ON BACH-FLAT GRADIENT SHRINKING RICCI SOLITONS

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Abstract

In this article, we classify n-dimensional ($n \geq 4$) complete Bach-flat gradient shrinking Ricci solitons. More precisely, we prove that any 4-dimensional Bach-flat gradient shrinking Ricci soliton is either Einstein, or locally conformally flat and hence a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^4$ or the round cylinder $\mathbb{S}^3 \times \mathbb{R}$. More generally, for $n \geq 5$, a Bach-flat gradient shrinking Ricci soliton is either Einstein, or a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^n$ or the product $N^{n-1} \times \mathbb{R}$, where N^{n-1} is Einstein.

1. The results

A complete Riemannian manifold (M^n, g_{ij}) is called a *gradient Ricci soliton* if there exists a smooth function f on M^n such that the Ricci tensor R_{ij} of the metric g_{ij} satisfies the equation

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}$$

for some constant ρ. For $\rho = 0$ the Ricci soliton is *steady*, for $\rho > 0$ it is *shrinking*, and for $\rho < 0$ *expanding*. The function f is called a *potential function* of the gradient Ricci soliton. Clearly, when f is a constant the gradient Ricci soliton is simply an Einstein manifold. Thus Ricci solitons are natural extensions of Einstein metrics. Gradient Ricci solitons play an important role in Hamilton's Ricci flow, as they correspond to self-similar solutions, and they often arise as singularity models. Therefore, it is important to classify gradient Ricci solitons or to understand their geometry.

In this article, we focus our attention on gradient shrinking Ricci solitons, which are possible Type I singularity models in the Ricci flow. We normalize the constant $\rho = 1/2$ so that the shrinking soliton equation is given by

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}. \tag{1.1}$$

DUKE MATHEMATICAL JOURNAL
Vol. 162, No. 6, © 2013 DOI 10.1215/00127094-2147649
Received 10 August 2011. Revision received 15 August 2012.
2010 *Mathematics Subject Classification*. Primary 53C21; Secondary 53C20, 53C25, 53C44.
Cao's work partially supported by National Science Foundation grant DMS-0909581.

In recent years, as a consequence of Perelman's groundbreaking work in [21] and [22], much research has been devoted to studying the geometry and classifications of gradient shrinking Ricci solitons. We refer the reader to the survey papers [4] and [5] by the first author, as well as to the references therein, for an overview of recent progress on the subject. In particular, it is known (see [7], [20], [22]) that any complete 3-dimensional gradient shrinking Ricci soliton is a finite quotient of either the round sphere $\mathbb{S}^3$, or of the Gaussian shrinking soliton $\mathbb{R}^3$, or of the round cylinder $\mathbb{S}^2 \times \mathbb{R}$. For higher dimensions, it has been proved that complete locally conformally flat gradient shrinking Ricci solitons are finite quotients of either the round sphere $\mathbb{S}^n$, or of the Gaussian shrinking soliton $\mathbb{R}^n$, or of the round cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ (this was first due to Zhang [25], based on the work of Ni and Wallach [20]; see also the works of Eminenti, La Nave, and Mantegazza [15], of Petersen and Wylie [23], of Cao, Wang, and Zhang [10], and of Munteanu and Sesum [19]). Moreover, it follows from the works of Fernández-López and García-Río [16] and of Munteanu and Sesum [19] that n-dimensional complete gradient shrinking solitons with harmonic Weyl tensor are rigid in the sense that they are finite quotients of the product of an Einstein manifold N^k with the Gaussian shrinking soliton $\mathbb{R}^{n-k}$.

Our aim in this paper is to investigate an interesting class of complete gradient shrinking Ricci solitons: those with vanishing Bach tensor. This well-known tensor was introduced by Bach [1] in the early 1920s to study conformal relativity. On any n-dimensional manifold (M^n, g_{ij}), $n \geq 4$, the Bach tensor is defined by

$$B_{ij} = \frac{1}{n-3}\nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R_{kl} W_i{}^k{}_j{}^l.$$

Here W_{ikjl} is the Weyl tensor. It is easy to see that if (M^n, g_{ij}) is either locally conformally flat (i.e., $W_{ikjl} = 0$) or Einstein, then (M^n, g_{ij}) is *Bach-flat*: $B_{ij} = 0$.

The case when $n = 4$ is the most interesting, as it is well known (see [2] or [14]) that on any compact 4-manifold (M^4, g_{ij}), Bach-flat metrics are precisely the critical points of the *conformally invariant* functional on the space of metrics,

$$\mathcal{W}(g) = \int_M |W_g|^2 \, dV_g,$$

where W_g denotes the Weyl tensor of g. Moreover, if (M^4, g_{ij}) is either *half conformally flat* (i.e., self-dual or anti-self-dual) or *locally conformal to an Einstein manifold*, then its Bach tensor vanishes. In this paper, we will show that the (stronger) converse holds for gradient shrinking solitons: *Bach-flat* 4*-dimensional gradient shrinking solitons are either Einstein or locally conformally flat*.

Our main results are the following classification theorems for Bach-flat gradient shrinking Ricci solitons.

THEOREM 1.1
Let (M^4, g_{ij}, f) be a complete Bach-flat gradient shrinking Ricci soliton. Then, (M^4, g_{ij}, f) is either
(i) *Einstein, or*
(ii) *locally conformally flat, and hence a finite quotient of either the Gaussian shrinking soliton $\mathbb{R}^4$ or the round cylinder $\mathbb{S}^3 \times \mathbb{R}$.*

More generally, for $n \geq 5$, we have the following.

THEOREM 1.2
Let (M^n, g_{ij}, f), $n \geq 5$, be a complete Bach-flat gradient shrinking Ricci soliton. Then, (M^n, g_{ij}, f) is either
(i) *Einstein, or*
(ii) *a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^n$, or*
(iii) *a finite quotient of $N^{n-1} \times \mathbb{R}$, where N^{n-1} is an Einstein manifold of positive scalar curvature.*

The basic idea in proving Theorem 1.1 and Theorem 1.2 is to explore the hidden relations between the Bach tensor B_{ij} and the Cotton tensor C_{ijk} on a gradient shrinking Ricci soliton. It turns out that the key link between these two classical tensors is provided by a third tensor, the covariant 3-tensor D_{ijk} defined by

$$D_{ijk} = \frac{1}{n-2}(A_{jk}\nabla_i f - A_{ik}\nabla_j f) + \frac{1}{(n-1)(n-2)}(g_{jk}E_{il} - g_{ik}E_{jl})\nabla_l f, \tag{1.2}$$

where A_{ij} is the Schouten tensor and E_{ij} is the Einstein tensor (see Section 3).

This tensor D_{ijk} (and its equivalent version in Section 3) was introduced by the authors in [8] to study the classification of locally conformally flat gradient steady solitons. On one hand, for any gradient Ricci soliton, it turns out that the Bach tensor B_{ij} can be expressed in terms of D_{ijk} and of the Cotton tensor C_{ijk}:

$$B_{ij} = -\frac{1}{n-2}\Big(\nabla_k D_{ikj} + \frac{n-3}{n-2}C_{jli}\nabla_l f\Big). \tag{1.3}$$

On the other hand, as shown in [8], D_{ijk} is closely related to the Cotton tensor and the Weyl tensor by

$$D_{ijk} = C_{ijk} + W_{ijkl}\nabla_l f. \tag{1.4}$$

By using (1.3), we are able to show that the vanishing of the Bach tensor B_{ij} implies the vanishing of D_{ijk} for gradient shrinking solitons (see Lemma 4.1). On the other hand, the norm of D_{ijk} is linked to the geometry of level surfaces of the potential function f by the following key identity (see Proposition 3.1): at any point $p \in M^n$ where $\nabla f(p) \neq 0$, we have

$$|D_{ijk}|^2 = \frac{2|\nabla f|^4}{(n-2)^2}\Big|h_{ab} - \frac{H}{n-1}g_{ab}\Big|^2 + \frac{1}{2(n-1)(n-2)}|\nabla_a R|^2, \tag{1.5}$$

where h_{ab} and H are the second fundamental form and the mean curvature for the level surface $\Sigma = \{f = f(p)\}$, and where g_{ab} is the induced metric on the level surface Σ. Thus, the vanishing of D_{ijk} and (1.5) tell us that the geometry of the shrinking Ricci soliton and the level surfaces of the potential function are very special (see Proposition 3.2); consequently, we deduce that $D_{ijk} = 0$ implies that the Cotton tensor $C_{ijk} = 0$ at all points where $|\nabla f| \neq 0$ (see Lemma 4.2 and Theorem 5.1). Furthermore, when $n = 4$, we can actually show, by using (1.4), that the Weyl tensor W_{ijkl} must vanish at all points where $|\nabla f| \neq 0$ (see Lemma 4.3). Then the main theorems follow immediately from the known classification theorem for locally conformally flat gradient shrinking Ricci solitons and the rigid theorem for gradient shrinking Ricci solitons with harmonic Weyl tensor, respectively.

Remark 1.1
Very recently, by cleverly using the tensor D_{ijk}, Chen and Wang [13] showed that 4-dimensional half-conformally flat gradient shrinking Ricci solitons are either Einstein or locally conformally flat. Since half-conformal flat implies Bach-flat in dimension 4, our Theorem 1.1 is clearly an improvement.

Note that by a theorem of Hitchin (see [2, Theorem 13.30]), a compact 4-dimensional half-conformally flat Einstein manifold (of positive scalar curvature) is $\mathbb{S}^4$ or $\mathbb{C}P^2$. Combining Hitchin's theorem and Theorem 1.1, we arrive at the following classification of 4-dimensional compact half-conformally flat gradient shrinking Ricci solitons, which was first obtained by Chen and Wang in [13].

COROLLARY 1.1 ([13, Theorem 1.2])
If (M^4, g_{ij}, f) is a compact half-conformally flat gradient shrinking Ricci soliton, then (M^4, g_{ij}) is isometric to the standard $\mathbb{S}^4$ or $\mathbb{C}P^2$.

Finally, in Section 5, we observe that for all gradient (shrinking, or steady, or expanding) Ricci solitons, the vanishing of D_{ijk} implies the vanishing of the Cotton tensor C_{ijk} at all points where $|\nabla f| \neq 0$ (see Theorem 5.1). This yields the classification of n-dimensional ($n \geq 4$) gradient shrinking Ricci solitons, as well as 4-dimensional gradient steady Ricci solitons, with vanishing D_{ijk}.

THEOREM 1.3
Let (M^n, g_{ij}, f), $n \geq 4$, *be a complete gradient shrinking Ricci soliton with* $D_{ijk} = 0$. *Then*

(i) (M^4, g_{ij}, f) *is either Einstein, or a finite quotient of* $\mathbb{R}^4$ *or* $\mathbb{S}^3 \times \mathbb{R}$*;*

(ii) *for* $n \geq 5$, (M^n, g_{ij}, f) *is either Einstein, or a finite quotient of the Gaussian shrinking soliton* $\mathbb{R}^n$*, or a finite quotient of* $N^{n-1} \times \mathbb{R}$*, where* N^{n-1} *is Einstein.*

THEOREM 1.4
Let (M^4, g_{ij}, f) *be a complete gradient steady Ricci soliton with* $D_{ijk} = 0$*; then* (M^4, g_{ij}, f) *is either Ricci flat or isometric to the Bryant soliton.*

2. Preliminaries

In this section, we fix our notation and we recall some basic facts and known results about gradient Ricci solitons that we need in the proofs of Theorem 1.1 and Theorem 1.2.

First of all, we recall that on any n-dimensional Riemannian manifold (M^n, g_{ij}) ($n \geq 3$), the Weyl curvature tensor is given by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}),$$

and the Cotton tensor by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R).$$

Remark 2.1
In terms of the Schouten tensor,

$$A_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}, \tag{2.1}$$

we have

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}A_{jl} - g_{il}A_{jk} - g_{jk}A_{il} + g_{jl}A_{ik}),$$
$$C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik}.$$

It is well known that, for $n = 3$, W_{ijkl} vanishes identically, while $C_{ijk} = 0$ if and only if (M^3, g_{ij}) is locally conformally flat; for $n \geq 4$, $W_{ijkl} = 0$ if and only if (M^n, g_{ij}) is locally conformally flat. Moreover, for $n \geq 4$, the Cotton tensor C_{ijk} is, up to a constant factor, the divergence of the Weyl tensor:

$$C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl}, \tag{2.2}$$

and hence the vanishing of the Cotton tensor $C_{ijk} = 0$ (in dimension $n \geq 4$) is also referred as being harmonic Weyl.

Moreover, for $n \geq 4$, the Bach tensor is defined by

$$B_{ij} = \frac{1}{n-3}\nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R_{kl} W_i{}^k{}_j{}^l.$$

By (2.2), we have

$$B_{ij} = \frac{1}{n-2}(\nabla_k C_{kij} + R_{kl} W_i{}^k{}_j{}^l). \tag{2.3}$$

Note that C_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices

$$C_{ijk} = -C_{jik} \qquad \text{and} \qquad g^{ij} C_{ijk} = g^{ik} C_{ijk} = 0. \tag{2.4}$$

Next we recall some basic facts about complete gradient shrinking Ricci solitons satisfying (1.1).

LEMMA 2.1 ([18, Section 20])
Let (M^n, g_{ij}, f) be a complete gradient shrinking Ricci soliton satisfying equation (1.1). Then we have

$$\nabla_i R = 2R_{ij}\nabla_j f, \tag{2.5}$$

and

$$R + |\nabla f|^2 - f = C_0$$

for some constant C_0. Here R denotes the scalar curvature.

Note that if we normalize f by adding the constant C_0 to it, then we have

$$R + |\nabla f|^2 = f. \tag{2.6}$$

LEMMA 2.2
Let (M^n, g_{ij}, f) be a complete gradient steady soliton. Then it has nonnegative scalar curvature $R \geq 0$.

Lemma 2.2 is a special case of a more general result of Chen [12], which states that $R \geq 0$ for any ancient solution to the Ricci flow (for an alternative proof of Lemma 2.2, see, e.g., the more recent work of Pigola, Rimoldi, and Setti [24]).

LEMMA 2.3 ([9, Theorems 1.1, 1.2])
Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton satisfying (1.1) and the normalization (2.6). Then,

(i) *the potential function f satisfies the estimates*

$$\frac{1}{4}\big(r(x)-c_1\big)^2 \le f(x) \le \frac{1}{4}\big(r(x)+c_2\big)^2,$$

where $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$ and where c_1 and c_2 are positive constants depending only on n and the geometry of g_{ij} on the unit ball $B(x_0, 1)$;

(ii) *there exists some constant $C > 0$ such that*

$$\mathrm{Vol}\big(B(x_0, s)\big) \le C s^n$$

for $s > 0$ sufficiently large.

3. The covariant 3-tensor D_{ijk}

In this section, we review the covariant 3-tensor D_{ijk} (introduced in our previous work [8]) along with its important properties.

For any gradient Ricci soliton satisfying the defining equation

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}, \tag{3.1}$$

the covariant 3-tensor D_{ijk} is defined as

$$D_{ijk} = \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) + \frac{1}{2(n-1)(n-2)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) - \frac{R}{(n-1)(n-2)}(g_{jk}\nabla_i f - g_{ik}\nabla_j f).$$

Note that, by using (2.5), D_{ijk} can also be expressed as

$$D_{ijk} = \frac{1}{n-2}(A_{jk}\nabla_i f - A_{ik}\nabla_j f) + \frac{1}{(n-1)(n-2)}(g_{jk}E_{il} - g_{ik}E_{jl})\nabla_l f, \tag{3.2}$$

where A_{ij} is the Schouten tensor in (2.1) and where $E_{ij} = R_{ij} - (R/2)g_{ij}$ is the Einstein tensor.

This 3-tensor D_{ijk} is closely tied to the Cotton tensor, and it played a significant role in our previous work [8] on classifying locally conformally flat gradient steady solitons, as well as in the subsequent works of Brendle [3] and Chen and Wang [13].

LEMMA 3.1
Let (M^n, g_{ij}, f) $(n \geq 3)$ be a complete gradient soliton satisfying (3.1). Then D_{ijk} is related to the Cotton tensor C_{ijk} and the Weyl tensor W_{ijkl} by

$$D_{ijk} = C_{ijk} + W_{ijkl}\nabla_l f.$$

Proof
From the soliton equation (3.1), we have

$$\nabla_i R_{jk} - \nabla_j R_{ik} = -\nabla_i \nabla_j \nabla_k f + \nabla_j \nabla_i \nabla_k f = -R_{ijkl}\nabla_l f.$$

Hence, using (2.5), we obtain

$$\begin{aligned} C_{ijk} &= \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) \\ &= -R_{ijkl}\nabla_l f - \frac{1}{(n-1)}(g_{jk}R_{il} - g_{ik}R_{jl})\nabla_l f \\ &= -W_{ijkl}\nabla_l f - \frac{1}{n-2}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) \\ &\quad + \frac{1}{2(n-1)(n-2)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{ik}\nabla_j f - g_{jk}\nabla_i f) \\ &= -W_{ijkl}\nabla_l f + D_{ijk}. \end{aligned}$$

□

Remark 3.1
By Lemma 3.1, D_{ijk} is equal to the Cotton tensor C_{ijk} in dimension $n = 3$. In addition, it is easy to see that

$$D_{ijk}\nabla_k f = C_{ijk}\nabla_k f.$$

Also, D_{ijk} vanishes if (M^n, g_{ij}, f) $(n \geq 3)$ is either Einstein or locally conformally flat. Moreover, like the Cotton tensor C_{ijk}, D_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices

$$D_{ijk} = -D_{jik} \qquad \text{and} \qquad g^{ij}D_{ijk} = g^{ik}D_{ijk} = 0. \tag{3.3}$$

What is so special about D_{ijk} is the following key identity, which links the norm of D_{ijk} to the geometry of the level surfaces of the potential function f. We refer readers to [8, Lemma 4.4] for its proof.

PROPOSITION 3.1 ([8, Lemma 4.1])
Let (M^n, g_{ij}, f) $(n \geq 3)$ be an n-dimensional gradient Ricci soliton satisfying (3.1). Then, at any point $p \in M^n$ where $\nabla f(p) \neq 0$, we have

$$|D_{ijk}|^2 = \frac{2|\nabla f|^4}{(n-2)^2}\Big|h_{ab} - \frac{H}{n-1}g_{ab}\Big|^2 + \frac{1}{2(n-1)(n-2)}|\nabla_a R|^2,$$

where h_{ab} and H are the second fundamental form and the mean curvature of the level surface $\Sigma = \{f = f(p)\}$ and where g_{ab} is the induced metric on Σ.

Finally, thanks to Proposition 3.1, the vanishing of D_{ijk} implies many nice properties about the geometry of the Ricci soliton (M^n, g_{ij}, f) and the level surfaces of the potential function f.

PROPOSITION 3.2
Let (M^n, g_{ij}, f) $(n \geq 3)$ be any complete gradient Ricci soliton with $D_{ijk} = 0$, let c be a regular value of f, and let $\Sigma_c = \{f = c\}$ be the level surface of f. Set $e_1 = \nabla f/|\nabla f|$, and pick any orthonormal frame $e_2, \ldots, e_n$ tangent to the level surface Σ_c. Then

(a) *$|\nabla f|^2$ and the scalar curvature R of (M^n, g_{ij}, f) are constant on Σ_c;*
(b) *$R_{1a} = 0$ for any $a \geq 2$, and $e_1 = \nabla f/|\nabla f|$ is an eigenvector of Rc;*
(c) *the second fundamental form h_{ab} of Σ_c is of the form $h_{ab} = Hg_{ab}/(n-1)$;*
(d) *the mean curvature H is constant on Σ_c;*
(e) *on Σ_c, the Ricci tensor of (M^n, g_{ij}, f) either has a unique eigenvalue λ or it has two distinct eigenvalues λ and μ of multiplicity 1 and $n-1$, respectively. In either case, $e_1 = \nabla f/|\nabla f|$ is an eigenvector of λ.*

Proof
Clearly (a) and (c) follow immediately from $D_{ijk} = 0$, Proposition 3.1, and (2.6); and (b) follows from (a) and (2.5): $R_{1a} = (1/(2|\nabla f|))\nabla_a R = 0$.

For (d), we consider the Codazzi equation

$$R_{1cab} = \nabla_a^{\Sigma_c} h_{bc} - \nabla_b^{\Sigma_c} h_{ac}, \quad a, b, c = 2, \ldots, n. \tag{3.4}$$

Tracing over b and c in (3.4), we obtain

$$R_{1a} = \nabla_a^{\Sigma_c} H - \nabla_b^{\Sigma_c} h_{ab} = \Big(1 - \frac{1}{n-1}\Big)\nabla_a H.$$

Then (d) follows since $R_{1a} = 0$.

Finally, the second fundamental form is given by

$$h_{ab} = \Big\langle \nabla_a \frac{\nabla f}{|\nabla f|}, e_b \Big\rangle = \frac{\nabla_a \nabla_b f}{|\nabla f|} = \frac{\rho g_{ab} - R_{ab}}{|\nabla f|}.$$

Combining this with (c), we see that

$$R_{ab} = \rho g_{ab} - |\nabla f| h_{ab} = \Big(\rho - \frac{H}{n-1}|\nabla f|\Big) g_{ab}.$$

But both H and $|\nabla f|$ are constant on Σ_c, so the Ricci tensor restricted to the tangent space of Σ_c has only one eigenvalue μ:

$$\mu = R_{aa} = \rho - H|\nabla f|/(n-1), \quad a = 2, \dots, n, \tag{3.5}$$

which is constant along Σ_c. On the other hand,

$$\lambda = R_{11} = R - \sum_{a=2}^{n} R_{aa} = R - (n-1)\rho + H|\nabla f|, \tag{3.6}$$

again a constant along Σ_c. This proves (e). □

Remark 3.2
In any neighborhood U of the level surface Σ_c where $|\nabla f|^2 \neq 0$, we can always express the metric g_{ij} as

$$ds^2 = \frac{1}{|\nabla f|^2(f,\theta)}(df)^2 + g_{ab}(f,\theta)\, d\theta^a\, d\theta^b. \tag{3.7}$$

Here $\theta = (\theta^2, \dots, \theta^n)$ denotes any local coordinates on Σ_c. It follows from Proposition 3.2 that, when $D_{ijk} = 0$, the metric g_{ij} is in fact a warped product metric on U of the form

$$ds^2 = dr^2 + \varphi^2(r)\bar{g}_{\Sigma_c}, \tag{3.8}$$

where $\bar{g}_{\Sigma_c}$ denotes the induced metric on Σ_c. Furthermore, $(\Sigma_c, \bar{g}_{\Sigma_c})$ is necessarily Einstein. The details can be found in [6].

4. The proofs of Theorem 1.1 and Theorem 1.2

Throughout this section, we assume that (M^n, g_{ij}, f), $n \geq 4$, is a complete gradient shrinking soliton satisfying (1.1).

First of all, we relate the Bach tensor B_{ij} to the Cotton tensor C_{ijk} and the tensor D_{ijk}, and then we show that the Bach-flatness implies that $D_{ijk} = 0$.

LEMMA 4.1
Let (M^n, g_{ij}, f) be a complete gradient shrinking soliton. If $B_{ij} = 0$, then $D_{ijk} = 0$.

Proof
By direct computations and by using (2.2), (2.3), and Lemma 3.1, we have

$$
\begin{aligned}
B_{ij} &= -\frac{1}{n-2}\nabla_k C_{ikj} + \frac{1}{n-2} R_{kl} W_{ikjl} \\
&= -\frac{1}{n-2}\nabla_k (D_{ikj} - W_{ikjl}\nabla_l f) + \frac{1}{n-2} R_{kl} W_{ikjl} \\
&= -\frac{1}{n-2}(\nabla_k D_{ikj} - \nabla_k W_{jlik}\nabla_l f) + \frac{1}{n-2}(R_{kl} + \nabla_k\nabla_l f) W_{ijkl}.
\end{aligned}
$$

Hence,

$$
B_{ij} = -\frac{1}{n-2}\Big(\nabla_k D_{ikj} + \frac{n-3}{n-2} C_{jli}\nabla_l f\Big). \tag{4.1}
$$

Next, we use (4.1) to show that Bach-flatness implies the vanishing of the tensor D_{ijk}. By Lemma 2.3, for each $r > 0$ sufficiently large, $\Omega_r = \{x \in M \mid f(x) \le r\}$ is compact. Now, by the definition of D_{ijk} and the identity (4.1), as well as properties (2.4) and (3.3), we have

$$
\begin{aligned}
&\int_{\Omega_r} B_{ij}\nabla_i f \nabla_j f \, dV \\
&= -\frac{1}{(n-2)} \int_{\Omega_r} \nabla_k D_{ikj} \nabla_i f \nabla_j f \, dV \\
&= \frac{1}{(n-2)}\Big(\int_{\Omega_r} D_{ikj}\nabla_i f \nabla_k \nabla_j f \, dV - \int_{\Omega_r} \nabla_k (D_{ikj}\nabla_i f \nabla_j f)\, dV \Big) \\
&= -\frac{1}{(n-2)}\Big(\int_{\Omega_r} D_{ikj}\nabla_i f R_{jk}\, dV + \int_{\partial\Omega_r} D_{ikj}\nabla_i f \nabla_j f \nu_k \, dS \Big) \\
&= -\frac{1}{2(n-2)} \int_{\Omega_r} D_{ikj}(\nabla_i f R_{jk} - \nabla_k f R_{ij})\, dV \\
&= -\frac{1}{2}\int_{\Omega_r} |D_{ikj}|^2 \, dV.
\end{aligned}
$$

Here we have used the fact, in view of (3.3), that

$$
\int_{\partial\Omega_r} D_{ikj}\nabla_i f \nabla_j f \nu_k \, dS = \int_{\partial\Omega_r} D_{ikj}\nabla_i f \nabla_j f \nabla_k f \frac{1}{|\nabla f|}\, dS = 0.
$$

By taking $r \to \infty$, we immediately obtain

$$
\int_M B_{ij}\nabla_i f \nabla_j f \, dV = -\frac{1}{2}\int_M |D_{ikj}|^2 \, dV.
$$

This completes the proof of Lemma 4.1. □

LEMMA 4.2
Let (M^n, g_{ij}, f), $n \geq 4$, be a complete gradient shrinking Ricci soliton with vanishing D_{ijk}. Then the Cotton tensor $C_{ijk} = 0$ at all points where $\nabla f \neq 0$.

Proof
First of all, $D_{ijk} = 0$ and Lemma 3.1 imply that

$$C_{ijk} = -W_{ijkl} \nabla_l f, \tag{4.2}$$

and hence

$$C_{ijk} \nabla_k f = -W_{ijkl} \nabla_k f \nabla_l f = 0. \tag{4.3}$$

Next, for any point $p \in M$ with $\nabla f(p) \neq 0$, we choose a local coordinates system $(\theta^2, \ldots, \theta^n)$ on the lever surface $\Sigma = \{f = f(p)\}$. In any neighborhood U of the level surface Σ where $|\nabla f|^2 \neq 0$, we use the local coordinates system

$$(x^1, x^2, \ldots, x^n) = (f, \theta^2, \ldots, \theta^n)$$

adapted to level surfaces. In the following, we use a, b, c to represent indices on the level sets which ranges from 2 to n, while i, j, k are used to represent indices on M ranging from 1 to n. Under the above chosen local coordinates system, the metric g can be expressed as

$$ds^2 = \frac{1}{|\nabla f|^2} df^2 + g_{ab}(f, \theta) \, d\theta^a \, d\theta^b.$$

Next, we denote $\nu = -\frac{\nabla f}{|\nabla f|}$. It is then easy to see that

$$\nu = -|\nabla f| \partial_f \qquad \text{or} \qquad \partial_f = \frac{1}{|\nabla f|^2} \nabla f.$$

Also, ∂_1 and ∂_f will be interchangeable below. We have

$$\nabla_1 f = 1 \qquad \text{and} \qquad \nabla_a f = 0 \quad \text{for } a \geq 2. \tag{4.4}$$

Then, in this coordinate, (4.3) implies that

$$C_{ij1} = 0.$$

CLAIM 1
$D_{ijk} = 0$ implies that $C_{abc} = 0$ for $a \geq 2, b \geq 2$, and $c \geq 2$.

To show that $C_{abc}=0$, we make use of Proposition 3.2 as follows. From the Codazzi equation (3.4) and $h_{ab}=Hg_{ab}/(n-1)$, we get

$$R_{1cab}=\nabla^{\Sigma}_a h_{bc}-\nabla^{\Sigma}_b h_{ac}=\frac{1}{n-1}\big(g_{bc}\partial_a(H)-g_{ac}\partial_b(H)\big). \tag{4.5}$$

But we also know that the mean curvature H is constant on the level surface Σ of f, so

$$R_{1abc}=0.$$

Moreover, since $R_{1a}=0$, we easily obtain

$$W_{1abc}=R_{1abc}=0.$$

By (4.2), we have

$$C_{abc}=-W_{abci}\nabla_j fg^{ij}=W_{1cab}\nabla_1 fg^{11}=0.$$

This finishes the proof of Claim 1.

CLAIM 2
$D_{ijk}=0$ *implies that* $C_{1ab}=C_{a1b}=0$.

To prove this, let us compute the second fundamental form in the preferred local coordinates system $(f,\theta^2,\dots,\theta^n)$:

$$h_{ab}=-\langle\nu,\nabla_a\partial_b\rangle=-\langle\nu,\Gamma^1_{ab}\partial_f\rangle=\frac{\Gamma^1_{ab}}{|\nabla f|}.$$

But the Christoffel symbol Γ^1_{ab} is given by

$$\Gamma^1_{ab}=\frac{1}{2}g^{11}\Big(-\frac{\partial g_{ab}}{\partial f}\Big)=\frac{1}{2}|\nabla f|\nu(g_{ab}).$$

Hence, we obtain

$$h_{ab}=\frac{1}{2}\nu(g_{ab}). \tag{4.6}$$

On the other hand, since $|\nabla f|$ is constant along level surfaces, we have

$$[\partial_a,\nu]=-\big[\partial_a,|\nabla f|\partial_f\big]=0.$$

Then using the fact that $\langle\nu,\nu\rangle=1$ and that $\langle\nu,\partial_a\rangle=0$, it is easy to see that

$$\nabla_\nu\nu=0. \tag{4.7}$$

By direct computations and using Proposition 3.2, we can compute the following component of the Riemannian curvature tensor:

$$
\begin{aligned}
Rm(\nu, \partial_a, \nu, \partial_b) &= \langle \nabla_\nu \nabla_a \partial_b - \nabla_a \nabla_\nu \partial_b, \nu\rangle \\
&= \big\langle \nabla_\nu (\nabla_a^\Sigma \partial_b + \nabla_a^\perp \partial_b), \nu\big\rangle - \langle \nabla_a \nabla_\nu \partial_b, \nu\rangle \\
&= \langle \nabla_a^\Sigma \partial_b, -\nabla_\nu \nu\rangle + \big\langle \nabla_\nu(-h_{ab}\nu), \nu\big\rangle + \langle \nabla_b \nu, \nabla_a \nu\rangle \\
&= -\nu(h_{ab}) + h_{ac}h_{cb} \\
&= -\frac{\nu(H)}{n-1} g_{ab} + \frac{H^2}{(n-1)^2} g_{ab}.
\end{aligned}
$$

Taking trace in a, b yields

$$
Rc(\nu, \nu) = -\nu(H) + \frac{H^2}{n-1}.
$$

Thus

$$
\begin{aligned}
Rm(\nu, \partial_a, \nu, \partial_b) &= -\frac{\nu(H)}{n-1} g_{ab} + \frac{H^2}{(n-1)^2} g_{ab} \\
&= \frac{Rc(\nu, \nu)}{n-1} g_{ab}.
\end{aligned}
$$

Finally, we are ready to compute C_{1ab}:

$$
C_{1ab} = -W_{1abi} \nabla_j f g^{ij} = W_{1a1b} |\nabla f|^2 = W(\nu, \partial_a, \nu, \partial_b). \tag{4.8}
$$

However, by using Proposition 3.2(e), we have

$$
\begin{aligned}
W(\nu, \partial_a, \nu, \partial_b) &= Rm(\nu, \partial_a, \nu, \partial_b) + \frac{R g_{ab}}{(n-1)(n-2)} - \frac{1}{n-2}\big(Rc(\nu, \nu) g_{ab} + R_{ab}\big) \\
&= \frac{Rc(\nu, \nu)}{n-1} g_{ab} + \frac{R g_{ab}}{(n-1)(n-2)} - \frac{1}{n-2}\big(Rc(\nu, \nu) g_{ab} + R_{ab}\big) \\
&= \frac{\lambda}{n-1} g_{ab} + \frac{(\lambda + (n-1)\mu) g_{ab}}{(n-1)(n-2)} - \frac{1}{n-2}(\lambda g_{ab} + \mu g_{ab}) \\
&= 0.
\end{aligned}
$$

Hence,

$$
C_{1ab} = W_{1a1b} = 0. \tag{4.9}
$$

This finishes the proof of Claim 2.

Therefore, we have shown that $C_{ij1} = 0$, $C_{abc} = 0$, and $C_{1ab} = 0$. This proves Lemma 4.2. □

For dimension $n = 4$, we can prove a stronger result with the following.

LEMMA 4.3
Let (M^4, g_{ij}, f) *be a complete gradient shrinking Ricci soliton with vanishing* D_{ijk}. *Then the Weyl tensor* $W_{ijkl} = 0$ *at all points where* $\nabla f \neq 0$.

Proof
From Lemma 4.2, we know that $D_{ijk} = 0$ implies that $C_{ijk} = 0$. Hence it follows from Lemma 3.1 that

$$W_{ijkl} \nabla_l f = 0$$

for all $1 \leq i, j, k, l \leq 4$. For any p where $|\nabla f| \neq 0$, we can attach an orthonormal frame at p with $e_1 = (\nabla f / |\nabla f|)$, and then we have

$$W_{1ijk}(p) = 0, \quad \text{for } 1 \leq i, j, k \leq 4. \tag{4.10}$$

Thus it remains to show that

$$W_{abcd}(p) = 0$$

for all $2 \leq a, b, c, d \leq 4$. However, this essentially reduces to showing that the Weyl tensor is zero in 3 dimensions (see [17, pp. 276–277])—observing that the Weyl tensor W_{ijkl} has all the symmetry of the R_{ijkl} and is trace-free in any two indices. Thus,

$$W_{2121} + W_{2222} + W_{2323} + W_{2424} = 0,$$

and so, by (4.10),

$$W_{2323} = -W_{2424}.$$

Similarly, we have

$$W_{2424} = -W_{3434} = W_{2323},$$

which implies that $W_{2323} = 0$. On the other hand,

$$W_{1314} + W_{2324} + W_{3334} + W_{4344} = 0,$$

so $W_{2324} = 0$. This shows that $W_{abcd} = 0$ unless a, b, c, d are all distinct. But there are only three choices for the indices a, b, c, d since they range from 2 to 4. ☐

Now we are ready to finish the proof of our main theorems.

Conclusion of the proof of Theorem 1.1

Let (M^4, g_{ij}, f) be a complete Bach-flat gradient shrinking Ricci soliton. Then, by Lemma 4.1, we know that $D_{ijk} = 0$. We divide the arguments into two cases.

Case 1: the set $\Omega = \{p \in M \mid \nabla f(p) \neq 0\}$ *is dense.* By Lemma 4.1 and Lemma 4.3, we know that $W_{ijkl} = 0$ on Ω. By continuity, we know that $W_{ijkl} = 0$ on M^4. Therefore, we conclude that (M^4, g_{ij}, f) is locally conformally flat. Furthermore, according to the classification result for locally conformally flat gradient shrinking Ricci solitons mentioned in the introduction, (M^4, g_{ij}, f) is a finite quotient of either $\mathbb{R}^4$ or $\mathbb{S}^3 \times \mathbb{R}$.

Case 2: $|\nabla f|^2 = 0$ *on some nonempty open set.* In this case, since any gradient shrinking Ricci soliton is analytic in harmonic coordinates, it follows that $|\nabla f|^2 = 0$ on M; that is, (M^4, g_{ij}) is Einstein.

This completes the proof of Theorem 1.1. □

Conclusion of the proof of Theorem 1.2

Let (M^n, g_{ij}, f), $n \geq 5$, be a Bach-flat gradient shrinking Ricci soliton. Then, by Lemma 4.1, Lemma 4.2, and the same argument as in the proof of Theorem 1.1 above, we know that (M^n, g_{ij}, f) either is Einstein or has harmonic Weyl tensor. In the latter case, by the rigidity theorem of Fernández-López and García-Río [16] and of Munteanu and Sesum [19] for harmonic Weyl tensor, (M^n, g_{ij}, f) is either Einstein or isometric to a finite quotient of $N^{n-k} \times \mathbb{R}^k$ $(k > 0)$, the product of an Einstein manifold N^{n-k} with the Gaussian shrinking soliton $\mathbb{R}^k$. However, Proposition 3.2(e) says that the Ricci tensor either has one unique eigenvalue or two distinct eigenvalues with multiplicity of 1 and $n-1$, respectively. Therefore, only $k = 1$ and $k = n$ can occur in $N^{n-k} \times \mathbb{R}^k$. □

5. Gradient Ricci solitons with vanishing D_{ijk}

First of all, we notice that the proofs of Lemma 4.2 and Lemma 4.3 are valid for gradient steady and expanding Ricci solitons. Hence we have the following general result.

THEOREM 5.1

Let (M^n, g_{ij}, f), $n \geq 4$, *be a complete nontrivial gradient Ricci soliton satisfying (3.1) and with* $D_{ijk} = 0$. *Then*

(i) *the Weyl tensor* $W_{ijkl} = 0$ *for* $n = 4$ *(i.e.,* (M^n, g_{ij}, f) *is locally conformally flat);*

(ii) *the Cotton tensor* $C_{ijk} = 0$ *for* $n \geq 5$ *(i.e.,* (M^n, g_{ij}, f) *has harmonic Weyl tensor).*

As an immediate consequence of Theorem 5.1, of the classification theorem for locally conformally flat gradient shrinking solitons and the rigidity theorem for gradient shrinking solitons with harmonic Weyl tensor mentioned in the introduction, and of Proposition 3.2(e), we have the following rigidity theorem for gradient shrinking Ricci solitons with vanishing D_{ijk}.

COROLLARY 5.1
Let (M^n, g_{ij}, f)*,* $n \geq 4$*, be a complete gradient shrinking Ricci soliton with* $D_{ijk} = 0$*. Then*

(i) (M^4, g_{ij}, f) *is either Einstein, or a finite quotient of* $\mathbb{R}^4$ *or* $\mathbb{S}^3 \times \mathbb{R}$*;*

(ii) *for* $n \geq 5$*,* (M^n, g_{ij}, f) *is either Einstein, or is a finite quotient of the Gaussian shrinking soliton* $\mathbb{R}^n$*, or is a finite quotient of* $N^{n-1} \times \mathbb{R}$*, where* N^{n-1} *is Einstein.*

Moreover, combining Theorem 5.1(i) and the 4-dimension classification theorem for locally conformally flat gradient steady Ricci solitons (see [8], [11]), we have the following.

COROLLARY 5.2
Let (M^4, g_{ij}, f) *be a complete gradient steady Ricci soliton with* $D_{ijk} = 0$*. Then* (M^4, g_{ij}, f) *is either Ricci flat or isometric to the Bryant soliton.*

Finally, let us further examine the relations among D_{ijk}, C_{ijk}, W_{ijkl}, and B_{ij}. Note that Theorem 5.1(ii) tells us that for any nontrivial gradient Ricci soliton, $D_{ijk} = 0$ implies that $C_{ijk} = 0$. On the other hand, the converse is not true because the product space $\mathbb{S}^k \times \mathbb{R}^{n-k}$ has $C_{ijk} = 0$ but not $D_{ijk} = 0$ by Proposition 3.2(e) for $k \geq 2$ and $n - k \geq 2$. So one naturally would wonder how much stronger is the condition $D_{ijk} = 0$ than $C_{ijk} = 0$? It turns out that we have several equivalent characterizations of $D_{ijk} = 0$.

THEOREM 5.2
Let (M^n, g_{ij}, f)*,* $n \geq 5$*, be a nontrivial gradient Ricci soliton satisfying (3.1). Then the following statements are equivalent:*

(a) $D_{ijk} = 0$*;*

(b) $C_{ijk} = 0$*, and* $W_{1ijk} = 0$ *for* $1 \leq i, j, k \leq n$*;*

(c) $\operatorname{div} B \cdot \nabla f = 0$ *and* $W_{1a1b} = 0$ *for* $2 \leq a, b \leq n$*.*

Proof
For cases (a) → (b), this follows from Theorem 5.1 and Lemma 3.1.

For cases (b) → (c), we see clearly that it suffices to show that $C_{ijk} = 0$ implies that $\operatorname{div} B \cdot \nabla f = 0$. In fact, $C_{ijk} = 0$ implies that $\operatorname{div} B = 0$ for $n \geq 5$. This follows from the following formula, which is well known at least for $n = 4$ among experts in conformal geometry and general relativity.

LEMMA 5.1
For $n \geq 4$, we have

$$\operatorname{div} B \equiv \nabla_j B_{ij} = \frac{n-4}{(n-2)^2} C_{ijk} R_{jk}.$$

Proof
Recall that we have

$$C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik}$$

and

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}A_{jl} - g_{il}A_{jk} - g_{jk}A_{il} + g_{jl}A_{ik}). \tag{5.1}$$

By using the expression of the Bach tensor in (2.3), we have

$$(n-2)\nabla_i B_{ij} = \nabla_i \nabla_k (\nabla_k A_{ij} - \nabla_i A_{kj}) + \nabla_k R_{kl} W_{ikjl} + R_{kl} \nabla_k W_{ikjl}.$$

But,

$$\begin{aligned} \nabla_i \nabla_k (\nabla_k A_{ij} - \nabla_i A_{kj}) &= (\nabla_i \nabla_k - \nabla_k \nabla_i) \nabla_k A_{ij} \\ &= -R_{il} \nabla_l A_{ij} + R_{kl} \nabla_k A_{lj} + R_{ikjl} \nabla_k A_{il} \\ &= R_{ikjl} \nabla_k A_{il}. \end{aligned}$$

Thus, by using (5.1),

$$\begin{aligned} \nabla_i \nabla_k (\nabla_k A_{ij} - \nabla_i A_{kj}) + \nabla_k R_{kl} W_{ikjl} &= (R_{ikjl} - W_{ikjl}) \nabla_k A_{il} \\ &= \frac{1}{n-2}(A_{jk} g_{il} C_{lki} + A_{ik} C_{kji}) \\ &= -\frac{1}{n-2} R_{ki} C_{jki}. \end{aligned}$$

Moreover, by (2.2), we know that

$$\nabla_k W_{ikjl} = \frac{n-3}{n-2} C_{jlk}.$$

Summing up, we obtain

$$(n-2)\nabla_i B_{ij} = \frac{n-4}{n-2} R_{kl} C_{jkl}.$$

□

For (c) → (a), by Lemma 5.1, Lemma 3.1, and (3.3), we have

$$\begin{aligned}\operatorname{div} B \cdot \nabla f &= \frac{n-4}{(n-2)^2} C_{ijk} R_{jk} \nabla_i f \\ &= \frac{n-4}{(n-2)^2} (D_{ijk} - W_{ijkl} \nabla_l f) R_{jk} \nabla_i f \\ &= \frac{n-4}{2(n-2)} |D_{ijk}|^2 + \frac{n-4}{(n-2)^2} W_{1a1b} R_{ab} |\nabla f|^2.\end{aligned}$$

Thus, $\operatorname{div} B \cdot \nabla f = 0$ and $W_{1a1a} = 0$ for $2 \le a \le n$ imply that $D_{ijk} = 0$ for all $1 \le i, j, k \le n$.

This completes the proof of Theorem 5.2. □

Acknowledgments. We are very grateful to S.-T. Yau for suggesting, in the summer of 2010, that we consider 4-dimensional self-dual gradient shrinking Ricci solitons, which in part inspired us to study Bach-flat gradient shrinking Ricci solitons. We would also like to thank Richard Hamilton for his interest in our work, and we thank the referees for very helpful comments and suggestions that helped improve the exposition of this article.

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