Iwasawa Main Conjecture for Non-Ordinary Modular Forms

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Abstract

Let $p > 2$ be a prime. Under mild assumptions, we prove the Iwasawa main conjecture of Kato, for modular forms with general weight and conductor prime to p. This generalizes an earlier work of the author on supersingular elliptic curves.

1 Introduction

Let $p > 2$ be a prime. The history of Iwasawa theory dates back to the 1950's, when Iwasawa studied the p-part of class groups of cyclotomic extensions of $\mathbb{Q}$ of p-power degree. It turns out that one can pass to the cyclotomic $\mathbb{Z}_p$ extension field $\mathbb{Q}_\infty$ and get a finitely generated torsion module structure over the so called Iwasawa algebra $\Lambda := \mathbb{Z}_p[[\Gamma]]$, where Γ is defined to be $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. In 1960's Kobota and Leopoldt discovered an analytic counterpart of Iwasawa's Λ-module namely the p-adic L-function which packages together the algebraic part of special values of L-functions of $\mathbb{Q}$ twisted by finite order cyclotomic characters. An old philosophy, which is explicitly formulated as the Iwasawa main conjecture in this context, is that such special L-values should give the size of some class groups with corresponding actions of the Galois group Γ. Later in the 1970's, Barry Mazur made a key observation that Iwasawa's idea can be applied to elliptic curves, or more generally, abelian varieties. This has important application to the Birch and Swinnerton-Dyer conjecture. Later on the formulation has been vastly generalized to motives, including the case of modular forms, by Greenberg and other people.

Towards a proof there are in principal two approaches. One is using Euler system. This idea originated in the work of Kolyvagin in late 1980's and later on axiomized by Rubin in a handful of different contexts. The Euler system method is especially useful in giving the *upper* bound of the arithmetic objects (namely Selmer groups). With the help of the class number formula, this is enough to imply the full equality in Iwasawa main conjecture for certain Hecke characters. The other approach is to use modular forms. Such idea first appeared in a work of Ribet in early 1980's (called the Ribet's lemma). This method is used to deduce the *lower* bound of Selmer groups and is employed by Mazur-Wiles and Wiles to prove the Iwasawa main conjecture for totally real fields. However to study Iwasawa theory for motives of rank larger than one, where one does not have the class number formula, one needs to apply both the Euler system method and the modular form method to give the full equality. This is illustrated by the recent work of Kato and Skinner-Urban in the proof of Iwasawa main conjecture for modular forms ordinary at p. Kato proved the upper bound for Selmer groups by constructing an Euler system using K-theory of modular curves, while Skinner-Urban used modular form method on the rank 4 unitary group $\mathrm{U}(2,2)$. Now we discuss

some details about Kato and Skinner-Urban's work, which is closely related to the present paper.
Strict Selmer Groups
Let

$$f = \sum_{n=1}^{\infty} a_n q^n$$

be a normalized cuspidal eigenform for $\mathrm{GL}_2/\mathbb{Q}$ with even weight k and conductor N. By work of Shimura, Deligne, Langlands and others, one can associate a two dimensional irreducible Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_L)$. Here L is a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_L$ is the integer ring of it. We choose the L so that it contains all the coefficients of $\Gamma_0(N)$ cusp forms. This Galois representation is determined by requiring that for all primes $\ell \nmid pN$,

$$\mathrm{tr}\rho_f(\mathrm{Frob}_\ell) = a_\ell.$$

Moreover $T_f|_{G_{\mathbb{Q}_p}}$ is crystalline in the sense of Fontaine with Hodge-Tate weights $(0, k-1)$. (We use the convention that the cyclotomic character has Hodge-Tate weight 1.) In this paper we define the Iwasawa algebra $\Lambda = \mathcal{O}_L[[\Gamma]]$. Let S be the set of primes dividing pN. We define

$$H^1_{\mathrm{cl,Iw}}(\mathbb{Q}^S/\mathbb{Q}, T_f(-\frac{k-2}{2})) := \varprojlim_n H^1(\mathbb{Q}_n^S/\mathbb{Q}_n, T_f(-\frac{k-2}{2}))$$

where $\mathbb{Q}_n$ is running over all intermediate field extensions between $\mathbb{Q}_\infty$ and $\mathbb{Q}$. Kato proved that it is a torsion-free rank one module over Λ, and defined a zeta element z_{Kato} in it. On the arithmetic side, we define the Selmer group

$$\mathrm{Sel}_{\mathrm{str},\mathbb{Q}_n}(f) := \ker\{H^1(\mathbb{Q}_n, T_f(-\frac{k-2}{2}) \otimes_{\mathcal{O}_L} \mathcal{O}_L^*) \to \prod_{v\nmid p} H^1(I_v, T_f(-\frac{k-2}{2}) \otimes_{\mathcal{O}_L} \mathcal{O}_L^*)$$
$$\times H^1(\mathbb{Q}_{n,p}, T_f(-\frac{k-2}{2}) \otimes_{\mathcal{O}_L} \mathcal{O}_L^*)\}.$$

Here the superscript $*$ means Pontryagin dual. Define also

$$\mathrm{Sel}_{\mathrm{str},\mathbb{Q}_\infty}(f) := \varinjlim \mathrm{Sel}_{\mathrm{str},\mathbb{Q}_n}(f)$$

and

$$X_{\mathrm{str}} := \mathrm{Sel}_{\mathrm{str},\mathbb{Q}_\infty}(f)^*.$$

Kato formulated the Iwasawa main conjecture as

Conjecture 1.1.

$$\mathrm{char}_\Lambda X_{\mathrm{str}} := \mathrm{char}(\frac{H^1_{\mathrm{cl,Iw}}(\mathbb{Q}^S/\mathbb{Q}, T_f(-\frac{k-2}{2}))}{\Lambda z_{\mathrm{Kato}}}).$$

Remark 1.2. *In [22] Kato used the Λ-module $H^2(\mathbb{Q}_S, T_f(-\frac{k}{2}) \otimes \Lambda)$, which is isomorphic to X_{str} here (see [28, Page 12]).*

Kato proved the following

Theorem 1.3. *We have*

$$\mathrm{char}_{\Lambda\otimes\mathbb{Q}_p}(X_{\mathrm{str}}) \supseteq \mathrm{char}_{\Lambda\otimes\mathbb{Q}_p}(\frac{H^1_{\mathrm{Iw}}(\mathbb{Q}, T_f(-\frac{k-2}{2}))}{\Lambda z_{\mathrm{Kato}}}).$$

If moreover $\mathrm{Im}(G_{\mathbb{Q}})$ *contains* $\mathrm{SL}_2(\mathbb{Z}_p)$*, then the above containment is true as ideals of* Λ.

In the case when f has CM the theorem depends on the result of Karl Rubin. Under some hypothesis Skinner and Urban [40] proved the other side containment in the Iwasawa main conjecture, in the case when the form f is good ordinary at p (meaning a_p is a p-adic unit). If f is not ordinary the situation is more complicated. Our goal in this paper is the prove the other side containment for general modular forms with $p \nmid N$.

We make the following assumption

(Irred) The residual Galois representation $\bar{T}$ is irreducible over $G_{\mathbb{Q}_p}$.

This assumption is made to ensure that the local Iwasawa cohomology group at p is free over the Iwasawa algebra, which simplifies the argument. We did not think about if there is any essential difficulty without this assumption. Our main theorem is the following

Theorem 1.4. *Assume* $2|k$*,* $p \nmid N$*,* (Irred)*, and that* $\bar{T}|_{\mathbb{Q}(\zeta_p)}$ *is absolutely irreducible. Assume moreover that the* p*-component of the automorphic representation* π_f *is a principal series representation with distinct Satake parameters. Then we have*

- *If there is an* $\ell||N$*, then*

$$\mathrm{char}_{\Lambda[1/p]}(X_{\mathrm{str}}) \subseteq \mathrm{char}_{\Lambda[1/p]}(\frac{H^1_{\mathrm{cl,Iw}}(\mathbb{Q}, T_f(-\frac{k-2}{2}))}{\Lambda z_{\mathrm{Kato}}}).$$

- *If there is an* $\ell||N$ *such that* π_ℓ *is the Steinberg representation twisted by* $\chi_{\mathrm{ur}}^{\frac{k}{2}}$ *for* χ_{ur} *being the unramified character sending* p *to* $(-1)^{\frac{k}{2}}p^{\frac{k}{2}-1}$*. Suppose moreover that the weight* k *is in the Fontaine-Laffaille range* $k \leq p-1$*, Then*

$$\mathrm{char}_{\Lambda}(X_{\mathrm{str}}) \subseteq \mathrm{char}_{\Lambda}(\frac{H^1_{\mathrm{cl,Iw}}(\mathbb{Q}, T_f(-\frac{k-2}{2}))}{\Lambda z_{\mathrm{Kato}}}).$$

Note that for the one side containment above we do *not* need to assume the image of the residual Galois representation $\bar{T}$ contains $\mathrm{SL}_2(\mathbb{Z}_p$. The assumption on the Satake parameter is conjecturally automatic. The assumption on the ℓ on the second part is due to using results of Hsieh in [17] and can probably be weakened if one has a general weight version of [7]. (It is used to ensure the local root numbers of f over the auxiliary quadratic field to be $+1$.) The reason for the Fontaine-Laffille assumption is two fold: one reason is to use the result of Faltings-Jordan [14] for the freeness of certain cohomology module over the local Hecke algebra to compare the periods Ω_f^{can} and $\Omega_f^+\Omega_f^-$. On the other hand we need an integral comparison theorem for p-adic etale and deRham cohomology for modular curves. Recently Bhargav-Morrow-Scholze [6] proved the general integral comparison result for cohomology with trivial coefficient sheaf, replacing the Fontaine-Laffaille functor by the Breuil-Kisin module functor. If general coefficient sheaf is allowed in Bhargav-Morrow-Scholze's result, we may remove this assumption on small weight for this part via a careful study of the corresponding Breuil-Kisin module. A corollary of part two is that if $L(f, \frac{k}{2}) \neq 0$ then the p-part of the Tamagawa number conjecture is true (see Corollary 4.30), for all but finitely many small primes p (i.e. smaller than $k+1$).

Remark 1.5. *If $k = 2$ then the assumptions* (Irred)*, the freeness of the Hecke module and that $\bar{T}|_{\mathbb{Q}(\zeta_p)}$ is absolutely irreducible are redundant. Moreover in part two we can alternatively assume that N is square-free and that there are at least two primes $\ell||N$ such that the modulo p representation $\bar{T}_f|_{G_\ell}$ is ramified. (It seems that in [7] it is enough to assume N^+ there is square-free. In that case the assumption N is square-free would be redundant).*

The proof for the lower bound of Selmer groups uses the relations to L-functions. In the ordinary case, Skinner-Urban studied the congruences between Eisenstein series and cusp forms on $\mathrm{U}(2,2)$, upon fixing some quadratic imaginary extension $\mathcal{K}/\mathbb{Q}$. One important reason why it is possible is in the ordinary case, the Iwasawa main conjecture can be formulated in Greenberg's style, making it convenient to use the lattice construction to produce enough elements in the Selmer groups. In the non-ordinary case such formulation is not available. In the finite slope case there is Iwasawa theory for trianguline representations developed by Pottharst (with the expense of inverting p for everything). However it seems still rather difficult to prove the conjecture directly. For example one challenging problem is to construct explicit families of finite slope Klingen-Eisenstein series (the strategy developed from *loc.cit* does not give an overconvergent family, thus cannot be used to construct a finite slope projection), and studying p-adic properties of the Fourier expansion. There are also other difficulties, including constructing families of triangulations around all points of interest.

The first general result in the non-ordinary case is the $\pm$-main conjecture for elliptic curves $E/\mathbb{Q}$, recently proved by the author [43]. To illustrate the idea, we first briefly discuss Greenberg-Iwasawa theory, which plays a crucial role in the argument. Taking a quadratic imaginary field $\mathcal{K}/\mathbb{Q}$ such that p splits as $v_0\bar{v}_0$. Let $\Gamma_\mathcal{K}$ be the Galois group of $\mathcal{K}_\infty/\mathcal{K}$ for $\mathcal{K}_\infty$ being the $\mathbb{Z}_p^2$ extension of $\mathcal{K}$. Then any form g with complex multiplication by $\mathcal{K}$ is ordinary at p. Suppose the weight of g is greater than the weight of f. Then the Iwasawa theory of the Rankin-Selberg product $f \otimes g$ has the same form as ordinary forms, since it satisfies the Panchishkin's condition. This makes the corresponding Iwasawa main conjecture more accessible. Moreover this motive is closely related to the Iwasawa theory of the original modular form f. We first give the precise formulation of the main conjecture. In application we suppose g is the Hida family corresponding to characters of $\Gamma_\mathcal{K}$ and we identify the Galois representation of g with the induced representation from $G_\mathcal{K}$ to $G_\mathbb{Q}$ of some character Ψ of $\Gamma_\mathcal{K}$. On the arithmetic side we defined

$$\begin{aligned}\mathrm{Sel}^{\mathrm{Gr}}_{\mathcal{K},f} = \ker\{H^1(\mathcal{K}, T_f(-\frac{k-2}{2}) \otimes \Lambda^*(\Psi)) \qquad &\to \prod_{v\nmid p} H^1(\mathcal{K}_v, T_f(-\frac{k-2}{2}) \otimes \Lambda^*(\Psi)) \\ &\times H^1(\mathcal{K}_{\bar{v}_0}, T_f(-\frac{k-2}{2}) \otimes \Lambda^*(\Psi)).\end{aligned}$$

$$X^{\mathrm{Gr}}_{\mathcal{K},f} := (\mathrm{Sel}^{\mathrm{Gr}}_{\mathcal{K},f})^*.$$

On the analytic side, there is a Greenberg p-adic L-function $\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}} \in \mathcal{O}^{\mathrm{ur}}_L[[\Gamma_\mathcal{K}]]$ with interpolation property given in Proposition 3.1. Here we write $\mathcal{O}^{\mathrm{ur}}_L$ for the completion of the maximal unramified extension of $\mathcal{O}_L$. The Greenberg-Iwasawa main conjecture is the following:

Conjecture 1.6. *(Greenberg Main Conjecture)*

$$\mathrm{char}_{\mathcal{O}^{\mathrm{ur}}_L[[\Gamma_\mathcal{K}]]}(X^{\mathrm{Gr}}_{\mathcal{K},f} \otimes_{\mathcal{O}_L} \mathcal{O}^{\mathrm{ur}}_L) = (\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}).$$

We will call this conjecture (GMC) in this paper. In our case the fact that $\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}$ is integral is explained in the text. Under some hypothesis we proved in [43], [44] that if f has weight two, then up to powers of p we have

$$\mathrm{char}_{\mathcal{O}_L^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]}(X^{\mathrm{Gr}}_{\mathcal{K},f}\otimes_{\mathcal{O}_L}\mathcal{O}_L^{\mathrm{ur}})\subseteq(\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}).$$

After this is proved, in [43] the author used the explicit reciprocity law for Beilinson-Flach element and Poitou-Tate exact sequence to deduce Conjecture 1.1 from Conjecture 1.6.

To prove the main theorem in this paper for general modular forms of any even weight, the difficulty is two-fold. First of all, the author only proved Greenberg's main conjecture when f has weight two. The obstacle is that if f has higher weight then one needs to compute the Fourier-Jacobi expansion for vector valued Eisenstein series, which seems formidable. Secondly we do not have an explicit local theory as in the $\pm$ case (except in the special case when $a_p=0$, and for elliptic curves over $\mathbb{Q}$ but $a_p\neq 0$ by Florian Sprung), while the work [43] used such theory in a crucial way.

Our first result is the following theorem on one containment of Conjecture 1.6,which generalizes the result in [44] to forms f of any even weight k.

Theorem 1.7. *Suppose there is at least one rational prime q where the automorphic representation π_f associated to f is not a principal series representation. Assume moreover that the p-component of the automorphic representation π_f is a principal series representation with distinct Satake parameters, and that the residual Galois representation $\bar{\rho}_f$ is irreducible over $G_{\mathcal{K}(\zeta_p)}$. Then up to powers of p we have*

$$\mathrm{char}_{\mathcal{O}_L^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]}(X^{\mathrm{Gr}}_{\mathcal{K},f}\otimes_{\mathcal{O}_L}\mathcal{O}_L^{\mathrm{ur}})\subseteq(\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}).$$

Moreover if for each $\ell|N$ non-split in $\mathcal{K}$, we have $\ell||N$ is ramified in $\mathcal{K}$ and π_ℓ is the Steinberg representation twisted by $\chi_{\mathrm{ur}}^{\frac{k}{2}}$ for χ_{ur} being the unramified character sending p to $(-1)^{\frac{k}{2}}p^{\frac{k}{2}-1}$. Then the above containment is true before inverting p.

The last part is by appealing to the result of Hsieh [17] on the vanishing of anti-cyclotomic μ-invariant of $\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}$. To prove this theorem, we use the full strength of our joint work with Eischen [12] on constructing vector-valued Klingen Eisenstein families on $\mathrm{U}(3,1)$, from pullbacks of nearly holomorphic Siegel Eisenstein series on $\mathrm{U}(3,3)$. We combine our earlier work in [43], [44] on the p-adic property for Fourier-Jacobi coefficients with the general theory of Ikeda that the Fourier-Jacobi coefficient of nearly holomorphic Siegel Eisenstein series can be C^∞-approximated by finite sums of products of Eisenstein series and theta functions on the Jacobi group containing $\mathrm{U}(2,2)$. (In the scalar valued case it is already such a finite sum since the Siegel Eisenstein series showing up is *holomorphic*.) We fix one Arichimedean weight and vary the p-adic nebentypus in families. While the Fourier-Jacobi expansion in the vector-valued case seems quite hard to compute, we can still use a conceptual argument to factor out a convergent infinite sum of Archimedean integrals for it and prove the factor is non-zero. After this we can apply the techniques we developed in our previous work to prove that certain Fourier-Jacobi coefficient is co-prime to the p-adic L-function we study. Along the way we determine the constant coming from the local pullback integrals for doubling methods at Archimedean places, by comparing our construction with Hida's construction using Rankin-Selberg method. This is crucial for the proof. (It seems such constants are hard to compute directly. But logically only with this in hand we can know that the family constructed in [12] is not zero in the vector-valued case!) Along the way we also remove the square-free conductor assumption for f in our previous works. We believe that besides proving the main theorem of this paper, the above theorem itself should have independent interest.

To prove the main theorem, we first prove the result after inverting p. Our idea is to use the analytic Iwasawa theory of Pottharst and Iwasawa theory for (φ, Γ)-modules (upgraded to a two variable setting), in the context of Nekovar's Selmer complexes [35]. It turns out that Pottharst's triangline-ordinary theory works in similar way as the classical ordinary case when working with the more flexible analytic Iwasawa theory. In the two-variable setting there are subtleties to take care of – for example there is a finite set of height one primes where the regulator map vanishes. Also we need to compare differently constructed analytic p-adic L-functions – although they agree on all arithmetic points, however these do not uniquely determine the analytic functions themselves.

After this, we only need to study powers of p. After carefully studying the control theorem, we only need to compute the cardinality (which is $< \infty$) of the Selmer group for $T_f(-\frac{k-2}{2})$ twisted by some finite order cyclotomic character. We use a different idea and work directly with Kato's zeta element. We avoid the search for nice integral local theory analogous to the $\pm$ theory. Instead we take one "generic" finite order cyclotomic character twist of T_f, and consider deformations of it along the one variable family which corresponds to the $\mathbb{Z}_p$-extension of $\mathcal{K}$ that is totally ramified at $\bar{v}_0$ and unramified at v_0. In this family we do have a nice integral local theory at v_0. The key fact is a uniform boundedness result for Bloch-Kato's logarithm map for families of unramified twists. We prove this by a careful study of Fontaine's rings B_{dR} and A_{cris}. Fortunately this theory is enough for our purposes.

We managed to make our proof to work as general as possible. In fact most part of the argument can be applied to the case when f has ramification at p (most interestingly when π_f is supercuspidal at p). However the essential difficulty is to prove the reciprocity law for Beilinson-Flach element at the arithmetic points corresponding to $L(f, \chi, \frac{k}{2})$ for finite order characters χ of Γ. These correspond to Rankin-Selberg products of f with weight one forms g, in which geometry does not give the required formula directly. In the crystalline case, Kings-Loeffler-Zerbes achieved this by constructing a big regulator map interpolating the Bloch-Kato $\exp^*$ map and log map, and do some analytic continuation. In the non-crystalline case it seems very hard to work out such a big regulator map and the interpolation formula explicitly. One also needs to understand certain p-adic L-function for such f, which has infinite slope. It seems there are some ongoing work on this using explicit description of p-adic local Langlands correspondence, follows an early idea of M.Emerton. We hope experts in such areas can shed some light on such problems. (And this paper provides more motivation for such investigations).

The paper is organized as follows: in Section 2 we recall and develop some p-adic local theory needed for the argument. One key result is to study the Iwasawa theory for the $\mathbb{Z}_p$-extension of $\mathcal{K}$ which is unramified in v_0 and totally ramified in $\bar{v}_0$. In Section 3 we prove the Greenberg type main conjecture for general weight modular forms f, by studying the local integral at the Archimedean place. In Section 4 we give the proof of the main theorem of this paper.

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Notations: In this paper we often write $T = T_f(-\frac{k-2}{2})$ for the Galois representation T_f associated to f. We write $\mathbb{Q}_\infty$ as the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let $\mathcal{K}$ be a quadratic imaginary extension of $\mathbb{Q}$ where p splits as $v_0\bar{v}_0$. Let $\mathcal{K}_\infty$ be the $\mathbb{Z}_p^2$ extension of $\mathcal{K}$. Let $\mathbb{Q}_{p,\infty}^{\mathrm{ur}}$ be the unramified $\mathbb{Z}_p$-

extension of $\mathbb{Q}_p$. Let $\Gamma = \Gamma_{\mathbb{Q}} = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, $\Gamma_{\mathcal{K}} = \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$ and $\Gamma_1 = \mathrm{Gal}(\mathbb{Q}_{p,\infty}^{\mathrm{ur}}/\mathbb{Q}_p)$. Let $L/\mathbb{Q}_p$ be a finite extension as before with integer ring $\mathcal{O}_L$. Define $\Lambda = \Lambda_{\mathbb{Q}} = \mathcal{O}_L[[\Gamma]]$ and $\Lambda_{\mathcal{K}} = \mathcal{O}_L[[\Gamma_{\mathcal{K}}]]$. Let $\mathcal{K}^{v_0}$ ($\mathcal{K}^{\bar{v}_0}$) be the $\mathbb{Z}_p$-extension of $\mathcal{K}$ unramified outside v_0 (unramified outside $\bar{v}_0$, respectively). Define $\Gamma_{v_0} = \mathrm{Gal}(\mathcal{K}^{v_0}/\mathcal{K})$ and $\Gamma_{\bar{v}_0} = \mathrm{Gal}(\mathcal{K}^{\bar{v}_0}/\mathcal{K})$. We write ϵ for the cyclotomic character.

2 (φ, Γ)-modules and Iwasawa Theory

To save notations we only present the case when the intersection of $\mathcal{K}_\infty$ and the narrow Hilbert class field is $\mathcal{K}$. The general case works in the same way (see [43]).

2.1 Iwasawa Cohomology Groups

Lemma 2.1. *We have* $H^1(\mathbb{Q}_p, T/pT) \simeq \mathbb{F}_p^2$.

Proof. It follows from the local Euler characteristic formula that

$$\frac{\sharp H^0(\mathbb{Q}_p, T/pT) \cdot \sharp H^2(\mathbb{Q}_p, T/pT)}{\sharp H^1(\mathbb{Q}_p, T/pT)} = p^{-2}.$$

By (Irred) and Tate local duality we have $H^0(\mathbb{Q}_p, T/pT) = H^2(\mathbb{Q}_p, T/pT) = 0$. So $\sharp H^1(\mathbb{Q}_p, T/pT) = p^2$. Thus $H^1(\mathbb{Q}_p, T/pT) \simeq \mathbb{F}_p^2$. $\square$

Lemma 2.2. *We have* $H^1(\mathbb{Q}_p, T) \simeq \mathbb{Z}_p^2$.

Proof. Using the local Euler characteristic formula we have

$$\sharp H^1(\mathbb{Q}_p, T/p^nT) = p^{2n}.$$

From the cohomology long exact sequence of

$$0 \longrightarrow T \xrightarrow{\times p^n} T \longrightarrow T/p^nT \longrightarrow 0$$

we see that $H^1(\mathbb{Q}_p, T)$ is a torsion-free $\mathbb{Z}_p$-modules and that $H^1(\mathbb{Q}_p, T)/p^nH^1(\mathbb{Q}_p, T) \simeq H^1(\mathbb{Q}_p, T/p^nT)$. Since $H^2(\mathbb{Q}_p, T) = 0$ by (Irred) and Tate local duality. These altogether gives the lemma. $\square$

We define the classical Iwasawa cohomology $H^1_{\mathrm{cl,Iw}}(\mathbb{Q}_{p,\infty}, T)$ to be the inverse limit with respect to the co-restriction map.

$$\varprojlim_{\mathbb{Q}_p \subseteq \mathbb{Q}_{p,n} \subset \mathbb{Q}_{p,\infty}} H^1(\mathbb{Q}_{p,n}, T).$$

Lemma 2.3. *We have*

$$H^1_{\mathrm{cl,Iw}}(\mathbb{Q}_{p,\infty}, T) \simeq \Lambda$$

and

$$H^1_{\mathrm{cl,Iw}}(\mathbb{Q}_{p,\infty}^{\mathrm{ur}}, T) \simeq \mathbb{Z}_p[[\Gamma_1]].$$

Proof. From (Irred) we know $H^0(K, T[p]) = 0$ for any K Abelian over $\mathbb{Q}_p$. As above for any $0 \neq f \in \Lambda$ it follows from the cohomological long exact sequence of

$$0 \longrightarrow T \otimes \Lambda \xrightarrow{\times f} T \otimes \Lambda \longrightarrow T \otimes \Lambda/f\Lambda \longrightarrow 0$$

and (Irred) that $H^1_{\mathrm{cl,Iw}}(\mathbb{Q}_{p,\infty}, T)$ is a torsion-free Λ-module, and that $H^1_{\mathrm{cl,Iw}}(\mathbb{Q}_{p,\infty}, T)/fH^1_{\mathrm{cl,Iw}}(\mathbb{Q}_{p,\infty}, T) \simeq H^1(\mathbb{Q}_p, T \otimes \Lambda/f\Lambda)$. So $H^1_{\mathrm{cl,Iw}}(\mathbb{Q}_{p,\infty}, T)/(T,p)H^1_{\mathrm{cl,Iw}}(\mathbb{Q}_{p,\infty}, T) \simeq H^1(\mathbb{Q}_p, T/pT)$. By Nakayama's lemma $H^1_{\mathrm{cl,Iw}}(\mathbb{Q}_{p,\infty}, T)$ is a Λ-module generated by two elements and is thus a quotient of $\Lambda \oplus \Lambda$. Taking ω_n as above, we see that $H^1(\mathbb{Q}_{p,n}, T)$ is a Λ_n-module generated by two elements. As before using Euler local characteristic formula we can show that the $\mathbb{Z}_p$-rank of $H^1(\mathbb{Q}_{p,n}, T)$ is the same as that of $\Lambda_n \oplus \Lambda_n$. So the two generators of $H^1_{\mathrm{cl,Iw}}(\mathbb{Q}_{p,\infty}, T)$ gives an isomorphism $H^1(\mathbb{Q}_{p,n}, T) \simeq \Lambda_n \oplus \Lambda_n$. Take inverse limit we get $H^1_{\mathrm{cl,Iw}}(\mathbb{Q}_{p,\infty}, T) \simeq \Lambda \oplus \Lambda$. For the second statement, again we have $H^1_{\mathrm{cl,Iw}}(\mathbb{Q}^{ur}_{p,\infty}, T)$ is a $\mathbb{Z}_p[[\Gamma_1]]$-module generated by two elements. Similarly we can prove that $H^1(\mathbb{Q}_{p,m,n}, T)$ is a $\Lambda_{m,n}$-module generated by two elements. The rest of the argument is the same as for the first statement. $\square$

2.2 (φ, Γ)-modules

We first recall some standard notions of p-adic Hodge theory. Let $\mathbb{C}_p$ be the p-adic completion of $\bar{\mathbb{Q}}_p$ and $\mathcal{O}_{\mathbb{C}_p}$ be the elements whose p-adic valuation is less than or equal to 1. Fix once for all p^n-th root of unity ζ_{p^n} with $\zeta^p_{p^{n+1}} = \zeta_{p^n}$. Let $\tilde{E}^+ = \varprojlim_{n\geq 0} \mathcal{O}_{\mathbb{C}_p}/p$ with respect to the p-th power map as the transition map. We define a valuation v on $\tilde{E}^+$ as follows. Suppose $x = (x_n)_n$ the define $v(x) = \lim_n p^n v(x_n)$. Here the valuation v is normalized so that $v(p) = 1$ and for n large we take a lifting of x_n in $\mathcal{O}_{\mathbb{C}}$ and use its valuation to define $v(x_n)$. This valuation on $\tilde{E}^+$ is easily seen to be well defined. Define $\tilde{E}$ as the fraction field of $\tilde{E}^+$. Let $\varepsilon := (\bar{\zeta}_{p^n})_{n\geq 0} \in \tilde{E}^+$ ($\bar{x}$ being the image of x). Let $\tilde{A}^+ := W(\tilde{E}^+)$ and $\tilde{A} := W(\tilde{E})$ be the ring of Witt vectors of $\tilde{E}^+$ and $\tilde{E}$ respectively. Write $[x]$ for the Techimuller lift of $x \in \tilde{E}^+$ or $\tilde{E}$. There is a surjective ring homomorphism $\theta : \tilde{A}^+ \to \mathcal{O}_{\mathbb{C}_p}$ with $\theta([\bar{x}_n]) := \lim_{n\to\infty} x_n^{p^n}$. Define $B^+_{\mathrm{dR}} := \varprojlim_{n\geq 0} \tilde{A}^+[1/p]/(\ker(\theta)[1/p])^n$. Define $t = \log([\varepsilon]) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}([\varepsilon]-1)^n \in B^+_{\mathrm{dR}}$. Then B^+_{dR} is a discrete valuation ring with maximal ideal (t) and residue field $\mathbb{C}_p$.

For $0 \leq r \leq s < \infty$, $r, s \in \mathbb{Q}$, let $\tilde{A}^{[r,s]}$ be the p-adic completion of $\tilde{A}^+[\frac{p}{[\varepsilon-1]^r}, \frac{[\varepsilon-1]^s}{p}]$. Let $\tilde{B}^{[r,s]} = \tilde{A}^{[r,s]}[1/p]$. Write $\tilde{B}^{\dagger,r}_{\mathrm{rig}} = \cap_{r\geq s>\infty} \tilde{B}^{[r,s]}$ and $\tilde{B}^{\dagger}_{\mathrm{rig}} := \cup_r \tilde{B}^{\dagger,r}_{\mathrm{rig}}$. So there is a nature injection $\tilde{B}^{[\frac{p-1}{p},\frac{p-1}{p}]} \hookrightarrow B^+_{\mathrm{dR}}$ and injection

$$\iota_n : \tilde{B}^{\dagger,p^{n-1}(p-1)} \to \tilde{B}^{\dagger,\frac{p-1}{p}}_{\mathrm{rig}} \hookrightarrow \tilde{B}^{[\frac{p-1}{p},\frac{p-1}{p}]} \hookrightarrow B^+_{\mathrm{dR}} \tag{1}$$

for each $n \geq 0$. For any finite extension $K/\mathbb{Q}_p$ there is an Robba ring $B^\dagger_{\mathrm{rig},K} \subseteq \tilde{B}^\dagger_{\mathrm{rig}}$ of it. If K is unramified over $\mathbb{Q}_p$ then

$$B^\dagger_{\mathrm{rig},K} = \cup_{r>0} B^{\dagger,r}_{\mathrm{rig},K}$$

for

$$B^{\dagger,r}_{\mathrm{rig},K} := \{f(T) = \sum_{n\in\mathbb{Z}} a_n T^n | a_n \in K, f(T) \text{ convergent in } p^{-1/r} < |T|_p < 1\}$$

with $T = [\varepsilon] - 1$. In this case we write

$$B^{+}_{\mathrm{rig},K} := \{f(T) = \sum_{n \in \mathbb{Z}} a_n T^n | a_n \in K, f(T) \text{ convergent in } 0 \leq |T|_p < 1\}.$$

If K is ramified over $\mathbb{Q}_p$ then the construction of $B^{\dagger}_{\mathrm{rig},K}$ requires the theory of norm fields. There is also a ψ operator $B^{\dagger}_{\mathrm{rig},K} \to B^{\dagger}_{\mathrm{rig},K}$ defined as follows: we have $B^{\dagger}_{\mathrm{rig},K} = \oplus_{i=1}^{p-1} (T+1)^i \varphi(B^{\dagger}_{\mathrm{rig},K})$. For any

$$x = \sum_{i=1}^{p-1} (1+T)^i \varphi(x_i)$$

define $\psi(x) = x_0$. The ι_n defined in (1) satisfies

$$\iota_n(B^{\dagger, p^{n-1}(p-1)}_{\mathrm{rig},K}) \to K_n[[t]]$$

with $K_n := K(\zeta_{p^n})$.

Definition 2.4. *A (φ, Γ_K)-module D of rank d over $B^{\dagger}_{\mathrm{rig},K}$ if*

- *D is a finite free $B^{\dagger}_{\mathrm{rig},K}$-module of rank d;*
- *D is equipped with a φ-semilinear map $\varphi : D \to D$ such that*

  $$\varphi^*(D) : B^{\dagger}_{\mathrm{rig},K} \otimes_{\varphi, B^{\dagger}_{\mathrm{rig},K}} D \to D : a \otimes x \to a\varphi(x)$$

  is an isomorphism;
- *D is equipped with a continuous semilinear action of Γ_K which commutes with φ.*

Bloch-Kato exponential maps

Recall the fundamental exact sequence in p-adic Hodge theory

$$0 \to \mathbb{Q}_p \to B^{\varphi=1}_{\mathrm{cris}} \oplus B^{+}_{\mathrm{dR}} \to B_{\mathrm{dR}} \to 0. \tag{2}$$

Tensoring with the Galois representation V and taking the K-Galois cohomology long exact sequence we define the Bloch-Kato's exponential map to be the coboundary map

$$\frac{D_{\mathrm{dR}}(V)}{D^{+}_{\mathrm{dR}}(V) + D^{\varphi=1}_{\mathrm{cris}}(V)} \to H^1(K, V).$$

In general Nakamura defined the exp and $\exp^*$ maps for deRham (φ, Γ)-modules.

For any $n > n(D)$ we define

$$D^{+}_{\mathrm{dif}}(D) := K_n[[t]] \otimes_{\iota_n, B^{\dagger, r_n}_{\mathrm{rig},K}} D^{(n)}$$

and

$$D_{\mathrm{dif}}(D) := K_n((t)) \otimes_{\iota_n, B^{\dagger, r_n}_{\mathrm{rig},K}} D^{(n)}.$$

We also define

$$D^{K}_{\mathrm{dR}}(D) = D_{\mathrm{dif}}(D)^{\Gamma_K = 1}, D^{K}_{\mathrm{crys}}(D) = D[1/t]^{\Gamma_K = 1}.$$

The filtration on $D^K_{\mathrm{dR}}(D)$ is given by

$$\mathrm{Fil}^i D^K_{\mathrm{dR}}(D) = D^K_{\mathrm{dR}}(D) \cap t^i D^+_{\mathrm{dif}}(D), i \in \mathbb{Z}.$$

We define a (φ, Γ)-module D to be crystalline (deRham) if the rank D is equal to the $\mathbb{Z}_p$-rank of D_{crys} (D_{dR}). If V is a representation over some finite extension L of $\mathbb{Q}_p$, we make all these definitions by regarding it as a $\mathbb{Q}_p$ representation.

2.3 Λ_∞ and Co-admissible modules

We summarize some facts and definitions in [37] for later use.

Definition 2.5. *([34, Definition 3.1]) The analytic Iwasawa algebra* $\Lambda_\infty := \varprojlim_n \Lambda[\hat{\mathfrak{m}^n/p}][1/p]$. *This is the ring of rigid analytic functions on the open unit disc and is a Bezout domain. The analytic Iwasawa cohomology*

$$H^q_{\mathrm{Iw}}(K, D) := \varprojlim_n H^q(K, D \hat{\otimes}_K \tilde{\Lambda}^\iota_n)$$

as a Λ_∞*-module.*

Definition 2.6. *A co-admissible* Λ_∞*-module* M *is the module of global sections of coherent analytic sheaves on* W*. That means, there is an inverse system* $(M_n)_n$ *of finitely generated* $\Lambda_n[1/p]$*-modules such that the map* $M_{n+1} \to M_n$ *induces isomorphisms* $M_{n+1} \otimes_{\Lambda_{n+1}[1/p]} \Lambda_n[1/p] \simeq M_n$*. Then*

$$M = \varprojlim_n M_n.$$

The following proposition is proved by Pottharst [37].

Proposition 2.7. *(1) The torsion submodule* M_{tors} *of a admissible* Λ_∞*-module* M *is also co-admissible, and* M/M_{tors} *is a finitely generated free* Λ_∞*-module.*

(2) The torsion co-adimissible Λ_∞*-modules are those isomorphic to* $\prod_{\alpha \in I} \Lambda_\infty \mathfrak{p}_\alpha^{n_\alpha}$ *for some collections* $\{\mathfrak{p}_\alpha\}_{\alpha \in I}$ *of closed points of* $\cup_n \mathrm{Spec}\Lambda_n[1/p]$ *(*n_α *are positive integers) such that for each* n *there are only finitely many* α *with* $\mathfrak{p}_\alpha \in \mathrm{Spec}\Lambda[1/p]$*.*

Definition 2.8. *(Pottharst) Let* M *as above be torsion. We define the divisor for* M *as the formal sum* $\sum_\alpha n_\alpha \mathfrak{p}_\alpha$*. We define the characteristic ideal* $\mathrm{char}_{\Lambda_\infty}(M)$ *to be the principal ideal generated by some* $f_M \in \Lambda$ *such that the divisor of* f_M *is the same as the divisor for* M*. (Such* f_M *exists by a well known result of Lazard).*

Then as in [37, Page 7]

$$H^q_{\mathrm{Iw}}(K, V) = H^q_{\mathrm{cl,Iw}}(K, V) \otimes_\Lambda \Lambda_\infty.$$

2.4 Unramified Iwasawa Theory

In this subsection we prove some key facts about uniform boundedness of Bloch-Kato logarithm map along unramified field extensions. Later on we are going to study the $\mathbb{Z}_p$-extension of $\mathcal{K}$ which is unramified at v_0 but totally ramified at $\bar{v}_0$, and the result proved here will be of crucial importance. We write in this subsection r the highest Hodge-Tate weight of T. Recall $T = [\varepsilon] - 1$. We also define $\gamma_n(x) = \frac{x^n}{n!}$. Let $q' = \varphi^{-1}(q)$ for $q = \sum_{a \in \mathbb{F}_p} [\varepsilon]^{[a]}$.

Lemma 2.9. *We have $\theta(q')=0$, but $\theta(q'/T)\neq 0$.*

Proof. It is clear that $\theta(q')=0$. Take $\varpi=(\varpi^0,\varpi^1,\cdots)\in\tilde{E}^+$ with $\varpi^0=-p$ and $\xi:=[\varpi]+p$. Then $\ker\{\theta: W(R)\to\mathcal{O}_{\mathbb{C}}\}$ is the principal ideal generated by ξ ([15, Proposition 5.12]). If $\theta(q'/T)=0$ then $q'=\xi^2\lambda$ for some $\lambda\in W(\tilde{E}^+)$. So $\bar{q}'=\varpi^2\bar{\lambda}$. But $v(\bar{q}')=1$ and $v(\varpi^2)=2$, a contradiction. $\square$

Definition 2.10.

$$\mathfrak{a}=\theta(q'/T).$$

Definition 2.11. *Let A^0_{cris} be the divided power envelop of $W(\tilde{E}^+)$ with respect to $\ker\theta$, that is, by adding all elements $a^m/m!$ for all $a\in\ker\theta$. Defining the ring $A_{\mathrm{cris}}=\varprojlim_n A^0_{\mathrm{cris}}/p^nA^0_{\mathrm{cris}}$ and $B^+_{\mathrm{cris}}:=A_{\mathrm{cris}}[1/p]$.*

Now let $\mathrm{Fil}^rA_{\mathrm{cris}}=A_{\mathrm{cris}}\cap\mathrm{Fil}^rB_{\mathrm{dR}}$ and $\mathrm{Fil}^r_pA_{\mathrm{cris}}=\{x\in\mathrm{Fil}^rA_{\mathrm{cris}}|\varphi x\in p^rA\}$. Then we have the following

Lemma 2.12. *For every $x\in\mathrm{Fil}^rA_{\mathrm{cris}}$, $p^a\cdot a!x\in\mathrm{Fil}^r_pA_{\mathrm{cris}}$ for a the largest integer such that $(p-1)a<r$. Moreover $\mathrm{Fil}^r_pA_{\mathrm{cris}}$ is the associated sub $W(\tilde{E}^+)$-module of A_{cris} generated by ${q'}^j\gamma_b(p^{-1}t^{p-1})$ for $j+(p-1)b\geq r$.*

This is just [15, Proposition 6.24].

Let T be a two dimensional Galois representation of $G_{\mathbb{Q}_p}$ over $\mathcal{O}_L$ with $V:=T\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ being deRham. Let v_1,v_2 be a basis of T. Suppose the Hodge-Tate weight of V is (r,s) with $r>0$, $s\leq 0$. Let ω_V be a generator of the one dimensional space $\mathrm{Fil}^0D_{\mathrm{dR}}(V)$ over L. Then $t^r\omega_V\in B^+_{\mathrm{dR}}\otimes T$. There is an element $z\in W(\tilde{E}^+)[\frac{1}{p}]\otimes T$ such that

$$t^r\omega_V-z\in\mathrm{Fil}^{r+1}B_{\mathrm{dR}}\otimes T. \tag{3}$$

Thus there is an n with $p^nz\in W(\tilde{E}^+)\otimes T$.

Write $U^1=\{x\in\tilde{E}|v(x-1)\geq 1\}$. Let m_r be such that $p^{m_r}\nmid p^a\cdot a!b!p^b/\mathfrak{a}^j$ for all $j+(p-1)b\geq r$. We define U^1_r to be the p-adic closure of the $\mathbb{Z}_p$-submodule of $\mathrm{Fil}^{-r}B^{\varphi=1}_{\mathrm{cris}}$ generated by elements of the form $\frac{a_1\cdots a_i}{t^i}$ where a_i are elements in $\log(U^1)$, $i\leq r$ (See [15, 6.1.3] for details about the log map). Take $m\geq r+(r-1)m_r+n$ we claim that $\mathrm{Im}(U^1_r)\otimes_{\mathbb{Z}_p}T$ contains $p^ma\omega_V$ for all $a\in W(\bar{\mathbb{F}}_p)$, where $\mathrm{Im}(U^1_r)$ is the image of $\log(U^1_r)$ in $\frac{B_{\mathrm{dR}}}{B^+_{\mathrm{dR}}}$.

We use induction. Take $\tilde{b}_{r-1}\in\log U^1$ such that

$$\theta(\tilde{b}_{r-1})\cdot\theta(\tilde{v})^{r-1}\cdot t^{-r}\equiv p^{1+n}a\omega_V(\mathrm{mod}\ \mathrm{Fil}^{1-r}B_{\mathrm{dR}}\otimes T).$$

This is possible because $\theta(\log(U^1))\supseteq p\mathcal{O}_{\mathbb{C}_p}$. Suppose we found $\tilde{b}_{r-1},\cdots,\tilde{b}_i\in\log(U^1)$ and $\tilde{c}_{r-1},\cdots,\tilde{c}_i\in\log(U^1)$ such that

$$p^{r-i+m_r(r-i-1)+n}a\cdot\omega_V\equiv(\frac{\tilde{b}_{r-1}}{t}\cdot(\frac{\tilde{v}}{t})^{r-1}+\cdots+\frac{\tilde{b}_i}{t}(\frac{\tilde{v}}{t})^i)v_1+(\frac{\tilde{c}_{r-1}}{t}\cdot(\frac{\tilde{v}}{t})^{r-1}+\cdots+\frac{\tilde{c}_i}{t}(\frac{\tilde{v}}{t})^i)v_2(\mathrm{mod}\ \mathrm{Fil}^{-i}B_{\mathrm{dR}}\otimes T).$$

Then by (3)

$$\begin{aligned}&t^r(p^{r-i+m_r(r-i-1)+n}a\cdot\omega_V-(\frac{\tilde{b}_{r-1}}{t}\cdot(\frac{\tilde{v}}{t})^{r-1}+\cdots+\frac{\tilde{b}_i}{t}(\frac{\tilde{v}}{t})^i)v_1-(\frac{\tilde{c}_{r-1}}{t}\cdot(\frac{\tilde{v}}{t})^{r-1}+\cdots+\frac{\tilde{c}_i}{t}(\frac{\tilde{v}}{t})^i))v_2\\&\in(\mathrm{Fil}^{r-i}A_{\mathrm{cris}}+\mathrm{Fil}^{r+1}B^+_{\mathrm{dR}})\otimes T.\end{aligned}$$

By Lemma 2.12 we know the image under θ of the coefficient of v_1 in

$$(t^i(p^{r-i+m_r(r-i-1)+n}a\cdot\omega_V-(\frac{\tilde{b}_{r-1}}{t}\cdot(\frac{\tilde{v}}{t})^{r-1}+\cdots+\frac{\tilde{b}_i}{t}(\frac{\tilde{v}}{t})^i))v_1)$$

is in $p^{-m_r}\mathcal{O}_{\mathbb{C}_p}$, and that the image under θ of the coefficient of v_2 in

$$(t^i(p^{r-i+m_r(r-i-1)+n}a\cdot\omega_V-(\frac{\tilde{c}_{r-1}}{t}\cdot(\frac{\tilde{v}}{t})^{r-1}+\cdots+\frac{\tilde{c}_i}{t}(\frac{\tilde{v}}{t})^i))v_2)$$

is in $p^{-m_r}\mathcal{O}_{\mathbb{C}_p}$. So there is a choice of $\tilde{b}_{r-1},\cdots,\tilde{b}_{i-1}$ and $\tilde{c}_{r-1},\cdots,\tilde{c}_{i-1}$ such that

$$p^{r-i+1+m_r(r-i)+n}a\cdot\omega_V-(\frac{\tilde{b}_{r-1}}{t}\cdot(\frac{\tilde{v}}{t})^{r-1}+\cdots+\frac{\tilde{b}_{i-1}}{t}(\frac{\tilde{v}}{t})^{i-1})v_1-(\frac{\tilde{c}_{r-1}}{t}\cdot(\frac{\tilde{v}}{t})^{r-1}+\cdots+\frac{\tilde{c}_{i-1}}{t}(\frac{\tilde{v}}{t})^{i-1})v_2$$
$$\in \mathrm{Fil}^{1-i}B_{\mathrm{dR}}\otimes T.$$

(The $\tilde{b}_{r-1},\cdots,\tilde{b}_i$ and $\tilde{c}_{r-1},\cdots,\tilde{c}_i$ are the previous $\tilde{b}_{r-1},\cdots,\tilde{b}_i$ and $\tilde{c}_{r-1},\cdots,\tilde{c}_i$ multiplied by p^{1+m_r}.)

Continuing this process we can find $\tilde{b}_{r-1},\cdots,\tilde{b}_0$ and $\tilde{c}_{r-1},\cdots,\tilde{c}_0$ such that

$$p^{r+m_r(r-1)+n}a\cdot\omega_V-(\frac{\tilde{b}_{r-1}}{t}(\frac{\tilde{v}}{t})^{r-1}+\cdots+\frac{\tilde{b}_0}{t})v_1-(\frac{\tilde{c}_{r-1}}{t}(\frac{\tilde{v}}{t})^{r-1}+\cdots+\frac{\tilde{c}_0}{t})v_2\in B_{\mathrm{dR}}^+\otimes T.$$

This proves the claim.

Another observation is that $U_r^1\cap\ker(B_{\mathrm{cris}}^{\varphi=1}\to\frac{B_{\mathrm{dR}}}{B_{\mathrm{dR}}^+})\subseteq p^{-m_r}\mathbb{Z}_p$. This can be seen by noting that for any element C in this intersection, $t^rC\in\mathrm{Fil}^rA_{\mathrm{cris}}$. So $\theta(C)\in p^{-m_r}\mathcal{O}_{\mathbb{C}_p}$. The following proposition follows immediately.

Proposition 2.13. *We consider the co-boundary map*

$$\exp:\frac{(B_{\mathrm{dR}}\otimes V)^{I_{\mathbb{Q}_p}}}{(B_{\mathrm{dR}}\otimes V)^{I_{\mathbb{Q}_p}}+(B_{\mathrm{cris}}\otimes V)^{I_{\mathbb{Q}_p},\varphi=1}}\to H^1(I_{\mathbb{Q}_p},V).$$

Then there is an $m>0$ such that for any $a\in W(\bar{\mathbb{F}}_p)$ we have $\exp(a\cdot\omega)\in p^{-m}H^1(I_{\mathbb{Q}_p},T)$.

Proof. To see this we just use the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker & \longrightarrow & U_r^1\oplus B_{\mathrm{dR}}^+ & \longrightarrow & U_r^1+B_{\mathrm{dR}}^+ & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_p & \longrightarrow & B_{\mathrm{cris}}^{\varphi=1}\oplus B_{\mathrm{dR}}^+ & \longrightarrow & B_{\mathrm{dR}} & \longrightarrow & 0
\end{array}.$$

□

We refer to Section 4.1 for some discussion of Yager modules and periods for unramfied representations, and the definitions of the d and $\rho(d)$.

Corollary 2.14. *Let ρ be an unramified p-adic character of $G_{\mathbb{Q}_p}$ such that $\rho(\mathrm{Frob}_p)=1+\mathfrak{m}\in\mathcal{O}_{\mathbb{C}_p}$ with $\mathrm{val}_p(\mathfrak{m})>0$. Then the map*

$$\exp:(\frac{B_{\mathrm{dR}}}{B_{\mathrm{dR}}^+}\otimes V(\rho))^{G_{\mathbb{Q}_p}}\to H^1(G_{\mathbb{Q}_p},V(\rho))$$

can be constructed as

$$\exp(\lim_n \sum_{\sigma\in U_n} d_n^\sigma \rho(\sigma)\cdot\omega_f) = \lim_n \sum_{\sigma\in U_n} \rho(\sigma)\exp(d_n^\sigma\cdot\omega_f).$$

The right hand side is well defined thanks to Proposition 2.13.

To see this, we consider the natural unramified rank one Galois representation of $U = \mathrm{Gal}(\mathbb{Q}_{p,\infty}^{\mathrm{ur}}/\mathbb{Q})$ over the Iwasawa algebra $\mathbb{Z}_p[[U]]$. We consider the map from $\mathbb{Z}_p[[U]]$ to ρ mapping u to $\rho(u)$. Tensoring this map with (2) and taking the long exact sequence of Galois cohomology, we get the required formula.

Corollary 2.15. *For some integer m we consider the inverse limit of the maps for $\mathbb{Q}_p \subseteq F_n \subset \mathbb{Q}_p^{\mathrm{ur}}$,*

$$\exp\mathcal{O}_{F_n}\cdot\omega_f \to p^{-m}H^1(F_n, T)$$

and get a map

$$\exp : (\varprojlim_n \mathcal{O}_{F_n})\cdot\omega_f \to p^{-m}H^1(\mathbb{Q}_p, T\otimes\mathbb{Z}_p[[U]]).$$

Then for some choice of such an integer m we have $p^m\exp(d\cdot\omega_f)$ generates a $\mathbb{Z}_p[[U]]$-direct summand of

$$H^1(\mathbb{Q}_p, T\otimes\mathbb{Z}_p[[U]]) \simeq \mathbb{Z}_p[[U]]\oplus\mathbb{Z}_p[[U]].$$

Proof. By Corollary 2.14 the specialization of $\exp(d\omega_f)$ to any $\phi \in \mathrm{Spec}\mathbb{Z}_p[[U]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ with u mapping to $1+\mathfrak{m}$ with $\mathrm{val}_p(\mathfrak{m}) > 0$ is non-zero (since the exp map for $V(\rho)$ is injective on $\frac{D_{\mathrm{dR}}(V(\rho))}{D^+_{\mathrm{dR}}(V(\rho))}$). The corollary follows by observing that $\mathbb{Z}_p[[U]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is a Bezout domain. $\square$

3 Iwasawa-Greenberg Main Conjecture

The Idea

Our goal in this section is to prove Theorem 1.7. The idea in [43] in the proof for weight two case is roughly summarized as follows. We first construct families of Klingen Eisenstein series on the unitary group $\mathrm{U}(3,1)$ using [12]. The Hida theory developed in [43, Section 3] enables us to construct a family of cusp forms, which is congruent to the Klingen Eisenstein family modulo $\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}$. Then we proved there is a functional (constructed via Fourier-Jacobi expansion map) acting on the space of families of semi-ordinary forms on $\mathrm{U}(3,1)$, which maps the Klingen Eisenstein family to an element which is a unit up in the coefficient ring $\mathcal{O}_L^{ur}[[\Gamma_\mathcal{K}]]$, up to multiplying by an element in $\bar{\mathbb{Q}}_p^\times$. (This is the hard part of the whole argument). With this in hand, this functional and the cuspidal family we mentioned above gives a map from the cuspidal Hecke algebra to $\mathcal{O}^{ur}[[\Gamma_\mathcal{K}]]$ which, modulo $\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}$ gives the Hecke eigenvalues acting on the Klingen Eisenstein family. Passing to the Galois side, such congruence enables us to construct enough elements in the Selmer groups from the “lattice construction”, proving the lower bound of the Selmer group.

Now we return to the situation in this paper (i.e. general weight), all the ingredients are available, except that we need to construct the corresponding functional using Fourier-Jacobi expansion map, so that its value on the Klingen Eisenstein family is an element in $\mathcal{O}_L^{\mathrm{ur}}[[\Gamma_\mathcal{K}]]^\times$, up to multiplying by a non-zero constant. Recall that in [43], since the Klingen Eisenstein series is realized using pullback formula under

$$\mathrm{U}(3,1)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(3,3)$$

for the Siegel Eisenstein series E_{sieg} on $\mathrm{U}(3,3)$, we computed that the 1-st Fourier-Jacobi coefficient $\mathrm{FJ}_1 E_{\mathrm{sieg}} = E\cdot\Theta$ where E is a Siegel Eisenstein series on $\mathrm{U}(2,2)$ and Θ is a theta function on $N\mathrm{U}(2,2)$. We constructed a theta function θ_1 on $N\mathrm{U}(2)$ and defined a functional l_{θ_1} by pairing with θ_1 along N, from forms on $D(\mathbb{Q})\backslash D(\mathbb{A})$ to forms on $\mathrm{U}(2)$. We also constructed auxiliary theta functions h and θ on $\mathrm{U}(2)$ and deformed them in Hida families $\mathbf{h}$ and θ. We used the doubling method for h under $\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$ to see that $\langle l_{\theta_1}\mathrm{FJ}_1(E),h\rangle$ is essentially the triple product integral $\int h(g)\theta^{\mathrm{low}}(g)f(g)dg$. Here the superscript *low* means the level group for θ at p is lower triangular. We also construct families of forms $\tilde{h}$ and $\tilde{\theta}$ in the dual space of h and θ, respectively. The product

$$\int h(g)\theta^{\mathrm{low}}(g)f(g)dg\int\tilde{h}(g)\tilde{\theta}^{\mathrm{low}}(g)\tilde{f}(g)dg$$

can be evaluated using Ichino's triple product formula. In [43] we have seen that both $\int h(g)\theta(g)f(g)dg$ and $\int\tilde{h}(g)\tilde{\theta}(g)\tilde{f}(g)dg$ are interpolated by elements in $\hat{\mathcal{O}}_L^{ur}[[\Gamma_{\mathcal{K}}]]$, and the product is in $(\hat{\mathcal{O}}_L^{ur}[[\Gamma_{\mathcal{K}}]])^\times$ up to multiplying by an element in $\bar{\mathbb{Q}}_p^\times$ (a number fixed throughout the whole family).

In this paper we use the construction in [12] of the Klingen Eisenstein series, using the pullback formula for

$$\mathrm{U}(3,1)\times\mathrm{U}(0,2)\hookrightarrow\mathrm{U}(3,3)$$

from the nearly holomorphic Siegel Eisenstein series.

Proposition 3.1. *Suppose the unitary automorphic representation $\pi=\pi_f$ generated by the weight k form f is such that π_p is an unramified principal series representation $\pi(\chi_1,\chi_2)$ with distinct Satake parameters. Let $\tilde{\pi}$ be the dual representation of π. Let Σ be a finite set of primes containing all the bad primes*

(i) There is an element $\mathcal{L}^\Sigma_{f,\mathcal{K}}\in\Lambda_{\mathcal{K},\mathcal{O}_L^{ur}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ such that for any character ξ_ϕ of $\Gamma_{\mathcal{K}}$, which is the avatar of a Hecke character of conductor p, infinite type $(\frac{\kappa_\phi}{2}+m_\phi,-\frac{\kappa_\phi}{2}-m_\phi)$ with κ_ϕ an even integer which is at least 6, $m_\phi\geq\frac{k-2}{2}$, we have

$$\phi(\mathcal{L}^\Sigma_{f,\mathcal{K}})=\frac{L^\Sigma(\tilde{\pi},\xi_\phi,\frac{\kappa_\phi-1}{2})\Omega_p^{4m_\phi+2\kappa_\phi}}{\Omega_\infty^{4m_\phi+2\kappa_\phi}}c'_\phi.p^{\kappa_\phi-3}\mathfrak{g}(\xi_{\phi,2})^2\prod_{i=1}^{2}(\chi_i^{-1}\xi_{\phi,2}^{-1})(p)$$

c'_ϕ is a constant coming from an Archimedean integral.

(ii) There is a set of formal q-expansions $\mathbf{E}_{f,\xi_0}:=\{\sum_\beta a^t_{[g]}(\beta)q^\beta\}_{([g],t)}$ for $\sum_\beta a^t_{[g]}(\beta)q^\beta\in\Lambda_{\mathcal{K},\mathcal{O}_L^{ur}}\otimes_{\mathbb{Z}_p}\mathcal{R}_{[g],\infty}$ where $\mathcal{R}_{[g],\infty}$ is some ring to be defined later, $([g],t)$ are p-adic cusp labels, such that for a Zariski dense set of arithmetic points $\phi\in\mathrm{Spec}_{\mathcal{K},\mathcal{O}_L}$, $\phi(\mathbf{E}_{f,\xi_0})$ is the Fourier-Jacobi expansion of the highest weight vector of the holomorphic Klingen Eisenstein series constructed by pullback formula which is an eigenvector for U_{t^+} with non-zero eigenvalue. The weight for $\phi(\mathbf{E}_{f,\xi_0})$ is $(m_\phi-\frac{k-2}{2},m_\phi+\frac{k-2}{2},0;\kappa_\phi)$.

(iii) The $a^t_{[g]}(0)$'s are divisible by $\mathcal{L}^\Sigma_{f,\mathcal{K},\xi_0}.\mathcal{L}^\Sigma_{\bar{\tau}'}$ where $\mathcal{L}^\Sigma_{\bar{\tau}'}$ is the p-adic L-function of a Dirichlet character as in [12].

We also refer to [12] for the convention of weights of automorphic forms on $\mathrm{U}(2)$ and $\mathrm{U}(3,1)$. This is just a translation of the main theorem of [12] to the situation here. We can recover the full p-adic L-function $\mathcal{L}_{f,\mathcal{K}}$ by putting back the Euler factors at primes in Σ. We can actually make the constant c'_ϕ precise.

Lemma 3.2. *The constant c'_ϕ above is given by*

$$\Gamma(\kappa_\phi + m_\phi - \frac{k}{2})\Gamma(\kappa_\phi + m_\phi + \frac{k}{2} - 1)2^{-3\kappa_\phi - 4m_\phi + 1}\pi^{1-2\kappa_\phi - 2m_\phi} i^{k-\kappa_\phi - 2m_\phi - 1}.$$

Proof. It is not easy to compute the c'_ϕ directly. We prove the lemma by a comparison of the above p-adic L-function and Hida's Rankin-Selberg p-adic L-function. We pick an auxiliary Hida family of ordinary forms $\mathbf{f}'$ and compare

- The product $\mathcal{L}^{\mathrm{Hida}}_{\mathbf{f}'\otimes\mathbf{g}} \cdot \mathcal{L}^{\mathrm{Katz}}_{\mathcal{K}} h_{\mathcal{K}}$, where $\mathcal{L}^{\mathrm{Katz}}_{\mathcal{K}} h_{\mathcal{K}}$ is the class number $h_{\mathcal{K}}$ of $\mathcal{K}$ times the Katz p-adic L-function, which interpolates the Petersson inner product of specializations of $\mathbf{g}_\phi$ (see [18]). The $\mathcal{L}^{\mathrm{Hida}}_{\mathbf{f}'\otimes\mathbf{g}}$ is the Rankin-Selberg p-adic L-function constructed by Hida in [16] interpolating algebraic part of the critical values of Rankin-Selberg L-functions for specializations of $\mathbf{f}'$ and $\mathbf{g}$, where the specializations of $\mathbf{g}$ has higher weight.

- The p-adic L-function $\mathcal{L}_{\mathbf{f}',\mathcal{K}}$ constructed using doubling method as above.

We first look at the arithmetic points where the Siegel Eisenstein series are of scalar weight. The computations are essentially done in [45] (although the ramifications in *loc.cit* is slightly different, however those assumptions are put for constructing the family of Klingen Eisenstein series. The computations in the doubling method construction of the p-adic L-function carries out in the same way in the situation here). We see that the above two items have the same value at these points. As these arithmetic points are Zariski dense, the two should be equal identically. Then we look at the arithmetic points considered in the above proposition. Comparing the interpolation formulas here and in [16, Theorem I], we get the formulas for c'_ϕ (note that the critical L-value is not zero since it is away from center). ☐

Now we still write $\mathcal{L}^{\mathrm{Hida}}_{f\otimes\mathbf{g}}$ for the Rankin-Selberg Hida p-adic L-function interpolating critical values of the Rankin-Selberg L-function for f and specializations of g whose weight is higher than f. Since the higher weight form g is ordinary, Hida's construction works in the same way even though f is not ordinary. We have the following

Corollary 3.3.

$$\mathcal{L}^{\mathrm{Hida}}_{f\otimes\mathbf{g}} \cdot \mathcal{L}^{\mathrm{Katz}}_{\mathcal{K}} h_{\mathcal{K}} = \mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}.$$

The corollary follows from above lemma and the interpolation formulas on both hand sides. From now on we write $\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}$ for the $\mathcal{L}_{f,\mathcal{K}}$ constructed above, since it corresponds to the Greenberg's main conjecture. We prove the following

Lemma 3.4. *The $\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}$ is in $\mathcal{O}^{\mathrm{ur}}_L[[\Gamma_{\mathcal{K}}]]$.*

Proof. From the construction the denominator of $\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}$ can only be powers of p times the product of the Euler factors of a finite number of primes of $\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}$. But by the argument in [43, Proposition 8.3] we know the denominator can at most be powers of U if we take $\mathbb{Z}_p[[U]]$ as the coefficient ring of $\mathbf{g}$. Thus the denominator must be a unit. ☐

Now let us return to the proof of Theorem 1.7. The main difficulty here is that the explicit local Fourier-Jacobi computation at Archimedean place is very hard to study if the weight is not scalar. In fact since E_{sieg} is only nearly holomorphic instead of holomorphic, from the general theory in [20] the Fourier-Jacobi coefficients of it is not necessarily a finite sum of products of Siegel Eisenstein series and theta functions, but can be infinitely approximated by them in $C^\infty(D(\mathbb{Q})\backslash D(\mathbb{A}))$. Our idea is not to compute such local Fourier-Jacobi integrals at ∞. Instead we fix the weight (k, κ_ϕ, m_ϕ) (notation as before) and varying its nebentypus at p. Such arithmetic points are Zariski dense in $\mathrm{Spec}\mathcal{O}_L[[\Gamma_\mathcal{K}]]$. We show that there is a number C_∞ depending only on the Archimedean data (and is thus the same number for all arithmetic points), which can be proved to be non-zero, and an element $\mathcal{L} \in \mathcal{O}_L^{\mathrm{ur}}[[\Gamma_\mathcal{K}]]^\times \cdot \bar{\mathbb{Q}}_p^\times$, such that at each of such arithmetic point we have

$$\langle l_{\theta_1}(\mathrm{FJ}_\beta(\phi(E))), h_\phi\rangle \cdot \int \tilde{h}_\phi(g)\tilde{\theta}_\phi^{\mathrm{low}}(g)\tilde{f}(g)dg = C_\infty \cdot \phi(\mathcal{L}).$$

<u>Families of Theta Lifting</u>

We pick up auxiliary Hecke characters χ_θ, χ_{aux} and $\chi_h = \chi_\theta^{-c}\chi_{\mathrm{aux}}$ of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$, similar as in [44, Section 8.2], except that we require the χ_{aux} to have infinite type $(-\frac{k-2}{2}, \frac{k-2}{2})$ instead of $(0,0)$. We refer to [1] for backgrounds of Fock models and Schrodinger models at Archimedean places. We construct families $\mathbf{h}$ of CM forms on $\mathrm{U}(2)$ of weight $(\frac{k-2}{2}, -\frac{k-2}{2})$ and varying p-part of nebentypus, and try to evaluate their Petersson inner products using Rallis inner product formula as in [44, 8.3]. The seesaw diagram is

$$\begin{tikzcd}
\mathrm{U}(1,1)(\omega_{\lambda^2}) & \mathrm{U}(2)(\omega_\lambda)\times\mathrm{U}(2)(\omega_\lambda) \\
\mathrm{U}(1)(\omega_{\lambda^2})\times\mathrm{U}(1)(\omega_{\lambda^2}) \arrow[u] & \mathrm{U}(2)(\omega_{\lambda^2}) \arrow[u]
\end{tikzcd}$$

To do so we refer to [12, Section 4.4.4] for the construction of a differential operator D_1^d (where $d = 2m_\phi$ here) on the space of modular forms on $\mathrm{U}(2,2)$ of weight 1 to the space of nearly holomorphic $(\mathrm{st}_{\mathrm{GL}_2}\boxtimes\mathrm{st}_{\mathrm{GL}_2})\otimes(1\boxtimes\det)$-valued forms, and write $\mathrm{Proj}^{L_{(k-2,0)}\boxtimes(L_{(k-2,0)}\otimes\det)}$ (here we use the standard notations as in [12] for algebraic representation $L_{(k-2,0)}$ of GL_2 of highest weight $(k-2,0)$, and $\mathrm{st}_{\mathrm{GL}_2}$ for the standard representation of GL_2) for the projection corresponding to the summand $L_{(k-2,0)}\boxtimes(L_{(k-2,0)}\otimes\det)$ as a subrepresentation of $(\mathrm{st}_{\mathrm{GL}_2}\boxtimes\mathrm{st}_{\mathrm{GL}_2})\otimes(1\boxtimes\det)$. Let ϕ_∞^0 be the Archimedean Schwartz function defined in [44, Section 8.3, case 1] and $\phi_{\infty,k-2} := \mathrm{Proj}^{L_{(k-2,0)}\boxtimes(L_{(k-2,0)}\otimes\det)}D_1^d\phi_\infty^0$. The $\phi_{\infty,k-2}$ is easily seen to be non-zero, by for example checking the q-expansions. It corresponds to polynomials in the Fock models. We first construct a theta function Θ on $\mathrm{U}(2,2)$ as in [44, Section 8.3, case 0 - case 5] by replacing the Archimedean theta kernel ϕ_∞^0 there by $\phi_{\infty,k-2}$, and keeping the theta kernel at all finite places. This corresponds to the dual reductive pair $(\mathrm{U}(2,2)\times\mathrm{U}(1))$. Pullback under

$$\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$$

we get theta function on $\mathrm{U}(2)$ (for the dual reductive pair $(\mathrm{U}(2),\mathrm{U}(1))$. (For more detailed backgrounds we refer to [44, Section 4.8.1].) We get families $\mathbf{h}$, $\boldsymbol{\theta}$, $\tilde{\mathbf{h}}$ and $\tilde{\boldsymbol{\theta}}$ of weight $(\frac{k-2}{(}2), -\frac{k-2}{2})$, $(0,0)$, $(\frac{2-k}{2}, \frac{k-2}{2})$, $(0,0)$ respectively (note that weight 0 forms on definite unitary corresponds to weight 2 forms on GL_2 under the Jacquet-Langlands correspondence) . Moreover for ϕ's varying in the set of arithmetic points, the number $\langle h_\phi, \tilde{h}_\phi\rangle p^{t_\phi}$ is interpolated by $c_{\infty,k}\cdot\mathcal{L}_h$ where $c_{\infty,\underline{k}}$ is a nonzero constant which depends only on k and $\mathcal{L}_h$ is a p-adic L-function of the CM character $\chi_h\chi_h^{-c}$. (That $c_{\infty,k}$ is non-zero can be see as follows. The Petersson inner product can alternative be evaluated using Rallis inner product formula and the following diagram as well

$$\begin{tikzcd} \mathrm{U}(2,2) & \mathrm{U}(1)\times\mathrm{U}(1) \\ \mathrm{U}(2)\times\mathrm{U}(2) \arrow[u] & \mathrm{U}(1) \arrow[u] \end{tikzcd}$$

Thus we are in the situation of doubling method construction of the p-adic L-function for $\mathbf{h}$ for

$$\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2).$$

This situation is slightly different from our previous discussion on double method, since we are starting from a weight one Siegel Eisenstein series on $\mathrm{U}(2,2)$ while in [44] we start with weight $\kappa>6$ Siegel Eisenstein series. However it is easy to see that the pullback integral at the Archimedean place is non-zero, since the Siegel Eisenstein section appearing here is a non-zero generator of the one-dimensional subspace of the induced representation for the Siegel Eisenstein series, with the given action of $\mathrm{U}(2)(\mathbb{R})\times\mathrm{U}(2)(\mathbb{R})$ (i.e. weight $(\frac{k-2}{2},-\frac{k-2}{2})$).

<u>Interpolating Inner Products</u>

We take a basis $\{v_1,v_2,\cdots,v_t\}$ of $V_{(\frac{k-2}{2},-\frac{k-2}{2})}$ and $\{v_1^\vee,v_2^\vee,\cdots,v_t^\vee\}$ of $V^\vee_{(\frac{k-2}{2},-\frac{k-2}{2})}$. If

$$f\in\mathcal{A}(\mathrm{U}(2),V_{(\frac{k-2}{2},-\frac{k-2}{2})})$$

(the space of $V_{(\frac{k-2}{2},-\frac{k-2}{2})}$-valued automorphic forms on the definite unitary group),

$$h\in\mathcal{A}(\mathrm{U}(2),V^\vee_{(\frac{k-2}{2},-\frac{k-2}{2})}).$$

Writing $f=f_1v_1+\cdots+f_tv_t$ and $h=h_1v_1^\vee+\cdots+h_tv_t^\vee$. Then define $\langle f,h\rangle:=\sum_{i=1}^t\langle f_i,h_i\rangle$.

We first develop a vector valued generalization in [44, Section 7.3] of pairing of a family of forms $\mathbf{f}$ of weight $V_{(\frac{k-2}{2},-\frac{k-2}{2})}$ (but with nebentypus at p varying) on $\mathrm{U}(2)$ to a Hida family of eigenforms $\mathbf{g}$ of weight $V^\vee_{(\frac{k-2}{2},-\frac{k-2}{2})}$. We define $V^\vee_{(\frac{k-2}{2},-\frac{k-2}{2})}$-valued measure $d\mu_{\mathbf{g}}$ as in *loc.cit* and use the above pairing to define

$$\int_{[\mathrm{U}(2)]}\mathbf{f}d\mu_{\mathbf{g}}$$

as an element in Λ, such that for each arithmetic points ϕ,

$$\phi(\int_{[\mathrm{U}(2)]}\mathbf{f}d\mu_{\mathbf{g}})=p^{t_\phi}\cdot\langle\mathbf{f}_\phi,\mathbf{g}_\phi^{\mathrm{low}}\rangle.$$

<u>Ikeda Theory</u>

Now we put ourselves in the context of [20]. We are in the $m=1$ and $n=2$ of [20, Section 2, case 2] (see the definitions of X,Y,Z,V there). Let ψ be an additive character of $\mathbb{A}$. Define

$$\varphi_{\phi_1,\phi_2}=\int_{X(\mathbb{A})}\phi_1(t-\frac{x}{2})\overline{\phi_2(t+\frac{x}{2})}\psi(-z-t^ty)dt \tag{4}$$

for $\phi_1, \phi_2 \in \mathcal{S}(X(\mathbb{A}))$. Then $\varphi_{\phi_1,\phi_2} \in \mathcal{S}_\psi(V(\mathbb{A}))$ and functions of this form generate a dense subspace of $\mathcal{S}_\psi(V_0(\mathbb{A}))$. For $\beta \in \mathbb{Q}$, from [20, Proposition 1.3] and its proof we can find a series of functions $\varphi_{i,\infty} \in \mathcal{S}_\beta(V_\infty)$, each are finite sums of functions of the form defined as (4), such that

$$\sum_{i=1}^{\infty} \rho(\varphi_i)\mathrm{FJ}_\beta(E)$$

is convergent to $\mathrm{FJ}_\beta(E)$ in $C^\infty(D(\mathbb{Q})\backslash D(\mathbb{A}))$. We write $\varphi_{i,\infty} = \sum_{j=1}^{n_i} \varphi_{\phi_{4,i,j,\infty},\phi_{2,i,j,\infty}}$. Let $\phi_{4,i,j} = \phi_{4,i,j,\infty} \prod_{v<\infty} \phi_{4,v}$ and $\phi_{2,i,j} = \phi_{2,i,j,\infty} \prod_{v<\infty} \phi_{2,v}$. An easy analysis sees it makes sense for the $\underline{k}$-component $\phi_{\infty,\underline{k}}$ (which is a Schwartz function) of the Archimedean Schwartz function ϕ_∞, such that the theta series of $\phi_{\infty,\underline{k}} \otimes \prod_{v<\infty} \phi_v$ is the $\underline{k}$-component of the theta series of $\phi_\infty \otimes \prod_{v>\infty} \phi_v$.

We deduce from [20, Proposition 1.3] (and its proof) that $\rho(\varphi_{\phi_{4,\infty},\phi_{2,\infty}})\mathrm{FJ}_\beta(E)$ can be written as finite sums of expressions as

$$\Theta_{\phi_{4,\infty}\otimes\prod_{v<\infty}\phi_v}(nh)E(R(f_\infty,\phi_{2,\infty})\cdot\prod_{v<\infty} f'_v)$$

for some Siegel section $\prod_{v<\infty} f'_v$ and Schwartz functions $\prod_{v<\infty} \phi_v$. But

$$E(R(f_\infty,\rho(\varphi^\iota_{i,j,\infty})\phi_{2,\infty})\cdot\prod_{v<\infty} f'_v) = \langle\rho(\varphi_{\phi_{4,\infty},\phi_{2,\infty}})\mathrm{FJ}_\beta(E),\Theta_{\phi_4^\vee}\rangle)$$

where $\phi^\vee_{4,\infty}$ is a Schwartz function such that

$$\langle\phi^\vee_{4,\infty},\phi_{4,\infty}\rangle = 1.$$

Now the computations in [44] implies for each $v < \infty$

$$\mathrm{FJ}_{\beta,v}(E) = \sum_{j_v=1}^{n_v} f_{j_v}\phi_{j_v}.$$

Then from the computation in [20, Page 628] on

$$\langle\mathrm{FJ}_\beta(E),\Theta_\phi\rangle$$

and choosing the test Schwartz function ϕ properly, we know that implies that

$$\rho(\varphi_{\phi_{4,\infty},\phi_{2,\infty}})\mathrm{FJ}_\beta(E) = (\prod_v\sum_{j_v})E(R(\rho(\varphi_{\phi_{4,\infty},\phi_{2,\infty}})f_\infty,\phi^\vee_{4,\infty})\cdot\prod_v f_{j_v},-)\cdot\Theta_{\phi_{4,\infty}\cdot\prod_{v<\infty}\phi_{j_v}}.$$

By the doubling method for h under $\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$ of the Siegel Eisenstein series

$$E(R(\rho(\varphi_{\phi_{4,\infty},\phi_{2,\infty}})f_\infty,\phi^\vee_{4,\infty})\cdot\prod_v f_{j_v},-)$$

above (see[44, Proposition 6.1] for details), we know there is a constant $C_{i,j,\infty}$ such that

$$\langle l_{\theta_1}(\mathrm{FJ}_\beta(E)),h\rangle = C_{i,j,\infty}\int_{[\mathrm{U}(2)]} h(g)\cdot\theta_3^{\mathrm{low}}(g)f(g)dg = C_{i,j,\infty}\int_{[\mathrm{U}(2)]} h(g)\theta_3^{\mathrm{low}}(g)f(g)dg.$$

Here as in [44] the superscript "low" means the level group for θ_3^{low} at p is lower triangular. That the "θ" part appearing is the eigen-component θ_3 of trivial weight comes from considering the central character. Since $\sum_{i=1}^{\infty}(\sum_{j=1}^{n_i} E(R(f,\phi_{2,i,j},g)\Theta_{\phi_{4,i,j}}(ng))$ is C^∞ convergent to $\mathrm{FJ}_\beta(E)$ in $C^\infty(D(\mathbb{Q})\backslash D(\mathbb{A}))$ (regarded as series with index i) we see that $\sum_i(\sum_j C_{i,j,\infty})$ is convergent as a series of i and we write C_∞ for the sum.

We can manage to make some choice of $\phi_{1,\infty}$ and $\phi_1 = \phi_{1,\infty}\times\prod_{v<\infty}\phi_{1,v}$, and define the functional l_{θ_1} as in [44, Sections 4.9, 8.5], such that $\theta_1=\theta_{\phi_1}$ is such that $l_{\theta_1}\in\mathrm{Hom}(-,\mathcal{O}_L)$ and that the $(\frac{k-2}{2},-\frac{k-2}{2})$-component of $l_{\theta_1}(\mathrm{FJ}_\beta(E))$ is non-zero for some $\beta\in\mathbb{Q}^\times\cap\mathbb{Z}_p^\times$. If not then the $(\frac{k-2}{2},-\frac{k-2}{2})$-component of E_{Kling} is a constant function on $X_{3,1}$. This contradicts the description of the boundary restriction of E_{Kling}, namely the $(\frac{k-2}{2},-\frac{k-2}{2})$-component is zero at some cusp while non-zero at other cusps.

Thus there must be a $\beta'\neq 0$ such that $\mathrm{proj}_{(\frac{k-2}{2},-\frac{k-2}{2})}(\mathrm{FJ}_{\beta'}=0)$. Let $\beta'=p^n\beta''$ for $\beta''\in\mathbb{Z}_p^\times$ and $n\in\mathbb{Z}$. Let y be an element in $\mathcal{K}^\times$ which is very close to $(p,1)$ in the p-adic topology of $\mathcal{K}_p$. Then $\mathrm{diag}(y\bar{y},y,y,1)^n\in\mathrm{U}(3,1)(\mathbb{Q})$. Set $\beta=\beta'(y\bar{y})^{-n}\in\mathbb{Z}_p^\times\cap\mathbb{Q}$ then

$$\mathrm{proj}_{(\frac{k-2}{2},-\frac{k-2}{2})}(\mathrm{FJ}_\beta\rho(\mathrm{diag}(y\bar{y},y,y,1)_p)E)$$

is not the zero function. So there must be a choice of θ_1 and some weight $(\frac{k-2}{2},-\frac{k-2}{2})$ form h such that the C_∞ above is non-zero. Note that the C_∞ only depends on our Archimedean datum. The reason of making sure that $\beta\in\mathbb{Z}_p^\times$ is that only for those β we did the Fourier-Jacobi coefficient computation at p for the Klingen Eisenstein series in [44].

As in [43] we consider triple product expression

$$\int\tilde{\mathbf{h}}\tilde{f}d\mu_{\tilde{\boldsymbol{\theta}}_3}.$$

(recall $\tilde{\mathbf{h}}$, $\tilde{f}$, $\tilde{\boldsymbol{\theta}}_3$ are in the dual automorphic representation space for $\mathbf{h}$, f and $\boldsymbol{\theta}_3$, respectively. This expression is interpolated by an element in $\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]$. As in [44, Sections 8.4, 8.5] we appeal to Ichino's formula to evaluate the product of the two triple product integrals above. This product turns out to be some constant $C\in\bar{\mathbb{Q}}_p^\times$ times a product of several p-adic L-functions (see [44, between Definition 8.15 to Lemma 8.16]), which are units in $\mathcal{O}_L^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$. Note that the representation L^{k-2} has dimension $k-1$, and the local triple product at the Archimedean place is $\frac{1}{k-1}\cdot(k-1)=1$ by Peter-Weyl Theorem. By our choices for χ_θ and χ_h, we arrive at the following

Proposition 3.5. *The*

$$\int l_{\theta_1}(\mathrm{FJ}_\beta(E))d\mu_{\mathbf{h}}$$

is a product of an element in $\mathcal{O}_L^{ur}[[\Gamma_{\mathcal{K}}]]^\times$ and an element in $\bar{\mathbb{Q}}_p^\times$.

Remark 3.6. *To see that*

$$F\mapsto\int l_{\theta_1}(\mathrm{FJ}_\beta(F))d\mu_{\mathbf{h}}$$

indeed gives a functional on the space of semi-ordinary families (over the two-dimensional weight space) of forms on $\mathrm{U}(3,1)$, we note that the weight k is fixed throughout the two-dimensional family. In the theory of p-adic semi-ordinary families (as developed in [43]) we are interpolating the highest weight vector of the automorphic forms. Thus each component of the L^{k-2}-projection of $\mathrm{FJ}_\beta(F)$ is interpolated p-adic analytically. So we can indeed define the integration of it with $d\mu_{\mathbf{h}}$.

We finally compute a local triple product integral, which enables us to remove the square-free conductor assumption in [44].

Proposition 3.7. *Suppose π_ℓ is supercuspidal representation with trivial character and conductor p^t, $t \geq 2$ and $\varphi_\ell \in \pi_\ell$ is a new vector. Let $\tilde{\pi}_\ell$ be the contragradient representation of π_ℓ and $\tilde{\varphi}_\ell \in \tilde{\pi}_\ell$ be the new vector. Consider the matrix coefficient $\Phi = \Phi_{\varphi_\ell,\tilde{\varphi}_\ell}(g) := \langle \pi(g)\varphi_\ell, \tilde{\varphi}_\ell \rangle$, normalized such that $\Phi_{\varphi_\ell,\tilde{\varphi}_\ell}(1) = 1$. Then for $g \in \mathrm{diag}(\ell^n, 1)\begin{pmatrix} 1 & \ell^{-n}\mathbb{Z}_\ell \\ & 1 \end{pmatrix} K_t$, $\Phi(g) \neq 0$ only when $n = 0$. In that case $\Phi(g) = 1$. For $g \in \mathrm{diag}(1, \ell^n)\begin{pmatrix} 1 & \\ \ell^{t-n}\mathbb{Z}_\ell & 1 \end{pmatrix} K_t$, $\Phi(g) \neq 0$ only when $n = 0$. In this case $\Phi(g) = 1$.*

This is an easy consequence of [19, Proposition 3.1].

Corollary 3.8. *Let π_ℓ be a supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ with trivial character and conductor p^t. Let $\varphi_\ell \in \pi_\ell$ and $\tilde{\varphi}_\ell \in \tilde{\varphi}_\ell$ be as above. Let $\chi_{h,1}$, $\chi_{h,2}$, $\chi_{\theta,1}$, $\chi_{\theta,2}$ be characters of $\mathbb{Q}_\ell^\times$ with conductors p^{t_1} and $t_1 > t$ such that $\chi_{h,1}\chi_{\theta,1}$ and $\chi_{h,2}\chi_{\theta,2}$ are both unramified. Then Ichino's local triple production integral*

$$I_\ell(\varphi_\ell \otimes f_{\chi_\theta} \otimes f_{\chi_h}, \tilde{\varphi}_\ell \otimes \tilde{f}_{\tilde{\chi}_\theta} \otimes \tilde{f}_{\tilde{\chi}_h}) = \mathrm{Vol}(K_{t_1}).$$

Now we state the main theorem of this section.

Theorem 3.9. *Suppose that*

- *There is at least one prime ℓ such that $\pi_{f,\ell}$ is not a principal series representation.*
- *The p component of π_f is an unramified principal series representation with distinct Satake parameters.*
- *The residual Galois representation $\bar{\rho}_f$ is absolutely irreducible over $\mathcal{K}[\sqrt{(-1)^{\frac{p-1}{2}}p}]$.*

Then the one containment

$$\mathrm{char}_{\mathcal{O}_L^{\mathrm{ur}}[[\Gamma_\mathcal{K}]]}(X^{\mathrm{Gr}}_{f,\mathcal{K}}) \subseteq (\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}})$$

up to powers of p. If moreover that

- *For each $\ell|N$ non-split in $\mathcal{K}$, the ℓ is ramified in $\mathcal{K}$ and the representation $\pi_{f,\ell}$ is the Steinberg representation twisted by the quadratic unramified character.*

Then we have

$$\mathrm{char}_{\mathcal{O}_L^{\mathrm{ur}}[[\Gamma_\mathcal{K}]]}(X^{\mathrm{Gr}}_{f,\mathcal{K}}) \subseteq (\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}).$$

Proof. The argument goes in the same way as [43, Theorem 5.3]. The assumption on $\bar{\rho}$ is made to apply the modularity lifting result as in *loc.cit.* (There the weight of f is assumed to be 2 and thus the assumption is redundant). □

4 Proof of Main Results

4.1 Beilinson-Flach Elements and Yager Modules

Now we reproduce some constructions in [43]. Recall $\mathbf{g}$ be the Hida family of normalized CM forms attached to characters of $\Gamma_{\mathcal{K}}$ with the coefficient ring $\Lambda_{\mathbf{g}} := \mathbb{Z}_p[[U]]$ (the trivial character of $\Gamma_{\mathcal{K}}$ is a specialization of this family). We write $\mathcal{L}_{\mathbf{g}}$ for the fraction ring of $\Lambda_{\mathbf{g}}$. As in [25] let $M(f)^*$ ($M(\mathbf{g})^*$) be the part of the cohomology of the modular curves which is the Galois representation associated to f ($\mathbf{g}$). The corresponding coefficients for $M(f)^*$ and $M(\mathbf{g})^*$ is $\mathbb{Q}_p$ and $\mathcal{L}_{\mathbf{g}}$. (Note that the Hida family $\mathbf{g}$ is not quite a Hida family considered in *loc.cit.* It plays the role of a branch $\mathbf{a}$ there). Note also that $\mathbf{g}$ is cuspidal (which is called "generically non-Eisenstein" in an earlier version) in the sense of [27]. We have $M(\mathbf{g})^*$ is a rank two $\mathcal{L}_{\mathbf{g}}$ vector space and, there is a short exact sequence of $\mathcal{L}_{\mathbf{g}}$ vector spaces with $G_{\mathbb{Q}_p}$ action:

$$0 \to \mathscr{F}_{\mathbf{g}}^+ \to M(\mathbf{g})^* \to \mathscr{F}_{\mathbf{g}}^- \to 0$$

with $\mathscr{F}_{\mathbf{g}}^{\pm}$ being rank one $\mathcal{L}_{\mathbf{g}}$ vector spaces such that the Galois action on $\mathscr{F}_{\mathbf{g}}^-$ is unramified. Since $\mathbf{g}$ is a CM family with p splits in $\mathcal{K}$, the above exact sequence in fact splits as $G_{\mathbb{Q}_p}$ representation. For an arithmetic specialization g_ϕ of $\mathbf{g}$ the Galois representation $M(f)^* \otimes M(g_\phi)^*$ is the induced representation from $G_{\mathcal{K}}$ to $G_{\mathbb{Q}}$ of $M(f)^* \otimes \xi_{\mathbf{g}_\phi}$ where $\xi_{\mathbf{g}_\phi}$ is the Hecke character corresponding to $\mathbf{g}_\phi$. This identification will be used implicitly later. We also write $D_{\mathrm{dR}}(f) = (M(f)^* \otimes B_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}$. The transition map is given by co-restriction. For f let $D_{\mathrm{dR}}(f)$ be the Dieudonne module for $M(f)^*$ and let $\eta_f^\vee$ be any basis of $\mathrm{Fil}^0 D_{\mathrm{dR}}(f)$. Let $\omega_f^\vee$ be a basis of $\frac{D_{\mathrm{dR}}(f)}{\mathrm{Fil}^0 D_{\mathrm{dR}}(f)}$ such that $\langle \omega_f^\vee, \omega_f \rangle = 1$.

We mainly follow [31] to present the theory of Yager modules. Let $K/\mathbb{Q}_p$ be a finite unramified extension. For $x \in \mathcal{O}_K$ we define $y_{K/\mathbb{Q}_p}(x) = \sum_{\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)} x^\sigma[\sigma] \in \mathcal{O}_{\mathcal{K}}[\mathrm{Gal}(K/\mathbb{Q}_p)]$ (note our convention is slightly different from [31]). Let $\mathbb{Q}_p^{ur}/\mathbb{Q}_p$ be an unramified $\mathbb{Z}_p$-extension with Galois group U. Then the above map induces an isomorphism of $\Lambda_{\mathcal{O}_F}(U)$-modules

$$y_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p} : \varprojlim_{\mathbb{Q}_p \subseteq K \subseteq \mathbb{Q}_p^{ur}} \mathcal{O}_F \simeq \mathcal{S}_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p} = \{f \in \hat{\mathbb{Z}}_p^{ur}[[U]] : f^u = [u]f\}$$

for any $u \in U$ a topological generator. Here the superscript means u acting on the coefficient ring while $[u]$ means multiplying by the group-like element u^{-1}. The module $S_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p}$ is called the Yager module. It is explained in *loc.cit* that the $S_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p}$ is a free rank one module over $\mathbb{Z}_p$. Let $\mathcal{F}$ be a $\mathbb{Z}_p$ representation of U then they defined a map $\rho : \hat{\mathbb{Z}}_p^{ur}[[U]] \to \mathrm{Aut}(\mathcal{F} \otimes \hat{\mathbb{Z}}_p^{ur})$ by mapping u to its action on $\mathcal{F}$ and extend linearly. As is noted in *loc.cit* the image of elements in the Yager module is in $(\mathcal{F} \otimes \hat{\mathbb{Z}}_p^{ur})^{G_{\mathbb{Q}_p}}$. We define as a generator of the Yager module for $\mathbb{Q}_p$. Then we can define $\rho(d)$ and let $\rho(d)^\vee$ be the element in $\hat{\mathbb{Z}}_p^{ur}[[U]]$ which is the inverse of $\varprojlim_m \sum_{\sigma \in U/p^m U} d_m^\sigma \cdot \sigma^{-1}$.

Now we recall some notations in [36]. Let $ES_p(D_{\mathcal{K}}) := \varprojlim_r H^1(X_1(D_{\mathcal{K}}p^r) \otimes \bar{\mathbb{Q}}, \mathbb{Z}_p)$ and $GES_p(D_{\mathcal{K}}) := \varprojlim_r H^1(Y_1(D_{\mathcal{K}}p^r) \otimes \bar{\mathbb{Q}}, \mathbb{Z}_p)$ which are modules equipped with Galois action of $G_{\mathbb{Q}}$. Here $X_1(D_{\mathcal{K}}p^r)$ and $Y_1(\mathcal{D}_{\mathcal{K}}p^r)$ are corresponding compact and non-compact modular curves. Recall in *loc.cit* there is an ordinary idempotent e^* associated to the covariant Hecke operator U_p. Let $\mathfrak{A}_\infty^* = e^* ES_p(D_{\mathcal{K}})^{I_p} = e^* GES_p(D_{\mathcal{K}})^{I_p}$ (see the Theorem in *loc.cit*). Let $\mathfrak{B}_\infty^*$ ($\tilde{\mathfrak{B}}_\infty^*$) be the quotient of $e^* ES_p(D_{\mathcal{K}})$ ($e^* GES_p(D_{\mathcal{K}})$) over $\mathfrak{A}_\infty^*$.

In an earlier version of [27] the authors defined elements $\omega_{\mathbf{g}}^{\vee} \in (\mathscr{F}_{\mathbf{g}}^{+}(\chi_{\mathbf{g}}^{-1}) \otimes \hat{\mathbb{Z}}_p^{ur})^{G_{\mathbb{Q}_p}}$ and $\eta_{\mathbf{g}}^{\vee} \in (\mathscr{F}_{\mathbf{g}}^{-} \otimes \hat{\mathbb{Z}}_p^{ur})^{G_{\mathbb{Q}_p}}$. Here the $\chi_{\mathbf{g}}$ is the central character for $\mathbf{g}$. We briefly recall the definitions since they are more convenient for our use (these notions are replaced by their dual in the current version of [27]). In the natural isomorphism

$$\mathfrak{A}_{\infty}^{*} \otimes_{\mathbb{Z}_p[[T]]} \hat{\mathbb{Z}}_p^{ur}[[T]] \simeq \mathrm{Hom}_{\hat{\mathbb{Z}}_p^{ur}}(S^{ord}(D_{\mathcal{K}}, \chi_{\mathcal{K}}, \hat{\mathbb{Z}}_p^{ur}[[T]]), \hat{\mathbb{Z}}_p^{ur}[[T]])$$

(see the proof of [36, Corollary 2.3.6], the $\omega_{\mathbf{g}}^{\vee}$ is corresponds to the functional which maps each normalized eigenform to 1. On the other hand $\eta_{\mathbf{g}}^{\vee}$ is defined to be the element in $\mathfrak{B}_{\infty}^{*}$ which, under the pairing in [36, Theorem 2.3.5], pairs with $\omega_{\mathbf{g}}^{\vee}$ to the product of local root numbers at primes to p places of $\mathbf{g}$. This product moves p-adic analytically and is a unit.

We take basis $v^{\pm}$ of $\mathscr{F}_{\mathbf{g}}^{\pm}$ with respect to which $\omega_{\mathbf{g}}^{\vee}$ and $\eta_{\mathbf{g}}^{\vee}$ are $\rho(d)^{\vee}v^{+}$ and $\rho(d)v^{-}$ (see the discussion for Yager Modules). Let $\Psi_{\mathbf{g}}$ be the $\Lambda_{\mathbf{g}}$-valued Galois character of $G_{\mathcal{K}}$ corresponding to the Galois representation associated to $\mathbf{g}$ (i.e. $\mathrm{Ind}_{G_{\mathbb{Q}}}^{G_{\mathcal{K}}}\Psi_{\mathbf{g}} = M(\mathbf{g})^{*}$). Since p splits as $v_0\bar{v}_0$ in $\mathcal{K}$, there is a canonical identification $(\mathrm{Ind}_{G_{\mathbb{Q}}}^{G_{\mathcal{K}}}\Psi_{\mathbf{g}})|_{G_{\mathbb{Q}_p}} \simeq \Psi_{\mathbf{g}}|_{G_{\mathcal{K}_{v_0}}} \oplus \Psi_{\mathbf{g}}|_{G_{\mathcal{K}_{\bar{v}_0}}}$ and can take a $\Lambda_{\mathbf{g}}$-basis of the right side as $\{v, c\cdot v\}$ where c is the complex conjugation. (Note that there are two choices for the $\Psi_{\mathbf{g}}$ and we choose the one so that $\Psi_{\mathbf{g}}|_{G_{\mathcal{K}_{v_0}}}$ corresponds to $\mathscr{F}_{\mathbf{g}}^{-}$).
Convention:
we use the basis $\{v^{+}, c\cdot v^{+}\}$ to identify the Galois representation of $\mathbf{g}$ with the induced representation $\mathrm{Ind}_{G_{\mathbb{Q}}}^{\mathcal{K}}\Psi_{\mathbf{g}}$.

In [30], the authors constructed Beilinson-Flach elements $\mathrm{BF} = \mathrm{BF}_{f_\alpha,\mathbf{g}} \in H^1_{\mathrm{cl,Iw}}(\mathbb{Q}_\infty, T_f \otimes \mathbf{T}_{\mathbf{g}})$. Recall that these classes are obtained from writing down classes in $H^3_{\mathrm{mot}}(Y_1(N)^2_{\mathbb{Q}_p}, T^{[k,k']}_{\mathrm{sym}}(\mathscr{H}_{\mathbb{Q}_p}(2-j)))$ explicitly for cuspidal eigenforms f and g with weights k and k' respectively and $j+1 < k, k'$, and consider the image under the map

$$\begin{aligned} H^3_{\mathrm{mot}}(Y_1(N)^2_{\mathbb{Q}_p}, T^{[k,k']}_{\mathrm{sym}}(\mathscr{H}_{\mathbb{Q}_p}(2-j))) &\xrightarrow{r^{\mathrm{t}}} H^3_{\mathrm{et}}(Y_1(N)^2_{\mathbb{Q}_p}, T^{[k,k']}_{\mathrm{sym}}(\mathscr{H}_{\mathbb{Q}_p}(2-j))) \\ &\xrightarrow{\mathrm{AJ}_{\mathrm{t},f,g}} \quad H^1(\mathbb{Q}_p, M_{\mathrm{t}}(f\otimes g)^{*}(-j)). \end{aligned}$$

Here $\mathrm{Tsym}^{[k,k']}$ means the $[k,k']$-component in the symmetric tensor product of the universal elliptic curve $\mathscr{H}$ over the modular curves. The r^{t} is the etale regulator map and $\mathrm{AJ}_{\mathrm{t},f,g}$ is the etale Abel-Jacobi map, followed by projecting to the $f\otimes g$-component. Deforming f in a Coleman family $\mathcal{F}$ and varying the k, k' and j in p-adic families one gets the three-variable Beilinson-Flach class, which specializes to the two-variable class under $\mathcal{F} \to f$.

4.2 Control Theorem of Selmer Groups

Let $P \in \mathrm{Spec}\Lambda$ be a generic arithmetic point (i.e. corresponding to a finite order character of Γ) and $\mathbf{v} \in H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, T\otimes\Lambda)$ an element of a Λ-basis of the latter such that the image of $\mathbf{v}$ is an Λ/P-basis of $H^1_f(\mathbb{Q}_p, T\otimes\Lambda/P)$. Then it is easy to see that $\mathbf{v}$ satisfies the following

(*) For all but finitely many integers m and $x := \gamma - (1+p)^m$, the map $\frac{H^1(\mathbb{Q}^S/\mathbb{Q},\mathbf{T})}{xH^1(\mathbb{Q}^S/\mathbb{Q},\mathbf{T})} \to \frac{H^1(\mathbb{Q}_p,\mathbf{T})}{\mathbb{Z}_p\bar{\mathbf{v}}}$ is injective.

Picking up such a $\mathbf{v}$ is important, especially when we prove that certain module of dual Selmer group has non pseudo-null submodules later on. (A pseudo-null submodule over Λ means it has finite cardinality). We consider the control theorem for $\mathbf{v}^\vee$-Selmer groups. This means the Selmer condition which is the usual one at primes outside p, but is the orthogonal complement of $\Lambda\mathbf{v}$ at p under Tate local pairing. We look at the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Sel}_{\mathbf{v}^\vee}(\mathbb{Q},\mathbf{A}[P]) & \longrightarrow & H^1(\mathbb{Q}^S/\mathbb{Q},\mathbf{A}[P]) & \longrightarrow & \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q},\mathbf{A}[P]) \\ & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & \mathrm{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}_\infty)^P & \longrightarrow & \varinjlim_n H^1(\mathbb{Q}_n^S/\mathbb{Q}_n,\mathbf{A})^P & \longrightarrow & \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}_\infty,\mathbf{A}) \end{array}$$

where $\mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q},\mathbf{A}) = \prod_{\ell\nmid p} H^1(G_\ell,\mathbf{A})\times\frac{H^1(G_p,\mathbf{A})}{(\Lambda\cdot\mathbf{v})^\vee}$ and $\mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q},\mathbf{A}) = \prod_{\ell\nmid p} H^1(G_\ell,\mathbf{A}[P])\times\frac{H^1(G_p,\mathbf{A}[P])}{((\Lambda/P)\cdot\mathrm{Im}(\mathbf{v}))^\vee}$. We define the Tamagawa number of $\mathbf{A}[P]$ at $\ell\neq p$ to be

$$c_{P,\ell} = \sharp\ker\{H^1(G_\ell,\mathbf{A}[P])\to H^1(I_\ell,\mathbf{A}[P])\}.$$

Also let $c'_{P,\ell}\in\mathcal{O}_L\otimes\Lambda/P$ be such that

$$c_{P,\ell} = \sharp(\mathcal{O}_L\otimes(\Lambda/P)/c'_{P,\ell}\mathcal{O}_L\otimes(\Lambda/P)).$$

We define a number $c_{P,p}$ at p as follows (up to a p-adic unit): let $\mathcal{V}$ be a $\mathcal{O}_P$-basis of $\frac{H^1(\mathbb{Q}_p,T\otimes\Lambda/P)}{H^1_f(\mathbb{Q}_p,T\otimes\Lambda/P)}$, then

$$\exp^*\mathcal{V} = c'_{P,p}\omega_f,\ c_{P,p} := \sharp(\frac{\mathcal{O}_L\otimes(\Lambda/P)}{c'_{P,p}\mathcal{O}_L\otimes(\Lambda/P)}). \qquad (5)$$

Here we identify the T_f with its realization in the cohomology of the modular curve.

Remark 4.1. *We discuss a little about the relations between this $c_{P,p}$ and Tamagawa numbers. We first note that in the Fontaine-Laffaille range $k<p$, the number is actually a local number. This can be seen using the integral comparison theorem between crystalline and deRham cohomology of modular curves. For details see Section 4.4. Keep this assumption, suppose P corresponds to the trivial character of Γ. Then $T\otimes\Lambda/P$ is crystalline. Then its Tamagawa number is defined in [29, (5.6)] as*

$$\sharp(\frac{\mathcal{O}_L}{c'_{P,p}/\det(1-\varphi|D_{\mathrm{cris}}(V))\mathcal{O}_L}).$$

For more backgrounds justifying this definition see [5].

For χ a character of $\mathrm{Gal}(\mathbb{Q}_{p,n}/\mathbb{Q}_p)$ with coefficient ring E, in literature people usually use the convention that

$$D_{\mathrm{dR}}(V\otimes\chi) = D_{\mathrm{dR}}(V)\otimes D_{\mathrm{dR}}(\chi) = D_{\mathrm{dR}}(V)\otimes(\mathbb{Q}_{p,n}\otimes E)^\chi.$$

Recall when defining the $\exp^*$ map for T_P, the pairing on both Galois cohomology and Dieudonne module come from

$$(T_P\otimes\mathbb{Q}_p)\times(T_{P^\iota}\otimes\mathbb{Q}_p)\to\mathbb{Q}_{p,n}(1)\to\mathbb{Q}_p(1)$$

where the last is the trace map. By the formula in [31, Lemma B.4], we see for any $a\in K_n$

$$\mathrm{tr}_{K_n/\mathbb{Q}_p}(a(\langle\exp^* x,\log y\rangle-(x,y))) = 0.$$

Thus $\langle \exp^* x, \log y\rangle = (x,y)_{K_n}$. We observe that the pairing of cup product

$$(,): \frac{H^1(\mathbb{Q}_p, T\otimes\Lambda/P)}{H^1_f(\mathbb{Q}_p, T\otimes\Lambda/P)} \times H^1_f(\mathbb{Q}_p, T\otimes\Lambda/P^\iota) \to \mathcal{O}_L$$

is surjective. Thus for some $\mathcal{O}_{P^\iota}$ basis $\bar{\mathcal{V}}$ of $H^1_f(\mathbb{Q}_p, T\otimes\Lambda/P^\iota)$, we have

$$\log \bar{\mathcal{V}} = \frac{1}{c'_{P,p}}\cdot \omega_f^\vee. \tag{6}$$

Remark 4.2. *In the case when f is ordinary at p, or f corresponds to a supersingular elliptic curve with $a_p = 0$, it is not hard to explicitly compute the $c_{P,p}$ using the local theory, e.g. in [43].*

The Poitou-Tate exact sequence implies that if $\sharp\mathrm{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}, \mathbf{A}[P]) < \infty$ and $H^1_{\mathbf{v}}(\mathbb{Q}^S/\mathbb{Q}, \mathbf{T}/P^\iota\mathbf{T}) = 0$, then

$$\prod_\ell c_{P,\ell}\mathrm{Fitt}\mathrm{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}, \mathbf{A}[P]) = \mathrm{Fitt}X_{\mathbf{v}^\vee}/PX_{\mathbf{v}^\vee}.$$

Lemma 4.3. *The cardinality $\sharp(H^2(\mathbb{Q}^S/\mathbb{Q}, \mathbf{T}))[x] < \infty$ for all but finitely many m's and $x = \gamma - (1+p)^m$.*

Proof. The $H^2(\mathbb{Q}^S/\mathbb{Q}, \mathbf{T})$ is a finitely generated Λ-module. Then the lemma follows from the well known structure theorem of finitely generated Λ-modules. □

Lemma 4.4. *The $\frac{H^1(\mathbb{Q}^S/\mathbb{Q},\mathbf{A})}{xH^1(\mathbb{Q}^S/\mathbb{Q},\mathbf{A})} = 0$ for all but finitely many m's and $x = \gamma - (1+p)^m$.*

Proof. We have

$$\frac{H^1(\mathbb{Q}^S/\mathbb{Q},\mathbf{A})}{xH^1(\mathbb{Q}^S/\mathbb{Q},\mathbf{A})} \hookrightarrow H^2(\mathbb{Q}^S/\mathbb{Q}, \mathbf{A}[x]).$$

From the Global duality the right side is dual to

$$\ker\{H^1(\mathbb{Q}^S/\mathbb{Q}, T_x) \to \prod_{v\in S} H^1(\mathbb{Q}_v, T_x)\}.$$

We claim this term is 0 for all but finitely many m. Indeed $H^1(\mathbb{Q}^S/\mathbb{Q}, T_x)$ is p-torsion free by (Irred). Moreover we have exact sequence

$$\frac{H^1(\mathbb{Q}^S/\mathbb{Q},\mathbf{T})}{xH^1(\mathbb{Q}^S/\mathbb{Q},\mathbf{T})} \hookrightarrow H^1(\mathbb{Q}^S/\mathbb{Q}, T_x) \to H^2(\mathbb{Q}^S/\mathbb{Q}, \mathbf{T})[x].$$

The last term is finite for all but finitely many m by lemma 4.3. The $H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{T})$ is a torsion-free rank one Λ-module such that the localization map $H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{T}) \to H^1(\mathbb{Q}_p, \mathbf{T})$ is injective. (Because by [39] the image of z_{Kato} under this map is non-zero). Now it is easy to see that

$$\ker\{H^1(\mathbb{Q}^S/\mathbb{Q}, T_x) \to H^1(\mathbb{Q}_p, T_x)\}$$

is 0 for all but finitely many m. The lemma follows readily. □

Proposition 4.5. *The $X_{\mathbf{v}^\vee}$ has no pseudo-null submodules.*

Proof. Let $x = \gamma - (1+p)^m$ for some integer m. Then we claim for all but finitely many integers m we have surjection

$$H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{A}[x]) \twoheadrightarrow \mathcal{P}_{\boldsymbol{v}^\vee}(\mathbb{Q}, \mathbf{A}[x]). \tag{7}$$

We first look at the exact sequence

$$H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{T}) \xrightarrow{\times x} H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{T}) \longrightarrow H^1(\mathbb{Q}^S/\mathbb{Q}, T_x) \longrightarrow H^2(\mathbb{Q}^S/\mathbb{Q}, \mathbf{T})[x].$$

From Lemma 4.3 the last term is torsion for all but finitely many m. By property (*) on $\mathbf{v}$ we get

$$H^1_{\mathbf{v}}(\mathbb{Q}^S/\mathbb{Q}, T_x) = 0$$

for these x. From Poitou-Tate exact sequence

$$H^1_{\mathbf{v}}(\mathbb{Q}^S/\mathbb{Q}, T_x) \to \mathcal{P}_{\mathbf{v}}(\mathbb{Q}, T_x) \to H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{A}[x])^\vee,$$

we get the claim.

It is also clear that the map $H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{A}[x]) \to H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{A})[x]$ is an isomorphism, and that the map

$$\mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, \mathbf{A}[x]) \to \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, \mathbf{A})[x]$$

is surjective. These altogether imply

$$H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{A})[x] \to \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, \mathbf{A})[x]$$

is surjective.

Then consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}, \mathbf{A}) \arrow[r] \arrow[d, "x"] & H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{A}) \arrow[r] \arrow[d, "x"] & \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, \mathbf{A}) \arrow[d, "x"] \\
0 \arrow[r] & \mathrm{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}, \mathbf{A}) \arrow[r] & H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{A}) \arrow[r] & \mathcal{P}_{\mathrm{v}^\vee}(\mathbb{Q}, \mathbf{A})
\end{tikzcd}$$

By Snake lemma and Lemma 4.4 the $\frac{\mathrm{Sel}_{\mathbf{v}^\vee}(\mathbb{Q},\mathbf{A})}{x\mathrm{Sel}_{\mathbf{v}^\vee}(\mathbb{Q},\mathbf{A})} = 0$ for all but finitely many m and $x = \gamma - (1+p)^m$. By Nakayama's lemma, there is no quotient of $\mathrm{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}, \mathbf{A})$ of finite cardinality. Thus $X_{\mathbf{v}^\vee}$ has no pseudo-null submodules. $\square$

Writing $\mathscr{F}$ for the characteristic polynomial of $X_{\mathbf{v}^\vee}$ and write ϕ for the arithmetic point corresponding to P. We also write $X_{\mathbf{v}^\vee,\phi}$ for the Selmer group for $\mathbf{A}[\phi]$. Note that the local Selmer condition of it at all primes outside p is $\{0\}$. Thus the control theorem as before implies that

$$\prod_{\ell \nmid p} c_{P,\ell}(f) \sharp(X_{\mathbf{v}^\vee,\phi}) = \sharp(X_{\mathbf{v}^\vee}/PX_{\mathbf{v}^\vee}). \tag{8}$$

Note for any finitely generated torsion Iwasawa module M of Λ, if (x) is a prime ideal of Λ with $\sharp(\frac{M}{xM}) < \infty$, and $\mathscr{F}$ is a generator of $\mathrm{char}_\Lambda(M)$, then

$$\sharp(\frac{M}{xM}) \geq \sharp(\frac{\Lambda}{(\mathscr{F}, x)}).$$

If M has no pseudo-null submodule then the above "$\geq$" is an "$=$". So the control theorem argument as before implies that

$$\prod_{\ell \nmid p} c_{P,\ell}(f) \sharp(X_{\mathbf{v}^\vee,\phi}) = \sharp(\mathcal{O}_\phi/(\mathscr{F}(\phi))). \tag{9}$$

4.3 Selmer Complexes and Iwasawa Main Conjecture

Before continuing we explain how we make choice for the quadratic imaginary field $\mathcal{K}$. We choose it to be ramified in the prime ℓ in the assumption of Theorem 1.4, split at p, and is split at all other primes of N. We also ensure that the $\bar{T}$ is still irreducible over $G_{\mathcal{K}(\zeta_p)}$.

We follow [37], [38] to present the analytic Iwasawa theory, in the framework of Nekovar's Selmer complex. Let $U \simeq \mathbb{Z}_p$ be the Galois group $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$. We define $\mathcal{A}$ to be the affinoid ring $\mathcal{O}_L\langle p^{-r}U\rangle$ for some $r > 0$ and $\Lambda_{\mathcal{A},\infty} = \Lambda_\infty \hat{\otimes} \mathcal{A}$. We fix a local condition, which means for any $v \in S$ a bounded complex of finite type $\Lambda_{\mathcal{A},\infty}$-modules $U_v^\bullet$ and a morphism

$$i_v : U_v^\bullet \to C^\bullet_{\mathrm{cont}}(G_v, T \otimes \Lambda_{\mathcal{A},\infty}).$$

Here we write $C^\bullet_{\mathrm{cont}}(G, M)$ for the space of continuous cochains of the G-module M. We define the Selmer complex for f over $\mathcal{K}$ to be the mapping cone

$$\mathrm{Cone}[C^\bullet_{\mathrm{cont}}(G_{\mathcal{K},S}, T \otimes_{\mathcal{O}_L} \Lambda_{\mathcal{A},\infty}) \oplus \oplus_{v\in S} U_v^\bullet \to \oplus_{v\in S} C^\bullet_{\mathrm{cont}}(G_v, T \otimes \Lambda_{\mathcal{A},\infty})][-1],$$

where the map is given by $\oplus_v(\mathrm{res}_v, -i_v)$. Throughout this paper, for each $v \in S$ not dividing p, we use the unramified local condition by

$$i_v : U_v^\bullet : C^\bullet_{\mathrm{cont}}(G_v/I_v, (T \otimes \Lambda_{\mathcal{A},\infty})_v^I) \to C^\bullet_{\mathrm{cont}}(G_v, T \otimes \Lambda_{\mathcal{A},\infty}).$$

We will make several different choices for the local Selmer conditions at p. We need some preparations.

Definition 4.6. *Write $\mathscr{R}$ for the Robba ring $B^\dagger_{\mathrm{rig},\mathbb{Q}_p}$ over $\mathbb{Q}_p$ and $\mathscr{R}^+$ for $B^+_{\mathrm{rig},\mathbb{Q}_p}$. We define a triangulation of a two-dimensional (φ,Γ)-modules D over $\mathscr{R}$ to be a short exact sequence $0 \to \mathcal{F}^+D \to D \to \mathcal{F}^-D \to 0$ where $\mathcal{F}^\pm D$ are free rank one (φ,Γ)-modules over $\mathscr{R}$. For any finite extension L of $\mathbb{Q}_p$, we define $\mathscr{R}_L = \mathscr{R} \otimes_{\mathbb{Q}_p} L$ and can talk about triangulations of (φ,Γ)-modules of rank two over $\mathscr{R}_L$.*

Definition 4.7. *If V is a two dimensional crystalline representation of $G_{\mathbb{Q}_p}$. A refinement of V is a full φ-stable filtration of $D_{\mathrm{cris}}(V)$:*

$$\mathcal{F}_0 = 0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 = D_{\mathrm{cris}}(V).$$

This is equivalent to an ordering of $\{\alpha, \beta\}$.

It is summarized in [4, 2.4] that there is a one-to-one correspondence between triangulations of $D(V)$ and refinements of V, given by $\mathcal{F}_1 = \mathcal{F}^+D[1/t] \cap D_{\mathrm{cris}}(V)$ and $\mathcal{F}_1 = \mathcal{F}^1[1/t]^\Gamma$. Let the Robba ring over $\mathcal{A}$ be $\mathscr{R}_\mathcal{A} := \mathscr{R}\hat{\otimes}\mathcal{A}$. We also write $\mathscr{R}^+_\mathcal{A} = \mathscr{R}^+\hat{\otimes}\mathcal{A}$. There is a nature action $U \hookrightarrow \mathcal{A}^\times$. Then we can define a (φ,Γ)-module $D_\mathcal{A}$ over $\mathscr{R}_\mathcal{A}$ by pulling back the action of Γ on D but twisting the action of φ on D by the Frobenius action via $1 + U$ as above. We define the analytic Iwasawa cohomology for $D_\mathcal{A}$ in the same way as 2.5.

Lemma 4.8. *The $H^i_{\mathrm{Iw}}(\mathbb{Q}_p, D_\mathcal{A})$ can be computed using the complex*

$$D_\mathcal{A} \xrightarrow{\psi-1} D_\mathcal{A}$$

concentrated at degrees 1 and 2.

This is just [23, Theorem 4.4.8]. Suppose D has the form $\mathscr{R}(\alpha^{-1})$, then we have

Proposition 4.9. *There is an exact sequence*

$$0 \to \oplus_{m=0}^{\infty}(t^m D_{\mathrm{cris}}(D_{\mathcal{A}}))^{\varphi=1} \to (\mathscr{R}_{\mathcal{A}}^+ \otimes D)^{\psi=1} \to (\mathscr{R}_{\mathcal{A}}^+ \otimes D)^{\psi=0} \to \bigoplus_{m=0}^{N} \frac{t^m \otimes D_{\mathrm{cris}}(D_{\mathcal{A}})}{(1-\varphi)(t^m \otimes D_{\mathrm{cris}}(D_{\mathcal{A}}))}$$

where the third arrow is given by $\varphi - 1$ and $N >> 0$. Note that α, β are Weil numbers of odd weight $k-1$. The second term above is easily seen to be 0.

This proposition is the family version of [34, Lemma 3.18]. The analogues in our setting of the results in Lemma 3.17 of *loc.cit* is proved in [8, Section 2] as well. For the above rank one (φ, Γ)-module D we consider a finite set of height one primes $S(D)$ of $\Lambda_{\mathcal{K},\infty}$, each generated by an element of the form $(U+1-\alpha p^{-m})$ for some non-negative integer m. Note that this is a finite set because for $m >> 0$, $(U+1-\alpha p^{-m})$ is invertible. For any height one prime P not in $S(D)$ the localized map at P

$$\varphi - 1 : (\mathscr{R}_{\mathcal{A}}^+ \otimes D)_P^{\psi=1} \to (\mathscr{R}_{\mathcal{A}}^+ \otimes D)_P^{\psi=0}$$

is an isomorphism. Now suppose D is the (φ, Γ)-module of $V_f = T_f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We fix a triangulation of D by requiring $\mathcal{F}^- := D/\mathcal{F}^+$ to be the (φ, Γ)-module $\mathscr{R}(\alpha^{-1})$. We thus define an induced triangulation of $D_{\mathcal{A}}$ in the obvious way. For any (φ, Γ)-module of the form $\mathscr{R}_{\mathcal{A}}(\alpha^{-1})$ for some $\alpha \in \mathcal{A}^*$, we define a regulator map as [25, (6.2.1)]:

$$\begin{aligned} \mathrm{Reg}_{\mathscr{R}_{\mathcal{A}}(\alpha^{-1})} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, \mathscr{R}_{\mathcal{A}}(\alpha^{-1}) &\xrightarrow{\simeq} \mathscr{R}_{\mathcal{A}}(\alpha^{-1})^{\psi=1} \\ &\xrightarrow{\simeq} \mathscr{R}_{\mathcal{A}}^+(\alpha^{-1})^{\psi=1} \xrightarrow{1-\varphi} \mathscr{R}_{\mathcal{A}}^+(\alpha^{-1})^{\psi=0} \xrightarrow{\simeq} \mathcal{A}\hat{\otimes}\Lambda_{\mathbb{Q},\infty}. \end{aligned}$$

The last map is the Mellin transform. As in [25, Section 6] since $\mathcal{F}^-$ has the form $\mathscr{R}_{\mathcal{A}}(\alpha^{-1})$ so we can define the regulator map $\mathrm{Reg}_{\mathcal{F}^-}$ as above. The $\mathcal{F}^+$ is of the form $\mathscr{R}(\beta^{-1})$ twisted by the $k-1$-th Tate twist, so as in *loc.cit* we can still define the regulator map $\mathrm{Reg}_{\mathcal{F}^+}$ by re-parameterizing the weight space. We have an exact sequence

$$0 \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p, \mathcal{F}^+(D_{\mathcal{A}})) \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p, D_{\mathcal{A}}) \to H^1_{\mathrm{Iw}}(\mathbb{Q}_P, \mathcal{F}^-(D_{\mathcal{A}})) \to 0.$$

(As noted in [37, Proof of Proposition 2.9], by the machinery developed in [38] we only need to check that for any specialization of $\mathcal{F}_{\mathcal{A}}^+$ twisted by a character of Γ, the H^2 becomes trivial. This is easily seen by the local duality.)

Definition 4.10. *We define the α local Selmer condition U_p at p as the second arrow above, using the identification [38, Theorem 2.8] of the derived category of Galois cohomology of Galois representations over an affinoid algebra and the corresponding (φ, Γ)-module.*

By Proposition 4.9 for any $P \notin S(D)$ the regulator map Reg gives isomorphisms

$$H^1(\mathbb{Q}_p, \mathcal{F}_{\mathcal{A}}^+)_P \simeq \Lambda_{\mathcal{A},\infty,P},$$

$$H^1(\mathbb{Q}_p, \mathcal{F}_{\mathcal{A}}^-)_P \simeq \Lambda_{\mathcal{A},\infty,P}.$$

We also have

$$H^1(\mathbb{Q}_p, D)_P \simeq \Lambda^2_{\mathcal{A},\infty,P}.$$

Denote $\mathrm{R}\tilde{\Gamma}(G_{\mathcal{K},S}, U_v^\bullet, T\otimes\Lambda_{\mathcal{A},\infty})$ as the image of the Selmer complex in the derived category of finite type $\Lambda_{\mathcal{A},\infty}$-modules. Let $\tilde{H}^i(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})$ be the cohomology groups of the Selmer complex, which are called extended Selmer groups. Replacing $\Lambda_{\mathcal{A},\infty}$ by $\Lambda_{\mathbb{Q},\infty}$ we similarly define $\tilde{H}^i(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathbb{Q},\infty})$, etc. We record here some properties of the Selmer complex

Proposition 4.11. *[37, Theorem 4.1]*

$$R\tilde{\Gamma}_\alpha(G_{\mathcal{K},S}, U_v^\bullet, T\otimes\Lambda_{\mathcal{A},\infty})\otimes^L_{\Lambda_{\mathcal{A},\infty}}\Lambda_{\mathbb{Q},\infty}\simeq R\tilde{\Gamma}_\alpha(G_{\mathcal{K},S}, U_v^\bullet, T\otimes\Lambda_{\mathbb{Q},\infty}).$$

In particular

$$0\to\tilde{H}^i_\alpha(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{K},\infty})\otimes\Lambda_{\mathbb{Q},\infty}\to\tilde{H}^i_\alpha(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathbb{Q},\infty})\to\mathrm{Tor}_1^{\Lambda_{\mathcal{K},\infty}}(\tilde{H}^{i+1}_\alpha(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathbb{Q},\infty}),\Lambda_{\mathbb{Q},\infty})\to 0.$$

Definition 4.12. *We define $X_{\alpha\alpha}$ to be $\tilde{H}^2(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})$ defined using the α-Selmer conditions at both v_0 and $\bar{v}_0$. Similarly we define $X_{0,\mathrm{rel}}$, $X_{\alpha,\mathrm{rel}}$, etc (here* rel *stands for "relaxed").*

It follows from the definition of the Selmer complex that

$$\begin{aligned}0\to\tilde{H}^1_{\alpha,\mathrm{rel}}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})\to H^1(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})&\to H^1(G_{v_0},\mathcal{F}^-)\to\tilde{H}^2_{\alpha,\mathrm{rel}}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{K},\infty})\\&\to H^2(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})\to\oplus_{v\nmid p}H^2(G_{\mathcal{K}_v}, T\otimes\Lambda_{\mathcal{A},\infty}).\end{aligned}\tag{10}$$

(See [37, Page 18], especially the computation of local Galois cohomology at $v\nmid p$. Note also that $0=H^2_{\mathrm{Iw}}(G_p, D_{\mathcal{A}})=H^2_{\mathrm{Iw}}(G_p,\mathcal{F}^+_{\mathcal{A}})=H^2_{\mathrm{Iw}}(G_p,\mathcal{F}^-_{\mathcal{A}})$.) Also

$$\begin{aligned}0\to\tilde{H}^1_{0,\mathrm{rel}}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})\to H^1(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})&\to H^1(G_{v_0},D)\to\tilde{H}^2_{0,\mathrm{rel}}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})\\&\to H^2(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})\to\oplus_{v\nmid p}H^2(G_{\mathcal{K}_v}, T\otimes\Lambda_{\mathcal{A},\infty}).\end{aligned}\tag{11}$$

Denote the third arrows of 10 and 11 as (A) and (B). Define Ker by the exact sequence

$$0\to\mathrm{Ker}\to\frac{H^1(G_{v_0},D)}{\mathrm{im}(B)}\to\frac{H^1(G_{v_0},\mathcal{F}^-)}{\mathrm{im}(A)}\to 0.$$

Then

$$\mathrm{Ker}=\frac{\mathrm{im}(B)+H^1(G_{v_0},\mathcal{F}^+)}{\mathrm{im}(B)}\simeq\frac{H^1(G_{v_0},\mathcal{F}^+)}{\mathrm{im}(B)\cap H^1(G_{v_0},\mathcal{F}^+)}\simeq\frac{H^1(G_{v_0},\mathcal{F}^+)}{\mathrm{im}(H^1_{\alpha,\mathrm{rel}}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty}))}.$$

Combining this with (10) and (11) we obtain

$$0\to\frac{H^1(G_{v_0},\mathcal{F}^+)}{\mathrm{im}(H^1_{\alpha,\mathrm{rel}}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty}))}\to\tilde{H}^2_{0,\mathrm{rel}}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})\to\tilde{H}^2_{\alpha,\mathrm{rel}}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})\to 0.\tag{12}$$

Similarly we get

$$0\to\frac{H^1(G_{v_0},\mathcal{F}^-)}{\mathrm{im}(H^1_{\alpha,\mathrm{rel}}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty}))}\to\tilde{H}^2_{\alpha,\alpha}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})\to\tilde{H}^2_{\alpha,\mathrm{rel}}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})\to 0.\tag{13}$$

Before continuing we need the following

Lemma 4.13. *The $H^1_{\alpha,\mathrm{rel}}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})$ has rank one over $\Lambda_{\mathcal{A},\infty}$.*

Proof. We only need to know that $H^1_{\alpha\alpha}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})=0$. We need to use the

Lemma 4.14. *[3, Proposition 5] Let χ be a finite order character of Γ. Then*

$$H^1_\alpha(\mathbb{Q}_p, T\otimes\chi)=H^1_f(\mathbb{Q}_p, T\otimes\chi).$$

Now if $H^1_{\alpha\alpha}(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})=0$ has positive rank, then for all ϕ we have the Bloch-Kato Selmer group of T_χ has positive rank, which contradicts the main theorem of [22]. □

We have

$$\mathrm{loc}_{v_0}\mathrm{BF}_\alpha\in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, \mathcal{F}^+D_{\mathcal{A}})$$

by [25, Section 7].

Let $\Lambda_{\mathbb{Q},n}$ be the p-adic completion of $\Lambda[\mathfrak{m}^n/p]$ and $\Lambda_{\mathcal{A},n}:=\Lambda_{\mathbb{Q},n}\hat{\otimes}\mathcal{A}$. Note that the ring $\Lambda_{\mathcal{A},\infty}$ and $\Lambda_{\mathbb{Q},\infty}$ are not Noetherian, while $\Lambda_{\mathbb{Q},n}$ and $\Lambda_{\mathcal{A},n}$ are. So in order to prove the main conjecture in terms of characteristic ideal, we first observe that we only need to prove the equality for ideals after tensoring with $\Lambda_{\mathbb{Q},n}$ for each n. We replace $\Lambda_{\mathcal{A},\infty}$ by $\Lambda_{\mathcal{A},n}$ and define the corresponding Selmer complex for $T\otimes\Lambda_{\mathcal{A},n}$, using the pullbacks of the local Selmer conditions U_v under the nature map $f_n:\Lambda_{\mathcal{A},\infty}\to\Lambda_{\mathcal{A},n}$. Note that by [38, Theorem 1.6] we have an isomorphism in the derived category

$$\mathbf{L}f_n^*\mathrm{R}\Gamma_{\mathrm{cont}}(G, T\otimes\Lambda_{\mathcal{K},\infty})\simeq\mathrm{R}\Gamma_{\mathrm{cont}}(G, T\otimes\Lambda_{\mathcal{K},n}).$$

Definition 4.15. *We define the congruence number c_f of f. Consider the localized Hecke algebra $\mathbb{T}_{\mathfrak{m}_f}$ acting on the space of $\mathcal{O}_L$-valued cusp forms with respect to $\Gamma_0(N)$, where $\mathfrak{m}_f$ is the maximal ideal corresponding to f. Then $\mathbb{T}_{\mathfrak{m}_f}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\simeq L\oplus B$ for some algebra B, where the L-corresponds to projecting to f-component. Let 1_f be the idempotent corresponding to this L. On the other hand we write ℓ_f for the generator of the rank one $\mathcal{O}_L$-module $\mathbb{T}_{\mathfrak{m}_f}\cap L$. Let $c_f:=\frac{\ell_f}{1_f}$.*

Now we define several p-adic L-functions.

Definition 4.16. *We define a Rankin-Selberg p-adic L-function $\mathcal{L}_{f\otimes\mathbf{g}}$ (here we fix one Hecke eigenvalue α of f at p). Notice the difference from the previously defined $\mathcal{L}^{\mathrm{Hida}}_{f\otimes\mathbf{g}}$ which interpolates critical Rankin-Selberg L-values where the specializations of $\mathbf{g}$ has weight higher than f. We multiply the "geometric" p-adic L-function for the Rankin-Selberg product $f\otimes\mathbf{g}$ constructed in [25, Appendix], interpolating the critical values of the Rankin-Selberg L-values of f and specializations of $\mathbf{g}$ whose weight is less than the weight of f. Then we multiply it by the congruence number of f and denote the product as $\mathcal{L}_{f\otimes\mathbf{g}}$. In the special case here where $\mathbf{g}$ comes from families of characters of $\Gamma_{\mathcal{K}}$ we also denote it as $\mathcal{L}_{\alpha\alpha}$. (Note that the period for the "geometric" p-adic L-function is the Petersson inner product of f with itself. Such Petersson inner product, divided by the congruence number of f is the so called canonical period of f.)*

We also define the p-adic L-function $\mathcal{L}_\alpha(f)$ of f_α over $\mathbb{Q}$ by requiring that for any finite order character χ of Γ with conductor p^n ($n\geq 2$) and $r+\frac{k-2}{2}\in[1,k-1]$,

$$\mathcal{L}_\alpha(f)(\epsilon^r\chi^{-1})=\frac{(r+\frac{k-2}{2}-1)!p^{n(r+\frac{k-2}{2})}\alpha^{-n}\mathfrak{g}(\chi)^{-1}}{(2\pi i)^{\frac{k-2}{2}+r}\Omega_f^{(-1)^{\frac{k}{2}-r}}}L^{\{p\}}(f,\chi,r+\frac{k-2}{2}).$$

The reciprocity law in *loc.cit* implies that under the convention at the end of Section 4.1,

$$\langle \mathrm{Reg}_{\bar{v}_0,\mathcal{F}^-}(\mathrm{BF}_\alpha), \eta_f\rangle = \mathcal{L}_{\alpha\alpha}/c_f, \tag{14}$$

$$\langle \mathrm{Reg}_{v_0,\mathcal{F}^+}(\mathrm{BF}_\alpha), \omega_f\rangle = \mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}} \cdot \frac{1}{h_{\mathcal{K}}\mathcal{L}^{\mathrm{Katz}}_{\mathcal{K}}} \cdot \frac{v^-}{c\cdot v^+}. \tag{15}$$

We say one word about different conventions about the $\omega^\vee_{\mathbf{g}}$ and $\eta^\vee_{\mathbf{g}}$: in [25] the $D(\mathcal{F}^-\mathcal{M}^*_{\mathbf{g}})$ is identified with the $(\mathcal{F}^-\mathcal{M}^*_{\mathbf{g}}\otimes\hat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}$ via a choice of the $\rho(d)$. (In fact there is a gap in the construction of Urban's Ranin-Selberg p-adic L-functions as noted in the appendix of [25]. But as the authors informed us, they have been able to resolve the problem [24] by proving interpolation formulas of their "geometric" p-adic L-functions at all points we are interested in.) We need the following

Lemma 4.17. *If we identify the coefficient ring of* $\mathbf{g}$ *with* $\mathbb{Z}_p[[U]]$*. Then up to some powers of* U *we have*

$$\frac{h_{\mathcal{K}}\mathcal{L}^{\mathrm{Katz}}_{\mathcal{K}} c\cdot v^+}{v^-}$$

is in $\mathbb{Z}_p^{\mathrm{ur}}[[U]]$.

This is explained in [44, Proposition 8.3].

Note that by an easy argument using global duality (see [35, 5.1.6]), the $\Lambda_{\mathcal{A},\infty}$-characteristic ideal of $\tilde{H}^2(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathcal{A},\infty})$ is exactly the base change to $\Lambda_{\mathcal{A},\infty}$ of that for $X^{\mathrm{Gr}}_{f,\mathcal{K}}$. So by (12) and (13) we arrived at the following

Proposition 4.18. *Let* P *be a height one prime of* $\Lambda_{\mathcal{K},\infty}$*, which is not in* $S_{v_0}(\mathscr{F}^+(D))$ *(the subscript* v_0 *means identifying* $\mathcal{K}_{v_0}$ *with* $\mathbb{Q}_p$*) or* $S_{\bar{v}_0}(\mathscr{F}^-(D))$*, then*

$$\mathrm{ord}_P \mathrm{char}_{\Lambda_{\mathcal{K},\infty}}(X_{\alpha,\alpha}) \geq \mathrm{ord}_P\mathcal{L}_{\alpha\alpha}.$$

Note that the pullback of U in Lemma 4.17 is a height one prime which does not contain $\mathcal{L}_{\alpha\alpha}$, since the latter is not identically 0 on the cyclotomic line.

Remark 4.19. *Note that the parameter* U *and* $\mathcal{A}$ *at* v_0 *and* $\bar{v}_0$ *are different as parameters in* $\Gamma_{\mathcal{K}}$*. In fact one can prove that the set* $S_{v_0}(\mathscr{F}^+(D))$ *and* $S_{\bar{v}_0}(\mathscr{F}^-(D))$ *are the same. But we will not need it in this paper.*

So we have

$$\mathrm{ord}_P\mathrm{Fitt}(X_{\alpha\alpha}) \geq \mathrm{ord}_P\mathcal{L}_{\alpha\alpha}$$

for those primes. Observe that when specializing to the cyclotomic line any prime P in $S_{v_0}(\mathscr{F}^+(D))$ or $S_{\bar{v}_0}(\mathscr{F}^-(D))$ specializes to the trivial ideal of Λ_∞. So by Proposition 4.11 we have the

Corollary 4.20.

$$\mathrm{char}_{\Lambda_{\mathbb{Q},\infty}}(X_{\alpha\alpha}) \subseteq (\mathcal{L}_{\alpha\alpha}).$$

To save notation we also write $\mathcal{L}_{\alpha\alpha}$ for the specialization of $\mathcal{L}_{\alpha\alpha}$ to the cyclotomic line. In order to relate this to Kato's main conjecture we need to study the relations between $\mathcal{L}_{\alpha\alpha}$ and $\mathcal{L}_\alpha(f)\cdot\mathcal{L}_\alpha(f^{\chi_{\mathcal{K}}})$. Note that although we know they are equal at all arithmetic points, however these interpolation formulas do not determine the element in $\Lambda_{\mathbb{Q},\infty}$ uniquely. So we need to prove the

Lemma 4.21. *Up to multiplying by a non-zero constant we have*

$$\mathcal{L}_{\alpha\alpha} = \mathcal{L}_\alpha(f) \cdot \mathcal{L}_\alpha(f^{\chi_\mathcal{K}}).$$

Proof. There are several ways of proving this and we only give one. The eigencurve machinery implies we can deform f_α into a Coleman family F. Then Bellaiche constructed [2] the corresponding two variable p-adic L-functions $\mathcal{L}_{F,\mathbb{Q}}$ and $\mathcal{L}_{F\otimes\chi_\mathcal{K},\mathbb{Q}}$ which specializes to $\mathcal{L}_\alpha(f)$ and $\mathcal{L}_\alpha(f\otimes\chi_\mathcal{K})$. Also there is the Rankin-Selberg three-variable p-adic L-function constructed by Loeffler-Zerbes [25] as the "geometric" p-adic L-function $\mathcal{L}(F\otimes\mathbf{g})$ (whose weigth space contains the weight space for $\mathcal{L}(F)$ and $\mathcal{L}(F\otimes\chi_\mathcal{K})$ as closed subspace) which specializes to $\mathcal{L}_{\alpha\alpha}$. Note that the Loeffler-Zerbes included the interpolation formulas when the specialization of $\mathbf{g}$ has weight one. There is a p-adically dense set of arithmetic points (crystalline points, see [25]) in the two-variable weight space for $\mathcal{L}_{\alpha\alpha}$ where the specializations of $\mathcal{L}(F\otimes\mathbf{g})$ and $\mathcal{L}(F)\cdot\mathcal{L}(F\otimes\chi_\mathcal{K})$ are equal. Here one small subtlety is the different period in each case – the canonical period for each specialization of F, or the product of the $\pm$ periods. However at least locally the ratio of such periods is interpolated as rigid analytic functions, say by taking the ratio of the two p-adic L-functions above. So we multiply one p-adic L-function by this ratio and this product should be equal to the second p-adic L-function. This specialization to f of this ratio is the constant mentioned in the lemma (which is just the non-zero number $\frac{\Omega_f^{\mathrm{can}}}{\Omega_f^+\Omega_f^-}$). So as rigid analytic functions the two variable p-adic L-functions $\mathcal{L}(F\otimes\mathbf{g})$ and $\mathcal{L}(F)\cdot\mathcal{L}(F\otimes\chi_\mathcal{K})$ must be equal up to multiplying by this constant. Thus we get the corollary. □

It is also easy to see that

$$\tilde{R\Gamma}_\alpha(G_{\mathcal{K},S}, T\otimes\Lambda_{\mathbb{Q},\infty}, f) \simeq \tilde{R\Gamma}_\alpha(G_{\mathbb{Q},S}, T\otimes\Lambda_{\mathbb{Q},\infty}, f) \oplus \tilde{R\Gamma}_\alpha(G_{\mathbb{Q},S}, T\otimes\Lambda_{\mathbb{Q},\infty}, f^{\chi_\mathcal{K}}).$$

The following theorem is proved by Pottharst in [37, Theorem 5.4], which is essentially a reformulation of Kato's theorem.

Theorem 4.22.

$$\mathrm{char}_{\Lambda_{\mathbb{Q},\infty}}(X_{\alpha\alpha}(f,\mathbb{Q})) \supseteq (\mathcal{L}_\alpha(f)).$$

Moreover the "=" is equivalent to Conjecture 1.1 after inverting p.

Combining what we have proved with Kato's theorem we have

Theorem 4.23.

$$\mathrm{char}_{\Lambda_{\mathbb{Q},\infty}}(X_{\alpha\alpha}(f,\mathbb{Q})) = (\mathcal{L}_\alpha(f)).$$

4.4 Powers of p

We briefly discuss the ω_f and η_f in [25]. We refer to [10, Section 2.1, 2.2] for the background of the motive $\mathcal{M}_k/_{\mathcal{O}_L}$ associated to the space of weight k modular forms. This comes from the $k-2$-th symmetric power of the universal elliptic curve over the modular curve. Suppose $k < p$ (i.e. the Fontaine-Laffaille range). Let D_{FL} be the Fontaine-Laffaille functor. Then by the comparison theorem of Faltings (as noted in *loc.cit*), we have

$$D_{\mathrm{FL}}(H^1_{\mathrm{t}}(\mathcal{M}_k)) = H^1_{\mathrm{dR}}(\mathcal{M}_k).$$

(See [30, Section 6.10]). The Galois representation T_f is realized as $H^1_{\mathrm{t}}(\mathcal{M}_k)[\lambda_f]$ (meaning the maximal submodule of $H^1_{\mathrm{t}}(\mathcal{M}_k)$ on which the Hecke algebra $\mathbb{T}_0(N)$ is acting via its action λ_f on f). Thus

$$D_{\mathrm{FL}}(T_f) = H^1(\mathcal{M}_k)[\lambda_f].$$

On the other hand we have an exact sequence of Hecke modules (localized at the maximal ideal $\mathfrak{m}_f \subset \mathbb{T}_0(N)$):

$$0 \to H^0(X_0(N), \omega^k/\mathcal{O}_L)_{\mathfrak{m}_f} \to H^1_{\mathrm{dR}}(\mathcal{M}_k)_{\mathfrak{m}_f} \to H^1(X_0(N), \omega^{2-k}/\mathcal{O}_L)_{\mathfrak{m}_f} \to 0.$$

Here ω^k is the weight k automorphic sheave. As Hecke modules the next to last term is free of rank one over $\mathbb{T}_0(N)_{\mathfrak{m}_f}$ and the second term is isomorphic to $S(X_0(N), \mathcal{O}_L)_{\mathfrak{m}_f}$ ($\mathcal{O}_L$-valued cusp forms). Unravelling the definitions the $\eta_f \in H^1(X_0(N), \omega^{2-k}/\mathcal{O}_L)_{\mathfrak{m}_f} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ in [25] corresponds to the f-component projector 1_f under the identification of $H^1(X_0(N), \mathbb{Z}_p)_{\mathfrak{m}_f} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with $\mathbb{T}_0(N)_{\mathfrak{m}_f}$, while

$$\frac{D_{\mathrm{FL}}(T_f)}{\mathrm{Fil}^0 D_{\mathrm{FL}}(T_f)} \simeq H^1(X_0(N), \omega^{2-k})_{\mathfrak{m}_f}[\lambda_f].$$

So by definition the ratio of a generator of $\frac{D_{\mathrm{FL}}(T_f)}{\mathrm{Fil}^0 D_{\mathrm{FL}}(T_f)}$ over η_f is the *congruence number* c_f of f (determined up to a p-adic unit).

Remark 4.24. *Note that up to multiplying by a p-adic unit, the canonical period Ω_f^{can} is the Petersson inner product period in the Rankin-Selberg p-adic L-function $\mathcal{L}_{f\otimes\mathbf{g}}$ divided by this c_f.*

Definition 4.25. *We say an arithmetic point $\tilde{\phi} \in \mathcal{X}$ is generic if $L(f, \chi_{\tilde{\phi}}, k/2) \neq 0$ and $\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}|_{\mathcal{Y}_{\tilde{\phi}}}$ is not identically* 0. *It is clear that all but finite many arithmetic points are generic.*

We write (v_1, v_2) for an $\mathcal{O}_{L_n}$-basis of $H^1(\mathbb{Q}_p, T_{\tilde{\phi}})$ such that v_1 is a generator of $H^1_f(\mathbb{Q}_p, T_{\tilde{\phi}})$. We sometimes write them as $v_{1,\bar{v}_0}, v_{2,\bar{v}_0}$. Let $\tilde{\phi}$ be such that $\tilde{\phi}$ and $\tilde{\phi}^{-1}$ are both generic. Write $\Gamma_{\mathcal{K}} = \Gamma_{cyc} \times \Gamma_{\bar{v}_0}$ and let $\mathrm{pr} : \mathcal{Y} := \mathrm{Spec}\ \mathbb{Z}_p[[\Gamma_{cyc} \times \Gamma_{\bar{v}_0}]] \to \mathcal{X} := \mathrm{Spec}\ [[\Gamma_{cyc}]]$ be the natural projection. For $\tilde{\phi} \in \mathcal{X}$ let

$$\mathcal{Y}_{\tilde{\phi}} = \mathcal{Y} \otimes_{\mathcal{X},\tilde{\phi}} A(\tilde{\phi}).$$

We consider $\mathcal{Y}_{\tilde{\phi}}$. Define an element $\mathcal{L}^1_{\tilde{\phi}} \in \mathcal{O}_L[[\Gamma_{v_0}]]$ such that

$$\mathrm{BF}|_{\mathcal{Y}_{\tilde{\phi}}} \equiv (\mathcal{L}^1_{\tilde{\phi}})(G(\chi^{-1}_{\tilde{\phi}})(\frac{\beta_f}{p^{1+j}})^r) v_{2,\bar{v}_0} (\mathrm{mod}\ v_{1,\bar{v}_0}). \tag{16}$$

Then $\mathcal{L}^1_{\tilde{\phi}}(0) \neq 0$. For any $\tilde{\phi}$ of conductor p^r and such that $\chi_{\tilde{\phi}}\epsilon^{1+j-\frac{k}{2}}$ has finite order, we have

$$\alpha_f^{2r} \cdot \frac{\tilde{\phi}(\mathcal{L}_{f_\alpha}\mathcal{L}_{f_\alpha^{\chi_{\mathcal{K}}}})}{G(\chi_{\tilde{\phi}})^2 p^{2rj}} = \frac{L_{\mathcal{K}}(f, 1+j, \chi^{-1}_{\tilde{\phi}})}{(2\pi i)^{2+2j}\Omega_f^{\mathrm{can}}}.$$

We also have for any $\phi \in \mathcal{Y}_{\tilde{\phi}}$

$$\log_{v_0} \phi(\mathrm{BF}) = \phi(\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}) G(\chi^{-1}_{\tilde{\phi}}) \cdot (\frac{\alpha_g\beta_f}{p^{1+j}})^r \omega_f^\vee. \tag{17}$$

Note that β_f, j, r only depends on $\tilde{\phi}$, $\alpha_{\mathbf{g}}$ is an element in $\mathbb{Z}_p[[\Gamma_{v_0}]]$ such that $\alpha_{\mathbf{g}}(0) = 1$. Recall also the remark right after (4.3) and (15) for the role played by $\omega_{\mathbf{g}}^\vee$ and $\eta_{\mathbf{g}}^\vee$. We have

$$\exp^*_{\bar{v}_0} \tilde{\phi}(\mathrm{BF}) = \frac{1}{c_f}\tilde{\phi}(\mathcal{L}_{f_\alpha}\mathcal{L}_{f_\alpha^{\chi_\mathcal{K}}}) \cdot G(\chi_{\tilde{\phi}}^{-1}) \cdot (\frac{\alpha_f \beta_g}{p^{1+j}})^r \cdot \omega_f$$

where c_f is the congruence number we defined before. So

$$\begin{aligned}\exp^*_{\bar{v}_0} \tilde{\phi}(\mathrm{BF}) &= \frac{L_\mathcal{K}(f, 1+j, \chi_{\tilde{\phi}}) \cdot G(\chi_{\tilde{\phi}}^{-1})}{c_f(2\pi i)^{2+2j}\Omega_f^{\mathrm{can}}} G^2(\chi_{\tilde{\phi}}^{-1})(\frac{\alpha_f}{p^{1+j}})^r \cdot \omega_f p^{2rj}\alpha_f^{-2r} \\ &= \frac{L_\mathcal{K}(f, 1+j, \chi_{\tilde{\phi}}^{-1})}{c_f(2\pi i)^{2+2j}\Omega_f^{\mathrm{can}}} G(\chi_{\tilde{\phi}}^{-1})\alpha_f^{-r}p^{rj}\omega_f\end{aligned}$$

Thus

$$\exp^*_{\bar{v}_0} \tilde{\phi}(\mathrm{BF}) = \frac{L_\mathcal{K}(f, 1+j, \chi_{\tilde{\phi}}^{-1})}{c_f \Omega_f^{\mathrm{can}}}(G(\chi_{\tilde{\phi}}^{-1})(\frac{\beta_f}{p^{1+j}})^r) \cdot (p^{2jr+r}/p^{(k-1)r})\omega_f. \tag{18}$$

Similarly for ϕ above

$$\log_{v_0} \phi^{-1}(\mathrm{BF}) = \phi^{-1}(\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}})G(\chi_{\tilde{\phi}})(\frac{\alpha_g\beta_f}{p^{1+j}})^r\omega_f^\vee, \tag{19}$$

and

$$\exp^*_{\bar{v}_0} \tilde{\phi}^{-1}(\mathrm{BF}) = \frac{L_\mathcal{K}(f, k-1-j, \chi_{\tilde{\phi}^{-1}}^{-1})}{c_f(2\pi i)^{2k-2-2j}\Omega_f^{\mathrm{can}}}(G(\chi_{\tilde{\phi}})(\frac{\beta_f}{p^{1+j}})^r)(p^{2r(k-2-j)+r}/p^{(k-1)r})\omega_f. \tag{20}$$

We have

$$\mathcal{L}^1_{\tilde{\phi}}(0) = \frac{L_\mathcal{K}(f, 1+j, \chi_{\tilde{\phi}}^{-1})p^{2jr+r}/p^{(k-1)r}\omega_f}{c_f(2\pi i)^{2+2j}\Omega_f^{\mathrm{can}} \exp^* \tilde{\phi}(v_{2,\bar{v}_0}) \exp^* \tilde{\phi}^{-1}(v_{2,\bar{v}_0})}.$$

By the results in Section 2.4 the

$$\varprojlim_{\mathbb{Q}_p \subseteq K \subseteq \mathbb{Q}_p^{\mathrm{ur}}} H^1_f(K, T_{\tilde{\phi}})$$

is an $\mathcal{O}_{L_n}[[\Gamma_{\bar{v}_0}]]$-direct summand of

$$\varprojlim_{\mathbb{Q}_p \subseteq K \subseteq \mathbb{Q}_p^{\mathrm{ur}}} H^1(K, T_{\tilde{\phi}}) \simeq \mathcal{O}_{L_n}[[\Gamma_{\bar{v}_0}]]^2.$$

We define the "unramified" local Selmer condition at v_0 (as well as its dual under Tate duality) by this

$$\varprojlim_{\mathbb{Q}_p \subseteq K \subseteq \mathbb{Q}_p^{\mathrm{ur}}} H^1_f(K, T_{\tilde{\phi}})$$

along the one-variable family over $\mathcal{O}_{L_n}[[\Gamma_{\bar{v}_0}]]$.

Now we look at the following exact sequences of $\mathcal{O}_{L_n}[[U]]$-modules

$$0 \to H^1_{\mathrm{ur,rel}}(\mathcal{K}_S, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}}) \to H^1_{\mathrm{ur}}(\mathcal{K}_{v_0}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}}) \to X_{0,\mathrm{rel}} \to X_{0,\mathrm{ur}} \to 0 \tag{21}$$

$$0 \to H^1_{\mathrm{ur,rel}}(\mathcal{K}_S, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}}) \to \frac{H^1(\mathcal{K}_{\bar{v}_0}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}})}{\mathcal{O}_{L_n}[[U]]v_1} \to X_{v_1,\mathrm{ur}} \to X_{0,\mathrm{ur}} \to 0. \tag{22}$$

By Corollaries 2.14 and 2.15 there is an isomorphism

$$H^1_{\mathrm{ur}}(\mathcal{K}_{v_0}, T \otimes \Lambda_{\mathcal{K}}|_{\mathcal{Y}_{\tilde{\phi}}}) \simeq \mathcal{O}_{L_n}[[U]]$$

which interpolates $c'_{\tilde{\phi},p}$ times the logarithmic map, divided by the specializations of $\rho(d)$ there. Now Theorem 1.7 and our (6), (17),(18),(19), (20) and (21) implies that

$$\begin{aligned} & c_f^2 c'_{\tilde{\phi},p}(f) \cdot c'_{\tilde{\phi}^{-1},p}(f) \mathrm{char}_{\mathcal{O}_{L_n}[[U]]}(X_{v_1,\mathrm{ur},\tilde{\phi}}) \mathrm{char}_{\mathcal{O}_{L_n}[[U]]}(X_{v_1,\mathrm{ur},\tilde{\phi}^{-1}}) \\ & \subseteq \mathrm{char}_{\mathcal{O}_{L_n}[[U]]}(\frac{H^1_{\mathrm{ur,rel}}(\mathcal{K}_S, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}})}{\mathcal{O}_{L_n}[[U]]\mathrm{BF}_{\alpha,\tilde{\phi}}}) \cdot \mathrm{char}_{\mathcal{O}_{L_n}[[U]]}(\frac{H^1_{\mathrm{ur,rel}}(\mathcal{K}_S, T_{\tilde{\phi}^{-1}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}^{-1}}})}{\mathcal{O}_{L_n}[[U]]\mathrm{BF}_{\alpha,\tilde{\phi}^{-1}}}). \end{aligned}$$

Here in applying Theorem 1.7 we have used the following argument: the two variable $\mathrm{char}(X^{\mathrm{Gr}}_{\mathcal{K},f}) \subseteq (\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}})$ implies two variable $\mathrm{Fitt}(X^{\mathrm{Gr}}_{\mathcal{K},f}) \subseteq (\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}})$, which implies one variable $\mathrm{Fitt}(X^{\mathrm{Gr}}_{\mathcal{K},f}) \subseteq (\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}})$, which implies the one variable $\mathrm{char}(X^{\mathrm{Gr}}_{\mathcal{K},f}) \subseteq (\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}})$ by that $\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]$ is a normal domain. Note that the powers of U in Lemma 4.17 does not have effect the argument since the specialization of BF at $\tilde{\phi}$ is not zero. The c_f comes from the discussion in Remark 4.24. Also pay attention to that the $p^{2jr+r}/p^{(k-1)r}$ at $\tilde{\phi}$ and $\tilde{\phi}^{-1}$ cancel out. We note that any prime v of $\mathcal{K}$ not dividing p is finitely decomposed in the $\mathbb{Z}_p$-extension $\mathcal{K}^{\bar{v}_0}/\mathcal{K}$, and is completely split in $\mathcal{K}_\infty/\mathcal{K}^{\bar{v}_0}$. Specializing to $U \to 0$, applying (22), and applying the control theorem as in Section 4.2 we obtain (the v may or may not divide p)

$$\begin{aligned} & \prod_v c'_{\mathcal{K},\tilde{\phi},v}(f) \prod_v c'_{\mathcal{K},\tilde{\phi}^{-1},v}(f) \mathrm{char}_{\mathcal{O}_L}(X_{v_1,v_1,\tilde{\phi}}) \mathrm{char}_{\mathcal{O}_L}(X_{v_1,v_1,\tilde{\phi}^{-1}}) \\ & \subseteq (\frac{L_{\mathcal{K}}(f, 1+j, \chi^{-1}_{\tilde{\phi}})}{(2\pi i)^{2+2j}\Omega^{\mathrm{can}}_f})(\frac{L_{\mathcal{K}}(f, k-1-j, \chi^{-1}_{\tilde{\phi}^{-1}})}{(2\pi i)^{2k-2-2j}\Omega^{\mathrm{can}}_f}). \end{aligned} \tag{23}$$

Here the subscript $\mathcal{K}$ means the local Tamagawa numbers over $\mathcal{K}$. It is known that $c_{\mathcal{K},\ell,\phi}(f) = c_{\ell,\phi}(f) \cdot c_{\ell,\phi}(f^{\chi_{\mathcal{K}}})$. The last preparation is the following

Lemma 4.26. *The $\Omega^{\mp}_{f^{\chi_{\mathcal{K}}}}$ is an $\mathcal{O}_L$-multiple of $\Omega^{\pm}_f$.*

Proof. We prove it following the first half of [42, Lemma 9.6] (note that the running assumption here is slightly different from [42]). The f is a new form of level N and $f^{\chi_{\mathcal{K}}}$ is a new form whose conductor is some M which is a divisor of ND^2. Consider the map

$$H^1(\Gamma_0(N), L^{k-2}_{/\mathcal{O}_L})_{\mathfrak{m}_f} \to H^1(\Gamma_0(ND^2), L^{k-2}_{/\mathcal{O}_L})_{\mathfrak{m}_f}$$

constructed in *loc.cit.* The $\omega^{\pm}_f$ is mapped to $\omega^{\mp}_{f^{\chi_{\mathcal{K}}}}$ under this map, and $\gamma^{\pm}_f$ is mapped to some multiple of $\omega^{\mp}_{f^{\chi_{\mathcal{K}}}}$. Note $\mathfrak{m}_f$ is non-Eisenstein. So the image of $\gamma^{\pm}_f$ are elements in $H^1(\Gamma_0(M), L^{k-2}_{/\mathcal{O}_L})^{\mp}_{\mathfrak{m},f}$. Thus $\mathrm{im}(\gamma^{\pm}_f)$ are $\mathcal{O}_L$-multiples of $\gamma^{\mp}_{f^{\chi_{\mathcal{K}}}}$. □

We can prove the following

Corollary 4.27. *With the assumptions of Theorem 1.4 part two, we have*

$$\mathrm{char}_\Lambda(\frac{H^1_{\mathrm{Iw}}(\mathbb{Q},T)}{\Lambda z_{\mathrm{Kato}}})\supseteq \mathrm{char}_\Lambda(X_{\mathrm{str}}).$$

Proof. This corollary follows from (23: with the notation as Section 4.2, let $A\in\Lambda$ be the image of $z_{\mathrm{Kato},f}$ in $\frac{H^1(\mathbb{Q}_p,\mathbf{T})}{\Lambda\mathbf{v}}$ upon the choice of a basis of the latter, and B the image of $z_{\mathrm{Kato},f\chi_{\mathcal{K}}}$ in $\frac{H^1(\mathbb{Q}_p,\mathbf{T})}{\Lambda\mathbf{v}}$. Let $\mathscr{F}_1=A\cdot B$. Then by Kato's result and what we proved, $(\mathscr{F}_1)=(\mathscr{F})$ as ideals of Λ up to powers of p. On the other hand Kato proved that

$$\exp^*\tilde{\phi}(z_{\mathrm{Kato}})=\frac{L_{\mathcal{K}}(f,1+j,\chi_{\tilde{\phi}}^{-1})}{(2\pi i)^{1+j}\Omega_f^{(-1)^j}}\omega_f.$$

We know that up to multiplying by a p-adic unit we have

$$\Omega_f^{\mathrm{can}}=\Omega_f^+\Omega_f^-$$

by the work in [14]. (One uses the pairing constructed in *loc.cit* and work of Hida, see e.g. [42, Lemma 9.5]. The key is the freeness over the local Hecke ring $\mathbb{T}_{\mathfrak{m}_f}$ of $H^1(\Gamma_0(N),L^{k-2}_{/\mathcal{O}_L})_{\mathfrak{m}_f}$, which is proved in [14] when the weight is in the Fontaine-Laffaille range). Therefore, the discussion at the end of Section 4.2 gives the corollary (Lemma 4.26 is also used here). □

Remark 4.28. *We do need to argue with the one variable family over $\mathcal{O}_{L_n}[[\Gamma_{\bar{v}_0}]]$ instead of only look at the one point $\tilde{\phi}$, because we cannot prove that $\mathcal{L}^{\mathrm{Gr}}_{f,\mathcal{K}}$ is not identically zero along the cyclotomic line.*

The following corollary reproves an early result of [41].

Corollary 4.29. *Assumptions are as part one of Theorem 1.4. If $L(f,k/2)=0$, then* $\mathrm{corank}\mathrm{Sel}_{f,\mathbb{Q}}\geq 1$.

Proof. Write ϕ_0 for the point in $\mathrm{Spec}\Lambda$ corresponding to $T\mapsto 0$. If the specialization of z_{Kato} to ϕ_0 is 0, then $T\in\mathrm{char}(\frac{H^1_{Iw}(\mathbb{Q},T)}{\Lambda z_{\mathrm{Kato}}})$, which implies $T\in\mathrm{char}(X_{str})$. By the control theorem of strict Selmer groups, we see $\mathrm{corank}\mathrm{Sel}_{E,\mathbb{Q}}\geq 1$. If the specialization of z_{Kato} to ϕ_0 is non-zero, then by our assumption that $L(f,k/2)=0$ it generates a non-zero element in $H^1_f(\mathbb{Q},V_f)$. Then it is easily seen that $\mathrm{corank}\mathrm{Sel}_{f,\mathbb{Q}}\geq 1$. □

Finally we prove the Tamagawa number conjecture of Bloch-Kato in the case when the central critical value of f is non-zero. Recall by our assumption (Irred) the module $H^1(\mathbb{Q}_p,T)$ is free of rank two over $\mathcal{O}_L$.

Combining Theorem 1.4, Kato's result and the control theorem of Selmer groups we get the following

Corollary 4.30. *Assumptions are as in part two of Theorem 1.4. Assume moreover that the image of $G_{\mathbb{Q}}$ in $\mathrm{Aut}(T_f)$ contains $\mathrm{SL}_2(\mathbb{Z}_p)$. If $L(f,\frac{k}{2})\neq 0$, then the full Iwasawa main conjecture for T is true, and up to multiplying by a p-adic unit we have*

$$\mathrm{Tam}_p(T)\prod_{l|N}c_l(T)\cdot\sharp\mathrm{Sel}_{p^\infty}(T)=\sharp(\frac{\mathcal{O}_L}{(\frac{L(f,\frac{k}{2})}{(2\pi i)^{\frac{k}{2}}\Omega_f^{(-1)^{\frac{k}{2}-1}}})\mathcal{O}_L}).$$

We remark that under the Fountain-Laffaille assumption $k < p$, the (Irred) is automatic thanks to a result of Edixhoven [11].

Proof. Recall that $\mathrm{Tam}_p(T)$ is defined by

$$\sharp(\frac{\mathcal{O}_L}{c'_{P,p}/\det(1-\varphi|D_{\mathrm{cris}}(V))\mathcal{O}_L})$$

for $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. On the other hand

$$\exp^* \phi_0(z_{\mathrm{Kato}}) = \frac{L^{\{p\}}(f, \frac{k}{2})}{(2\pi i)^{\frac{k}{2}} \Omega_f^{(-1)^{\frac{k}{2}-1}}} \omega_f$$

where the $\{p\}$ means removing the Euler factor at p. We use our argument in the Section on control theorems, taking the P there to correspond to the trivial character and take some auxiliary $\mathbf{v}$ as in there, and prove the $X_{\mathbf{v}^\vee}$ has no pseudo-null submodules. These altogether gives the result. $\square$

References

[1] Adams, Jeffrey. “The theta correspondence over $\mathbb{R}$.” Harmonic Analysis, Group Representations, Automorphic Forms, and Invariant Theory (2007): 1-39.

[2] Joël Bellaïche. Critical p-adic L-functions, Inventiones Mathematicae, 2012, 189(1): pages. 1-60.

[3] Bellaïche, Joël. “Ranks of Selmer groups in an analytic family.” Trans. Amer. Math. Soc. 364. 9 (2012): 4735âĂŞ4761.

[4] Joël Bellaïche, Gaetan Chenevier, p-adic families of Galois representations and higher rank Selmer groups arXiv:math/0602340.

[5] Berger, Laurent. “Nombres de Tamagawa de Certaines Representations Cristallines.” Preprint, `http://perso.ens-lyon.fr/laurent.berger/autrestextes/tamagawa.pdf`.

[6] Bhargav Bhatt, Matthew Morrow, Peter Scholze, Integral p-adic Hodge Theory, preprint, arXiv:1602.03148.

[7] Ashay Burungale, On the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p, II: Shimura curves, arXiv:1504.02342.

[8] G.Chenevier, Sur la densite des representations cristallines de $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, Math. Ann. (2012).

[9] Delbourgo, Daniel. “On the p-Adic Birch, Swinnerton-Dyer Conjecture for Non-semistable Reduction.” Journal of Number Theory 95.1 (2002): 38-71.

[10] Fred Diamond , Matthias Flach , Li Guo, Adjoint motives of modular forms and the Tamagawa number conjecture, Annales Scientifiques De LâĂŹecole Normale Superieure, 2004, 37(5):663-727.

[11] Edixhoven, Bas. The weight in Serre’s conjectures on modular forms. Inventiones mathematicae 109.1 (1992): 563-594.

[12] E. Eischen and X. Wan, p-adic L-Functions of Finite Slope Forms on Unitary Groups and Eisenstein Series, to appear in Journal of the Institute of Mathematics of Jussieu.

[13] Gerd Faltings, Crystalline Cohomology and p-adic Etale Cohomology, in Algebraic Analysis, Geometry, and Number Theory, the Johns Hopkins University Press (1989), 25-80.

[14] G.Faltings, B.Jordan, Crystalline Cohomology and $\mathrm{GL}(2,\mathbb{Q})$, Israel Journal of Mathematics, October 1995, Volume 90, Issue 1, pp 1âĂŞ66

[15] Fontaine, Jean-Marc, and Yi Ouyang. "Theory of p-adic Galois representations." preprint, http://staff.ustc.edu.cn/ yiouyang/galoisrep.pdf.

[16] Hida, H: On p-adic L-functions of $\mathrm{GL}(2) \times \mathrm{GL}(2)$ over totally real fields. Ann. Inst. Fourier 40, 311-391 (1991).

[17] M.-L. Hsieh, Special values of anticyclotomic Rankin-Selbeg L-functions. Documenta Mathematica, 19 (2014), 709-767.

[18] H. Hida, J. Tilouine, Anti-cyclotomic Katz p-adic L-functions and congruence modules, Ann. Sci. Ecole Norm. Sup. (4) 26 (2) (1993) 189-259.

[19] Hu, Yueke. "Triple product formula and mass equidistribution on modular curves of level N." arXiv preprint arXiv:1409.8173 (2014).

[20] Ikeda, Tamotsu. "On the theory of Jacobi forms and Fourier-Jacobi coefficients of Eisenstein series." Journal of Mathematics of Kyoto University 34.3 (1994): 615-636.

[21] D.Jetchev, C.Skinner and X.Wan, The Birch-Swinnerton-Dyer Formula For Elliptic Curves of Analytic Rank One, Joint with Dimitar Jetchev and Christopher Skinner, arXiv:1512.06894.

[22] K. Kato, p-adic Hodge theory and values of zeta functions of modular forms, Cohomologies p-adiques et applications arithmetiques. III. Asterisque No. 295 (2004), ix, p. 117-290.

[23] KS Kedlaya, J Pottharst, L Xiao, Cohomology of arithmetic families of (φ, Γ)-modules, Journal of the American Mathematical Society, 2014, 27(4):1043-1115.

[24] David Loeffler, and Sarah Livia Zerbes, work in preparation.

[25] David Loeffler, and Sarah Livia Zerbes, Rankin-Eisenstein classes in Coleman families, preprint, arXiv: 1506.06703 (2015).

[26] Kings, Guido, David Loeffler, and Sarah Livia Zerbes. "Rankin-Eisenstein classes for modular forms." arXiv preprint arXiv:1501.03289 (2015).

[27] Kings, Guido, David Loeffler, and Sarah Livia Zerbes. "Rankin–Eisenstein classes and explicit reciprocity laws." arXiv preprint arXiv:1503.02888 (2015).

[28] Kobayashi, Shin-ichi. "Iwasawa theory for elliptic curves at supersingular primes." Inventiones mathematicae 152.1 (2003): 1-36.

[29] Lei, Antonio, Bounds on the Tamagawa numbers of a crystalline representation over towers of cyclotomic extensions, arXiv:1510.06450.

[30] Lei, Antonio, David Loeffler, and Sarah Livia Zerbes. "Euler systems for Rankin-Selberg convolutions of modular forms." Annals of Mathematics, Vol 180(2), 653-771.

[31] Loeffler, David, and Sarah Livia Zerbes. "Iwasawa theory and p-adic L-functions over $\mathbb{Z}_p^2$-extensions." arXiv preprint arXiv:1108.5954 (2011).

[32] Loeffler, David, and Sarah Livia Zerbes. "Rankin–Eisenstein classes in Coleman families." arXiv preprint arXiv:arXiv:1506.06703 (2015).

[33] Milne, James S. Arithmetic duality theorems. Charleston: BookSurge, 2006.

[34] K.Nakamura, Iwasawa Theory of de Rham (φ, Γ)-modules over the Robba Ring, Journal of the institute of mathematics of Jussieu 13.01 (2014): 65-118.

[35] Jan Nekovar, Selmer Complexes, Astérisque, vol. 310 (2006), 559 pages.

[36] Ohta, Masami. "Ordinary p-adic etale cohomology groups attached to towers of elliptic modular curves. II." Mathematische Annalen 318.3 (2000): 557-583.

[37] J.Pottharst, Cyclotomic Iwasawa Theory of Motives, preprint 2012.

[38] Pottharst, Jonathan. "Analytic families of finite-slope Selmer groups." Algebra and Number Theory 7.7 (2013): 1571-1612.

[39] Rohrlich, David E. "L-functions and division towers." Mathematische Annalen 281.4 (1988): 611-632.

[40] Skinner, Christopher, and Eric Urban. "The Iwasawa main conjectures for GL_2." Inventiones mathematicae 195.1 (2014): 1-277.

[41] C.Skinner and E.Urban, Vanishing of L-functions and Ranks of Selmer Groups, ICM 2006.

[42] C.Skinner, W.Zhang, Indivisibility of Heegner points in the multiplicative case, arXiv:1407.1099.

[43] X.Wan, Iwasawa Main Conjecture for Supersingular Elliptic Curves, preprint, 2015.

[44] X.Wan, Iwasawa Main Conjecture for Rankin-Selberg p-adic L-functions, submitted, arXiv:1408.4044, 2014.

[45] X. Wan, Families of Nearly Ordinary Eisenstein Series on Unitary Groups, to appear in Algebra and Number Theory.

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