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ON LOCALLY CONFORMALLY FLAT GRADIENT STEADY RICCI SOLITONS

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ABSTRACT. In this paper, we classify n-dimensional ($n \geq 3$) complete noncompact locally conformally flat gradient steady solitons. In particular, we prove that a complete noncompact nonflat locally conformally flat gradient steady Ricci soliton is, up to scaling, the Bryant soliton.

1. The results

A complete Riemannian metric g_{ij} on a smooth manifold M^n is called a *gradient steady Ricci soliton* if there exists a smooth function F on M^n such that the Ricci tensor R_{ij} of the metric g_{ij} is given by the Hessian of F:

$$R_{ij} = \nabla_i \nabla_j F. \tag{1.1}$$

The function F is called a *potential function* of the gradient steady soliton. Clearly, when F is a constant the gradient steady Ricci soliton is simply a Ricci flat manifold. Thus Ricci solitons are natural extensions of Einstein metrics. Gradient steady solitons play an important role in Hamilton's Ricci flow as they correspond to translating solutions, and often arise as Type II singularity models. Thus one is interested in classifying them and understanding their geometry.

It turns out that compact gradient steady solitons must be Ricci flat. In dimension $n = 2$, Hamilton [14] discovered the first example of a complete noncompact gradient steady soliton on $\mathbb{R}^2$, called the *cigar soliton*, where the metric is given by

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

The cigar soliton has positive curvature and is asymptotic to a cylinder of finite circumference at infinity. Furthermore, Hamilton [14] showed that *the only complete steady soliton on a two-dimensional manifold with bounded (scalar) curvature R which assumes its maximum at an origin is, up to scaling, the cigar soliton.* For $n \geq 3$, Bryant [2] proved that *there exists, up to scaling, a unique complete rotationally symmetric gradient Ricci soliton on* $\mathbb{R}^n$, see, e.g., Chow et al. [8] for details. The Bryant soliton has positive sectional curvature, linear curvature decay and volume growth on the order of $r^{(n+1)/2}$. In the Kähler case, the first author [3] constructed a complete gradient steady Kähler-Ricci soliton on $\mathbb{C}^m$, for $m \geq 2$, with positive sectional curvature and $U(m)$ symmetry.

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A well-known conjecture is that, in dimension $n = 3$, the Bryant soliton is the only complete noncompact (κ-noncollapsed) gradient steady soliton with positive sectional curvature.[1] We remark that S.-C. Chu [9] and H. Guo [12] have studied the geometry of 3-dimensional gradient steady solitons with positive sectional curvature and the scalar curvature R attaining its maximum at some origin. For $n \geq 4$, it is also natural to ask if the Bryant soliton is the only complete noncompact, positively curved, locally conformally flat gradient steady soliton. In this paper, we classify n-dimensional ($n \geq 3$) complete noncompact locally conformally flat gradient steady solitons and give an affirmative answer to the latter question. Our main results:

Theorem 1.1. *Let (M^n, g_{ij}, F), $n \geq 3$, be an n-dimensional complete noncompact locally conformally flat gradient steady Ricci soliton with positive sectional curvature. Then, (M^n, g_{ij}, F) is isometric to the Bryant soliton.*

Theorem 1.2. *Let (M^n, g_{ij}, F), $n \geq 3$, be an n-dimensional complete noncompact locally conformally flat gradient steady Ricci soliton. Then, (M^n, g_{ij}, F) is either flat or isometric to the Bryant soliton.*

Our work was motivated in part by the works of physicists Israel [16] and Robinson [21] concerning the uniqueness of the Schwarzschild black hole among all static, asymptotically flat vacuum space-times. In their setting, the Einstein field equations take the form

$$R_{ij} + V^{-1}\nabla_i\nabla_j V = 0 \tag{1.2}$$

and

$$\Delta V = 0$$

for a certain positive potential function V on a three-dimensional space-like hypersurface (N^3, g_{ij}). They proved that such (N^3, g_{ij}, V) must be rotationally symmetric without assuming locally conformal flatness. In fact, they were able to prove that such (N^3, g_{ij}, V) is locally conformally flat. However, it remains a challenge to do the same for 3-dimensional gradient steady Ricci solitons.

2. Preliminaries

In this section, we recall some basic facts and collect several known results about gradient steady solitons.

Let (M^n, g_{ij}, F) be a gradient steady Ricci soliton so that the Ricci tensor R_{ij} of the metric g_{ij} is given by the Hessian of the potential function F:

$$R_{ij} = \nabla_i\nabla_j F.$$

Taking the covariant derivatives and using the commutating formula for covariant derivatives, we obtain

$$\nabla_i R_{jk} - \nabla_j R_{ik} = R_{ijkl}\nabla_l F. \tag{2.1}$$

Taking the trace on j and k, and using the contracted second Bianchi identity

$$\nabla_j R_{ij} = \frac{1}{2}\nabla_i R,$$

we get

$$\nabla_i R = -2R_{ij}\nabla_j F.$$

[1] Perelman ([20], **11.9**) claimed that the conjecture is true but didn't give any detail, or sketch, of a proof.

Thus

$$\nabla_i(R+|\nabla F|^2) = -2(R_{ij} - \nabla_i\nabla_j F)\nabla F = 0.$$

Hence we have

Lemma 2.1 (Hamilton [15]). *Let (M^n, g_{ij}, F) be a complete gradient steady soliton satisfying* (1.1). *Then we have*

$$\nabla_i R = -2R_{ij}\nabla_j F \tag{2.2}$$

and

$$R + |\nabla F|^2 = C_0 \tag{2.3}$$

for some constant C_0. Here R denotes the scalar curvature.

Taking the trace in (1.1), we get

$$R = \Delta F. \tag{2.4}$$

Combining (2.4) with (2.3), we have

$$\Delta F + |\nabla F|^2 = C_0. \tag{2.5}$$

If M is compact, then it follows that

$$0 = \int_M \Delta(e^F) = \int_M (\Delta F + |\nabla F|^2)e^F = C_0 \int_M e^F.$$

Thus $C_0 = 0$. Now integrating (2.5), with $C_0 = 0$, over M yields

$$\int_M |\nabla F|^2 = 0,$$

implying that F is a constant. Therefore, we get

Proposition 2.1 (cf. Hamilton [15], Ivey [17]). *On a compact manifold M^n, a gradient steady Ricci soliton is necessarily a Ricci flat Einstein metric.*

Complete noncompact steady solitons do exist and here are some basic examples:

Example 2.1 (**The cigar soliton**). In dimension two, Hamilton [14] discovered the first example of a complete noncompact steady soliton on $\mathbb{R}^2$, called the *cigar soliton*, where the metric is given by

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

with potential function

$$F = \log(1 + x^2 + y^2).$$

The (scalar) curvature of the cigar soliton is given by

$$R = \frac{1}{1 + x^2 + y^2}.$$

Hence it is positive, attains its maximum at the origin, and decays to zero exponentially fast (in terms of the geodesic distance) at space infinity. Furthermore, the cigar soliton has linear volume growth and is asymptotic to a cylinder of finite circumference at ∞.

Example 2.2 (**The Bryant soliton).** In the Riemannian case, higher-dimensional examples of noncompact gradient steady solitons were found by Robert Bryant [2] on $\mathbb{R}^n$ ($n \geq 3$). They are rotationally symmetric and have positive sectional curvature. Furthermore, the geodesic sphere S^{n-1} of radius s has a diameter of the order $\sqrt{s}$. Thus the volume of the geodesic balls $B_r(0)$ grows according to the order of $r^{(n+1)/2}$.

Example 2.3 (**Noncompact gradient steady Kähler solitons).** In the Kähler case, the first author [3] found two examples of complete rotationally symmetric noncompact gradient steady Kähler-Ricci solitons

(a) On $\mathbb{C}^n$ (for $n = 1$ it is just the cigar soliton). These examples are $U(n)$ invariant and have positive sectional curvature. It is interesting to point out that the geodesic sphere S^{2n-1} of radius s is an S^1-bundle over $\mathbb{C}P^{n-1}$ where the diameter of S^1 is of the order 1, while the diameter of $\mathbb{C}P^{n-1}$ is of the order $\sqrt{s}$. Thus the volume of the geodesic balls $B_r(0)$ grows according to the order of r^n, n being the complex dimension. Also, the curvature $R(x)$ decays like $1/r$.

(b) On the blowup of $\mathbb{C}^n/\mathbb{Z}_n$ at the origin. This is the same space on which Eguchi-Hansen ($n = 2$) and Calabi ($n \geq 2$) constructed examples of hyper-Kähler metrics. For $n = 2$, the underlying space is the canonical line bundle over $\mathbb{CP}^1$.

Now a Ricci flat metric is clearly a stationary solution of Hamilton's Ricci flow

$$\frac{\partial g_{ij}(t)}{\partial t} = -2R_{ij}(t).$$

This happens, for example, on a flat torus or on any $K3$-surface with a Calabi-Yau metric.

On the other hand, suppose that we have a complete steady Ricci soliton g_{ij} on a smooth manifold M^n with potential function F. As observed by Z.-H. Zhang [22], the gradient vector field $V = \nabla F$ is a complete vector field on M. Let φ_t denote the one-parameter group of diffeomorphisms of M^n generated by $-V$. Then it easily follows that

$$\tilde{g}_{ij}(t) = \varphi_t^* g_{ij}$$

is a solution to the Ricci flow for $-\infty < t < \infty$, with $\tilde{g}_{ij}(0) = g_{ij}$.

Next we present a useful result, which was implicitly proved by B.-L. Chen [7]. For the reader's convenience, we include a proof here (see also Proposition 5.5 in [4]).

Proposition 2.2. *Let $g_{ij}(t)$ be a complete ancient solution to the Ricci flow on a noncompact manifold M^n. Then the scalar curvature R of $g_{ij}(t)$ is nonnegative for all t.*

Proof. Suppose $g_{ij}(t)$ is defined for $-\infty < t \leq T$ for some $T > 0$. We divide the argument into two steps:

Step 1. Consider any complete solution $g_{ij}(t)$ defined on $[0, T]$. For any fixed point $x_0 \in M$, pick $r_0 > 0$ sufficiently small so that

$$|Rc|(\cdot, t) \leq (n-1)r_0^{-2} \qquad \text{on } B_t(x_0, r_0)$$

for all $t \in [0, T]$. Then for any positive number $A > 2$, pick $K_A > 0$ such that $R \geq -K_A$ on $B_0(x_0, Ar_0)$ at $t = 0$. We claim that there exists a universal constant

$C > 0$ (depending on the dimension n) such that

$$R(\cdot,t) \geq \min\{-\frac{n}{t+\frac{1}{K_A}}, -\frac{C}{Ar_0^2}\} \qquad \text{on } B_t(x_0, \frac{3A}{4}r_0) \tag{2.6}$$

for each $t \in [0,T]$.

Indeed, take a smooth nonnegative decreasing function ϕ on $\mathbb{R}$ such that $\phi = 1$ on $(-\infty, 7/8]$, and $\phi = 0$ on $[1,\infty)$. Consider the function

$$u(x,t) = \phi(\frac{d_t(x_0,x)}{Ar_0})R(x,t),$$

where $d_t(x_0,x)$ is the distance function from x_0 to x at time t. Then we have

$$(\frac{\partial}{\partial t} - \Delta)u = \frac{\phi' R}{Ar_0}(\frac{\partial}{\partial t} - \Delta)d_t(x_0,x) - \frac{\phi'' R}{(Ar_0)^2} + 2\phi|Rc|^2 - 2\nabla\phi\cdot\nabla R$$

at smooth points of $d_t(x_0,\cdot)$.

Let $u_{\min}(t) = \min_M u(\cdot,t)$. Whenever $u_{\min}(t_0) \leq 0$, assume $u_{\min}(t_0)$ is achieved at some point $\bar{x} \in B_{t_0}(x_0,r_0)$; then $\phi' R(\bar{x},t_0) \geq 0$. On the other hand, by Lemma 8.3(a) of Perelman [20] (see also Lemma 3.4.1 (i) of [5]), we know that

$$(\frac{\partial}{\partial t} - \Delta)d_t(x_0,x) \geq -\frac{5(n-1)}{3r_0}$$

outside $B_t(x_0,r_0)$. Following (Section 3, Hamilton [13]), we define

$$\left.\frac{d}{dt}\right|_{t=t_0} u_{\min} = \liminf_{h\to 0^+} \frac{u_{\min}(t_0+h) - u_{\min}(t_0)}{h},$$

the lim inf of all forward difference quotients. Then

$$\left.\frac{d}{dt}\right|_{t=t_0} u_{\min} \geq -\frac{5(n-1)}{3Ar_0^2}\phi' R + \frac{2}{n}\phi R^2 + \frac{1}{(Ar_0)^2}(\frac{2\phi'^2}{\phi} - \phi'')R.$$

Hence,

$$\left.\frac{d}{dt}\right|_{t=t_0} u_{\min} \geq \frac{1}{n}u_{\min}^2(t_0) - \frac{C^2}{(Ar_0^2)^2}$$

provided $u_{\min}(t_0) \leq 0$. Now integrating the above inequality, we get

$$u_{\min}(t) \geq \min\{-\frac{n}{t+\frac{1}{K_A}}, -\frac{C}{Ar_0^2}\} \qquad \text{on } B_t(x_0, \frac{3A}{4}r_0),$$

and the inequality (2.6) in our claim follows.

Step 2. Now if our solution $g_{ij}(t)$ is ancient, we can replace t by $t-\alpha$ in (2.6) and get

$$R(\cdot,t) \geq \min\{-\frac{n}{t-\alpha+\frac{1}{K_A}}, -\frac{C}{Ar_0^2}\} \qquad \text{on } B_t(x_0, \frac{3A}{4}r_0).$$

Letting $A \to \infty$ and then $\alpha \to -\infty$, we see that $R(\cdot,t) \geq 0$ for all t. This completes the proof of Proposition 2.2. □

As an immediate corollary, we have

Lemma 2.2. *Let (M^n, g_{ij}, F) be a complete gradient steady soliton. Then it has nonnegative scalar curvature $R \geq 0$.*

We remark that Munteanu-Sesum [19] recently proved a Liouville-type theorem for gradient steady solitons; namely, a gradient steady soliton does not admit any nontrivial harmonic function with finite Dirichlet energy. As a consequence, a gradient steady soliton has at most one nonparabolic end.

For steady solitons with nonnegative Ricci curvature, we have the following.

Lemma 2.3 (Hamilton [15])**.** *Let (M^n, g_{ij}, F) be a complete gradient steady soliton with nonnegative Ricci curvature $Rc \geq 0$ and assume the scalar curvature R attains its maximum at some point x_0. Then the potential function F is weakly convex and attains its minimum at x_0.*

Moreover, if the Ricci curvature of (M^n, g_{ij}, F) is assumed to be positive, then Lemma 2.3 can be strengthened to the following.[2]

Proposition 2.3. *Let (M^n, g_{ij}, F) be a complete noncompact gradient steady soliton with positive Ricci curvature $Rc > 0$. Assume the scalar curvature R attains its maximum at some origin x_0. Then, there exist some constants $0 < c_1 \leq \sqrt{C_0}$ and $c_2 > 0$ such that the potential function F satisfies the estimates*

$$c_1 r(x) - c_2 \leq F(x) \leq \sqrt{C_0}\, r(x) + |F(x_0)|, \tag{2.7}$$

where $r(x) = d(x_0, x)$ is the distance function from x_0, and $C_0 = R_{max}$ is the constant in (2.2)*. In particular, F is a strictly convex exhaustion function achieving its minimum at the only critical point x_0, and the underlying manifold M^n is diffeomorphic to $\mathbb{R}^n$.*

Proof. It is clear that the upper estimate in (2.7) in fact holds for complete gradient steady solitons in general, because $|\nabla F|^2 \leq C_0$ by (2.3) and Lemma 2.2.

To prove the lower estimate, we consider any minimizing normal geodesic $\gamma(s)$, $0 \leq s \leq s_0$ for large $s_0 > 0$, starting from the origin $x_0 = \gamma(0)$. Denote by $X(s) = \dot{\gamma}(s)$ the unit tangent vector along γ, and $\dot{F} = \nabla_X F(\gamma(s))$. By (1.1), we have

$$\nabla_X \dot{F} = \nabla_X \nabla_X F = Rc(X, X). \tag{2.8}$$

Integrating (2.8) along γ and noting that x_0 is the (unique) minimum point of F, we get, for $s \geq 1$,

$$\dot{F}(\gamma(s)) = \int_0^s Rc(X, X) ds \geq \int_0^1 Rc(X, X) ds \geq c_1,$$

where $c_1 > 0$ is the least eigenvalue of Rc on the unit geodesic ball $B_{x_0}(1)$. Thus,

$$F(\gamma(s_0)) = \int_1^{s_0} \dot{F}(\gamma(s)) ds + F(\gamma(1)) \geq c_1 s_0 - c_1 + F(\gamma(1)). \qquad \square$$

Now we turn our attention to locally conformally flat steady Ricci solitons. For any Riemannian manifold (M^n, g), let

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk})$$

[2]See also Remark 5.5 in [4], as well as Lemma 3.2 in [9] .

be the Weyl tensor, and denote by

$$C_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{ij}\nabla_k R - g_{ik}\nabla_j R) \tag{2.9}$$

the Cotton tensor. It is well known that, for $n = 3$, W_{ijkl} vanishes identically while $C_{ijk} = 0$ if and only if (M^3, g_{ij}) is locally conformally flat; for $n \geq 4$, $W_{ijkl} = 0$ if and only if (M^n, g_{ij}) is locally conformally flat. Moreover, for $n \geq 4$, the vanishing of the Weyl tensor W_{ijkl} implies the vanishing of the Cotton tensor C_{ijk}, while $C_{ijk} = 0$ corresponds to the Weyl tensor being harmonic.

In proving Theorem 1.2, we need the following important facts due to B.-L. Chen [7] (for $n = 3$) and Z.-H. Zhang [23] (for $n \geq 4$):

Proposition 2.4. *Let (M^n, g_{ij}, F) be a complete gradient steady Ricci soliton. Then (M^n, g_{ij}, F) has nonnegative curvature operator $Rm \geq 0$, provided either*
(a) *$n = 3$, or*
(b) *$n \geq 4$ and (M^n, g_{ij}, F) is locally conformally flat.*

Remark 2.1. Part (a) is a special case of a more general result due to B.-L. Chen [7] for 3-dimensional complete ancient solutions. Part (b) was essentially proved by Z.-H. Zhang [23]. In fact, the same arguments in [23] imply that a complete ancient solution $g_{ij}(t)$ to the Ricci flow with vanishing Weyl tensor for each time t is necessarily of nonnegative curvature operator.

Combining Proposition 2.4 with (2.3) in Lemma 2.1, we have

Proposition 2.5. *Let (M^n, g_{ij}, F) be a complete gradient steady soliton such that either $n = 3$, or $n \geq 4$ and (M^n, g_{ij}, F) is locally conformally flat. Then (M^n, g_{ij}, F) has bounded and nonnegative curvature operator $0 \leq Rm \leq C$.*

3. The proofs of Theorem 1.1 and Theorem 1.2

Throughout this section, we assume that (M^n, g_{ij}, F) $(n \geq 3)$ is a complete noncompact locally conformally flat gradient steady soliton. We are going to prove Theorem 1.1, which is the same as Proposition 3.1 below, and Theorem 1.2.

Proposition 3.1. *If (M^n, g_{ij}, F), $n \geq 3$, is an n-dimensional complete noncompact locally conformally flat gradient steady soliton with positive sectional curvature, then (M^n, g_{ij}, F) is a rotationally symmetric gradient steady soliton on $\mathbb{R}^n$, hence isometric to the Bryant soliton.*

Proof. First of all, since the sectional curvature of (M^n, g_{ij}, F) is positive we know that M^n is diffeomorphic to $\mathbb{R}^n$ by Gromoll-Meyer [11] (or by Proposition 2.3 if R attains its maximum at some origin x_0). Moreover, since the Ricci curvature is positive, the potential function F is strictly convex, thus having at most one critical point. Secondly, if we denote by $G = |\nabla F|^2$, then in any neighborhood, where $G \neq 0$, of the equipotential hypersurface

$$\Sigma_c =: \{x \in M : F(x) = c\}$$

of a regular value c of F, we can express the metric $ds^2 = g_{ij}(x)dx^i dx^j$ as

$$ds^2 = \frac{1}{G(F,\theta)}dF^2 + g_{ab}(F,\theta)d\theta^a d\theta^b, \tag{3.1}$$

where $\theta = (\theta^2, \cdots, \theta^n)$ denotes intrinsic coordinates for Σ_c. It is clear that the key step in proving Proposition 3.1 is to show that in (3.1) we have $G = G(F)$, $g_{ab} = g_{ab}(F)$, and that (Σ_c, g_{ab}) is a space form of positive curvature.

Note that an n-dimensional rotationally symmetric metric is of the form

$$ds^2 = \psi^2(t)dt^2 + \varphi^2(t)\bar{g},$$

where $\bar{g}$ is the standard metric on the unit sphere $\mathbb{S}^{n-1}$, and the Ricci tensor of such a metric is given by

$$Rc = (n-1)(-\frac{\varphi_{tt}}{\varphi} + \frac{\varphi_t\psi_t}{\varphi\psi})dt^2 - (\frac{\varphi\varphi_{tt}}{\psi^2} + \frac{(n-2)\varphi_t^2}{\psi^2} - \frac{\varphi\varphi_t\psi_t}{\psi^3} + 2 - n)\bar{g}.$$

In particular, it has at most two distinct eigenvalues depending only on t.

We shall first derive a useful formula for the norm square of the Cotton tensor for steady solitons with vanishing Weyl tensor. This formula plays an important role in our derivation of the desired property of the Ricci tensor for the steady soliton metric. Moreover, as $W_{ijkl} = 0$ always when $n = 3$, this formula is of particular interest in the three-dimensional case.

Lemma 3.1. *For any n-dimensional gradient steady soliton ($n \geq 3$) with vanishing Weyl tensor $W_{ijkl} = 0$, we have*

$$|C_{ijk}|^2 = \frac{1}{(n-2)^2}(|R_{ik}\nabla_j F - R_{ij}\nabla_k F|^2 - \frac{2}{(n-1)}|R\nabla F - \frac{1}{2}\nabla G|^2). \tag{3.2}$$

Here C_{ijk} is the Cotton tensor defined by (2.9).

Proof. Notice that, by (1.1) and Lemma 2.1, we have

$$\nabla G = 2Rc(\nabla F, \cdot) = -\nabla R. \tag{3.3}$$

Using (2.1), (3.3), and the assumption $W_{ijkl} = 0$, we can express

$$\begin{aligned} C_{ijk} &= \nabla_k R_{ij} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{ij}\nabla_k R - g_{ik}\nabla_j R) \\ &= R_{kjil}\nabla_l F + \frac{1}{2(n-1)}(g_{ij}\nabla_k G - g_{ik}\nabla_j G) \\ &= \frac{1}{n-2}(R_{ik}\nabla_j F - R_{ij}\nabla_k F) + \frac{1}{2(n-1)(n-2)}(g_{ik}\nabla_j G - g_{ij}\nabla_k G) \\ &\quad - \frac{R}{(n-1)(n-2)}(g_{ik}\nabla_j F - g_{ij}\nabla_k F). \end{aligned}$$

Hence, by direct computations, we have

$$\begin{aligned} |C_{ijk}|^2 =& \frac{1}{(n-2)^2}|R_{ik}\nabla_j F - R_{ij}\nabla_k F|^2 + \frac{1}{2(n-1)(n-2)^2}|\nabla G|^2 \\ &+ \frac{2R^2}{(n-1)(n-2)^2}|\nabla F|^2 + \frac{2}{(n-1)(n-2)^2}[R\nabla G\cdot\nabla F - Rc(\nabla F, \nabla G)] \\ &- \frac{4R}{(n-1)(n-2)^2}[R|\nabla F|^2 - Rc(\nabla F, \nabla F)] - \frac{2R}{(n-1)(n-2)^2}\nabla G\cdot\nabla F. \end{aligned}$$

On the other hand, by (3.3),

$$Rc(\nabla F, \nabla G) = \frac{1}{2}|\nabla G|^2$$

and

$$Rc(\nabla F, \nabla F) = \frac{1}{2}\nabla G \cdot \nabla F.$$

Thus,

$$\begin{aligned}|C_{ijk}|^2 &= \frac{1}{(n-2)^2}|R_{ik}\nabla_j F - R_{ij}\nabla_k F|^2 \\ &\quad - \frac{1}{2(n-1)(n-2)^2}(|\nabla G|^2 - 4R\nabla G \cdot \nabla F + 4R^2|\nabla F|^2) \\ &= \frac{1}{(n-2)^2}(|R_{ik}\nabla_j F - R_{ij}\nabla_k F|^2 - \frac{2}{(n-1)}|R\nabla F - \frac{1}{2}\nabla G|^2). \qquad \square\end{aligned}$$

Next, using Lemma 3.1, we show that the Ricci tensor of our (M^n, g_{ij}, F) has, at least pointwisely, the desired property that it has at most two distinct eigenvalues.

Lemma 3.2. *At any point $p \in \Sigma_c$, the Ricci tensor of (M^n, g_{ij}, F) either has a unique eigenvalue λ, or has two distinct eigenvalues λ and μ of multiplicity 1 and $n-1$ respectively. In either case, $e_1 = \nabla F/|\nabla F|$ is an eigenvector with eigenvalue λ. In other words, for any orthonormal basis $e_2, \cdots, e_n$ tangent to the level surface Σ_c at p, the Ricci tensor has the following properties:*

(i) $Rc(e_1, e_1) = R_{11}$,
(ii) $Rc(e_1, e_b) = R_{1b} = 0,\ b = 2, \cdots, n$,
(iii) $Rc(e_a, e_b) = R_{aa}\delta_{ab},\ a, b = 2, \cdots, n$,

where either $R_{11} = \cdots = R_{nn} = \lambda$ or $R_{11} = \lambda$ and $R_{22} = \cdots = R_{nn} = \mu$.

Proof. For any regular value c of F, pick an orthonormal basis $\{E_1, \cdots, E_n\}$ of the tangent space T_pM at $p \in \Sigma_c = \{x \in M : F(x) = c\}$ so that $Rc(E_i, E_j) = \lambda_i\delta_{ij}$. Then, we have

$$|R_{ik}\nabla_j F - R_{ij}\nabla_k F|^2 = 2\sum_{j=1}^{n}|\nabla_j F|^2 \sum_{i\neq j}\lambda_i^2$$

and

$$|R\nabla F - \frac{1}{2}\nabla G|^2 = \sum_{j=1}^{n}|\nabla_j F|^2(\sum_{i\neq j}\lambda_i)^2.$$

Plugging the above two identities into (3.2) and noticing that $C_{ijk} = 0$ by assumption, it follows that

$$\sum_{j=1}^{n}|\nabla_j F|^2(\sum_{i\neq j, k\neq j}(\lambda_i - \lambda_k)^2) = 0. \tag{3.4}$$

Since c is a regular value of F and $p \in \Sigma_c$, $\nabla F(p) \neq 0$. On the other hand, from (3.4) it is easy to see that if $\nabla F(p)$ has two or more nonzero components with respect to $\{E_i\}_{i=1}^n$, then $\lambda_1 = \cdots = \lambda_n$, so the Ricci tensor has a unique eigenvalue. Otherwise, taking $\nabla_1 F \neq 0$ and $\nabla_i F = 0$ for $i = 2, \cdots, n$, then $\nabla F = |\nabla F|E_1$, with $E_1 = \nabla F/|\nabla F|$, is an eigenvector of Rc, and $\lambda_2 = \cdots = \lambda_n$. In either case, we conclude that ∇F is an eigenvector, and the Ricci tensor has the desired properties. $\square$

Remark 3.1. Fernández-López and García-Río [10] showed, by a different argument, that for shrinking solitons with harmonic Weyl tensor, the gradient of any potential function is an eigenvector of the Ricci tensor.

Finally we show that the eigenvalues of the Ricci tensor and the scalar curvature R are constant on level surfaces $\Sigma_c = \{F = c\}$.

Lemma 3.3. *Let c be a regular value of F and $\Sigma_c = \{F = c\}$. Then:*
(a) *The function $G = |\nabla F|^2$ is constant on Σ_c; i.e. G is a function of F only.*
(b) *The scalar curvature R of (M^n, g_{ij}, F) is constant on Σ_c.*
(c) *The second fundamental form h_{ab} of Σ_c is of the form $h_{ab} = \frac{H}{n-1} g_{ab}$.*
(d) *The mean curvature H is constant on Σ_c.*
(e) *Σ_c, with the induced metric g_{ab}, is of constant sectional curvature.*

Proof. Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal frame with $e_1 = \nabla F/|\nabla F|$ and $e_2, \cdots, e_n$ tangent to Σ_c.

First of all, by (3.3) and Lemma 3.1 (ii),

$$\nabla_a G = 2Rc(\nabla F, e_a) = 0, \ a = 2, \cdots, n.$$

Hence G is constant on Σ_c, and so is R by (2.3). This proves (a) and (b).

Next, since $R_{ab} = R_{aa} g_{ab}$ and $R_{22} = \cdots = R_{nn}$ by Lemma 3.1, we have

$$h_{ab} = G^{-1/2} R_{ab} = \frac{H}{n-1} g_{ab}, \ a, b = 2, \cdots, n,$$

where

$$H = G^{-1/2}(R - R_{11}).$$

Moreover, the Codazzi equation says that, for $a, b, c = 2, \cdots, n$,

$$R_{1cab} = \nabla_a^{\Sigma_c} h_{bc} - \nabla_b^{\Sigma_c} h_{ac}.$$

Tracing over b and c, we obtain

$$0 = R_{1a} = \nabla_a^{\Sigma_c} H - \nabla_b^{\Sigma_c} h_{ab} = (1 - \frac{1}{n-1}) e_a H,$$

proving (c) and (d).

Finally, by the Gauss equation, the sectional curvatures of (Σ_c, g_{ab}) are given by

$$R^{\Sigma_c}_{abab} = R_{abab} + h_{aa} h_{bb} - h_{ab}^2 = R_{abab} + \frac{H^2}{(n-1)^2}. \tag{3.5}$$

On the other hand, since $W_{ijkl} = 0$,

$$\begin{aligned} R_{abab} &= \frac{2}{n-2} R_{aa} - \frac{R}{(n-1)(n-2)} \\ &= \frac{2G^{1/2} H - R}{(n-1)(n-2)}. \end{aligned}$$

But we already showed that G, H, R are constant on Σ_c. Therefore (Σ_c, g_{ab}) has constant sectional curvature

$$K_c = \frac{2G^{1/2} H - R}{(n-1)(n-2)} + \frac{H^2}{(n-1)^2},$$

proving (e). □

With Lemma 3.3 in our hands, we now conclude the proof of Proposition 3.1. Recall that in any neighborhood, where $G \neq 0$, of the equipotential hypersurface

$$\Sigma_c =: \{x \in M : F(x) = c\}$$

of a regular value c of F, we can express the metric $ds^2 = g_{ij}(x)dx^i dx^j$ as

$$ds^2 = G^{-1}(F,\theta)dF^2 + g_{ab}(F,\theta)d\theta^a d\theta^b,$$

where $\theta = (\theta^2, \cdots, \theta^n)$ denotes the intrinsic coordinates for Σ_c. Then Lemma 3.3 tells us that $G = G(F)$, $g_{ab} = g_{ab}(F)$, and that (Σ_c, g_{ab}) is a space form, with positive curvature (this follows from (3.5) and the assumption that (M^n, g_{ij}, F) has positive sectional curvature). Also, F has exactly one minimum at some origin x_0; otherwise (M^n, g_{ij}, F) would split out a flat factor, which is impossible. Hence, on $M \setminus \{x_0\}$ we have

$$ds^2 = G^{-1}(F)dF^2 + \varphi^2(F)\bar{g}, \tag{3.6}$$

where $\bar{g}$ denotes the standard metric on the unit sphere $\mathbb{S}^{n-1}$, and ϕ is some smooth function on M^n depending only on F and vanishing only at x_0. Thus, (M^n, g_{ij}, F) is a rotationally symmetric gradient steady soliton on $\mathbb{R}^n$. Therefore, it is the Bryant soliton. □

Proof of Theorem 1.2. Now, by assumption, (M^n, g_{ij}, F) is a complete noncompact locally conformally flat gradient steady soliton. By Proposition 2.5, we know that (M^n, g_{ij}, F) has bounded and nonnegative curvature operator $0 \le Rm \le C$. From Hamilton's strong maximum principle (see [13] and [15]), (M^n, g_{ij}, F) is either of positive curvature operator, or its holonomy group reduces.

If (M^n, g_{ij}, F) has positive curvature operator $Rm > 0$, then by Proposition 3.1/Theorem 1.1, (M^n, g_{ij}, F) must be the Bryant soliton.

On the other hand, if (M^n, g_{ij}, F) has reduced holonomy, then (M^n, g_{ij}, F) is either a Riemannian product, or a locally symmetric space, or irreducible but not locally symmetric.

• Case (a): (M^n, g_{ij}, F) is a Riemannian product.

It is known (cf. p.61 in [1]) that the only conformally flat Riemannian products are either the product of a space form N^{n-1} with $\mathbb{S}^1$ or $\mathbb{R}^1$, or the product of two Riemannian manifolds, one with constant sectional curvature -1 and the other with constant sectional curvature 1. However, since our (M^n, g_{ij}) is a steady Ricci soliton with nonnegative sectional curvature, this implies that the latter case cannot occur and that in the former case N^{n-1} must be flat. Hence (M^n, g_{ij}, F) is flat in case (a).

• Case (b): (M^n, g_{ij}, F) is a locally symmetric space.

In case (b), (M^n, g_{ij}, F) is necessarily Einstein. However, (M^n, g_{ij}, F) is noncompact and of nonnegative sectional curvature. Therefore it follows that it must be Ricci flat and hence flat.

• Case (c): (M^n, g_{ij}, F) is irreducible and not locally symmetric.

In this case the flatness of (M^n, g_{ij}, F) follows from the holonomy classification theorem of Berger and Simons (cf. [1]), the fact that

$$|W_{ijkl}|^2 \ge \frac{3(m-1)}{m(m+1)(2m-1)}R^2 \tag{3.7}$$

for Kähler manifolds of complex dimension $m > 1$ (cf. Proposition 2.68 in [1]), and that our (M^n, g_{ij}, F) is noncompact, conformally flat and of nonnegative sectional curvature.

Thus we have shown that in all of the above three cases, (M^n, g_{ij}, F) are flat. This completes the proof of Theorem 1.2. □

Remark 3.2. Shortly after our work appeared on the arXiv, Catino and Mantegazza [6] obtained a similar result to our Theorem 1.1 for dimension $n \geq 4$ by studying the evolution of the Weyl tensor under the Ricci flow. However, as pointed out in [6], their argument does not work for $n = 3$.

4. Further remarks

In this section, we point out that several lemmas in the proof of Theorem 1.1 for gradient steady Ricci solitons also hold for gradient shrinking and expanding Ricci solitons satisfying

$$R_{ij} = \nabla_i \nabla_j F + \rho g_{ij}, \tag{4.1}$$

with $\rho = 1/2$ for shrinkers and $\rho = -1/2$ for expanders, respectively. In particular, our method, when combined with a result of Kotschwar [18], yields another proof of the classification theorem for n-dimensional ($n \geq 4$) complete locally conformally flat shrinking gradient solitons (see Proposition 4.1).

Remark 4.1. A similar version of Lemma 3.1 is valid for locally conformally flat gradient shrinkers and expanders.

First of all, similar arguments to those in Lemma 2.1 imply that

$$\nabla_i R = -2R_{ij}\nabla_j F$$

and

$$R + |\nabla F|^2 + 2\rho F = C_0$$

for some constant C_0. Hence, by defining

$$G_\rho =: G + 2\rho F = |\nabla F|^2 + 2\rho F,$$

we have

$$R + G_\rho = C_0$$

and

$$\nabla G_\rho = -\nabla R = 2Rc(\nabla F, \cdot). \tag{4.2}$$

Therefore, by replacing $G = |\nabla F|^2$ by $G_\rho = G + 2\rho F$ and carrying out the same arguments as in the proof of Lemma 3.1, we obtain

Lemma 4.1. *For any n-dimensional gradient shrinking or expanding soliton ($n \geq 3$) with vanishing Weyl tensor $W_{ijkl} = 0$, the Cotton tensor C_{ijk} has the property that*

$$|C_{ijk}|^2 = \frac{1}{(n-2)^2}(|R_{ik}\nabla_j F - R_{ij}\nabla_k F|^2 - \frac{2}{(n-1)}|R\nabla F - \frac{1}{2}\nabla G_\rho|^2). \tag{4.3}$$

Consequently, Lemma 3.2 and Lemma 3.3 hold for locally conformally flat gradient shrinkers and expanders as well.

Lemma 4.2. *Let (M^n, g_{ij}, F) be any n-dimensional, $n \geq 3$, complete locally conformally flat gradient shrinking or expanding Ricci soliton satisfying* (4.1). *Then, at any regular point $p \in \Sigma_c$, the Ricci tensor of (M^n, g_{ij}, F) either has a unique eigenvalue λ or has two distinct eigenvalues λ and μ of multiplicity 1 and $n-1$, respectively. In either case, $e_1 = \nabla F/|\nabla F|$ is an eigenvector with eigenvalue λ. In other words, for any orthonormal basis $e_2, \cdots, e_n$ tangent to the level surface Σ_c at*

p, the Ricci tensor has the following properties:

(i) $Rc(e_1, e_1) = R_{11}$,

(ii) $Rc(e_1, e_b) = R_{1b} = 0,\ b = 2, \cdots, n$,

(iii) $Rc(e_a, e_b) = R_{aa}\delta_{ab},\ a, b = 2, \cdots, n$,

where either $R_{11} = \cdots = R_{nn} = \lambda$ *or* $R_{11} = \lambda$ *and* $R_{22} = \cdots = R_{nn} = \mu$.

Lemma 4.3. *Let (M^n, g_{ij}, F) be any n-dimensional, $n \geq 3$, complete locally conformally flat gradient shrinking or expanding Ricci soliton satisfying* (4.1)*, and let c be a regular value of F and $\Sigma_c = \{F = c\}$. Then:*

(a) *The function $G = |\nabla F|^2$ is constant on Σ_c; i.e., G is a function of F only.*

(b) *R, the scalar curvature of (M^n, g_{ij}, F), is constant on Σ_c.*

(c) *The second fundamental form h_{ab} of Σ_c is of the form $h_{ab} = \frac{H}{n-1} g_{ab}$.*

(d) *The mean curvature H is constant on Σ_c.*

(e) *(Σ_c, g_{ab}) is of constant sectional curvature $K_{c,\rho}$.*

Note that for shrinking solitons, we have $K_{c,1/2} \geq 0$, according to [23].

Remark 4.2. There is another formula for $|C_{ijk}|^2$ which is valid for all n-dimensional ($n \geq 3$) complete gradient *shrinking*, *steady*, and *expanding* Ricci solitons with vanishing Weyl tensor. This new formula relates $|C_{ijk}|^2$ more explicitly to the geometry of the level surfaces of F and immediately implies Lemma 3.2/4.2 and Lemma 3.3/4.3 (a)-(d).

Lemma 4.4. *For any n-dimensional gradient shrinking, or steady, or expanding soliton (M^n, g_{ij}, F) ($n \geq 3$) with vanishing Weyl tensor $W_{ijkl} = 0$, we have*

$$|C_{ijk}|^2 = \frac{2G^2}{(n-2)^2}|h_{ab} - \frac{H}{n-1}g_{ab}|^2 + \frac{1}{2(n-1)(n-2)}|\nabla_a G|^2, \tag{4.4}$$

where h_{ab} and H are, as in Lemma 3.3*, the second fundamental form and the mean curvature for the level surface $\Sigma_c = \{F = c\}$ at any regular value c of F.*

Proof. Denote again

$$G = |\nabla F|^2 \quad \text{and} \quad G_\rho = G + 2\rho F$$

so that $G_0 = G$. Let c be any regular value of the potential function F and consider the corresponding equipotential hypersurface

$$\Sigma_c =: \{x \in M : F(x) = c\}.$$

Let $\{e_1, e_2, \cdots, e_n\}$ be any orthonormal frame, with $e_1 = \nabla F/|\nabla F| = \nabla F/\sqrt{G}$ and $e_2, \cdots, e_n$ tangent to Σ_c. Then the second fundamental form h_{ab} and the mean curvature H are given respectively by

$$\begin{aligned} h_{ab} =& \langle \nabla_a e_1, e_b\rangle = \langle \nabla_a \frac{\nabla F}{\sqrt{G}}, e_b\rangle \\ =& \frac{1}{\sqrt{G}} R_{ab}, \quad a, b = 2, \cdots, n \end{aligned}$$

and

$$H = \frac{1}{\sqrt{G}}(R - R_{11}).$$

By (4.2),

$$R_{11} = \frac{1}{G} Rc(\nabla F, \nabla F) = \frac{1}{2G}\nabla F \cdot \nabla G_\rho$$

and

$$R_{1a} = \frac{1}{\sqrt{G}} Rc(\nabla F, e_a) = \frac{1}{2\sqrt{G}} \nabla_a G_\rho.$$

Moreover,

$$\begin{aligned} |\nabla_{\Sigma_c} G_\rho|^2 =& \sum_{a=2}^{n} |\nabla_a G_\rho|^2 \\ =& |\nabla G_\rho|^2 - \frac{1}{G} |\nabla G_\rho \cdot \nabla F|^2. \end{aligned}$$

Thus, by direct computations, we obtain

$$\begin{aligned} |h_{ab} - \frac{H}{n-1} g_{ab}|^2 &= |h_{ab}|^2 - \frac{H^2}{n-1} \\ &= \frac{1}{G}(|Rc|^2 - 2\sum_{a=2}^{n} R_{1a}^2 - R_{11}^2) - \frac{(R - R_{11})^2}{(n-1)G} \\ &= \frac{1}{G}|Rc|^2 + \frac{R}{(n-1)G^2} \nabla F \cdot \nabla G_\rho - \frac{1}{2G^2} |\nabla G_\rho|^2 \\ &\quad + \frac{n-2}{4(n-1)G^3} |\nabla F \cdot \nabla G_\rho|^2 - \frac{R^2}{(n-1)G}. \end{aligned}$$

Now noticing that, by Lemma 4.1, we also have

$$\begin{aligned} |C_{ijk}|^2 = & \frac{2G}{(n-2)^2} |Rc|^2 - \frac{1}{2(n-2)^2} |\nabla G_\rho|^2 \\ & - \frac{1}{2(n-1)(n-2)^2} (|\nabla G_\rho|^2 - 4R \nabla G_\rho \cdot \nabla F + 4R^2 G), \end{aligned}$$

one can therefore verify directly that

$$\frac{2G^2}{(n-2)^2} |h_{ab} - \frac{H}{n-1} g_{ab}|^2 = |C_{ijk}|^2 - \frac{1}{2(n-1)(n-2)} |\nabla_{\Sigma_c} G_\rho|^2.$$

This completes the proof of Lemma 4.4. □

Remark 4.3. Based on Lemma 4.3 for shrinkers, it is not hard to see that the universal cover $(\tilde{M}^n, \tilde{g}_{ij}, \tilde{F})$ of a complete locally conformally flat gradient shrinking Ricci soliton is a rotationally symmetric gradient shrinking Ricci soliton and $\tilde{M}^n$ is diffeomorphic to either $\mathbb{R}^n$, or $\mathbb{S}^n$, or $\mathbb{S}^{n-1} \times \mathbb{R}$. Combining this with the result of Kotschwar [18], one obtains another proof of the following classification theorem for n-dimensional ($n \geq 4$) complete locally conformally flat shrinking gradient solitons (which is first due to the combined works of Z.-H. Zhang and Ni-Wallach, and also the combination of more recent work of Munteanu-Sesum and the earlier works of either Petersen-Wylie, or X. Cao, B. Wang and Z. Zhang).

Proposition 4.1. *Any n-dimensional, $n \geq 4$, complete locally conformally flat gradient shrinking Ricci soliton is a finite quotient of $\mathbb{R}^n$, or $\mathbb{S}^n$, or $\mathbb{S}^{n-1} \times \mathbb{R}$.*

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