Orlicz affine isoperimetric inequalities for star bodies

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Abstract. The original goal of this paper is to extend the affine isoperimetric inequality and Steiner type inequality of Orlicz projection bodies (which originated to Lutwak, Yang, and Zhang [36]), from convex bodies to *Lipschitz star bodies* (whose radial functions are locally Lipschitz).

In order to achieve it, we investigate the graph functions of the given Lipschitz star body K: Along almost all directions u, we can define the graph functions on an open dense subset of the orthogonal projection of K onto $u^{\perp}$.

Keywords and Phrases. Affine isoperimetric inequality, Lipschitz star bodies, Orlicz Petty projection body, graph functions.

1 Introduction

The classical isoperimetric inequality is formulated by

$$S(K) \geq n\omega_n^{1/n}|K|^{(n-1)/n},$$

where $S(K)$ and $|K|$ denote respectively the surface area and volume of a domain $K \subset \mathbb{R}^n$, and ω_n denotes the volume of the unit ball B. If K happens to be a *convex body* in $\mathbb{R}^n$, then *Cauchy's integral formula* of surface area tells us

$$S(K) = \frac{1}{\omega_{n-1}}\int_{S^{n-1}} |P_{u^{\perp}}K|du. \tag{1.1}$$

Here $|P_{u^{\perp}}K|$ denotes the $(n-1)$-dimensional volume of the orthogonal projection of K onto $u^{\perp}$. While the volume is *affine invariant* in the sense that $|AK| = |K|$ for any $A \in SL(n)$, the surface area $S(K)$ is not. Motivated by (1.1), the *integral affine surface area* $\Phi(K)$ is defined by

$$\Phi(K) = \frac{c_n}{\omega_{n-1}}\Big(\int_{S^{n-1}} |P_{u^{\perp}}K|^{-n}du\Big)^{-1/n},$$

where $c_n = (n\omega_n)^{(n+1)/n}$ is a normalizing constant so that $\Phi(B) = S(B)$. This quantity is proved to be affine invariant (see e.g., [29, 39, 48]), and is different from the affine surface area via a differential geometric viewpoint [20, 31, 41, 44].

Petty [39] proved the affine isoperimetric inequality

$$\Phi(K) \geq n\omega_n^{1/n}|K|^{(n-1)/n}. \tag{1.2}$$

For convex bodies, (1.2) is stronger than the classical one, since one can easily see that

$$S(K) \geq \Phi(K).$$

Note that the classical isoperimetric inequality holds for general *non-convex* sets, and it seems natural to watch the affine counterparts.

The integral affine surface area $\Phi(K)$ can be computed as the volume of the so-called *polar projection body* by Minkowski, which dates back to the turn of 20th century. For a quick review of its history, we would like to refer the interested readers to [33]. Two fundamental affine inequalities related to the projection body are the Petty and Zhang projection inequalities [46], which respectively characterized the ellipsoids and simplexes. Utilising the Petty projection inequality and a *convexification* method (from the solution of Minkowski problem), Zhang [47] established the affine Sobolev inequality, which connects affine convex geometry and functional analysis and is stronger than the classical Sobolev inequality. Lutwak [29] conjectured in 1984 that the generalizations of (1.2) hold, for the *k-th affine quermassintegrals*, which is not confirmed until the recent work of Milman & Yehudayoff [38].

During the past four decades, extensive studies related to the Petty projection inequality and its functional analogs sprang out, including [7, 16, 17, 22, 24, 25, 27, 33, 34, 36, 42]. Ludwig [25, 28], and Haberl [15] (see Li & Leng [21] for the L_∞ case) characterized the L_p projection body by affine contravariant valuation properties, and these leads to a study of nonsymmetric projection bodies, see e.g. [16]. Ludwig-Xiao-Zhang [27] studied the affine inequality of L_p projection body for star bodies with Lipschitz boundary. In 2010, Lutwak-Yang-Zhang [35, 36] respectively evolved the centroid and projection inequalities into the Orlicz setting. After that the Orlicz Brunn-Minkowski theory developed rapidly, see e.g. [3, 11–14, 18, 22, 24, 45, 49, 50].

Suppose K is a convex body containing the origin in its interior. Its *Orlicz projection body* $\Pi_\phi K$ is a convex body whose support function is

$$h_{\Pi_\phi K}(z) = \inf\left\{\lambda > 0 : \int_{\partial K} \phi\left(\frac{z\cdot\nu_K(x)}{\lambda x\cdot\nu_K(x)}\right) x\cdot\nu_K(x)d\mathcal{H}^{n-1}(x) \leq n|K|\right\}. \tag{1.3}$$

Here $\phi : \mathbb{R} \to [0, \infty)$ is a convex function such that $\phi(0) = 0$, and ν_K is the unit normal of K at x. This means that ϕ must be decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. We will assume throughout the article that one of these is strictly so; i.e., ϕ is either strictly decreasing on $(-\infty, 0]$ or strictly increasing on $[0, \infty)$.

It is straightforward to extend definition (1.3) to non-convex sets, provided with "nice" boundary conditions. However, the Orlicz Petty projection inequality were merely studied for convex bodies until now. Technically, to extend definition (1.3), we need that the normal ν_K needs to be well-defined, and the quantity $x \cdot \nu_K$ needs to be positive. The *Lipschitz star body* (whose *radial function* is locally Lipschitz, see Section 2 for the detailed definition of radial function) naturally comes to our sight. Note that it is a natural extension since the radial function of a convex body is always locally Lipschitz. Now we state our main result.

Theorem 1.1. *If K is a Lipschitz star body in $\mathbb{R}^n$, then the volume ratio*

$$|\Pi_\phi^* K|/|K|$$

is maximized when K is an ellipsoid centered at the origin. If ϕ is strictly convex, then the ellipsoids centered at the origin are the only maximizers.

For other extensions of the classical and L_p projection inequalities [27, 42], the authors used the idea of "convexification" mentioned above. It seems that this convexification method cannot be applied to the Orlicz case.

Our Theorem 1.1 is proved by using the *Steiner symmetrization*, which is a well-known powerful tool for the isoperimetric problem as well as its affine analogue. For the *sets of finite perimeter*, Chlebík-Cianchi-Fusco [5] characterized the equality cases of the Steiner's inequality

$$S(E) \geq S(S_u E).$$

Here $S_u E$ denotes the Steiner symmetrization of E (see Section 2). In [2], Barchiesi-Cagnetti-Fusco studied its stability for general Steiner symmetrization. However, it is still unknown if there are the affine isoperimetric inequalities and Steiner's inequalities of general affine surface areas, for non-convex sets.

After an investigation of the *graph functions* of star bodies (see the text below), for almost all $u \in S^{n-1}$, we establish the Steiner type affine inequality in Section 7

$$|\Pi_\phi^* S_u K| \geq |\Pi_\phi^* K|.$$

In our humble opinion, one technical obstacle to the Orlicz affine inequalities and Steiner type inequalities for non-convex sets in a geometric way might be that there is no corresponding "graphfunctions". Suppose that K is a convex body. Given a direction u, we denote K' to be the orthogonal projection of K onto $u^{\perp}$, and (x', t) to be the point $x' + tu$. Then, there are graph functions f and g representing the convex body K, such that

$$K = \Big\{(x', t) : f(x') \leq t \leq g(x'), \quad x' \in K'\Big\}.$$

The *representation* means that, given an $(n-1)$-dimensional convex body $K' \subset u^{\perp}$, and concave functions g and $-f$, we can recover the convex body K by

$$\bigcup_{x' \in K'} \big[(x', f(x')), (x', g(x'))\big], \tag{1.4}$$

and for almost all direction u, the boundary of K is the union of coordinate surfaces

$$\partial K = \Big\{\big(x, f(x')\big) : x' \in K'\Big\} \cup \Big\{\big(x, g(x')\big) : x' \in K'\Big\}. \tag{1.5}$$

Here $\big[(x', f(x')), (x', g(x'))\big]$ denotes the line segment connecting the two points. These facts were important in [16, 33, 36]. Because of these, we dive into the following

Question: Does a set of finite perimeter and its reduced boundary have the geometric representations in the form of (1.4) *and* (1.5) *via graph functions?*

If no condition is assumed, the structures of the reduced boundary and the projection of a set of finite perimeter can be very wild! Lipschitz star bodies can simplify the situation to some extent, as we will see in sections 2-4. Our main result regarding to the graph functions reads as follows.

Theorem 1.2. *Let K be a Lipschitz star body in $\mathbb{R}^n$. Then, for almost all $u \in S^{n-1}$, there is a sequence of disjoint open subsets $G_m \subset K'$, and two sequences of graph functions*

$$f_j, g_j : \bigcup_{m=j}^{\infty} G_m \to \mathbb{R},$$

satisfying

(i) $\bigcup_{m=1}^{\infty} G_m$ is open dense in K', and $f_1 < g_1 < \cdots < f_j < g_j$;

(ii) K has the representation (if we neglect an $\mathcal{H}^n$-null set)

$$K = \bigcup_{m=1}^{\infty} \bigcup_{\substack{x' \in G_m \\ 1 \leq j \leq m}} \big[(x', f_j(x')), (x', g_j(x'))\big],$$

and ∂K has the representation (if we neglect an $\mathcal{H}^{n-1}$-null set)

$$\partial K = \bigcup_{m=1}^{\infty} \bigcup_{j=1}^{m} \Big\{ \big(x, f_j(x')\big) : x' \in G_m \Big\} \cup \Big\{ \big(x, g_j(x')\big) : x' \in G_m \Big\}.$$

Theorem 1.2 is obtained by combining Theorem 3.1 and Corollary 4.2.

The remaining part of this paper is organized as follows. Section 2 collects some notations, definitions, and basic facts. The construction of graph functions appears in section 3. In section 4, we obtain the canonical area formulas of Lipschitz star bodies in regard to the graph functions, thanks to applications of geometric measure theory. Section 5 studies the radial convergence of Steiner symmetrization via "admissible" directions. In Section 6, we show the properties of the polar Orlicz projection body, including the affine covariant property (Lemma 6.4), and the "weak convergence" for Lipschitz star bodies (Lemma 6.3). Finally, Theorem 1.1 is proved in Section 7.

2 Preliminaries

We collect some basic definitions and facts of convex geometry and measure theory. The books [8–10, 20, 37, 40] are good general references.

2.1 Basic Notation

Let B and S^{n-1} respectively denote the Euclidean unit ball and unit sphere. Suppose $u \in S^{n-1}$ is a certain direction. We will always denote K' to be the orthogonal projection of K onto $u^{\perp}$; and for an $x \in \mathbb{R}^n$, x' always denotes its projection onto $u^{\perp}$. In this sense, we will write $x = (x', t)$, which is understood as

$$x = x' + tu, \quad \text{where } x' \in u^{\perp}.$$

For $x' \in u^{\perp}$, $l_u(x')$ denotes the ling passing through x' and parallel to u. If E is a Borel subset of $\mathbb{R}^n$ and E is of *Hausdorff dimension* k, then we may use $|E|$ to denote $\mathcal{H}^k(E)$. Let ∂E and int E denote the boundary of E and the interior of E, respectively.

Let K, L be two compact sets in $\mathbb{R}^n$. Their *Minkowski sum* $K + L$ is defined by

$$K + L = \{x + y : \ x \in K, \ y \in L\}.$$

For $\lambda > 0$, the *scalar multiplication* λK is given by

$$\lambda K = \{\lambda x : \ x \in K\}.$$

The *Hausdorff distance* between the compact sets is defined by

$$\delta_H(K, L) = \min\{t \geq 0 : \; K \subset L + tB, \; L \subset K + tB\}.$$

If a sequence of compact sets $\{K_i\}$ satisfies $\delta_H(K_i, K_0) \to 0$, then we say that K_i converges to K_0 in Hausdorff metric. We shall make use of a fact: The Hausdorff limit of a convergent sequence of compact sets is unique.

Let $\mathcal{C}$ be the class of convex function $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(0) = 0$ and such that ϕ is either strictly decreasing on $(-\infty, 0]$ or ϕ is strictly increasing on $[0, \infty)$. The subclass of $\mathcal{C}$ consisting of those $\phi \in \mathcal{C}$ that are strictly convex will be denoted by $\mathcal{C}_s$. Define c_ϕ by

$$c_\phi = \max\{c > 0 : \; \max\{\phi(c), \phi(-c)\} \leq 1\}. \tag{2.1}$$

We shall make use of a basic fact for convex functions: If $\phi \in \mathcal{C}$, for $a, b \in \mathbb{R}$ and $a \neq 0$, then the function

$$\Psi(t) := \phi(at - b) + \phi(-at - b), \; t > 0 \tag{2.2}$$

is increasing. Moreover, if ϕ is strictly convex, then $\Psi(t)$ is strictly increasing.

2.2 Convex bodies

Suppose K is a convex body. Let $h_K : \mathbb{R}^n \to \mathbb{R}$ denote its *support function*, namely,

$$h_K(x) = \max\{x \cdot y : \; y \in K\}.$$

It is *sublinear* (positively homogeneous and convex). An important fact is that one can uniquely construct a convex body from a sublinear function of $\mathbb{R}^n$ into $\mathbb{R}$.

It follows immediately that for any general linear transform $A \in GL(n)$, there is

$$h_{AK}(x) = h_K(A^t x), \;\; x \in \mathbb{R}^n.$$

If K contains the origin in its interior, then its *polar body* K^* is defined by

$$K^* = \{x \in \mathbb{R}^n : \; x \cdot y \leq 1 \text{ for all } y \in K\}.$$

It is easy to see that

$$(AK)^* = A^{-t} K^*, \quad \forall \; A \in GL(n). \tag{2.3}$$

2.3 Star bodies: Representing functions, radial distance, and Lipschitz constants

A compact set $K \subset \mathbb{R}^n$ is *star-shaped* (with respect to the origin) if the intersection of every line through the origin with K is a line segment. Denote $\rho_K : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ to be its *radial function*, namely

$$\rho_K(x) = \max\{\lambda \geq 0 : \ \lambda x \in K\}. \tag{2.4}$$

K is called a *star body* if ρ_K is positive and continuous. Let $\mathcal{S}_o^n$ denote the class of star bodies in $\mathbb{R}^n$. Clearly,

$$\rho_{cK}(x) = c\rho_K(x), \quad \rho_K(cx) = c^{-1}\rho_K(x), \quad \forall c > 0.$$

Throughout this paper, we set

$$r_K = \min_{u \in S^{n-1}} \rho_K(u), \ \text{ and } \ R_K = \max_{u \in S^{n-1}} \rho_K(u). \tag{2.5}$$

Define the *gauge function* $p_K : \mathbb{R}^n \to [0, \infty)$ by

$$p_K(x) = \rho_K(x)^{-1} \quad \text{when } x \neq 0, \quad \text{and} \quad p_K(0) = 0, \tag{2.6}$$

which is 1 homogeneous. Clearly, if ρ_K is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$, then p_K is locally Lipschitz on $\mathbb{R}^n$. If K happens to be a convex body, then

$$p_K(x) = h_{K^*}(x) = \rho_K(x)^{-1}, \ \ x \in \mathbb{R}^n \setminus \{0\}.$$

For two star bodies K and L, define their *radial distance* by

$$\tilde{\delta}(K, L) := |\rho_K - \rho_L|_\infty = \max_{u \in S^{n-1}} |\rho_K(u) - \rho_L(u)|. \tag{2.7}$$

A sequence $\{K_i\} \subset \mathcal{S}_o^n$ is said to be converging to a star body K with respect to radial distance, if $\tilde{\delta}(K_i, K) \to 0$. A useful fact is that

$$\delta_H(K, L) \leq \tilde{\delta}(K, L),$$

and hence the radial convergence implies the Hausdorff convergence.

If ρ_K is locally Lipschitz continuous on $\mathbb{R}^n \setminus \{0\}$, then we say K is a *Lipschitz star body*. The class of Lipschitz star bodies will be denoted by $\bar{\mathcal{S}}_o^n$. We remark that a convex body containing the origin in its interior is always in $\bar{\mathcal{S}}_o^n$. Let $\Omega \subset\subset \mathbb{R}^n \setminus \{0\}$ (compactly contained). We denote $L_K(\Omega)$ to be the Lipschitz constant of ρ_K on Ω:

$$L_K(\Omega) = \sup\Big\{\frac{|\rho_K(x) - \rho_K(y)|}{|x - y|} : x, y \in \Omega, \ \text{and } x \neq y\Big\}. \tag{2.8}$$

Since ρ_K is -1-homogeneous, the following statements are clearly equivalent

1. K is a Lipschitz star body.

2. ρ_K is Lipschtz on the sphere S^{n-1}.

3. ρ_K is Lipschtz on $(RK) \setminus \big(\mathrm{int}(rK)\big)$ with $0 < r < R$.

For convenience, we choose a large enough set and denote

$$L_K = L_K\Bigg(\Big(\frac{2R_K}{r_K}K\Big\backslash\frac{r_K}{2R_K}\mathrm{int}K\Big)\bigcup\Big(\frac{2}{r_K}K\Big\backslash\frac{1}{2R_K}\mathrm{int}K\Big)\Bigg). \tag{2.9}$$

Note that both S^{n-1} and ∂K are compactly contained in the interior of the set in (2.9). Therefore, $L_K(S^{n-1}) \leq L_K$. We will use the constants L_K and $L_K(S^{n-1})$ in this paper.

2.4 Boundary structure of the Lipschitz star body

Let $K \in \bar{\mathcal{S}}^n_o$. Since ρ_K is Lipschitz on S^{n-1}, it can be seen that ∂K is $\mathcal{H}^{n-1}$-*rectifiable*, and hence K is a *set of finite perimeter*. We refer to [1, 9, 37] for good and general literature in regard to this topic.

Denote its *reduced boundary* by

$$\partial^* K = \{x \in \partial K : \text{ the gradient } \nabla\rho_K(x) \text{ exists at } x\},$$

and it coincides with the reduced boundary in the theory of sets of finite perimeter. For $x \in \partial^* K$, the *unit outer normal vector* is given by

$$\nu_K(x) = -\frac{\nabla\rho_K(x)}{|\nabla\rho_K(x)|}. \tag{2.10}$$

It coincides with the outer normal vector in the meaning of differential geometry as well as in the measure theory. The computation in Section 4 indicates that $\nu_K(x)$ can equivalently be represented by gauge function p_K.

The following lemma shows that $|\nabla\rho_K|$ never vanishes, and $x \cdot \nu_K(x)$ is strictly positive and uniformly bounded from below on $\partial^* K$. Note that $x \cdot \nu_K(x)$ may be zero or negative if K is supposed to be a general set, e.g., a set of finite perimeter. This partly explains why we consider Lipschitz star bodies in current paper, when studying the Orlicz Petty projection bodies.

Lemma 2.1. *If $K \in \bar{\mathcal{S}}^n_o$, then $\mathcal{H}^{n-1}(\partial K)$ is finite, and*

$$\mathcal{H}^{n-1}(\partial K \backslash \partial^* K) = 0. \tag{2.11}$$

Moreover, for $x \in \partial^* K$, we have

$$\frac{1}{L_K} \leq x \cdot \nu_K(x) \leq R_K. \tag{2.12}$$

Proof. 1. The statement $\mathcal{H}^{n-1}(\partial K) < \infty$ follows from the fact that $\rho_K(u)u : S^{n-1} \to \partial K$ is a Lipschitz map.

Since $\rho_K : \mathbb{R}^n \backslash \{0\} \to (0, \infty)$ is locally Lipschitz, the gradient $\nabla \rho_K(x)$ exists for $\mathcal{H}^n$-a.e. $x \in \mathbb{R}^n$. Since ρ_K is -1 homogeneous,

$$\rho_K(tx) = \frac{1}{t}\rho_K(x), \quad t > 0, \quad x \neq 0, \tag{2.13}$$

$\nabla \rho_K(x)$ exists if and only if $\nabla \rho_K(tx)$ exists for any $t > 0$. Therefore, $\nabla \rho_K(u)$ exists $\mathcal{H}^{n-1}$-a.e. on S^{n-1}. This, together with the fact that $\rho_K(u)u : S^{n-1} \to \partial K$ is a Lipschitz map, implies that $\nabla \rho_K(x)$ exists for $\mathcal{H}^{n-1}$-a.e. $x \in \partial K$.

2. Differentiating the equation (2.13) at $t = 1$ gives

$$-x \cdot \nabla \rho_K(x) = \rho_K(x). \tag{2.14}$$

Therefore, for $x \in \partial^* K$, we have

$$R_K |\nabla \rho_K(x)| \geq |x||\nabla \rho_K(x)| \geq \rho_K(x) = 1. \tag{2.15}$$

Recall the Lipschitz constant L_K given by (2.9). For $x \in \partial^* K$ and sufficiently small t, we have

$$|\rho_K(x + tu) - \rho_K(x)| \leq L_K |t|, \quad \forall u \in S^{n-1},$$

and hence

$$|\nabla \rho_K(x)| \leq L_K. \tag{2.16}$$

By(2.10), (2.14), (2.15) and (2.16), for $x \in \partial^* K$, we have

$$\frac{1}{L_K} \leq x \cdot \nu(x) = \frac{1}{|\nabla \rho_K(x)|} \leq R_K.$$

$\square$

For $u \in S^{n-1}$, let

$$\partial_u^* K = \{x \in \partial^* K : \nu(x) \cdot u = 0\}. \tag{2.17}$$

The following lemma is an analogue of the Ewald-Larman-Rogers theorem of convex bodies, and it is a corollary of [24, Lemma 5.9]. We note that the openness of Ω in [24] was indeed not necessary. For completeness, we restate the results in the Appendix of current paper.

Lemma 2.2. *Suppose $K \in \bar{\mathcal{S}}_o^n$. There is a set $T(K) \subset S^{n-1}$, such that $\mathcal{H}^{n-1}\big(S^{n-1} \setminus T(K)\big) = 0$ and for any $u \in T(K)$,*

$$\partial_u^* K = \{x \in \partial^* K : \nu(x) \cdot u = 0\}$$

is an $\mathcal{H}^{n-1}$-null set.

2.5 Steiner symmetrization

Steiner symmetrization is a classical and well-known device, which has seen a number of applications to problems of geometric and functional nature, see, e.g., [4–6, 16, 22, 24].

Let K be a compact subset of $\mathbb{R}^n$, and $u \in S^{n-1}$. The *Steiner symmetral* $S_u K$ along with u is defined by

$$S_u K := \Big\{(x', t) : \ |t| \leq \frac{|l_u(x') \cap K|}{2}, x' \in K'\Big\}.$$

If K happens to be a convex body, $S_u K$ turns out to be a new convex body whose graph functions are $(g - f)/2$ and $(f - g)/2$. Here we recall that the definitions of the graph functions f and g are in Section 1.

When K happens to a Lipschitz star body, as studied in the next whole section, we can define its j-th graph functions on an open dense subset of K', $\bigcup_{m=1}^{\infty} G_m$ say, and the new body $S_u K$ can be formulated by (3.6).

We recall some basic facts of Steiner symmetrization. If K is a star body, then

$$S_u K + \varepsilon B \subset S_u(K + \varepsilon B), \quad \forall \varepsilon > 0. \tag{2.18}$$

A basic fact is that the radial distance between K and a centered Ball is non-increasing under Steiner symmetrization. Actually, if K is a star body, such that $\tilde{\delta}(K, rB) \leq a$ with $0 < a < r$, then one can deduce from the definition that

$$(r - a)B \subset S_u K \subset (r + a)B. \tag{2.19}$$

It is easy to see from the definition that the radial distance of a star body K and a centered Ball is non-increasing under Steiner symmetrization. The following tool developed in [36] will also be used.

Lemma 2.3. *[36, Lemma 1.1] Suppose K, M are convex bodies containing the origin in their interiors, and consider $K, M \subset \mathbb{R}^{n-1} \times \mathbb{R}$. Then $S_{e_n} K^* \subset M^*$ if and only if*

$$h_K(x', t) = 1 = h_K(x', -s), \ \text{ with } t \neq -s \implies h_M\left(x', \frac{1}{2}t + \frac{1}{2}s\right) \leq 1.$$

In addition, if $S_{e_n}K^* = M^*$, then $h_K(x', t) = 1 = h_K(x', -s)$, with $t \neq -s$, implies that $h_M(x', \frac{1}{2}t + \frac{1}{2}s) = 1$.

3 The graph functions of Lipschitz star bodies

In order to locally define the graph functions (Lemma 3.1), it is quite natural to consider the implicit function theorem. Since we do not have the continuous differentiability, we would like to invoke the implicit function theorem in [19, Theorem E]. Since the [19, Theorem E] was proved by using Brouwer's fixed point theorem, we do not directly obtain the uniqueness of the implicit function. We consider it in Lemma 3.3.

Lemma 3.1. *Suppose $x_0 = (x_0', t_0) \in \partial^* K$, and $u \cdot \nu(x_0) \neq 0$. Then, for each small neighbourhood $I_0 = (t_0 - \epsilon, t_0 + \epsilon)$, there exists a neighborhood of x_0', say $N_{x_0'}$, and there is a continuous function $g(x')$ defined on $N_{x_0'}$, such that $g(N_{x_0'}) \subset I_0$, g is differentiable at x_0' and*

$$\nu(x_0) = \operatorname{sgn}(u \cdot \nu(x_0)) \frac{(-\nabla g(x_0'), 1)}{\sqrt{1 + |\nabla g(x_0')|^2}}. \tag{3.1}$$

Here ∇g means the gradient of g in $u^\perp$, and is also understood as a vector in $\mathbb{R}^n$ in the sense $u^\perp \subset \mathbb{R}^n$.

Proof. Since $x_0 \in \partial^* K$ and $u \cdot \nu(x_0) \neq 0$, we know that ρ_K is differentiable at $x_0 = (x_0', t_0)$, and $u \cdot \nabla \rho_K(x_0', t_0) \neq 0$. Then, by the implicit function theorem without continuous differentiability assumption (see e.g. [19, Theorem E]), the equation

$$\rho_K(x', t) = 1$$

determines a continuous implicit function $t = g(x')$, in a neighborhood of x_0'. And the function g is differentiable at x_0', such that $g(N_{x_0'}) \subset I_0$ and

$$\rho_K(x', g(x')) = 1.$$

This means $(x', g(x')) \in \partial K$, and the computation of $\nu(x_0)$ follows immediately from (2.10) and the implicit function theorem. $\square$

Now we construct a equivalent subset of $\mathrm{int} K'$, on which we can define the graph functions.

Lemma 3.2. *Suppose $u \in S^{n-1}$ and $\mathcal{H}^{n-1}(\partial_u^* K) = 0$. Let $\partial^0 K = \partial^* K \setminus \partial_u^* K$, and let*

$$K_0' = K' \setminus \Big(P_{u^\perp}(\partial K \setminus \partial^0 K)\Big).$$

be the relative complement of the orthogonal projection of $\partial K \setminus \partial^0 K$ onto $u^\perp$. Then, we have

(i) $\partial^0 K$ is equivalent to ∂K, and K_0' is equivalent to K'. That is to say,

$$\mathcal{H}^{n-1}(\partial K \setminus \partial^0 K) = 0, \quad \text{and} \quad \mathcal{H}^{n-1}(K' \setminus K_0') = 0.$$

(ii) For any $x' \in K_0'$, the intersection $l_{x'} \cap \partial K$ is contained in $\partial^* K$ and has finite elements.

(iii) $K_0' \subset \mathrm{int} K'$, and $\mathcal{H}^{n-1}(\mathrm{rel}\partial K_0') = 0$.

Proof. (i) It follows from Lemma 2.1 that $\mathcal{H}^{n-1}(\partial K \setminus \partial^* K) = 0$. This together with the assumption $\mathcal{H}^{n-1}(\partial_u^* K) = 0$ implies $\mathcal{H}^{n-1}(\partial K \setminus \partial^0 K) = 0$. Since the projection map is a contraction, we also have $\mathcal{H}^{n-1}(K' \setminus K_0') = 0$.

(ii) Let $x' \in K_0'$ and $(x', t) \in l_{x'} \cap \partial K$. That $l_{x'} \cap \partial K \subset \partial^* K$ is obvious from the definition of K_0'. We claim that (x', t) must be an isolated point of the compact set $l_{x'} \cap \partial K$. To see this, we observe from our construction that (x', t) lies in $\partial^0 K$, and hence $\rho_K(x', t)$ is differentiable. If (x', t) is not isolated, there is a disjoint sequence $t_k \to t$ so that $\rho_K(x', t_k) = 1$, and then the directional derivative $u \cdot \nabla \rho_K(x', t) = 0$, which contradicts to the fact $(x', t) \notin \partial_u^* K$.

Now we have shown our claim. Therefore, $l_{x'} \cap \partial K$, as a compact set of isolated points, must be finite.

(iii) Let $x' \in K_0'$, and $(x', t) \in l_{x'} \cap \partial K$. If $x' \in \mathrm{rel}\partial K_0'$, since (x', t) is a differential point of ρ_K, we must have $\nu_K(x', t) \cdot u = 0$, which is a contradiction. For the statement that $\mathcal{H}^{n-1}(\mathrm{rel}\partial K_0') = 0$, we only need to notice that K' is an $(n-1)$ dimensional star body. $\square$

Note that our local graph function obtained in Lemma 3.1 used the implicit function theorem in [19], and it may not be unique. But for the Lipschitz star bodies, we have the following.

Lemma 3.3. *Let $x' \in K_0'$, and $(x', t) \in \partial^* K$. Suppose there is an open ball $N_0 \subset u^\perp$ containing x', and there is a continuous function $f : N_0 \to \mathbb{R}$, such that*

$$f(x') = t, \quad \text{and} \quad (y', f(y')) \in \partial K, \quad y' \in N_0.$$

Then, the function satisfying conditions above is unique on N_0.

Proof. Suppose there are two functions f_1 and f_2 defined on N_0, such that for $j = 1, 2$,

$$f_j(x') = t, \quad \text{and} \quad (y', f_j(y')) \in \partial K, \quad y' \in N_0.$$

Let $M_0 \subset N_0$ be the set of all points $y' \in N_0$ so that $f_1(y') = f_2(y')$. We will show that M_0 is open and closed in N_0. The closeness is obvious, and we only have to show the openness.

Define two open subsets of ∂K by

$$D_j = \{(y', f_j(y')) : y' \in N_0\}.$$

Let $\omega_j \subset S^{n-1}$ be open subsets defined by

$$\omega_j = \left\{ \frac{x}{|x|} \in S^{n-1} : x \in D_j \right\}.$$

Since ρ_K is continuous and positive, $|x|$ never vanishes on ∂K. Now, let $\omega = \omega_1 \cap \omega_2$. Then ω is open in S^{n-1}. Denote C_ω to be the open cone generated by ω:

$$C_\omega = \{\lambda v : \lambda > 0, \ v \in \omega\}. \tag{3.2}$$

If $y' \in M_0$, then $(y', f_j(y')) \in D_1 \cap D_2$, and then $(y', f_j(y')) \in C_\omega \cap \partial K$. If $\lambda v \in C_\omega \cap \partial K$, then $\lambda = \rho_K(v)$, and $v \in \omega_1 \cap \omega_2$. This implies the existence of y_1', y_2', such that

$$v = \frac{(y_1', f_1(y_1'))}{|(y_1', f_1(y_1'))|} = \frac{(y_2', f_2(y_2'))}{|(y_2', f_2(y_2'))|}.$$

And since K is a star body, we must have

$$|(y_1', f_1(y_1'))| = |(y_2', f_2(y_2'))| = \rho_K(v), \quad \text{and} \quad (y_1', f_1(y_1')) = (y_2', f_2(y_2')).$$

Therefore, M_0 is exactly the orthogonal projection of $C_\omega \cap \partial K$, and hence it is open. Since $x' \in M_0$, and M_0 is open and closed in N_0, M_0 must be N_0 itself. □

Lemma 3.4. *The set K_0' in Lemma 3.2 can be split into subsets $G_1^*, ..., G_m^*, ...,$ $m = 1, 2, ...,$ such that for each $x' \in G_m^*$, we have the following:*

(i) The intersection $l_{x'} \cap \partial K$ has $2m$ elements;

(ii) For each $x' \in G_m^*$, there is an open neighborhood N_0 containing x', and there are $2m$ graph functions defined on N_0, such that

$$f_1(y') < g_1(y') < ... < f_m(y') < g_m(y'), \quad y' \in N_0;$$

(iii) There is an open ball $N_{x'} \subset N_0$ and for each $y' \in N_{x'}$, the intersection $l_{y'} \cap K$ is exactly

$$\bigcup_{k=1}^{m}[(y', f_k(y')), (y', g_k(y'))];$$

(iv) The sets G_m^* are relative open.

Proof. For $x' \in K_0'$, we know that it satisfies all the conditions in Lemma 3.2. Thus, we can suppose that the elements of $l_{x'} \cap \partial K$ are exactly

$$(x', t_1), ..., (x', t_k) \in \partial^* K, \quad \text{and } t_1 < ... < t_k,$$

and they satisfy $\nabla \rho_K(x', t_i) \cdot u \neq 0$. We claim that

$$\nabla \rho_K(x', t_1) \cdot u > 0,\ \nabla \rho_K(x', t_2) \cdot u < 0,\ \cdots,\ \nabla \rho_K(x', t_{k-1}) \cdot u > 0,\ \nabla \rho_K(x', t_k) \cdot u < 0. \tag{3.3}$$

To prove the first inequality, we observe that t_1 is the smallest number t so that $(x', t) \in K$. Thus, $\rho_K(x', t) < 1, \forall t < t_1$. By the differentiability of ρ_K at (x', t_1), we have

$$\nabla \rho_K(x', t_1) \cdot u = \lim_{t \to t_1^-} \frac{\rho_K(x', t) - \rho_K(x', t_1)}{t - t_1} \geq 0.$$

This together with $\nabla \rho_K(x', t_1) \cdot u \neq 0$ implies that $\nabla \rho_K(x', t_1) \cdot u > 0$. By this, and the definition of directional derivative, we see that in a small right neighborhood of t_1, $\rho_K(x', t)$ is larger than 1. This means $\rho_K(x', t) > 1$ for any $t_1 < t < t_2$. Successively using similar argument as above, we will have

$$\nabla \rho_K(x', t_2) \cdot u < 0,\ \cdots,\ \nabla \rho_K(x', t_{k-1}) \cdot u > 0,\ \nabla \rho_K(x', t_k) \cdot u < 0.$$

Now we proved our claim, and hence k is always even. Moreover,

$$l_{x'} \cap K = [(x', t_1), (x', t_2)] \cup \cdots \cup [(x', t_{k-1}), (x', t_k)]. \tag{3.4}$$

This explains the statement (i), and we can define G_m^* to be the set of $x' \in K_0'$ so that $\mathcal{H}^0(l_{x'} \cap \partial K) = 2m$. Clearly $K_0' = \bigcup_{m=1}^{\infty} G_m^*$.

For $x' \in G_m^*$, and for each t_i, we can choose its neighborhood I_i, such that $t_j \notin I_i$, $\forall j \neq i$. By Lemma 3.1, there is a small open ball containing x', say N_0, such that continuous functions f_i are defined on N_0, and

$$y' \mapsto (y', f_i(y')) \in \partial K.$$

By the local uniqueness provided by Lemma 3.3, the graphs

$$\{(y', f_i(y')) : y' \in N_0\}$$

must be disjoint. This implies the statement (ii).

Now we trun to prove (iii). Denote R_0 to be the open cylinder generated by N_0:

$$R_0 = \{y' + tu : t \in \mathbb{R},\ y' \in N_0\}.$$

We define the boundary pieces D_j and the open cone C_j, respectively, by

$$D_j = \{(y', f_i(y')) : y' \in N_0\},\ \text{when}\ j = 2i-1\,, \qquad D_j = \{(y', g_i(y')) : y' \in N_0\},\ \text{when}\ j = 2i,$$

and

$$C_j = \{\lambda x : \lambda > 0,\ x \in D_j\}.$$

Note that $R_0 \cap \partial K$ may not be the union of D_j, and actually our aim is to choose a smaller open set in N_0, such that it follows.

Since $R_0 \cap C_j$ is always open, there is an closed bounded cylinder Q_j in the form

$$Q_j = \{(y', t) :\ |t - t_j| \le h_j,\ 0 \le |y' - x'| \le r_j\}, \qquad \text{with}\ r_j, h_j > 0,$$

such that $Q_j \subset R_0 \cap C_j$.

On one hand, since K is a star body, the construction of C_j imply that $Q_j \subset C_j$ will not contain any other boundary points except the points in D_j.

On the other hand, by (3.4) and the continuity of ρ_K, there exists $\epsilon > 0$ so that

$$\rho_K(x', t) > 1+\epsilon, \quad t \in [t_1+h_1, t_2-h_2] \cup [t_3+h_3, t_4-h_4] \cup \cdots \cup [t_{2m-1}+h_{2m-1}, t_{2m}-h_{2m}],$$

and

$$\rho_K(x', t) < 1-\epsilon, \quad t \in [t_2+h_2, t_3-h_3] \cup [t_4+h_4, t_5-h_5] \cup \cdots \cup [t_{2m-2}+h_{2m-2}, t_{2m-1}-h_{2m-1}].$$

Since ρ_K is locally Lipschitz, we can choose $r_0 > 0$ so that

$$\rho_K(y', t) > 1+\frac{\epsilon}{2}, \quad t \in [t_1+h_1, t_2-h_2] \cup [t_3+h_3, t_4-h_4] \cup \cdots \cup [t_{2m-1}+h_{2m-1}, t_{2m}-h_{2m}],$$

and

$$\rho_K(y', t) < 1-\frac{\epsilon}{2}, \quad t \in [t_2+h_2, t_3-h_3] \cup [t_4+h_4, t_5-h_5] \cup \cdots \cup [t_{2m-2}+h_{2m-2}, t_{2m-1}-h_{2m-1}],$$

for all $|y' - x'| < r_0$.

Let $r_{x'} = \min\{r_0, r_1, ..., r_{2m}\}$, and let $N_{x'}$ be the open ball centered at x' with radius $r_{x'}$ in $u^\perp$. Then statement (iii) follows, and statement (iv) is just a corollary of it. $\square$

By Lemma 3.4, we can define a bit larger open set G_m by

$$G_m = \bigcup_{x' \in G_m^*} N_{x'}, \tag{3.5}$$

such that the $2m$ graph functions are well-defined on G_m.

Theorem 3.1. *Let $K \in \bar{\mathcal{S}}_o^n$, and let $u \in S^{n-1}$ satisfying $\mathcal{H}^{n-1}(\partial_u^* K) = 0$. There are open subsets $G_m \subset K'$, measurable subsets $G_m^* \subset G_m$, and functions*

$$f_j, g_j : \bigcup_{m=j}^{\infty} G_m \to \mathbb{R},$$

satisfying the following properties.

(i) $\mathcal{H}^{n-1}(K' \backslash \bigcup_{m=1}^{\infty} G_m) = 0$ and $\mathcal{H}^{n-1}(G_m \backslash G_m^*) = 0$, and $f_1 < g_1 < \cdots < f_m < g_m$;

(ii) K has the representation (if we neglect an $\mathcal{H}^n$-null set)

$$\bigcup_{m=1}^{\infty} \bigcup_{\substack{x' \in G_m \\ 1 \le j \le m}} [(x', f_j(x')), (x', g_j(x'))];$$

(iii) f_j, g_j are differentiable at each $x' \in G_m^*$, and

$$\nu_K(x', f_j(x')) = \frac{(\nabla f_j(x'), -1)}{\sqrt{1 + |\nabla f_j(x')|^2}}, \quad \nu_K(x', g_j(x')) = \frac{(-\nabla g_j(x'), 1)}{\sqrt{1 + |\nabla g_j(x')|^2}}.$$

Proof. Combine Lemmas 3.1-Lemma 3.4 with the definition of G_m in (3.5). $\square$

By Theorem 3.1, the Steiner symmetral of K in direction u turns out to be

$$S_u K := \bigcup_{m=1}^{\infty} \bigcup_{x' \in G_m} \Big[\big(x', \sum_{j=1}^{m} \frac{f_j(x') - g_j(x')}{2}\big), \big(x', \sum_{j=1}^{m} \frac{g_j(x') - f_j(x')}{2}\big) \Big]. \tag{3.6}$$

Moreover, Lemma 3.3 and the fact that $S_u K$ is also a Lipschitz star body implies that $\sum_j (f_j - g_j)/2$ and $\sum_j (g_j - f_j)/2$ are precisely its graph functions.

4 Area and coarea formulas

In regard to the graph functions, we would like to show the following area formula for the Lipschitz star body K. Although it has fine geometric explanations, it still requires a proof since we have no idea on that whether or not f_j, g_j are locally Lipschitz. Its proof is based on a result in geometric measure theory (see Theorem A below).

Theorem 4.1. *If $u \in S^{n-1}$ is a direction satisfying $\mathcal{H}^{n-1}(\partial_u^* K) = 0$, then for any integrable Borel function F, we have*

$$\begin{aligned}\int_{\partial K} F(x) d\mathcal{H}^{n-1}(x) =& \sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{G_m} F(x', f_j(x'))\sqrt{1+|\nabla f_j(x')|^2} d\mathcal{H}^{n-1}(x') \\ &+ \sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{G_m} F(x', g_j(x'))\sqrt{1+|\nabla g_j(x')|^2} d\mathcal{H}^{n-1}(x').\end{aligned}$$

Theorem 4.1 together with Theorem 3.1 implies the following corollary, which states the relation of areas of the surface and projection.

Corollary 4.1. *If $u \in S^{n-1}$ is a direction satisfying $\mathcal{H}^{n-1}(\partial_u^* K) = 0$, then*

$$\begin{aligned}\int_{\partial K} \big(u \cdot \nu_K(x)\big)_+ d\mathcal{H}^{n-1}(x) =& \int_{\partial K} \big(u \cdot \nu_K(x)\big)_- d\mathcal{H}^{n-1}(x) \\ =& \sum_{m=1}^{\infty} m \cdot \mathcal{H}^{n-1}(G_m) \\ \geq& \mathcal{H}^{n-1}(K').\end{aligned}$$

Since K is contained in the cylinder $K' \times [-R_K u, R_K u]$, by Corollary 4.1, we have

$$\int_{\partial K} \big(u \cdot \nu_K(x)\big)_+ d\mathcal{H}^{n-1}(x) = \int_{\partial K} \big(u \cdot \nu_K(x)\big)_- d\mathcal{H}^{n-1}(x) \geq \mathcal{H}^{n-1}(K') \geq \frac{|K|}{2R_K}. \quad (4.1)$$

Taking $F = 1$ in Theorem 4.1, and recalling Theorem 3.1, we obtain the following.

Corollary 4.2. *If $u \in S^{n-1}$ is a direction satisfying $\mathcal{H}^{n-1}(\partial_u^* K) = 0$, then ∂K has the representation (if we neglect an $\mathcal{H}^{n-1}$-null set)*

$$\partial K = \bigcup_{m=1}^{\infty}\bigcup_{j=1}^{m} \Big\{ \big(x, f_j(x')\big) : x' \in G_m \Big\} \cup \Big\{ \big(x, g_j(x')\big) : x' \in G_m \Big\}.$$

The following Theorem A is a variation of [5, Theorem F], which applied to the reduced boundary of a set of finite perimeter. More precisely, both are consequences of [1, Theorem 2.93, (2.72)], which takes the Lipschitz map f to be the projection map $P_{u^\perp}$ restricted on the rectifiable set E, where E and f are as in [1, Theorem 2.93].

Theorem A. (Area formula.) *Let $E \subset \mathbb{R}^n$ be an $\mathcal{H}^{n-1}$-rectifiable set, and let G be a nonnegative Borel function on $\mathbb{R}^n$. Then*

$$\int_E G(x)|u \cdot \nu_E(x)| d\mathcal{H}^{n-1}(x) = \int_{u^\perp}\int_{E \cap l_u(x')} G(x', t) d\mathcal{H}^0(t) d\mathcal{H}^{n-1}(x'). \quad (4.2)$$

Proof of Theorem 4.1. In order to obtain Theorem 4.1, we notice that ∂K is actually $\mathcal{H}^{n-1}$-rectifiable. Let $p : u^\perp \to S^{n-1} \setminus \{u\}$ be the reverse stereographic projection map. Since p is Lipschitz, and ρ_K restricted on S^{n-1} is also Lipschitz, we infer that the map

$$\psi(x) = \rho_K(p(x))p(x) : u^\perp \to \partial K \setminus \{\rho_K(u)u\}$$

is Lipschitz. Then, in Theorem A, we take $E = \partial K$ and $G(x) = F(x)/|u \cdot \nu_K(x)|$. Here we notice that $\mathcal{H}^{n-1}(\partial_u^* K) = 0$, and hence $|u \cdot \nu_K(x)| > 0$ for almost all $x \in \partial K$. Combining these with Theorem 3.1, we complete the proof. $\square$

We also refer to [37, Theorem 11.6] for the area formula. The following coarea formula can be found in [1, (2.74)]. A remark is that both Theorem A and B are special cases of [1, Theorem 2.93].

Theorem B. (Coarea formula.) *Let G be a nonnegative Borel function on $\mathbb{R}^n$, and let $E \subset \mathbb{R}^n$ be a Borel set. If $F : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, then*

$$\int_E G(x)|\nabla F(x)|dx = \int_{-\infty}^{\infty} \int_{E\cap\{F=t\}} G(x)d\mathcal{H}^{n-1}(x)dt. \tag{4.3}$$

Taking $F(x)$ to be the gauge function $p_K(x)$ (see (2.6)), $E = K$, a direct computation leads to

$$\nabla p_K(x) = -\frac{\nabla \rho_K(x)}{\rho_K(x)^2} = -\nabla \rho_K\big(\rho_K(x)x\big).$$

$\nabla p_K(x)$ is 0-homogeneous, and is parallel to the normal vector at the boundary point $\bar{x} = \rho_K(x)x$. By (2.14), $|\nabla p_K(x)| = \bar{x} \cdot \nu_K(\bar{x})$. Note that $\bar{x} \cdot \nu_K(\bar{x})$ never vanishes.

Let H be a nonnegative Borel function on $\mathbb{R}^n$. Taking $G(x) = H\big(\rho_K(x)x\big)/|\nabla p_K(x)|$ in Theorem B, we obtain the cone-volume type formula

$$\int_K H\big(\rho_K(x)x\big)dx = \frac{1}{n}\int_{\partial K} H(x)x \cdot \nu_K(x)d\mathcal{H}^{n-1}(x). \tag{4.4}$$

Specially, if $H(x) \equiv 1$, we have the cone-volume formula for Lipschitz star body

$$|K| = \frac{1}{n}\int_{\partial K} x \cdot \nu_K(x)d\mathcal{H}^{n-1}(x).$$

The other useful application of (4.4) is that we can obtain an equivalent representation of the definition (1.3). Since $\nu_K(x) = \nabla p_K(x)/|\nabla p_K(x)|$ and $|\nabla p_K(x)|^{-1} = x \cdot \nu_K(x)$, for $\lambda > 0$, we have

$$\phi\Big(\frac{u \cdot \nu_K(x)}{\lambda x \cdot \nu_K(x)}\Big) = \phi\Big(\frac{u \cdot \nabla p_K(x)}{\lambda}\Big),$$

which is 0-homogeneous. Therefore, by (4.4) and (1.3), we have

$$\int_{\partial K} \phi\left(\frac{u \cdot \nu_K(y)}{\lambda x \cdot \nu_K(x)}\right) x \cdot \nu_K(x)d\mathcal{H}^{n-1}(x) = n\int_K \phi\Big(\frac{u \cdot \nabla p_K(x)}{\lambda}\Big)dx. \tag{4.5}$$

5 The convergence of Steiner symmetrization

So far the graph functions have been defined with respect to somewhat "admissible" directions. Given a star body K, we denote

$$T(K) = \{u \in S^{n-1} : \mathcal{H}^{n-1}(\partial_u^* K) = 0\}.$$

Although it is inferred from Lemma 2.2 that $T(K)$ is dense in S^{n-1}, it can vary a lot with the change of K. This explains why we need to prove the following convergence result.

We also note that the Hausdorff convergence is not enough in studying the Orlicz projection operator for star bodies.

Theorem 5.1. *For any $K \in \bar{\mathcal{S}}_o^n$, let $T(K) = \{u \in S^{n-1} : \ \mathcal{H}^{n-1}(\partial_u^* K) = 0\}$. Then there exists $\{u_i\}_{i=1}^{\infty} \subset S^{n-1}$ such that $u_1 \in T(K)$,*

$$u_{i+1} \in T(K_i), \ \text{ where } \ K_i = S_{u_i} \dots S_{u_1} K, \ i \geq 1, \tag{5.1}$$

and

$$K_i \to B_K \ \text{ with respect to radial distance.} \tag{5.2}$$

Here B_K means the centered Euclidean ball having the same volume as K.

The following lemma guarantees that the bodies K_i are always in $\bar{\mathcal{S}}_o^n$.

Lemma 5.1. *If $K \in \bar{\mathcal{S}}_o^n$, then $S_u K \in \bar{\mathcal{S}}_o^n$ and $L_{S_u K} \leq L_K$.*

Proof. We will prove a stronger result: For arbitrary $0 < s_1 < s_2$, there is

$$L_{S_u K}\Big((s_2 S_u K)\backslash \mathrm{int}(s_1 S_u K)\Big) \leq L_K\Big((s_2 K)\backslash \mathrm{int}(s_1 K)\Big). \tag{5.3}$$

Let $x, y \in (s_2 S_u K) \backslash \mathrm{int}\,(s_1 S_u K)$ be such that

$$\frac{\rho_{S_u K}(y) - \rho_{S_u K}(x)}{|x-y|} = L_{S_u K}\Big((s_2 S_u K)\backslash \mathrm{int}(s_1 S_u K)\Big).$$

Denote $\rho_{S_u K}(x) = h_1$, $\rho_{S_u K}(y) = h_2$ with $1/s_2 \leq h_1 < h_2 \leq 1/s_1$. Then $x \in \frac{1}{h_1}\partial(S_u K)$ and $y \in \frac{1}{h_2}\partial(S_u K)$.

Let $\varepsilon > 0$. On one hand, we have the following observation

$$\frac{1}{h_2}K + \varepsilon B \subset \frac{1}{h_1}K \quad \Longleftrightarrow \quad dist\Big(\frac{1}{h_2}\partial K, \frac{1}{h_1}\partial K\Big) \geq \varepsilon.$$

On the other hand, if $\frac{1}{h_2}K+\varepsilon B\subset \frac{1}{h_1}K$, then it follows from the basic fact (2.18) that

$$\frac{1}{h_2}S_uK+\varepsilon B\subset S_u\left(\frac{1}{h_2}K+\varepsilon B\right)\subset \frac{1}{h_1}S_uK.$$

Recalling that $x\in\frac{1}{h_1}\partial(S_uK)$ and $y\in\frac{1}{h_2}\partial(S_uK)$, we conclude

$$|x-y|\geq \operatorname{dist}\Big(\frac{1}{h_1}\partial(S_uK),\frac{1}{h_2}\partial(S_uK)\Big)\geq \operatorname{dist}\Big(\frac{1}{h_1}\partial K,\frac{1}{h_2}\partial K\Big).$$

Then, there are $\bar{x}\in\frac{1}{h_1}\partial K$ and $\bar{y}\in\frac{1}{h_2}\partial K$, such that

$$\frac{h_2-h_1}{|\bar{x}-\bar{y}|}\geq\frac{h_2-h_1}{|x-y|}=L_{S_uK}\Big((s_2S_uK)\backslash\operatorname{int}(s_1S_uK)\Big).$$

This implies (5.3). By this, recalling the definition of the Lipschitz constant (2.9), and by

$$r_K\leq r_{S_uK}\leq R_{s_uK}\leq R_K,$$

we complete the proof of this Lemma. □

Since L_K is the Lipschitz constant chosen on a large enough set containing the sphere S^{n-1}, Lemma 5.1 immediately implies the following.

Corollary 5.1. *Let K_i be as in Theorem 5.1. Then there exists an absolute constant L_0 such that*

$$L_{K_i}(S^{n-1})\leq L_{K_i}\leq L_0.$$

Lemma 5.2. *If $K_i\in\bar{\mathcal{S}}^n_o$, $i=0,1,\ldots,\infty$, satisfying*

(i) There are absolute constants $0<r<R$ such that $rB\subset K_i\subset RB$;

(ii) There is an absolute constant $L_0>0$ bounds the Lipschitz constants $L_{K_i}(S^{n-1})$

$$L_{K_i}(S^{n-1})\leq L_0;$$

(iii) K_i converges to a compact set K_0 in Hausdorff metric.

Then, we have

1. *$K_0\in\bar{\mathcal{S}}^n_o$ and K_i converges to K_0 with respect to radial distance;*

2. *If $u\in S^{n-1}$, then S_uK_i converges to S_uK_0 with respect to radial distance.*

Proof. 1. Our proof of this statement follows from the following two points.

(a) By (i) and (ii), the sequence $\{\rho_{K_i}\}$ is uniformly bounded and has bounded Lipschitz constant on the sphere. Let $\rho_{K_{i_j}}$ be an arbitrary subsequence of ρ_{K_i}. It follows directly from the Arzelà-Ascoli theorem that $\rho_{K_{i_j}}$ has a uniformly convergent subsequence, converging to a Lipschitz continuous function $\rho \in C(S^{n-1})$. By condition (i), ρ is also positive on S^{n-1}.

(b) Let K be the star body whose radial function is ρ. Since the radial convergence implies the Hausdorff convergence, K_{i_j} must converge to K in Hausdorff metric. Now condition (iii) tells us $K = K_0$.

Combining (a) and (b), we deduce that any subsequence of ρ_{K_i} has a uniformly convergent subsequence converging to ρ_{K_0}. Hence $\rho_{K_i} \to \rho_{K_0}$ uniformly on S^{n-1}.

2. By (i), for any $\varepsilon > 0$, there exists a positive integer N such that $i > N$

$$(1-\varepsilon)K_0 \subset K_i \subset (1+\varepsilon)K_0.$$

Therefore,

$$(1-\varepsilon)S_u K_0 \subset S_u K_i \subset (1+\varepsilon)S_u K_0,$$

which implies that $S_u K_i$ converges to $S_u K_0$ with respect to radial distance.

□

Proof of Theorem 5.1. Define

$$\Theta_K = \{S_{u_k} \dots S_{u_1} K : \ k \in \mathbb{N},\ u_1 \in T(K),\ u_2 \in T(K_1), \dots, u_k \in T(K_{k-1})\},$$

where $K_j = S_{u_j} \dots S_{u_1} K$, $j = 1, \dots, k-1$.

For $K \in \bar{\mathcal{S}}_o^n$, let $R_K = \max_{x \in K} |x|$ to be the outer radius of K, i.e., the smallest $r > 0$ such that $K \subset rB$. Set $R_1 = \inf_{C \in \Theta_K} R_C$, where $C \in \Theta_K$.

Since R_1 is the infimum, there is a sequence of $\{C_i\} \subset \Theta_K$ so that $R_{C_i} \to R_1$. The sequence $\{C_i\}$ is clearly bounded, because each C_i is contained in $R_K B$. By Blaschke's selection theorem (see e.g., [40, Theorem 1.8.5]), there is a subsequence C_{i_j} converging to a compact set $\bar{K}$ in Hausdorff metric. By Lemma 5.2, $\bar{K} \in \bar{\mathcal{S}}_o^n$ and $R_{\bar{K}} = R_1$. Denote $B_1 = R_1 B$, and clearly $\bar{K} \subset B_1$. We *claim* that

$$R_1 = \frac{|K|^{\frac{1}{n}}}{|B|^{\frac{1}{n}}}, \quad \text{and} \qquad C_{i_j} \to B_1 \text{ in Hausdorff metric.} \qquad (Claim)$$

Assume the contrary that $R_1 > |K|^{\frac{1}{n}}/|B|^{\frac{1}{n}}$ and $B_1 \neq \bar{K}$. Then, there exists an open spherical cap $U \subset \partial B_1$, such that $\mathrm{cl}U \cap \bar{K} = \emptyset$. For any line ξ such that

$\xi \cap U \neq \emptyset$, we have $|\xi \cap B_1| > |\xi \cap \bar{K}|$. It suggests that in any direction $v \in S^{n-1}$, the Steiner symmetral $S_v \bar{K}$ fails to intersect both U and the new cap U_v, which denotes the reflection of U with respect to the hyperplane $v^{\perp}$. Since ∂B_1 is compact, there is a finite set of directions $\{v_1, \ldots, v_m\} \subset S^{n-1}$ such that $\bigcup_{i=1}^m U_{v_i}$ covers the whole sphere ∂B_1. This suggests that the output star body $\tilde{K} = S_{v_m} \ldots S_{v_1} \bar{K}$ is strictly contained in B_1, and hence $R_{\tilde{K}} < R_1$.

Moreover, since a reflection is a continuous map, we can choose a sufficiently small $\delta > 0$ such that $S^{n-1} \subset \bigcup_{i=1}^m U_{u_i}$, for any $u_i \in B(v_i, \delta) \cap S^{n-1}$. Here $B(v_i, \delta_i)$ denotes the open ball centered at v_i with radius δ_i, and $i = 1, ..., m$. Same as shown above, the outer radius of $S_{u_m} \ldots S_{u_1} \bar{K}$ is also strictly smaller than R_1.

Now we are going to construct a sequence of convergent bodies in Θ_K. There exists directions $u_1, ..., u_m$, such that

$$u_1 \in B(v_1, \delta) \cap S^{n-1} \cap \bigcap_{j=1}^{\infty} T(C_{i_j}),$$

and

$$u_k \in B(v_k, \delta) \cap S^{n-1} \cap \bigcap_{j=1}^{\infty} T(S_{u_{k-1}} \ldots S_{u_1} C_{i_j}),$$

for $k = 2, ..., m$. Denote $\tilde{C}_{i_j} = S_{u_m} \cdots S_{u_1} C_{i_j}$. Since $C_{i_j} \to \bar{K}$, by Lemma 5.2, we have

$$\tilde{C}_{i_j} = S_{u_m} \cdots S_{u_1} C_{i_j} \to S_{u_m} \cdots S_{u_1} \bar{K} =: \bar{K}_1.$$

Then, for sufficiently large j, we have $R_{\tilde{C}_{i_j}} < R_1$. But this contradicts with the assumption that R_1 is the infimum.

Now we have confirmed our claim, and we are going to apply it consecutively. Let ε_k be a sequence of positive numbers so that $\varepsilon_k \to 0^+$. By the claim for K, there is

$$D_1 := S_{u_{i_1}} \cdots S_{u_1} K \in \Theta_K, \quad \text{such that} \quad \delta_H(D_1, B_K) < \varepsilon_1.$$

Applying (claim) for D_1, there is

$$D_2 := S_{u_{i_2}} \ldots S_{u_{i_1+1}} D_1 \in \Theta_{D_1}, \quad \text{such that} \quad \delta_H(D_2, B_K) < \varepsilon_2.$$

Continue the process, and we get a sequence $\{D_j\}_{j=1}^{\infty}$

$$D_j := S_{u_{i_j}} \ldots S_{u_{i_{j-1}+1}} D_{k-1} \in \Theta_{D_{j-1}}, \quad \text{such that} \quad \delta_H(D_j, B_K) < \varepsilon_j.$$

Therefore $D_j \to B_K$ in Hausdorff metric. By Lemma 5.2, $D_j \to B_K$ with respect to radial distance. Denote $K_l = S_{u_l} \cdots S_{u_1} K$, and notice the basic fact that the radial distance of a star body and a centered ball is non-increasing under Steiner symmetrization (see (2.19)). Now $K_l \to B_K$, and we complete the proof. $\square$

6 Properties of Orlicz projection bodies for Lipschitz star bodies

Let $\phi \in \mathcal{C}$. Recall that the definition of Orlicz projection operator Π_ϕ is given by

$$h_{\Pi_\phi K}(x) = \inf\left\{\lambda > 0 : \int_{\partial^* K} \phi\left(\frac{x \cdot \nu(y)}{\lambda y \cdot \nu(y)}\right) y \cdot \nu(y) d\mathcal{H}^{n-1}(y) \leq n|K|\right\}.$$

The following Lemma 6.1 states some basic properties of $\Pi_\phi K$. Since its proof is canonical and is same as its convex counterpart in [36, Lemma 2.1-2.2], we omit it here.

Lemma 6.1. *Suppose $\phi \in \mathcal{C}$ and $K \in \bar{\mathcal{S}}_o^n$. Then, we have*

1. *If $z \in \mathbb{R}^n \backslash \{0\}$, then $h_{\Pi_\phi K}(z) = \lambda_0$ if and only if*
$$\frac{1}{n|K|} \int_{\partial^* K} \phi\left(\frac{z \cdot \nu_K(x)}{\lambda_0 x \cdot \nu_K(x)}\right) x \cdot \nu_K(x) d\mathcal{H}^{n-1}(x) = 1.$$
2. *$h_{\Pi_\phi K}$ is sublinear. Namely, it defines a convex body whose support function is given by $h_{\Pi_\phi K}$.*

The following lemma gives the explicit upper and lower bounds of $\Pi_\phi K$.

Lemma 6.2. *Let $\phi \in \mathcal{C}$ and $K \in \bar{\mathcal{S}}_o^n$. Then*

$$\frac{1}{2nc_\phi R_K} \leq h_{\Pi_\phi K}(u) \leq \frac{L_K}{c_\phi}, \qquad \forall u \in S^{n-1}.$$

Proof. Since $T(K)$ is dense in S^{n-1} and $h_{\Pi_\phi K}$ is continuous, we may assume $u \in T(K)$. Suppose $h_{\Pi_\phi K}(u) = \lambda_0$. Then

$$\frac{1}{n} \int_{\partial^* K} \phi\left(\frac{u \cdot \nu_K(x)}{\lambda_0 x \cdot \nu_K(x)}\right) \frac{x \cdot \nu_K(x)}{|K|} d\mathcal{H}^{n-1}(x) = 1.$$

1. The lower bound. By (2.1), the definition of c_ϕ, either $\phi(c_\phi) = 1$ or $\phi(-c_\phi) = 1$. Without loss of generality, suppose $\phi(c_\phi) = 1$. By the fact that ϕ is non-negative and increasing on $[0, \infty)$, estimate (2.12), cone-volume formula (4.4) and Jensen's inequality, and finally (4.1), we have
$$\begin{aligned}
\phi(c_\phi) = 1 &= \frac{1}{n} \int_{\partial^* K} \phi\left(\frac{u \cdot \nu_K(x)}{\lambda_0 x \cdot \nu_K(x)}\right) \frac{x \cdot \nu_K(x)}{|K|} d\mathcal{H}^{n-1}(x) \\
&\geq \frac{1}{n} \int_{\partial^* K} \phi\left(\frac{(u \cdot \nu_K(x))_+}{\lambda_0 x \cdot \nu_K(x)}\right) \frac{x \cdot \nu_K(x)}{|K|} d\mathcal{H}^{n-1}(x) \\
&\geq \phi\left(\frac{1}{n} \int_{\partial^* K} \frac{(u \cdot \nu_K(x))_+}{\lambda_0 |K|} d\mathcal{H}^{n-1}(x)\right) \\
&\geq \phi\left(\frac{1}{2n\lambda_0 R_K}\right).
\end{aligned}$$

Since ϕ is monotone increasing on $[0, +\infty)$, we obtain $1/(2nc_\phi R_K) \leq \lambda_0$.

2. The upper bound. By (2.12), we have

$$\left| \frac{u \cdot \nu_K(x)}{\lambda_0 x \cdot \nu_K(x)} \right| \leq \frac{L_K}{\lambda_0}.$$

It follows by the properties of ϕ and (4.4) that

$$1 = \int_{\partial^* K} \phi\left(\frac{u \cdot \nu_K(x)}{\lambda_0 x \cdot \nu_K(x)}\right) \frac{x \cdot \nu_K(x)}{n|K|} d\mathcal{H}^{n-1}(x) \leq \max\left\{\phi\left(\frac{L_K}{\lambda_0}\right), \phi\left(\frac{-L_K}{\lambda_0}\right)\right\}.$$

Since the even function $t \mapsto \max\{\phi(t), \phi(-t)\}$ is monotone increasing on $[0, \infty)$, we conclude $\lambda_0 \leq L_K/c_\phi$.

□

The following lemma provides the weak continuity of the Orlicz projection operator $\Pi_\phi^* : \bar{\mathcal{S}}_o^n \to \mathcal{K}_o^n$.

Lemma 6.3. *Let K_i, K, and B_K be as in Theorem 5.1. Then, there exists a subsequence of $\Pi_\phi K_i$, denoted by $\Pi_\phi K_{i_j}$, such that $\Pi_\phi K_{i_j}$ converges to a convex body $\bar{Q}$ in Hausdorff metric. Moreover, $\bar{Q}$ contains the origin, and satisfies*

$$\bar{Q}^* \subset \Pi_\phi^* B_K. \tag{6.1}$$

Proof. By Lemma 5.1, there is a constant L_0 so that

$$L_{K_i} \leq L_0, \quad \forall i.$$

It follows from this and Lemma 6.2 that $\{\Pi_\phi K_i\}$ is uniformly bounded, so that we can apply Blaschke's selection theorem. After passing to a subsequence, we assume that $\Pi_\phi K_i$ converges to a convex body $\bar{Q}$.

Recall the definition of the gauge function (2.6), which is locally Lipschitz on $\mathbb{R}^n$. When $x \neq 0$, we have

$$\nabla p_{K_i}(x) = \frac{\nabla \rho_{K_i}(x)}{\rho_{K_i}(x)^2} = \nabla \rho_{K_i}\big(\rho_{K_i}(x) x\big).$$

Since $\rho_{K_i}(x) x$ always lies on ∂K_i, we have

$$|\nabla p_{K_i}(x)| = |\nabla \rho_{K_i}\big(\rho_{K_i}(x) x\big)| \leq L_{K_i} \leq L_0.$$

Let $\mathring{B}_R$ denote $\mathrm{int}(RB)$. Since $\rho_{K_i} \to \rho_{B_K}$ uniformly, there exists $R_0 > 0$, such that $K_i \subset \mathring{B}_{R_0}$ for any $i \geq 0$. Set

$$\bar{p}_{K_i}(x) = \min\{p_{K_i}(x), 1\}.$$

Then $\bar{p}_{K_i}$ is Lipschitz on $\mathbb{R}^n$, and hence in the Sobolev space $W_{1,q}(\mathring{B}_{R_0})$ for any $q \geq 1$. Moreover, since $|\nabla \bar{p}_{K_i}|$ is uniformly bounded, by [1, Proposition 2.5 (b)], we have

$$\bar{p}_{K_i} \to \bar{p}_{B_K} \quad \text{weakly in } W_{1,q}(\mathring{B}_{R_0}).$$

Let $u \in S^{n-1}$, and let λ be an arbitrary real number satisfying $\lambda < h_{\Pi_\phi B_K}(u)$. Since ϕ is a convex function, the integral

$$\int_{\mathring{B}_{R_0}} \phi\Big(\frac{u \cdot \nabla \bar{p}_{K_i}(x)}{\lambda}\Big) dx$$

is lower semicontinuous with respect to weak convergence in $W_{1,q}(\mathring{B}_{R_0})$ (see e.g., [8, Theorem 1 in P.19]). That is to say,

$$\int_{\mathring{B}_{R_0}} \phi\Big(\frac{u \cdot \nabla \bar{p}_{B_K}(x)}{\lambda}\Big) dx \leq \liminf_{i \to \infty} \int_{\mathring{B}_{R_0}} \phi\Big(\frac{u \cdot \nabla \bar{p}_{K_i}(x)}{\lambda}\Big) dx. \tag{6.2}$$

Since $\nabla p_K(x)$ is 0-homogenous, by the cone-volume formula (4.4), we have

$$\begin{aligned}
&\int_{\partial^* K_i} \phi\Big(\frac{u \cdot \nu_{K_i}(x)}{\lambda x \cdot \nu_{K_i}(x)}\Big) x \cdot \nu_{K_i}(x) d\mathcal{H}^{n-1}(x) \\
=& n \int_{K_i} \phi\Big(\frac{u \cdot \nabla p_{K_i}(x)}{\lambda}\Big) dx, \\
=& n \int_{\mathring{B}_{R_0}} \phi\Big(\frac{u \cdot \nabla \bar{p}_{K_i}(x)}{\lambda}\Big) dx.
\end{aligned}$$

Combining this, (6.2), the fact that $|B_K| = |K_i|$ and $\lambda < h_{\Pi_\phi(B_K)}(u)$, we have

$$1 < \int_{\partial^* B_K} \phi\Big(\frac{u \cdot \nu_{B_K}(x)}{\lambda x \cdot \nu_{B_K}(x)}\Big) \frac{x \cdot \nu_{B_K}(x)}{n|B_K|} d\mathcal{H}^{n-1}(x) \leq \liminf_{i \to \infty} \int_{\partial^* K_i} \phi\Big(\frac{u \cdot \nu_{K_i}(x)}{\lambda x \cdot \nu_{K_i}(x)}\Big) \frac{x \cdot \nu_{K_i}(x)}{n|K_i|} d\mathcal{H}^{n-1}(x).$$

By the inequalities above, we can select a subsequence $\{K_{i_j}\}$, such that for sufficiently large j, there is

$$\int_{\partial^* K_{i_j}} \phi\Big(\frac{u \cdot \nu_{K_{i_j}}(x)}{\lambda x \cdot \nu_{K_{i_j}}(x)}\Big) \frac{x \cdot \nu_{K_{i_j}}(x)}{n|K_{i_j}|} d\mathcal{H}^{n-1}(x) > 1.$$

Since $\lim_i \Pi_\phi K_i = \bar{Q}$, we have $h_{\bar{Q}}(u) = \lim_{j \to \infty} h_{\Pi_\phi K_{ij}}(u) \geq \lambda$. Since λ is an arbitrary number satisfying $\lambda < h_{\Pi_\phi B_K}(u)$, we obtain $h_{\bar{Q}} \geq h_{\Pi_\phi B_K}$. Hence $\bar{Q}^* \subset \Pi^*_\phi B_K$. $\square$

We now demonstrate the affine invariance of the polar of the Orlicz projection body.

Lemma 6.4. *If $K \in \bar{\mathcal{S}}_o^n$ and $A \in \mathrm{GL}(n)$, then*

$$\Pi_\phi^*(AK) = A(\Pi_\phi^* K). \tag{6.3}$$

Proof. Let $x \in \partial K$, and put $\bar{x} = Ax$. By the definition of gauge function, we have $p_{AK}(\bar{x}) = p_K(x)$. Clearly p_{AK} is also locally Lipschitz, and p_{AK} is differentiable at $\bar{x}$ if and only if p_K is differentiable at x. Thus

$$\nabla p_{AK}(\bar{x}) = \nabla p_{AK}(Ax) = A^{-t}\nabla p_K(x). \tag{6.4}$$

By equations (1.3) and (4.5), then equation (6.4) and that $|AK| = |\det A||K|$, we have

$$\begin{aligned} h_{\Pi_\phi(AK)}(u) =& \inf\left\{\lambda > 0 : \int_{AK} \phi\Big(\frac{u \cdot \nabla p_{AK}(\bar{x})}{\lambda}\Big) d\bar{x} \le |AK|\right\} \\ =& \inf\left\{\lambda > 0 : \int_K \phi\Big(\frac{(A^{-1}u) \cdot \nabla p_K(x)}{\lambda}\Big) dx \le |K|\right\} \\ =& h_{\Pi_\phi K}(A^{-1}u) = h_{A^{-t}(\Pi_\phi K)}(u), \end{aligned}$$

which implies $\Pi_\phi AK = A^{-t}\Pi_\phi K$. This together with (2.3) yields the desired result. $\square$

7 Proof of the Orlicz Petty projection inequality for star bodies

For an open set $\Omega \subset \mathbb{R}^{n-1}$ and a function $f : \Omega \to \mathbb{R}$ whose gradient exists a.e. $x' \in \Omega$, we shall use the symbol $\langle f\rangle : \Omega \to \mathbb{R}$ which is defined by

$$\langle f\rangle = f(x') - x' \cdot \nabla f(x').$$

We shall often make use of the fact that $\langle\cdot\rangle$ is a linear operator; i.e., for $f_1, f_2 : \Omega \to \mathbb{R}$ whose gradient exists in Ω, and $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\langle \alpha_1 f_1 + \alpha_2 f_2\rangle = \alpha_1\langle f_1\rangle + \alpha_2\langle f_2\rangle.$$

Let $K \in \bar{\mathcal{S}}_o^n$ and $\mathcal{H}^{n-1}(\partial_u^* K) = 0$. Using the symbol $\langle\cdot\rangle$, the area formula in Theorem 4.1 can be written as

$$\begin{aligned} \int_{\partial K} \phi\left(\frac{(y', t)\cdot \nu(x)}{x \cdot \nu(x)}\right) x \cdot \nu(x) d\mathcal{H}^{n-1}(x) =& \sum_{m=1}^{\infty}\sum_{j=1}^{m} \int_{G_m} \phi\left(\frac{t - y' \cdot \nabla g_j(x')}{\langle g_j\rangle(x')}\right)\langle g_j\rangle(x') dx' \\ &+ \sum_{m=1}^{\infty}\sum_{j=1}^{m} \int_{G_m} \phi\left(\frac{-t + y' \cdot \nabla f_j(x')}{\langle -f_j\rangle(x')}\right)\langle -f_j\rangle(x') dx'. \end{aligned} \tag{7.1}$$

By (3.6), the formula of S_uK turns to

$$\begin{aligned}
&\int_{\partial(S_uK)} \phi\left(\frac{(y',t)\cdot\nu(x)}{x\cdot\nu(x)}\right) x\cdot\nu(x)d\mathcal{H}^{n-1}(x)\\
=&\sum_{m=1}^{\infty}\int_{G_m}\phi\left(\frac{t-\sum_{j=1}^{m} y'\cdot\nabla\left(\frac{g_j-f_j}{2}\right)(x')}{\sum_{j=1}^{m}\left\langle\frac{g_j-f_j}{2}\right\rangle(x')}\right)\sum_{j=1}^{m}\left\langle\frac{g_j}{2}-\frac{f_j}{2}\right\rangle(x')dx'\\
&+\sum_{m=1}^{\infty}\int_{G_m}\phi\left(\frac{-t-\sum_{j=1}^{m} y'\cdot\nabla\left(\frac{g_j-f_j}{2}\right)(x')}{\sum_{j=1}^{m}\left\langle\frac{g_j-f_j}{2}\right\rangle(x')}\right)\sum_{j=1}^{m}\left\langle\frac{g_j}{2}-\frac{f_j}{2}\right\rangle(x')dx'.
\end{aligned}\tag{7.2}$$

We now prove that the Steiner symmetrization decreases the volume of Orlicz projection body.

Theorem 7.1. *Suppose $\phi\in\mathcal{C}$. If $K\in\bar{\mathcal{S}}_o^n$ and $\mathcal{H}^{n-1}(\partial_u^*K)=0$, then*

$$S_u\Pi_\phi^*K\subset\Pi_\phi^*(S_uK).\tag{7.3}$$

If $\phi\in\mathcal{C}_s$ and $S_u(\Pi_\phi^*K)=\Pi_\phi^*(S_uK)$, then $l_u(x')\cap K$ is a line segment $\forall x'\in K'$, and all the midpoints of $l_u(x')\cap K$ lie in a common hyperplane passing through the origin.

Proof. Assume $h_{\Pi_\phi K}(y',t)=1$ and $h_{\Pi_\phi K}(y',-s)=1$, with $t\neq -s$. By Lemma 6.1, we have

$$\frac{1}{n|K|}\int_{\partial^*K}\phi\left(\frac{(y',t)\cdot\nu(x)}{x\cdot\nu(x)}\right)x\cdot\nu(x)d\mathcal{H}^{n-1}(x)=1\tag{7.4}$$

and

$$\frac{1}{n|K|}\int_{\partial^*K}\phi\left(\frac{(y',-s)\cdot\nu(x)}{x\cdot\nu(x)}\right)x\cdot\nu(x)d\mathcal{H}^{n-1}(x)=1.\tag{7.5}$$

Now we invoke Lemma 2.3 to prove (7.3), and it suffices to show

$$h_{\Pi_\phi(S_uK)}\Big(y',\frac{1}{2}t+\frac{1}{2}s\Big)\leq 1.\tag{7.6}$$

It follows from (7.2), the inequality (2.2), Jensen's inequality, and (7.1) that

$$
\begin{aligned}
&\int_{\partial^*(S_uK)} \phi\left(\frac{(y',\frac{t+s}{2})\cdot\nu(x)}{x\cdot\nu(x)}\right) x\cdot\nu(x)d\mathcal{H}^{n-1}(x)\\
=&\sum_{m=1}^{\infty}\int_{G_m}\phi\left(\frac{\frac{t+s}{2}-y'\cdot\sum_{j=1}^m\nabla(\frac{g_j}{2}-\frac{f_j}{2})(x')}{\sum_{j=1}^m\left\langle\frac{g_j}{2}-\frac{f_j}{2}\right\rangle(x')}\right)\sum_{j=1}^m\left\langle\frac{g_j}{2}-\frac{f_j}{2}\right\rangle(x')dx'\\
&+\sum_{m=1}^{\infty}\int_{G_m}\phi\left(\frac{-\frac{t+s}{2}-y'\cdot\sum_{j=1}^m\nabla(\frac{g_j}{2}-\frac{f_j}{2})(x')}{\sum_{j=1}^m\left\langle\frac{g_j}{2}-\frac{f_j}{2}\right\rangle(x')}\right)\sum_{j=1}^m\left\langle\frac{g_j}{2}-\frac{f_j}{2}\right\rangle(x')dx'\\
\leq&\sum_{m=1}^{\infty}\int_{G_m}\phi\left(\frac{m\cdot\frac{t+s}{2}-y'\cdot\sum_{j=1}^m\nabla(\frac{g_j}{2}-\frac{f_j}{2})(x')}{\sum_{j=1}^m\left\langle\frac{g_j}{2}-\frac{f_j}{2}\right\rangle(x')}\right)\sum_{j=1}^m\left\langle\frac{g_j}{2}-\frac{f_j}{2}\right\rangle(x')dx'\\
&+\sum_{m=1}^{\infty}\int_{G_m}\phi\left(\frac{-m\cdot\frac{t+s}{2}-y'\cdot\sum_{j=1}^m\nabla(\frac{g_j}{2}-\frac{f_j}{2})(x')}{\sum_{j=1}^m\left\langle\frac{g_j}{2}-\frac{f_j}{2}\right\rangle(x')}\right)\sum_{j=1}^m\left\langle\frac{g_j}{2}-\frac{f_j}{2}\right\rangle(x')dx'\\
\leq&\frac{1}{2}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\left(\int_{G_m}\phi\left(\frac{t-y'\cdot\nabla g_j(x')}{\langle g_j\rangle(x')}\right)\langle g_j\rangle(x')dx'+\int_{G_m}\phi\left(\frac{-t+y'\cdot\nabla f_j(x')}{\langle -f_j\rangle(x')}\right)\langle -f_j\rangle(x')dx'\right)\\
&+\frac{1}{2}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\left(\int_{G_m}\phi\left(\frac{-s-y'\cdot\nabla g_j(x')}{\langle g_j\rangle(x')}\right)\langle g_j\rangle(x')dx'+\int_{G_m}\phi\left(\frac{s+y'\cdot\nabla f_j(x')}{\langle -f_j\rangle(x')}\right)\langle -f_j\rangle(x')dx'\right)\\
=&\frac{1}{2}\int_{\partial^*K}\phi\left(\frac{(y',t)\cdot\nu(x)}{x\cdot\nu(x)}\right)x\cdot\nu(x)d\mathcal{H}^{n-1}(x)+\frac{1}{2}\int_{\partial^*K}\phi\left(\frac{(y',-s)\cdot\nu(x)}{x\cdot\nu(x)}\right)x\cdot\nu(x)d\mathcal{H}^{n-1}(x).
\end{aligned}
\tag{7.7}
$$

This, together with (7.4), (7.5) and the fact $|S_uK|=|K|$, implies that

$$
\frac{1}{n|SK|}\int_{\partial^*(SK)}\phi\left(\frac{(y',\frac{1}{2}t+\frac{1}{2}s)\cdot\nu(x)}{x\cdot\nu(x)}\right)x\cdot\nu(x)d\mathcal{H}^{n-1}(x)\leq 1.
$$

This and a glance at definition (1.3), gives (7.6), and thus (7.3) is proved.

Suppose that ϕ is strictly convex and $S_u(\Pi_\phi^*K)=\Pi_\phi^*(S_uK)$. Lemma 2.3 forces

$$
h_{\Pi_\phi(S_uK)}\left(y',\frac{1}{2}t+\frac{1}{2}s\right)=1,
$$

and equality holds in (7.7). The strict convexity of ϕ together with the inequality (2.2) implies that the open sets G_m must be empty unless $m=1$. Then, $\mathrm{relint}K'=G_1$, and there are only two graph functions f_1,g_1. It also suggests that $l_u(x')\cap K$ must be a line segment, for each $x'\in K'$.

Moreover, since equality holds in the second inequality of (7.7), we have

$$
\frac{t-y'\cdot\nabla g_1(x')}{\langle g_1\rangle(x')}=\frac{s+y'\cdot\nabla f_1(x')}{\langle -f_1\rangle(x')}\quad\text{and}\quad\frac{-s-y'\cdot\nabla g_1(x')}{\langle g_1\rangle(x')}=\frac{-t+y'\cdot\nabla f_1(x')}{\langle -f_1\rangle(x')}
\tag{7.8}
$$

for all $x' \in G_1^*$. Let $y' = 0$ and note that s and t cannot vanish at the same time. We deduce that $\langle g_1 \rangle(x') = \langle -f_1 \rangle(x')$. Namely,

$$\Big(x', g_1 + f_1\Big) \cdot \Big(\nabla(g_1 + f_1)(x'), -1\Big) = 0. \tag{7.9}$$

The fact that $\langle g_1 \rangle(x') = \langle -f_1 \rangle(x')$, together with (7.8) also shows that

$$y' \cdot \nabla(g_1 + f_1)(x') = t - s. \tag{7.10}$$

Since y' can be chosen in any direction in $u^\perp$, $\nabla(g_1 + f_1)(x')$ must be a constant vector in $u^\perp$, say $v_0 \in u^\perp$. Substituting this into (7.9) shows that the midpoints of $l_u(x') \cap K$ lie in a common hyperplane, for all $x' \in G_1^*$. Since K is a star body and $G_1 = \mathrm{relint} K'$, we complete the proof. $\square$

Proof of Theorem 1.1. By the monotonicity of the Orlicz projection operator (Theorem 7.1), the convergence of Steiner symmetrizations (Theorem 5.1), and the weak continuity of the Orlicz projection operator (Lemma 6.3), we have

$$|\Pi_\phi^* K| \le |\bar{Q}^*| \le |\Pi_\phi^* B_K|.$$

By the affine invariance of the polar Orlicz projection operator (Lemma 6.4), the volume ratio $|\Pi_\phi^* K|/|K|$ is maximized when K is an ellipsoid centered at the origin.

If $\phi \in \mathcal{C}_s$ and $|\Pi_\phi^* K|/|K| = |\Pi_\phi^* B_K|/|B_K|$, then $S_u \Pi_\phi^* K = \Pi_\phi^*(S_u K)$, for any $u \in T(K)$. By Theorem 7.1, $l_u(x') \cap K$ is a line segment for every $x' \in K'$, and all of the midpoints of $l_u(x') \cap K$ lie in a common hyperplane that passes through the origin. Since $T(K)$ is dense in S^{n-1}, K must be an ellipsoid centered at the origin (see e.g., [24, Theorem 5.4]) for the characterization of ellipsoid). $\square$

8 Appendix

We restate the [24, Lemma 5.8] and [24, Lemma 5.9] after a little modification. The proof of them are completely same as [24], and we would like to refer the reader to the the updated version "arXiv:2008.07026" for a detailed proof.

Lemma 8.1. *Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$. Let $D_u \in \Sigma$ be a measurable subset of X for every $u \in S^{n-1}$. If there exists a Borel set $S \subset S^{n-1}$ such that $\mathcal{H}^{n-1}(S) > 0$ and $\mu(D_u) > 0$ for every $u \in S$, then there exist n linearly independent vectors $u_1, u_2, \ldots, u_n \in S$ such that*

$$\mu\left(\bigcap_{i=1}^{n} D_{u_i}\right) > 0. \tag{8.1}$$

Lemma 8.2. *Let Ω be a set of finite perimeter in $\mathbb{R}^n$ and let*

$$D_u = \{x \in \partial^*\Omega : u \cdot v^{\Omega}(x) = 0\}. \tag{8.2}$$

Then there exists a set $T_1 \subset S^{n-1}$ such that $\mathcal{H}^{n-1}(S^{n-1} \setminus T_1) = 0$ and $\mathcal{H}^{n-1}(D_u) = 0$ for any $u \in T_1$.

Acknowledgement. The authors appreciate the anonymous referees' helpful suggestions for improving this manuscript. The authors would like to thank Qun Wang and Gaoyong Zhang for their comments on a previous version of this manuscript. Research of Lin is supported by NSFC 11971080 and the funds of the Basic and Advanced Research Project of CQ CSTC cstc2018jcyjAX0790. Research of Xi is supported by NSFC 12071277, and STSCM program 20JC1412600.

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