# GENERAL AFFINE INVARIANCES RELATED TO MAHLER VOLUME

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Abstract. General affine invariances related to Mahler volume are introduced. We establish their affine isoperimetric inequalities by using a symmetrization scheme that involves a total of $2n$ elaborately chosen Steiner symmetrizations at a time. The necessity of this scheme, as opposed to the usual Steiner symmetrization, is demonstrated with an example (see the Appendix).

## 1. Introduction

The study of affine isoperimetric inequalities is fundamental in modern convex geometry. Different from their classical Euclidean relatives, affine isoperimetric inequalities compare two *affine invariant* (invariant under special linear transforms) functionals associated with convex bodies (or more general sets). The survey articles of Lutwak [32] and Zhang [54] are good references related to this topic.

An affine invariant quantity of immense interest is the *Mahler volume*

$$|K||K^*|.$$

Here $K$ is an $o$-symmetric (symmetric with respect to the origin $o$) convex body (compact convex set with non-empty interior), $|K|$ denotes its volume and $K^*$ its polar body defined as $\{x \in \mathbb{R}^n : x \cdot y \leq 1, \forall y \in K\}$. The Mahler volume describes how "round" an $o$-symmetric convex body is. Its sharp upper bound was obtained by Santaló [46] and is known as the Blaschke-Santaló inequality. The sharp lower bound of the Mahler volume is known as the Mahler conjecture [40]. A breakthrough was made in 2017 by Iriyeh and Shibata [21] where they solved the three dimensional case of Mahler conjecture. The proof has since then been simplified by Fradelizi, Hubard, Meyer, Roldán-Pensado & Zvavitch [11]. The conjecture is still open for higher dimensions. Its history and related studies can be seen in, for instance, [4, 23, 24, 49].

Mahler volume is closely related to many other important affine objects, including centroid bodies and projection bodies. The classical centroid body dates back at least to Dupin. If $K$ is an $o$-symmetric convex body, then the *centroid body* of $K$ is the body whose boundary consists of the locus of the centroids of the halves of $K$ formed when $K$ is cut by codimension 1 subspaces. The affine isoperimetric inequality describing the lower bound of the volume of the centroid body in terms of the volume of the convex body itself is known as *the Busemann-Petty centroid inequality*, see Petty [43] and Schneider [48]. The *projection bodies* were introduced at the turn of the previous century by Minkowski. The affine isoperimetric inequality related to its polar body is called the Petty projection inequality, see Petty [44]. The detailed history can also be found in the books of Schneider [48], Gardner [12], and Leichtweiß [25].

There are two ways to extend these classical affine objects. One way is to view them as convex bodies generated by cosine transforms. In particular, Lutwak & Zhang [39] defined the

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$L_p$ centroid body $\Gamma_p K$ (see also [34, 37]) whose support function is

$$h_{\Gamma_p K}(u) = \Big(\frac{1}{c_{n,p}|K|}\int_K |x\cdot u|^p dx\Big)^{\frac{1}{p}}.$$

Lutwak & Zhang [39] revealed that the Blaschke-Santaló inequality can in fact be recovered as the limiting case ($p\to\infty$) of the affine isoperimetric inequality for $p$-th polar centroid body

$$|K||\Gamma_p^* K| \le \omega_n^2, \tag{1.1}$$

The above inequality is now known as the $L_p$ Blaschke-Santaló inequality. In [34], Lutwak, Yang, & Zhang established two families of affine isoperimetric inequalities, one for $L_p$ centroid bodies and the other for $L_p$ projection bodies. All these affine isoperimetric inequalities are crucial ingredients of the $L_p$ Brunn-Minkowski theory (introduced by Lutwak [31, 33]), which is a highly non-trivial extension of the Brunn-Minkowski theory. Since its introduction, the $L_p$ Brunn-Minkowski theory has become the center in the field of convex geometry, see, for example, [1–3, 6, 8, 14, 16, 18, 19, 22, 26, 26–29, 51, 52, 56–58]. Utilising the Petty projection inequality and the solution of Minkowski problem, Zhang [55] established the affine Sobolev-Zhang inequality, which is stronger than the classical Sobolev inequality. After that, Lutwak, Yang & Zhang [35] established a family of functional affine isoperimetric inequalities in the $L_p$ setting. These inequalities have since then been extended to more general cases and inspired many functional isoperimetric inequalities, see, for example, [15, 17, 50].

Another way to extend the classical Busemann-Petty centroid inequality, is to view the volume of classical centroid body as the expected value of the volume of random simplices with vertices sampled from $K$,

$$\frac{1}{|K|^n}\int_K\cdots\int_K |[o,x_1,...,x_n]|dx_1\ldots dx_n.$$

Here $[o,x_1,...,x_n]$ denotes the simplex whose vertices are $o,...,x_n$. In this setting, the Busemann-Petty centroid inequality was extended by Paouris & Pivovarov [41] into quite general cases. See also Dann, Paouris & Pivovarov [10] for other functional versions of Busemann's random simplex inequality. These results, including rearrangement inequalities for functions, can also be seen in the surveys of Lutwak [32] and Paouris & Pivovarov [42].

The difference between the aforementioned two ways of extensions can be summarized as: one is from "inner product", and the other is from "exterior product".

The Orlicz-Brunn-Minkowski theory stems from the two papers by Lutwak, Yang & Zhang [37, 38] and the papers by Ludwig & Reitzner [28] and Ludwig [27]. A systematic study on the framework of Orlicz Brunn-Minkowski theory was initiated by Gardner, Hug & Weil [13]. The results for convex bodies were obtained independently Xi, Jin & Leng [53]. See also [42] for a probabilistic approach common to the $L_p$ and Orlicz settings. Over the years, a great deal of effort has gone into trying to extend things from the $L_p$ Brunn-Minkowski theory to the Orlicz theory. Among these results are the affine isoperimetric inequalities for centroid bodies, projection bodies, and the random simplices. These extensions are often non-trivial. The fact that one loses homogeneity when replacing $|t|^p$ by a generic convex function often makes the proofs fundamentally different and challenging.

Our main effort in this paper is to give a general version of the $L_p$ Blaschke-Santaló inequality.

This article is dedicated to study the following functional

$$N_\phi(K,L) := \inf\left\{\lambda > 0 : \frac{1}{|K||L|}\int_K\int_L \phi\Big(\frac{x\cdot y}{\lambda}\Big)dxdy \le 1\right\},$$

where $K$ and $L$ are convex bodies, and $\phi : \mathbb{R} \to [0,+\infty)$ is an even convex function such that $\lim_{x\to\infty}\phi(x) = \infty$. We remark that an application of Fatou's lemma together with the continuity of $\phi$ immediately imply that the above infimum can be obtained.

It is easy to see that $N_\phi(K,L)$ satisfies

$$N_\phi(K,L) = N_\phi(AK, A^{-t}L)$$

for all $A \in \mathrm{GL}(n)$. Our first main result can be written as follows. Note that the $L_p$ version of the quantity $N_\phi(K,L)$ was studied by Lutwak, Yang & Zhang [36] as well as Campi and Gronchi [7].

**Theorem 1.1.** *Let $K, L \subset \mathbb{R}^n$ be convex bodies and $\phi$ be an even convex function defined on $\mathbb{R}$ with $\lim_{x\to\infty}\phi(x) = \infty$. Then,*

$$N_\phi(K,L) \ge C_\phi |K|^{\frac{1}{n}}|L|^{\frac{1}{n}}, \tag{1.2}$$

*where $C_\phi = N_\phi(B,B)/\omega_n^{\frac{2}{n}}$ is a constant depending only on $\phi$ and $n$, and $B$ is the Euclidean unit ball. Moreover, if $\phi$ is strictly convex, equality holds in (1.2) if and only if $K$ and $L$ are dilates of a pair of polar reciprocal origin-symmetric ellipsoids.*

The to-be-established inequality contains the inequality (1.1). In fact, choosing $\phi(t) = |t|^p$ and $L = (\Gamma_p K)^*$ recovers it. If we take $L = K^*$ and write for abbreviation that

$$N_\phi(K) = N_\phi(K,K^*),$$

then (1.2) becomes

$$N_\phi(K) \ge C_\phi \cdot |K|^{\frac{1}{n}}|K^*|^{\frac{1}{n}} = N_\phi(B_K),$$

where $B_K$ is the $o$-symmetric ball that has the same volume as $K$. Clearly, $N_\phi(K)$ is affine invariant, i.e., $N_\phi(AK) = N_\phi(K)$, $\forall A \in \mathrm{GL}(n)$.

Note that, the appearance of "inner product" in the affine invariance $N_\phi(K,L)$ makes this extension non-trivial. We also obtain the following rearrangement inequality.

**Theorem 1.2.** *Let $f, g$ be two non-negative, quasi-concave, and integrable functions on $\mathbb{R}^n$. Let $\phi$ be an even convex function on $\mathbb{R}$. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi(x\cdot y)f(x)g(y)dxdy \ge \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi(x\cdot y)f^*(x)g^*(y)dxdy, \tag{1.3}$$

*Moreover, if $\phi$ is strictly convex, then equality holds if and only if the closures of $\{x \in \mathbb{R}^n : f(x) > t\}$ and $\{y \in \mathbb{R}^n : g(y) > s\}$ are dilates of a common pair of polar reciprocal origin-symmetric ellipsoids, for almost all $t, s > 0$.*

Here $f^*$ and $g^*$ are the *symmetric decreasing rearrangement* (see Section 2 for the definition) of $f$ and $g$ respectively. When the quantity $\phi(x\cdot y)$ is replaced by $\phi(x-y)$, inequality (1.3) with its sign reversed is the well known Riesz rearrangement inequality; see the work of C. A. Rogers [45], and Brascamp-Lieb-Luttinger [5] for a general version. The main difficulty in establishing Theorem 1.1 is that the quantity

$$\int_K\int_L \phi(x\cdot y)dxdy \tag{1.4}$$

could increase under one single application of Steiner symmetrization (see the example in the Appendix). A new symmetrization scheme which involves a total of $2n$ elaborately chosen Steiner symmetrizations at a time to $K$ and $L$ is introduced in the current work. It will be shown that the quantity (1.4) is non-increasing with respect to this symmetrization scheme. See Lemma 4.2. Properties of Steiner symmetrization, particularly regarding the newly introduced symmetrization scheme, are included in Section 3.

A quantity related to (1.4) is

$$\int_{\partial K}\int_{\partial L}|\sigma_K(x)\cdot\sigma_L(y)|dxdy, \tag{1.5}$$

where $\sigma_K$ and $\sigma_L$ are the Gauss maps of $K$ and $L$, respectively. When $\partial K$ and $\partial L$ are sufficiently smooth with everywhere positive Gauss curvature, (1.5) may be reformulated as

$$\int_{S^{n-1}}\int_{S^{n-1}}|u\cdot v|f_K(u)f_L(v)dudv,$$

where $f_K$ and $f_L$ are the reciporical Gauss curvature of $K$ and $L$ (viewed as functions on the normal sphere). This can be viewed as a spherical analog of the integral in Theorem 1.2. Another spherical analog can be found in [20]. Quantity (1.5) is closely related to Petty's conjecture (see Page 570 in [48]), which is one of the major problems in the area of affine isoperimetric inequality for volume of projection bodies. Lutwak [30] showed that the conjecture that the minimum of (1.5) for $K$ and $L$ with fixed volume is attained at a pair of polar-reciprocal ellipsoids is equivalent to Petty's conjecture. The volume of projection body shares a common feature with the central quantity (1.4) considered in the current paper: it does not necessarily decrease under the usual Steiner symmetrization. An example of this was provided in Theorem 3 in [47]. It is unknown whether the symmetrization scheme adopted in the current paper could be developed to deal with the quantity (1.5).

## 2. Basic notations

At times, we will use $x^{(i)}$ to denote the $i$-th component of a point $x\in\mathbb{R}^n$.

Throughout the paper, by convex body, we mean a compact convex subset of $\mathbb{R}^n$ with non-empty interior. We will write $\mathcal{K}^n$ for the set of all convex bodies in $\mathbb{R}^n$. For $K\in\mathcal{K}^n$, we shall write $B_K$ for the ball in $\mathbb{R}^n$ centered at the origin with the same volume as $K$.

Given a convex body $K$ contains the origin in its interior, it is not hard to see that for a linear transformation $\phi$, we have $(\phi K)^*=\phi^{-t}K^*$. Thus, the polar body of an origin-symmetric ellipsoid is also an origin-symmetric ellipsoid. In particular, if $E=\phi B$, then $E^*=\phi^{-t}B$. Here $B$ is the unit ball in $\mathbb{R}^n$. Such a pair of ellipsoids are said to be *polar reciprocal* to each other.

For $u\in S^{n-1}$, denote by $K_u$ the image of the orthogonal projection of $K$ onto $u^\perp$. We write $\overline{\ell}_u(K;y'):K_u\to\mathbb{R}$ and $\underline{\ell}_u(K;y'):K_u\to\mathbb{R}$ for the *overgraph* and *undergraph functions* of $K$ in the direction $u$; i.e.

$$K=\{y'+tu:-\underline{\ell}_u(K;y')\le t\le\overline{\ell}_u(K;y')\text{ for }y'\in K_u\}.$$

Clearly, they are concave functions if $K$ is a convex body.

The *Steiner symmetral* $S_uK$ of $K\in\mathcal{K}^n$ in the direction $u$ can be defined as the body whose orthogonal projection onto $u^\perp$ is identical to that of $K$ and whose overgraph and undergraph functions are given by

$$\bar{\ell}_u(S_uK;y') = \underline{\ell}_u(S_uK;y') = \frac{1}{2}[\bar{\ell}_u(K;y') + \underline{\ell}_u(K;y')].$$

Let $f$ be an integrable function on $\mathbb{R}^n$. The *symmetric decreasing rearrangement* of $f$, denoted by $f^*$, is the radial symmetric and decreasing function such that for each $t \in \mathbb{R}$,

$$\mathcal{H}^n(\{x \in \mathbb{R}^n : f(x) > t\}) = \mathcal{H}^n(\{x \in \mathbb{R}^n : f^*(x) > t\}).$$

Here, by radial symmetric, we mean the superlevel sets of $f^*$ are origin-centered balls.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *quasi-concave*, if

$$f\big((1-\phi)x + \phi y\big) \geq \min\{f(x), f(y)\}, \quad \forall \phi \in [0,1], \quad \forall x, y \in \mathbb{R}^n.$$

## 3. Steiner symmetrization and its properties

By the definition of Steiner symmetrization, we have $(S_u\,K)_u = K_u$ for each $u \in S^{n-1}$. Moreover, the Steiner symmetral $S_u\,K$ is symmetric with respect to the hyperplane $u^\perp$. Also obvious is the fact that if $K \subset L$, then

$$S_uK \subset S_uL, \tag{3.1}$$

for each $u \in S^{n-1}$.

**Lemma 3.1.** *Let $K \subset \mathbb{R}^n$ be a convex body and $u \in S^{n-1}$. Suppose $v \in u^\perp \cap S^{n-1}$. If $K$ is symmetric with respect to $v^\perp$, then $S_u\,K$ is also symmetric with respect to $v^\perp$.*

*Proof.* For each $x \in \mathbb{R}^n$, write $x$ as

$$x = tu + sv + y'',$$

where $t, s \in \mathbb{R}$ and $y'' \in u^\perp \cap v^\perp$.

Suppose $x_0 \in S_u\,K$ and $x_0 = t_0u + s_0v + y_0''$. Let $y_0' = s_0v + y_0'' \in u^\perp$ and $z_0' = -s_0v + y_0''$. Since $K$ is symmetric with respect to $v^\perp$, the orthogonal image $K_u$ is also symmetric with respect to $v^\perp$. Hence $z_0' \in K_u$. Also, since $K$ is symmetric with respect to $v^\perp$, the point $tu + sv + y_0'' \in K$ if and only if $tu - sv + y_0'' \in K$, where $t, s \in \mathbb{R}$. Hence $\underline{l}_u(K;y_0') = \underline{l}_u(K;z_0')$ and $\bar{l}_u(K;y_0') = \bar{l}_u(K;z_0')$. Thus, we have

$$t_0u - s_0v + y_0'' \in S_u\,K.$$

Hence $S_u\,K$ is symmetric with respect to $v^\perp$. $\square$

The next corollary follows immediately from the previous lemma and that the Steiner symmetral $S_u\,K$ is symmetric with respect to $u^\perp$.

**Corollary 3.2.** *Let $K \subset \mathbb{R}^n$ be a convex body and $e_1, \cdots, e_n$ be an orthonormal basis. Define*

$$Q = S_{e_1}\,S_{e_2}\,\cdots S_{e_n}K.$$

*The convex body $Q$ is 1-unconditional; i.e., $Q$ is symmetric with respect to $e_i^\perp$ for all $i = 1, 2, \cdots, n$.*

For each convex body $K$, write $B_K$ for the ball centered at the origin with the same volume as $K$. Let $u \in S^{n-1}$. We claim that if $d_H(K, B_K)$ is less than the radius of $B_K$, then

$$d_H(S_uK, B_K) \leq d_H(K, B_K). \tag{3.2}$$

To see this, write $r_0$ to be the radius of $B_K$. By the definition of Hausdorff distance, for any $d_H(K, B_K) < \varepsilon < r_0$, we have

$$K \subset B_K + \varepsilon B, \tag{3.3}$$

and

$$B_K \subset K + \varepsilon B,$$

or, equivalently by the fact that $\varepsilon < r_0$,

$$(r_0 - \varepsilon)B \subset K. \tag{3.4}$$

Applying Steiner symmetrization $S_u$ to both sides of (3.3) and (3.4), and using (3.1), we have

$$S_u K \subset B_K + \varepsilon B, \tag{3.5}$$

and

$$(r_0 - \varepsilon)B \subset S_u K,$$

or, equivalently by the fact that $\varepsilon < r_0$,

$$B_K \subset S_u K + \varepsilon B. \tag{3.6}$$

Equations (3.5) and (3.6), definition of Hausdorff distance, and the fact that $\varepsilon$ can be arbitrarily close to $d_H(K, B_K)$, immediately imply (3.2).

The proof of the following lemma is modified from the proof of Theorem 10.3.2 in [48].

**Lemma 3.3.** *Let $K$ be a convex body in $\mathbb{R}^n$. There exists a sequence of ordered orthonormal bases $e_1^i, \ldots, e_n^i$ such that*

$$K^i \text{ converges to } B_K \text{ in Hausdorff metric,}$$

*where $B_K$ is the ball centered at the origin with $V(B_K) = V(K)$. Here, $K^0 = K$ and $K^i = S_{e_n^i} \ldots S_{e_1^i} K^{i-1}$.*

*Proof.* Let $I : e_1, \ldots, e_n$ be an ordered orthonormal basis for $\mathbb{R}^n$. We shall write for simplicity that $S_I = S_{e_n} \cdots S_{e_1}$.

Define the set

$$\mathcal{Q} = \{S_{I_k} S_{I_{k-1}} \ldots S_{I_1} K : \text{ orthonormal bases } I_1, \ldots, I_k \text{ and } k > 0\}.$$

For each $Q \in \mathcal{K}_o^n$, write $r_Q$ as the outer radius of $Q$, i.e., the smallest $r > 0$ such that $Q \subset rB$. Set $r_0 = \inf_{Q \in \mathcal{Q}} r_Q$. Let $Q_i$ be a sequence in $\mathcal{Q}$ such that $r_{Q_i} \to r_0$. Since the set $\mathcal{Q}$ is uniformly bounded as a result of (3.1), we can invoke Blaschke's selection theorem and assume (by possibly taking a subsequence) that $Q_i$ converges in Hausdorff metric to a non-empty compact convex set $Q_0$.

By the choice of $r_0$, it is apparent that $Q_0 \subset r_0 B$. We claim that $Q_0 = r_0 B$. To see this, we prove by contradiction. Assume that $Q_0$ is strictly contained in $r_0 B$ and $Q_0 \neq r_0 B$. Therefore, there exists $x_0 \in \partial(r_0 B)$ and a neighborhood $U$ of $x_0$ such that $U \cap \partial(r_0 B)$ contains non-empty interior with respect to the induced topology on $\partial(r_0 B)$ and $U \cap Q_0 = \emptyset$. Note that for any line $\xi$ passing through $U \cap \partial(r_0 B)$ and not tangent to $r_0 B$, the length of the line segment $\xi \cap r_0 B$ is strictly larger than the length of the line segment $\xi \cap Q_0$. This suggests that for each ordered orthonormal basis $I : e_1, \ldots, e_n$, the convex body $S_I K$ will not intersect $U \cap \partial(r_0 B)$ and $C_i$, where $C_i$ is the reflection of $U \cap \partial(r_0 B)$ with respect to $e_i^\perp$. Since $\partial(r_0 B)$ is compact, we may choose a finite number of orthonormal bases, say $I_1, \ldots, I_k$, so that $U \cap \partial(r_0 B)$ together with the reflections generated by it with respect to $u^\perp$ for $u \in \cup_k I_k$ will form a finite cover of $\partial(r_0 B)$. Therefore $S_{I_k} \cdots S_{I_1} Q_0 \subset \operatorname{int} Q_0$ and as a result, the outer radius of $S_{I_k} \cdots S_{I_1} Q_0$ is strictly smaller than $r_0$.

Towards this end, choose a subsequence of $\{Q_i\}$ so that $S_{I_k}\cdots S_{I_1}Q_i$ converges to a non-empty convex compact set $Q_0'$. It is not hard to see that this implies that $Q_0'\subset S_{I_k}\cdots S_{I_1}Q_0$, see, for example, Lemma 10.3.1 in [48]. This, combined with what we concluded about $S_{I_k}\cdots S_{I_1}Q_0$, implies that the outer radius of $Q_0'$ is strictly smaller than $r_0$. Thus, there must exist $i_0$ such that $S_{I_k}\cdots S_{I_1}Q_{i_0}\in\mathcal{Q}$ has its outer radius strictly smaller than $r_0$. This is a contradiction to the choice of $r_0$.

Hence, there exists a sequence $Q_i\in\mathcal{Q}$ such that $Q_i\to r_0B$ in Hausdorff metric. Moreover, it can be easily seen that $r_0B=B_K$ since Steiner symmetrization preserves volume.

Towards this end, let $\varepsilon_k$ be a sequence of sufficiently small positive numbers (less than $r_0$) and $\varepsilon_k\to 0$. Choose $K^1\in\mathcal{Q}$ such that $d_H(K^1,B_K)<\varepsilon_1$. Now, applying the above argument again but this time on $K^1$ instead of on $K$ allows us to conclude the existence of $I_1,I_2,\dots,I_m$ and $K^2=S_{I_m}\cdots S_{I_1}K^1\in\mathcal{Q}$ such that $d_H(K^2,B_{K^1})<\varepsilon_2$. Notice that $B_{K^1}=B_K$ since Steiner symmetrization preserves volume. Hence $d_H(K^2,B_K)<\varepsilon_2$. Carrying on this process, we can find a sequence $K^i\in\mathcal{Q}$ such that $d_H(K^i,B_K)<\varepsilon_i$.

To reach the desired result, we only need to use (3.2) to conclude that the Hausdorff distance is non-increasing after applying each Steiner symmetrization. □

**Lemma 3.4.** *Let $K$ and $L$ be two convex bodies in $\mathbb{R}^n$. There exists a sequence of orthonormal bases $e_1^i,\dots,e_n^i$ such that*

*$K^i$ and $L^i$ converges to $B_K$ and $B_L$ in Hausdorff metric respectively,*

*where $B_K$ and $B_L$ are the balls centered at the origin with $V(B_K)=V(K)$ and $V(B_L)=V(L)$. Here, $K^0=K$, $L^0=L$ and*

$$K^i=S_{e_n^i}\cdots S_{e_1^i}K^{i-1},\qquad L^i=S_{e_1^i}\cdots S_{e_n^i}L^{i-1}.$$

*Proof.* Suppose $I:e_1,\dots,e_n$. For simplicity, we shall write $S_I=S_{e_n}\cdots S_{e_1}$ and $S_{-I}=S_{e_1}\cdots S_{e_n}$. Let $\varepsilon_k$ be a sequence of sufficiently positive numbers such that $\varepsilon_k\to 0$.

By Lemma 3.3, there exists orthonormal bases $I_1,\dots,I_{k_1}$ such that $d_H(\widetilde{K_1},B_K)<\varepsilon_1$ for $\widetilde{K_1}=S_{I_{k_1}}\cdots S_{I_1}K$.

Let $\widetilde{L_1}=S_{-I_{K_1}}\cdots S_{-I_1}L$. Applying Lemma 3.3 to $\widetilde{L_1}$, we have that there exists orthonormal bases $I_{k_1+1},\dots,I_{k_1+k_2}$ such that $d_H(\widetilde{L_2},B_{\widetilde{L_1}})=d_H(\widetilde{L_2},B_L)<\varepsilon_2$ for $\widetilde{L_2}=S_{-I_{k_1+k_2}}\cdots S_{-I_{k_1+1}}\widetilde{L_1}$.

Set $\widetilde{K_2}=S_{I_{k_1+k_2}}\cdots S_{I_{k_1+1}}\widetilde{K_1}$. We continue in this fashion, by applying Lemma 3.3 alternatively to the sequences $\widetilde{K_i}$ and $\widetilde{L_i}$. This allows us to conclude a sequence of orthonormal bases $I_i$ and sequences $\widetilde{K_i}$, $\widetilde{L_i}$ such that $\widetilde{K_i}\to B_K$ and $\widetilde{L_i}\to B_L$.

Equation (3.2) now allows us to conclude that $I_i$ is the desired sequence of orthonormal bases. □

The following lemma is a direct consequence of Lemma 3.4 and the fact that convergence in Hausdorff metric implies convergence of characteristic functions in $L_1$ norm.

**Lemma 3.5.** *Let $K$ and $L$ be two convex bodies in $\mathbb{R}^n$. There exists a sequence of ordered orthonormal bases $e_1^i,\dots,e_n^i$ such that*

$$\lim_{i\to\infty}\|1_{K^i}-1_{B_K}\|_1=0,\qquad \lim_{i\to\infty}\|1_{L^i}-1_{B_L}\|_1=0, \tag{3.7}$$

*where $K_0 = K$, $L_0 = L$, and*

$$K^i = S_{e_n^i} \dots S_{e_1^i} K^{i-1}, \qquad L^i = S_{e_1^i} \dots S_{e_n^i} L^{i-1}. \tag{3.8}$$

For any fixed convex body $K$ and $u \in S^{n-1}$, write $x \in K$ as

$$x = y' + tu,$$

where $y' \in u^\perp$ and $t \in \mathbb{R}$. Define $\phi_{K,u} : K \to S_u K$ by

$$\phi_{K,u}(x) = x - \frac{1}{2}(\bar{l}_u(K; y') - \underline{l}_u(K; y'))u,$$

where $x \in K$ and $x = y' + tu$. Intuitively, the map $\phi_{K,u}$ moves each point $x$ in $K$ in the direction of $u$ so that $\phi_{K,u}(K \cap \{y' + tu : t \in \mathbb{R})$ is a line segment symmetric about the hyperplane $u^\perp$.

Note that $\phi_{K,u}$ is one-to-one and onto. Let $\psi_{K,u} : S_u K \to K$ be the inverse of $\phi_{K,u}$; i.e., for each $x \in S_u K$,

$$\psi_{K,u}(x) = x + \frac{1}{2}(\bar{l}_u(K; y') - \underline{l}_u(K; y'))u, \tag{3.9}$$

where $x = y' + tu$.

**Lemma 3.6.** *Let $K \subset \mathbb{R}^n$ be a convex body and $u \in S^{n-1}$. The map $\psi_{K,u}$ as defined in (3.9) is Lipschitz continuous on any compact subset of $\operatorname{int} S_u K$. Moreover, if $x \in \operatorname{int} S_u K$ is a differentiable point for $\psi_{K,u}$, then the Jacobian matrix of $\psi_{K,u}$ at $x$ has determinant 1.*

*Proof.* That $\psi_{K,u}$ is Lipschitz continuous on any compact subset of $\operatorname{int} S_u K$ is immediate from the fact that both $\underline{l}_u(K; \cdot)$ and $\bar{l}_u(K; \cdot)$ are concave.

That the Jacobian matrix of $\psi_{K,u}$ has determinant 1 at each differentiable point $x \in S_u K$ comes from the fact that

$$\psi_{K,u}(x) \cdot v = x \cdot v,$$

for each $v \in u^\perp$. □

Let $K \subset \mathbb{R}^n$ be a convex body and $I : e_1, \dots, e_n$ be an ordered orthonormal basis for $\mathbb{R}^n$. Define $K_0 = K$ and

$$K_i = S_{e_i} K_{i-1},$$

for $i = 1, \dots, n$. Define $\Psi_{K,I} : K_n \to K_0 = K$ as

$$\Psi_{K,I} = \psi_{K_0,e_1} \circ \psi_{K_1,e_2} \circ \dots \psi_{K_{n-1},e_n} \tag{3.10}$$

where the $\psi$'s are as defined in (3.9). Note that by Corollary 3.2, the convex body $K_n$ is 1-unconditional.

The map $\Psi_{K,I}$ may be expressed using the following lemma.

**Lemma 3.7.** *Let $K \subset \mathbb{R}^n$ be a convex body and $I : e_1, \dots, e_n$ be an ordered orthonormal basis for $\mathbb{R}^n$. Define $\Psi_{K,I}$ as in (3.10). Then, for each $i = 1, \dots n$, there exists $l_K^{(i)} : K_n \to \mathbb{R}$ such that $l_K^{(i)}$ is symmetric in its first $i$ arguments and the $i$-th coordinate of $\Psi_{K,I}$ may be expressed as*

$$\left[\Psi_{K,I}(x^{(1)}, \dots, x^{(n)})\right]^{(i)} = x^{(i)} + l_K^{(i)}(|x^{(1)}|, \dots, |x^{(i)}|, x^{(i+1)}, \dots, x^{(n)}), \tag{3.11}$$

*for each $x = (x^{(1)}, \dots, x^{(n)}) \in K_n$. Moreover, the map $\Psi_{K,I}$ is Lipschitz continuous on any compact subset of $\operatorname{int} K_n$ and its Jacobian is 1 wherever it is defined.*

*Proof.* Note that by the definition of Steiner symmetrization, the $i$-th coordinate of a point $x \in K_n$ can only be changed by $\psi_{K_{i-1},e_i}$. Hence,

$$\left[\Psi_{K,I}(x^{(1)},\ldots,x^{(n)})\right]^{(i)} = \psi_{K_{i-1},e_i} \circ \psi_{K_i,e_{i+1}} \circ \cdots \circ \psi_{K_{n-1},e_n}(x^{(1)},\ldots,x^{(n)}).$$

The same observation shows that the first $i$ coordinates remain unchanged under $\psi_{K_i,e_{i+1}} \circ \cdots \circ \psi_{K_{n-1},e_n}$; that is,

$$\psi_{K_i,e_{i+1}} \circ \cdots \circ \psi_{K_{n-1},e_n}(x^{(1)},\ldots,x^{(n)}) = (x^{(1)},\ldots,x^{(i)},\tilde{x}^{(i+1)},\ldots,\tilde{x}^{(n)}),$$

where $\tilde{x}^{(j)} = \tilde{x}^{(j)}(x^{(1)},\ldots,x^{(n)})$ are functions of $(x^{(1)},\ldots,x^{(n)})$. Note that by Lemma 3.1, the convex bodies $K_i,\ldots,K_n$ are symmetric with respect to $e_1^\perp,\ldots,e_i^\perp$. This implies that $\tilde{x}^{(j)}(x^{(1)},\ldots,x^{(n)})$ is symmetric with respect to its first $i$ arguments; that is,

$$\tilde{x}_j = \tilde{x}_j(x^{(1)},\ldots,x^{(n)}) = \tilde{x}_j(|x^{(1)}|,\ldots,|x^{(i)}|,x^{(i+1)},\ldots,x^{(n)}). \tag{3.12}$$

By (3.9),

$$\begin{aligned}
&\left[\psi_{K_{i-1},e_i} \circ \cdots \circ \psi_{K_{n-1},e_n}(x^{(1)},\ldots,x^{(n)})\right]^{(i)}\\
=&\left[\psi_{K_{i-1},e_i}(x^{(1)},\ldots,x^{(i)},\tilde{x}^{(i+1)},\ldots,\tilde{x}^{(n)})\right]^{(i)}\\
=&x^{(i)} + \frac{1}{2}(\bar{l}_{e_i}(K_{i-1};(x^{(1)},\ldots,x^{(i-1)},0,\tilde{x}^{(i+1)},\ldots,\tilde{x}^{(n)})) - \underline{l}_{e_i}(K_{i-1};(x^{(1)},\ldots,x^{(i-1)},0,\tilde{x}^{(i+1)},\ldots,\tilde{x}^{(n)})))
\end{aligned} \tag{3.13}$$

Note that $K_{i-1}$ symmetric with respect to $e_1^\perp,\ldots,e_{i-1}^\perp$. Hence, both $\bar{l}_{e_i}(K_{i-1};\cdot)$ and $\underline{l}_{e_i}(K_{i-1};\cdot)$ are symmetric with respect to the first $(i-1)$ arguments. Define $l_K^{(i)}$ as

$$\begin{aligned}
&l_K^{(i)}(x^{(1)},\ldots,x^{(n)})\\
=&\frac{1}{2}(\bar{l}_{e_i}(K_{i-1};(x^{(1)},\ldots,x^{(i-1)},0,\tilde{x}^{(i+1)},\ldots,\tilde{x}^{(n)})) - \underline{l}_{e_i}(K_{i-1};(x^{(1)},\ldots,x^{(i-1)},0,\tilde{x}^{(i+1)},\ldots,\tilde{x}^{(n)}))).
\end{aligned} \tag{3.14}$$

By (3.12) and the symmetry property we observed about $\bar{l}_{e_i}(K_{i-1};\cdot)$ and $\underline{l}_{e_i}(K_{i-1};\cdot)$, we conclude that $l_K^{(i)}$ is symmetric with respect to its first $i$ arguments; that is,

$$l_K^{(i)}(x^{(1)},\ldots,x^{(n)}) = l_K^{(i)}(|x^{(1)}|,\ldots,|x^{(i)}|,x^{(i+1)},\ldots,x^{(n)}). \tag{3.15}$$

Equations (3.13), (3.14), and (3.15) imply (3.11).

The facts that $\Psi_{K,I}$ is Lipschitz continuous on any compact subset of $\operatorname{int} K_n$ and its Jacobian is 1 wherever it is defined follow immediately from its definition, Lemma 3.6, and the fact that Steiner symmetrization is volume preserving. □

## 4. Proof of the main results

Let $I: e_1,\ldots,e_n$ be an ordered orthonormal basis for $\mathbb{R}^n$. Denote by $-I$ the orthonormal basis in reversed order; that is $-I: e_n,\ldots,e_1$. Let $K$ and $L$ be two convex bodies in $\mathbb{R}^n$. For each $i = 1,\ldots,m$, define $K_0 = K$, $L_0 = L$, and

$$K_i = S_{e_i}K_{i-1}, \qquad L_i = S_{e_{n-i+1}}L_{i-1}, \tag{4.1}$$

for $i = 1,\ldots,n$. Consider $\Psi_{K,I}$ and $\Psi_{L,-I}$ as in (3.10). In particular,

$$\begin{aligned}
\Psi_{K,I} &= \psi_{K_0,e_1} \circ \psi_{K_1,e_2} \circ \cdots \circ \psi_{K_{n-1},e_n},\\
\Psi_{L,-I} &= \psi_{L_0,e_n} \circ \psi_{L_1,e_{n-1}} \circ \cdots \circ \psi_{L_{n-1},e_1}.
\end{aligned}$$

By Lemma 3.7, there exist $l_K^{(k)}: K_n \to \mathbb{R}$ and $l_L^{(k)}: L_n \to \mathbb{R}$ such that

$$\begin{aligned} \Psi_{K,I}^{(k)}(x) &= x^{(k)} + l_K^{(k)}(x) = x^{(k)} + l_K^{(k)}(|x^{(1)}|, \dots, |x^{(k)}|, x^{(k+1)}, \dots, x^{(n)}), \\ \Psi_{L,-I}^{(k)}(y) &= y^{(k)} + l_L^{(k)}(y) = y^{(k)} + l_L^{(k)}(y^{(1)}, \dots, y^{(k-1)}, |y^{(k)}|, \dots, |y^{(n)}|). \end{aligned} \tag{4.2}$$

For notational simplicity, write

$$d_K^{(k)}(x) = \frac{1}{2} l_K^{(k)}(x) - \frac{1}{2} l_K^{(k)}(-x), \tag{4.3}$$

and

$$d_L^{(k)}(x) = \frac{1}{2} l_L^{(k)}(x) - \frac{1}{2} l_L^{(k)}(-x). \tag{4.4}$$

Denote by $d_K : K_n \to \mathbb{R}^n$ and $d_L : L_n \to \mathbb{R}^n$ the maps whose $k$-th coordinate function are precisely $d_K^{(k)}$ and $d_L^{(k)}$.

By (4.2), $d_K^{(k)}$ is symmetric with respect to its first $k$ arguments and $d_L^{(k)}$ is symmetric with respect to its last $(n-k+1)$ arguments; that is

$$\begin{aligned} d_K^{(k)}(x^{(1)}, \dots, x^{(n)}) &= d_K^{(k)}(|x^1|, \dots, |x^k|, x^{(k+1)}, \dots, x^{(n)}), \\ d_L^{(k)}(y^{(1)}, \dots, y^{(n)}) &= d_L^{(k)}(y^{(1)}, \dots, y^{(k-1)}, |y^{(k)}|, \dots, |y^{(n)}|). \end{aligned} \tag{4.5}$$

Obviously, it follows from (4.3) and (4.4) that the functions $d_K^{(k)}$ and $d_L^{(k)}$ are odd. In particular,

$$d_K^{(n)} = 0 = d_L^{(1)}. \tag{4.6}$$

Moreover, by Lemma 3.7

$$\begin{aligned} \Psi_{K,I}(x) - \Psi_{K,I}(-x) &= 2x + 2d_K(x), \\ \Psi_{L,-I}(y) - \Psi_{L,-I}(-y) &= 2y + 2d_L(y). \end{aligned}$$

Denote by $\Omega$ the set of $n$ by $n$ diagonal matrices whose diagonal entries are either 1 or $-1$.

**Lemma 4.1.** *For each $x, y$, we have*

$$\frac{1}{2^n} \sum_{A \in \Omega} (Ax + d_K(Ax)) \cdot (Ay + d_L(Ay)) = x \cdot y. \tag{4.7}$$

*Proof.* To see this, write

$$\begin{aligned} f_{k,1}(x, y) &= x^{(k)} y^{(k)}, \\ f_{k,2}(x, y) &= x^{(k)} d_L^{(k)}(y), \\ f_{k,3}(x, y) &= y^{(k)} d_K^{(k)}(x), \\ f_{k,4}(x, y) &= d_K^{(k)}(x) d_L^{(k)}(y). \end{aligned}$$

Using the new notation, we only need to show

$$\frac{1}{2^n} \sum_{A \in \Omega} \sum_{k=1}^{n} (f_{k,1}(Ax, Ay) + f_{k,2}(Ax, Ay) + f_{k,3}(Ax, Ay) + f_{k,4}(Ax, Ay)) = x \cdot y.$$

We shall show that $\sum_{A \in \Omega} f_{k,2}(Ax, Ay) = 0$ for each $k \in \{1, .., n\}$. Notice that by (4.5),

$$d_L^{(k)}(-y^{(1)}, \dots, -y^{(k-1)}, y^{(k)}, \dots, y^{(n)}) = -d_L^{(k)}(y^{(1)}, \dots, y^{(n)}),$$

which implies that

$$f_{k,2}(A_0Ax, A_0Ay) + f_{k,2}(Ax, Ay) = 0,$$

for each $A \in \Omega$ and $A_0 = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1)$ where there are $(k-1)$ many $(-1)$'s. Hence,

$$\sum_{A\in\Omega} f_{k,2}(Ax, Ay) = 0.$$

The same argument, but this time with $A_0 = \mathrm{diag}(1, \dots, 1, -1, \dots, -1)$ where there are $k$ many 1's, will imply

$$\sum_{A\in\Omega} f_{k,3}(Ax, Ay) = 0.$$

Now, take $A_0 = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1)$ again, where there are $(k-1)$ many $(-1)$'s. For any $A \in \Omega$, by (4.5), we have

$$d_K^{(k)}(A_0Ax) = d_K^{(k)}(Ax), \quad d_L^{(k)}(A_0Ay) = -d_L^{(k)}(Ay),$$

and hence

$$f_{k,4}(A_0Ax, A_0Ay) + f_{k,4}(Ax, Ay) = 0.$$

Taking summation over all $A \in \Omega$, we have

$$\sum_{A\in\Omega} f_{k,4}(Ax, Ay) = 0.$$

On the other side, it is trivial to see that

$$\sum_{A\in\Omega}\sum_{k=1}^{n} f_{k,1}(Ax, Ay) = 2^n x \cdot y.$$

□

**Lemma 4.2.** *Let $K, L \subset \mathbb{R}^n$ be two convex bodies and $\phi$ be an even convex function defined on $\mathbb{R}$. Suppose $I : e_1, \dots, e_n$ is an ordered orthonormal basis for $\mathbb{R}^n$ and $K_i$ and $L_i$ are as defined in (4.1). Then*

$$\int_K \int_L \phi(x \cdot y)dxdy \geq \int_{K_n} \int_{L_n} \phi(x \cdot y)dxdy. \tag{4.8}$$

*Proof.* By Lemma 3.7, we have

$$\int_K \int_L \phi(x \cdot y)dxdy = \int_{K_n} \int_{L_n} \phi(\Psi_{K,I}(x) \cdot \Psi_{L,-I}(y))dxdy. \tag{4.9}$$

To see that the change of variable formula works, one notes first that since $\Psi_{K,I}$ and $\Psi_{L,-I}$ are Lipschitz on any compact subsets of $\mathrm{int}\, K_n$ and $\mathrm{int}\, L_n$ (by Lemma 3.7), respectively, the change of variable formula can be applied to any compact subset of $\mathrm{int}\, K_n$ and $\mathrm{int}\, L_n$. Since $\phi$ is an even convex function, it is bounded from below. Now, the change of variable in (4.9) holds because one can take advantage of the monotone convergence theorem, and the fact that the $n$ dimensional Hausdorff measure of the boundary of a convex body is zero.

Since $K_n$ and $L_n$ are 1-unconditional, the following four integrals are identical:

$$\int_{K_n}\int_{L_n}\phi(\Psi_{K,I}(x)\cdot\Psi_{L,-I}(y))dxdy, \qquad \int_{K_n}\int_{L_n}\phi(\Psi_{K,I}(-x)\cdot\Psi_{L,-I}(y))dxdy,$$
$$\int_{K_n}\int_{L_n}\phi(\Psi_{K,I}(-x)\cdot\Psi_{L,-I}(-y))dxdy, \qquad \int_{K_n}\int_{L_n}\phi(\Psi_{K,I}(x)\cdot\Psi_{L,-I}(-y))dxdy.$$

Taking advantage that $\phi$ is even and convex, we have

$$\begin{aligned}
&\int_{K_n}\int_{L_n}\phi(\Psi_{K_I}(x)\cdot\Psi_{L,-I}(y))dxdy\\
\geq&\int_{K_n}\int_{L_n}\phi\left(\frac{1}{4}(\Psi_{K,I}(x)-\Psi_{K,I}(-x))\cdot(\Psi_{L,-I}(y)-\Psi_{L,-I}(-y))\right)dxdy\\
=&\int_{K_n}\int_{L_n}\phi\left(\big(x+d_K(x)\big)\cdot\big(y+d_L(y)\big)\right)dxdy.
\end{aligned}\tag{4.10}$$

Recall that $\Omega$ is the set of all $n$ by $n$ diagonal matrices whose diagonal entries are either 1 or -1. By the fact that $K_n$ and $L_n$ are symmetric with respect to each $e_i^\perp$, that $\phi$ is even and convex, and Lemma 4.1, we have

$$\begin{aligned}
&\int_{K_n}\int_{L_n}\phi\left(\big(x+d_K(x)\big)\cdot\big(y+d_L(y)\big)\right)dxdy\\
=&\frac{1}{2^n}\sum_{A\in\Omega}\int_{K_n}\int_{L_n}\phi\left(\big(Ax+d_K(Ax)\big)\cdot\big(Ay+d_L(Ay)\big)\right)dxdy\\
\geq&\int_{K_n}\int_{L_n}\phi\left(\frac{1}{2^n}\sum_{A\in\Omega}\big(Ax+d_K(Ax)\big)\cdot\big(Ay+d_L(Ay)\big)\right)dxdy\\
=&\int_{K_n}\int_{L_n}\phi\left(x\cdot y\right)dxdy.
\end{aligned}\tag{4.11}$$

The desired inequality can now be obtained by combining the above inequalities. ☐

**Lemma 4.3.** *If $\phi:\mathbb{R}\to\mathbb{R}$ is strictly convex, then equality holds in (4.8) if and only if there is a linear transform $T\in SL(n)$ such that $\Psi_{K,I}(x)=Tx$ and $\Psi_{L,I}(y)=T^{-t}y$.*

*Proof.* Suppose equality holds in (4.8). Then equality must hold in (4.11). For simplicity, we write

$$f_i(x,y)=f_{i,1}(x,y)+f_{i,2}(x,y)+f_{i,3}(x,y)+f_{i,4}(x,y).$$

Since $\phi$ is strictly convex, this implies that

$$\sum_{i=1}^n f_i(Ax,Ay)=\sum_{i=1}^n f_i(x,y), \qquad \text{a.e. for } x\in K_n, y\in L_n,$$

for all $A\in\Omega$. This, together with (4.7), shows that

$$\sum_{i=1}^n f_i(x,y)=\sum_{i=1}^n x^{(i)}y^{(i)}, \qquad \text{a.e. for } x\in K_n, y\in L_n.$$

Or, equivalently by the definition of $f_i$,

$$\sum_{i=1}^{n}\left(x^{(i)}d_L^{(i)}(y)+y_i d_K^{(i)}(x)+d_K^{(i)}(x)d_L^{(i)}(y)\right)=0, \tag{4.12}$$

for almost all $x\in K_n$ and $y\in L_n$. By continuity of $d_K^{(i)}$ and $d_L^{(i)}$, (4.12) is valid for all $x\in K_n$ and $y\in L_n$.

On the other side, equality in (4.8) implies equality in (4.10), which implies that

$$\Psi_{K,I}(x)\cdot\Psi_{L,-I}(y)=\frac{1}{2}(\Psi_{K,I}(x)-\Psi_{K,I}(-x))\cdot\Psi_{L,-I}(y),$$

for almost all $x\in K_n$ and $y\in L_n$. This implies that

$$\Psi_{K,I}(x)\cdot y=\frac{1}{2}(\Psi_{K,I}(x)-\Psi_{K,I}(-x))\cdot y,$$

for almost all $x\in K_n$ and $y\in L$. Since $L$ contains interior points, we conclude that

$$\Psi_{K,I}(x)=\frac{1}{2}\left(\Psi_{K,I}(x)-\Psi_{K,I}(-x)\right).$$

By this, and the continuity of the map $\Psi_{K,I}$, we have

$$\Psi_{K,I}(x)=-\Psi_{K,I}(-x), \tag{4.13}$$

for all $x\in K_n$. This, in turn, implies that $K$ is origin-symmetric. To see this, suppose $y\in K$, then there exists $x\in K_n$ such that $y=\Psi_{K,I}(x)$. Since $K_n$ is 1-unconditional, $-x\in K_n$. Hence $-y=-\Psi_{K,I}(x)=\Psi_{K,I}(-x)\in K$.

The same argument for $L$ implies that $L$ is also origin-symmetric. Therefore, there exists $r>0$ such that

$$r\sqrt{n}B_n\subset \operatorname{int}(K\cap L).$$

Towards this end, for each $k=1,2,\dots,n$, let $x_k=(r,r,\dots,r,0,\dots,0)\in\mathbb{R}^n$ where $r$ appears $k$ times. By (4.5) and the fact that $d_K^{(i)}$ is odd (from (4.3)), we have

$$d_K^{(i)}(x_k)=0, \tag{4.14}$$

for $i\geq k$.

We will show, by induction (on $i$), that there exists constants $c_{i,j}$ with $2\leq i\leq n$ and $1\leq j\leq i-1$ such that

$$d_L^{(i)}(y)=c_{i,1}y^{(1)}+\ldots+c_{i,i-1}y^{(i-1)}, \tag{4.15}$$

for $y\in L_n$.

Consider the case $i=2$. Inserting $x=x_2$ in (4.12) and using (4.14), we have

$$rd_L^{(1)}(y)+rd_L^{(2)}(y)+y_1d_K^{(1)}(x_2)+d_K^{(1)}(x_2)d_L^{(1)}(y)=0.$$

This, together with (4.6), implies

$$d_L^{(2)}(y)=-\left(d_K^{(1)}(x_2)/r\right)\cdot y^{(1)},$$

which proves (4.15) for the case $i=2$ by choosing $c_{2,1}=-d_K^{(1)}(x_2)/r$.

For the inductive step, assume (4.15) is valid for $i \leq k \leq n-1$. For the case $i = k+1$, insert $x = x_k$ into (4.12). By (4.6), We have

$$r\sum_{i=2}^{k+1} d_L^{(i)}(y) + \sum_{i=1}^{k} y_i d_K^{(i)}(x_k) + \sum_{i=2}^{k} d_K^{(i)}(x_k) d_L^{(i)}(y) = 0,$$

or,

$$d_L^{(k+1)}(y) = -\left( r\sum_{i=2}^{k} d_L^{(i)}(y) + \sum_{i=1}^{k} y^{(i)} d_K^{(i)}(x_k) + \sum_{i=2}^{k} d_K^{(i)}(x_k) d_L^{(i)}(y) \right) \Big/ r.$$

This and (4.15) for the cases $i \leq k$ show that $d_L^{(k+1)}(y)$ is a linear combination of $y_1, \ldots, y_k$, thus establishing (4.15) for the case $i = k$.

Equations (4.15) and (4.6) immediately implies the existence of an $n \times n$ matrix $M_L$ such that

$$(d_L^{(1)}(y), \ldots, d_L^{(n)}(y))^t = M_L (y^{(1)}, \ldots, y^{(n)})^t.$$

The same argument applied to $K$ will imply the existence of an $n \times n$ matrix $M_K$ such that

$$(d_K^{(1)}(x), \ldots, d_K^{(n)}(x))^t = M_K (x^{(1)}, \ldots, x^{(n)})^t.$$

This, (4.13), the definition of $\Psi_{K,I}$ (see (3.10)), and the definition of $d_K^{(i)}$ (see (4.3)) imply that

$$\Psi_{K,I}(x) = \frac{1}{2}\left(\Psi_{K,I}(x) - \Psi_{K,I}(-x)\right) = (\mathrm{I_n} + M_K)x,$$

where I is the identity matrix. Similarly,

$$\Psi_{L,I}(y) = (\mathrm{I_n} + M_L)y.$$

Now, since (4.12) is valid for all $x \in K$ and and $y \in L$, we have

$$M_L^t + M_K + M_K M_L^t = 0,$$

or equivalently

$$(\mathrm{I_n} + M_K)(\mathrm{I_n} + M_L)^t = \mathrm{I_n}.$$

This implies $\Psi_{L,I}(y) = (\mathrm{I_n} + M_K)^{-t}y$. To see that $(\mathrm{I_n} + M_K) \in \mathrm{SL}(n)$, we simply use the fact that Steiner symmetrization preserves volume. This settles the "only if" part of the lemma.

To see the "if" part, assume there is $T \in \mathrm{SL}(n)$ such that $\Psi_{K,I}(x) = Tx$ and $\Psi_{L,I}(y) = T^{-t}y$. Then $K_n = T^{-1}K$ and $L_n = T^t L$. That the equality holds in (4.8) follows trivially from a change of variable in integral. □

**Theorem 4.4.** *Let $K, L \subset \mathbb{R}^n$ be convex bodies and $\phi$ be an even convex function defined on $\mathbb{R}$. Then,*

$$\int_K \int_L \phi(x \cdot y) dx dy \geq \int_{B_K} \int_{B_L} \phi(x \cdot y) dx dy, \tag{4.16}$$

*Moreover, if $\phi$ is strictly convex, equality holds in (4.16) if and only if $K$ and $L$ are dilates of a pair of polar reciprocal origin-symmetric ellipsoids.*

*Proof.* By Lemma 3.5, there exists a sequence of ordered orthornomal bases $e_1^i, \ldots, e_n^i$ such that (3.7) holds. Let $K^i$ and $L^i$ be as defined in (3.8). Repeated use of Lemma 4.2 shows that

$$\int_K \int_L \phi(x \cdot y) dx dy \geq \int_{K^i} \int_{L^i} \phi(x \cdot y) dx dy.$$

Let $i$ go to $\infty$. By (3.7), we have

$$\int_K \int_L \phi(x \cdot y) dx dy \geq \int_{B_K} \int_{B_L} \phi(x \cdot y) dx dy.$$

The rest of the proof is dedicated to show the equality condition.

Suppose equality holds in (4.16). By Lemma 4.2,

$$\int_{K^{i-1}} \int_{L^{i-1}} \phi(x \cdot y) dx dy = \int_{K^i} \int_{L^i} \phi(x \cdot y) dx dy,$$

for each $i$. This and Lemma 4.3 imply that there exists $T_i \in \mathrm{SL}(n)$ such that

$$K^{i-1} = T_i(K^i), \qquad L^{i-1} = T_i^{-t}(L^i).$$

Let $G_i = T_1 \cdots T_i$. Hence $K = G_i(K^i)$ and $L = G_i^{-t}(L^i)$. This implies that $K$ and $L$ are $o$-symmetric and there are $r_0, R_0 > 0$ such that

$$r_0 B_n \subset K, L \subset R_0 B_n.$$

Equation (3.1) implies

$$r_0 B_n \subset K^i, L^i \subset R_0 B_n$$

for all $i \geq 1$. Hence,

$$G_i x, G_i^{-t} x \in R_0 B_n$$

for each $x \in r_0 B_n$, which implies that the sequences of linear transformations $G_i$ and $G_i^{-t}$ are uniformly bounded. Thus, there exists a convergent subsequence, which we also denote by $G_i$, such that

$$G_i \to \bar{G} \in \mathrm{SL}(n), \qquad \text{and } G_i^{-t} \to \bar{G}^{-t} \in \mathrm{SL}(n).$$

By the properties of the support function,

$$\begin{aligned}
|h_{G_i K^i}(u) - h_{\bar{G} B_K}(u)| &\leq |h_{G_i K^i}(u) - h_{G_i B_K}(u)| + |h_{G_i B_K}(u) - h_{\bar{G} B_K}(u)| \\
&= |h_{K^i}(G_i^t u) - h_{B_K}(G_i^t u)| + |h_{B_K}(G_i^t u) - h_{B_K}(\bar{G}^t u)| \\
&\leq \|h_{K_i} - h_{B_K}\|_\infty \cdot |G_i^t u| + |h_{B_K}(G_i^t u) - h_{B_K}(\bar{G}^t u)|.
\end{aligned}$$

This, the facts that the quantity $|G_i^t u|$ is bounded, and $G_i^t u \to \bar{G}^t u$ for each $u \in S^{n-1}$, imply that

$$h_{G_i K^i}(u) \to h_{\bar{G} B_K}(u),$$

for each $u \in S^{n-1}$. Note that $G_i K^i = K$. Hence $K = \bar{G} B_K$.

Using the same argument (but this time to $L$), we have $L = \bar{G}^{-t} B_L$. Since both $B_K$ and $B_L$ are Euclidean balls, we conclude that $K$ is an ellipsoid centered at the origin and that $L$ is a dilation of its polar.

To see that equality holds when $K$ is an ellipsoid centered at the origin and that $L$ is a dilation of its polar, one simply needs to use the change of variable formula for integrals. □

When $\phi(t) = |t|^p$, an immediate consequence is:

**Theorem 4.5.** *Let $K, L \subset \mathbb{R}^n$ be two convex bodies and $p \geq 1$. Then,*

$$\int_K \int_L |x \cdot y|^p dx dy \geq c_p |K|^{\frac{n+p}{n}} |L|^{\frac{n+p}{n}}, \tag{4.17}$$

*where $c_p$ is an easy-to-compute constant when $K = L = B$. Moreover, when $p > 1$, equality holds in (4.17) if and only if $K$ and $L$ are dilates of a pair of polar reciprocal origin-symmetric ellipsoids.*

*Proof.* Let $\phi(t) = |t|^p$. Note that when $p \geq 1$, the function $\phi$ is even and convex. By Theorem 1.1,

$$\int_K \int_L |x \cdot y|^p dxdy \geq \int_{B_K} \int_{B_L} |x \cdot y|^p dxdy.$$

Equation (4.17) follows immediately by homogeneity.

When $p > 1$, the function $\phi$ is strictly convex and the equality condition follows directly from the equality condition of Theorem 1.1. □

It is important to note that Theorem 4.5 is due to Lutwak, Yang & Zhang [36] and relies on the polar $L_p$ polar centroid body inequality established by Lutwak & Zhang [39]. See also Campi & Gronchi [7]. We would also like to refer the readers to [9], where stronger stochastic polar $L_p$-centroid body inequalities for measures with spherically decreasing densities (not just Lebesgue measure) were presented. The corresponding Orlicz versions follow similarly. A concrete special case can be found in Example 8 in Section 5 of [42] where level sets of the logarithmic Laplace transform are discussed.

We shall now show that Theorem 4.4 directly implies Theorem 1.1.

*Proof of Theorem 1.1.* By Theorem 4.4, if $\lambda > 0$ is such that

$$\frac{1}{|K||L|} \int_K \int_L \phi\left(\frac{x \cdot y}{\lambda}\right) dxdy \leq 1,$$

then, it must be the case that

$$\frac{1}{|B_K||B_L|} \int_{B_K} \int_{B_L} \phi\left(\frac{x \cdot y}{\lambda}\right) dxdy \leq 1. \tag{4.18}$$

Write $r_K = \omega_n^{-\frac{1}{n}} |B_K|^{\frac{1}{n}}$ and $r_L = \omega_n^{-\frac{1}{n}} |B_L|^{\frac{1}{n}}$. A change of variable $x = r_K x'$ and $y = r_L y'$ in (4.18) immediately implies

$$\begin{aligned} 1 &\geq \frac{1}{|B_K||B_L|} \int_{B_K} \int_{B_L} \phi\left(\frac{x \cdot y}{\lambda}\right) dxdy \\ &= \frac{1}{\omega_n^2} \int_B \int_B \phi\left(\frac{x' \cdot y'}{r_K^{-1} r_L^{-1} \lambda}\right) dxdy. \end{aligned}$$

This immediately implies that $N_\phi(K, L) \geq C_\phi |K|^{\frac{1}{n}} |L|^{\frac{1}{n}}$.

For the equality condition, assume now that $\phi$ is strictly convex. Equality holds in (1.2), if and only if there exists a common $\lambda_0 > 0$ such that

$$\frac{1}{|K||L|} \int_K \int_L \phi\left(\frac{x \cdot y}{\lambda_0}\right) dxdy = 1 = \frac{1}{|B_K||B_L|} \int_{B_K} \int_{B_L} \phi\left(\frac{x \cdot y}{\lambda_0}\right) dxdy.$$

The desired equality condition follows immediately from the equality condition in Theorem 4.4. □

Using layer-cake representation, we may now prove the promised Theorem 1.2.

*Proof of Theorem 1.2.* For each $t, s > 0$, define

$$K_t = \{x \in \mathbb{R}^n : f(x) > t\}, \qquad L_s = \{y \in \mathbb{R}^n : g(y) > s\}.$$

Since $f$ is integrable, then $K_t$ are bounded for $t > 0$. Similarly, $L_s$ are bounded for $s > 0$.

Since $f$ and $g$ are quasi-concave, both $K_t$ and $L_s$ are convex, and hence their boundary are of measure 0 with respect to the Lebesgue measure. By this, the layer-cake representation, Theorem 4.4, and the definition of rearrangement, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \lambda(x\cdot y)f(x)g(y)dxdy &= \int_0^\infty ds\int_0^\infty dt\int_{K_t}\int_{L_s}\lambda(x\cdot y)dxdy\\ &= \int_0^\infty ds\int_0^\infty dt\int_{\mathrm{cl}K_t}\int_{\mathrm{cl}L_s}\lambda(x\cdot y)dxdy\\ &\geq \int_0^\infty ds\int_0^\infty dt\int_{B_{K_t}}\int_{B_{L_s}}\lambda(x\cdot y)dxdy\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\lambda(x\cdot y)f^*(x)g^*(y)dxdy,\end{aligned} \tag{4.19}$$

where $\mathrm{cl}K_t$ and $\mathrm{cl}L_s$ are the closure of $K_t$ and $L_s$ respectively.

To see the equality condition when $\lambda$ is strictly convex, assume the equality holds. Then, for almost all $t \in [0,\infty)$ and almost all $s \in [0,\infty)$,

$$\int_{\mathrm{cl}K_t}\int_{\mathrm{cl}L_s}\lambda(x\cdot y)dxdy = \int_{K_t^*}\int_{L_s^*}\lambda(x\cdot y)dxdy.$$

By the equality condition in Theorem 4.4, there is a linear tansform $T \in \mathrm{SL}(n)$, such that for almost all $t \in [0,\infty)$ and almost all $s \in [0,\infty)$, $\mathrm{cl}K_t = TB_{K_t}$ and $\mathrm{cl}L_s = T^{-t}B_{L_s}$. This shows the "only if" part of the equality condition.

To see the "if" part of the equality condition, one only needs to use the equality condition in Theorem 4.4 to conclude that equality holds in (4.19).

□

## 5. Appendix

The following example in $\mathbb{R}^2$ shows precisely why we need the new symmetrization scheme in Lemma 4.4.

Let $m$ be an arbitrary integer. Consider the convex bodies

$$K = \left\{(x^{(1)}, x^{(2)}) \in \mathbb{R}^2 : -m \leq x^{(1)} \leq m, -x^{(1)} - \frac{1}{m} \leq x^{(2)} \leq -x^{(1)} + \frac{1}{m}\right\},$$

and

$$L = \left\{(y^{(1)}, y^{(2)}) \in \mathbb{R}^2 : -m \leq y^{(1)} \leq m, y^{(1)} - \frac{1}{m} \leq y^{(2)} \leq y^{(1)} + \frac{1}{m}\right\}.$$

Note that $K$ is the convex hull generated by $\{-m\}\times[m-\frac{1}{m}, m+\frac{1}{m}]$ and $\{m\}\times[-m-\frac{1}{m}, -m+\frac{1}{m}]$ and $L$ is the convex hull generated by $\{-m\}\times[-m-\frac{1}{m}, -m+\frac{1}{m}]$ and $\{m\}\times[m-\frac{1}{m}, m+\frac{1}{m}]$.

Let $e = (0,1) \in S^1$. Then by the definition of Steiner symmetrization (given in Section 3), it is simple to show that $S_eK = S_eL = [-m,m]\times[-\frac{1}{m}, \frac{1}{m}]$.

**Example.** For sufficiently large $m$, the convex bodies $K$ and $L$ as defined above satisfy

$$\int_K \int_L |x\cdot y| dy dx < \int_{S_e K}\int_L |x\cdot y|dydx, \tag{5.1}$$

and

$$\int_K \int_L |x\cdot y| dy dx < \int_{S_e K}\int_{S_e L} |x\cdot y|dydx. \tag{5.2}$$

The statements above follow from direct computation. By definition of $K$ and $L$,

$$\int_K\int_L |x\cdot y|dydx = \int_{-m}^{m}\int_{-m}^{m}\int_{-x^{(1)}-\frac{1}{m}}^{-x^{(1)}+\frac{1}{m}}\int_{y^{(1)}-\frac{1}{m}}^{y^{(1)}+\frac{1}{m}} |x^{(1)}y^{(1)}+x^{(2)}y^{(2)}|dy^{(2)}dx^{(2)}dy^{(1)}dx^{(1)}$$

By the change of variable $u_1 = x^{(1)}/m$, $v_1 = y^{(1)}/m$, $u_2 = m(x^{(2)}+x^{(1)})$, and $v_2 = m(y^{(2)}-y^{(1)})$, we have

$$\begin{aligned}
\int_K\int_L |x\cdot y|dxdy &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left|\frac{1}{m^2}u_2v_2 + v_1u_2 - u_1v_2\right| dv_2du_2dv_1du_1\\
&\le \frac{4}{m^2}\int_{-1}^{1}\int_{-1}^{1}|u_2v_2|du_2dv_2 + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}|v_1u_2-u_1v_2|\,dv_2du_2dv_1du_1\\
&\to \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}|v_1u_2-u_1v_2|\,dv_2du_2dv_1du_1,
\end{aligned}$$

as $m\to\infty$.

Similarly,

$$\begin{aligned}
\int_{S_eK}\int_L |x\cdot y|dydx &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left|m^2u_1v_1 + \frac{1}{m^2}u_2v_2 + v_1u_2\right| dv_2du_2dv_1du_1\\
&\ge 4m^2\int_{-1}^{1}\int_{-1}^{1}|u_1v_1|\,dv_1du_1 - \frac{4}{m^2}\int_{-1}^{1}\int_{-1}^{1}|u_2v_2|\,dv_2du_2\\
&\quad - 4\int_{-1}^{1}\int_{-1}^{1}|v_1u_2|\,du_2dv_1\\
&\to\infty,
\end{aligned}$$

as $m\to\infty$.

Also,

$$\begin{aligned}
\int_{S_eK}\int_{S_eL} |x\cdot y|dydx &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left|m^2u_1v_1 + \frac{1}{m^2}u_2v_2\right| dv_2du_2dv_1du_1\\
&\ge 4m^2\int_{-1}^{1}\int_{-1}^{1}|u_1v_1|\,dv_1du_1 - \frac{4}{m^2}\int_{-1}^{1}\int_{-1}^{1}|u_2v_2|\,dv_2du_2\\
&\to\infty,
\end{aligned}$$

as $m\to\infty$.

Hence, for sufficiently large $m$, both (5.1) and (5.2) are valid.

The above example shows that in $\mathbb{R}^2$, applying Steiner symmetrization once is not good enough to show Theorem 4.4. Similar counterexample can be constructed in $n$-dimensional case, and it will imply that $(n-1)$-times Steiner symmetrization is not enough to prove Theorem 4.4.

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