\mathcal{K}$, the above exact sequence in fact splits as $G_{\mathbb{Q}_p}$. For an arithmetic specialization g_ϕ of $\mathbf{g}$ the Galois representation $M(f)^* \otimes M(g_\phi)^*$ is the induced representation from $G_{\mathcal{K}}$ to $G_{\mathbb{Q}}$ of $M(f)^* \otimes \xi_{\mathbf{g}_\phi}$ where $\xi_{\mathbf{g}_\phi}$ is the Hecke character corresponding to $\mathbf{g}_\phi$. This identification will be used implicitly later. We also write $D_{dR}(f) = (M(f)^* \otimes B_{dR})^{G_{\mathbb{Q}_p}}$. We will write $H^1_{\mathrm{Iw}}(\mathcal{K}_\infty, -) := \varprojlim_{\mathcal{K} \subseteq \mathcal{K}' \subseteq \mathcal{K}_\infty} H^1(\mathcal{K}', -)$. The transition map is given by co-restriction. For f let $D_{dR}(f)$ be the Dieudonne module for $M(f)^*$ and let $\eta_f^\vee$ be any basis of $\mathrm{Fil}^0 D_{dR}(f)$. Let $\omega_f^\vee$ be a basis of $\frac{D_{dR}(f)}{\mathrm{Fil}^0 D_{dR}(f)}$ such that $\langle \omega_f^\vee, \omega_f \rangle = 1$.

7.2 Yager modules

We mainly follow [38] to present the theory of Yager modules. Let $K/\mathbb{Q}_p$ be a finite unramified extension. For $x \in \mathcal{O}_K$ we define $y_{K/\mathbb{Q}_p}(x) = \sum_{\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)} x^\sigma[\sigma] \in \mathcal{O}_{\mathcal{K}}[\mathrm{Gal}(K/\mathbb{Q}_p)]$ (note our convention is slightly different from [38]). Let $\mathbb{Q}_p^{ur}/\mathbb{Q}_p$ be an unramified $\mathbb{Z}_p$-extension with Galois group U. Then the above map induces an isomorphism of $\Lambda_{\mathcal{O}_F}(U)$-modules

$$y_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p} : \varprojlim_{\mathbb{Q}_p \subseteq K \subseteq \mathbb{Q}_p^{ur}} \mathcal{O}_F \simeq \mathcal{S}_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p} = \{f \in \hat{\mathbb{Z}}_p^{ur}[[U]] : f^u = [u]f\}$$

for any $u \in U$ a topological generator. Here the superscript means u acting on the coefficient ring while $[u]$ means multiplying by the group-like element u^{-1}. The module $S_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p}$ is called the Yager module. It is explained in *loc.cit* that the $S_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p}$ is a free rank one module over $\mathbb{Z}_p$. Let $\mathcal{F}$ be a representation of U then they defined a map $\rho : \hat{\mathbb{Z}}_p^{ur}[[U]] \to \mathrm{Aut}(\mathcal{F} \otimes \hat{\mathbb{Z}}_p^{ur})$ by mapping u to its action on $\mathcal{F}$ and extend linearly. As is noted in *loc.cit* the image of elements in the Yager module is in $(\mathcal{F} \otimes \hat{\mathbb{Z}}_p^{ur})^{G_{\mathbb{Q}_p}}$. Recall also that

$$d := \varprojlim_m d_m \in \varprojlim_m \mathcal{O}_{k^m}^\times$$

defined in the proof of Lemma 6.2 is a generator of the Yager module for $\mathbb{Q}_p$. Then we can define $\rho(d)$ and let $\rho(d)^\vee$ be the element in $\hat{\mathbb{Z}}_p^{ur}[[U]]$ which is the inverse of $\varprojlim_m \sum_{\sigma \in U/p^m U} d_m^\sigma \cdot \sigma^{-1}$. We have the following

Lemma 7.1. *(1)*

$$\frac{1}{\varprojlim_m \sum_{\sigma \in U/p^m U} d_m^\sigma \cdot \sigma^{-1}} \in \mathcal{S}_\infty.$$

(2)

$$\varprojlim_m \sum_{\sigma\in U/p^mU} d_m^\sigma \cdot \sigma^2 \in (\varprojlim_m \sum_{\sigma\in U/p^mU} d_m^\sigma \cdot \sigma)^2 \cdot \mathbb{Z}_p[[U]]^\times.$$

Proof. Straightforward computation on the Galois action. □

7.3 Beilinson-Flach elements

Unlike Kato's zeta element, the Beilinson-Flach elements constructed in [30] are not in the Iwasawa cohomology (in fact they are unbounded classes). So we need to construct from them a bounded family of classes. Our construction can be viewed as a Galois cohomology analogue of Pollack's construction of the $\pm$ p-adic L-function.

We first define

$$\log_p^-(X) := \frac{1}{p}\prod_{m=1}^{\infty}\frac{\Phi_{2m-1}(1+X)}{p},$$

$$\log_p^+(X) := \frac{1}{p}\prod_{m=1}^{\infty}\frac{\Phi_{2m}(1+X)}{p}.$$

Write $X_{v_0} = \gamma_{v_0} - 1$ and $X_{\bar{v}_0} = \gamma_{\bar{v}_0} - 1$. We write $\log^\pm_{\bar{v}_0}$ for $\log_p^\pm(X_{\bar{v}_0})$ and $\log^\pm_{v_0}$ for $\log_p^\pm(X_{v_0})$ as elements of $\Lambda = \mathbb{Q}_p[[\Gamma_{v_0}\times U]] = \mathbb{Q}_p[[\Gamma_{\bar{v}_0}\times U]]$. We use $\mathbb{Z}_p[[U]] \simeq \mathbb{Z}_p[[Y]]$ mapping u to $1+Y$.

Definition 7.2. *Let $r = \frac{1}{2}$ and define $\mathcal{H}_r(X)$ to be power serie in $\mathbb{Q}_p[[X]]$ of growth $O(\log_p^{\frac{1}{2}})$ consisting of $\sum_{n=0}^\infty a_nX^n$ such that $\max\{p^{-[\frac{1}{2}\ell(n)]}|a_n|_p\}_n < \infty$ where $\ell(n)$ is the smallest integer m such that $p^m > n$ (see [38]). This is equipped with a norm on it: $\sum_{n=0}^\infty a_nX^n$ has norm $\max\{p^{-[\frac{1}{2}\ell(n)]}|a_n|_p\}_n$. Our $\mathcal{H}_r$ is the Mellin Transform*

$$\int_{t\in\mathbb{Z}_p}(1+X)^t d\mu$$

of r-admissible distributions $d\mu$ defined in loc.cit. *Let $\mathcal{H}_{r,0} := \mathbb{Z}_p[[Y]]\otimes\mathcal{H}_r(X_{v_0})$. We also define $\mathcal{H}_{0,r}$ to be the completed tensor product $\mathbb{Z}_p[[Y]]\hat{\otimes}\mathcal{H}_r(X_{\bar{v}_0})$ with respect to the obvious norm on $\mathbb{Z}_p[[X]]$ and the norm of $\mathcal{H}_r$ mentioned above (note that the definitions for $\mathcal{H}_{r,0}$ and $\mathcal{H}_{0,r}$ are not symmetric).*

We see that $\log^-_{v_0} \in \mathcal{H}_{r,0}$ and $\log^-_{\bar{v}_0} \in \mathcal{H}_{0,r}$. In [36] the authors defined Beilinson-Flach elements BF_α and $BF_{-\alpha}$ for f_α and $f_{-\alpha}$, as elements in $\mathcal{H}_{r,0}\otimes H^1_{\mathrm{Iw}}(\mathbb{Q}_\infty, M(f)^*\otimes M(\mathbf{g})^*)$. It is easily seen that the module $H^1_{\mathrm{Iw}}(\mathbb{Q}_\infty, M(f)^*\otimes M(\mathbf{g})^*)$ can be identified with $H^1_{\mathrm{Iw}}(\mathcal{K}_\infty, M(f)^*)$.

Now we recall some notations in [43]. Let $ES_p(D_\mathcal{K}) := \varprojlim_r H^1(X_1(D_\mathcal{K}p^r)\otimes\bar{\mathbb{Q}},\mathbb{Z}_p)$ and $GES_p(D_\mathcal{K}) := \varprojlim_r H^1(Y_1(D_\mathcal{K}p^r)\otimes\bar{\mathbb{Q}},\mathbb{Z}_p)$ which are modules equipped with Galois action of $G_\mathbb{Q}$. Here $X_1(D_\mathcal{K}p^r)$ and $Y_1(\mathcal{D}_\mathcal{K}p^r)$ are corresponding compact and non-compact modular curves. Recall in *loc.cit* there is an ordinary idempotent e^* associated to the covariant Hecke operator U_p. Let $\mathfrak{A}^*_\infty = e^*ES_p(D_\mathcal{K})^{I_p} = e^*GES_p(D_\mathcal{K})^{I_p}$ (see the Theorem in *loc.cit*). Let $\mathfrak{B}^*_\infty$ ($\tilde{\mathfrak{B}}^*_\infty$) be the quotient of $e^*ES_p(D_\mathcal{K})$ ($e^*GES_p(D_\mathcal{K})$) over $\mathfrak{A}^*_\infty$.

In an earlier version of [32] the authors defined elements $\omega^\vee_\mathbf{g} \in (\mathscr{F}^+_\mathbf{g}(\chi_\mathbf{g}^{-1})\otimes\hat{\mathbb{Z}}_p^{ur})^{G_{\mathbb{Q}_p}}$ and $\eta^\vee_\mathbf{g} \in (\mathscr{F}^-_\mathbf{g}\otimes\hat{\mathbb{Z}}_p^{ur})^{G_{\mathbb{Q}_p}}$. Here the $\chi_\mathbf{g}$ is the central character for $\mathbf{g}$. We briefly recall the definitions since they are more convenient for our use (these notions are replaced by their dual in the current version of [32]). In the natural isomorphism

$$\mathfrak{A}^*_\infty \otimes_{\mathbb{Z}_p[[T]]} \hat{\mathbb{Z}}_p^{ur}[[T]] \simeq \mathrm{Hom}_{\hat{\mathbb{Z}}_p^{ur}}(S^{ord}(D_\mathcal{K}, \chi_\mathcal{K}, \hat{\mathbb{Z}}_p^{ur}[[T]]), \hat{\mathbb{Z}}_p^{ur}[[T]])$$

(see the proof of [43, Corollary 2.3.6], the $\omega^\vee_{\mathbf{g}}$ is corresponds to the functional which maps each normalized eigenform to 1. On the other hand $\eta^\vee_{\mathbf{g}}$ is defined to be the element in $\mathfrak{B}^*_\infty$ which, under the pairing in [43, Theorem 2.3.5], pairs with $\omega^\vee_{\mathbf{g}}$ to the product of local root numbers at prime to p places of $\mathbf{g}$. This product moves p-adic analytically and is a unit.

Let v_1, v_2 be a Λ basis of $H^1(\mathcal{K}_{\bar{v}_0}, M(f)^* \otimes \Lambda(-\Psi))$. Then there are $f_1, f_2 \in \mathcal{H}_r(X_{v_0}) \otimes_{\mathbb{Z}_p[[X_{v_0}]]} \mathbb{Z}_p[[X_{v_0}, T]]$ and some $f_0 \in \mathrm{Frac}(\mathbb{Z}_p[[T]])\backslash\{0\}$ such that $BF_\alpha - BF_{-\alpha} = \alpha \cdot f_0(f_1v_1 + f_2v_2)$. Let

$$\mathcal{L} = \mathcal{L}^G_{V_f} : H^1(G_{\mathbb{Q}_p}, M(f)^* \otimes \Lambda(-\Psi)) \to \oplus_{i=1}^{p^t}(\mathcal{H}_{0,r} \otimes D_{\mathrm{cris}}(V_f)) \cdot \gamma_i$$

be the regulator map defined in [38, Theorem 4.7]. (We know $\mathcal{L}(v_i) \in \mathcal{H}_{0,r}$ by [38, Proposition 4.8]). We write Pr^α and $\mathrm{Pr}^{-\alpha}$ for the projection map from $D_{\mathrm{cris}}(V_f)$ to the α or $-\alpha$ eigenspace for Frobenius action φ (as numbers, with respect to the basis given by the image of the Neron differential ω_E in the $\pm\alpha$-eigenspaces of $D_{\mathrm{cris}}(V_f)$). Let

$$\mathcal{L}^+ = \frac{\mathrm{Pr}^\alpha - \mathrm{Pr}^{-\alpha}}{2\alpha} \circ \mathcal{L},\ \mathcal{L}^+ = \frac{\mathrm{Pr}^\alpha + \mathrm{Pr}^{-\alpha}}{2} \circ \mathcal{L}.$$

Then by Proposition 7.5 in the following, we have

$$f_0f_1\mathcal{L}^+(v_1) + f_0f_2\mathcal{L}^+(v_2) = \log^-_{v_0}\log^-_{\bar{v}_0} L^{++}_{f,p},$$

$$f_0f_1\mathcal{L}^-(v_1) + f_0f_2\mathcal{L}^-(v_2) = \log^-_{v_0}\log^+_{\bar{v}_0} L^{+-}_{f,p}.$$

We need the following

Lemma 7.3. *The $L^{+-}_{f,p}$ and $L^{-+}_{f,p}$ are not identically zero.*

Proof. We just need to know that the $L_\mathcal{K}(E, \chi, 1)$ is non zero for some character χ of $\Gamma_\mathcal{K}$ whose conductor at v_0 is a even power of p and whose conductor at $\bar{v}_0$ is an odd power of p. This is just [51, Theorem 2]. □

We have the following

Lemma 7.4. *We have $f_1, f_2 \in \log^-_{v_0} \cdot\mathrm{Frac}(\mathbb{Z}_p[[X_{v_0}, T]])$.*

Proof. We first claim that $\det\begin{pmatrix}\mathcal{L}^+(v_1) & \mathcal{L}^+(v_2)\\ \mathcal{L}^-(v_1) & \mathcal{L}^-(v_2)\end{pmatrix}$ is not identically zero. Suppose it is not the case. Then we have

$$\log^-_{\bar{v}_0} L^{++}_{f,p} \cdot \mathcal{L}^-(v_1) - \log^+_{\bar{v}_0} L^{+-}_{f,p} \cdot \mathcal{L}^+(v_1) = 0,$$

$$\log^-_{\bar{v}_0} L^{++}_{f,p} \cdot \mathcal{L}^-(v_2) - \log^+_{\bar{v}_0} L^{+-}_{f,p} \cdot \mathcal{L}^+(v_2) = 0.$$

Let $\zeta_1, \cdots, \zeta_s$ be the zeros of $\log^-_p$ such that $X_{\bar{v}_0} - \zeta_i$ is a divisor of $L^{+-}_{f,p}$ (easily seen to be a finite set since $L^{+-}_{f,p}$ is not identically zero). Then for any other root ζ of $\log^-_p$, $\mathcal{L}^+(v_1)$ restricts to the zero function at the line $X_{\bar{v}_0} = \zeta$. If we expand $\mathcal{L}^+(v_1)$ as a power series in $X_{\bar{v}_0}$ and U, then by Weierstrass preparation theorem (see [70, Theorem 7.3]), the coefficient for U^m (any m) is $\log^-_{\bar{v}_0}/\prod(X_{\bar{v}_0}-\zeta_i)$ times some element in $\mathbb{Z}_p[[X_{\bar{v}_0}]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ whose coefficients are uniformly bounded (bound also independent of m). (See [48, Proof of Theorem 5.1]). Thus $\mathcal{L}^+(v_1)\in\log^-_{\bar{v}_0}\cdot\mathrm{Frac}(\mathbb{Z}_p[[X_{\bar{v}_0},U]])$ We can apply the same argument to all $\mathcal{L}^{\pm}(v_i)$ and get

$$\mathcal{L}^+(v_i)\in\log^-_{\bar{v}_0}\cdot\mathrm{Frac}(\mathbb{Z}_p[[X_{\bar{v}_0},U]]),$$

$$\mathcal{L}^-(v_i)\in\log^+_{\bar{v}_0}\cdot\mathrm{Frac}(\mathbb{Z}_p[[X_{\bar{v}_0},U]]).$$

But $\det\begin{pmatrix}\mathcal{L}^+(v_1) & \mathcal{L}^+(v_2)\\ \mathcal{L}^-(v_1) & \mathcal{L}^-(v_2)\end{pmatrix}=0$. This contradicts the fact that $\mathcal{L}$ is injective, which is proved in [38, Proposition 4.11].

Now let us return to the proof of the lemma. Fix $k=1$ or 2. If f_k is identically 0 then nothing is needed. If not, recall we fixed representatives $\gamma_1,\cdots,\gamma_{p^t}$ of $\Gamma_{\mathcal{K}}/\Gamma_p$. Then from the claim there is an $a\in\mathbb{C}_p$, $|a|_p<1$ such that

$$0\neq f_k|_{X_{\bar{v}_0}=a}\in\oplus_{i=1}^{p^t}\log^-_{v_0}\cdot\mathrm{Frac}(\mathbb{Z}_p[[X_{v_0}]])\cdot\gamma_i. \tag{6}$$

It is possible to write $f_k=\sum_j f_{kj}(X_{v_0})\cdot g_{kj}(X_{v_0},T)$ (finite sum) where $f_{kj}(X_{v_0})\in\mathcal{H}_r(X_{v_0})\otimes_{\mathbb{Z}_p[[X_{v_0}]]}\mathrm{Frac}(\mathbb{Z}_p[[X_{v_0}]])$ and $g_{kj}(X_{v_0}),T)\in\mathbb{Z}_p[[X_{v_0},T]]$ such that either $f_{k1}(X_{v_0})\in\log^-_{v_0}\cdot\mathrm{Frac}(\mathbb{Z}_p[[X_{v_0}]])$ or $\{\log^-_{v_0}\}\cup\{f_{kj}\}$ forms a linearly independent set over $\mathrm{Frac}(\mathbb{Z}_p[[X_{v_0}]])$. Then (6) implies we must have $j=1$ and $f_{k1}(X_{v_0})\in\log^-_{v_0}\cdot\mathrm{Frac}(\mathbb{Z}_p[[X_{v_0}]])$. Thus the lemma is true. □

So there is an element $0\neq h\in\mathbb{Z}_p[[X_{v_0},Y]]$ such that $h\cdot f_i\in\log^-_{v_0}\cdot\Lambda$. We define the bounded cohomology class

$$BF^+:=\frac{f_1h}{2\log^-_{v_0}}v_1+\frac{f_2h}{2\log^-_{v_0}}v_2\in H^1_{\mathrm{Iw}}(\mathcal{K}_\infty,M(f)^*) \tag{7}$$

It follows from that the Galois cohomology image of Beilinson-Flach element is geometric that the BF^+ maps to $H^1_+(G_{v_0},M(f)^*\otimes\Lambda(-\Psi))\subseteq H^1(G_{v_0},M(f)^*\otimes\Lambda(-\Psi))$, since for any arithmetic point ϕ such that $\log^-_{v_0}|_\phi\neq 0$, the class is in the finite part $H^1_f(G_{v_0},-)$.

We take basis $v^\pm$ of $\mathscr{F}^\pm_{\mathbf{g}}$ with respect to which $\omega^\vee_{\mathbf{g}}$ and $\eta^\vee_{\mathbf{g}}$ are $\rho(d)^\vee v^+$ and $\rho(d)v^-$ (see the discussion for Yager Modules). We use this basis to give $\Lambda_{\mathbf{g}}$-integral structure for $M(\mathbf{g})^*$. With this integral structure we can talk about specializing BF^+ to arithmetic points ϕ, provided we remove the set of ϕ's in a lower dimensional subspace (the zeroes of the denominator for BF^+ with respect to the basis). The following propositions are proved in [30]. Let $\Psi_{\mathbf{g}}$ be the $\Lambda_{\mathbf{g}}$-valued Galois character of $G_{\mathcal{K}}$ corresponding to the Galois representation associated to $\mathbf{g}$ (i.e. $\mathrm{Ind}^{G_{\mathcal{K}}}_{G_{\mathbb{Q}}}\Psi_{\mathbf{g}}=M(\mathbf{g})^*$). Since p splits as $v_0\bar{v}_0$ in $\mathcal{K}$, there is a canonical identification $(\mathrm{Ind}^{G_{\mathcal{K}}}_{G_{\mathbb{Q}}}\Psi_{\mathbf{g}})|_{G_{\mathbb{Q}_p}}\simeq\Psi_{\mathbf{g}}|_{G_{\mathcal{K}_{v_0}}}\oplus\Psi_{\mathbf{g}}|_{G_{\mathcal{K}_{\bar{v}_0}}}$ and can take a $\Lambda_{\mathbf{g}}$-basis of the right side as $\{v,c\cdot v\}$ where c is the complex conjugation. (Note that there are two choices for the $\Psi_{\mathbf{g}}$ and we choose the one so that $\Psi_{\mathbf{g}}|_{G_{\mathcal{K}_{v_0}}}$ corresponds to $\mathscr{F}^-_{\mathbf{g}}$).

Convention: we use the basis $\{v^+,c\cdot v^+\}$ to identify the Galois representation of $\mathbf{g}$ with the induced representation $\mathrm{Ind}^{\mathcal{K}}_{G_{\mathbb{Q}}}\Psi_{\mathbf{g}}$.

In the following we define ϕ in a generic set of arithmetic points corresponding to a finite order character of $\Gamma_{\mathcal{K}}$ to mean all such ϕ outside a proper closed sub-scheme of $\mathrm{Spec}\Lambda$.

Proposition 7.5. *For some $H_0 \in \bar{\mathbb{Q}}_p^\times$ and ϕ in a generic set of arithmetic points corresponding to a primitive character of $\Gamma_n \times U_m$ with n an even number (as local Galois group at $\bar{v}_0$), for any $\alpha, \beta \in \{\pm\sqrt{-p}\}$,*

$$H_0 \cdot \mathrm{Pr}^{\mathscr{F}_{\mathbf{g}}^+}\mathrm{Pr}^{\beta}(\exp^*(\phi(BF_\alpha))) = \frac{\phi(L_{p,\alpha,\beta})\varepsilon(\chi_\phi^{-1})}{(-p)^{\frac{n}{2}}}\eta_{f,\beta}^\vee \otimes \phi(\omega_{\mathbf{g}}^\vee).$$

Here Pr^β and $\eta_{f,\beta}^\vee$ denote projecting to the β-eigenspace of $D_{dR}(f)$, $\exp^*$ is the Bloch-Kato dual exponential map. The χ_ϕ means composing Ψ with ϕ. By class field theory the χ_ϕ can be considered as a character of $\mathbb{Q}_p^\times$. For the $\mathrm{Pr}^{\mathscr{F}_{\mathbf{g}}^+}$, we recall that $M(\mathbf{g})^*$ is split as the direct sum of $\mathscr{F}_{\mathbf{g}}^+$ and $\mathscr{F}_{\mathbf{g}}^-$ as Galois modules which are rank one vector spaces over $\mathcal{L}_{\mathbf{g}}$. So if we exclude the set of ϕ's in a lower dimensional space it makes sense to talk about projection to $(\mathscr{F}_{\mathbf{g}}^- \otimes \hat{\mathbb{Z}}_p^{ur})^{G_{\mathbb{Q}_p}}$ or $(\mathscr{F}_{\mathbf{g}}^+(\chi_{\mathbf{g}}^{-1}) \otimes \hat{\mathbb{Z}}_p^{ur})^{G_{\mathbb{Q}_p}}$ components at ϕ.

Proof. It follows from the explicit reciprocity law in [30, Theorem 7.1.4, Theorem 7.1.5] together with the interpolation property of the big regulator map [38, Theorem 4.15]. Note that $\alpha^n = \beta^n = (-p)^{\frac{n}{2}}$ and that the β_g in *loc.cit* corresponds to the $\chi_\phi(u)^n$ part (u being the arithmetic Frobenius) of $\varepsilon(\chi_\phi^{-1})$. □

The proposition has the following corollary using Proposition 6.11.

Corollary 7.6. *We use the convention before Proposition 7.5. Then for some $H_1 \in \mathbb{Q}_p^\times$ we have*

$$\mathrm{Col}_{\bar{v}_0}^+(BF^+) = h \cdot H_1 \cdot L_p^{++}.$$

(the h is defined in (7)).

Proof. First recall that $\omega_{\mathbf{g}}^\vee$ is the $\rho(d)^\vee v^+$ for the basis $v^\pm$ we have chosen. If we take H_1 to be $\frac{\Omega_E^+\Omega_E^-}{\langle f,f\rangle}$ times some element in $\bar{\mathbb{Q}}_p^\times$ (recall $\eta_f^\vee$ is defined up to a scalar). Then the corollary follows. □

Proposition 7.7. *There is a non-zero element $0 \neq H_2 \in \hat{\mathbb{Z}}_p^{ur}[[T]]$, such that for ϕ in a generic set of arithmetic points corresponding to a primitive character of $\Gamma_n \times U_m$ (as Galois group at v_0) with n an even number and for $\alpha \in \{\pm\sqrt{-p}\}$,*

$$\phi(H_2)\mathrm{Pr}^{\mathscr{F}_{\mathbf{g}}^-}\log_{v_0}\phi(BF_\alpha) = \frac{1}{\alpha \cdot (-p)^{\frac{n}{2}}}\phi(\mathcal{L}_{f,\mathcal{K}}^2) \cdot \varepsilon(\chi_\phi^{-1})\omega_f^\vee \otimes \phi(\eta_{\mathbf{g}}^\vee).$$

Here $\log_{v_0}$ is the Bloch-Kato logarithm map at v_0.

Proof. This again follows from [30, Theorem 7.1.4, Theorem 7.1.5]. Note that the arithmetic points at which the interpolation formulas are proved there are not quite the ϕ's considered here. In fact those points in *loc.cit* correspond to the product of a finite order character of Γ and some character of U which is not of finite order. We may use the lemma below to get the result we need. We also need to compare the p-adic L-function in *loc.cit* with the one in [67]. In [67] we used the Σ-primitive p-adic L-function which is in Λ for Σ a finite set of primes. The original p-adic L-function is obtained by putting back the Euler factors at Σ. We only know a priory it is in the fraction field of Λ. There is another construction of this p-adic L-function $\mathcal{L}_{f,\mathcal{K}}^{Urban} \in \Lambda \otimes_{\Lambda_{\mathbf{g}}} \mathcal{L}_{\mathbf{g}}$ by E.Urban [66] using Rankin-Selberg method. This is the p-adic L-function used in [30]. However the period there is the Petersson inner product of the normalized eigenforms in $\mathbf{g}$ instead of the CM period. The ratio of these periods is given by $h_{\mathcal{K}} \cdot \mathcal{L}_{\mathcal{K}}^{Katz} \in \hat{\mathbb{Z}}_p^{ur}[[T]]$ (see [22]). So we may choose $h_{\mathcal{K}} \cdot \mathcal{L}_{\mathcal{K}}^{Katz}$ times some constant in $\bar{\mathbb{Q}}_p^\times$ as the H_2. The proposition follows. □

Recall F_n is the unramified extension of $\mathbb{Q}_p$ of degree p^n.

Lemma 7.8. *Let χ be a primitive character of $\Gamma/p^n\Gamma_n$ and let K_χ be the field obtained from $\mathbb{Q}_p$ by joining in the coefficient ring of χ. We let*

$$\sum_{\tau\in\Gamma/p^n\Gamma}\zeta^\tau\omega_f\otimes\chi(\tau):=\omega_{f,\chi}\in\frac{D_{\mathrm{dR}}^{K_\chi}(T)}{\mathrm{Fil}^0 D_{\mathrm{dR}}^{K_\chi}(T)}\otimes_{\mathbb{Q}_p}K_\chi.$$

The $\varprojlim_n H^1_f(F_n,T(\chi))$ is free of rank one over $\mathcal{O}_{K_\chi}[[U]]$. Moreover if $\varprojlim_n H^1_f(F_n,T(\chi))\rightarrow(\varprojlim_n\mathcal{O}_{F_n}\cdot\omega_{f,\chi})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is a homomorphism of $\mathcal{O}_{K_\chi}[[U]]$-modules and for a Zariski dense set points $\phi\in\mathrm{Spec}\mathcal{O}_{K_\chi}[[U]]$ with the associated unramified Galois character denoted as ρ_ϕ, the specialization to $\phi:\mathcal{O}_{K_\chi}[[U]]\rightarrow\mathbb{C}_p$ of this map is the Bloch-Kato logarithm map

$$H^1_f(\mathbb{Q}_p,T(\chi\rho_\phi))\rightarrow\frac{D_{\mathrm{dR}}(T(\chi\rho_\phi))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(T(\chi\rho_\phi))}.$$

Proof. Let $K_{\chi,n}$ be the composed field of F_n and K_χ and let $K_{\chi,\infty}$ be the union of all $K_{\chi,n}$ and $\hat{\mathcal{O}}_{K_{\chi,\infty}}$ be the p-adic completion of the integer ring of $K_{\chi,\infty}$. We first observe that there is an integer m'' such that the image of $\hat{\mathcal{O}}_{K_{\chi,n}}\cdot\omega_f$ under the Bloch-Kato exponential map lies in $p^{-m''}H^(K_{\chi,\infty},T)$. This follows from the explicit formula for the logarithm map of the formal group E of $\hat{E}$. From this we know there is an integer m' with

$$\exp:\hat{\mathcal{O}}_{F_\infty}\cdot\omega_{f,\chi}\rightarrow p^{-m'}H^1(F_\infty,T(\chi)). \tag{8}$$

and

$$\varprojlim_n\mathcal{O}_{F_n}\cdot\omega_{f,\chi}\rightarrow p^{-m'}\varprojlim_n H^1(F_n,T(\chi)).$$

Suppose ρ is an unramified character of $G_{\mathbb{Q}_p}$ with $\rho(u)=1+\mathfrak{m}$ so that $|\mathfrak{m}|_p<1$ is in a finite extension $L/\mathbb{Q}_p$. Recall that $d\in\varprojlim_n\mathcal{O}_{F_n}$ and $\rho(d):=\varprojlim_n\sum_{\sigma\in U/p^nU}\rho(\sigma)d_n^\sigma$. Define

$$\omega_{f,\chi\rho}:=\rho(d)\cdot\omega_{f,\chi}=\varprojlim_n\sum_\sigma d_n^\sigma\cdot\omega_{f,\chi}\otimes\rho(\sigma)\in(\varprojlim_n\mathcal{O}_{F_n})\omega_f\otimes_{\mathbb{Z}_p}\mathcal{O}_{L_\chi}$$

for $L_\chi=K_\chi(\mathfrak{m})$. Then the boundedness (8) implies

$$\rho(d)\cdot\omega_{f,\chi}\rightarrow\varprojlim_n\sum_{\sigma\in U/p^nU}\rho(\sigma)\exp(d_n^\sigma\cdot\omega_{f,\chi})$$

gives the Bloch-Kato exponential map for $V(\chi\rho)$. Now it follows from the fact that the exponential mpa for $V(\chi\rho)$ is an isomorphism from $\mathbb{Q}_p\omega_{f,\chi\rho}\otimes_{\mathbb{Q}_p}L_\chi$ to $H^1_f(\mathbb{Q}_p,V(\chi\rho))$ and some elementary theory of $\mathcal{O}_{K_\chi}[[U]]$-module structures that there is an integer m such that

$$\exp(\varprojlim_n\mathcal{O}_{F_n}\cdot\omega_{f,\chi})$$

is p^m times a rank one $\mathcal{O}_{K_\chi}[[U]]$-direct summand of the free rank two $\mathcal{O}_{K_\chi}[[U]]$ module $\varprojlim_n H^1(F_n,T(\chi))$. These altogether give the lemma. $\square$

Corollary 7.9. *We use the convention before Proposition 7.5. Then for some* $0 \neq H_3 \in \mathrm{Frac}\Lambda_{\mathbf{g}}$ *we have*

$$\mathrm{LOG}^+_{v_0}(BF^+) = h \cdot H_3 \cdot (\mathcal{L}^2_{f,\mathcal{K}}).$$

Proof. Take H_3 as $-\frac{1}{H_2} \cdot \frac{v^-}{c \cdot v^+}$ and use Proposition 6.11. □

Remark 7.10. *Both* H_2 *and* $\frac{v^-}{c \cdot v^+}$ *are elements in the fraction field of* $\Lambda_{\mathbf{g}}$. *In the next section we are going to carefully study them to get a refined main theorem. In particular we will use Rubin's work on CM main conjecture to prove that* $1/H_3$ *is "almost" integral.*

8 Proof of Main Results

In this section we first prove a weak version of the two-variable ++ main conjecture, which can be used to deduce the one variable main conjecture of Kobayashi after inverting p. To take care of powers of p, we need to study the ratio $\frac{c \cdot \omega^\vee_{\mathbf{g}}}{\eta^\vee_{\mathbf{g}}}$ ($c \in G_{\mathbb{Q}}$ is the complex conjugation, will make precise definition for the c-action later on), which boils down to studying certain congruence modules. Our idea is appeal to the main conjecture for CM fields proved by Rubin, and an argument of Hida-Tilouine [18] constructing elements in certain anticyclotomic Selmer groups from congruence modules.

8.1 The Two Variable Main Conjecture

We first prove the weak version of one side (lower bound for Selmer groups) of Conjecture 6.7. We define a couple of Selmer groups

$$H^1_3(\mathcal{K}, M(f)^* \otimes \Lambda_{\mathcal{K}}(-\Psi)) :=$$
$$\ker\{H^1(\mathcal{K}, M(f)^* \otimes \Lambda_{\mathcal{K}}(-\Psi)) \to \prod_{v \nmid p} H^1(I_v, M(f)^* \otimes \Lambda_{\mathcal{K}}(-\Psi)) \times \frac{H^1(G_{v_0}, M(f)^* \otimes \Lambda_{\mathcal{K}}(-\Psi))}{H^1_+(G_{v_0}, M(f)^* \otimes \Lambda_{\mathcal{K}}(-\Psi))}\},$$

and

$$\mathrm{Sel}_{v_0,+} := \varinjlim_{\mathcal{K} \subseteq \mathcal{K}' \subseteq \mathcal{K}_\infty} \ker\{H^1(\mathcal{K}', M(f)^* \otimes \Lambda_{\mathcal{K}}(\Psi) \otimes (\Lambda_{\mathcal{K}})^*) \to \prod_{v \nmid p} H^1(I_v, M(f)^* \otimes \Lambda_{\mathcal{K}}(\Psi) \otimes (\Lambda_{\mathcal{K}})^*)$$
$$\times \frac{H^1(G_{v_0}, M(f)^* \otimes \Lambda_{\mathcal{K}}(\Psi) \otimes (\Lambda_{\mathcal{K}})^*)}{E^+(\mathcal{K}'_{v_0}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \times H^1(G_{\bar{v}_0}, M(f)^* \times \Lambda_{\mathcal{K}}(\Psi) \otimes (\Lambda_{\mathcal{K}})^*)\},$$

$$X_{v_0,+} := \mathrm{Sel}^*_{v_0,+}.$$

Recall that BF^+ is in $H^1_3(\mathcal{K}, M(f) \otimes \Lambda_{\mathcal{K}})$.

Conjecture 8.1. *For any height one prime* P *of* $\Lambda_{\mathcal{K}}$ *we have the length of*

$$H^1_3(\mathcal{K}, M(f) \otimes \Lambda_{\mathcal{K}})/\Lambda_{\mathcal{K}} \cdot BF^+$$

at Λ_P *is the same as that of* $X_{v_0,+}$. *We also make the weak version and "one divisibility" version of the above conjecture. (We will see in the proof of next theorem that* $H^1_3(\mathcal{K}, M(f) \otimes \Lambda_{\mathcal{K}})$ *is a torsion free rank one* $\Lambda_{\mathcal{K}}$*-module).*

Theorem 8.2. *The weak version of both Conjecture 6.7 and the main conjecture in [67] are equivalent to the conjecture above. Moreover the inequality* $\mathrm{length}_P X^+_{f,\mathcal{K}} \geq \mathrm{ord}_P L^{++}_{f,p}$ *is true under the assumption of Theorem 6.13.*

Proof. Note that $H^1_3(G_\mathcal{K}, T_f \otimes \Lambda_\mathcal{K}(-\Psi))$ is torsion-free of rank one over $\Lambda_\mathcal{K}$. This can be seen as follows: the torsion-freeness is obvious. If the rank is at least two, then the kernel of the map from $H^1_3(G_\mathcal{K}, T_f \otimes \Lambda_\mathcal{K}(-\Psi))$ to $\frac{H^1(G_{\bar{v}_0}, T_f \otimes \Lambda_\mathcal{K}(-\Psi))}{H^1_+(G_{\bar{v}_0}, T_f \otimes \Lambda_\mathcal{K}(-\Psi))}$ has rank at least one, thus not torsion. Specialize to the cyclotomic line $\gamma^- - 1 = 0$, we see this is impossible by [33, Theorem 7.3 (i)]. So the rank has to be at most one. Now recall that by Corollaries 7.6 and 7.9, the image of BF^+ by $\mathrm{Col}^+_{\bar{v}_0}$ and $\mathrm{LOG}^+_{v_0}$ are certain p-adic L-functions which are not identically zero. It then follows that the kernels of $H^1_3(G_\mathcal{K}, T_f \otimes \Lambda_\mathcal{K}(-\Psi)) \to \frac{H^1(G_{\bar{v}_0}, T_f \otimes \Lambda_\mathcal{K}(-\Psi))}{H^1_+(G_{\bar{v}_0}, T_f \otimes \Lambda_\mathcal{K}(-\Psi))}$ and $H^1_3(G_\mathcal{K}, T_f \otimes \Lambda_\mathcal{K}(-\Psi)) \to H^1_+(G_{v_0}, T_f \otimes \Lambda_\mathcal{K}(-\Psi))$ must be 0.

The above discussion gives the following exact sequences (Poitou-Tate long exact sequence):

$$0 \to H^1_3(G_\mathcal{K}, T_f \otimes \Lambda_\mathcal{K}(-\Psi)) \to \frac{H^1(G_{\bar{v}_0}, T_f \otimes \Lambda_\mathcal{K}(-\Psi))}{H^1_+(G_{\bar{v}_0}, T_f \otimes \Lambda_\mathcal{K}(-\Psi))} \to X^{++} \to X_{v_0,+} \to 0$$

and

$$0 \to H^1_3(G_\mathcal{K}, T_f \otimes \Lambda_\mathcal{K}(-\Psi)) \to H^1_+(G_{v_0}, T_f \otimes \Lambda_\mathcal{K}(-\Psi)) \to X_{v_0} \to X_{v_0,+} \to 0.$$

We know $X^+_{E,\mathbb{Q}}$ is torsion by [33]. So the control theorem Proposition 8.7 in the following implies that X^{++} is torsion over $\Lambda_\mathcal{K}$. So the rank of $H^1_3(\mathcal{K}, T_f \otimes \Lambda_\mathcal{K}(-\Psi))$ must be one. Then the argument is the same as [33, Theorem 7.4], using Corollaries 7.6 and 7.9 and the above exact sequences. □

Note that at the moment we can only treat height one primes of Λ which are not pullbacks of height one primes of $\Lambda_\mathbf{g}$ and thus can only prove the weak version of the theorem. In order to get a refined result we need to study the relations between v^+ and $c \cdot v^-$ we discussed before Proposition 7.5. In fact we can prove the strong version of Conjecture 6.7 by applying Rubin's work on the main conjecture for $\mathcal{K}$. We first study certain Eisenstein components of the modular curve cohomology. Let $\mathbb{T}$ be the Hecke algebra generated by T_ℓ's for $\ell \nmid pD_\mathcal{K}$ and U_ℓ's for $\ell | pD_\mathcal{K}$, acting on the space of ordinary cuspidal forms with tame level group $\Gamma_1(D_\mathcal{K})$. Let $\mathbb{T}_{\mathfrak{m}_\mathbf{g}}$ be the localization of $\mathbb{T}$ at the maximal ideal corresponding to $\mathbf{g}$. These Hecke algebras are reduced since $\mathrm{cond}(\chi_\mathcal{K}) = D_\mathcal{K}$ and the nebentypus of forms congruent to $\mathbf{g}$ must be congruent to $\chi_\mathcal{K}$ modulo p and thus conductor must be $\mathcal{D}_\mathcal{K}$ as well. Then the family $\mathbf{g}$ is a component of it. We write the non-CM component $\mathbb{T}_{NCM}$ for the quotient of $\mathbb{T}_{\mathfrak{m}_\mathbf{g}}$ corresponding to $\mathrm{Spec}(\mathbb{T}_{\mathfrak{m}_\mathbf{g}})$ with all irreducible components corresponding to families of $\mathcal{K}$-CM forms deleted. Let $\mathcal{C}_{CM} \subset \mathbb{T}_{NCM}$ be the congruence ideal generated by $\{t - t_\mathbf{g}\}_t$'s for t running over all Hecke operators (including the U_p operator) and $t_\mathbf{g}$ is the Hecke eigenvalue for t on $\mathbf{g}$. Then the map $\Lambda_\mathbf{g} \to \mathbb{T}_{NCM}/C_{CM}$ is surjective. We let I_{CM} be the kernel of this map.

Proposition 8.3. *We have* $\mathrm{ord}_P \mathcal{L}^{Katz}_\mathcal{K} \geq \mathrm{Length}_P(\Lambda_\mathbf{g}/I_{CM})$ *for any height one prime* P *of* $\Lambda_\mathcal{K}$*, unless* P *is the pullback to* $\Lambda_\mathcal{K} = \mathbb{Z}_p[[\Gamma \times \Gamma^-]]$ *of the augmentation ideal* $(\gamma - 1)\mathbb{Z}_p[[\Gamma^-]]$ *of* $\mathbb{Z}_p[[\Gamma^-]]$ *(we call these primes "exceptional").*

Proof. We note that each irreducible component B of $\mathbb{T}_{NCM}$ the Galois representation $\rho_B : G_\mathbb{Q} \to \mathrm{GL}_2(\mathrm{Frac}(B))$ has irreducible restriction to $G_\mathcal{K}$. This is because there exists classical specialization at that component which is not a CM form with respect to $\mathcal{K}$. Let

$$X_{CM} := H^1_f(\mathcal{K}, \Lambda_\mathbf{g}(\chi_\mathbf{g}\chi_\mathbf{g}^{-c}) \otimes_{\Lambda_\mathbf{g}} \Lambda^*_\mathbf{g})^*$$

where $\chi_{\mathbf{g}}$ denotes the family of CM character corresponding to the family of CM form $\mathbf{g}$. The Selmer condition "f" is defined by restricting trivially to $H^1(I_v, -)$ at all primes $v \neq v_0$. Then the "lattice construction" (see [18, Corollary 3.3.6], and see [68] for the construction in the situation here) gives that for any non-exceptional height one prime P of $\Lambda_{\mathbf{g}}$,

$$\mathrm{length}_{\Lambda_{\mathbf{g},P}}(\Lambda_{\mathbf{g}}/I_{CM})_P \leq \mathrm{ord}_P X_{CM}.$$

This construction works unless P corresponds to the pullback of the augmentation ideal in the anticyclotomic line (these cases do not satisfy the [18, (SEP.P)] on page 32 of *loc.cit*). On the other hand, Rubin [52], [53] proved that we have $(\mathcal{L}_{\mathcal{K}}^{Katz}/F_d) = \mathrm{char}(X_{CM})$ (note that F_d is a unit in $\hat{\mathbb{Z}}_p^{ur}[[T]]$). In fact Rubin proved a two-variable main conjecture and we easily have that the two variable dual Selmer group specializes exactly to the one variable anticyclotomic dual Selmer group here. These together imply the proposition. □

Recall we defined basis $v^{\pm}$ of $\mathscr{F}_{\mathbf{g}}^{\pm}$. Let c be the complex conjugation in $G_{\mathbb{Q}}$. Recall that $\omega_{\mathbf{g}}^{\vee} = \rho(d)^{\vee} \cdot v^{+}$. If $c \in G_{\mathbb{Q}}$ is the complex conjugation we define

$$c \cdot \omega_{\mathbf{g}}^{\vee} = \rho(d)^{\vee}(c \cdot v^{+}) \in (\mathscr{F}_{\mathbf{g}}^{-} \otimes \hat{\mathbb{Z}}_p^{ur})^{G_{\mathbb{Q}_p}}.$$

We have the following

Lemma 8.4. *We have*

$$\mathrm{ord}_P \mathcal{L}_{\mathcal{K}}^{Katz} + \mathrm{ord}_P \frac{c \cdot \omega_{\mathbf{g}}^{\vee}}{\eta_{\mathbf{g}}^{\vee}} \geq 0$$

for any height one prime P which is not (p) and not "exceptional" as defined in Proposition 8.3. (Note that we have to exclude the prime (p) due to CM components other than $\mathbf{g}$.)

Proof. There is a Hecke operator $1_{\mathbf{g}}$ in $\mathbb{T}_{\mathfrak{m}_{\mathbf{g}}} \otimes_{\Lambda_{\mathbf{g}}} F_{\Lambda_{\mathbf{g}}}$, the non-integral Hecke operator which cuts off the $\mathbf{g}$-part of any Hida family (See [61, 12.2] for details. Note also that $\mathbf{g}$ is generically non-Eisentein meaning that the generic specialization of it is cuspidal). From [43, Theorem and Corollary 2.3.6] we know $\mathfrak{B}_{\infty}^{*} \otimes \hat{\mathbb{Z}}_p^{ur} \simeq S^{ord}(\Gamma_1(D_{\mathcal{K}}), \hat{\mathbb{Z}}_p^{ur}[[T]])$ (the space of $\hat{\mathbb{Z}}_p^{ur}[[T]]$-coefficient ordinary families with tame level $D_{\mathcal{K}}$) as Hecke modules under which $\eta_{\mathbf{g}}^{\vee}$ maps to the normalized eigenform $\mathbf{g}$ (See the choice for them in [32, Theorem 7.4.10]). Note that $\rho(d)$ and $\rho(d)^{\vee}$ are invertible elements in $\hat{\mathbb{Z}}_p^{ur}[[T]]$. Note also that $c \cdot \omega_{\mathbf{g}}^{\vee}$ is in the cuspidal part $\mathfrak{B}_{\infty}^{*} \otimes \hat{\mathbb{Z}}_p^{ur} \subset \hat{\mathfrak{B}}_{\infty}^{*} \otimes \hat{\mathbb{Z}}_p^{ur}$ of the cohomology. So we just need to prove that for any $F \in S^{ord}(\Gamma_1(D_{\mathcal{K}}), \Lambda_{\mathbf{g}})$,

$$\mathrm{ord}_P \mathcal{L}_{\mathcal{K}}^{Katz} + \mathrm{ord}_P \frac{1_{\mathbf{g}} \cdot F}{\mathbf{g}} \geq 0 \tag{9}$$

for any non-exceptional primes $P \neq (p)$. This follows from Proposition 8.3: first of all, the $\mathcal{K}$-CM components other than $\mathbf{g}$ corresponds to characters of the Hilbert class group of $\mathcal{K}$. So it is easy to see that there is a $t_1 \in \mathbb{T}_{\mathfrak{m}_{\mathbf{g}}}$ such that $t_1 \mathbf{g} = a_{t_1} \cdot g$ for a_{t_1} being the product of an element in $\bar{\mathbb{Q}}_p^{\times}$ and an element of $\mathbb{Z}_p[[T]]^{\times}$, and such that t_1 kills $\mathcal{K}$-CM components of $\mathbb{T}_{\mathfrak{m}_{\mathbf{g}}}$ other than $\mathbf{g}$. Proposition 8.3 implies that there is an $\ell_{\mathbf{g}} \in \mathbb{T}_{\mathfrak{m}_{\mathbf{g}}}$ such that $t_1 \ell_{\mathbf{g}} \cdot F = a\mathbf{g}$ for $a \in \Lambda$ and $\ell_{\mathbf{g}} \cdot \mathbf{g} = b\mathbf{g}$ with $\mathrm{ord} \mathcal{L}_{\mathcal{K}}^{Katz} \geq \mathrm{ord}_P b$. But $t_1 \ell_{\mathbf{g}} F = t_1 \ell_{\mathbf{g}} 1_{\mathbf{g}} F = t_1 b 1_{\mathbf{g}} F = a_{t_1} a\mathbf{g}$. So $\mathrm{ord}_P b + \mathrm{ord}_P \frac{1_{\mathbf{g}} F}{\mathbf{g}} \geq 0$ and we get (9). □

Now we are ready to prove our theorem.

Theorem 8.5. *For any height one prime $P \neq (p)$ of $\Lambda_{\mathcal{K}}$ which is not exceptional, we have*

$$\mathrm{length}_P X^{++}_{f,\mathcal{K}} \geq \mathrm{ord}_P L^{++}_{f,p}. \tag{10}$$

Proof. Completely the same as the proof of Theorem 8.2 except that we also take Lemma 8.4 into consideration (see proof of Proposition 7.7 for where $\mathcal{L}^{Katz}_{\mathcal{K}}$ plays a role). □

To take care of the prime (p) we use the following proposition of Pollack-Weston.

Proposition 8.6. *Suppose N is square-free, $a_p = 0$. Suppose moreover that any prime divisor of N either splits in $\mathcal{K}$ or is inert. Assume for any such inert prime q we have $\bar{\rho}|_{G_q}$ is ramified and there are odd number of such inert primes. Then*

$$\mathrm{ord}_{(p)} L^{++}_{f,p} \leq 0.$$

Proof. We may assume that $L^{++}_{f,p} \in \Lambda$. By [50] the anticyclotomic μ-invariant for the specialization of $L^{++}_{f,p}$ to anticyclotomic line is 0. Note that the period used in [50] is Ω^{can} which, up to multiplying by a p-adic unit is $\Omega^+_E\Omega^-_E$. Note also that in *loc.cit* they assumed moreover that

- $\mathrm{Im}(G_{\mathbb{Q}}) = \mathrm{Aut}(T_E)$.
- The anticyclotomic $\mathbb{Z}_p$-extension of $\mathcal{K}$ is totally ramified at p.

But these assumptions are not necessary: the surjectivity of the Galois representation can be replaced by irreducibility (See [29]). The second assumption is needed only for the vanishing of the algebraic μ-invariant and not needed for the analytic μ-invariant. (We thank Chan-Ho Kim for discussing these with us). □

Now we prove the lower bound for Selmer groups in Conjecture 6.7. Note that the pullback of the augmentation ideal of the anticyclotomic line does not contain $L^{++}_{f,p}$ since the specialization of the latter to the cyclotomic line is not identically zero. We conclude that under the assumption of Theorem 8.5 and Proposition 8.6 the full one-side inequality for (10) is true.

8.2 Kobayashi's Main Conjecture

Now we prove a control theorem for Selmer groups and deduce Kobayashi's one-variable main conjecture from the two variable one.

Proposition 8.7. *Let P be the prime of $\Lambda_{\mathcal{K}}$ generated by $T-1$ then*

$$X^{++} \otimes \Lambda_{\mathcal{K}}/P \simeq X^+_{E,\mathcal{K}_{cyc}}$$

where the last term is the + dual Selmer group of E over $\mathcal{K}_{cyc}$ defined similar as X^{++}.

Proof. This theorem is proved in the same way as [33, Theorem 9.3]. One first proves that $\hat{E}(\mathfrak{m}_{m,n})$ has no p-power torsion points as in [33, Proposition 8.7]. This implies that

$$\varprojlim_m \varprojlim_n H^1(k_{m,n}, T) \to H^1(k_{m_0,n_0}, T)$$

is surjective. Then the control theorem follows in the same way as Proposition 9.2 of *loc.cit.* □

Proof. (of Theorem 1.4) The above proposition implies the cyclotomic main conjecture over $\mathcal{K}$ under the assumption of Proposition 8.6. Note that since N is square-free, there must be a prime q such that $E[p]|_{G_q}$ is ramified (since otherwise by Ribet's level lowering there will be a weight two cuspidal eigenform with level 1, which can not exist). To prove Theorem 1.4, we just need to choose the auxiliary $\mathcal{K}$. We take $\mathcal{K}$ such that p and all prime divisors of N except q are split in $\mathcal{K}$ while q is inert. This main conjecture over $\mathcal{K}$ together with one divisibility over $\mathbb{Q}$ proved in [33] gives the proof of the main Theorem. Note that in [25] it is assumed that the image of $G_{\mathbb{Q}}$ is $\mathrm{Aut}(T_E) = \mathrm{GL}_2(\mathbb{Z}_p)$. However under our assumption that N is square-free it is enough to assume $E[p]|_{G_{\mathbb{Q}}}$ is absolutely irreducible, as explained in [59, Page 15-16]. The irreducibility of $E[p]|_{G_{\mathbb{Q}_p}}$ is proved in [7]. □

Finally we prove the following refined BSD formula.

Corollary 8.8. *Suppose E is an elliptic curve with square-free conductor N and supersingular reduction at p such that $a_p = 0$. If $L(E,1) \neq 0$ then we have the following refined BSD formula*

$$\frac{L(E,1)}{\Omega_E} = \sharp Ш_{E/\mathbb{Q}} \cdot \prod_{\ell | N} c_\ell$$

up to a p-adic unit. Here c_ℓ is the Tamagawa number of E at ℓ. Note that by irreducibility of the Galois representation we know the p-part of the Mordell-Weil group is trivial.

Proof. This is proved as in [12, Theorem 4.1], replacing the argument for the prime p by [33, Proposition 9.2] for $\mathcal{L}^+_{E,\mathbb{Q}}$. (In fact, all we need to do is to show that the p-adic component of the map g_n in the commutative diagram on top of [33, Page 27] is injective, which follows from that (9.33) of *loc.cit* is injective. This is nothing but the Pontryagin dual of Proposition 9.2 there). We use the interpolation formula [33, (3.6)] on the analytic side. Note also the fact that the Iwasawa module of dual Selmer group has no non-trivial subgroup of finite cardinality is also deduced within the proof of [12, Theorem 4.1] and can be obtained in the same way in our situation. This argument is also given in details in [27].

□

References

[1] Joël Bellaïche, Gaetan Chenevier, p-adic families of Galois representations and higher rank Selmer groups arXiv:math/0602340.

[2] Breuil, C., Conrad, B., Diamond, F., and Taylor, R. (2001). On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises. Journal of the American Mathematical Society, 843-939.

[3] Bertolini, Massimo, and Henri Darmon. "Iwasawa's main conjecture for elliptic curves over anticyclotomic Zp-extensions." Annals of mathematics (2005): 1-64.

[4] Bloch, S., and Kato, K., L-functions and Tamagawa numbers of motives. In The Grothendieck Festschrift, Vol. I, Progr. Math. 86, Birkhauser, Boston, MA, 1990,333-400.

[5] Cartier, Pierre. "Representations of p-adic groups: a survey." Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part. Vol. 1. 1979.

[6] Darmon, Henri, and Adrian Iovita. "The anticyclotomic Main Conjecture for elliptic curves at supersingular primes." Journal of the Institute of Mathematics of Jussieu 7.02 (2008): 291-325.

[7] Edixhoven, Bas. The weight in Serre's conjectures on modular forms. Inventiones mathematicae 109.1 (1992): 563-594.

[8] E.Eischen, M.Harris, J.Li, C.Skinner, p-adic L-functions for Unitary Shimura Varieties (II), in preparation.

[9] E. Eischen and X. Wan, p-adic L-Functions of Finite Slope Forms on Unitary Groups and Eisenstein Series, to appear in Journal of the Institute of Mathematics of Jussieu

[10] G. Faltings, C.-L. Chai, Degeneration of Abelian Varieties, Ergebnisse der Math. 22, Springer Verlag, 1990.

[11] R. Greenberg, Iwasawa theory and p-adic Deformations of Motives, Proc. on Motives held at Seattle, 1994.

[12] Greenberg, Ralph. "Iwasawa theory for elliptic curves." Arithmetic theory of elliptic curves. Springer Berlin Heidelberg, 1999. 51-144.

[13] Greenberg, Ralph, and Vinayak Vatsal. "On the Iwasawa invariants of elliptic curves." Inventiones mathematicae 142.1 (2000): 17-63.

[14] Harris, Michael. "Eisenstein series on Shimura varieties." The Annals of Mathematics 119.1 (1984): 59-94.

[15] Hida, H.: On p-adic L-functions of $\mathrm{GL}(2) \times \mathrm{GL}(2)$ over totally real fields. Ann. Inst. Fourier 40, 311-391 (1991).

[16] Hida, Haruzo. "Control theorems of p-nearly ordinary cohomology groups for $\mathrm{SL}(n)$." Bulletin de la Societe Mathematique de France 123.3 (1995): 425.

[17] Haruzo Hida, p-adic automorphic forms on Shimura varieties, Springer Monographs in Mathematics, Springer-Verlag, New York, 2004.

[18] Hida, Haruzo, and Jaques Tilouine. "On the anticyclotomic main conjecture for CM fields." Inventiones mathematicae 117.1 (1994): 89-147.

[19] Hsieh, Ming-Lun. "Eisenstein congruence on unitary groups and Iwasawa main conjectures for CM fields." Journal of the American Mathematical Society 27.3 (2014): 753-862.

[20] M.-L. Hsieh, On the non-vanishing of Hecke L-values modulo p, American Journal of Mathematics 134 (2012), no. 6, 1503-1539..

[21] M.-L. Hsieh, Special values of anticyclotomic Rankin-Selbeg L-functions, Documenta Mathematica, 19 (2014), 709-767.

[22] H. Hida, J. Tilouine, Anti-cyclotomic Katz p-adic L-functions and congruence modules, Ann. Sci. Ecole Norm. Sup. (4) 26 (2) (1993) 189-259.

[23] A. Ichino, Trilinear forms and the central values of triple product L-functions, Duke Math. J., 145 (2008), 281-307

[24] T. Ikeda, On the theory of Jacobi forms and Fourier-Jacobi coefficients of Eisenstein series, J. Math. Kyoto Univ. 34-3 (1994).

[25] K. Kato, p-adic Hodge theory and values of zeta functions of modular forms, Cohomologies p-adiques et applications arithmetiques. III. Asterisque No. 295 (2004), ix, p. 117-290.

[26] Kim, Byoung Du. "Signed-Selmer Groups over the $\mathbb{Z}_p^2$-extension of an Imaginary Quadratic Field." Canadian Journal of Mathematics, 66.4 (2014): 826-843.

[27] Kim, Byoung Du. "THE PLUS/MINUS SELMER GROUPS FOR SUPERSINGULAR PRIMES." Journal of the Australian Mathematical Society 95.02 (2013): 189-200.

[28] Kisin, Mark. "Moduli of finite flat group schemes, and modularity." Ann. of Math.(2) 170.3 (2009): 1085-1180.

[29] Kim, Chan Ho, Anticyclotomic Iwasawa invariants and congruences of modular forms, preprint.

[30] David Loeffler, and Sarah Livia Zerbes, Rankin-Eisenstein classes in Coleman families, preprint, arXiv: 1506.06703 (2015).

[31] Kings, Guido, David Loeffler, and Sarah Livia Zerbes. "Rankin-Eisenstein classes for modular forms." arXiv preprint arXiv:1501.03289 (2015).

[32] Kings, Guido, David Loeffler, and Sarah Livia Zerbes. "Rankin–Eisenstein classes and explicit reciprocity laws." arXiv preprint arXiv:1503.02888 (2015).

[33] Kobayashi, Shin-ichi. "Iwasawa theory for elliptic curves at supersingular primes." Inventiones mathematicae 152.1 (2003): 1-36.

[34] K.-W. Lan, Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties, preprint, J. Reine Angew. Math. 664 (2012), pp. 163–228.

[35] K.-W. Lan, Arithmetic compactifications of PEL-type Shimura varieties, Harvard PhD thesis, 2008 (revised 2010).

[36] Lei, Antonio, David Loeffler, and Sarah Livia Zerbes. "Euler systems for Rankin-Selberg convolutions of modular forms." Annals of Mathematics, Vol 180(2), 653-771.

[37] Liu, Ruochuan. "Triangulation of refined families." arXiv preprint arXiv:1202.2188 (2012).

[38] Loeffler, David, and Sarah Livia Zerbes. "Iwasawa theory and p-adic L-functions over $\mathbb{Z}_p^2$-extensions." arXiv preprint arXiv:1108.5954 (2011).

[39] E. Lapid and S. Rallis, On the local factors of representations of classical groups in Automorphic representations, L-functions and applications: progress and prospects, 309-359, Ohio State Univ. Math. Res. Inst. Publ. 11, de Gruyter, Berlin, 2005.

[40] S. Morel, On the Cohomology of Certain Non-compact Shimura Varieties, Annals of Math. Studies 173, Princeton University Press, princeton, 2010.

[41] C. Moglin and J.-L. Waldspurger, Spectral decomposition and Eisenstein series. Une paraphrase de l'ecriture, Cambridge Tracts in Mathematics 113, Cambridge University Press, Cambridge,1995.

[42] B. Mazur and A. Wiles, Class fields of abelian extensions of $\mathbb{Q}$, Invent. Math. 76, p. 179-330,1984.

[43] Ohta, Masami. "Ordinary p-adic etale cohomology groups attached to towers of elliptic modular curves. II." Mathematische Annalen 318.3 (2000): 557-583.

[44] Perrin-Riou, Bernadette. p-adic L-Functions and p-Adic Representations. American Mathematical Society, 2000.

[45] Andreatta F, Iovita A, Pilloni V. p-adic families of Siegel modular cuspforms. Annals of mathematics, 2015, 181(2): 623-697.

[46] Bijakowski S, Pilloni V, Stroh B. ClassicitÃľ de formes modulaires surconvergentes. Annals of Mathematics, 2016, 183: 975-1014.

[47] J.Pottharst, Cyclotomic Iwasawa theory of motives, preprint. (`http://vbrt.org/jay/math/`).

[48] Pollack, Robert. "On the p-adic L-function of a modular form at a supersingular prime." Duke Mathematical Journal 118.3 (2003): 523-558.

[49] Pollack, Robert, and Karl Rubin. "The main conjecture for CM elliptic curves at supersingular primes." Annals of mathematics (2004): 447-464.

[50] Pollack, Robert, and Tom Weston. "On anticyclotomic Îij-invariants of modular forms." Compositio Mathematica 147.05 (2011): 1353-1381.

[51] Rohrlich, David E. "*L*-functions and division towers." Mathematische Annalen 281.4 (1988): 611-632.

[52] Rubin, Karl. "The "main conjectures" of Iwasawa theory for imaginary quadratic fields." Inventiones mathematicae 103.1 (1991): 25-68.

[53] Rubin, Karl. "More "main conjectures" for imaginary quadratic fields." Centre de Recherches Mathematiques 4 (1994): 23-28.

[54] G. Shimura, Euler products and Eisenstein series, CBMS Regional Conference Series in Mathematics 93, American Mathematical Society, Providence, RI, 1997.

[55] G. Shimura, Arithmeticity in the theory of automorphic forms, Mathematical Surveys and Monographs, 82. American Mathematical Society, Providence, RI, 2000. x+302 pp.

[56] S.-W. Shin, Galois representations arising from some compact Shimura varieties, to appear in Annals of Math.

[57] C. Skinner, A Converse of Gross, Zagier and Kolyvagin, preprint, 2013.

[58] Skinner, Christopher. "Galois Representations Associated to Unitary Groups over $\mathbb{Q}$." Algebra and Number Theory, 2012, 8(8):1697-1717.

[59] Christopher Skinner, "Multiplicative reduction and the cyclotomic main conjecture for GL_2", arXiv:1407.1093.

[60] Florian Sprung, Iwasawa theory for elliptic curves at supersingular primes: A pair of main conjectures, Journal of Number Theory, Volume 132, Issue 7, July 2012, Pages 1483-1506.

[61] Skinner, Christopher, and Eric Urban. "The Iwasawa main conjectures for GL2." Inventiones mathematicae 195.1 (2014): 1-277.

[62] C.Skinner, W.Zhang, Indivisibility of Heegner points in the multiplicative case, arXiv:1407.1099.

[63] Taylor, Richard. "On the meromorphic continuation of degree two L-functions." Doc. Math 729779 (2006).

[64] Tilouine, Jacques, and Eric Urban. "Several-variable p-adic families of Siegel-Hilbert cusp eigensystems and their Galois representations." Annales Scientifiques de lÂĄ'Ecole Normale SupÂĄerieure. Vol. 32. No. 4. No longer published by Elsevier, 1999.

[65] Eric Urban, Groupes de Selmer et Fonctions L p-adiques pour les representations modulaires adjointes (E. Urban), preprint (2006).

[66] Eric Urban, Nearly overconvergent modular forms, Proceedings of the Conference "Iwasawa 2012", to appear, 2013.

[67] X. Wan, Iwasawa Main Conjecture for Rankin-Selberg p-adic L-functions, submitted, arXiv:1408.4044, 2014.

[68] X. Wan, Introduction to Skinner-Urban's Work on Iwasawa Main Conjecture, to appear in Proceedings of Iwasawa 2012 conference, Heidelberg, Germany.

[69] X. Wan, Families of Nearly Ordinary Eisenstein Series on Unitary Groups, to appear in Algebra and Number Theory.

[70] Washington, Lawrence C. Introduction to cyclotomic fields. Vol. 83. Springer Science and Business Media, 1997.

[71] A. Wiles, The Iwasawa conjecture for totally real fields, Ann. of Math. 131, 1990.

[72] A. Wiles, Elliptic curves and Fermat's Last Theorem Ann. of Math. 141, p. 443-551, 1995.

[73] Woodbury, Michael. "Explicit trilinear forms and the triple product L-functions." Preprint (2012).

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