THE GEOMETRY ON SMOOTH TOROIDAL COMPACTIFICATIONS OF SIEGEL VARIETIES

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Abstract. We study smooth toroidal compactifications of Siegel varieties thoroughly from the viewpoints of mixed Hodge theory and Kähler-Einstein metric. We observe that any cusp of a Siegel space can be identified as a set of certain weight one polarized mixed Hodge structures. We then study the infinity boundary divisors of toroidal compactifications, and obtain a global volume form formula of an arbitrary smooth Siegel variety $\mathcal{A}_{g,\Gamma}(g>1)$ with a smooth toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}$ such that $D_\infty := \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ is normal crossing. We use this volume form formula to show that the unique group-invariant Kähler-Einstein metric on $\mathcal{A}_{g,\Gamma}$ endows some restraint combinatorial conditions for all smooth toroidal compactifications of $\mathcal{A}_{g,\Gamma}$. Again using the volume form formula, we study the asymptotic behaviour of logarithmical canonical line bundle on any smooth toroidal compactification of $\mathcal{A}_{g,\Gamma}$ carefully and we obtain that the logarithmical canonical bundle degenerate sharply even though it is big and numerically effective.

Contents

0. Introduction

Throughout this paper, the number g is an integer more than two.

Siegel varieties are locally symmetric varieties. They are important and interesting in algebraic geometry and number theory because they arise as moduli spaces for Abelian varieties with a polarization and a level structure.

The purpose of this paper is to study smooth toroidal compactifications of Siegel varieties and their applications, we also try to understand the Kähler-Einstein metrics on Siegel varieties through the compactifications. We discuss the geometric aspects of the theory after the works of Ash-Mumford-Rapoport-Tai and Faltings-Chai. Later advances in algebraic geometry have given us many very effective tools for studying these varieties and their toroidal compactifications.

There is a general theory of compactifications of all locally symmetric varieties D/Γ(D a bounded symmetric domain, $\Gamma \subset \mathrm{Aut}(D)$ an arithmetic subgroup). Every variety D/Γ has its Stake-Baily-Borel compactification, which is a canonical minimal compactification. But this compactification has rather bad singularities. In another direction, Ash, Mumford, Rapoport and Tai, in their collaborated book [1], use the theory of toroidal embedding to construct a whole class of compactifications with mild singularities, including, when Γ is neat, smooth compactifications. Faltings and Chai use purely algebraic method to construct arithmetic toroidal compactifications of Siegel varieties.

A toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}$ of a Siegel variety $\mathcal{A}_{g,\Gamma} := \mathfrak{H}_g/\Gamma$(here $\mathfrak{H}_g$ is the Siegel space of genus g and $\Gamma \subset \mathrm{Aut}(\mathfrak{H}_g)$ is an arithmetic subgroup) is totally determined by a combinatorial condition : an admissible family of polyhedral decompositions of certain positive cones. As well known, the natural Bergman metric on $\mathfrak{H}_g$ is Kähler-Einstein. The first author believes that the intrinsic Kähler-Einstein metric on a quasi-projective manifold M should be helpful for finding a nice compactification $\overline{M}$ of M, and he has thought this problem for a long time. In this paper, we can assert that the Kähler-Einstein metric on $\mathcal{A}_{g,\Gamma}$ endows some restraint combinatorial conditions for all toroidal smooth compactifications of $\mathcal{A}_{g,\Gamma}$(Theorem 3.2, Theorem 3.7 in Section 3). Let us explain this result : Let $\sigma_{\max}$ be an admissible top-dimensional polyhedral cone with $N(= \dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma})$ edges $\rho_1, \cdots \rho_N$. Each edge ρ_i of $\sigma_{\max}$ corresponds to an irreducible components D_i of the boundary divisor $D_\infty := \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$. Assume that D_∞ is normal crossing. For every $i = 1, \cdots, N$, let s_i be the global section of the line bundle $[D_i]$ defining D_i. Then, the $(s_1, \cdots, s_N)$ give us a global coordinate system on $\mathcal{A}_{g,\Gamma}$ and we can choose a suitable Hermitian metric $||\cdot||_i$ on each $[D_i]$ such that the volume form on $\mathcal{A}_{g,\Gamma}$ is represented by

$$\Phi_{g,\Gamma} = \frac{2^{\frac{g(g-1)}{2}} \mathrm{vol}_\Gamma(\sigma_{\max})^2 d\mathcal{V}_g}{(\prod_{j=1}^N ||s_i||_i^2) F_{\sigma_{\max}}^{g+1}(\log ||s_1||_1, \cdots, \log ||s_N||_N)}. \tag{0.0.1}$$

where $d\mathcal{V}_g$ is a global smooth volume form on $\overline{\mathcal{A}}_{g,\Gamma}$, $F_{\sigma_{\max}}$ is a homogenous rational polynomial of degree g and $\mathrm{vol}_\Gamma(\sigma_{\max})$ is the lattice volume of $\sigma_{\max}$, moreover the coefficients of $F_{\sigma_{\max}}$ are totally determined by $\sigma_{\max}$ with marking order of edges and Γ(Statement (1) of Theorem 3.2). An interesting observation is that the unique Kähler-Einstein metric on $\mathcal{A}_{g,\Gamma}$ guarantees a real Monge-Ampére equation of elliptic type

$$\det(\frac{\partial^2 H}{\partial x_i \partial x_j})_{i,j} = 2^{\frac{g(g-1)}{2}} \mathrm{vol}_\Gamma(\sigma_{\max})^2 \exp((g+1)H) \tag{0.0.2}$$

for $H := -\log F_{\sigma_{\max}}$ on the domain $\{(x_1, \cdots, x_{\frac{g(g+1)}{2}}) \in \mathbb{R}^{\frac{g(g+1)}{2}} \mid x_i \leq -C < 0 \forall i\}$ (Statement (3) of Theorem 3.2). This Monge-Ampére equation 0.0.2 defines a system of rational polynomials, and the system of all coefficients of $F_{\sigma_{\max}}$ gives a nature solution to that system of rational polynomials. Moreover, this system defines an affine variety $\mathfrak{Q}_g$ over $\mathbb{Q}$, which is dependent only on $\mathfrak{H}_g$. The important thing is that the set of all admissible top-dimensional polyhedral cones has an injection into the set $\mathfrak{Q}_g(\mathbb{Z})$ of all integral point of $\mathfrak{Q}_g$(Theorem 3.7). Furthermore, we give a remark in 3.8 that the real elliptic Monge-Ampére equation 0.0.2 and Theorem 3.7 are always true for all smooth toroidal compactifications whether D_∞ is normal crossing or not.

As an important application of the formula 0.0.1 in Algebraic geometry, we study the asymptotic behaviour of logarithmical canonical line bundles on smooth toroidal compactifications of $\mathcal{A}_{g,\Gamma}$(Theorem 4.7, Theorem 4.13 and Theorem 4.15 in Section 4). We find all logarithmical cotangent bundles degenerate sharply even though $K_{\overline{\mathcal{A}}_{g,\Gamma}} + D_\infty$'s are big and numerically effective(cf.[25]). For convenience, we fix a compactification $\overline{\mathcal{A}}_{g,\Gamma}$ and write $D_\infty = \bigcup_j D_j$, $D_{i,\infty} := \bigcup_{j \neq i} D_j \cap D_i$ and $D_i^* := D_i \setminus D_{i,\infty}$. Mumford also shows that the form $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Phi_{g,\Gamma}$ on $\mathcal{A}_{g,\Gamma}$ is a current on $\overline{\mathcal{A}}_{g,\Gamma}$ representing $c_1([K_{\overline{\mathcal{A}}_{g,\Gamma}} + D_\infty])$ in cohomology class. Using the formula 0.0.1, we obtain that $\partial\overline{\partial}\log\Phi_{g,\Gamma}$ can be extended to a continuous form on $\mathcal{A}_{g,\Gamma} \cup \bigcup_j D_j^*$, and that the restriction of this continuous form to each D_i^*(denote by $\mathrm{Res}_{D_i}(\partial\overline{\partial}\log\Phi_{g,\Gamma})$) is a smooth form on D_i^*. Moreover, the key point is that the form $\mathrm{Res}_{D_i}(\partial\overline{\partial}\log\Phi_{g,\Gamma})$ on D_i^* has Poincaré growth on $D_{i,\infty}$ by Mumford's goodness property. Therefore, we can regard

$\frac{\sqrt{-1}}{2\pi}\mathrm{Res}_{D_i}(\partial\overline{\partial}\log\Phi_{g,\Gamma})$ as a positive closed current on D_i. Let $||\cdot||_i$ be an arbitrary Hermitian metric on the line bundle $[D_i]$ for each D_i, we get

$$\begin{aligned}&(K_{\overline{\mathcal{A}}_{g,\Gamma}}+D_\infty)^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}-d}\cdot D_{i_1}\cdots D_{i_d}\\ =\ &(\frac{\sqrt{-1}}{2\pi})^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}-d}\int_{D_{i_l}}\mathrm{Res}_{D_{i_l}}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}-d})\wedge(\bigwedge_{1\leq j\leq d,j\neq l}c_1([D_{i_j}],||\cdot||_{i_j}))\end{aligned}$$

for any $d(1\leq d\leq N-1)$ irreducible components $D_{i_1},\cdots,D_{i_d}$ of the boundary divisor D_∞ and any integer $l\in[1,d]$(Theorem 4.13). Furthermore, we observe that irreducible components of D_∞ are all from lower genus Siegel varieties and the type of $\mathrm{Res}_{D_i}(\partial\overline{\partial}\log\Phi_{g,\Gamma})$ is similar with the type of $\partial\overline{\partial}\log\Phi_{g,\Gamma}$. Due to this structure of self-similarity, we use the method of recursion to deduce an integral formula : For any $d(1\leq d\leq N-1)$ different irreducible components $D_1,\cdots,D_d$ of D_∞ satisfying that $\bigcap_{l=1}^{d}D_l\neq\emptyset$, there is

$$\begin{aligned}&(K_{\overline{\mathcal{A}}_{g,\Gamma}}+D_\infty)^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}-d}\cdot D_1\cdots D_d\\ =\ &\int_{\bigcap_{k=1}^{d}D_k}\mathrm{Res}_{\bigcap_{i=1}^{d}D_i}(\mathrm{Res}_{\bigcap_{i=1}^{d-1}D_i}\cdots(\mathrm{Res}_{D_1}((\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Phi_{g,\Gamma})^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}-d})\cdots)).\end{aligned}$$

A direct consequence is that if $d\geq g-1$ then the intersection number

$$(K_{\overline{\mathcal{A}}_{g,\Gamma}}+D_\infty)^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}-d}\cdot D_1\cdots D_d=0$$

for any d different irreducible components $D_1,\cdots,D_d$ of D_∞. Therefore, the divisor $K_{\overline{\mathcal{A}}_{g,\Gamma}}+D_\infty$ on $\overline{\mathcal{A}}_{g,\Gamma}$ is never ample(Theorem 4.15).

In general, the boundary divisors of smooth toroidal compactifications may have self-intersections(cf.[1] and [14]). However, in most geometric applications, we would like to have a nice toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}$ of $\mathcal{A}_{g,\Gamma}$ such that the added infinity boundary $D_\infty=\overline{\mathcal{A}}_{g,\Gamma}\backslash\mathcal{A}_{g,\Gamma}$ is a normal crossing divisor, for example, in Mumford's work of Hirzebruch's proportionality theorem in the non-compact case(cf.[25]). In Section 2, we study the boundaries of smooth toroidal compactifications explicitly and we actually obtain a sufficient and necessary combinatorial condition for toroidal compactifications with normal crossing boundary divisor(Theorem 2.20 and Theorem 2.22 in Section 2).

On the other hand, the degenerate limits of Abelian varieties have been studied by Mumford,Oda-Seshadri,Nakamura and Namikawa. Deligne's Theorem(cf.[10]) shows that the nth cohomology group of an arbitrary complex variety X carries a canonical mixed Hodge structure, and that the structure is reduced to an ordinary Hodge structure of pure weight n if X is a complete nonsingular variety. Thus, toroidal compactifications of Siegel varieties can be related back to degenerations of Abelian varieties or to degenerations of weight one Hodge structures. Roughly, there is a correspondence between the category of degenerations of Abelian varieties and the category of limits of weight one Hodge structures. The Hodge-theoretic interpretation of the boundary $\overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ of toroidal compactification is given by Carlson, Cattani and Kaplan in [4]. Thus, we believe that any rational boundary component(cusp) of a Siegel variety must parameterize some class of mixed Hodge structures. That is the motivation for our studying cusps of Siegel varieties. Recently, Kato and Usui generalize the work of Carlson-Cattani-Kaplan, and use the idea of logarithmic geometry to give toroidal compactifications of period domains from view of mixed Hodge theory(cf.[22]). We explore this topic, and obtain that the Hodge-theoretic interpretation of the boundary of Siegel varieties coincides with the classic description given by Satake-Baily-Borel in [30] and [2]. Actually, any cusp of Siegel space $\mathfrak{H}_g$ can be identified with a set of certain weight one polarized mixed Hodge structures(Theorem 1.17 in Section 1).

The results of this paper, the methods and the techniques in this paper, are essential to all locally symmetric varieties. Thus, the results of this paper can be generalized to general locally symmetric varieties by our methods and techniques in this paper.

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Notation.

For any real Lie group $\mathcal{P}$, $\mathcal{P}^+$ is the identity component of $\mathcal{P}$ for the real topology. For any linear space L_k over a field k, a finite field extension $k \subset K$ allows we to define a K-linear space $L_K := L_k \otimes K$.

Throughout this paper, we fix a real vector space $V_{\mathbb{R}}$ of dimensional $2g$ and fix a standard symplectic form $\psi = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ on $V_{\mathbb{R}}$. For any non-degenerate skew-symmetric bilinear form $\widetilde{\psi}$ on $V_{\mathbb{R}}$, it is known that there is an element $T \in \mathrm{GL}(V_{\mathbb{R}})$ such that ${}^tT\widetilde{\psi}T = \psi$. We now fix a symplectic basis $\{e_i\}_{1\leq i\leq 2g}$ of the symplectic space $(V_{\mathbb{R}}, \psi)$ such that $\psi(e_i, e_{g+i}) = -1$ for $1 \leq i \leq g$ and $\psi(e_i, e_j) = 0$ for $|j - i| \neq g$.

- Denote by $V_{\mathbb{Z}} := \oplus_{1\le i\le 2g}\mathbb{Z}e_i$, then $V_{\mathbb{R}} = V_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{R}$ and $V_{\mathbb{Z}}$ is a standard lattice in $V_{\mathbb{R}}$. In this paper, we fix the lattice $V_{\mathbb{Z}}$ and fix the rational space $V_{\mathbb{Q}} := V_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Q}$.
- Let $V_{\mathbb{Q}}^{(k)}$ be the rational subspace of $V_{\mathbb{Q}}$ spanned by $\mathbb{Q}$-vectors $\{e_{k+1},\cdots,e_g\}$ for $0\le k\le g-1$, and $V_{\mathbb{Q}}^{(g)} := \{0\}$. Let $V^{(k)} := V_{\mathbb{Q}}^{(k)}\otimes\mathbb{R}$ for $0\le k\le g$.
- Define $\mathrm{Sp}(g,\mathfrak{R}) := \{h\in \mathrm{GL}(V_{\mathfrak{R}})\,|\,\psi(hu,hv)=\psi(u,v)\,\forall u,v\in V_{\mathfrak{R}}\}$ where $V_{\mathfrak{R}} := V_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathfrak{R}$ for any $\mathbb{Z}$-algebra $\mathfrak{R}$. Let $\Gamma_g(n) := \{\gamma\in\mathrm{Sp}(g,\mathbb{Z})\mid \gamma\equiv I_{2g} \mod n\}$ for any integer $n\ge 2$ and $\Gamma_g=\Gamma_g(1) := \mathrm{Sp}(g,\mathbb{Z})$. Thus each congruent group $\Gamma_g(n)$ is a normal subgroup of $\mathrm{Sp}(g,\mathbb{Z})$ with finite index.

For any free $\mathbb{Z}$-module $W_{\mathbb{Z}}$, we use W to represent a linear space over $\mathbb{Z}$(i.e, $W(\mathfrak{R}) := W_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathfrak{R}$ for any $\mathbb{Z}$-algebra $\mathfrak{R}$), and we define $\mathrm{GL}(W)$ to be the algebraic group over $\mathbb{Q}$ representing the functor ($\mathbb{Q}-$ algebras $\longrightarrow$ Groups, $\mathfrak{R}\longmapsto \mathrm{GL}(W_{\mathfrak{R}})$) (cf.[23]). We always write $\mathrm{GL}(n)$ for $\mathrm{GL}(W)$ if $\mathrm{rank}W_{\mathbb{Z}}=n$. For the fixed free $\mathbb{Z}$-module $V_{\mathbb{Z}}$, we also define $\mathrm{Sp}(V,\psi)$ to be the algebraic group over $\mathbb{Q}$ representing the functor

$$\mathbb{Q}\text{-algebras}\longrightarrow \text{Groups},\quad \mathfrak{R}\longmapsto \mathrm{Sp}(V,\psi)(\mathfrak{R}) := \mathrm{Sp}(g,\mathfrak{R}).$$

We know that $\mathrm{Sp}(V,\psi)$ is an algebraic subgroup of $\mathrm{GL}(V)$.

- Two subgroups S_1 and S_2 of $\mathrm{Sp}(g,\mathbb{Q})$ are commensurable if $S_1\cap S_2$ has finite index in both S_1 and S_2. A subgroup $\Gamma\subset\mathrm{Sp}(g,\mathbb{Q})$ is arithmetic if $\rho(\Gamma)$ is commensurable with $\rho(\mathrm{Sp}(g,\mathbb{Q}))\cap\mathrm{GL}(n,\mathbb{Z})$ for some embedding $\rho:\mathrm{Sp}(V,\psi)\overset{\hookrightarrow}{\longrightarrow}\mathrm{GL}(n)$. By a result of Borel, a subgroup $\Gamma\subset\mathrm{Sp}(g,\mathbb{Q})$ is arithmetic if and only if that $\rho'(\Gamma)$ is commensurable with $\rho'(\mathrm{Sp}(g,\mathbb{Q}))\cap\mathrm{GL}(n',\mathbb{Z})$ for every embedding $\rho':\mathrm{Sp}(V,\psi)\overset{\hookrightarrow}{\longrightarrow}\mathrm{GL}(n')$(cf.Chap. VI. [23]). Thus a subgroup $\Gamma\subset\mathrm{Sp}(g,\mathbb{Z})$ is arithmetic if and only if $[\mathrm{Sp}(g,\mathbb{Z}):\Gamma]<\infty$.
- Let k' be a subfield of $\mathbb{C}$ and $\widetilde{V}_{k'}$ a k'-vector space. Let $\mathrm{GL}(\widetilde{V})$ be an algebraic group defined over k' as above. An automorphism α of a k'-vector space is defined to be neat (or torsion free) if its eigenvalues in $\mathbb{C}$ generate a torsion free subgroup of $\mathbb{C}$. An element $h\in\mathrm{Sp}(g,\mathbb{Q})$ is said to be neat(or torsion free) if $\rho(h)$ is neat for one faithful representation $\rho:\mathrm{Sp}(V,\psi)\overset{\hookrightarrow}{\longrightarrow}\mathrm{GL}(\widetilde{V})$. A subgroup $\Gamma\subset\mathrm{Sp}(g,\mathbb{R})$ is said to be neat if all elements of Γ are torsion free. We have that if $h\in\mathrm{Sp}(g,\mathbb{Q})$ is neat then $\rho'(h)$ is neat for every representation ρ' of $\mathrm{Sp}(V,\psi)$ defined over k'(cf.[23]). For example, the $\Gamma_g(n)$ is a neat arithmetic subgroup of $\mathrm{Sp}(g,\mathbb{Q})$ if $n\ge 3$.
- For any arithmetic subgroup $\Gamma\subset\mathrm{Sp}(g,\mathbb{Q})(g\ge 2)$, we can find a neat subgroup $\Gamma'\subset\Gamma$ of finite index. In fact, the neat subgroup Γ' can be given by congruence conditions(cf.[23]).

The Siegel space $\mathfrak{H}_g$ of degree g is a set of all symmetric matrices over $\mathbb{C}$ of degree g whose imaginary parts are positive defined. The simple Lie group $\mathrm{Sp}(g,\mathbb{R})$ acts transitively on $\mathfrak{H}_g$ as $\begin{pmatrix} A & B\\ C & D\end{pmatrix}\bullet\tau := \frac{A\tau+B}{C\tau+D}$. Let $o := \sqrt{-1}I_g$ be a fixed point on $\mathfrak{H}_g$.

- The stabilizer of o is isomorphic to the unitary group $\mathrm{U}(g)$. We identify o with $\pi(e)$ where $\pi : \mathrm{Sp}(g,\mathbb{R}) \to \mathfrak{H}_g$ is the standard projection and $e \in \mathrm{Sp}(g,\mathbb{R})$ is the identity. The element $s_o := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ in $\mathrm{Sp}(g,\mathbb{R})$ acts as an involution of $\mathfrak{H}_g$ leaving o as the only isolated fixed point. Therefore the Siegel space $\mathfrak{H}_g$ is an non-compact Hermitian symmetric space.
- From now on, let $G_{\mathbb{R}} = \mathrm{Sp}(g,\mathbb{R})$ be the real Lie group with Lie algebra $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} := \mathrm{Lie}(G_{\mathbb{R}})$, and regard $K_{\mathbb{R}} := \mathrm{U}(g)$ as a real Lie group with Lie algebra $\mathfrak{k} = \mathfrak{k}_{\mathbb{R}} = \mathrm{Lie}(K_{\mathbb{R}})$. With respect to the standard symplectic basis, we have :

$$\begin{aligned}
\mathfrak{g}_{\mathbb{R}} &= \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(2g,\mathbb{R}) \mid D = -{}^tA,\ {}^tB = B,\ {}^tC = C\}, \\
\mathfrak{k}_{\mathbb{R}} &= \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in M(2g,\mathbb{R}) \mid {}^tA = -A,\ {}^tB = B\} \cong u(g)
\end{aligned}$$

A **Siegel variety** is defined to be $\mathcal{A}_{g,\Gamma} := \Gamma\backslash\mathfrak{H}_g$, where Γ is an arithmetic subgroup of $\mathrm{Sp}(g,\mathbb{Q})$. Any Siegel variety is a normal quasi-project variety.

- Any neat arithmetic subgroup Γ of $\mathrm{Sp}(g,\mathbb{Q})$ acts freely on the Siegel Space $\mathfrak{H}_g$, so that the induced $\mathcal{A}_{g,\Gamma}$ is a regular quasi-projective complex variety of dimension $g(g+1)/2$. A **Siegel variety of degree** g **with level** n is defined to be $\mathcal{A}_{g,n} := \Gamma_g(n)\backslash\mathfrak{H}_g$. Thus, the Siegel varieties $\mathcal{A}_{g,n}$ $n \geq 3$ are quasi-projective complex manifolds.

1. Cusps of Siegel varieties from the viewpoint of mixed Hodge theory

1.1. **Typical homogenous rational polarized VHS on Siegel space.** Define

$$\mathrm{U}^1 := \mathrm{U}(1) = \{|z| = 1 \mid z \in \mathbb{C}\}.$$

Let $\tau \in \mathfrak{H}_g$ be an arbitrary point. Let $T_\tau(\mathfrak{H}_g)$ be the real tangent space at τ and J_τ the complex structure on $T_\tau(\mathfrak{H}_g)$ induced by the global complex structure J of $\mathfrak{H}_g$.

The Siegel space $\mathfrak{H}_g$ has a natural $\mathrm{Sp}(g,\mathbb{R})$-invariant Kähler metric(Bergman metric). Regard $\mathfrak{H}_g$ as a Riemannian symmetric space, we have a basic fact:

Lemma 1.1 (Cf.[17]&[11])**.** *For any $z = a + \sqrt{-1}b \in \mathrm{U}^1$, there is a unique isometric automorphism $u_\tau(z) : \mathfrak{H}_g \to \mathfrak{H}_g$ such that*

- *$u_\tau(z)$ maps τ to τ,*
- *$du_\tau(z) : T_\tau(\mathfrak{H}_g) \to T_\tau(\mathfrak{H}_g)$ is given by $v \mapsto z \cdot v := av + bJ_\tau(v)$.*

The uniqueness of $u_\tau(z)$ ensures that $u_\tau(1) = \mathrm{Id}$ and

$$u_\tau(z_1z_2) = u_\tau(z_1) \circ u_\tau(z_2) = u_\tau(z_2) \circ u_\tau(z_1),\ \forall z_1, z_2 \in \mathrm{U}^1.$$

Therefore, we obtain a group homomorphism

$$u_\tau : \mathrm{U}^1 \longrightarrow \mathrm{Aut}(\mathfrak{H}_g) = \mathrm{Sp}(g,\mathbb{R})/\{\pm I_{2g}\},\ z \mapsto u_\tau(z). \tag{1.1.1}$$

Furthermore, we can lift u_τ^2 to be a group homomorphism $h_\tau : \mathrm{U}^1 \longrightarrow \mathrm{Sp}(g,\mathbb{R})$. Since $\mathrm{Sp}(g,\mathbb{R})$ is a simply-connected topological space. The uniqueness of $u_\tau(z)$ guarantees that

$$h_{M(\tau)} = M h_\tau M^{-1},\ \text{for } \forall \tau \in \mathfrak{H}_g, \forall M \in \mathrm{Sp}(g,\mathbb{R}). \tag{1.1.2}$$

The $u_o^2(\sqrt{-1})$ is the involution of $\mathfrak{H}_g$ fixing the point $o = \sqrt{-1}I_g$, so $h_o(\sqrt{-1})$ is one of $\{\pm \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}\}$. Since that $h_o : \mathrm{U}^1 \to \mathrm{Sp}(g,\mathbb{R})$ is a group homomorphism and that $h_o(\mathrm{U}^1)$ is a commutative group in $U(g)$, we must have $h_o(\sqrt{-1}) = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

Let $\mathbb{G}_m := \mathrm{GL}(1)$ be an algebraic torus. A priori $\mathbb{G}_m = \mathbb{G}_{m/\mathbb{Q}}$ is defined over $\mathbb{Q}$, and thus $\mathbb{G}_m(k) = k^\times$ for any field k containing $\mathbb{Q}$. Following 1.4.4 in [11], we define $\mathrm{GSp}(V,\psi)$ to be the quotient $\mathrm{Sp}(V,\psi) \times \mathbb{G}_m$ by the central subgroup $\{e, (\epsilon, -1)\}$, it is an algebraic group over $\mathbb{Q}$. Let ι be the composed homomorphism

$$\iota : \mathrm{Sp}(V,\psi) \xrightarrow{\hookrightarrow} \mathrm{Sp}(V,\psi) \times \mathbb{G}_m \longrightarrow \mathrm{GSp}(V,\psi),$$

and $t : \mathrm{GSp}(V,\psi) \to \mathbb{G}_m$ the homomorphism given by

$$\mathrm{GSp}(V,\psi)(\mathbb{R}) \to \mathbb{G}_m(\mathbb{R}),\ \ [(g,\lambda)] \mapsto \lambda^2.$$

We then have a split exact sequence

$$1 \longrightarrow \mathrm{Sp}(V,\psi) \xrightarrow{\iota} \mathrm{GSp}(V,\psi) \xrightarrow{t} \mathbb{G}_m \longrightarrow 1. \tag{1.1.3}$$

According to Deligne's Hodge theory(cf.[10],[11],[12]), each h_τ actually corresponds to a rational Hodge structure on $V_\mathbb{R}$ of pure weight one given by a Hodge filtration

$$F_\tau^\bullet = \left(F_\tau^2 \subset F_\tau^1 \subset F_\tau^0\right) = \left(0 \subset F_\tau^1 \subset V_\mathbb{C}\right)$$

on $V_\mathbb{C}$ such that

$$F_\tau^1 = \text{ the subspace of } V_\mathbb{C} \text{ spanned by the column vectors of } \begin{pmatrix} \tau \\ I_g \end{pmatrix}.$$

Moreover, we observe that h_τ is polarized automatically by the standard symplectic form ψ, i.e., $F_\tau^\bullet$ satisfies the following two Riemann-Hodge bilinear relations:

a) $\psi(F_\tau^p, F_\tau^q) = 0$ for $p + q > 1$.

b) The Hermitian form $V_\mathbb{C} \times V_\mathbb{C} \longrightarrow \mathbb{C}$ $(x,y) \mapsto \psi(C_\tau(x), \overline{y})$ is positive definite.

Proposition 1.2 (Satake-Deligne [30]&[11])**.** *Define*

$$\mathfrak{S}_g = \mathfrak{S}(V_\mathbb{R},\psi) := \{F^1 \in \mathrm{Grass}(g, V_\mathbb{C})\ ) \mid \psi(F^1,F^1) = 0, \sqrt{-1}\psi(F^1, \overline{F^1}) > 0\}.$$

The map $h : \mathfrak{H}_g \xrightarrow{\cong} \mathfrak{S}_g \quad \tau \longmapsto F^1_\tau$ *identifies the Siegel space* $\mathfrak{H}_g$ *with the period domain* $\mathfrak{S}_g$*. Moreover, the map* h *is biholomorphic.*

Let $\pi : \mathrm{Sp}(g,\mathbb{R}) \to \mathfrak{H}_g \cong \mathrm{Sp}(g,\mathbb{R})/\mathrm{U}(g)$ be the standard projection mapping the identity e to the point $o = \sqrt{-1}I_g$. Then, we have the following commutative diagrams of differentials

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}h_o(z)} & \mathfrak{g} \\ {\scriptstyle d\pi}\downarrow & & \downarrow{\scriptstyle d\pi} \\ T_o(\mathfrak{H}_g) & \xrightarrow{d(h_o(z))|_o} & T_o(\mathfrak{H}_g), \end{array} \quad \forall z \in \mathrm{U}^1. \tag{1.2.1}$$

In particular, we obtain that $\sigma := \mathrm{Ad}(h_o(\sqrt{-1}))$ is an involution of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{Sp}(g,\mathbb{R}))$, and there is a decomposition of $\mathfrak{g}$(orthogonal under the no-degenerate Killing form) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{k} = \mathrm{Lie}(K) \cong \mathrm{Lie}(\mathrm{U}(g))$ and $\mathfrak{p} \cong T_o(\mathfrak{H}_g)$. Since $h_o(\mathrm{U}^1)$ is a commutative subgroup in $\mathrm{Sp}(g,\mathbb{R})$, both $\mathfrak{k}$ and $\mathfrak{p}$ are $\mathrm{Ad}(h_o(\mathrm{U}^1))$-invariant. Moreover, $\mathrm{Ad}(h_o(\exp(\frac{2\pi\sqrt{-1}}{8})))|_{\mathfrak{p}}$ is compatible with the complex structure J_o of the tangent space $T_o(\mathfrak{H}_g)$. Thus, the collection $\{J_\tau\}_{\tau\in\mathfrak{H}_g}$ is $\mathrm{Sp}(g,\mathbb{R})$-invariant and is same as the original global complex structure J on $\mathfrak{H}_g$.

Since $J_{\mathfrak{o}}$ gives a decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ into $\pm\sqrt{-1}$-eigenspaces, we have :

Corollary 1.3. *For every* $z \in \mathrm{U}^1$, *the adjoint homomorphism* $\mathrm{Ad}(h_o(z)) : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ *is given by*

$$\begin{aligned} \mathrm{Ad}(h_o(z))|_{\mathfrak{p}^+} &: \mathfrak{p}^+ \longrightarrow \mathfrak{p}^+ \qquad v \longmapsto z^2 v, \\ \mathrm{Ad}(h_o(z))|_{\mathfrak{p}^-} &: \mathfrak{p}^- \longrightarrow \mathfrak{p}^- \qquad v \longmapsto z^{-2} v, \\ \mathrm{Ad}(h_o(z))|_{\mathfrak{k}_{\mathbb{C}}} &: \ \mathfrak{k}_{\mathbb{C}} \longrightarrow \mathfrak{k}_{\mathbb{C}} \qquad v \longmapsto v. \end{aligned}$$

Now, we have the following composite homomorphism

$$\mathrm{U}^1 \xrightarrow{h_o} \mathrm{Sp}(g,\mathbb{R}) \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathfrak{g}). \tag{1.3.1}$$

It determines a weight zero real Hodge structure on $\mathfrak{g}$(cf.[11],[12]). Actually, this real Hodge structure on $\mathfrak{g}$ has type of $\{(-1,1),(0,0),(1,-1)\}$ by the corollary 1.3 :

$$\mathfrak{g}^{0,0} = \mathfrak{k}_{\mathbb{C}} \ , \ \mathfrak{g}^{-1,1} = \mathfrak{p}^+ \text{ and } \mathfrak{g}^{1,-1} = \mathfrak{p}^-.$$

Denote by

$$F^1_o(\mathfrak{g}) := \mathfrak{g}^{1,-1}, \ F^0_o(\mathfrak{g}) := \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1} \text{ and } F^{-1}_o(\mathfrak{g}) = \mathfrak{g}_{\mathbb{C}}.$$

Lemma 1.4. *For* $\mathfrak{g} = \mathrm{Lie}(\mathrm{Sp}(g,\mathbb{R}))$, *we have that*

$$F^p_o(\mathfrak{g}) = \{X \in \mathfrak{g}_{\mathbb{C}} \,|\, X(F^s_o) \subset F^{s+p}_o \ \forall s\} \ p = -1,0,1,$$

where $F^\bullet_o$ *is the Hodge filtration on* $V_{\mathbb{C}}$ *given by* h_o.

Proof. The composite homomorphism $\mathrm{U}^1 \xrightarrow{h_o} \mathrm{Sp}(g,\mathbb{R}) \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{End}(V_{\mathbb{R}}))$ determines a weight zero real Hodge structure on $\mathrm{End}(V_{\mathbb{R}}) = \mathrm{Lie}(\mathrm{GL}(V_{\mathbb{R}}))$. Also, the composite homomorphism $\mathrm{U}^1 \xrightarrow{h_o} \mathrm{Sp}(g,\mathbb{R}) \xrightarrow{\mathrm{Ad}} \mathfrak{g}$ gives a weight zero real Hodge structure on $\mathfrak{g}$. The inclusion $\mathfrak{g} \subset \mathrm{End}(V_{\mathbb{R}})$ is a morphism of Hodge structures by the following commutative diagrams

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\subset} & \mathrm{End}(V_{\mathbb{R}}) \\ {\scriptstyle \mathrm{Ad}(g)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}(g)} \\ \mathfrak{g} & \xrightarrow{\subset} & \mathrm{End}(V_{\mathbb{R}}) \end{array} \qquad \forall g \in \mathrm{Sp}(g,R). \tag{1.4.1}$$

Write $H_o^{s,1-s} := F_o^s/F_o^{s+1}$ for $s = 0, 1$. We obtain that

$$\mathfrak{g}^{i,-i} = \{X \in \mathfrak{g}_{\mathbb{C}} \,|\, X(H_o^{s,1-s}) \subset H_o^{s+i,1-s-i}\ \forall s\} \ \text{ for } i = -1, 0, 1.$$

as a subset of $\mathrm{End}(V_{\mathbb{R}})^{i,-i}$. □

Remark. $F_o^0(\mathfrak{g})$ and $F_o^1(\mathfrak{g})$ are then Lie subalgebras of $\mathfrak{g}_{\mathbb{C}}$. In particular, both $\mathfrak{g}^{-1,1}$ and $\mathfrak{g}^{1,1}$ are commutative complex Lie subalgebras of $\mathfrak{p}_{\mathbb{C}}$.

Corollary 1.5 (Deligne [12])**.** *Gluing Hodge structures $h_\tau\ \forall \tau \in \mathfrak{H}_g$ altogether, the local system $\mathbb{V} := V_{\mathbb{Q}} \times \mathfrak{H}_g$ underlies a homogenous rational variation of polarized Hodge structure of weight one on $\mathfrak{H}_g$.*

Proof. The relation $\mathfrak{g}^{-1,1} = \{X \in \mathfrak{g}_{\mathbb{C}} \,|\, X(H_o^{s,1-s}) \subset H_o^{s-1,2-s}\ \forall s\}$ in the lemma 1.4 shows that the holomorphic tangent bundle of $\mathfrak{H}_g$ is horizontal(cf [28]). □

Suppose that Γ is a neat arithmetic subgroup of $\mathrm{Sp}(g,\mathbb{Q})$, we immediately have:

- Since $\mathfrak{H}_g$ is simply connected, the fundamental group of $\mathcal{A}_{g,\Gamma}$ has $\pi_1(\mathcal{A}_{g,\Gamma}, o) \cong \Gamma$.
- There is a natural local system $\mathbb{V}_{g,\Gamma} := V_{\mathbb{Q}} \times_\Gamma \mathfrak{H}_g$ on $\mathcal{A}_{g,\Gamma}$ given by the fundamental representation $\rho : \pi_1(\mathcal{A}_{g,\Gamma}, o) \longrightarrow \mathrm{GSp}(V,\psi)(\mathbb{Q})$.

Proposition 1.6. *Let Γ be a neat arithmetic subgroup of* $\mathrm{Sp}(g,\mathbb{Q})$*. We have :*

1. The local system $\mathbb{V}_{g,\Gamma}$ underlies a rational variation of polarized Hodge structure on $\mathcal{A}_\Gamma$. Moreover, the associated period map

$$h_\Gamma : \mathcal{A}_{g,\Gamma} \xrightarrow{\cong} \Gamma\backslash\mathfrak{S}_g. \tag{1.6.1}$$

is induced by the isomorphism h in the proposition 1.2.

2. Let $\widetilde{\mathcal{A}}_{g,\Gamma}$ be an arbitrary smooth compactification of $\mathcal{A}_{g,\Gamma}$ with simple normal crossing divisor $D_\infty := \widetilde{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$. Around the boundary divisor D_∞, all local monodromies of any rational PVHS $\widetilde{\mathbf{H}}$ on $\mathcal{A}_{g,\Gamma}$ are unipotent.

Proof. By the corollary 1.5, the local system $\mathbb{V} = V_{\mathbb{Q}} \times \mathfrak{H}_g$ admits a homogenous rational variation of polarized Hodge structure on $\mathfrak{H}_g$. The arguments in Section 4 of [36] show that the VHS attached to $\mathbb{V}$ can induce a locally homogenous rational variation of polarized Hodge structure on the local system $\mathbb{V}_\Gamma$, and so the period map h_Γ is given by the $\mathrm{Sp}(g,\mathbb{R})$-equivariant isomorphism $h : \mathfrak{H}_g \xrightarrow{\cong} \mathfrak{S}_g$ in the proposition 1.2.

It is well-known that all local monodromies of the rational PVHS $\widetilde{\mathbf{H}}$ around D_∞ are quasi-unipotent(i.e., all eigenvalues of monodromies are roots of the unit). Since $\mathfrak{H}_g$ is simply-connected and Γ is neat, all eigenvalues of monodromies must be the unity, and so these monodromies are unipotent. □

1.2. **Cusps on Siegel spaces.** Let $F_o^0(G_{\mathbb{C}})$(resp. $F_o^1(G_{\mathbb{C}})$) be the subgroup of $G_{\mathbb{C}}$ satisfying that $\mathrm{Lie}(F_o^0(G_{\mathbb{C}})) = F_o^0(\mathfrak{g})$ (resp. $\mathrm{Lie}(F_o^1(G_{\mathbb{C}})) = F_o^1(\mathfrak{g})$). The lemma 1.4 shows immediately that $F_o^1(G_{\mathbb{C}})$ is the parabolic subgroup preserving the Hodge filtration $F_o^\bullet$, and that $F_o^1(G_{\mathbb{C}})$ is the unipotent radical of $F_o^0(G_{\mathbb{C}})$.

Proposition 1.7 (Harish-Chandra Embedding Theorem cf.[11])**.** *The set $\mathfrak{S}_g$ is contained in the largest cell of $\check{\mathfrak{S}}_g$. Precisely, the map*

$$\zeta : \mathfrak{S}_g \longrightarrow \overline{F_o^1(\mathfrak{g})} \;\; h \longmapsto n_h$$

identifies $\mathfrak{S}_g$ with a bounded open subset of $\overline{F_o^1(\mathfrak{g})}$, where the space $\overline{F_o^1(\mathfrak{g})}$ is the closure of $F_o^1(\mathfrak{g})$ in $\mathfrak{g} = \mathrm{Lie}(\mathrm{Sp}(g,\mathbb{R}))$ and the element $n_h \in \overline{F_o^1(\mathfrak{g})}$ is determined by $h = \exp(n_h)h_o$.

This Harish-Chandra embedding allows us to define the closure $\overline{\mathfrak{S}}_g := \exp(\overline{\zeta(\mathfrak{S}_g)}^{cl})h_o$ and the boundary $\partial\mathfrak{S}_g := \overline{\mathfrak{S}}_g \setminus \mathfrak{S}_g$, where $\overline{\zeta(\mathfrak{S}_g)}^{cl}$ is the closure of $\zeta(\mathfrak{S}_g)$ in $\overline{F_o^1(\mathfrak{g})}$.

Corollary 1.8.

$$\begin{aligned} \overline{\mathfrak{S}}_g &= \{F^1 \in \mathrm{Grass}(g, V_{\mathbb{C}}) \mid \psi(F^1, F^1) = 0, \sqrt{-1}\psi(F^1, \overline{F^1}) \geq 0\}, \\ \partial\mathfrak{S}_g &= \{F^1 \in \check{\mathfrak{S}}_g \mid \sqrt{-1}\psi(F^1, \overline{F^1}) \geq 0, \psi(\cdot, \bar{\cdot}) \textit{ is degenerate on } F^1\}. \\ &= \{F^1 \in \overline{\mathfrak{S}}_g \mid F^1 \cap \overline{F^1} \textit{ is a non trivial isotropic space }\} \end{aligned}$$

A **boundary component** of the space $\mathfrak{S}_g = \mathfrak{S}(V, \psi)$ is a subset in $\partial\mathfrak{S}_g$ of type

$\mathfrak{F}(W_{\mathbb{R}}) := \{F^1 \in \overline{\mathfrak{S}}_g \mid F^1 \cap \overline{F^1} = W_{\mathbb{R}} \otimes \mathbb{C}$ where $W_{\mathbb{R}}$ is an isotropic real subspace of $V_{\mathbb{R}}$ $\}$.

A **k-th boundary component** of $\mathfrak{S}(V_{\mathbb{R}}, \psi)$ is a $\mathfrak{F}(W_{\mathbb{R}})$ with $\dim_{\mathbb{R}} W_{\mathbb{R}} = k$. We note $\mathfrak{F}(\{0\}) = \mathfrak{S}_g$ and other boundary components are subsets of $\partial\mathfrak{S}_g$. The group $\mathrm{Sp}(g,\mathbb{R})$ has a natural action on the set of all boundary components as $M \bullet \mathfrak{F}(W_{\mathbb{R}}) := \mathfrak{F}(M(W_{\mathbb{R}}))$ $\forall M \in \mathrm{Sp}(g,\mathbb{R})$. The compact space $\overline{\mathfrak{S}}_g$ is a disjoint union of all boundary components, and

$$\partial\mathfrak{S}_g = \overset{\circ}{\bigcup_{\text{nontrivial isotropic } W_{\mathbb{R}} \subset V_{\mathbb{R}}}} \mathfrak{F}(W_{\mathbb{R}}).$$

The **normalizer** of a boundary component $\mathfrak{F}$ is the subgroup $\mathcal{N}(\mathfrak{F})$ of $\mathrm{Sp}(g,\mathbb{R})$ containing of those g such that $g\mathfrak{F}=\mathfrak{F}$. A boundary component $\mathfrak{F}$ is said to be **rational** if its normalizer $\mathcal{N}(\mathfrak{F})$ is defined over $\mathbb{Q}$(i.e., there is an algebraic subgroup $N^{\mathfrak{F}}\subset \mathrm{Sp}(V,\psi)$ defined over $\mathbb{Q}$ such that $\mathcal{N}(\mathfrak{F})^+ = N^{\mathfrak{F}}(\mathbb{R})^+$ cf.[23]). For convenience, a(k-th) rational boundary component is called to be a(k-th) **cusp**(or a cusp of depth k). We note that for any two isotropic rationally-defined subspaces $W_2\subset W_1$ of $V_{\mathbb{R}}$, the set $\{F/W_2 \mid F\in\mathfrak{F}(W_1)\}$ is a cusp of $\mathfrak{S}(W_2^{\perp}/W_2,\psi)$. A k-th cusp of $\mathfrak{H}_g$ always corresponds to a Siegel space of genus $g-k$, in particular $\mathfrak{F}(V^{(k)})\cong\mathfrak{H}_k$.

Remark 1.9 (Cf.(4.15)-(4.16) §5 [26])**.** The following conditions are equivalent for a k-th boundary component $\mathfrak{F}(W)$:

i. $\mathfrak{F}(W)$ is rational;
ii. W is an isotropic rationally-defined subspaces, i.e., $W=W_{\mathbb{Q}}\otimes\mathbb{R}$ where $W_{\mathbb{Q}}$ is an isotropic subspace of $V_{\mathbb{Q}}$(and so $W_{\mathbb{R}}=(W_{\mathbb{R}}\cap V_{\mathbb{Q}})\otimes\mathbb{R}$);
iii. $\exists M\in\mathrm{Sp}(g,\mathbb{Q})$ such that $W=M(V^{(k)})$;
iv. $\exists M\in\mathrm{Sp}(g,\mathbb{Z})$ such that $W=M(V^{(k)})$.

Since the symplectic group $\mathrm{Sp}(V,\psi)$ is simple, we have :

Proposition 1.10 (Baily-Borel cf.[2],[11],[1])**.** *The map $\mathfrak{F}\mapsto N^{\mathfrak{F}}$ is a bijection between the set of proper boundary components of $\mathfrak{S}_g$ to the set of maximal parabolic algebraic subgroups of* $\mathrm{Sp}(V,\psi)$*. Moreover, the boundary component $\mathfrak{F}$ is rational if and only if $N^{\mathfrak{F}}$ is a maximal rational parabolic algebraic subgroup of* $\mathrm{Sp}(V,\psi)$.

For any proper boundary component $\mathfrak{F}$ of $\mathfrak{S}_g$, there is an isotropic subspace $V_{\mathfrak{F}}$ of $(V_{\mathbb{R}},\psi)$ with $\mathfrak{F}(V_{\mathfrak{F}})=\mathfrak{F}$, and so there will be an increasing filtration of $V_{\mathbb{R}}$

$$W^{\mathfrak{F}}_{\bullet}(\mathbb{R})=\big(0\subset W^{\mathfrak{F}}_0\subset W^{\mathfrak{F}}_1\subset W^{\mathfrak{F}}_2\big):=\big(0\subset V_{\mathfrak{F}}\subset (V_{\mathfrak{F}})^{\perp}\subset V_{\mathbb{R}}\big). \tag{1.10.1}$$

This filtration 1.10.1 corresponds to a unique morphism $w'_{\mathfrak{F}}:\mathbb{G}_m\to\mathrm{GSp}(V,\psi)$ defined over $\mathbb{R}$. We also define the cocharacter $w_{\mathfrak{F}}:=w'_{\mathfrak{F}}w_0^{-1}:\mathbb{G}_m\to\mathrm{Sp}(V,\psi)$ where

$$w_0^{-1}:\mathbb{G}_m\xrightarrow{(e,(\cdot)^{-1})}\mathrm{Sp}(V,\psi)\times\mathbb{G}_m\longrightarrow\mathrm{GSp}(V,\psi).$$

We note that $w_{\mathfrak{F}}$ is defined over $\mathbb{R}$, and that $w_{\mathfrak{F}}$ is defined over $\mathbb{Q}$ if and only if the boundary component $\mathfrak{F}$ is rational. We have the following composed homomorphism :

$$\mathbb{G}_m(\mathbb{R})\xrightarrow{w_{\mathfrak{F}}}\mathrm{Sp}(g,\mathbb{R})\xrightarrow{\mathrm{Ad}}\mathrm{GL}(\mathfrak{g})\subset\mathrm{GL}(\mathrm{End}(V_{\mathbb{R}})). \tag{1.10.2}$$

Define $\mathrm{End}(V_{\mathbb{R}})^i:=\{X\in\mathrm{End}(V_{\mathbb{R}})\mid \mathrm{Ad}(w_{\mathfrak{F}}(\lambda))X=\lambda^iX,\forall\lambda\in\mathbb{G}_m(\mathbb{R})\}$ and

$$\mathfrak{g}^i:=\mathrm{End}(V_{\mathbb{R}})^i\cap\mathfrak{g}=\{X\in\mathfrak{g}\mid \mathrm{Ad}(w_{\mathfrak{F}}(\lambda))X=\lambda^iX,\forall\lambda\in\mathbb{R}^{\times}\}. \tag{1.10.3}$$

We then have

$$\begin{array}{rcl}\mathfrak{g} & = & \mathfrak{g}^{-2}\oplus\mathfrak{g}^{-1}\oplus\mathfrak{g}^{-0}\oplus\mathfrak{g}^{1}\oplus\mathfrak{g}^{2} \\ & \subset & \mathrm{End}(V_{\mathbb{R}})^{-2}\oplus\mathrm{End}(V_{\mathbb{R}})^{-1}\oplus\mathrm{End}(V_{\mathbb{R}})^{0}\oplus\mathrm{End}(V_{\mathbb{R}})^{1}\oplus\mathrm{End}(V_{\mathbb{R}})^{2}=\mathrm{End}(V_{\mathbb{R}})\end{array}$$

by 1.4.1 and 1.10.2. Thus the weight morphism $w_{\mathfrak{F}}$ endows an increasing filtration on Lie algebra $\mathfrak{g}$(respectively on $\mathrm{End}(V_{\mathbb{R}})$) $W^{\mathfrak{F}}_{\bullet}(\mathfrak{g}) = (0 \subset W^{\mathfrak{F}}_{-2}(\mathfrak{g}) \subset \cdots \subset W^{\mathfrak{F}}_{2}(\mathfrak{g}) = \mathfrak{g})$ with

$$W^{\mathfrak{F}}_{-2}(\mathfrak{g})=\mathfrak{g}^{-2}, W^{\mathfrak{F}}_{-1}(\mathfrak{g})=\mathfrak{g}^{-2}\oplus\mathfrak{g}^{-1} \text{ and } W^{\mathfrak{F}}_{0}(\mathfrak{g})=\mathfrak{g}^{-2}\oplus\mathfrak{g}^{-1}\oplus\mathfrak{g}^{0}. \tag{1.10.4}$$

Lemma 1.11. *For* $\mathfrak{g}=\mathrm{Lie}(\mathrm{Sp}(g,\mathbb{R}))$,

$$W^{\mathfrak{F}}_{s}(\mathfrak{g})=\{X\in\mathfrak{g}\mid X(W^{\mathfrak{F}}_{l})\subset W^{\mathfrak{F}}_{s+l}\ \forall l\}$$

and

$$[W^{\mathfrak{F}}_{s}(\mathfrak{g}),W^{\mathfrak{F}}_{t}(\mathfrak{g})]\subset W^{\mathfrak{F}}_{s+t}(\mathfrak{g}).$$

Proof. Choose subspaces V^j of $V_{\mathbb{R}}$ such that we can write $W^{\mathfrak{F}}_{i}=\oplus_{j\le i}V^j$, and define $V^j=\{0\}$ if $j\notin[-2,2]$. Similarly as in the proof of the lemma 1.4, we obtain

$$\mathfrak{g}^{i}=\{X\in\mathfrak{g}\mid X(V^{j})\subset V^{j+i}\ \forall j\}\ \text{ for any integer } i\in[-2,2].$$

by the commutative diagram 1.4.1 and the definition of $\mathrm{End}(V_{\mathbb{R}})^i$ show that

Therefore, the first statement is true and the second statement follows it. □

Corollary 1.12 (Cf.[11],[1],[24])**.** *Let* $\mathfrak{F}=\mathfrak{F}(W)$ *be a boundary component of the Siegel space* $\mathfrak{H}_g$. *We have :*

- $W^{\mathfrak{F}}_{0}(\mathfrak{g})$, $W^{\mathfrak{F}}_{-1}(\mathfrak{g})$ *and* $\mathfrak{g}^0$ *are Lie subalgebras of* $\mathfrak{g}=\mathrm{Lie}(\mathrm{Sp}(g,\mathbb{R}))$;
- $W^{\mathfrak{F}}_{-2}(\mathfrak{g})$ *and* $\mathfrak{g}^2$ *are commutative Lie subalgebras of* $\mathfrak{g}$.

Let $\big(W_{-1}\subset W_0\subset W_1\subset W_2\big):=\big(0\subset W\subset W^{\perp}\subset V_{\mathbb{R}}\big)$ *be the filtration corresponding to* $\mathfrak{F}$. *For each integer i in* $[-2,0]$, *we define* $W^{\mathfrak{F}}_{i}(\mathrm{Sp}(V,\psi))$ *to be the algebraic subgroup of* $\mathrm{Sp}(V,\psi)$ *of elements acting as the identity map on* $\bigoplus_p W_p/W_{p-i}$. *We have :*

- $W^{\mathfrak{F}}_{0}(\mathrm{Sp}(V,\psi))=N^{\mathfrak{F}}$ *is a parabolic subgroup of* $\mathrm{Sp}(V,\psi)$ *with Lie algebra* $W^{\mathfrak{F}}_{0}(\mathfrak{g})$;
- $\mathcal{W}^{\mathfrak{F}}:=W^{\mathfrak{F}}_{-1}(\mathrm{Sp}(V,\psi))$ *has Lie algebra* $W^{\mathfrak{F}}_{-1}(\mathfrak{g})$, *and it is the unipotent radical of* $N^{\mathfrak{F}}$;
- $U^{\mathfrak{F}}:=W^{\mathfrak{F}}_{-2}(\mathrm{Sp}(V,\psi))$ *is the center of* $\mathcal{W}^{\mathfrak{F}}$, *its Lie algebra* $W^{\mathfrak{F}}_{-2}(\mathfrak{g})=\mathfrak{g}^{-2}$ *is commutative;*
- $Z^{\mathfrak{F}}:=$ *the centralizer of the morphism* $w_{\mathfrak{F}}$ *in* $N^{\mathfrak{F}}$, *it has Lie algebra* $\mathfrak{g}^0$;
- $V^{\mathfrak{F}}:=\mathcal{W}^{\mathfrak{F}}/U^{\mathfrak{F}}$ *is an Abelian group whose Lie algebra identifies with the space* $\mathfrak{g}^{-1}$.

All above algebraic subgroups will be defined over $\mathbb{Q}$ *if* $\mathfrak{F}$ *is rational. Similarly, we can define Lie subgroups* $W^{\mathfrak{F}}_{-2}(G), W^{\mathfrak{F}}_{-1}(G), W^{\mathfrak{F}}_{0}(G)$ *of* $G=\mathrm{Sp}(g,\mathbb{R})$.

1.3. **Polarized Mixed Hodge structures attached to cusps.** For an isotropic real subspace $L_{\mathbb{R}}$ of $(V_{\mathbb{R}}, \psi)$(resp. rational subspace $L_{\mathbb{Q}}$ of $(V_{\mathbb{Q}}, \psi)$), let

$$L_{\mathbb{R}}^{\perp} := \{v \in V_{\mathbb{R}} \mid \psi(v, L_{\mathbb{R}}) = 0\}(\text{resp. } L_{\mathbb{Q}}^{\perp} := \{v \in V_{\mathbb{Q}} \mid \psi(v, L_{\mathbb{Q}})\}\ ),$$

let $L_{\mathbb{R}}^{\vee}$ be the dual space of $L_{\mathbb{R}}$ in $V_{\mathbb{R}}$ with respect to $(V_{\mathbb{R}}, \psi)$(resp. $L_{\mathbb{Q}}^{\vee}$ the dual space of $L_{\mathbb{Q}}$ in $V_{\mathbb{Q}}$ with respect to $(V_{\mathbb{Q}}, \psi)$). For any $N \in \mathfrak{g}$, there is a symmetric bilinear form $\psi_N : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ given by

$$\psi_N(v, u) := \psi(v, N(u)). \tag{1.12.1}$$

Lemma 1.13. *Let $\mathfrak{F} = \mathfrak{F}(W)$ be a cusp of $\mathfrak{H}_g$ and let N be an arbitrary nonzero element in $W_{-2}^{\mathfrak{F}}(\mathfrak{g}) = \mathrm{Lie}(U^{\mathfrak{F}}(\mathbb{R}))$. Let $W_{\bullet}^{\mathfrak{F}}(\mathbb{R}) = \big(0 \subset W_{\mathbb{R},0}^{\mathfrak{F}} \subset W_{\mathbb{R},1}^{\mathfrak{F}} \subset W_{\mathbb{R},2}^{\mathfrak{F}}\big) := \big(0 \subset W \subset W^{\perp} \subset V_{\mathbb{R}}\big)$ be the weight filtration associated to the cusp $\mathfrak{F}$.*

1. *The inclusions $\mathrm{Im}(N) \subset W$ and $W^{\perp} \subset \mathrm{Ker}(N)$ are held. The element N induces a weight filtration $W_{\bullet}(N) = (0 \subset W_{-1}(N) \subset W_0(N) \subset W_1(N)$ by setting $W_1(N) := V_{\mathbb{R}}, W_{-1}(N) := \mathrm{Im}(N : V_{\mathbb{R}} \to V_{\mathbb{R}})$ and $W_0(N) := \mathrm{Ker}(N : V_{\mathbb{R}} \to V_{\mathbb{R}})$.*
2. *For any two $N_1, N_2 \in W_{-2}^{\mathfrak{F}}(\mathfrak{g})$, $N_1N_2 = N_2N_1 = 0$.*
3. *The ψ_N can be regarded as a symmetric bilinear form on $V_{\mathbb{R}}/W^{\perp}$. If ψ_N is non-degenerate on $V_{\mathbb{R}}/W^{\perp}$ then $W_{\bullet}^{\mathfrak{F}}(\mathbb{R}) = (W(N)[-1])_{\bullet}$, where $(W(N)[-1])_{\bullet}$ is the weight filtration given by $(W(N)[-1])_j := W_{j-1}(N)\ \forall j$.*

Proof. 1. The lemma 1.11 shows that $\mathrm{Im}(N) \subset W$ as $N \in W_{-2}^{\mathfrak{F}}(\mathfrak{g})$. It is easy to obtain that $\mathrm{Im}(N) \subset W \Longleftrightarrow W^{\perp} \subset \mathrm{Ker}(N)$.
2. It is obvious by that $\mathrm{Im}(N_2) \subset W \subset W^{\perp} \subset \mathrm{Ker}(N_1)$.
3. If there is a vector $0 \neq v \in W_0(N) \setminus W^{\perp}$ then $N(v) = 0$, and so $\psi(v, N(v)) = 0$. Thus, we must have $W^{\perp} = W_0(N) = \mathrm{Ker}(N : V_{\mathbb{R}} \to V_{\mathbb{R}})$.
Claim: $\mathrm{Im}(N) = W \Longleftrightarrow W^{\perp} = \mathrm{Ker}(N)$.
Proof of the claim.
   - "$\Rightarrow$" : Suppose $\mathrm{Im}(N) = W$. Then, $\psi(y, W) = \psi(y, \mathrm{Im}(N)) = -\psi(N(y), V_{\mathbb{R}}) = 0$ for any $y \in \mathrm{Ker}(N)$. Thus $\mathrm{Ker}(N) \subset W^{\perp}$.
   - "$\Leftarrow$" : Suppose $W^{\perp} = \mathrm{Ker}(N)$. Since $N : \frac{V_{\mathbb{R}}}{\mathrm{Ker}(N)} \xrightarrow{\cong} \mathrm{Im}(N)$, we get

$$\dim_{\mathbb{R}} \mathrm{Im}(N) = \dim_{\mathbb{R}} V_{\mathbb{R}} - \dim_{\mathbb{R}} \mathrm{Ker}(N) = \dim_{\mathbb{R}} V_{\mathbb{R}} - \dim_{\mathbb{R}} W^{\perp} = \dim_{\mathbb{R}} W.$$

□

Given a rational boundary component $\mathfrak{F}(W)$, we have a convex cone in $\mathfrak{g}$ which does not contain any linear subspace :

$$C(\mathfrak{F}(W)) := \{N \in W_{-2}^{\mathfrak{F}(W)}(\mathfrak{g}) \mid \psi_N > 0 \text{ on } \frac{V_{\mathbb{R}}}{W^{\perp}}\} \tag{1.13.1}$$

where ψ_N is a symmetric bilinear form on $V_{\mathbb{R}}$ defined in 1.12.1(cf.[1],[4]), we always call $C(\mathfrak{F}(W))$ positive cone in $W_{-2}^{\mathfrak{F}(W)}(\mathfrak{g})$. With respect to the cusp $\mathfrak{F}_k := \mathfrak{F}(V^{(k)})$, the positive cone is

$$C(\mathfrak{F}_k) = \{\begin{pmatrix} 0_k & 0 & 0 & 0 \\ 0 & 0_{g-k} & 0 & u \\ 0 & 0 & 0_k & 0 \\ 0 & 0 & 0 & 0_{g-k} \end{pmatrix} \mid u \in \mathrm{Sym}^+_{g-k}(\mathbb{R})\},$$

where $\mathrm{Sym}^+_{g-k}(\mathbb{R})$ is defined to be the set of positive-definite matrices in $\mathrm{Sym}_{g-k}(\mathbb{R})$.

Corollary 1.14. *Let $\mathfrak{F} = \mathfrak{F}(W)$ be a cusp of $\mathfrak{H}_g$ and N an arbitrary nonzero elements in $C(\mathfrak{F})$. We have $N^2 = 0$ and*

$$W(N)[-1])_\bullet = W^{\mathfrak{F}}_\bullet(\mathbb{R}) = \big(0 \subset W^{\mathfrak{F}}_{\mathbb{R},0}(= W) \subset W^{\mathfrak{F}}_{\mathbb{R},1}(= W^\perp) \subset W^{\mathfrak{F}}_{2,\mathbb{R}}(= V_{\mathbb{R}})\big).$$

Proof. It is obvious by the lemma 1.13.

□

For any isotropic rationally-defined subspace W of $V_{\mathbb{R}}$, the dual space $W^\vee$ of W is also an isotropic rationally-defined subspace of $V_{\mathbb{R}}$ satisfying $\dim_{\mathbb{R}} W^\vee = \dim_{\mathbb{R}} W$. For any cusp $\mathfrak{F} = \mathfrak{F}(W)$ of the Siegel space $\mathfrak{H}_g$,

$$\mathfrak{F}^\vee := \mathfrak{F}(W^\vee). \tag{1.14.1}$$

is defined to be the **dual cusp** of $\mathfrak{F}$.

We observe that the space $\check{\mathfrak{S}}_g$ can be identified with a set of Hodge filtrations

$$\{F^\bullet = (0 \subset F^1 \subset V_{\mathbb{C}}) \mid F^1 \in \mathrm{Grass}(g, V_{\mathbb{C}}),\ \psi(F^1, F^1) = 0\}$$

and any cusp $\mathfrak{F}$ can also be identified with a set of certain Hodge filtrations.

Lemma 1.15. *Let $\mathfrak{F} = \mathfrak{F}(W)$ be a cusp of the Siegel space $\mathfrak{H}_g$ and $W^{\mathfrak{F}}_\bullet = (0 \subset W_{\mathbb{Q}} \subset W^\perp_{\mathbb{Q}} \subset V_{\mathbb{Q}})$ the corresponding weight filtration on $V_{\mathbb{Q}}$ where $W_{\mathbb{Q}}$ is an isotropic subspace of $V_{\mathbb{Q}}$ given by $W = W_{\mathbb{Q}} \otimes \mathbb{R}$. Let $N \in C(\mathfrak{F})$ be an element in the positive cone $C(\mathfrak{F}) \subset W^{\mathfrak{F}}_{-2}(\mathfrak{g})$ and $F^\bullet = (0 \subset F^1 \subset V_{\mathbb{C}})$ a filtration in $\mathfrak{F}$.*

1. There are following direct sum decompositions

$$\begin{aligned} W^\perp_{\mathbb{Q}} &= W_{\mathbb{Q}} \bigoplus ((W^\vee_{\mathbb{Q}})^\perp \cap W^\perp_{\mathbb{Q}}), \\ (W^\vee_{\mathbb{Q}})^\perp &= W^\vee_{\mathbb{Q}} \bigoplus ((W^\vee_{\mathbb{Q}})^\perp \cap W^\perp_{\mathbb{Q}}), \\ F^1 &= W_{\mathbb{C}} \bigoplus ((W^\vee_{\mathbb{C}})^\perp \cap W^\perp_{\mathbb{C}} \cap F^1). \end{aligned}$$

Each above decomposition is orthogonal under the form ψ.

Define

$$\check{F}^1 := W^\vee_{\mathbb{C}} \bigoplus ((W^\vee_{\mathbb{C}})^\perp \cap W^\perp_{\mathbb{C}} \cap F^1).$$

The dual filtration $\check{F}^\bullet := (0 \subset \check{F}^1 \subset V_{\mathbb{C}})$ *is in* $\mathfrak{F}^\vee$*, and there is a bijection*

$$\mathfrak{F} \longrightarrow \mathfrak{F}^\vee \;\; F^\bullet \longmapsto \check{F}^\bullet.$$

2. *That* $N(V_{\mathbb{R}}) = N(W_{\mathbb{R}}^\vee) = W_{\mathbb{R}}$*,* $N(\check{F}^1) = W_{\mathbb{C}}$ *and* $N(F^1) = 0$*. Also, there is an isomorphism* $N|_{W_{\mathbb{R}}^\vee} : W_{\mathbb{R}}^\vee \xrightarrow{\cong} W_{\mathbb{R}}$*.*
3. *There holds* $\exp(\sqrt{-1}N)\check{F}^1 \in \mathfrak{S}_g$*.*

Proof. There exists a $M \in \mathrm{Sp}(g, V_{\mathbb{Q}})$ such that $\mathfrak{F}_k := \mathfrak{F}(V^{(k)}) = M(\mathfrak{F})$ for some k, we then have $w_{\mathfrak{F}_k} = M w_{\mathfrak{F}} M^{-1}$ and

$$W_\bullet^{\mathfrak{F}_k} = M W_\bullet^{\mathfrak{F}} M^{-1},\; W_\bullet^{\mathfrak{F}_k}(\mathfrak{g}) = M W_\bullet^{\mathfrak{F}}(\mathfrak{g}) M^{-1} \text{ and } C(\mathfrak{F}_k) = M C(\mathfrak{F}) M^{-1}.$$

In particular, we have $(V^{(k)_{\mathbb{Q}}})^\vee = M(W_{\mathbb{Q}}^\vee)$ and $((V_{\mathbb{Q}}^{(k)})^\vee)^\perp = M((W_{\mathbb{Q}}^\vee)^\perp)$.

Thus, it is sufficient to prove the statements in the case of $\mathfrak{F} = \mathfrak{F}_k$. Now, we obtain :

$$(V_{\mathbb{Q}}^{(k)})^\vee = \mathrm{span}_{\mathbb{Q}}\{e_{g+k+1}, \cdots, e_{2g}\}, (V_{\mathbb{Q}}^{(k)})^\perp = \mathrm{spac}_{\mathbb{Q}}\{e_1, \cdots, e_g, e_{g+1}, \cdots, e_{g+k}\}$$

and $((V_{\mathbb{Q}}^{(k)})^\vee)^\perp \cap (V_{\mathbb{Q}}^{(k)})^\perp = \mathrm{span}_{\mathbb{Q}}\{e_1, \cdots, e_k, e_{g+1}, \cdots, e_{g+k}\}$.

1. Obviously, there is a direct sum decomposition

$$(V_{\mathbb{Q}}^{(k)})^\perp = V_{\mathbb{Q}}^{(k)} \oplus (((V_{\mathbb{Q}}^{(k)})^\vee)^\perp \cap (V_{\mathbb{Q}}^{(k)})^\perp).$$

   This decomposition is orthogonal with respect to the form ψ. By duality, we also get the second equality in the statement (1). Since $\psi(F^1, F^1) = 0$, we have $F^1 \subset W_{\mathbb{C}}^\perp$. By the first equality, any $f \in F^1$ can be written as $f = v_1 + v_2$ where $v_1 \in W_{\mathbb{C}}$ and $v_2 \in (W_{\mathbb{C}}^\vee)^\perp \cap W_{\mathbb{C}}^\perp$. Due to $W_{\mathbb{C}} \subset F^1$, we have $v_2 = f - v_1 \in F^1$ and so $v_2 \in (W_{\mathbb{C}}^\vee)^\perp \cap W_{\mathbb{C}}^\perp \cap F^1$. Thus, the third equality in the statement (1) is true.
2. Since

$$N = \begin{pmatrix} 0_k & 0 & 0 & 0 \\ 0 & 0_{g-k} & 0 & B \\ 0 & 0 & 0_k & 0 \\ 0 & 0 & 0 & 0_{g-k} \end{pmatrix}$$

   where B is a positive-definite matrix in $\mathrm{Sym}_k(\mathbb{R})$, We must have

$$N(W_{\mathbb{R}}^\vee) = W_{\mathbb{R}} \;\text{ and }\; N(W_{\mathbb{R}}^\perp) = 0.$$

   The other equalities in the statement (2) are obvious.
3. Let f be a an arbitrary nontrivial vector in $\check{F}^1$. By the statement (1), we can write

$$f = w_1 + w_2 \text{ where } w_1 \in W_{\mathbb{C}}^\vee \text{ and } w_2 \in (W_{\mathbb{C}}^\vee)^\perp \cap W_{\mathbb{C}}^\perp \cap F^1.$$

Then we have :

$$\begin{aligned}
\sqrt{-1}\psi(\exp(\sqrt{-1}N)f, \overline{\exp(\sqrt{-1}N)f})) &= \sqrt{-1}\psi(\exp(\sqrt{-1}N)f, \exp(-\sqrt{-1}tN)\overline{f}) \\
&= \sqrt{-1}\psi(f, \exp(-2\sqrt{-1}N)\overline{f}) \\
&= \sqrt{-1}\psi(f, \overline{f}) + 2\psi(f, N\overline{f}) \\
&= \sqrt{-1}\psi(f, \overline{f}) + 2\psi(w_1, N\overline{w_1}) + 2\psi(w_2, N\overline{w_1}) \\
&= \sqrt{-1}\psi(f, \overline{f}) + 2\psi(w_1, N\overline{w_1}) + 2\psi(\overline{w_1}, Nw_2) \\
&= \sqrt{-1}\psi(w_2, \overline{w_2}) + 2\psi(w_1, B\overline{w_1}).
\end{aligned}$$

We get $\sqrt{-1}\psi(w_2, \overline{w_2}) \geq 0$, and obtain that $\sqrt{-1}\psi(w_2, \overline{w_2}) = 0$ if and only if $w_2 = 0$. Also, we get $\psi(w_1, B\overline{w_1}) \geq 0$ by $N \in C(\mathfrak{F})$, and obtain that $\psi(w_1, B\overline{w_1}) = 0$ if and only if $w_1 = 0$. Therefore $\sqrt{-1}\psi(\exp(\sqrt{-1}N)f, \overline{\exp(\sqrt{-1}N)f})) > 0$ and so $\sqrt{-1}\psi(\cdot, \bar{\cdot})$ is Hermitian positive on $\exp(\sqrt{-1}N)\check{F}^1$. Now we can finish the proof of the statement (3) by the fact that a point $F \in \overline{\mathfrak{S}}_g$ is in $\mathfrak{S}_g$ if and only if $\sqrt{-1}\psi(\cdot, \bar{\cdot})$ is Hermitian positive on F.

□

Some calculations in the lemma 1.15 are taken from [4] and [13] with minor modification. In [4], Carlson, Cattani and Kaplan originally use Hodge theory to construct toroidal compactifications of Siegel varieties. In the following theorem 1.17, we further observe that the Hodge-theoretic interpretation of boundary components naturally coincides with the classic description given by Satake-Baily-Borel in [30] and [2]: Any cusp of Siegel space $\mathfrak{H}_g$ can be identified as a set of certain weight one polarized mixed Hodge structures.

A **mixed Hodge structure**(MHS) on $V_{\mathbb{Q}}$(resp.$V_{\mathbb{R}}$) consists of two filtrations, an increasing filtration on $V_{\mathbb{Q}}$(resp. $V_{\mathbb{R}}$)-the weight filtration $W_\bullet$, and a decreasing filtration $F^\bullet$ on $V_{\mathbb{C}}$-the Hodge filtration, such that the filtration $F^\bullet$ induces a Hodge structure on each $\mathrm{Gr}_r^W V_{\mathbb{Q}} := W_r/W_{r-1}$ (resp. $\mathrm{Gr}_r^W V_{\mathbb{R}} := W_r/W_{r-1}$) of pure weight r, where $F^p(\mathrm{Gr}_r^W V_{\mathbb{C}}) := \frac{F^p\cap(W_r\otimes\mathbb{C})}{F^p\cap(W_{r-1}\otimes\mathbb{C})}$ $\forall p$(cf.[10],[28]).

Definition 1.16 (Cf.[5]&[6])**.** A **polarized mixed Hodge structure**(PMHS) of weight l on $V_{\mathbb{R}}$ consists of a MHS $(W_\bullet, F^\bullet)$ on $V_{\mathbb{R}}$, a nilpotent element $N \in F^{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{R}}$ and a non-degenerate bilinear form Q such that

- $N^{l+1} = 0$, and $W_\bullet = (W(N)[-l])_\bullet$ where $W(N)[-l]_j := W_{j-l}(N)$ $\forall j$;
- $Q(F^p, F^{l-p+1}) = 0$ $\forall p$;
- $N(F^p) \subset F^{p-1}$ $\forall p$;
- the weight $l+r$ Hodge structure induced by $F^\bullet$ on $P_{l+r} := \ker(N^{r+1} : \mathrm{Gr}^{W_\bullet}_{l+r} \to \mathrm{Gr}^{W_\bullet}_{l-r-2})$ is polarized by $Q(\cdot, N^r(\cdot))$, i.e., that $Q(\cdot, N^r(\cdot))$ is $(-1)^{l+r}$-symmetric on

P_{l+r} and

$$\begin{aligned} Q(P^{p_1,q_1}_{l+r,\mathbb{C}}, N^r(P^{p_2,q_2}_{l+r,\mathbb{C}})) &= 0 \quad \text{unless } p_1 = q_2 \text{ and } p_2 = q_1 \\ (\sqrt{-1})^{p-q} Q(v, N^r(\overline{v})) &> 0 \quad \text{for any nonzero } v \in P^{p,q}_{l+r,\mathbb{C}}. \end{aligned}$$

Theorem 1.17. *Let $\mathfrak{F}$ be a rational component of Siegel space $\mathfrak{H}_g$ and*

$$W^{\mathfrak{F}}_\bullet = \big(0 \subset W^{\mathfrak{F}}_0 \subset W^{\mathfrak{F}}_1 \subset W^{\mathfrak{F}}_2 (= V_\mathbb{Q})\big)$$

the corresponding weight filtration on the rational space $V_\mathbb{Q}$. Let ψ be the standard symplectic form on $V_\mathbb{R} = V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{R}$. We have :

1. *For any Hodge filtration $F^\bullet \in \mathfrak{F}^\vee$ and any element $N \in C(\mathfrak{F})$, the quadruple $(F^\bullet_\tau, W^{\mathfrak{F}}_\bullet, N, \psi)$ determines a polarized mixed Hodge structure of weight one on $V_\mathbb{R}$.*
2. *Any pair $(F^\bullet_\tau, W^{\mathfrak{F}}_\bullet)$ with $F^\bullet_\tau \in \mathfrak{S}_g$ is a mixed Hodge structure of weight one on $V_\mathbb{Q}$.*
3. *Moreover,*

$$\mathfrak{F}^\vee = \{F^\bullet \in \check{\mathfrak{S}}_g \mid (F^\bullet, W^{\mathfrak{F}}_\bullet, N, \psi) \text{ is a PMHS of weight one for all } N \in C(\mathfrak{F})\}.$$

Proof. Let $G = \mathrm{Sp}(g, \mathbb{R})$.

1. By the corollary 1.14, we get that

$$(W(N)[-1])_\bullet = (W^{\mathfrak{F}} \otimes \mathbb{R})_\bullet := \big(0 \subset W^{\mathfrak{F}}_{0,\mathbb{R}} \subset W^{\mathfrak{F}}_{1,\mathbb{R}} \subset W^{\mathfrak{F}}_{2,\mathbb{R}} (= V_\mathbb{R})\big) \ \forall N \in C(\mathfrak{F}).$$

Let $F^\bullet = (0 \subset F^1 \subset F^0 = V_\mathbb{C})$ be a filtration $\mathfrak{F}^\vee$. By definition, we have that

$$F^1 \cap \overline{F^1} = \mathrm{Im}(N)^\vee_\mathbb{C} = (W^{\mathfrak{F}}_0)^\vee \otimes \mathbb{C} \ \forall N \in C(\mathfrak{F}).$$

Since $\psi(F^1, F^1) = \psi(\overline{F^1}, \overline{F^1}) = 0$, we have that

$$F^1 \subset (\mathrm{Im}(N)^\vee_\mathbb{C})^\perp \text{ and } \overline{F^1} \subset (\mathrm{Im}(N)^\vee_\mathbb{C})^\perp.$$

Claim i. For any integer r in $[0, 2]$, we have

$$F^p(\mathrm{Gr}^{W^{\mathfrak{F}}}_r V_\mathbb{C}) \oplus \overline{F^{r-p+1}(\mathrm{Gr}^{W^{\mathfrak{F}}}_r V_\mathbb{C})} \xrightarrow{\cong} \mathrm{Gr}^{W^{\mathfrak{F}}}_r V_\mathbb{C} \ \forall p.$$

The claim i is easy to check. Then, we obtain the decreasing filtration $F^\bullet(\mathrm{Gr}^{W^{\mathfrak{F}}}_r V_\mathbb{C})$ on $\mathrm{Gr}^{W^{\mathfrak{F}}}_r V_\mathbb{Q}$ arises as a Hodge structure on $\mathrm{Gr}^{W^{\mathfrak{F}}}_r V_\mathbb{Q}$ of pure weight r for $r = 0, 1, 2$(cf.[16]&[28]), i.e., the pair $(F^\bullet, W^{\mathfrak{F}}_\bullet)$ is a mixed Hodge structure on $V_\mathbb{Q}$. The polarization is automatically true by the definition of $C(\mathfrak{F})$(cf.1.13.1).

2. Let $\check{F}^\bullet = (0 \subset \check{F}^1 \subset \check{F}^0 (= V_\mathbb{C}))$ be a fixed decreasing filtration in $\mathfrak{F}^\vee$. The lemma 1.15 shows that

$$F^1_\delta := \exp(\sqrt{-1}N)\check{F}^1 \in \mathfrak{S}_g.$$

Write $F^\bullet_\delta := (0 \subset F^1_\delta \subset F^0_\delta (= V_\mathbb{C}))$. We begin to show that the pair $(F^\bullet_\delta, W^{\mathfrak{F}}_\bullet)$ is a mixed Hodge structure on $V_\mathbb{Q}$ with same Hodge numbers as the MHS $(\check{F}^\bullet, W^{\mathfrak{F}}_\bullet)$.

Since $F_\delta^p \oplus \overline{F_\delta^{2-p}} = V_{\mathbb{C}}\, \forall p$, we get $F_\delta^p \bigcap \overline{F_\delta^{2+k-p}} \subset F_\delta^p \bigcap \overline{F_\delta^{2-p}} = \{0\}\, \forall k \geq 0\ \forall p$ and so

$$F_\delta^p \mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}} \bigcap \overline{F_\delta^{1+a-p} \mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}}} = \{0\}\ \forall a \geq 1\ \forall p. \tag{1.17.1}$$

Let $M := \exp(\sqrt{-1}N)$. Since $M \in W_{-2}^{\mathfrak{F}}(G)$ is unipotent, M respects the weight filtration $(W^{\mathfrak{F}} \otimes \mathbb{C})_\bullet$ and acts as identity on $\mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}}$ for all a. Moreover, M induces a complex linear isomorphisms $M : \check{F}^p \mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}} \xrightarrow{\cong} F_\delta^p \mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}}\, \forall a, p$. Since

$$F_\delta^1 \mathrm{Gr}_0^{W^{\mathfrak{F}}} V_{\mathbb{C}} \cong \check{F}^1 \mathrm{Gr}_0^{W^{\mathfrak{F}}} V_{\mathbb{C}} = \{0\} \text{ and } F_\delta^0 \mathrm{Gr}_0^{W^{\mathfrak{F}}} V_{\mathbb{C}} \cong \check{F}^0 \mathrm{Gr}_0^{W^{\mathfrak{F}}} V_{\mathbb{C}} \cong \mathrm{Gr}_0^{W^{\mathfrak{F}}} V_{\mathbb{C}},$$

by the above 1.17.1 we have

$$F_\delta^p \mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}} \bigoplus \overline{F_\delta^{1+a-p} \mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}}} \cong \mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}}\ \forall a, p.$$

The following claim ii guarantees the statement 2 is true.

Claim ii. Let τ be an arbitrary point in $\mathfrak{H}_g$. If $(F_\tau^\bullet, W_\bullet^{\mathfrak{F}})$ is a mixed Hodge structure on $V_{\mathbb{Q}}$ then for any $\tau' \in \mathfrak{H}_g$, $(F_{\tau'}^\bullet, W_\bullet^{\mathfrak{F}})$ is again a MHS on $V_{\mathbb{Q}}$ and there is an isomorphism of MHSs $(F_\tau^\bullet, W_\bullet^{\mathfrak{F}}) \xrightarrow{\cong} (F_{\tau'}^\bullet, W_\bullet^{\mathfrak{F}})$.

Proof of the claim ii. It is known that G acts transitively on $\mathfrak{S}_g$, and

$$\mathfrak{S}_g \cong G/K_\tau, \quad \text{where } K_\tau = \{M \in G \mid MF_\tau^\bullet = F_\tau^\bullet\}.$$

There always holds $G = W_0^{\mathfrak{F}}(G)K_\tau$(cf.Page 242(5.24) in [28]). Thus the group $W_0^{\mathfrak{F}}(G)$ acts transitively on $\mathfrak{S}_g$ and there is a $M \in W_0^{\mathfrak{F}}(G)$ such that

$$\tau' = M(\tau) \text{ and } F_{\tau'}^\bullet = MF_\tau^\bullet,\ \overline{F_{\tau'}^\bullet} = \overline{MF_\tau^\bullet} = M\overline{F_\tau^\bullet}.$$

Since M respects $(W^{\mathfrak{F}} \otimes \mathbb{C})_\bullet$, we have isomorphisms

$$\begin{aligned} M &: F_\tau^\bullet \mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}} \xrightarrow{\cong} F_{\tau'}^\bullet \mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}},\ \forall a \\ M &: \overline{F_\tau^\bullet \mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}}} \xrightarrow{\cong} \overline{F_{\tau'}^\bullet \mathrm{Gr}_a^{W^{\mathfrak{F}}} V_{\mathbb{C}}}\ \forall a. \end{aligned}$$

Thus, $(F_{\tau'}^\bullet, W_\bullet^{\mathfrak{F}})$ is a MHS on $V_{\mathbb{Q}}$ and M induces an isomorphism of MHSs

$$M : (F_\tau^\bullet, W_\bullet^{\mathfrak{F}}) \xrightarrow{\cong} (F_{\tau'}^\bullet, W_\bullet^{\mathfrak{F}}).$$

3. Let $F^\bullet = (0 \subset F^1 \subset F^2(= V_{\mathbb{C}}))$ be a Hodge filtration in $\check{\mathfrak{S}}_g$ and $N \in C(\mathfrak{F})$.

Claim iii. Suppose that $(F^\bullet, W_\bullet^{\mathfrak{F}}, N, \psi)$ is a PMHS, then $F^\bullet$ is in $\mathfrak{F}^\vee$.

After the claim iii, the statement 3 will be true by using the corollary 1.14 and the statement 1 in this theorem : We have the fact that if $(F^\bullet, W_\bullet^{\mathfrak{F}}, N_1, \psi)$ is a PMHS for one certain $N_1 \in C(\mathfrak{F})$ then $(F^\bullet, W_\bullet^{\mathfrak{F}}, N, \psi)$ is again a PMHS for any $N \in C(\mathfrak{F})$. We now begin to prove the claim iii by the following steps (A) to (C).

(A): From the MHS $(F^\bullet, W_\bullet^{\mathfrak{F}})$, we obtain the following facts :

- $(\mathrm{Gr}_0^{W^{\mathfrak{F}}} V_{\mathbb{Q}}, F^\bullet \mathrm{Gr}_0^{W^{\mathfrak{F}}} V_{\mathbb{C}})$ is a Hodge structure of pure weight zero, and so
$$F^1 \cap \mathrm{Im}(N)_{\mathbb{C}} = \overline{F^1} \cap \mathrm{Im}(N)_{\mathbb{C}} = \{0\}.$$
- $(\mathrm{Gr}_1^{W^{\mathfrak{F}}} V_{\mathbb{Q}}, F^\bullet \mathrm{Gr}_1^{W^{\mathfrak{F}}} V_{\mathbb{C}})$ is a Hodge structure of pure weight one. By definition,
$$F^1 \mathrm{Gr}_1^{W^{\mathfrak{F}}} V_{\mathbb{C}}) \cong F^1 \cap \mathrm{Ker}(N)_{\mathbb{C}}.$$
- $(\mathrm{Gr}_2^{W^{\mathfrak{F}}} V_{\mathbb{Q}}, F^\bullet \mathrm{Gr}_2^{W^{\mathfrak{F}}} V_{\mathbb{C}})$ has a rational Hodge structure of pure type $(1,1)$.Thus,
$$F^1 \mathrm{Gr}_2^{W^{\mathfrak{F}}} V_{\mathbb{C}} = \overline{F^1 \mathrm{Gr}_2^{W^{\mathfrak{F}}} V_{\mathbb{C}}} = \mathrm{Gr}_2^{W^{\mathfrak{F}}} V_{\mathbb{C}},$$
and so $N(F^1) = N(\overline{F^1}) = \mathrm{Im}(N)_{\mathbb{C}}$, $\ker(N)_{\mathbb{C}} + F^1 = \mathrm{Ker}(N) + \overline{F^1} = V_{\mathbb{C}}$.

(B): We show that $F^1 \cap \overline{F^1} = \mathrm{Im}(N)_{\mathbb{C}}^\vee = (W_0^{\mathfrak{F}} \otimes \mathbb{C})^\vee$.

- $\mathrm{Im}(N)_{\mathbb{C}}^\vee \subseteq F^1 \cap \overline{F^1}$: Since $(\mathrm{Gr}_2^{W^{\mathfrak{F}}} V_{\mathbb{Q}}, F^\bullet \mathrm{Gr}_2^{W^{\mathfrak{F}}} V_{\mathbb{C}})$ is polarized $\psi(\cdot, N(\cdot))$, we have that $\mathrm{Im}(N)_{\mathbb{C}}^\vee = (N(\overline{F^1}))^\vee \subset F^1$ and $\mathrm{Im}(N)_{\mathbb{C}}^\vee = (N(F^1))^\vee \subset \overline{F^1}$. Then
$$\mathrm{Im}(N)_{\mathbb{C}}^\vee \subseteq F^1 \cap \overline{F^1} \text{ and } F^1 \subset (\mathrm{Im}(N)_{\mathbb{C}}^\vee)^\perp.$$
- $F^1 \cap \overline{F^1} \subseteq \mathrm{Im}(N)_{\mathbb{C}}^\vee$: Let v be an arbitrary vector in $E := F^1 \cap \overline{F^1}$. By the second equality in(1) of the lemma 1.15, v can be written uniquely as
$$v = v_1 + v_2, \text{ where } v_1 \in \mathrm{Im}(N)_{\mathbb{C}}^\vee, v_2 \in \mathrm{Ker}(N)_{\mathbb{C}}.$$
As $\mathrm{Im}(N)_{\mathbb{C}}^\vee \subset E$, we have $v_2 = v - v_1 \in E$ and so $v_2 \in \mathrm{Ker}(N)_{\mathbb{C}} \cap E$. Since the weight one Hodge structure $(\mathrm{Gr}_1^{W^{\mathfrak{F}}} V_{\mathbb{Q}}, F^\bullet \mathrm{Gr}_1^{W^{\mathfrak{F}}} V_{\mathbb{C}})$ is polarized by the form ψ, we must have that $\sqrt{-1}\psi(F^1 \cap \mathrm{Ker}(N)_{\mathbb{C}}, \overline{F^1 \cap \mathrm{Ker}(N)_{\mathbb{C}}}) > 0$. On the other hand, the $\sqrt{-1}\psi(v_2, \overline{v_2})$ is zero by that $\psi(E, \overline{E}) = \psi(E, E) = 0$. Thus $v_2 = 0$ and $v \in F^1 \cap \overline{F^1}$.

(C): Due to $\mathrm{Im}(N)_{\mathbb{C}}^\vee \cap \mathrm{Ker}(N)_{\mathbb{C}} = \{0\}$, we get that
$$V_{\mathbb{C}} = \mathrm{Im}(N)_{\mathbb{C}}^\vee \oplus \mathrm{Ker}(N)_{\mathbb{C}}, \quad F^1 = \mathrm{Im}(N)_{\mathbb{C}}^\vee \bigoplus F^1 \cap \mathrm{Ker}(N)_{\mathbb{C}}.$$
Since that $\psi(F^1, F^1) = \psi(\overline{F^1}, \overline{F^1}) = 0$ and the form $\sqrt{-1}\psi(\cdot, \bar{\cdot})$ is positive on the space $F^1 \cap \mathrm{Ker}(N)_{\mathbb{C}}$, we have that $\sqrt{-1}\psi(F^1, \overline{F^1}) \geq 0$, which is equivalent to that $F^\bullet \in \overline{\mathfrak{S}}_g$.

□

2. Toroidal compactifications and their infinity boundary divisors

Let $\Gamma \subset \mathrm{Sp}(g, \mathbb{Z})$ be an arithmetic subgroup.Denote by $\mathfrak{F}_k$ the cusp $\mathfrak{F}(V^{(k)})$, which is a cusp of depth $g-k$ isomorphic to $\mathfrak{H}_k$.

According to the corollary 1.12, we define $u^{\mathfrak{F}} := \mathrm{Lie}(U^{\mathfrak{F}}(\mathbb{R})) = W^{\mathfrak{F}}_{-2}(\mathfrak{g})$ and $v^{\mathfrak{F}} := \mathrm{Lie}(V^{\mathfrak{F}}(\mathbb{R})) = W^{\mathfrak{F}}_{-2}(\mathfrak{g})$ for any cusp $\mathfrak{F}$. We note that the Lie algebra $v^{\mathfrak{F}}$ is identified with the space $\mathfrak{g}^{-1}$ and that $\dim_{\mathbb{R}}(C(\mathfrak{F})) = k(k+1)/2$ if $\mathfrak{F}$ is a k-th cusp.

2.1. **Equivalent toroidal embedding.** Let N be a lattice, i.e., a free $\mathbb{Z}$-module of finite rank and $M = N^\vee := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual. We fix isomorphisms $M \cong \mathbb{Z}^d,\ N \cong \mathbb{Z}^d$. The lattice N can be regarded as the group of 1-parameter subgroups of the algebraic torus $T_N := \mathrm{Spec}\mathbb{C}[M]$. Actually, any $a = (a_1, \cdots, a_d) \in N \cong \mathbb{Z}^d$ corresponds to a unique one-parameter subgroup $\lambda_a : \mathbb{G}_m \to T_N$ given by $\lambda_a(t) = (t_1^{a_1}, \cdots, t_d^{a_d}) \in T_N(\mathbb{C}) \forall t \in \mathbb{G}_m(\mathbb{C})$. We also note that $T_N \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{G}_m)$. On the other hand, the dual lattice M can be regarded as $X(T_N)$(the group of characters of T_N). Any $m = (m_1, \cdots, m_d) \in M$ corresponds to a unique $\chi^m \in X(T_N)$ given by

$$\chi^m(t_1, \cdots, t_d) = t_1^{m_1} \cdots t_d^{m_d} \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^* \,\forall (t_1, \cdots, t_d) \in T_N(\mathbb{C}) \cong (\mathbb{C}^*)^d.$$

Obviously, the lattices M, N are related by a non-degenerated canonical pairing

$$M \times N \longrightarrow \mathbb{Z}\ (m, a) \longmapsto < m, a >,$$

where $< m, a >$ is determined by $\chi^m(\lambda_a(t)) = t^{<m,a>}\ \ t \in \mathbb{C}^*$.

A **convex rational polyhedral cone** in $N_{\mathbb{R}}$ is a subset $\sigma \subset N_{\mathbb{R}}$ such that

$$\sigma = \{\sum_{i=1}^{t} \lambda_i y_i | \lambda_i \in \mathbb{R}_{\geq 0},\ \ i = 1, \cdots, t\}$$

for a finite number of vectors $y_i \in N_{\mathbb{Q}}, i = 1, \cdots, t$; its dual of σ is defined to be

$$\sigma^\vee = \{m \in M_{\mathbb{R}} \mid\ < m, u > \geq 0\ \forall u \in \sigma\},$$

which in fact is a convex rational polyhedral cone in $M_{\mathbb{R}}$; $\dim \sigma$ is defined to be the dimension of the smallest subspace of $N_{\mathbb{R}}$ containing σ; a **face** of σ is a convex rational polyhedral cone σ' in $N_{\mathbb{R}}$ such that $\sigma' = \sigma \cap \{v \in N_{\mathbb{R}} | \lambda(x) = 0\}$ for some $\lambda \in \sigma^\vee \cap M_{\mathbb{Q}}$, denoted by $\sigma' \prec \sigma$. A 1-dimensional convex rational polyhedral cone is called an **edge**. Any convex rational polyhedral cone σ endows an **affine toric variety** $X_\sigma = \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap M]$ where $\mathbb{C}[\sigma^\vee \cap M] := \{\sum\limits_{m=(m_1,\cdots,m_d)\in\sigma^\vee\cap M} a_m x_1^{m_1} \cdots x_d^{m_d} \mid a_m \in \mathbb{C}\}$. A face τ of σ induces an open immersion $X_\tau \hookrightarrow X_\sigma$ of affine varieties, so that X_τ can be identified with an open subvariety of X_σ.

A convex rational polyhedral cone σ of $N_{\mathbb{R}}$ is called **strong** if and only if $\sigma \cap (-\sigma) = \{0\}$. A strong rational convex polyhedral cone σ in $N_{\mathbb{R}}$ is said **regular**(with respect to N) provided that σ is generated by part of a $\mathbb{Z}$-basis of N. A **fan** of N is defined to be a nonempty collection $\Sigma = \{\sigma_\alpha\}$ of convex rational polyhedral cones in $N_{\mathbb{R}}$ such that

- all cones in Σ are strong;
- if σ' is a face of a cone $\sigma \in \Sigma$, then $\sigma' \in \Sigma$;
- for any two $\sigma, \tau \in \Sigma$, the intersection $\sigma \cap \tau$ is a face of both σ and τ.

The set of all edges in a fan Σ is denoted by $\Sigma(1)$. A fan Σ is said **regular**(with respect to N) if all $\sigma \in \Sigma$ are regular. A fan $\Sigma = \{\sigma_\alpha\}$ of N determines a separate scheme $X_\Sigma := \bigcup_{\sigma\in\Sigma} X_\sigma$ by patching together the X_{σ_α}'s along the $X_{\sigma_\alpha\cap\sigma_\beta}$'s.

Let Σ be an arbitrary fan of N. In general, the associated scheme X_Σ is normal and locally of finite type over $\mathbb{C}$. The X_Σ is smooth if and only if Σ is regular, moreover X_Σ is of finite type over $\mathbb{C}$ if and only if Σ is a finite collection of convex rational polyhedral cones. Let σ_α be any cone in Σ. Since $\sigma_\alpha^\vee$ spans $M_\mathbb{R}$, there is an open immersion of the algebraic torus $T_N := \mathrm{Spec}\mathbb{C}[M]$ in $X_\sigma = \mathrm{Spec}k[\sigma_\alpha^\vee \cap M]$(We call $T_N \subset X_\sigma$ a **toric embedding**). The action $T_N \times T_N \to T_N$ given by the translation in T_N can be extended to an action $T_N \times X_{\sigma_\alpha} \to X_{\sigma_\alpha}$. The open immersion $X_\tau \hookrightarrow X_{\sigma_\alpha}$ induced by a face $\tau \prec \sigma_\alpha$ is certainly equivariant with respect to the actions of T_N. Therefore, there is a natural open immersion $T_N \overset{\hookrightarrow}{\longrightarrow} X_\Sigma$ with the unique action T_N on X_Σ extending T_N's action on each X_{σ_β}.

Proposition 2.1 (Cf.[1],[19]&[15])**.** *Let $\Sigma = \{\sigma_\alpha\}$ be a fan of a lattice N.*

1. *There is a bijection between the set of cones in Σ and the set of orbits in X_Σ, and there holds $\dim \sigma_\alpha + \dim_\mathbb{C} \mathcal{O}^{\sigma_\alpha} = \dim_\mathbb{C} T_N$. Moreover, $\sigma_\alpha \subsetneq \sigma_\beta$ if and only if $\mathcal{O}^{\sigma_\beta} \subset \overline{\mathcal{O}^{\sigma_\alpha}}^{\mathrm{cl}}$, where $\overline{\mathcal{O}^{\sigma_\alpha}}^{\mathrm{cl}}$ denotes the closure in both the classical and Zariski topologies of X_Σ. In particular, each edge $\rho \in \Sigma$ gives a codimension one closed subscheme $D_\rho := \overline{\mathcal{O}^{\rho}}^{\mathrm{cl}}$ in X_Σ, which actually is a T_N-invariant prime divisor of X_Σ.*
2. *The collection $\{\mathcal{O}^{\sigma_\alpha}\}$ is a stratification of X_Σ in the classical analytic topology. Furthermore, $X_\tau = \coprod_{\delta\prec\tau} \mathcal{O}^\delta$, $\overline{\mathcal{O}^{\tau}}^{\mathrm{cl}} = \coprod_{\delta\in\Sigma,\delta\succeq\tau} \mathcal{O}^\delta$ for any cone $\tau \in \Sigma$.*

Corollary 2.2. *Let Σ be a fan of a lattice N and X_Σ the associated scheme with torus embedding $T_N := \mathrm{Spec}\mathbb{C}[N^\vee]$.*

1. *For any cone $\sigma \in \Sigma$, $\overline{\mathcal{O}^{\sigma}}^{\mathrm{cl}}$ is a closed subscheme of X_Σ with only normal singularities; moreover, $\overline{\mathcal{O}^{\sigma}}^{\mathrm{cl}}$ is smooth if Σ is a regular fan.*
2. *Assume that every low-dimensional cone $\tau \in \Sigma$($\dim\tau < \mathrm{rank}N$) is a face of some top-dimensional cone $\sigma_{\max} \in \Sigma$(i.e., $\dim\sigma_{\max} = \mathrm{rank}N$). If Σ is a regular fan then the infinity boundary $D_\infty := X_\Sigma \setminus T_N$ is simple normal crossing, i.e., all irreducible components of D_∞ are smooth and they intersect each other transversely.*

Proof. It is a direct consequence of the proposition 2.1. □

2.2. **Admissible families of polyhedral decompositions.** In [1], Mumford and his coworkers have constructed explicitly a class of toroidal compactifications of D/Γ for each bounded symmetric domain D with an arithmetic subgroup $\Gamma \subset \mathrm{Aut}(D)$. Actually, the compactification is determined by a certain combinatorial Γ-admissible rational polyhedral cone decompositions.

We define a partial order on the set of cusps of $\mathfrak{H}_g$: For any two cusps $\mathfrak{F}(W_1)$ and $\mathfrak{F}(W_2)$, we say $\mathfrak{F}(W_1) \prec \mathfrak{F}(W_2)$ if and only if $W_2 \subset W_1$. According to this partial order, $\mathfrak{F}(\{0\}) = \mathfrak{H}_g$ is the unique maximal element, and a cusp of depth g is called a **minimal cusp**(or **minimal rational boundary component**) of $\mathfrak{H}_g$. We call $\mathfrak{F}_0$ the **standard minimal cusp** of $\mathfrak{H}_g$.

Definition 2.3 (Cf.[26]). Suppose that C is an open cone in a real vector space $E_{\mathbb{R}}$, where $E_{\mathbb{R}}$ has an underlying integral structure $E_{\mathbb{Z}}$, i.e.,$E_{\mathbb{R}} = E_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. A **(rational) boundary component of** C **is a cone** $C' = (\overline{C} \cap E')^{\circ}$(denote by $C' \prec C$) given by a linear (rationally-defined) subspace E' of $E_{\mathbb{R}}$ with $E' \cap C = \emptyset$, where $\overline{C}$ is the closure of the cone C in $E_{\mathbb{R}}$. The **rational closure** $\overline{C}^{\mathrm{rc}}$ of C is the union of all rational boundary components of C.

We note that any proper rational boundary component of $C(\mathfrak{F}_0)$ is of form $C(\mathfrak{F}')$ where $\mathfrak{F}'$ is a cusp with $\mathfrak{F}_0 \prec \mathfrak{F}'$(cf.Theorem 3 in §4.4 of Chap.III [1]).

Lemma 2.4. *Let* $\mathfrak{F}(W_1)$ *and* $\mathfrak{F}(W_2)$ *be two cusps of* $\mathfrak{H}_g$. *The cusp* $\mathfrak{F}(W_1 \cap W_2)$ *has following properties :*

1. *There is* $u^{\mathfrak{F}(W_1 \cap W_2)} \subset u^{\mathfrak{F}(W_1)} \cap u^{\mathfrak{F}(W_2)}$. *Moreover, if there is a maximal isotropic subspace* W *of* $V_{\mathbb{R}}$ *containing* $W_1 \cup W_2$ *then* $u^{\mathfrak{F}(W_1 \cap W_2)} = u^{\mathfrak{F}(W_1)} \cap u^{\mathfrak{F}(W_2)}$.
2. *If* $W_1 \cap W_2$ *is a proper subspace of* W_1 *then*
$$u^{\mathfrak{F}(W_1 \cap W_2)} \cap C(\mathfrak{F}(W_1)) = \emptyset \text{ and } C(\mathfrak{F}(W_1 \cap W_2)) \cap C(\mathfrak{F}(W_1)) = \emptyset$$
3. *The equalities*
$$\overline{C(\mathfrak{F}(W_1 \cap W_2))} = \overline{C(\mathfrak{F}(W_1))} \cap u^{\mathfrak{F}(W_1 \cap W_2)} = \overline{C(\mathfrak{F}(W_2))} \cap u^{\mathfrak{F}(W_1 \cap W_2)}$$
are held. If there is a maximal isotropic subspace W *of* $V_{\mathbb{R}}$ *containing* $W_1 \cup W_2$ *then*
$$\overline{C(\mathfrak{F}(W_1))} \cap \overline{C(\mathfrak{F}(W_2))} = \overline{C(\mathfrak{F}(W_1 \cap W_2))}.$$

Remark. The lemma implies that $C(\mathfrak{F}(W_1 \cap W_2))$ is a rational boundary component of both $C(\mathfrak{F}(W_1))$ and $C(\mathfrak{F}(W_2))$. The equalities in (3) are also true even if we replace $\overline{C(\mathfrak{F})}$ with $\overline{C(\mathfrak{F})}^{\mathrm{rc}}$.

Proof of the lemma 2.4. Let $U = W_1 \cap W_2$. Then $U^{\vee} \subset W_1^{\vee} \cap W_2^{\vee}$ and $W_1^{\perp} + W_2^{\perp} \subset U^{\perp}$. For any cusp $\mathfrak{F}(W)$, we have

$$u^{\mathfrak{F}(W)} = W_{-2}^{\mathfrak{F}(W)} = \{H \in \mathfrak{g} \mid H(W^{\vee}) \subset W \text{ and } H(W^{\perp}) = 0\}. \tag{2.4.1}$$

1. Since $W_1^{\vee} \oplus W_1^{\perp} = W_2^{\vee} \oplus W_2^{\perp} = U^{\vee} \oplus U^{\perp} = V_{\mathbb{R}}$, the equality 2.4.1 says there holds
$$u^{\mathfrak{F}(U)} \subset u^{\mathfrak{F}(W_1)} \cap u^{\mathfrak{F}(W_2)}.$$

Because there is a maximal isotropic subspace W of $V_{\mathbb{R}}$ containing both W_1 and W_2, the space $W_1 + W_2$ is isotropic in W so that $\dim(W_1 + W_2)^{\perp} = 2g - \dim(W_1 + W_2)$ and $U^{\perp} = W_1^{\perp} + W_2^{\perp}$. Let $N \in u^{\mathfrak{F}(W_1)} \cap u^{\mathfrak{F}(W_2)}$. Then we have

$$N(U^{\vee}) \subset N(W_1^{\vee} \cap W_2^{\vee}) \subset N(W_1^{\vee}) \cap N(W_2^{\vee}) \subset W_1 \cap W_2 = U$$

and $N(U^{\perp}) = N(W_1^{\perp} + W_2^{\perp}) = 0$. Thus $u^{\mathfrak{F}(W_1)} \cap u^{\mathfrak{F}(W_2)} \subset u^{\mathfrak{F}(U)}$.

2. Recall $C(\mathfrak{F}(W)) := \{N \in u^{\mathfrak{F}(W)} \mid \psi(\cdot, N(\cdot)) > 0 \text{ on } \frac{V_{\mathbb{R}}}{W^{\perp}}\}$. Let $N \in u^{\mathfrak{F}(U)}$ be an arbitrary element. Consider the filtration $0 \subset U \subsetneq W_1 \subset W_1^{\perp} \subset U^{\perp} \subsetneq V_{\mathbb{R}}$, we obtain that $\psi(\cdot, N(\cdot))$ is semi-positive but not positive on $V_{\mathbb{R}}/W_1^{\perp}$ since $N(U^{\perp}) = 0$. Thus, $u^{\mathfrak{F}(U)} \cap C(\mathfrak{F}(W_1)) = \emptyset$.
3. It is sufficient to prove that $\overline{C(\mathfrak{F}(U))} = \overline{C(\mathfrak{F}(W_1))} \cap u^{\mathfrak{F}(U)} = \overline{C(\mathfrak{F}(W_2))} \cap u^{\mathfrak{F}(U)}$ for any rational defined subspace U of W_1 with $U \subsetneq W_1, U \subsetneq W_2$. Let U be an arbitrary rational defined subspace U of W_1 such that $U \subsetneq W_1, U \subsetneq W_2$. We have that $\overline{C(\mathfrak{F}(W))} = \{N \in u^{\mathfrak{F}(W)} \mid \psi(\cdot, N(\cdot)) \text{ is semi-positive on } \frac{V_{\mathbb{R}}}{W^{\perp}}\}$. Let $N \in \overline{C(\mathfrak{F}(U))}$ be an arbitrary element. Using the above argument in (2), we get that the bilinear form $\psi(\cdot, N(\cdot))$ is semi-positive on $V_{\mathbb{R}}/W_1^{\perp}$ and $N \in \overline{C(\mathfrak{F}(W_1))} \cap u^{\mathfrak{F}(U)}$. On the other hand, $\overline{C(\mathfrak{F}(W_1))} \cap u^{\mathfrak{F}(U)} \subset \overline{C(\mathfrak{F}(U))}$ is clear.

□

Let $\mathfrak{F}$ be an arbitrary cusp of $\mathfrak{H}_g$. Since the Lie group $U^{\mathfrak{F}}(\mathbb{C})$ is connected and $N^2 = 0$ for any $N \in u_{\mathbb{C}}^{\mathfrak{F}}$ by the lemma 1.13, the exponential map $\exp : u_{\mathbb{C}}^{\mathfrak{F}} \xrightarrow{\cong} U^{\mathfrak{F}}(\mathbb{C})\ \ \zeta \mapsto I_{2g} + \zeta$ is an isomorphism. We can identify $U^{\mathfrak{F}}(\mathbb{C})$ with its Lie algebra $u_{\mathbb{C}}^{\mathfrak{F}}$ by this isomorphism and regard $U^{\mathfrak{F}}(\mathbb{C})$ as a complex space. Moreover, for any ring $\mathfrak{R}$ in $\{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, $U^{\mathfrak{F}}(\mathfrak{R})$(the set of all $\mathfrak{R}$-points of the algebraic group $U^{\mathfrak{F}}$) can be regarded as an $\mathfrak{R}$-module by

$$U^{\mathfrak{F}}(\mathfrak{R}) \cong M_{k(k+1)/2}(\mathfrak{R}) \cap u_{\mathbb{C}}^{\mathfrak{F}}, \tag{2.4.2}$$

Therefore $U^{\mathfrak{F}}(\mathbb{C})$ has a natural integer structure $U^{\mathfrak{F}}(\mathbb{Z})$ and for any ring $\mathfrak{R}$ in $\{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, there is an isomorphism $U^{\mathfrak{F}}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathfrak{R} = U^{\mathfrak{F}}(\mathfrak{R})$. The corollary 1.12 ensures that any element $\gamma \in N^{\mathfrak{F}}(\mathbb{Z})$ defines an automorphism $\overline{\gamma} : U^{\mathfrak{F}}(\mathbb{Z}) \longrightarrow U^{\mathfrak{F}}(\mathbb{Z}),\ \ u \mapsto \gamma u \gamma^{-1}$. Thus we obtain a group morphism $j_{\mathfrak{F}} : N^{\mathfrak{F}} \to \mathrm{Aut}(U^{\mathfrak{F}})$ such that there is

$$j_{\mathfrak{F}} : N^{\mathfrak{F}}(\mathfrak{R}) \to \mathrm{Aut}(U^{\mathfrak{F}}(\mathfrak{R}))\ \gamma \mapsto \overline{\gamma} := ((\cdot) \mapsto \gamma(\cdot)\gamma^{-1})$$

for any $\mathbb{Z}$-algebra $\mathfrak{R}$. We see that if $\gamma \in U^{\mathfrak{F}}$ then $\overline{\gamma}$ is the identity in $\mathrm{Aut}(U^{\mathfrak{F}})$.

In general, there is the Levi-decomposition of $N^{\mathfrak{F}}$, i.e., a semi-product of rational algebraic groups(cf.[11]&[1]): $N^{\mathfrak{F}} = \underbrace{(G_h^{\mathfrak{F}} \times G_l^{\mathfrak{F}})}_{\text{direct product}} \cdot \mathcal{W}^{\mathfrak{F}}$. Moreover, we have :

- The $p_{h,\mathfrak{F}} : N^{\mathfrak{F}} \to G_h^{\mathfrak{F}}$ and $p_{l,\mathfrak{F}} : N^{\mathfrak{F}} \to G_l^{\mathfrak{F}}$ are surjective and defined over $\mathbb{Q}$,
- the $G_l^{\mathfrak{F}}(\mathbb{R}) \cdot \mathcal{W}^{\mathfrak{F}}(\mathbb{R})$ acts trivially on $\mathfrak{F}$ and the $G_h^{\mathfrak{F}}$ is semi-simple,

- the $G_h^{\mathfrak{F}} \cdot \mathcal{W}^{\mathfrak{F}}$ centralizes $U^{\mathfrak{F}}$ and the $G_l^{\mathfrak{F}}$ is reductive without compact factors.

Example 2.5 (Cf.[8] and [26])**.** Consider the cusp $\mathfrak{F}_k$, we compute that

$$\mathcal{N}(\mathfrak{F}_k) = \{ \begin{pmatrix} A_{11} & 0_{k,g-k} & A_{12} & * \\ * & f & * & * \\ A_{21} & 0_{k,g-k} & A_{22} & * \\ 0_{g-k,k} & 0_{g-k,g-k} & 0_{g-k,k} & -{}^t f^{-1} \end{pmatrix} \in \mathrm{Sp}(g,\mathbb{R}) \mid \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathrm{Sp}(k,\mathbb{R}) \\ f \in \mathrm{GL}(g-k,\mathbb{R}) \}.$$

$$\begin{aligned} G_l^{\mathfrak{F}_k}(\mathbb{R})^+ &= \{ \begin{pmatrix} I_k & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & {}^t f^{-1} \end{pmatrix} \mid f \in \mathrm{GL}(g-k,\mathbb{R}), \det f > 0 \}, \\ &\cong \mathrm{GL}(g-k,\mathbb{R})^+ \\ G_h^{\mathfrak{F}_k}(\mathbb{R}) = G_h^{\mathfrak{F}_k}(\mathbb{R})^+ &= \{ \begin{pmatrix} A_{11} & 0_{k,g-k} & A_{12} & 0 \\ 0 & I_{g-k} & 0 & 0 \\ A_{21} & 0_{k,g-k} & A_{22} & 0 \\ 0_{g-k,k} & 0_{g-k,g-k} & 0_{g-k,k} & I_{g-k} \end{pmatrix} \mid \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathrm{Sp}(k,\mathbb{R}) \} \\ &\cong \mathrm{Sp}(k,\mathbb{R}). \end{aligned}$$

Thus, the action $G_l^{\mathfrak{F}_k}(\mathbb{R}) \times C(\mathfrak{F}_k) \longrightarrow C(\mathfrak{F}_k)$ $(M,A) \longmapsto MAM^{-1}$ is equivalent to the action $\mathrm{GL}(g-k,\mathbb{R}) \times \mathrm{Sym}^+_{g-k}(\mathbb{R}) \longrightarrow \mathrm{Sym}^+_{g-k}(\mathbb{R})$ $(f,u) \longmapsto f \cdot u \cdot^t f$.

Lemma 2.6. *Let $\mathfrak{F} = \mathfrak{F}(W_{\mathbb{R}})$ be a cusp of $\mathfrak{H}_g$ where $W_{\mathbb{R}} = W_{\mathbb{Z}} \otimes \mathbb{R}$ is a rationally define subspace of $V_{\mathbb{R}}$. Let W represent a linear space over $\mathbb{Z}$ given by $W(\mathfrak{R}) := W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathfrak{R}$ for any $\mathbb{Z}$-algebra $\mathfrak{R}$. Both $G_l^{\mathfrak{F}}$ and $G_h^{\mathfrak{F}}$ are algebraic group defined over $\mathbb{Q}$, and there are two isomorphisms $G_l^{\mathfrak{F}} \cong \mathrm{GL}(W)$ and $G_h^{\mathfrak{F}} \cong \mathrm{Sp}(W^{\perp}/W, \psi)$.*

Lemma 2.7. *The homomorphism $j_{\mathfrak{F}} : N^{\mathfrak{F}}(\mathbb{R}) \to \mathrm{Aut}(U^{\mathfrak{F}}(\mathbb{R}))$ induces a homomorphism $j_{C(\mathfrak{F})} : N^{\mathfrak{F}}(\mathbb{R}) \to \mathrm{Aut}(C(\mathfrak{F}))$. Moreover, $j_{C(\mathfrak{F})}$ factors through $p_{l,\mathfrak{F}} : N^{\mathfrak{F}}(\mathbb{R}) \longrightarrow G_l^{\mathfrak{F}}(\mathbb{R})$, i.e, there is a commutative diagram*

$$\begin{tikzcd} & G_l^{\mathfrak{F}}(\mathbb{R}) \arrow[dr, "j_{C(\mathfrak{F})}"] & \\ N^{\mathfrak{F}}(\mathbb{R}) \arrow[ur, "p_{l,\mathfrak{F}}"] \arrow[rr, "j_{C(\mathfrak{F})}"] & & \mathrm{Aut}(C(\mathfrak{F})) \end{tikzcd}.$$

Proof. It is sufficient to prove the statements in case of $\mathfrak{F} = \mathfrak{F}_k$. Let $\Omega_{\mathfrak{F}}$ be a fixed point in $C(\mathfrak{F})$. Since $G_h^{\mathfrak{F}} \cdot \mathcal{W}^{\mathfrak{F}}$ centralizes $U^{\mathfrak{F}}$, We obtain that

$$\begin{aligned} C(\mathfrak{F}) &= \text{ the orbit of } \Omega_{\mathfrak{F}} \text{ for the adjoint action of } G_l^{\mathfrak{F}}(\mathbb{R}) \text{ on } U^{\mathfrak{F}}(\mathbb{R}) \\ &= \text{ the orbit of } \Omega_{\mathfrak{F}} \text{ for the adjoint action of } N^{\mathfrak{F}}(\mathbb{R}) \text{ on } U^{\mathfrak{F}}(\mathbb{R})\ . \end{aligned}$$

The first equality is given by the computation in the example 2.5. Therefore, we get the $j_{C(\mathfrak{F})} : N^{\mathfrak{F}}(\mathbb{R}) \to \mathrm{Aut}(C(\mathfrak{F}))$ and $j_{C(\mathfrak{F})}$ factors through the morphism $p_{l,\mathfrak{F}}$. □

Define $\Gamma_{\mathfrak{F}} := \Gamma \cap \mathcal{N}(\mathfrak{F})$, $\overline{\Gamma_{\mathfrak{F}}} := j_{C(\mathfrak{F})}(\Gamma_{\mathfrak{F}})$. Since $\mathcal{N}(\mathfrak{F}) \subset N^{\mathfrak{F}}(\mathbb{R})$ and $\mathcal{N}(\mathfrak{F})^+ = N^{\mathfrak{F}}(\mathbb{R})^+$, the group $\Gamma_{\mathfrak{F}}$ is a discrete subgroup of the real Lie group $\mathcal{N}(\mathfrak{F})$(cf. [23]), and there is an inclusion $\overline{\Gamma_{\mathfrak{F}}} = j_{C(\mathfrak{F})}(\Gamma \cap G_l^{\mathfrak{F}}(\mathbb{R})) \xrightarrow{\subset} \mathrm{Aut}(U^{\mathfrak{F}}(\mathbb{Z})) \cap \mathrm{Aut}(C(\mathfrak{F}))$.

Definition 2.8 (Cf.[1]&[14])**.** Let $\mathfrak{F}$ be a cusp of $\mathfrak{H}_g$ and $\mathbb{G}$ an arithmetic subgroup of $\mathrm{Sp}(g,\mathbb{Q})$ with an action on $C(\mathfrak{F})$. A **$\mathbb{G}$-admissible polyhedral decomposition** of $C(\mathfrak{F})$ is a collection of convex rational polyhedral cones $\Sigma_{\mathfrak{F}} = \{\sigma_\alpha^{\mathfrak{F}}\}_\alpha \subset \overline{C(\mathfrak{F})}$ satisfying that $\Sigma_{\mathfrak{F}}$ is a fan, $\overline{C(\mathfrak{F})}^{\mathrm{rc}} = \bigcup\limits_\alpha \sigma_\alpha^{\mathfrak{F}}$ and $\mathbb{G}$ has an action on the set $\Sigma_{\mathfrak{F}}$ with finitely many orbits. A $\mathbb{G}$-admissible polyhedral decompositions $\Sigma_{\mathfrak{F}}$ of $C(\mathfrak{F})$ is **regular** with respect to an arithmetic subgroup $\Gamma' \subset \mathrm{Sp}(g,\mathbb{Z})$ if $\Sigma_{\mathfrak{F}}$ is regular with respect to the lattice $\Gamma' \cap U^{\mathfrak{F}}(\mathbb{Z})$.

Remark. Each convex rational polyhedral cone in $C(\mathfrak{F})$ is automatically strong since $C(\mathfrak{F})$ is non-degenerate. Moreover, the construction in [1] implies that every cone in a $\overline{\Gamma_{\mathfrak{F}}}$-admissible polyhedral decomposition $\Sigma_{\mathfrak{F}} = \{\sigma_\alpha\}$ of $C(\mathfrak{F})$ is actual a face of one top-dimensional cone in $\Sigma_{\mathfrak{F}}$(i.e., a cone $\sigma_{\max}^{\mathfrak{F}} \in \Sigma_{\mathfrak{F}}$ with $\dim \sigma_{\max}^{\mathfrak{F}} = \dim_{\mathbb{R}} C(\mathfrak{F})$).

Definition 2.9 (Cf.[1])**.** Let $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ be an arithmetic subgroup.

1. A **Γ-admissible family of polyhedral decompositions** is a collection $\{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$ of $\overline{\Gamma_{\mathfrak{F}}}$-admissible polyhedral decompositions $\Sigma_{\mathfrak{F}} = \{\sigma_\alpha^{\mathfrak{F}}\}$ of $C(\mathfrak{F})$, $\mathfrak{F}$ running over the cusps of $\mathfrak{H}_g$ such that
   - if $\mathfrak{F}^2 = \gamma\mathfrak{F}^1$ for some $\gamma \in \Gamma$ then $\Sigma_{\mathfrak{F}^2} = \gamma(\Sigma_{\mathfrak{F}^1})$,
   - if $\mathfrak{F}^1 \prec \mathfrak{F}^2$ then $\Sigma_{\mathfrak{F}^2} = \{\sigma_\alpha^{\mathfrak{F}^1} \cap \overline{C(\mathfrak{F}^2)} \mid \sigma_\alpha^{\mathfrak{F}^1} \in \Sigma_{\mathfrak{F}^1}\}$.
2. A Γ-admissible family of polyhedral decompositions $\{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$ is called **regular** if for any cusp $\mathfrak{F}$ the $\overline{\Gamma_{\mathfrak{F}}}$-admissible polyhedral decompositions $\Sigma_{\mathfrak{F}}$ of $C(\mathfrak{F})$ is regular with respect to Γ.

Lemma 2.10 (Cf.[7] and &[26])**.** *Let $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ be an arithmetic subgroup. Let $\Sigma_{\mathfrak{F}_0} := \{\sigma_\alpha^{\mathfrak{F}_0}\}$ be a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$, where $\mathfrak{F}_0$ is the standard minimal cusp of $\mathfrak{H}_g$. The $\Sigma_{\mathfrak{F}_0}$ endows a Γ-admissible family of polyhedral decompositions $\{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$ as follows :*

Step 1: *For any minimal cusp $\mathfrak{F}_{\min} = M(\mathfrak{F}_0)$ with $M \in \mathrm{Sp}(g, \mathbb{Z})$, we define*

$$\Sigma_{\mathfrak{F}_{\min}} := M(\Sigma_{\mathfrak{F}_0}) = \{M\sigma_\alpha^{\mathfrak{F}_0} M^{-1} \mid \sigma_\alpha^{\mathfrak{F}_0} \in \Sigma_{\mathfrak{F}_0}\}.$$

Step 2: *For any cusp $\mathfrak{F}$, if $\mathfrak{F}_{\min}$ is a minimal cusp with $\mathfrak{F}_{\min} \prec \mathfrak{F}$ then we define*

$$\Sigma_{\mathfrak{F}} := \Sigma_{\mathfrak{F}_{\min}}|_{\overline{C(\mathfrak{F})}} = \{\sigma_\alpha^{\mathfrak{F}_{\min}} \cap \overline{C(\mathfrak{F})} \mid \sigma_\alpha^{\mathfrak{F}_{\min}} \in \Sigma_{\mathfrak{F}_{\min}}\}.$$

Moreover, we have :

1. The $\Sigma_{\mathfrak{F}_0}$ is regular with respect to Γ if $\Sigma_{\mathfrak{F}_0}$ is regular with respect to $\mathrm{Sp}(g, \mathbb{Z})$.
2. If $\Sigma_{\mathfrak{F}_0}$ is regular with respect to Γ then the family $\{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$ is regular.

2.3. **General toroidal compactifications of $\mathcal{A}_{g,\Gamma}$.** Let $D(\mathfrak{F}) := \bigcup_{\alpha \in \mathrm{U}^{\mathfrak{F}}(\mathbb{C})} \alpha\mathfrak{S}_g$ for any cusp $\mathfrak{F}$. Actually, $D(\mathfrak{F}) = \bigcup_{C \in u^{\mathfrak{F}}} \exp(\sqrt{-1}C)\mathfrak{S}_g \subset \check{\mathfrak{S}}_g$ since the group $W_0^{\mathfrak{F}}(G)$ acts transitively on $\mathfrak{S}_g$. Here is another version of the lemma 1.15:

Proposition 2.11 (Cf.[30],[1],[21],[13])**.** *Let $\mathfrak{F} = \mathfrak{F}(W)$ be a cusp of $\mathfrak{S}_g$. We have that $D(\mathfrak{F}) = \{F \in \check{\mathfrak{S}}_g \mid \sqrt{-1}\psi(v, \overline{v}) > 0 \ \forall\, 0 \neq v \in F \cap W^\perp\}$ and a diffeomorphism*

$$\varphi : u_{\mathbb{C}}^{\mathfrak{F}} \times v_{\mathbb{R}}^{\mathfrak{F}} \times \mathfrak{F} \xrightarrow{\cong} D(\mathfrak{F}) \ \ (a + \sqrt{-1}b, c, F) \longmapsto \exp(a + \sqrt{-1}b)\exp(c)(\check{F})$$

such that $\varphi^{-1}(\mathfrak{S}_g) = (u^{\mathfrak{F}} + \sqrt{-1}C(\mathfrak{F})) \times v_{\mathbb{R}}^{\mathfrak{F}} \times \mathfrak{F}$, where $\check{F}$ is defined in the lemma 1.15.

Corollary 2.12. *Let $\mathfrak{F} = \mathfrak{F}(W)$ be a cusp of $\mathfrak{S}_g$ with $\dim_{\mathbb{R}} W = k$. The space $V^{\mathfrak{F}}(\mathbb{R})$ has a natural complex structure such that it is isomorphic to $M_{g-k,k}(\mathbb{C})$. Moreover, there is an isomorphism $\Phi : U^{\mathfrak{F}}(\mathbb{C}) \times M_{g-k,k}(\mathbb{C}) \times \mathfrak{F} \xrightarrow{\cong} D(\mathfrak{F})$.*

We sketch the construction of a general toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}^{\mathrm{tor}}$ of $\mathcal{A}_{g,\Gamma}$ by following [1] in the complex analytic topology. Let $\Sigma_{\mathrm{tor}}^{\Gamma} = \{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$ be a general Γ-admissible family of polyhedral decompositions. Let $\mathfrak{F}$ be an arbitrary cusp of $\mathfrak{H}_g$. Define $L_{\mathfrak{F}} := \Gamma \cap U^{\mathfrak{F}}(\mathbb{Q})$, $M_{\mathfrak{F}} = L_{\mathfrak{F}}^\vee := \mathrm{Hom}_{\mathbb{Z}}(L_{\mathfrak{F}}, \mathbb{Z})$. We have $L_{\mathfrak{F}} = \Gamma \cap U^{\mathfrak{F}}(\mathbb{Z})$ as $\Gamma \subset \mathrm{Sp}(g, \mathbb{Z})$. Using 2.4.2, $L_{\mathfrak{F}}$ is a full lattice in the vector space $U^{\mathfrak{F}}(\mathbb{C}) \cong u_{\mathbb{C}}^{\mathfrak{F}}$. The algebraic torus $T_{\mathfrak{F}} := \mathrm{Spec}\mathbb{C}[M_{\mathfrak{F}}]$ is isomorphic analytically to $\frac{U^{\mathfrak{F}}(\mathbb{C})}{\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q})} = (\mathbb{C}^\times)^{\dim u^{\mathfrak{F}}}$. Then, there is an analytic isomorphism $\frac{D(\mathfrak{F})}{\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q})} \cong T_{\mathfrak{F}} \times v_{\mathbb{R}}^{\mathfrak{F}} \times \mathfrak{F}$ by the embedding of Siegel domain of third type $D(\mathfrak{F}) \cong u_{\mathbb{C}}^{\mathfrak{F}} \times v_{\mathbb{R}}^{\mathfrak{F}} \times \mathfrak{F}$, so that there is a principal $T_{\mathfrak{F}}$-bundle $\frac{D(\mathfrak{F})}{\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q})} \to \frac{D(\mathfrak{F})}{u_{\mathbb{C}}^{\mathfrak{F}}}$. We note that if $\mathfrak{F}_{\min}$ is a minimal cusp of $\mathfrak{H}_g$ then $D(\mathfrak{F})/u_{\mathbb{C}}^{\mathfrak{F}}$ is a space of one single point. For any cone $\sigma \in \Sigma_{\mathfrak{F}}$, we replace $T_{\mathfrak{F}}$ with X_σ by the open embedding $T_{\mathfrak{F}} \xrightarrow{\hookrightarrow} X_\sigma$, and obtain a fiber bundle $P_{\mathfrak{F},\sigma} : X_\sigma \times_{T_{\mathfrak{F}}} \frac{D(\mathfrak{F})}{\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q})} \to \frac{D(\mathfrak{F})}{u_{\mathbb{C}}^{\mathfrak{F}}}$. Define

$$\widetilde{\Delta}_{\mathfrak{F},\sigma} = \text{the interior of the closure of } \frac{\mathfrak{H}_g}{\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q})} \text{ in } X_\sigma \times_{T_{\mathfrak{F}}} \frac{D(\mathfrak{F})}{\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q})}. \tag{2.12.1}$$

Using the similar method that gluing X_σ's to construct the scheme $X_{\Sigma_{\mathfrak{F}}}$, we glue all $P_{\mathfrak{F},\sigma}: X_\sigma \times_{T_{\mathfrak{F}}} \frac{D(\mathfrak{F})}{\Gamma\cap U^{\mathfrak{F}}(\mathbb{Q})} \to \frac{D(\mathfrak{F})}{u^{\mathfrak{F}}_{\mathbb{C}}}$'s to obtain a fiber bundle $P_{\mathfrak{F}}: X_{\Sigma_{\mathfrak{F}}} \times_{T_{\mathfrak{F}}} \frac{D(\mathfrak{F})}{\Gamma\cap U^{\mathfrak{F}}(\mathbb{Q})} \to \frac{D(\mathfrak{F})}{u^{\mathfrak{F}}_{\mathbb{C}}}$ with fiber $X_{\Sigma_{\mathfrak{F}}}$, and we also glue $\Delta_{\mathfrak{F},\sigma}$'s altogether to obtain an analytic space $Z'_{\mathfrak{F}}$. We call $Z'_{\mathfrak{F}}$ the **partial compactification in the direction** $\mathfrak{F}$ of the Siegel variety $\mathcal{A}_{g,\Gamma}$. For any cusp $\mathfrak{F}$ and any element $\gamma\in\Gamma$ there is an analytic isomorphism $\Pi'_{\mathfrak{F},\gamma\mathfrak{F}}: Z'_{\mathfrak{F}} \xrightarrow{\cong} Z'_{\gamma\mathfrak{F}}$; and for any two cusps $\mathfrak{F}_1 \prec \mathfrak{F}_2$ there is an analytic étale morphism $\Pi'_{\mathfrak{F}_2,\mathfrak{F}_1}: Z'_{\mathfrak{F}_2} \to Z'_{\mathfrak{F}_1}$ (cf.Lemma 1 in §5 Chap.III [1]).

Example 2.13 (Cf.[8])**.** Let $\mathfrak{F}=\mathfrak{F}(W)$ be a cusp of $\dim_{\mathbb{R}} W = k > 0$. Assume Γ is neat. We can describe $Z_{\mathfrak{F},\sigma}$ in a local coordinate system : Let σ be a regular cone in $C(\mathfrak{F})$ of top-dimension $k(k+1)/2$. As in [1] and [25], we take a $\mathbb{Z}$-basis $\{\zeta_\alpha\}_1^{k(k+1)/2}$ of $\Gamma\cap U^{\mathfrak{F}}(\mathbb{Q})$ such that $\mathbb{R}_+\zeta_1,\cdots,\mathbb{R}_+\zeta_{k(k+1)/2}$ are all edges of σ. Any $u\in U^{\mathfrak{F}}(\mathbb{C})$ can be written as $u=\sum_\alpha u_\alpha\zeta_\alpha$. Then we get an open embedding

$$\delta_{\mathfrak{F}}: \mathfrak{H}_g \xrightarrow{\hookrightarrow} U^{\mathfrak{F}}(\mathbb{C}) \times M_{g-k,k}(\mathbb{C}) \times \mathfrak{F}(W) \xrightarrow{\cong} \mathbb{C}^{k(k+1)/2} \times M_{g-k,k}(\mathbb{C}) \times \mathfrak{S}(W^\perp/W,\psi_W),$$

so that $(u_\alpha, s_i, t_j)\in \mathbb{C}^{k(k+1)/2}\times M_{g-k,k}(\mathbb{C})\times\mathfrak{S}(W^\perp/W,\psi_W)$ endows a coordinate system of $\mathfrak{H}_g$. Thus, we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{H}_g & \xrightarrow{\subset} & \mathbb{C}^{k(k+1)/2} \times M_{k,g-k}(\mathbb{C}) \times \mathfrak{S}(W^\perp/W,\psi_W) \\ \big\downarrow & & \big\downarrow{\scriptstyle (z_\alpha:=\exp(2\pi\sqrt{-1}u_\alpha),\, s_i,\, t_j)} \\ \frac{\mathfrak{H}_g}{\Gamma\cap U^{\mathfrak{F}}(\mathbb{Q})} & \xrightarrow{\subset} & (\mathbb{C}^*)^{k(k+1)/2} \times M_{k,g-k}(\mathbb{C}) \times \mathfrak{S}(W^\perp/W,\psi_W) \end{array} \tag{2.13.1}$$

and there holds $\bigcup\limits_\alpha \{(z_\alpha,s_i,t_j)\in\widetilde{\Delta}_{\mathfrak{F},\sigma} \mid z_\beta = 0\} \xrightarrow{\subset} \widetilde{\Delta}_{\mathfrak{F},\sigma}\setminus\frac{\mathfrak{H}_g}{\Gamma\cap U^{\mathfrak{F}}(\mathbb{Q})}$.

Lemma 2.14. *Let $\Sigma^\Gamma_{\text{tor}}=\{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$ be a Γ-admissible family of polyhedral decompositions.*

1. *Let $\mathfrak{F}$ be a cusp of $\mathfrak{H}_g$. The collection $\{\mathcal{S}(\mathfrak{F},\sigma)\}_{\sigma\in\Sigma_{\mathfrak{F}}}$ is a stratification of $Z'_{\mathfrak{F}}$. In particular, $\overline{\mathcal{S}(\mathfrak{F},\sigma)}^{\text{cl}}=\coprod_{\delta\in\Sigma_{\mathfrak{F}},\delta\succeq\sigma}\mathcal{S}(\mathfrak{F},\delta)\ \forall\sigma\in\Sigma_{\mathfrak{F}}$, where $\overline{\mathcal{S}(\mathfrak{F},\sigma)}^{\text{cl}}$ is the closure of $\mathcal{S}(\mathfrak{F},\sigma)$ in $Z'_{\mathfrak{F}}$. Moreover, the open embedding $\mathbb{U}_{\mathfrak{F}}(:=\frac{\mathfrak{H}_g}{\Gamma\cap U^{\mathfrak{F}}(\mathbb{Q})}) \xrightarrow{\subset} Z'_{\mathfrak{F}}$ is a toroidal embedding without self-intersections, i.e., every irreducible component of $Z'_{\mathfrak{F}}\setminus\mathbb{U}_{\mathfrak{F}}$ is normal.*
2. *For any two cusps $\mathfrak{F}^1,\mathfrak{F}^2$ with $\mathfrak{F}^1\prec\mathfrak{F}^2$, $\Pi'_{\mathfrak{F}^2,\mathfrak{F}^1}:(Z'_{\mathfrak{F}^2},\mathbb{U}_{\mathfrak{F}^2})\to(Z'_{\mathfrak{F}^1},\mathbb{U}_{\mathfrak{F}^1})$ is a toroidal morphism.*

Proof. By carefully reading [1], one can get the proof easily. □

The disjoint union $\widetilde{\mathcal{A}_{g,\Gamma}} := \overset{\circ}{\bigsqcup\limits_{\mathfrak{F}}} Z'_{\mathfrak{F}}$ has a natural Γ-action. An equivalent relation R is defined on $\widetilde{\mathcal{A}_{g,\Gamma}}$: We say $x\sim^R y$ for $x\in Z'_{\mathfrak{F}_1}, y\in Z'_{\mathfrak{F}_2}$ if and only if there exists a cusp

$\mathfrak{F}_3$, a point $z \in Z'_{\mathfrak{F}_3}$ and a $\gamma \in \Gamma$ such that $\mathfrak{F}_1 \preceq \mathfrak{F}_3, \gamma\mathfrak{F}_2 \preceq \mathfrak{F}_3$ and
(2.14.1)

$$(\Pi'_{\mathfrak{F}_3,\mathfrak{F}_1} : Z'_{\mathfrak{F}_3} \to Z'_{\mathfrak{F}_1}) \text{ maps } z \text{ to } x, \quad (\Pi'_{\mathfrak{F}_3,\gamma\mathfrak{F}_2} : Z'_{\mathfrak{F}_3} \to Z'_{\gamma\mathfrak{F}_2}) \text{ maps } z \text{ to } \Pi'_{\mathfrak{F}_2,\gamma\mathfrak{F}_2}(y).$$

Shown in §5 – §6 Chap. III. [1], the transitivity condition of the relation R holds and the relation graph in $\widetilde{\mathcal{A}_{g,\Gamma}} \times \widetilde{\mathcal{A}_{g,\Gamma}}$ is closed. We then obtain a compact Hausdorff analytic variety $\overline{\mathcal{A}}_{g,\Gamma}^{\mathrm{tor}} := \frac{\widetilde{\mathcal{A}_{g,\Gamma}}}{\sim R}$, which is called a **toroidal compactification** of $\mathcal{A}_{g,\Gamma}$. The $\overline{\mathcal{A}}_{g,\Gamma}^{\mathrm{tor}}$ is an algebraic space, but not projective in general. However, a theorem of Tai(cf.Chap.IV [1]) shows that if $\Sigma_{\mathrm{tor}}^{\Gamma}$ is projective(cf. Chap. IV. of [1]) then $\overline{\mathcal{A}}_{g,\Gamma}^{\mathrm{tor}}$ is a projective variety. The main theorem I in [1] shows that $\overline{\mathcal{A}}_{g,\Gamma}^{\mathrm{tor}}$ is the unique Hausdorff analytic variety containing $\mathcal{A}_{g,\Gamma}$ as an open dense subset such that $\overline{\mathcal{A}}_{g,\Gamma}^{\mathrm{tor}} = \bigcup_{\mathfrak{F}} \pi'_{\mathfrak{F}}(Z'_{\mathfrak{F}})$ and for every cusp $\mathfrak{F}$ of $\mathfrak{H}_g$ there is an open morphisms $\pi'_{\mathfrak{F}}$ making the following diagram commutative

(2.14.2)
$$\begin{tikzcd}
\frac{\mathfrak{H}_g}{\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q})} \arrow[r, hook] \arrow[d] & Z'_{\mathfrak{F}} \arrow[d, "\pi'_{\mathfrak{F}}"] \\
\mathcal{A}_{g,\Gamma} \arrow[r, hook] & \overline{\mathcal{A}}_{g,\Gamma}^{\mathrm{tor}}.
\end{tikzcd}$$

2.4. **Infinity boundary divisors on toroidal compactifications.** For a polyhedral decomposition $\Sigma_{\mathfrak{F}_0} := \{\sigma_\alpha^{\mathfrak{F}_0}\}$ of $C(\mathfrak{F}_0)$, all edges in $\Sigma_{\mathfrak{F}_0}$ are taken into two disjoint sets :

- Interior-edge=$\{\rho \in \Sigma_{\mathfrak{F}_0}(1) \mid \mathrm{Int}(\rho) \subset C(\mathfrak{F}_0)\}$,
- Boundary-edge=$\{\rho \in \Sigma_{\mathfrak{F}_0}(1) \mid \mathrm{Int}(\rho) \cap C(\mathfrak{F}_0) = \emptyset\}$,

where $\mathrm{Int}(\sigma)$ is defined to be the set of relative interior points of $\sigma \in \Sigma_{\mathfrak{F}_0}$. These two sets are both $\Gamma_{\mathfrak{F}_0}(= \Gamma \cap \mathcal{N}(\mathfrak{F}_0))$-invariant for any arithmetic subgroup Γ of $\mathrm{Sp}(g, \mathbb{Z})$.

Lemma 2.15. *Let Γ be a neat arithmetic subgroup of $\mathrm{Sp}(g, \mathbb{Q})$ and $\Sigma_{\mathfrak{F}_0} = \{\sigma_\alpha^{\mathfrak{F}_0}\}_\alpha$ a $\overline{\Gamma_{\mathfrak{F}_0}}$-admissible polyhedral decomposition of $C(\mathfrak{F})$. Let ρ be an edge in the set Boundary-edge.*

Assume that $\Sigma_{\mathfrak{F}_0}$ is regular with respect to Γ. There is a unique rationally-defined one dimensional isotropic subspace W_ρ of $V(= V^{(0)})$ such that $\mathrm{Int}(\rho) = C(\mathfrak{F}(W_\rho))$. Moreover, for any cone $\sigma \in \Sigma_{\mathfrak{F}_0}$ there exists a unique cusp $\mathfrak{F}_\sigma$ such that $\mathfrak{F}_0 \preceq \mathfrak{F}_\sigma$ and $\mathrm{Int}(\sigma) \subset C(\mathfrak{F}_\sigma)$.

Proof. By Theorem 3 in §4.4 of Chap.III [1], any proper rational boundary component of $C(\mathfrak{F}_0)$(cf.Definition 2.3) is of form $C(\mathfrak{F}_1)$ by a cusp $\mathfrak{F}_1$ with $\mathfrak{F}_0 \prec \mathfrak{F}_1$. Thus, there is a cusp $\mathfrak{F}'$ different with $\mathfrak{F}_0$ such that $\mathfrak{F}_0 \prec \mathfrak{F}'$ and $\rho \in \overline{C(\mathfrak{F}')}$.

Suppose $\mathfrak{F}' = \mathfrak{F}(W')$ has $\dim_{\mathbb{Q}} W' \geq 2$. By the lemma 2.10, $\Sigma_{\mathfrak{F}_0}$ endows a Γ-admissible family of polyhedral decompositions $\{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$ and the decomposition $\Sigma_{\mathfrak{F}'}$ is regular to with respect to Γ. Since $\overline{C(\mathfrak{F}_0)}^{\mathrm{rc}} = \bigcup_\alpha \sigma_\alpha^{\mathfrak{F}_0}$, there exists a top-dimensional cone $\sigma_{\max} \in \Sigma_{\mathfrak{F}_0}$ and a face τ of $\sigma_{\max}$ such that ρ is an edge of τ and $\tau \in \Sigma_{\mathfrak{F}'}$ with $\dim \tau = \dim C(\mathfrak{F}')$. Since the cone $\sigma_{\max}$ is regular, there is a face δ of σ satisfying $\rho \in \delta$ and $\delta \notin \Sigma_{\mathfrak{F}'}$. Thus, we have another cusp $\mathfrak{F}'' = \mathfrak{F}(W'')$ that W'' is not a subspace of W' and $\delta \in \Sigma_{\mathfrak{F}''}$. Then, $\rho \in u^{\mathfrak{F}(W' \cap W'')}$ by the lemma 2.4, and so $\mathrm{Int}(\rho)$ is in a proper rational boundary component of $C(\mathfrak{F}')$.

By recursion, we obtain that $\mathrm{Int}(\rho)$ is a proper rational boundary component of $C(\mathfrak{F}_0)$ and there is a rationally-defined one dimensional isotropic subspace W_ρ of V such that $\mathrm{Int}(\rho) = C(\mathfrak{F}(W_\rho))$. The uniqueness is due to(3) of the lemma 2.4.

The rest can be obtained by similar method. □

Definition 2.16. Let $\Gamma \subset \mathrm{Sp}(g, \mathbb{Z})$ be an arithmetic subgroup and $\mathfrak{F}$ a cusp of $\mathfrak{H}_g$. A $\overline{\Gamma_{\mathfrak{F}}}$-admissible polyhedral decomposition $\Sigma_{\mathfrak{F}}$ of $C(\mathfrak{F})$ is **Γ-separable** if a $\gamma \in \overline{\Gamma_{\mathfrak{F}}}$ satisfies $\gamma(\sigma) \cap \sigma \neq \{0\}$ for a cone $\sigma \in \Sigma_{\mathfrak{F}}$ then γ acts as the identity on the cone σ.

Remark. Note that any $\overline{\Gamma_{\mathfrak{F}}}$-admissible polyhedral decomposition $\Sigma_{\mathfrak{F}}$ of $C(\mathfrak{F})$ can be subdivided into another regular $\overline{\Gamma_{\mathfrak{F}}}$-admissible polyhedral decomposition $\widetilde{\Sigma}_{\mathfrak{F}}$(cf.[1],[14]), and it is obvious that the regular refinement $\widetilde{\Sigma}_{\mathfrak{F}}$ is also Γ-separable provided that $\Sigma_{\mathfrak{F}}$ is Γ-separable.

In fact, our definition of a $\overline{\Gamma_{\mathfrak{F}}}$-admissible polyhedral decomposition with Γ-separability is compatible with the condition (ii) in §2.4 Chap IV [14].The following is easy :

Lemma 2.17. *Let Γ be an arithmetic subgroup of $\mathrm{Sp}(g, \mathbb{Q})$. Let $\Sigma_{\mathfrak{F}}$ be a $\overline{\Gamma_{\mathfrak{F}}}$-admissible polyhedral decomposition of $C(\mathfrak{F})$, where $\mathfrak{F}$ is a cusp of $\mathfrak{H}_g$.*

Assume that the decomposition $\Sigma_{\mathfrak{F}}$ is regular with respect to Γ. The following two conditions are equivalent : (i) $\Sigma_{\mathfrak{F}}$ is Γ-separable; (ii) if an element $\gamma \in \overline{\Gamma_{\mathfrak{F}}}$ satisfies $\gamma(\sigma) \cap \sigma \neq \{0\}$ for a cone $\sigma \in \Sigma_{\mathfrak{F}}$ then γ acts as the identity on $C(\mathfrak{F})$.

Proof. A regular top-dimensional cone $\sigma_{\max}^{\mathfrak{F}} = \{\sum_{i=1}^{t} \lambda_i y_i | \lambda_i \in \mathbb{R}_{\geq 0},\ i = 1, \cdots, t\}$ has $\{y_1, \cdots, y_t\}$ as a $\mathbb{Z}$-basis of $\Gamma \cap U^{\mathfrak{F}}(\mathbb{Z})$. Then that $\Sigma_{\mathfrak{F}}$ is Γ-separable if and only if that any $\gamma \in \overline{\Gamma_{\mathfrak{F}}}$ satisfying $\gamma(\sigma) \cap \sigma \neq \{0\}$ for some cone $\sigma \in \Sigma_{\mathfrak{F}}$ acts as the identity on all cones in $\{\tau \in \Sigma_{\mathfrak{F}} \mid \tau \cap \gamma(\sigma) \cap \sigma \neq \{0\}\}$. So if γ acts as the identity on $\sigma_{\max}^{\mathfrak{F}}$ then γ must act as the identity on $C(\mathfrak{F})$.

□

Definition 2.18. Let $\Gamma \subset \mathrm{Sp}(g, \mathbb{Z})$ be an arithmetic subgroup. A **symmetric Γ-admissible family of polyhedral decompositions** is the Γ-admissible family of polyhedral decompositions induced by a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g, \mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$ as in the lemma 2.10. A **symmetric toroidal compactification**

of a Siegel variety $\mathcal{A}_{g,\Gamma}$ is a compactification constructed by a symmetric admissible family of polyhedral decompositions.

When we say a toroidal compactification constructed by some admissible polyhedral decomposition of $C(\mathfrak{F}_0)$, we always mean a symmetric toroidal compactification.

Due to the lemma 2.10, we have :

Lemma 2.19. *Let $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ be an arithmetic subgroup. Let $\Sigma_{\mathfrak{F}_0}$ be a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$. Let $\{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$ be the symmetric Γ-admissible family of polyhedral decompositions induced by $\Sigma_{\mathfrak{F}_0}$(cf.Lemma 2.10).*

1. *For any cusp $\mathfrak{F}$ of $\mathfrak{H}_g$, the induced $\overline{\Gamma_{\mathfrak{F}}}$-admissible polyhedral decomposition $\Sigma_{\mathfrak{F}}$ of $C(\mathfrak{F})$ is Γ-separable.*
2. *For any subgroup Γ' of Γ with finite index, the decomposition $\Sigma_{\mathfrak{F}_0}$ is Γ'-separable as a $\overline{\Gamma'_{\mathfrak{F}_0}}$-admissible polyhedral decomposition.*

In general, given a toroidal compactification $\overline{D/\Gamma}$ of a locally symmetric variety D/Γ, the toroidal embedding $D/\Gamma \subset \overline{D/\Gamma}$ is not without self-intersection, the main theorems II in [1] says that $D/\Gamma \subset \overline{D/\Gamma}$ is without monodromy in sense that for each stratum $\mathcal{O}$ the branches of $\overline{D/\Gamma} \setminus D/\Gamma$ through $\mathcal{O}$ are not permuted by going around loops in $\mathcal{O}$. For more efficient applications of toroidal compactifications in geometry, we continue to exploit the infinity boundary divisors explicitly.

Theorem 2.20. *Let $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ be an arithmetic subgroup and let $\Sigma_{\mathfrak{F}_0} := \{\sigma_\alpha^{\mathfrak{F}_0}\}$ be a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$ where $\mathfrak{F}_0$ is the standard minimal cusp of $\mathfrak{H}_g$. Let $\overline{\mathcal{A}}_{g,\Gamma}$ be the toroidal compactification of $\mathcal{A}_{g,\Gamma}$ constructed by $\Sigma_{\mathfrak{F}_0}$ and $D_\infty := \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ the boundary divisor.*

Assume that the decomposition $\Sigma_{\mathfrak{F}_0}$ is regular with respect to Γ.

1. *The number of irreducible components of D_∞ is equal to*
$$[\mathrm{Sp}(g,\mathbb{Z}):\Gamma] + [\mathrm{Sp}(g,\mathbb{Z}):\Gamma] \times \#\{\Gamma_{\mathfrak{F}_0}\text{-}\textit{orbits in Interior-edge}\}.$$
2. *The compactification $\overline{\mathcal{A}}_{g,\Gamma}$ is a smooth compact analytic variety with simple normal crossing boundary divisor D_∞, if the group Γ is neat and the decomposition $\Sigma_{\mathfrak{F}_0}$ is Γ-separable.*

Proof. We define an equivalent relation on the set of cusps : $\mathfrak{F} \sim^{\Gamma} \mathfrak{F}'$ if and only if there exists an element $\gamma \in \Gamma$ such that $\mathfrak{F}' = \gamma\mathfrak{F}$. The equivalent class is denoted by $[\cdot]$.

Let $\{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$ be the symmetric Γ-admissible family of polyhedral decompositions induced by the given decomposition $\Sigma_{\mathfrak{F}_0}$. For any cusp $\mathfrak{F}$, we also define an equivalent relation on $\Sigma_{\mathfrak{F}}$: $\sigma_\alpha^{\mathfrak{F}} \sim^{\Gamma_{\mathfrak{F}}} \sigma_\beta^{\mathfrak{F}}$ if and only if there exists an element $\gamma \in \Gamma_{\mathfrak{F}}$ such that $\sigma_\beta^{\mathfrak{F}} = \gamma(\sigma_\alpha^{\mathfrak{F}})$. Denoted this equivalent class by $[\cdot]_{\mathfrak{F}}$.

For each cusp $\mathfrak{F}$, a basic fact is that the group $\Gamma_{\mathfrak{F}}/\Gamma \cap U^{\mathfrak{F}}(\mathbb{R})$ acts properly discontinuously on $Z'_{\mathfrak{F}}$(cf.Proposition 1 in §6.3 Chap.III[1]), thus the morphism $\pi'_{\mathfrak{F}}$ factors through a morphism $\pi_{\mathfrak{F}} : Z_{\mathfrak{F}} \to \overline{\mathcal{A}}_{g,\Gamma}$ so that there is a commutative diagram :

$$\begin{tikzcd} Z'_{\mathfrak{F}} \arrow[r, "\mathrm{pr}_{\mathfrak{F}}"] \arrow[dr, "\pi'_{\mathfrak{F}}"'] & Z_{\mathfrak{F}} \arrow[d, "\pi_{\mathfrak{F}}"] \\ & \overline{\mathcal{A}}_{g,\Gamma}, \end{tikzcd} \tag{2.20.1}$$

where $Z_{\mathfrak{F}}$ is the quotient of $Z'_{\mathfrak{F}}$ by $\Gamma_{\mathfrak{F}}/\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q})$ and $\mathrm{pr}_{\mathfrak{F}}$ is a quotient morphism.

Let $\mathfrak{F}^1, \mathfrak{F}^2$ be two arbitrary cusps with $\mathfrak{F}^2 \prec \mathfrak{F}^1$. We actually have

$$\overline{\Gamma_{\mathfrak{F}^1}} - \text{orbit of } \sigma_\alpha^{\mathfrak{F}^1} \text{ in } \Sigma_{\mathfrak{F}^1} = (\overline{\Gamma_{\mathfrak{F}^2}} - \text{orbit of } \sigma_\alpha^{\mathfrak{F}^1} \text{ in } \Sigma_{\mathfrak{F}^2}) \bigcap \Sigma_{\mathfrak{F}^1} \quad \forall \sigma_\alpha^{\mathfrak{F}^1} \in \Sigma_{\mathfrak{F}^1}$$

by the fact of $\Sigma_{\mathfrak{F}^1} = \{\sigma_\beta \in \Sigma_{\mathfrak{F}^2} \mid \sigma_\beta \subset \overline{C(\mathfrak{F}^1)}\}$. Then, we get an induced morphism $\Pi_{\mathfrak{F}^1,\mathfrak{F}^2} : Z_{\mathfrak{F}^1} \longrightarrow Z_{\mathfrak{F}^2}$. Moreover, $\Pi_{\mathfrak{F}^1,\mathfrak{F}^2}$ is a local isomorphism satisfying the following commutative diagram

$$\begin{tikzcd} Z'_{\mathfrak{F}^1} \arrow[r, "\Pi'_{\mathfrak{F}^1,\mathfrak{F}^2}"] \arrow[d, "\mathrm{pr}_{\mathfrak{F}^1}"'] & Z'_{\mathfrak{F}^2} \arrow[d, "\mathrm{pr}_{\mathfrak{F}^2}"] \\ Z_{\mathfrak{F}^1} \arrow[r, "\Pi_{\mathfrak{F}^1,\mathfrak{F}^2}"] & Z_{\mathfrak{F}^2} \end{tikzcd}$$

Recall the construction of a general toroidal compactification of $\mathfrak{H}_g/\Gamma$, the condition 2.14.1 of the relation R ensures that the following diagram is commutative :

$$\begin{tikzcd} Z_{\mathfrak{F}_2} \arrow[r, "\Pi_{\mathfrak{F}_2,\mathfrak{F}_1}"] \arrow[dr, "\pi_{\mathfrak{F}_2}"'] & Z_{\mathfrak{F}_1} \arrow[d, "\pi_{\mathfrak{F}_1}"] \\ & \overline{\mathcal{A}}_{g,\Gamma} \end{tikzcd} \tag{2.20.2}$$

for any two cusps $\mathfrak{F}_1, \mathfrak{F}_2$ with $\mathfrak{F}_1 \prec \mathfrak{F}_2$.

Let $\mathfrak{F}_{\min}$ be an arbitrary minimal cusp. Denote by $\mathfrak{U}_{[\mathfrak{F}_{\min}]} := \pi'_{\mathfrak{F}_{\min}}(Z'_{\mathfrak{F}_{\min}})$. It is well-defined as $\pi'_{\mathfrak{F}}(Z'_{\mathfrak{F}}) = \pi'_{\gamma\mathfrak{F}}(Z'_{\gamma\mathfrak{F}})$ for $\gamma \in \Gamma$, $\forall \mathfrak{F}$. Since $\pi'_{\mathfrak{F}_{\min}}$ is an open morphism, the set $\mathfrak{U}_{[\mathfrak{F}_{\min}]}$ is open in $\overline{\mathcal{A}}_{g,\Gamma}$. There are useful properties F1-F4 :

F1: Let $\mathfrak{F}$ be an arbitrary cusp and γ an arbitrary element in Γ. It is obvious that there is an induced isomorphism $\Pi_{\mathfrak{F},\gamma\mathfrak{F}} : Z_{\mathfrak{F}} \to Z_{\gamma\mathfrak{F}}$ such that the following diagram is commutative

$$\begin{array}{ccc} Z'_{\mathfrak{F}} & \xrightarrow{\Pi'_{\mathfrak{F},\gamma\mathfrak{F}}} & Z'_{\gamma\mathfrak{F}} \\ {\scriptstyle \mathrm{pr}_{\mathfrak{F}}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{\gamma\mathfrak{F}}} \\ Z_{\mathfrak{F}} & \xrightarrow{\Pi_{\mathfrak{F},\gamma\mathfrak{F}}} & Z_{\gamma\mathfrak{F}} \end{array}$$

Thus, it is easy to get $\mathrm{pr}_{\mathfrak{F}}(\mathcal{S}(\mathfrak{F},\sigma_\alpha^{\mathfrak{F}})) = \mathrm{pr}_{\mathfrak{F}}(\mathcal{S}(\mathfrak{F},\kappa(\sigma_\alpha^{\mathfrak{F}})))\ \forall\sigma_\alpha^{\mathfrak{F}} \in \Sigma_{\mathfrak{F}},\ \forall\kappa \in \Gamma_{\mathfrak{F}_0}$. For any cone $\sigma^{\mathfrak{F}} \in \Sigma_{\mathfrak{F}}$, we define $\mathcal{Y}(\mathfrak{F},[\sigma^{\mathfrak{F}}]_{\mathfrak{F}}) := \mathrm{pr}_{\mathfrak{F}}(\mathcal{S}(\mathfrak{F},\sigma^{\mathfrak{F}}))$. The collection $\{\mathcal{Y}(\mathfrak{F},[\sigma_\alpha^{\mathfrak{F}}]_{\mathfrak{F}})\}_{\sigma_\alpha^{\mathfrak{F}}\in\Sigma_{\mathfrak{F}}}$ is then a stratification of $Z_{\mathfrak{F}}$.

F2: Let $\mathfrak{F}^1,\mathfrak{F}^2$ be two arbitrary cusps with $\mathfrak{F}^2 \prec \mathfrak{F}^1$. By the lemma 2.14, we obtain

$$\Pi^{-1}_{\mathfrak{F}^1,\mathfrak{F}^2}(\mathcal{Y}(\mathfrak{F}^2,[\sigma_\alpha^{\mathfrak{F}^2}]_{\mathfrak{F}^2})) = \begin{cases} \mathcal{Y}(\mathfrak{F}^1,[\tau]_{\mathfrak{F}^1}) & \text{if } \exists\tau \in [\sigma_\alpha^{\mathfrak{F}^2}]_{\mathfrak{F}^2} \text{ such that } \tau \in \Sigma_{\mathfrak{F}^1} \\ \emptyset & \text{others} \end{cases} \tag{2.20.3}$$

since $\Sigma_{\mathfrak{F}^1} = \{\sigma_\beta \in \Sigma_{\mathfrak{F}^2} \mid \sigma_\beta \subset \overline{C(\mathfrak{F}^1)}\}$.

F3: For any cusp $\mathfrak{F}(W)$ and for any element γ in Γ, $\mathfrak{F}(W\cap\gamma(W))$ is the unique minimal one in the set of cusps $\{\mathfrak{F} \mid \mathfrak{F}(W) \prec \mathfrak{F},\ \mathfrak{F}(\gamma(W)) \prec \mathfrak{F}\}$, and so we can glue $Z'_{\mathfrak{F}(W)}$ and $Z'_{\mathfrak{F}(\gamma(W))}$ along $Z'_{\mathfrak{F}(W\cap\gamma(W))}$.

On the other hand, if we restrict the action of the relation R on $Z'_{\mathfrak{F}_{\min}}$ then this relation R is reduced to the action of the $\Gamma_{\mathfrak{F}_{\min}}$ on $Z'_{\mathfrak{F}_{\min}}$. We indeed obtain an analytic isomorphism

$$\pi_{\mathfrak{F}_{\min}} : Z_{\mathfrak{F}_{\min}} \xrightarrow{\cong} \mathfrak{U}_{[\mathfrak{F}_{\min}]}. \tag{2.20.4}$$

Therefore all $\pi_{\mathfrak{F}} : Z_{\mathfrak{F}} \longrightarrow \overline{\mathcal{A}}_{g,\Gamma}\ \forall\mathfrak{F}$ are local isomorphism by the diagram 2.20.2.

F4: Let $\mathfrak{F}$ be an arbitrary cusp. We define

$$\mathcal{O}(\mathfrak{F},[\sigma^{\mathfrak{F}}]_{\mathfrak{F}}) := \pi_{\mathfrak{F}}(\mathcal{Y}(\mathfrak{F},[\sigma^{\mathfrak{F}}]_{\mathfrak{F}})) = \pi'_{\mathfrak{F}}(\mathcal{S}(\mathfrak{F},\sigma^{\mathfrak{F}}))\ \forall\sigma^{\mathfrak{F}} \in \Sigma_{\mathfrak{F}}.$$

Since $\pi_{\mathfrak{F}}$ is a local isomorphism, $\{\mathcal{O}(\mathfrak{F},[\sigma_\alpha^{\mathfrak{F}}]_{\mathfrak{F}})\}_{\sigma_\alpha^{\mathfrak{F}}\in\Sigma_{\mathfrak{F}}}$ is also a stratification of $\pi_{\mathfrak{F}}(Z_{\mathfrak{F}})$. In particular, $\{\mathcal{O}(\mathfrak{F}_{\min},[\sigma_\alpha^{\mathfrak{F}_{\min}}]_{\mathfrak{F}_{\min}})\}_{\sigma_\alpha^{\mathfrak{F}_{\min}}\in\Sigma_{\mathfrak{F}_{\min}}}$ is a stratification of $\mathfrak{U}_{\mathfrak{F}_{\min}}$. Furthermore, we have isomorphisms

$$\pi_{\mathfrak{F}_{\min}} : \mathcal{Y}(\mathfrak{F}_{\min},[\sigma_\alpha^{\mathfrak{F}_{\min}}]_{\mathfrak{F}_{\min}}) \xrightarrow{\cong} \mathcal{O}(\mathfrak{F}_{\min},[\sigma_\alpha^{\mathfrak{F}_{\min}}]_{\mathfrak{F}_{\min}})\ \forall\sigma_\alpha^{\mathfrak{F}_{\min}} \in \Sigma_{\mathfrak{F}_{\min}}. \tag{2.20.5}$$

For any two cusps $\mathfrak{F}_{\min}$ and $\mathfrak{F}'_{\min}$, we observe that $\mathfrak{U}_{[\mathfrak{F}_{\min}]} \neq \mathfrak{U}_{[\mathfrak{F}'_{\min}]}$ if and only if $[\mathfrak{F}_{\min}] \neq [\mathfrak{F}'_{\min}]$. Thus, the toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}$ is covered by $[\mathrm{Sp}(g,\mathbb{Z}):\Gamma]$ different open sets, i.e., $\overline{\mathcal{A}}_{g,\Gamma} = \bigcup_{i=1}^{[\mathrm{Sp}(g,\mathbb{Z}):\Gamma]} \mathfrak{U}_{[\mathfrak{F}^i_{\min}]}$ where $\mathfrak{F}^i_{\min}$'s are minimal cusps such that $[\mathfrak{F}^i_{\min}] \neq [\mathfrak{F}^j_{\min}]$ if $i \neq j$. Therefore, to study D_∞ on $\overline{\mathcal{A}}_{g,\Gamma}$, it is sufficient to study all codimension-one strata on $\mathfrak{U}_{\mathfrak{F}_0}$.

We always fixed $\mathfrak{F}^1_{\min}$ as the standard minimal cusp $\mathfrak{F}_0$. Let ρ be an edge in $\Sigma_{\mathfrak{F}_0}$ and $\mathcal{O}(\mathfrak{F}_0, [\rho]_{\mathfrak{F}_0})$ the associated stratum in $\mathfrak{U}_{[\mathfrak{F}_0]}$.

i. **Claim 1.** *Suppose that* $[\mathfrak{F}_{\min}] \neq [\mathfrak{F}_0]$. *That* $\mathcal{O}(\mathfrak{F}_0, [\rho]_{\mathfrak{F}_0}) \bigcap \mathfrak{U}_{[\mathfrak{F}_{\min}]} \neq \emptyset$ *if and only if there is an element* $\gamma \in \Gamma$ *such that*

$$\rho \subset (\overline{C(\mathfrak{F}_0)} \setminus C(\mathfrak{F}_0)) \bigcap (\overline{C(\gamma\mathfrak{F}_{\min})} \setminus C(\gamma\mathfrak{F}_{\min})).$$

Proof of Claim 1.*The "if" part : By the lemma 2.15, there is a cusp* $\mathfrak{F}_\rho$ *of depth one such that* $\mathfrak{F}_0 \prec \mathfrak{F}_\rho$ *and* $\mathrm{Int}(\rho) = C(\mathfrak{F}_\rho)$. *Clearly,* $C(\mathfrak{F}_\rho)$ *is also a rational boundary component of* $C(\gamma\mathfrak{F}_{\min})$. *Thus,* $\mathfrak{F}_0 \prec \mathfrak{F}_\rho$ *and* $\gamma\mathfrak{F}_{\min} \prec \mathfrak{F}_\rho$. *The gluing condition 2.14.1, together with 2.20.3 and 2.20.5 shows that* $\mathcal{O}(\mathfrak{F}_0, [\rho]_{\mathfrak{F}_0}) \cap \mathfrak{U}_{[\mathfrak{F}_{\min}]} \neq \emptyset$.

The "only if" part : We have a $z \in \mathcal{O}(\mathfrak{F}_0, [\rho]_{\mathfrak{F}_0})$ *such that* $z \in \mathfrak{U}_{[\mathfrak{F}_0]} \cap \mathfrak{U}_{[\mathfrak{F}_{\min}]}$. *By the gluing condition 2.14.1, there exists a* $\gamma \in \Gamma$, *a cusp* $\mathfrak{F}'$ *and a point in* $x \in Z'_{\mathfrak{F}'}$ *such that* $\mathfrak{F}_0 \prec \mathfrak{F}', \gamma\mathfrak{F}_{\min} \prec \mathfrak{F}'$ *and* $\pi'_{\mathfrak{F}'}(x) = z$. *Thus the edge* ρ *is in* $\Sigma_{\mathfrak{F}'}$ *by the lemma 2.14 and so* $\rho \in \Sigma_{\gamma\mathfrak{F}_{\min}} = \gamma(\Sigma_{\mathfrak{F}_{\min}})$. *The cusp* $\mathfrak{F}'$ *can not be* $\mathfrak{F}_0$ *by the condition* $[\mathfrak{F}_{\min}] \neq [\mathfrak{F}_0]$. *Since* $\rho \in \Sigma_{\mathfrak{F}'} \subset \Sigma_{\mathfrak{F}_0} \cap \Sigma_{\gamma\mathfrak{F}_{\min}}$, *we obtain* $\rho \subset (\overline{C(\mathfrak{F}_0)} \setminus C(\mathfrak{F}_0)) \bigcap (\overline{C(\gamma\mathfrak{F}_{\min})} \setminus C(\gamma\mathfrak{F}_{\min}))$.

ii. We construct a global irreducible divisor D_ρ in $\overline{\mathcal{A}}_{g,\Gamma}$ by the edge ρ as follows :

- Suppose that ρ is in the set Interior-edge. The claim 1 shows that

$$\mathcal{O}(\mathfrak{F}_0, [\rho]_{\mathfrak{F}_0}) \subset \overline{\mathcal{A}}_{g,\Gamma} \setminus \bigcup_{i=2}^{[\mathrm{Sp}(g,\mathbb{Z}):\Gamma]} \mathfrak{U}_{[\mathfrak{F}^i_{\min}]}.$$

  Let D_ρ be the closure of $\mathcal{O}(\mathfrak{F}_0, [\rho]_{\mathfrak{F}_0})$ in $\overline{\mathcal{A}}_{g,\Gamma}$. D_ρ is a global divisor in $\overline{\mathcal{A}}_{g,\Gamma}$, and

$$D_\rho \subset \overline{\mathcal{A}}_{g,\Gamma} \setminus \bigcup_{i=2}^{[\mathrm{Sp}(g,\mathbb{Z}):\Gamma]} \mathfrak{U}_{[\mathfrak{F}^i_{\min}]} \subset \mathfrak{U}_{[\mathfrak{F}^1_{\min}]} = \mathfrak{U}_{[\mathfrak{F}_0]}.$$

- Suppose that ρ is in the set boundary-edge. By the above case we also have that $D_{\rho'} \subset \mathfrak{U}_{[\mathfrak{F}_0]}$ if and only if ρ' is in the set Interior-edge. Thus we can rearrange the order of $\mathfrak{F}^i_{\min}$'s and get an integer $l \geq 2$ such that

$$\begin{cases} \mathcal{O}(\mathfrak{F}_0, [\rho]_{\mathfrak{F}_0}) \cap \mathfrak{U}_{[\mathfrak{F}^i_{\min}]} \neq \emptyset, & \text{for } i = 1, \cdots l; \\ \mathcal{O}(\mathfrak{F}_0, [\rho]_{\mathfrak{F}_0}) \cap \mathfrak{U}_{[\mathfrak{F}^i_{\min}]} = \emptyset, & \text{for other } i. \end{cases}$$

  Due to the claim 1, we let $\rho \subset \Sigma_{\mathfrak{F}^i_{\min}}$ and ρ in $\overline{C(\mathfrak{F}^i_{\min})} \setminus C(\mathfrak{F}^i_{\min})$ only for $i = 1, \cdots, l$. Then, there is a cusp $\mathfrak{F}_\rho$ of depth one such that $\rho \in \Sigma_{\mathfrak{F}_\rho}$ and $\mathfrak{F}^i_{\min} \prec \mathfrak{F}_\rho$ for all $i = 1, \cdots, l$ by the lemma 2.15. For each integer i in $[1, \cdots, l]$, the equality 2.20.3 says that $\mathcal{O}_\rho := \pi_{\mathfrak{F}_\rho}(\mathcal{Y}(\mathfrak{F}_\rho, [\rho]_{\mathfrak{F}_\rho}))$ is in the stratum $\mathcal{O}_i := \mathcal{O}(\mathfrak{F}^i_{\min}, [\rho]_{\mathfrak{F}^i_{\min}})$ of $\mathfrak{U}_{[\mathfrak{F}^i_{\min}]}$, and $\mathcal{O}_\rho$ is an open subset in each $\mathcal{O}_i$ since each $\pi_{\mathfrak{F}_\rho}$

is a local isomorphism. We glue all $\mathcal{O}_1, \cdots, \mathcal{O}_l$ together along $\mathcal{O}_\rho$ to obtain an analytic subspace $\mathcal{S}_\rho$. Therefore, the closure D_ρ of $\mathcal{S}_\rho$ is a global divisor in $\overline{\mathcal{A}}_{g,\Gamma}$.

Now we begin to prove the statements (1) and(2) in the theorem.

1. Define a set

$$\text{All-boundary-edge} := \bigcup_{\rho \in \text{Boundary-edge}} \bigcup_{\gamma \in \mathrm{Sp}(g,\mathbb{Z})} \gamma(\rho).$$

From the construction of the divisor by an edge in $\Sigma_{\mathfrak{F}_0}$, we immediately obtain :

$$\begin{aligned} & \text{the number of irreducible components of } D_\infty \\ = \ & \#\{\Gamma\text{-orbits in All-boundary-edge}\} + [\mathrm{Sp}(g,\mathbb{Z}) : \Gamma] \times \#\{\Gamma_{\mathfrak{F}_0}\text{-orbits in Interior-edge}\} \\ = \ & [\mathrm{Sp}(g,\mathbb{Z}) : \Gamma] + [\mathrm{Sp}(g,\mathbb{Z}) : \Gamma] \times \#\{\Gamma_{\mathfrak{F}_0}\text{-orbits in Interior-edge}\}. \end{aligned}$$

The last equality is due to the lemma 2.15 and the fact that every two cusps of depth one are $\mathrm{Sp}(g,\mathbb{Z})$-equivalent(cf.Remark(4.16) in §5 [26]).

2. Suppose that the decomposition of $\Sigma_{\mathfrak{F}_0}$ is Γ-separable. Let $\mathfrak{F}$ be an arbitrary cusp. By the lemma 2.19 the induced $\overline{\Gamma_{\mathfrak{F}}}$-admissible polyhedral decomposition $\Sigma_{\mathfrak{F}} = \{\sigma^{\mathfrak{F}}\}$ of $C(\mathfrak{F})$ is Γ-separable.

Claim 2. *For any $\gamma \in \Gamma_{\mathfrak{F}}$ and nontrivial $\sigma^{\mathfrak{F}} \in \Sigma_{\mathfrak{F}}$, the following are equivalent :*
i. $\overline{\mathcal{S}(\mathfrak{F}, \sigma^{\mathfrak{F}})}^{\mathrm{cl}} \bigcap \overline{\mathcal{S}(\mathfrak{F}, \gamma(\sigma^{\mathfrak{F}}))}^{\mathrm{cl}} \neq \emptyset$,
ii. γ acts as the identity on any cone in the set $\{\tau \in \Sigma_{\mathfrak{F}} \mid \tau \succeq \sigma^{\mathfrak{F}}\}$.

Proof of Claim 2. *Suppose $\overline{\mathcal{S}(\mathfrak{F}, \sigma^{\mathfrak{F}})}^{\mathrm{cl}} \bigcap \overline{\mathcal{S}(\mathfrak{F}, \gamma(\sigma^{\mathfrak{F}}))}^{\mathrm{cl}} \neq \emptyset$. We have*

$$\overline{\mathcal{S}(\mathfrak{F}, \sigma^{\mathfrak{F}})}^{\mathrm{cl}} = \coprod_{\delta \in \Sigma_{\mathfrak{F}}, \delta \succeq \sigma^{\mathfrak{F}}} \mathcal{S}(\mathfrak{F}, \delta), \quad \textit{and} \quad \overline{\mathcal{S}(\mathfrak{F}, \gamma(\sigma^{\mathfrak{F}}))}^{\mathrm{cl}} = \coprod_{\delta \in \Sigma_{\mathfrak{F}}, \delta \succeq \gamma(\sigma^{\mathfrak{F}})} \mathcal{S}(\mathfrak{F}, \delta)$$

by the lemma 2.14. Because the collection $\{\mathcal{S}(\mathfrak{F}, \sigma)\}_{\sigma \in \Sigma_{\mathfrak{F}}}$ is a stratification of $Z'_{\mathfrak{F}}$, we a cone $\delta \in \Sigma_{\mathfrak{F}}$ such that $\delta \succeq \sigma^{\mathfrak{F}}$ and $\delta \succeq \gamma(\sigma^{\mathfrak{F}})$. Thus $\sigma^{\mathfrak{F}} \subset \delta \cap \gamma^{-1}(\delta)$. Since $\Sigma_{\mathfrak{F}}$ is Γ-separable, γ acts as the identity on any cone $\tau \in \Sigma_{\mathfrak{F}}$ containing $\sigma^{\mathfrak{F}}$.

Using similar arguments in Claim 2, we also obtain

Claim 3. *Let $\gamma \in \Gamma_{\mathfrak{F}}$ and let $\sigma^{\mathfrak{F}} \neq \{0\}$ be a cone in $\Sigma_{\mathfrak{F}}$. If $\mathcal{S}(\mathfrak{F}, \sigma^{\mathfrak{F}}) \cap S(\mathfrak{F}, \gamma(\sigma^{\mathfrak{F}})) \neq \emptyset$ then $\mathcal{S}(\mathfrak{F}, \sigma^{\mathfrak{F}}) = S(\mathfrak{F}, \gamma(\sigma^{\mathfrak{F}}))$ and the restriction $\Pi'_{\mathfrak{F},\gamma\mathfrak{F}}|_{\mathcal{S}(\mathfrak{F},\sigma^{\mathfrak{F}})} : \mathcal{S}(\mathfrak{F}, \sigma^{\mathfrak{F}}) \longrightarrow S(\mathfrak{F}, \gamma(\sigma^{\mathfrak{F}}))$ is just the identification on $\mathcal{S}(\mathfrak{F}, \sigma^{\mathfrak{F}})$.*

By the claims 2 and 3, we have $\mathrm{pr}_{\mathfrak{F}} : \overline{\mathcal{S}(\mathfrak{F}, \sigma^{\mathfrak{F}})}^{\mathrm{cl}} \xrightarrow{\cong} \overline{\mathcal{Y}(\mathfrak{F}, [\sigma^{\mathfrak{F}}]_{\mathfrak{F}})}^{\mathrm{cl}}$ $\forall \{0\} \neq \sigma^{\mathfrak{F}} \in \Sigma_{\mathfrak{F}}$. Moreover, The statement(1) of the corollary 2.2 guarantees that $\overline{\mathcal{Y}(\mathfrak{F}, [\sigma^{\mathfrak{F}}]_{\mathfrak{F}})}^{\mathrm{cl}}$ has only normal singularities for any nonzero cone $\sigma^{\mathfrak{F}} \in \Sigma_{\mathfrak{F}}$. By the symmetry, we only need to consider singularities in the open set $\mathfrak{U}_{[\mathfrak{F}_0]}$. Since $\pi_{\mathfrak{F}_0} : Z_{\mathfrak{F}_0} \xrightarrow{\cong} \mathfrak{U}_{[\mathfrak{F}_0]}$ is an isomorphism, there is an isomorphism $\pi_{\mathfrak{F}_0} : \overline{\mathcal{Y}(\mathfrak{F}_0, [\sigma_\alpha^{\mathfrak{F}_0}]_{\mathfrak{F}_0})}^{\mathrm{cl}} \xrightarrow{\cong} \overline{\mathcal{O}(\mathfrak{F}_0, [\sigma_\alpha^{\mathfrak{F}_0}]_{\mathfrak{F}_0})}^{\mathrm{cl}}$ for any cone $\sigma_\alpha^{\mathfrak{F}_0} \in \Sigma_{\mathfrak{F}_0}$, where $\overline{\mathcal{O}(\mathfrak{F}_0, [\sigma_\alpha^{\mathfrak{F}_0}]_{\mathfrak{F}_0})}^{\mathrm{cl}}$ is the closure of $\mathcal{O}(\mathfrak{F}_0, [\sigma_\alpha^{\mathfrak{F}_0}]_{\mathfrak{F}_0})$ in $\mathfrak{U}_{[\mathfrak{F}_0]}$.

For each edge ρ in $\Sigma_{\mathfrak{F}_0}$, the global divisor D_ρ has that

$$D_\rho \cap \mathfrak{U}_{[\mathfrak{F}_0]} = \overline{\mathcal{O}(\mathfrak{F}_0, [\rho]_{\mathfrak{F}_0})}^{\mathrm{cl}} \cong \overline{\mathcal{Y}(\mathfrak{F}_0, [\rho]_{\mathfrak{F}_0})}^{\mathrm{cl}}.$$

Thus D_ρ is a normal variety. In particular D_ρ has non self-intersections.

Since $\Sigma_{\mathfrak{F}_0}$ is regular with respect to Γ, the statement (2) of the corollary 2.2 guarantees that $Z'_{\mathfrak{F}_0} = \overline{\mathcal{S}(\mathfrak{F}, \{0\})}^{\mathrm{cl}}$ is smooth and $\overline{\mathcal{O}(\mathfrak{F}, [\sigma^{\mathfrak{F}}]_{\mathfrak{F}})}^{\mathrm{cl}} (\cong \overline{\mathcal{S}(\mathfrak{F}, \sigma^{\mathfrak{F}})}^{\mathrm{cl}})$ is also smooth for any $\sigma^{\mathfrak{F}} \in \Sigma_{\mathfrak{F}}$ with $\sigma^{\mathfrak{F}} \neq \{0\}$. Again by the statement (2) of the corollary 2.2, we obtain that the irreducible components of D_∞ intersect transversely.

Now we suppose that Γ is neat. The fundamental group of $A_{g,\Gamma}$ is then isomorphic to Γ, and so $\overline{\Gamma_{\mathfrak{F}}}/U^{\mathfrak{F}} \cap \Gamma$ acts freely on $Z'_{\mathfrak{F}}$ for any cusp $\mathfrak{F}$, thus the morphism $\pi'_{\mathfrak{F}} : Z'_{\mathfrak{F}} \to \overline{\mathcal{A}}_{g,\Gamma}$ is étale. Therefore, $\mathfrak{U}_{[\mathfrak{F}_0]}$ is smooth since that $Z'_{\mathfrak{F}_0}$ is smooth.

□

Remark. Assume the condition that Γ is neat and the decomposition $\Sigma_{\mathfrak{F}_0}$ is Γ-separable. When an edge $\rho \in \Sigma_{\mathfrak{F}_0}$ is exactly in the set Boundary-edge, by using the argument in Theorem 2.2 of [32] we can assert that the associated irreducible boundary divisor $D_\rho \subset \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ is actually a smooth toroidal compactification $\overline{X}$ of some locally symmetric variety X. The variety X actually has a direct factor like a Siegel variety $\mathcal{A}_{g-1,\Gamma'}$ for some arithmetic subgroup $\Gamma' \subset \mathrm{Sp}(g-1,\mathbb{Z})$ induced by Γ. The toroidal compactification $\overline{X}$ is then constructed by a Γ-admissible family which is induced by the decomposition $\Sigma_{\mathfrak{F}_0}$. Moreover, if two edges ρ_2, ρ_2 in $\Sigma_{\mathfrak{F}_0}$ are both in the set Boundary-edge then $D_{\rho_1} \cong D_{\rho_2}$.

Definition 2.21. Let $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ be an arithmetic subgroup. Let $\Sigma_{\mathfrak{F}_0} := \{\sigma_\alpha^{\mathfrak{F}_0}\}$ be an $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$, and let $\overline{\mathcal{A}}_{g,\Gamma}$ be the symmetric toroidal compactification of $\mathcal{A}_{g,\Gamma}$ constructed by $\Sigma_{\mathfrak{F}_0}$.

With respect to the open morphism $\pi'_{\mathfrak{F}_0} : Z'_{\mathfrak{F}_0} \to \overline{\mathcal{A}}_{g,\Gamma}$, we define :

1. A top-dimensional cone $\sigma_{\max}$ in $\Sigma_{\mathfrak{F}_0}$ is said to be **Γ-fine** if the restriction $\pi'_{\mathfrak{F}_0}|_{B_\rho}$ is an isomorphism onto its image for every edge ρ of itself, where B_ρ is the divisor constructed by ρ on $Z'_{\mathfrak{F}_0}$.
2. The constructed symmetric toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}$ of $\mathcal{A}_{g,\Gamma}$ is called **geometrically Γ-fine** if the following condition is satisfied : The restriction $\pi'_{\mathfrak{F}_0}|_{B_\rho}$ is an isomorphism onto its image, where B_ρ is the divisor on $Z'_{\mathfrak{F}_0}$ constructed by ρ, ρ running over the edges of $\Sigma_{\mathfrak{F}_0}$.

The proof of the theorem 2.20 tells us that a Γ-separable decomposition $\Sigma_{\mathfrak{F}_0}$ will induce a geometrically Γ-fine symmetric toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}$ of $\mathcal{A}_{g,\Gamma}$.

Theorem 2.22. *Let $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ be a neat arithmetic subgroup and let $\Sigma_{\mathfrak{F}_0}$ be a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$. Let $\overline{\mathcal{A}}_{g,\Gamma}$ be the toroidal compactification of $\mathcal{A}_{g,\Gamma}$ constructed by $\Sigma_{\mathfrak{F}_0}$.*

Assume that the decomposition $\Sigma_{\mathfrak{F}_0}$ is regular with respect to Γ. The following four conditions are equivalent :

i. Every irreducible component of $D_\infty = \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ has non self-intersections;
ii. The compactification $\overline{\mathcal{A}}_{g,\Gamma}$ is geometrically Γ-fine;
iii. the decomposition $\Sigma_{\mathfrak{F}_0}$ is Γ-separable;
iv. The infinity boundary divisor $D_\infty = \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ is simple normal crossing.

Proof. By the corollary 2.2, that (i)$\Longleftrightarrow$(iv) is obviously true. The proof of the theorem 2.20 actually shows that (i)$\Longleftrightarrow$(ii) and (iii)$\Longrightarrow$(ii).

We now begin to show that (ii)$\Longrightarrow$(iii).

Since $\mathfrak{F}_0$ is a minimal cusp of $\mathfrak{H}_g$, $Z'_{\mathfrak{F}_0}$ is isomorphic to the toroidal variety $X_{\mathfrak{F}_0}$. Let σ be an arbitrary cone in $\Sigma_{\mathfrak{F}_0}$. Suppose that $\gamma \in \overline{\Gamma_{\mathfrak{F}}}$ is an element such that $\gamma(\sigma)\cap\sigma \neq \{0\}$.

We know that $\tau := \gamma(\sigma) \cap \sigma$ is a face of σ. Let ρ be an arbitrary edge of τ. Then, $\rho_1 := \gamma^{-1}(\rho)$ is also an edge of σ. Let B_ρ(resp. B_{ρ_1}) be the divisor on $Z'_{\mathfrak{F}_0}$ corresponding to the edge ρ(resp. ρ_1). We must have $\pi'_{\mathfrak{F}_0}(B_\rho) = \pi'_{\mathfrak{F}_0}(B_{\rho_1})$.

Claim(*) $\gamma(\rho) = \rho$: *Otherwise, there is a top-dimensional cone $\sigma_{\max} \in \Sigma_{\mathfrak{F}_0}$ containing σ and so B_ρ intersects with B_{ρ_1} transversely by the corollary 2.2. It contradicts the condition that $\overline{\mathcal{A}}_{g,\Gamma}$ is geometrically Γ-fine.*

Since γ is an automorphism of the lattice $\Gamma \cap U^{\mathfrak{F}_0}(\mathbb{Z})$, γ acts as the identity on the edge ρ so that γ acts as the identity on the cone τ.

Claim (**) $\tau = \sigma$: *Otherwise, $\gamma(\sigma)$ and σ are two different cone in $\Sigma_{\mathfrak{F}_0}$. Let $\mathcal{O}^\sigma$ (resp. $\mathcal{O}^{\gamma(\sigma)}$) be the orbit in $Z'_{\mathfrak{F}_0}$ corresponding to the cone σ(resp. $\gamma(\sigma)$). Let δ be an edge of τ. Then $\mathcal{O}^\sigma \cup \mathcal{O}^{\gamma(\sigma)} \subset B_\delta$, $\mathcal{O}^\sigma \cap \mathcal{O}^{\gamma(\sigma)} = \emptyset$. On the other hand, we also have $\pi'_{\mathfrak{F}_0}(\mathcal{O}^\sigma) = \pi'_{\mathfrak{F}_0}(\mathcal{O}^{\gamma(\sigma)})$, so that the image $\pi'_{\mathfrak{F}_0}(B_\delta)$ must have self-intersections. It is a contradiction.*

Therefore the γ acts as the identity on the cone σ. $\square$

Example 2.23 (Central cone decomposition)**.** In [18] and [26], Igusa and Namikawa introduce a projective $\mathrm{GL}(g,\mathbb{Z})$-admissible rational polyhedral decomposition Σ_{cent}(**central cone decomposition**) of $C(\mathfrak{F}_0)$ containing **principal cone**

$$\sigma_0 := \{X = (x_{ij}) \in \mathrm{Sym}_g(\mathbb{R}) \mid x_{ij} \leq 0 (i \neq j), \sum_{j=1}^{g} x_{ij} \geq 0 (\forall i)\},$$

which is top-dimensional regular cone with respect to the lattice basis of $U^{\mathfrak{F}_0}(\mathbb{Z})$. If $g \leq 3$ then the following properties are satisfied:

- That Σ_{cent} is regular with respect to $\text{Sp}(g, Z)$, and all edges of top-dimensional cones in the decomposition Σ_{cent} are on the boundary of $C(\mathfrak{F}_0)$;
- the principal cone σ_0 is the unique maximal cone in Σ_{cent} up to $\text{GL}(g, \mathbb{Z})$.

Therefore, we obtain that *if the genus* $g \leq 3$ *then the central cone decomposition* Σ_{cent} *can not be* $\text{Sp}(g, \mathbb{Z})$*-separable so that the boundary divisor of the induced toroidal compactification is not normal crossing.*

Corollary 2.24. *Let* $\Gamma \subset \text{Sp}(g, \mathbb{Z})$ *be a neat arithmetic subgroup and let* $\overline{\mathcal{A}}_{g,\Gamma}$ *be a geometrically* Γ*-fine toroidal compactification of the Siegel variety* $\mathcal{A}_{g,\Gamma} := \mathfrak{H}_g/\Gamma$ *constructed by a* $\overline{\Gamma_{\mathfrak{F}_0}}$*(or* $\text{GL}(g, \mathbb{Z})$*)-admissible polyhedral decomposition* $\Sigma_{\mathfrak{F}_0} := \{\sigma_\alpha^{\mathfrak{F}_0}\}$ *of* $C(\mathfrak{F}_0)$ *regular with respect to* Γ*, where* $\mathfrak{F}_0$ *is the standard minimal cusp of* $\mathfrak{H}_g$*.*

Let $D_1, \cdots, D_d$ *be* d *different irreducible components of the simple normal crossing boundary divisor* $D_\infty = \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$*. We have :*

1. *That* $D_1 \cap \cdots \cap D_d \neq \emptyset$ *if and only if that* $d \leq \dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma}$ *and there exists a minimal cusp* $\mathfrak{F}_{\min}$ *of* $\mathfrak{H}_g$ *and a top-dimensional cone* $\sigma_{\max}$ *in* $\Sigma_{\mathfrak{F}_{\min}}$ *with* d *differential edges* ρ_i $i = 1, \cdots, d$ *such that* $D_i = D_{\rho_i}$ $i = 1, \cdots, d$*, where* D_ρ *is the divisor constructed by an edge* ρ*.*
2. *Assume that* $d = \dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma}$*. There are only two cases:*
   - $D_1 \cap \cdots \cap D_d = \emptyset$ *and so* $D_1 \cdot D_2 \cdots D_d = 0$*.*
   - $D_1 \cap \cdots \cap D_d \neq \emptyset$ *and the intersection number* $D_1 \cdot D_2 \cdots D_d = 1$*.*

Proof. It is straightforward by the intersection theory on toric geometry(cf.[15]). □

3. Volume forms related to compactifications and constrained conditions of decompositions of cones from the viewpoint of Kähler-Einstein metric

We still denote $\mathfrak{F}_k$ the cusp $\mathfrak{F}(V^{(g-k)})$ of $\mathfrak{H}_g$ for $1 \leq k \leq g$. We take coordinate system $\tau = (\tau_{ij})_{1 \leq i,j \leq g} \in \mathfrak{H}_g$ of the Siegel space $\mathfrak{H}_g = \{\tau \in M_g(\mathbb{C}) \mid \tau = {}^t\tau, \text{Im}(\tau) > 0\}$. The Bergman metric on $\mathfrak{H}_g$ is

$$ds^2 = \sum_{1 \leq i \leq j \leq g, 1 \leq k \leq l \leq g} g_{ij,\overline{kl}} d\tau_{ij} d\overline{\tau_{kl}} := \text{Tr}(d\tau \text{Im}(\tau)^{-1} d\overline{\tau} \text{Im}(\tau)^{-1})$$

and its Kähler form is $\omega_{\mathrm{can}} = \frac{\sqrt{-1}}{2} \sum_{1\leq i\leq j\leq g, 1\leq k\leq l\leq g} g_{ij,\overline{kl}} d\tau_{ij} \wedge d\overline{\tau_{kl}}$. Then the volume form is

$$\begin{aligned}
\frac{1}{(\frac{g(g+1)}{2})!}\omega_{\mathrm{can}}^{\frac{g(g+1)}{2}} &= (\frac{\sqrt{-1}}{2})^{\frac{g(g+1)}{2}} \det(g_{ij,\overline{kl}}) dV_g \\
&= (\frac{\sqrt{-1}}{2})^{\frac{g(g+1)}{2}} \frac{dV_g}{(\det \mathrm{Im}(\tau))^{g+1}} =: \Phi_g(\tau),
\end{aligned}$$

where

$$\begin{aligned}
dV_g(\tau) &:= 2^{\frac{g(g-1)}{2}} \bigwedge_{1\leq i\leq j\leq g} d\tau_{ij} \wedge d\overline{\tau_{ij}} \\
&= (-1)^{\frac{(g-1)g(g+1)(g+2}{8}} 2^{\frac{g(g-1)}{2}} (\bigwedge_{1\leq i\leq j\leq g} d\tau_{ij}) \wedge (\bigwedge_{1\leq i\leq j\leq g} d\overline{\tau_{ij}})
\end{aligned}$$

(cf.[29]). The Bergman metric is Kähler-Einstein, i.e.,

$$\sqrt{-1}\partial\overline{\partial} \log \det(g_{ij,\overline{kl}}) = -\sqrt{-1}\partial\overline{\partial} \log(\det \mathrm{Im}(\tau))^{g+1} = \frac{g+1}{2}\omega_{\mathrm{can}}. \tag{3.0.1}$$

Let $\Gamma \subset \mathrm{Sp}(g, \mathbb{Z})$ be a neat arithmetic subgroup. Since$(\mathfrak{H}_g, ds^2)$ is $\mathrm{Sp}(g, \mathbb{R})$-invariant, it induces a canonical metric on the smooth Siegel variety $\mathcal{A}_{g,\Gamma} = \mathfrak{H}_g/\Gamma$. The canonical metric is a complete Kähler-Einstein metric with negative Ricci curvature. The volume form Φ_g is also $\mathrm{Sp}(g, \mathbb{R})$-invariant, and so we have an induced volume form $\Phi_{g,\Gamma}$ on $\mathcal{A}_{g,\Gamma}$. It is known that $\Phi_{g,\Gamma}$ is singular at the boundary divisor $D_\infty := \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ for any smooth toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}$.

3.1. **Volume forms of the Siegel space $\mathfrak{H}_g$ associated to cusps.** Associated to a cusp $\mathfrak{F}_{g-k}$, We can write the volume form Φ_g in the coordinate system explicitly.

Now we identify the Siegel Space $\mathfrak{H}_g$ with $\mathfrak{S}_g$ as in the proposition 1.2. According to the embedding

$$\varphi : u_{\mathbb{C}}^{\mathfrak{F}_{g-k}} \times v_{\mathbb{R}}^{\mathfrak{F}_{g-k}} \times \mathfrak{F}_{g-k} \xrightarrow{\cong} D(\mathfrak{F}_{g-k}) \ \ (u_1 + \sqrt{-1}u_2, v, F) \mapsto \exp(u_1 + \sqrt{-1}u_2) \exp(v)(\check{F})$$

and the isomorphism $\varphi : (u^{\mathfrak{F}_{g-k}} + \sqrt{-1}C(\mathfrak{F}_{g-k})) \times v_{\mathbb{R}}^{\mathfrak{F}_{g-k}} \times \mathfrak{F}_{g-k} \xrightarrow{\cong} \mathfrak{S}_g$ in the proposition 2.11,we obtain that

$$\begin{aligned}
\mathfrak{S}_g = \{ & \begin{pmatrix} I_{g-k} & 0 & 0 & 0 \\ 0 & I_k & 0 & Z \\ 0 & 0 & I_{g-k} & 0 \\ 0 & 0 & 0 & I_k \end{pmatrix} \cdot \begin{pmatrix} I_{g-k} & 0 & 0 & A \\ -{}^tB & I_k & {}^tA & \frac{{}^tAB - {}^tBA}{2} \\ 0 & 0 & I_{g-k} & B \\ 0 & 0 & 0 & I_k \end{pmatrix} \check{F} \\
& \mid Z = X + \sqrt{-1}Y \in (\mathrm{Sym}_k(\mathbb{R}) + \sqrt{-1}\mathrm{Sym}_k^+(\mathbb{R})), A + \sqrt{-1}B \in M_{g-k,k}(\mathbb{C}), \check{F} \in \mathfrak{F}_k^\vee \}
\end{aligned}$$

by the corollary 2.12, where $\mathrm{Sym}_k^+(\mathbb{R}) = \{Y \in \mathrm{Sym}_k(\mathbb{R}) \mid Y > 0\}$. Suppose $\tau' \in \mathfrak{H}_{g-k}$ and $Z = X + \sqrt{-1}Y \in (\mathrm{Sym}_k(\mathbb{R}) + \sqrt{-1}\mathrm{Sym}_k^+(\mathbb{R}))(=\mathfrak{H}_k)$ now. We note that each $\check{F} \in \mathfrak{F}_k^\vee$ can be written as :

$$(!) \qquad F_{\tau'} = \text{ subspace of } V_{\mathbb{C}} \text{ spanned by the column vectors of } \begin{pmatrix} \tau' & 0 \\ 0 & 0_k \\ I_{g-k} & 0 \\ 0 & I_k \end{pmatrix} \text{ for } \tau' \in \mathfrak{H}_{g-k}.$$

Thus, we get

$$\begin{aligned}
\mathfrak{S}_g \quad &\ni \quad \begin{pmatrix} I_{g-k} & 0 & 0 & 0 \\ 0 & I_k & 0 & Z \\ 0 & 0 & I_{g-k} & 0 \\ 0 & 0 & 0 & I_k \end{pmatrix} \cdot \begin{pmatrix} I_{g-k} & 0 & 0 & A \\ -{}^tB & I_k & {}^tA & \frac{{}^tAB - {}^tBA}{2} \\ 0 & 0 & I_{g-k} & B \\ 0 & 0 & 0 & I_k \end{pmatrix} F_{\tau'} \\
&= \quad \text{subspace of } V_{\mathbb{C}} \text{ spanned by the column vectors of } \begin{pmatrix} \tau' & A \\ {}^t(A - \tau' B) & Z + \frac{{}^tAB - {}^tBA}{2} \\ I_{g-k} & B \\ 0 & I_k \end{pmatrix} \\
&= \quad \text{subspace of } V_{\mathbb{C}} \text{ spanned by the column vectors of} \\
&\quad \begin{pmatrix} \tau' & (A - \tau' B) \\ {}^t(A - \tau' B) & Z +^t B\tau' B - \frac{({}^tAB + {}^tBA)}{2} \\ I_{g-k} & 0 \\ 0 & I_k \end{pmatrix} \\
&=: \quad F_\tau^1.
\end{aligned}$$

Thus this F_τ^1 corresponds to a point $\tau := \begin{pmatrix} \tau' & (A - \tau' B) \\ {}^t(A - \tau' B) & Z +^t B\tau' B - \frac{({}^tAB + {}^tBA)}{2} \end{pmatrix}$ in $\mathfrak{H}_g$ as we describe in the proposition 1.2. We also have that

$$\mathrm{Im}(\tau) = \begin{pmatrix} \mathrm{Im}(\tau') & -\mathrm{Im}(\tau')B \\ -{}^tB\mathrm{Im}(\tau') & \mathrm{Im}(Z) +^t B\mathrm{Im}(\tau')B \end{pmatrix}$$

and $\det \mathrm{Im}(\tau) = \det \mathrm{Im}(\tau') \det \mathrm{Im}(Z)$. Write $\tau' = (t_{ij}),\ A = (a_{ij}),\ B = (b_{ij}),\ Z = (c_{ij}),\ \ S := A + \sqrt{-1}B = (s_{ij})$, and $U := A - \tau' B = (u_{ij})$. The $((c_{ij}), (s_{ij}), (t_{i,j}))$

becomes a coordinate system of $\mathfrak{H}_g$ associated to the cusp $\mathfrak{F}_{g-k}$. Then,

$$\begin{aligned}
du_{ij} &= da_{ij} + d(\sum_{\alpha} t_{i\alpha}b_{\alpha j}) \\
&= da_{ij} + \sum_{\alpha} t_{i\alpha}db_{\alpha j} + \text{ forms containing } dt \\
d\overline{u_{ij}} &= da_{ij} + \sum_{\alpha} \overline{t_{i\alpha}}db_{\alpha j} + \text{ forms containing } d\bar{t} \\
du_{ij}\wedge d\overline{u_{ij}} &= -2\sqrt{-1}\sum_{\alpha}\mathrm{Im}(t_{i\alpha})da_{ij}\wedge db_{\alpha j} + \sum_{\alpha,\beta} t_{i\alpha}\overline{t_{i\beta}}db_{\alpha j}\wedge db_{\beta j} \\
&\qquad + \text{forms containing } dt \text{ or } d\bar{t}.
\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
&(\frac{\sqrt{-1}}{2})^{k(g-k)}\bigwedge_{i,j} du_{ij}\wedge d\overline{u_{ij}} \\
= \;& \det\mathrm{Im}(t_{ij})(\bigwedge_{i,j} da_{ij}\wedge db_{ij}) + \text{ forms containing } dt \text{ or } d\bar{t} \\
= \;& (\frac{\sqrt{-1}}{2})^{k(g-k)}\det\mathrm{Im}(\tau^{'})(\bigwedge_{i,j} ds_{ij}\wedge d\overline{s_{ij}}) + \text{ forms containing } dt \text{ or } d\bar{t}\ .
\end{aligned}$$

Write $R := Z +^t B\tau^{'}B - \frac{(^tAB+^tBA)}{2} = (r_{ij})$, we calculate the volume form in coordinate system (c_{ij}, s_{ij}, t_{ij}) of $\mathfrak{H}_g$:

$$\begin{aligned}
dV_g(\tau) &= 2^{\frac{g(g-1)}{2}}(\bigwedge_{1\le i\le j\le g-k} dt_{ij}\wedge d\overline{t_{ij}})\wedge(\bigwedge_{ij} du_{ij}\wedge d\overline{u_{ij}})\wedge(\bigwedge_{1\le i\le j\le k} dr_{ij}\wedge d\overline{r_{ij}}) \\
&= 2^{\frac{g(g-1)}{2}}\det\mathrm{Im}(\tau^{'})(\bigwedge_{1\le i\le j\le g-k} dt_{ij}\wedge d\overline{t_{ij}})\wedge(\bigwedge_{ij} ds_{ij}\wedge d\overline{s_{ij}})\wedge(\bigwedge_{1\le i\le j\le k} dc_{ij}\wedge d\overline{c_{ij}}).
\end{aligned}$$

Define

$$d\mathrm{Vol}(\tau^{'}) := 2^{\frac{(g-k)(g-k-1)}{2}}\bigwedge_{1\le i\le j\le g-k} dt_{ij}\wedge d\overline{t_{ij}},\ d\mathrm{Vol}(Z) := 2^{\frac{k(k-1)}{2}}\bigwedge_{1\le i\le j\le k} dc_{ij}\wedge d\overline{c_{ij}},$$

and $d\mathrm{Vol}(S) := 2^{k(g-k)} \bigwedge_{1\leq i\leq g-k, 1\leq j\leq k} ds_{ij}\wedge d\overline{s_{ij}}$. Since $d\mathrm{Vol}(\tau^{'})$ is just the standard Euclidian volume form dV_{g-k} on $\mathfrak{H}_{g-k}$, we have

$$\begin{aligned}\Phi_g(\tau) &= (\frac{\sqrt{-1}}{2})^{\frac{g(g+1)}{2}}\frac{dV_g(\tau)}{(\det\mathrm{Im}(\tau))^{g+1}}\\ &= (\frac{\sqrt{-1}}{2})^{\frac{g(g+1)}{2}}\frac{\det\mathrm{Im}(\tau^{'})dV_{g-k}\wedge d\mathrm{Vol}(S)\wedge d\mathrm{Vol}(Z)}{(\det\mathrm{Im}(\tau^{'})\det\mathrm{Im}(Z))^{g+1}}\\ &= \Phi_{g-k}(\tau^{'})\wedge(\frac{\sqrt{-1}}{2})^{k(g-k)}\frac{d\mathrm{Vol}(S)}{(\det\mathrm{Im}(\tau^{'}))^{k-1}}\wedge(\frac{\sqrt{-1}}{2})^{\frac{k(k+1)}{2}}\frac{d\mathrm{Vol}(Z)}{(\det\mathrm{Im}(Z))^{g+1}}.\end{aligned}$$

Proposition 3.1. *Let $\mathfrak{F}=\mathfrak{F}(V^{(g-k)})$ be a k-th cusp of $\mathfrak{H}_g$.*
1. The Siegel space can be written as

$$\begin{aligned}\mathfrak{H}_g &= \Big\{\tau := \begin{pmatrix}\tau^{'} & (A-\tau^{'}B)\\ {}^t(A-\tau^{'}B) & Z+{}^tB\tau^{'}B-\frac{({}^tAB+{}^tBA)}{2}\end{pmatrix}\in M_g(\mathbb{C})\\ &\qquad \mid \tau^{'}=(t_{ij})\in\mathfrak{H}_{g-k},\ Z=(c_{ij})\in\mathfrak{H}_k,\ S=(s_{ij}):=A+\sqrt{-1}B\in M_{g-k,k}(\mathbb{C})\Big\},\end{aligned}$$

and $((c_{ij},(s_{ij}),(t_{ij}))$ becomes a coordinate system of $\mathfrak{H}_g$ associated to $\mathfrak{F}$.
2. We have the following formula of volume form :

$$\Phi_g(\tau)=\begin{cases}\Phi_{g-k}(\tau^{'})\bigwedge(\frac{\sqrt{-1}}{2})^{k(g-k)}\frac{d\mathrm{Vol}(S)}{(\det\mathrm{Im}(\tau^{'}))^{k-1}}\bigwedge(\frac{\sqrt{-1}}{2})^{\frac{k(k+1)}{2}}\frac{d\mathrm{Vol}(Z)}{(\det\mathrm{Im}(Z))^{g+1}}, & 1\leq k<g\\ (\frac{\sqrt{-1}}{2})^{\frac{g(g+1)}{2}}\frac{d\mathrm{Vol}(Z)}{(\det\mathrm{Im}(Z))^{g+1}}, & k=g.\end{cases}$$

3.2. **Local volume forms of low-degree Siegel varieties.** For any two integers $1\leq i,j\leq g$, let $E_{ij}=(a_{\alpha\beta})$ be a $(g\times g)$-matrix of $a_{\alpha\beta}=\begin{cases}1, & (\alpha,\beta)=(i,j);\\ 0, & \text{others.}\end{cases}$

We compute volume forms of Siegel varieties $\mathcal{A}_{g,n}$ of low genus g with respect to certain special compactifications. Let $g=2$ or 3 in this subsection. Let Σ_{cent} be the central cone decomposition of $C(\mathfrak{F}_0)$ and σ_0 the principal cone in Σ_{cent} defined in the example 2.23. We can write down the cone σ_0 clearly(cf.[18],[26]) : $\sigma_0=\{\sum_{1\leq i\leq j\leq g}\lambda_{i,j}\zeta_{i,j}\mid\lambda_{i,j}\in\mathbb{R}_{\geq0}\}$ such that every edge $\mathbb{R}_{\geq0}\zeta_{i,j}$ is in $\overline{C(\mathfrak{F}_0)}^{\mathrm{rc}}\setminus C(\mathfrak{F}_0)$, where $\begin{cases}\zeta_{i,i}:=E_{i,i}, & 1\leq i\leq g;\\ \zeta_{i,j}:=-E_{i,j}-E_{j,i}+E_{i,i}+E_{j,j}, & 1\leq i<j\leq g.\end{cases}$ Let $\overline{\mathcal{A}}_{g,n}^{\mathrm{cent}}$ be the projective smooth toroidal compactification constructed by the central cone decomposition Σ_{cent} of $\overline{C(\mathfrak{F}_0)}$. The induced volume form $\Phi_{g,n}:=\Psi_{g,\Gamma(n)}$ on $\mathcal{A}_{g,n}$ is singular at the boundary divisor $D_{\infty,n}:=\overline{\mathcal{A}}_{g,n}^{\mathrm{cent}}\setminus\mathcal{A}_{g,n}$.

We now calculate the volume form on Siegel space $\mathfrak{H}_g$ associated to the cusp $\mathfrak{F}_0$. Let

$$(\star)\qquad \begin{cases}\zeta_{i,i}^n:=nE_{i,i}, & \text{for } 1\leq i\leq g;\\ \zeta_{i,j}^n:=n(-E_{i,j}-E_{j,i}+E_{i,i}+E_{j,j}) & \text{for } 1\leq i<j\leq g.\end{cases}$$

The $\{\zeta^n_{i,j}\}_{1\le i\le j\le g}$ is a basis of $\mathrm{Sym}_g(\mathbb{R})$, and it also can be regarded as a lattice basis of $\Gamma(n)\cap U^{\mathfrak{F}_0}(\mathbb{Q})$. We note that $\mathbb{R}_{\ge 0}\zeta^n_{i,j}$ $1\le i\le j\le g$ are all edges of the principal cone σ_0 in Σ_{cent}. Since the $Z=(c_{ij})_{1\le i,j\le g}$ in the proposition 3.1 can be written as $Z=\sum\limits_{1\le i\le j\le g} z_{ij}\zeta^n_{i,j}$, we get

$$\begin{aligned} Z &= \sum_{1\le i<j\le g} z_{ij}\zeta^n_{i,j}+\sum_{j=1}^{g} z_{jj}\zeta^n_{j,j}\\ &= n\Big(\sum_{1\le i<j\le g} z_{ij}(-E_{i,j}-E_{j,i})+\sum_{j=1}^{g}\big(z_{jj}+\sum_{l=1}^{j-1} z_{lj}+\sum_{l=j+1}^{g} z_{jl}\big)E_{j,j}\Big)\end{aligned}$$

On the other hand, $Z=\sum\limits_{1\le i<j\le g} c_{ij}(E_{i,j}+E_{j,i})+\sum\limits_{l=1}^{g} c_{ll}E_{l,l}$. Thus, we have

$$\begin{cases} c_{ij}=c_{ji}=-nz_{ij}, & \text{for } 1\le i<j\le g;\\ c_{jj}=n(z_{jj}+\sum\limits_{l=1}^{j-1} z_{lj}+\sum\limits_{l=j+1}^{g} z_{jl}) =: n(z_{jj}+m_j) & \text{for } 1\le j\le g.\end{cases}$$

Let $\widetilde{Z}:=Z/n$. We calculate the volume form Φ_g in the coordinate system (z_{ij}) :

$$dZ:=\bigwedge_{1\le i\le j\le g} dc_{ij}=\pm n^{g(g+1)/2}\bigwedge_{1\le i\le j\le g} dz_{ij},$$

$$\Phi_g(\tau)=\Big(\frac{\sqrt{-1}}{2}\Big)^{\frac{g(g+1)}{2}}\frac{2^{\frac{g(g-1)}{2}}\bigwedge\limits_{1\le i\le j\le g} dz_{ij}\wedge d\overline{z_{ij}}}{(\det \mathrm{Im}(\widetilde{Z}))^{g+1}}. \tag{3.1.1}$$

where

$$\mathrm{Im}(\widetilde{Z})=\mathrm{Im}\begin{pmatrix} z_{11}+m_1 & \cdots & -z_{1j} & \cdots & -z_{1g}\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ -z_{1j} & \cdots & z_{jj}+m_j & \cdots & -z_{jg}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -z_{1g} & \cdots & -z_{jg} & \cdots & z_{gg}+m_g\end{pmatrix}_{g\times g}.$$

Recall the example 2.13, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{H}_g & \xrightarrow{\subset} & U^{\mathfrak{F}_0}(\mathbb{C})\cong\mathbb{C}^{g(g+1)/2}\\ \downarrow & & \downarrow{\scriptstyle w_{ij}:=\exp(2\pi\sqrt{-1}z_{ij})}\\ \mathfrak{H}_g/\Gamma\cap U^{\mathfrak{F}}(\mathbb{Q}) & \xrightarrow{\subset} & (\mathbb{C}^*)^{g(g+1)/2}\end{array}$$

and the partial compactification

$$\begin{aligned}\widetilde{\Delta}_{\mathfrak{F}_0,\sigma_0} &= \big\{x=(w_{ij})_{1\leq i\leq j\leq g}\in\mathbb{C}^{g(g+1)/2} \mid \text{ there exists a neighborhood}\\ &\qquad \Delta_x \text{ of } x \text{ such that } \Delta_x\cap(\mathbb{C}^*)^{g(g+1)/2}\subset\frac{\mathfrak{H}_g}{\Gamma(l)\cap U^{\mathfrak{F}_0}(\mathbb{Q})}\big\}.\end{aligned}$$

Define $\widetilde{\Delta}^*_{\mathfrak{F}_0,\sigma_0} := \widetilde{\Delta}_{\mathfrak{F}_0,\sigma_0} - \bigcup_{1\leq i\leq j\leq g}\big\{(w_{ij})\in\widetilde{\Delta}_{\mathfrak{F}_0,\sigma_0} \mid w_{ij}=0\big\}$. The volume on $\widetilde{\Delta}^*_{\mathfrak{F}_0,\sigma_0}$ becomes

$$\Phi_{\sigma_0}(w_{ij}) = (\frac{\sqrt{-1}}{2})^{\frac{g(g+1)}{2}}\frac{2^{\frac{g(g-1)}{2}}\bigwedge_{1\leq i\leq j\leq g} dw_{ij}\wedge d\overline{w_{ij}}}{(\prod_{1\leq i\leq j\leq g}|w_{ij}|^2)(\det\log|W|)^{g+1}} \tag{3.1.2}$$

where

$$\log|W| := \begin{pmatrix} \log|w_{11}|+q_1 & \cdots & -\log|w_{1j}| & \cdots & -\log|w_{1g}| \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -\log|w_{1j}| & \cdots & \log|w_{jj}|+q_j & \cdots & -\log|w_{jg}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\log|w_{1g}| & \cdots & -\log|w_{jg}| & \cdots & \log|w_{gg}|+q_g \end{pmatrix}_{g\times g}$$

$$q_j := -2\pi\mathrm{Im}(m_j) = \sum_{l=1}^{j-1}\log|w_{lj}| + \sum_{l=j+1}^{g}\log|w_{jl}| \;\; 1\leq j\leq g.$$

For genus $g=2$, Wang has already obtained the volume form of this type in [32]. However, we should be careful that if $g\leq 3$ then (w_{ij}) can not be a local coordinate system of $\overline{\mathcal{A}}^{\mathrm{cent}}_{g,n}$ with respect to the central cone decomposition Σ_{cent} of $C(\mathfrak{F}_0)$ as we point out in the example 2.23.

3.3. **Global volume forms on Siegel varieties $\mathcal{A}_{g,\Gamma}$.** Let $\Sigma_{\mathfrak{F}_0} := \{\sigma^{\mathfrak{F}_0}_\alpha\}$ be a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$ regular with respect to an arithmetic subgroup $\Gamma\subset\mathrm{Sp}(g,\mathbb{Z})$. Let $\sigma_{\max}$ be a top-dimensional cone in $\Sigma_{\mathfrak{F}_0}$. The cone $\sigma_{\max}$ is generated by a lattice basis of $U^{\mathfrak{F}_0}\cap\Gamma$, and then we can endows a marking order on this lattice basis. We have $\sigma_{\max} = \sum_{\mu=1}^{\frac{g(g+1)}{2}}\mathbb{R}_{\geq 0}\overrightarrow{l(\mu)}$ where $\{\overrightarrow{l(\mu)}\}_{\mu=1}^{\frac{g(g+1)}{2}}$ is the marking basis of $U^{\mathfrak{F}_0}\cap\Gamma$. We write $\overrightarrow{l(\mu)} = \sum_{1\leq i\leq j\leq g} l^\mu_{i,j}\delta_{i,j}$ for $\mu=1,\cdots,\frac{g(g+1)}{2}$, where $\{\delta_{i,j}\}_{1\leq i\leq j\leq g}$ is a $\mathbb{Z}$-basis of $\mathrm{Sym}_g(\mathbb{Z})$ given by $\begin{cases}\delta_{i,i}=E_{i,i}, & 1\leq i\leq g;\\ \delta_{i,j}=(E_{i,j}+E_{j,i}) & 1\leq i<j\leq g.\end{cases}$

Then, we have a $(\frac{g(g+1)}{2} \times \frac{g(g+1)}{2})$ integral matrix

$$L_\Gamma(\sigma_{\max},, \{\overrightarrow{l(\mu)}\}) := \begin{pmatrix} l^1_{1,1} & l^1_{1,2} & \cdots & l^1_{g-1,g} & l^1_{g,g} \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ l^{\frac{g(g+1)}{2}}_{1,1} & l^{\frac{g(g+1)}{2}}_{1,2} & \cdots & l^{\frac{g(g+1)}{2}}_{g-1,g} & l^{\frac{g(g+1)}{2}}_{g,g} \end{pmatrix}. \tag{3.1.3}$$

We define lattice volume of the top-dimensional cone $\sigma_{\max}$ to be

$$\mathrm{vol}_\Gamma(\sigma_{\max}) := |\det(L_\Gamma(\sigma_{\max}), \{\overrightarrow{l(\mu)}\})|, \tag{3.1.4}$$

which is a positive integer independent of the marking order of the basis $\{\overrightarrow{l(\mu)}\}_{\mu=1}^{\frac{g(g+1)}{2}}$.

Theorem 3.2. *Let $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ be a neat arithmetic subgroup and $\Sigma_{\mathfrak{F}_0}$ a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$ regular with respect to Γ. Let $\overline{\mathcal{A}}_{g,\Gamma}$ be the toroidal compactification of $\mathcal{A}_{g,\Gamma} := \mathfrak{H}_g/\Gamma$ constructed by $\Sigma_{\mathfrak{F}_0}$.*

Assume that the boundary divisor $D_\infty := \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ is simple normal crossing. For each irreducible component D_i of $D_\infty = \bigcup_i D_i$, let s_i be the global section of the line bundle $[D_i]$ defining D_i. Let $\sigma_{\max}$ be an arbitrary top-dimensional cone in $\Sigma_{\mathfrak{F}_0}$ and renumber all components D_i's of D_∞ such that $D_1, \cdots, D_{\frac{g(g+1)}{2})}$ corresponds to the edges of $\sigma_{\max}$ with marking order.

1. The volume $\Phi_{g,\Gamma}$ on $\mathcal{A}_{g,\Gamma}$ can be represented by

$$\Phi_{g,\Gamma} = \frac{2^{\frac{g(g-1)}{2}}\mathrm{vol}_\Gamma(\sigma_{\max})^2 d\mathcal{V}_g}{(\prod_{j=1}^{\frac{g(g+1)}{2}} ||s_i||_i^2) F^{g+1}_{\sigma_{\max}}(\log||s_1||_1, \cdots, \log||s_{\frac{g(g+1)}{2}}||_{\frac{g(g+1)}{2}})}, \tag{3.2.1}$$

where $d\mathcal{V}_g$ is a global smooth volume form on $\overline{\mathcal{A}}_{g,\Gamma}$, each $||\cdot||_i$ is a suitable Hermitian metric on the line bundle $[D_i]$ and $F_{\sigma_{\max}} \in \mathbb{Z}[x_1, \cdots, x_{g(g+1)/2}]$ is a homogenous polynomial of degree g. Moreover, the coefficients of $F_{\sigma_{\max}}$ only depend on both Γ and $\sigma_{\max}$ with marking order of edges.

2. Let $\mathcal{G}_\Gamma := F_{\sigma_{\max}}(\log||s_1||_1, \cdots, \log||s_{\frac{g(g+1)}{2}}||_{\frac{g(g+1)}{2}})$. Then

(3.2.2)

$$(\sqrt{-1})^{\frac{g(g+1)}{2}}(\frac{g(g+1)}{2})!2^{\frac{g(g-1)}{2}} d\mathcal{V}_g = \prod_{j=1}^{\frac{g(g+1)}{2}} ||s_i||_i^2 \mathcal{G}_\Gamma^{g+1}(\partial\overline{\partial}\log\mathcal{G}_\Gamma^{g+1})^{\frac{g(g+1)}{2}} \text{ on } \mathcal{A}_{g,\Gamma},$$

3. Moreover, the polynomial $F_{\sigma_{\max}}(x_1,\cdots,x_{\frac{g(g+1)}{2}})$ *satisfies the following equation*

$$\begin{aligned}&\det\Big(F_{\sigma_{\max}}(\frac{\partial^2 F_{\sigma_{\max}}}{\partial x_i\partial x_j})_{i,j}-\begin{pmatrix}\frac{\partial F_{\sigma_{\max}}}{\partial x_1}\\ \vdots\\ \frac{\partial F_{\sigma_{\max}}}{\partial x_{\frac{g(g+1)}{2}}}\end{pmatrix}\begin{pmatrix}\frac{\partial F_{\sigma_{\max}}}{\partial x_1}, & \cdots, & \frac{\partial F_{\sigma_{\max}}}{\partial x_{\frac{g(g+1)}{2}}}\end{pmatrix}\Big)\\ =\ & (-1)^{\frac{g(g+1)}{2}}2^{\frac{g(g-1)}{2}}\mathrm{vol}_\Gamma(\sigma_{\max})^2F_{\sigma_{\max}}^{(g+1)(g-1)}.\end{aligned}$$

Remark 3.3. Let $H_{\sigma_{\max}}(x_1,\cdots,x_{\frac{g(g+1)}{2}})=-\log F_{\sigma_{\max}}(-x_1,\cdots,-x_{\frac{g(g+1)}{2}})$. The equation in the statement (3) becomes

$$\det(\frac{\partial^2 H_{\sigma_{\max}}}{\partial x_i\partial x_j})_{i,j}=2^{\frac{g(g-1)}{2}}\mathrm{vol}_\Gamma(\sigma_{\max})^2\exp((g+1)H_{\sigma_{\max}}). \tag{3.3.1}$$

By the formula 4.0.2 in the next section, it is a real Monge-Ampére of elliptic type on the domain $\{(x_1,\cdots,x_{\frac{g(g+1)}{2}})\in\mathbb{R}^{\frac{g(g+1)}{2}}\mid x_i\geq C\forall i\}$ for some positive number C.

Proof of the theorem 3.2. Let $N=\frac{g(g+1)}{2}$. Let $\mathfrak{F}_{\min}$ be an arbitrary minimal cusp of $\mathfrak{H}_g$, and let σ be any top-dimensional cone in the decomposition $\Sigma_{\mathfrak{F}_{\min}}$ induced by $\Sigma_{\mathfrak{F}_0}$(cf.Lemma 2.10). Recall the local chart $(\widetilde{\Delta}_{\mathfrak{F}_{\min},\sigma},(w_1^\sigma,\cdots,w_N^\sigma))$ in 2.12.1 and 2.13, we have

$$\begin{array}{ccc}\frac{\mathfrak{H}_g}{\Gamma\cap U^{\mathfrak{F}_{\min}}(\mathbb{Q})} & \xrightarrow{\subset} & \widetilde{\Delta}_{\mathfrak{F}_{\min},\sigma}\\ \Big\downarrow & & \pi'_{\mathfrak{F}_{\min}}\Big\downarrow \text{étale}\\ \mathfrak{H}_g/\Gamma & \xrightarrow{\subset} & \overline{\mathcal{A}}_{g,\Gamma},\end{array}$$

with a toroidal embedding $\frac{\mathfrak{H}_g}{\Gamma\cap U^{\mathfrak{F}_{\min}}(\mathbb{Q})}\xrightarrow{\subset}\widetilde{\Delta}_{\mathfrak{F}_{\min},\sigma}$(cf.Lemma 2.14). There are facts :

i. The morphism $\pi'_{\mathfrak{F}_{\min}}:\frac{\mathfrak{H}_g}{\Gamma\cap U^{\mathfrak{F}_{\min}}(\mathbb{Q})}\to\mathfrak{H}_g/\Gamma$ is surjective.
ii. Define $W_{\mathfrak{F}_{\min},\sigma}:=\pi'_{\mathfrak{F}_{\min}}(\widetilde{\Delta}_{\mathfrak{F}_{\min},\sigma})$. Since $\overline{\mathcal{A}}_{g,\Gamma}$ is geometrically fine, the restriction map $\pi'_{\mathfrak{F}_{\min}}|_{\{w_i^\sigma=0\}}$ is an isomorphism onto its image for each w_i^σ. Thus, $(W_{\mathfrak{F}_{\min},\sigma},(w_1^\sigma,\cdots,w_N^\sigma))$ becomes a coordinate neighborhood of $\overline{\mathcal{A}}_{g,\Gamma}$.
iii. That $W^*_{\mathfrak{F}_{\min},\sigma}=W_{\mathfrak{F}_{\min},\sigma}\setminus D_\infty=\mathfrak{H}_g/\Gamma$ where $W^*_{\mathfrak{F}_{\min},\sigma}:=W_{\mathfrak{F}_{\min},\sigma}\setminus\bigcup_{i=1}^N\{w_i^\sigma=0\}$.
iv. The compactification $\overline{\mathcal{A}}_{g,\Gamma}$ is covered by finitely many open sets of the form $W_{\mathfrak{F},\delta}$, where $\mathfrak{F}$ is a minimal cusp of $\mathfrak{H}_g$ and δ is a top-dimensional cone in the decomposition $\Sigma_{\mathfrak{F}}$.

Now, we begin to prove the statements $1-3$:

Let $\sigma_{\max}$ be an arbitrary top-dimensional cone in the decomposition $\Sigma_{\mathfrak{F}_0}$. We take a coordinate chart $(W^*_{\mathfrak{F}_0,\sigma_{\max}},(w_1,\cdots,w_N))$ on $\mathcal{A}_{g,\Gamma}$ constructed by $\sigma_{\max}$ as above such that $D_i\cap W_{\mathfrak{F}_0,\sigma_{\max}}=\{w_i=0\}$ for any integer $i\in[1,\frac{g(g+1)}{2}]$.

1. By Theorem 4.1 in [32] or by similar calculations as in 3.1.2, the volume form $\Phi_{g,\Gamma}$ on the chart $(W^*_{\mathfrak{F}_0,\sigma_{\max}},(w_1,\cdots,w_N))$ can be written as

$$\Phi_{\sigma_{\max}}=\frac{(\frac{\sqrt{-1}}{2})^N 2^{\frac{g(g-1)}{2}}\mathrm{vol}_\Gamma(\sigma_{\max})^2\bigwedge\limits_{1\le i\le N}dw_i\wedge d\overline{w_i}}{(\prod\limits_{1\le i\le N}|w_i|^2)(F_{\sigma_{\max}}(\log|w_1|,\cdots,\log|w_N|))^{g+1}}\tag{3.3.2}$$

where $F_{\sigma_{\max}}\in\mathbb{Z}[x_1,\cdots,x_N]$ is a homogenous polynomial of degree g. It is obvious that the coefficients of $F_{\sigma_{\max}}$ only depend on Γ and $\sigma_{\max}$ with marking order of edges.

For each integer $i\in[1,N]$, we choose a Hermitian metrics $||\cdot||_i$ on $[D_i]$ by setting

$$||s_i||_i=|w_i| \text{ on } W_{\mathfrak{F}_0,\sigma_{\max}}.$$

We also choose a global smooth volume form $d\mathcal{V}_g$ on $\overline{\mathcal{A}}_{g,\Gamma}$ such that

$$d\mathcal{V}_g|_{W_{\mathfrak{F}_0,\sigma_{\max}}}=(\frac{\sqrt{-1}}{2})^N\bigwedge_{1\le i\le N}dw_i\wedge d\overline{w_i}.$$

Then, we obtain that the form $\Phi_{g,\Gamma}$ on $\mathcal{A}_{g,\Gamma}$ can be represented by

$$\Phi_{g,\Gamma}=\frac{2^{\frac{g(g-1)}{2}}\mathrm{vol}_\Gamma(\sigma_{\max})^2 d\mathcal{V}_g}{(\prod\limits_{j=1}^N||s_i||_i^2)F^{g+1}_{\sigma_{\max}}(\log||s_1||_1,\cdots,\log||s_N||_N)}.$$

2. Since ω_{can} is Kähler-Einstein, as the formula 3.0.1 there holds

$$\sqrt{-1}\partial\overline{\partial}\log\big((\prod_{j=1}^N||s_i||_i^2)F^{g+1}_{\sigma_{\max}}(\log||s_1||_1,\cdots,\log||s_N||_N)\big)=-\omega_{\mathrm{can}} \text{ on } \mathcal{A}_{g,\Gamma}.$$

Since $\partial\overline{\partial}\log(\prod_{j=1}^N||s_i||_i^2)=0$ on $\mathcal{A}_{g,\Gamma}(=W^*_{\mathfrak{F}_0,\sigma_{\max}})$, we obtain

$$-\sqrt{-1}\partial\overline{\partial}\log F^{g+1}_{\sigma_{\max}}(\log||s_1||_1,\cdots,\log||s_N||_N)=\frac{g+1}{2}\omega_{\mathrm{can}} \text{ on } \mathcal{A}_{g,\Gamma}.$$

On $\mathcal{A}_{g,\Gamma}$, we then get

$$\begin{aligned}(\sqrt{-1})^N(\partial\overline{\partial}\log F^{g+1}_{\sigma_{\max}}(\log||s_1||_1,\cdots,\log||s_N||_N))^N &= (\frac{-g-1}{2}\omega_{\mathrm{can}})^N\\ &= (\frac{-g-1}{2})^N N!\Phi_{g,\Gamma}.\end{aligned}$$

3. With respect to the coordinate chart $(W^*_{\mathfrak{F}_0,\sigma_{\max}},(w_1,\cdots,w_N))$ on $\mathcal{A}_{g,\Gamma}$, we define $G(w_1,\cdots,w_N):=F_{\sigma_{\max}}(\log|w_1|,\cdots,\log|w_N|)$. By the equality 3.2.2, we get that
$$\begin{aligned}&(\frac{-g-1}{2})^N 2^{-N}N!2^{\frac{g(g-1)}{2}}\mathrm{vol}_\Gamma(\sigma_{\max})^2\bigwedge_{1\le i\le N}dw_i\wedge d\overline{w_i}\\ =\;& G^{g+1}(\prod_{j=1}^N|w_i|^2)(\partial\overline{\partial}\log G^{g+1})^N\\ =\;&(g+1)^N G^{g+1}(\prod_{j=1}^N|w_i|^2)(\frac{\partial\overline{\partial}G}{G}-\frac{\partial G\wedge\overline{\partial}G}{G^2})^N\\ =\;&(g+1)^N G^{g+1}(\prod_{j=1}^N|w_i|^2)(\sum_{i,j}\frac{GG_{w_i\overline{w_j}}-G_{w_i}G_{\overline{w_j}}}{G^2}dw_i\wedge d\overline{w_j})^N.\end{aligned}$$
Let $F_\alpha:=\frac{\partial F_{\sigma_{\max}}(x_1,\cdots,x_N)}{\partial x_\alpha}$ and $F_{\alpha\beta}:=\frac{\partial^2 F_{\sigma_{\max}}(x_1,\cdots,x_N)}{\partial x_\alpha\partial x_\beta}$ for all $1\le\alpha,\beta\le N$. On the Siegel variety $\mathcal{A}_{g,\Gamma}$, we have :
$$\begin{aligned}G_{w_i}(w_1,\cdots,w_N)&=\frac{F_i(\log|w_1|,\cdots,\log|w_N|)}{2w_i},\\ G_{\overline{w_i}}(w_1,\cdots,w_N)&=\frac{F_i(\log|w_1|,\cdots,\log|w_N|)}{2\overline{w_i}},\\ G_{w_i\overline{w_j}}(w_1,\cdots,w_N)&=\frac{F_{ij}(\log|w_1|,\cdots,\log|w_N|)}{4w_i\overline{w_j}},\end{aligned}$$
$$\Big(\det(F_{\sigma_{\max}}F_{ij}-F_iF_j)_{i,j}-(-1)^N 2^{\frac{g(g-1)}{2}}\mathrm{vol}_\Gamma(\sigma_{\max})^2F_{\sigma_{\max}}^{(g+1)(g-1)}\Big)(\log|w_1|,\cdots,\log|w_N|)=0.$$
We also have
$$(\frac{\partial^2\log F_{\sigma_{\max}}}{\partial x_i\partial x_j})_{i,j}=(F_{\sigma_{\max}}F_{ij}-F_iF_j)_{i,j}=F_{\sigma_{\max}}(F_{ij})_{i,j}-\begin{pmatrix}F_1\\ \vdots\\ F_N\end{pmatrix}\begin{pmatrix}F_1, & \cdots, & F_N\end{pmatrix}.$$
□

3.4. **Constrained combinational conditions of decompositions of cones.** Let $N=g(g+1)/2$, and let S_N be the group of permutations of the set $\{1,\cdots,N\}$.

For any integer $i\in[1,N]$, let $Y(i)$ be a $g\times g$ symmetric matrix $\begin{pmatrix}y_{11}(i)&\cdots&y_{1g}(i)\\ \vdots&\cdots&\vdots\\ y_{1g}(i)&\cdots&y_{gg}(i)\end{pmatrix}$ with N-variables. Each $Y(i)$ can be identified with a $1\times N$ matrix as
$$\widetilde{Y}(i):=(y_{11}(i),\cdots,y_{1g}(i),y_{22}(i),\cdots,y_{2g}(i),\cdots,y_{jj}(i),\cdots,y_{jg}(i),\cdots,y_{gg}(i)).$$

Define $Y := \begin{pmatrix} \widetilde{Y}(1) \\ \vdots \\ \widetilde{Y}(i) \\ \vdots \\ \widetilde{Y}(N) \end{pmatrix}$. We know that Y is a $N \times N$ matrix with N^2-variables. Define $D(Y) := \det(Y)$, it is a homogenous polynomial of degree N in $\mathbb{Z}[Y] := \mathbb{Z}[(y_{kl}(i))_{1\leq k\leq l\leq g, 1\leq i\leq N}]$. For any $\varsigma \in S_N$, we define $\varsigma(Y) := \begin{pmatrix} \widetilde{Y}(\varsigma(1)) \\ \vdots \\ \widetilde{Y}(\varsigma(i)) \\ \vdots \\ \widetilde{Y}(\varsigma(N)) \end{pmatrix}$.

We begin to show that there is a characteristic variety $\mathfrak{Q}_g$ by the unique group-invariant Kähler-Einstein metric on $\mathfrak{H}_g$. Define $F = \det(\sum_{i=1}^N x_i Y(i))$. We have

$$F = \sum_{i_1+\cdots+i_N=g, i_k\geq 0} t_{i_1\cdots i_N}(Y) x_1^{i_1}\cdots x_N^{i_N}$$

and each $t_{i_1\cdots i_g}(Y) \in \mathbb{Z}[Y]$ is a homogenous polynomial of degree g. Let

$$\begin{aligned} \mathcal{C}_1 &:= \det\big(F(\frac{\partial^2 F}{\partial x_i \partial x_i})_{i,j} - \begin{pmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_N} \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial x_1}, & \cdots, & \frac{\partial F}{\partial x_N} \end{pmatrix}\big), \\ \mathcal{C}_2 &:= (-1)^{\frac{g(g+1)}{2}} 2^{\frac{g(g-1)}{2}} F^{(g+1)(g-1)} D(Y)^2. \end{aligned}$$

We then write $\mathcal{C}_1 := \mathcal{C}_1 - \mathcal{C}_2$ as

$$\mathcal{C} = \sum_{j_1+\cdots+j_N=g(g^2-1), j_k\geq 0} C_{j_1\cdots j_N}(Y) x_1^{j_1}\cdots x_N^{j_N} \tag{3.3.3}$$

such that each $C_{j_1\cdots j_N}(Y) \in \mathbb{Q}[Y]$ is a homogenous polynomial of degree $g^2(g+1)$.

Lemma 3.4. *For any tuple $(j_1, \cdots, j_N)$ of non-negative integers with $\sum_{\alpha=1}^N j_\alpha = g(g^2-1)$, let $C_{j_1\cdots j_N}(Y) \in \mathbb{Q}[Y]$ be homogenous polynomials defined in 3.3.3. We have*

$$C_{j_1\cdots j_N}(\varsigma(Y)) = C_{j_{\varsigma^{-1}(1)}\cdots j_{\varsigma^{-1}(N)}}(Y)\ \forall \varsigma \in S_N.$$

Proof. Let ς be an arbitrary element in the group S_N. We get

$$\det(\sum_{i=1}^N x_i Y(\varsigma(i))) = \sum_{i_1+\cdots+i_N=g, i_k\geq 0} t_{i_1\cdots i_N}(\varsigma(Y)) x_1^{i_1}\cdots x_N^{i_N}.$$

On the other hand,

$$\begin{aligned}
\det(\sum_{i=1}^{N} x_i Y(\varsigma(i))) &= \det(\sum_{i=1}^{N} x_{\varsigma^{-1}(i)} Y(i)) \\
&= \sum_{i_1+\cdots+i_N=g, i_k\geq 0} t_{i_1\cdots i_N}(Y) x_{\varsigma^{-1}(1)}^{i_1}\cdots x_{\varsigma^{-1}(N)}^{i_N} \\
&= \sum_{i_{\varsigma(1)}+\cdots+i_{\varsigma(N)}=g, i_k\geq 0} t_{i_1\cdots i_N}(Y) x_1^{i_{\varsigma(1)}}\cdots x_N^{i_{\varsigma(N)}} \\
&= \sum_{j_1+\cdots+j_N=g, j_k\geq 0} t_{j_{\varsigma^{-1}(1)}\cdots j_{\varsigma^{-1}(N)}}(Y) x_1^{j_1}\cdots x_N^{j_N}.
\end{aligned}$$

Therefore, we obtain $t_{i_1\cdots i_N}(\varsigma(Y)) = t_{i_{\varsigma^{-1}(1)}\cdots i_{\varsigma^{-1}(N)}}(Y)\ \ \forall\, i_1+\cdots+i_N = g$ with $i_k \geq 0$. Since $D(\varsigma(Y))^2 \equiv D(Y)^2$, we prove the statement as well.

□

The group S_N has a natural action on the set of $(N\times N)$-matrices $M_{N\times N}(\mathbb{C})$ as

$$\varsigma(B) := \begin{pmatrix} \widetilde{B}(\varsigma(1)) \\ \vdots \\ \widetilde{B}(\varsigma(i)) \\ \vdots \\ \widetilde{B}(\varsigma(N)) \end{pmatrix} \quad \text{for } \varsigma\in S_N, B = (\widetilde{B}(1)^T, \cdots \widetilde{B}(i)^T, \cdots, \widetilde{B}(N)^T) \in M_{N\times N}(\mathbb{C}).$$

Since the S_N acts freely on $\mathrm{GL}(g,\mathbb{C}) = \{A \in M_{N\times N}(\mathbb{C}) \mid \det A \neq 0\}$, the quotient $\mathfrak{P}_g := \mathrm{GL}(g,\mathbb{C})/S_N$ is a smooth affine variety.

Lemma 3.5. *Let $\Sigma_{\mathfrak{F}_0} := \{\sigma_\alpha^{\mathfrak{F}_0}\}$ be a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$ regular with respect to an arithmetic subgroup $\Gamma\subset \mathrm{Sp}(g,\mathbb{Z})$. There is an injective map of sets $\nu_\Gamma : \{$top-dimensional cones in $\Sigma_{\mathfrak{F}_0}\} \stackrel{\hookrightarrow}{\longrightarrow} \mathfrak{P}_g(\mathbb{Z})$, where $\mathfrak{P}_g(\mathbb{Z})$ is the set of all integral points of the variety $\mathfrak{P}_g$.*

Proof. Let $\sigma_{\max}$ be an arbitrary top-dimensional cones in $\Sigma_{\mathfrak{F}_0}$. Let $\{\overrightarrow{l(\mu)}\}_{\mu=1}^N$ be any marking basis of $U^{\mathfrak{F}_0}(\mathbb{Z})\cap\Gamma$ such that $\sigma_{\max} = \sum_{\mu=1}^{N}\mathbb{R}_{\geq 0}\overrightarrow{l(\mu)}$. The $L_\Gamma(\sigma_{\max},\{\overrightarrow{l(\mu)}\})$ in 3.1.3 is an element in $\mathrm{GL}(g,\mathbb{Z})$, and its projective image $[L_\Gamma(\sigma_{\max},\{\overrightarrow{l(\mu)}\})]$ in $\mathrm{GL}(g,\mathbb{Z})/S_N$ is independent of the marking order of the basis $\{\overrightarrow{l(\mu)}\}_{\mu=1}^N$. Therefore we can define an injective map

$$\nu_\Gamma : \{\text{top-dimensional cones in } \Sigma_{\mathfrak{F}_0}\} \longrightarrow \mathfrak{P}_g(\mathbb{Z})$$

by sending $\sigma_{\max}$ to the equivalent class of $L_\Gamma(\sigma_{\max},\{\overrightarrow{l(\mu)}\})$ in $\mathfrak{P}_g(\mathbb{Z})$. □

Lemma 3.6. *Define*

$$\mathfrak{A}_g = \{Z \in \mathrm{GL}(g,\mathbb{C}) \mid C_{j_1\cdots j_N}(Z) = 0\ \forall j_1 + \cdots + j_N = g(g^2-1) \ \textit{with}\ j_k \geq 0\},$$

where $C_{j_1\cdots j_N}$*'s are polynomials defined in 3.3.3. The permutation group* S_N *acts freely on the affine variety* $\mathfrak{A}_g$.

Proof. By the lemma 3.4, we obtain that if $Z \in \mathfrak{A}_g$ then $\varsigma(Z) \in \mathfrak{A}_g$ for all $\varsigma \in S_N$. Thus the group S_N has a free action on $\mathfrak{A}_g$. □

Define

$$\mathfrak{Q}_g := \mathfrak{A}_g/S_N. \tag{3.6.1}$$

It is obvious that $\mathfrak{Q}_g$ is an affine variety defined over $\mathbb{Q}$ dependent only on $\mathfrak{H}_g$. We call $\mathfrak{Q}_g$ the g-**KE-characteristic variety**.

Theorem 3.7. *Let* $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ *be a neat arithmetic subgroup. Let* $\Sigma_{\mathfrak{F}_0} := \{\sigma_\alpha^{\mathfrak{F}_0}\}$ *be a* $\overline{\Gamma_{\mathfrak{F}_0}}$ *(or* $\mathrm{GL}(g,\mathbb{Z})$*)-admissible polyhedral decomposition of* $C(\mathfrak{F}_0)$ *regular with respect to* Γ, *where* $\mathfrak{F}_0$ *is the standard minimal cusp of the Siegel space* $\mathfrak{H}_g$. *Let* $\overline{\mathcal{A}}_{g,\Gamma}$ *be the toroidal compactification of* $\mathcal{A}_{g,\Gamma} := \mathfrak{H}_g/\Gamma$ *constructed by* $\Sigma_{\mathfrak{F}_0}$.

Assume that the boundary divisor $D_\infty := \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ *is simple normal crossing. There is an injective map of sets*

$$\nu_\Gamma : \{\textit{top-dimensional cones in } \Sigma_{\mathfrak{F}_0}\} \xrightarrow{\hookrightarrow} \mathfrak{Q}_g(\mathbb{Z}),$$

where $\mathfrak{Q}_g(\mathbb{Z})$ *is the set of all integral points of the g-KE-characteristic variety* $\mathfrak{Q}_g$.

Proof. It is straightforward by the theorem 3.2 and the lemma 3.5. □

Remark 3.8. Actually, the assumption of normal crossing $D_\infty := \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ in the theorem 3.7 is not necessary. The theorem 3.7 is true for all smooth toroidal compactifications. Consider the partial compactification given by the diagram

$$\begin{array}{ccc} \mathfrak{H}_g & \xrightarrow{\subset} & U^{\mathfrak{F}_0}(\mathbb{C}) \cong \mathbb{C}^{\frac{g(g+1)}{2}} \\ \downarrow & & \downarrow{\scriptstyle w_i := \exp(2\pi\sqrt{-1}z_i)} \\ \mathfrak{H}_g/\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q}) & \xrightarrow{\subset} & (\mathbb{C}^*)^{g(g+1)/2} \end{array}$$

with respect to an arbitrary regular top-dimensional cone $\sigma \in \Sigma_{\mathfrak{F}_0}$, the $(w_1, \cdots, w_{\frac{g(g+1)}{2}})$ is always a local coordinate system of the partial compactification even though it can not be a local coordinate system of $\overline{\mathcal{A}}_{g,\Gamma}$, the quotient manifold $\mathfrak{H}_g/\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q})$ also has an induced Kähler-Einstein metric with volume form 3.3.2. Therefore, the function $H_\sigma(x_1, \cdots, x_{\frac{g(g+1)}{2}}) := -\log F_\sigma(-x_1, \cdots, -x_{\frac{g(g+1)}{2}})$ must satisfy the elliptic real Monge-Ampére equation 3.3.1.

4. Asymptotic behaviours of logarithmical canonical line bundles

Let $N = g(g+1)/2$. For any positive integer n, we define a constant $C_n = (\frac{\sqrt{-1}}{2\pi})^n$.

In this section, we fix a neat subgroup $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ and a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition $\Sigma_{\mathfrak{F}_0} := \{\sigma_\alpha^{\mathfrak{F}_0}\}$ of $C(\mathfrak{F}_0)$ regular with respect to Γ such that the constructed symmetric toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}$ of $\mathcal{A}_{g,\Gamma} := \mathfrak{H}_g/\Gamma$ is geometrically Γ-fine, i.e., $D_\infty := \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ is a simple normal crossing divisor. Let $K_{\mathcal{A}_{g,\Gamma}}$ be the canonical divisor on $\mathcal{A}_{g,\Gamma}$ and h_B the metric on the canonical line bundle $\mathcal{O}_{\mathcal{A}_{g,\Gamma}}(K_{\mathcal{A}_{g,\Gamma}})$ induced by the Bergman metric ω_{can} of $\mathcal{A}_{g,\Gamma}$.

We define $D_\infty(\epsilon)$ to be tube neighborhood of D_∞ with radius ϵ for suitable real number $\epsilon > 0$. For every irreducible component Y of D_∞, we define

$$Y_\infty := \bigcup_{D_j \neq Y} (D_j \cap Y) \ \text{ and } \ Y^* := Y \setminus Y_\infty.$$

Then Y_∞ is a simple normal crossing divisor of Y. In the theorem 3.2, we show that there is a system $\{(U_\alpha, (w_1^\alpha, \cdots, w_N^\alpha))\}_\alpha$ of finitely many coordinate charts of the compactification $\overline{\mathcal{A}}_{g,\Gamma}$ such that

$$U_\alpha^* := U_\alpha \setminus D_\infty = \mathcal{A}_{g,\Gamma} \ \text{ and } \ U_\alpha \cap D_\infty = \bigcup_{i=1}^N \{w_i = 0\}. \tag{4.0.1}$$

On any such coordinate chart $(U_\alpha^*, (w_1^\alpha, \cdots, w_N^\alpha))$, the volume form $\Phi_{g,\Gamma}$ becomes

$$\Phi_\alpha = \frac{(\frac{\sqrt{-1}}{2})^N 2^{\frac{g(g-1)}{2}} \mathrm{vol}_\Gamma(\sigma_{\max})^2 \bigwedge_{1\leq i\leq N} dw_i^\alpha \wedge d\overline{w_i^\alpha}}{(\prod_{1\leq i\leq N} |w_i^\alpha|^2)(F^\alpha(\log|w_1^\alpha|, \cdots, \log|w_N^\alpha|))^{g+1}}.$$

where $F^\alpha \in \mathbb{R}[x_1, \cdots, x_N]$ is a homogenous polynomial in of degree g. We call this F^α the **local volume function** with respect to the local chart U_α^*. Write

$$F_i^\alpha := \frac{\partial F^\alpha(x_1, \cdots, x_N)}{\partial x_i}, F_{ij}^\alpha := \frac{\partial^2 F^\alpha(x_1, \cdots, x_N)}{\partial x_i \partial x_j} \ 1 \leq i, j \leq N.$$

Define

$$T_{i,j}^\alpha := F^\alpha F_{ij}^\alpha - F_i^\alpha F_j^\alpha \ \ 1 \leq i, j \leq N.$$

We have a $N \times N$ matrix $T^\alpha := (T_{i,j}^\alpha)$ such that each $T_{i,j}^\alpha$ is a homogenous polynomial of degree $2g-2$ in $\mathbb{R}[x_1, \cdots, x_N]$.

Now we begin to compute $\partial\overline{\partial} \log \Phi_\alpha$ as a distribution form on U_α :

$$
\begin{aligned}
& \partial\overline{\partial}\log\Phi_\alpha \\
= & -\partial\overline{\partial}\log\prod_{1\leq i\leq N}|w_i^\alpha|^2-(g+1)\partial\overline{\partial}\log F^\alpha(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|) \\
= & -\partial\overline{\partial}\log\prod_{1\leq i\leq N}|w_i^\alpha|^2 \\
& -(g+1)\{\frac{\partial\overline{\partial}F^\alpha(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|)}{F^\alpha(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|)}-\frac{(\partial F^\alpha\wedge\overline{\partial}F^\alpha)(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|)}{F^\alpha(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|)^2}\} \\
= & -\sum_{i=1}^N(2+(g+1)\frac{F_i^\alpha}{F^\alpha}(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|))\partial\overline{\partial}\log|w_i^\alpha| \\
& -(g+1)\sum_{1\leq i,j\leq N}\frac{T_{i,j}^\alpha}{(F^\alpha)^2}(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|)\partial\log|w_i^\alpha|\wedge\overline{\partial}\log|w_j^\alpha|.
\end{aligned}
$$

Particularly,

$$
\partial\overline{\partial}\log\Phi_\alpha=\sum_{1\leq i,j\leq N}K_{ij}(w_1^\alpha,\cdots,w_N^\alpha)dw_i^\alpha\wedge d\overline{w_j^\alpha}\quad\text{on } U_\alpha^*. \tag{4.0.2}
$$

is a smooth form on U_α^*, where $K=(K_{i,j})$ is a new $N\times N$ matrix given by

$$
K_{i,j}(w_1^\alpha,\cdots,w_N^\alpha):=\frac{-(g+1)}{4}\frac{T_{i,j}^\alpha(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|)}{w_i^\alpha\overline{w_j^\alpha}(F^\alpha)^2(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|)}\ \forall i,j.
$$

In general, with respect to an arbitrary smooth toroidal compactification $\overline{D/\Gamma}$ of any locally symmetric manifold D/Γ with normal crossing boundary divisor, Mumford has shown that any group-invariant Hermitian metric on the homogeneous holomorphic cotangent bundle $\Omega^1_{D/\Gamma}$ is good on $\overline{D/\Gamma}$ such that the good extension of $\Omega^1_{D/\Gamma}$ to $\overline{D/\Gamma}$ is just the logarithmical cotangent bundle on $\overline{D/\Gamma}$ (cf. Main Theorem 3.1 and Proposition 3.4 in Section 1 of [25]).

Lemma 4.1. *For any positive integer p, the smooth (p,p)-form $(\partial\overline{\partial}\log\Phi_{g,\Gamma})^p$ on U_α^* has Poincaré growth on $D_\infty\cap U_\alpha$(cf.[25] for the definition), and $(\sqrt{-1}\partial\overline{\partial}\log\Phi_{g,\Gamma})^p$ is a positive current on $\overline{\mathcal{A}}_{g,\Gamma}$.*

Proof. The Main Theorem 3.1 and Proposition 3.4 in [25] guarantee that $[K_{\overline{\mathcal{A}}_{g,\Gamma}}+D_\infty]$ is the unique extension of $\mathcal{O}_{\mathcal{A}_{g,\Gamma}}(K_{\mathcal{A}_{g,\Gamma}})$ to $\overline{\mathcal{A}}_{g,\Gamma}$ such that h_B is singular metric good on $\overline{\mathcal{A}}_{g,\Gamma}$. Thus $c_1(\mathcal{O}_{\mathcal{A}_{g,\Gamma}}(K_{\mathcal{A}_{g,\Gamma}}),h_B)$ has Poincaré growth on D_∞. By the Kähler-Einstein equality 3.0.1, we get that

$$(\frac{g+1}{2}\omega_{\mathrm{can}})^p = (\sqrt{-1}\partial\overline{\partial}\log\Phi_{g,\Gamma})^p = (2\pi c_1(\mathcal{O}_{\mathcal{A}_{g,\Gamma}}(K_{\mathcal{A}_{g,\Gamma}}), h_B))^p.$$

□

We can make an improvement on the above lemma : By the generalized Schwarz lemma(cf.[34],[9] and [27]), the lemma is true for not only smooth toroidal compactifications but also a general compactification with normal crossings boundary divisor.

Proposition 4.2. *Let $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ be a neat arithmetic subgroup and $\Sigma_{\mathfrak{F}_0}$ a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$ regular with respect to Γ. Let $\overline{\mathcal{A}}_{g,\Gamma}$ be a toroidal compactification of $\mathcal{A}_{g,\Gamma} := \mathfrak{H}_g/\Gamma$ constructed by $\Sigma_{\mathfrak{F}_0}$.*

Assume that the boundary divisor $D_\infty := \overline{\mathcal{A}}_{g,\Gamma}\setminus\mathcal{A}_{g,\Gamma}$ is simple normal crossing. Let $\Phi_{g,\Gamma}$ be the standard volume form on $\mathcal{A}_{g,\Gamma}$. Let p be a positive integer in $[1, \frac{g(g+1)}{2}]$ and let $\Psi := (\partial\overline{\partial}\log\Phi_{g,\Gamma})^p$. Write $D_\infty = \bigcup_{j=1}^{l} D_j$, we have :

1. *The Ψ can be extended to a continuous (p,p)-form $\widetilde{\Psi}$ on $(D^*\cup\bigcup_{j=1}^{l} D_j^*)$. Define*

$$\mathrm{Res}_{D_i}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p) := \widetilde{\Psi}|_{D_i^*} \tag{4.2.1}$$

   for each irreducible component D_i of D_∞. For each D_i, the form $\mathrm{Res}_{D_i}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)$ is a smooth form on D_i^*
2. *For each irreducible component D_i of D_∞, the form $\mathrm{Res}_{D_i}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)$ has Poincaré growth on the simple normal crossing divisor $D_{i,\infty}$ of D_i. That $\mathrm{Res}_{D_i}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)$ becomes a current on D_i in sense that the following integral*

$$\int_{D_i}\mathrm{Res}_{D_i}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)\wedge\alpha := \lim_{\varepsilon\to 0}\int_{D_i\setminus T_i(\varepsilon)}\mathrm{Res}_{D_i}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)\wedge\alpha \tag{4.2.2}$$

   is finite for each smooth $(2N-2p)$-form α, where $T_i(\varepsilon)$ is a tube neighborhood of $D_{i,\infty}$ with radius ε.
3. *For each irreducible component D_i of D_∞, the form $\mathrm{Res}_{D_i}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)$ is closed on D_i^* and $(\frac{\sqrt{-1}}{2\pi})^p\mathrm{Res}_{D_i}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)$ is a positive closed current on D_i.*

Proof. 1. The statement (1) is a consequence of (4) of the lemma 4.5 and the lemma 4.3 in the next subsection.
2. By symmetry, we prove the statement (2) for case D_1 only. It is a local problem and it is sufficient to prove this statement for $p = 1$.

Taking a local chart $(U_\alpha, (w_1^\alpha, w_2^\alpha, \cdots, w_N^\alpha))$ of $\overline{\mathcal{A}}_{g,\Gamma}$ as in 4.0.1, we have the smooth form

$$\partial\overline{\partial}\log\Phi_\alpha = -(g+1)\sum_{1\le i,j\le N}\frac{T_{i,j}^\alpha}{(F^\alpha)^2}(\log|w_1^\alpha|, \cdots, \log|w_N^\alpha|)\frac{dw_i^\alpha\wedge d\overline{w_j^\alpha}}{4w_i^\alpha\overline{w_j^\alpha}} \quad \text{on } U_\alpha^*.$$

Thus, on $D_1^*\cap U_\alpha$, the $\mathrm{Res}_{D_1}(\partial\overline{\partial}\log\Phi_{g,\Gamma})$ can be written as

$$\begin{aligned}\mathrm{Res}_{D_1}(\partial\overline{\partial}\log\Phi_{g,\Gamma}) &= -(g+1)\sum_{2\le i,j\le N}\frac{T_{i,j}^\alpha}{(F^\alpha)^2}(\log|w_1^\alpha|, \cdots, \log|w_N^\alpha|)|_{D_1^*\cap U_\alpha}\frac{dw_i^\alpha\wedge d\overline{w_j^\alpha}}{4w_i^\alpha\overline{w_j^\alpha}}\\ &=: \sum_{2\le i,j\le N} a_{i,j}dw_i^\alpha\wedge d\overline{w_j^\alpha}.\end{aligned}$$

Let $V_1\subset D_1$ be a small neighborhood in containing the origin point and U_1 a small neighborhood in U_α such that $U_1\cap D_1 = V_1$. Let i, j be two arbitrary integers with $2\le i,j\le N$. Since $\partial\overline{\partial}\log\Phi_{g,\Gamma}$ has Pincaré growth on $U_\alpha\cap D_\infty$ we have

$$\left|\frac{T_{i,j}^\alpha}{(F^\alpha)^2}(\log|w_1^\alpha|, \cdots, \log|w_N^\alpha|)\frac{1}{4w_i^\alpha\overline{w_j^\alpha}}\right| \le \frac{C}{|w_i^\alpha w_j^\alpha||\log|w_i^\alpha|\log|w_j^\alpha||} \quad \text{on } \ U_1\cap U_\alpha^*$$

for a suitable constant. Let $w_1^\alpha\to 0$, we get

$$|a_{ij}| \le \frac{C}{|w_i^\alpha w_j^\alpha||\log|w_i^\alpha|\log|w_j^\alpha||} \quad \text{on } \ V_1\cap D_1^*\cap U_\alpha^*.$$

Therefore, $\mathrm{Res}_{D_1}(\partial\overline{\partial}\log\Phi_{g,\Gamma})$ has Poincaré growth on $D_{1,\infty}\cap U_\alpha$, and the integral 4.2.2 is finite.

3. It is sufficient to prove the statement(3) in case of D_1 for $p=1$. It is also a local problem. Take a local chart U_α of $\overline{\mathcal{A}}_{g,\Gamma}$. Let V be an open neighborhood in $D_1^*\cap U_\alpha$. For a sufficiently small ε, we define a sub-complex manifold in D^*

$$V_\varepsilon := \{(\varepsilon, w_2^\alpha, \cdots, w_N^\alpha) \mid (0, w_2^\alpha, \cdots, w_N^\alpha)\in V\}.$$

On V_ε, we have

$$\begin{aligned}& d(\widetilde{\Psi}|_{V_\varepsilon})(\varepsilon, w_2^\alpha, \cdots, w_N^\alpha)\\ =\ & d(\Psi|_{V_\varepsilon})(\varepsilon, w_2^\alpha, \cdots, w_N^\alpha)\\ =\ & \sum_{2\le k,i,j\le N} Q_{k,i,j}(\log|\varepsilon|, \log|w_2^\alpha|, \cdots, \log|w_N^\alpha|)\frac{(\overline{w_k^\alpha}dw_k^\alpha + w_k^\alpha d\overline{w_k^\alpha})\wedge dw_i^\alpha\wedge d\overline{w_j^\alpha}}{|w_k^\alpha|^2 w_i^\alpha\overline{w_j^\alpha}},\end{aligned}$$

where $Q_{k,i,j}(x_1, x_2, \cdots, x_N)$'s are rational functions.

On V, we also have

$$
\begin{aligned}
& d\mathrm{Res}_{D_1}(\partial\overline{\partial}\log\Phi_{g,\Gamma})(w_2^\alpha,\cdots,w_N^\alpha) \\
= & \sum_{2\leq k,i,j\leq N} P_{k,i,j}(\log|w_2^\alpha|,\cdots,\log|w_N^\alpha|)\frac{(\overline{w_k^\alpha}dw_k^\alpha + w_k^\alpha d\overline{w_k^\alpha})\wedge dw_i^\alpha\wedge d\overline{w_j^\alpha}}{|w_k^\alpha|^2 w_i^\alpha\overline{w_j^\alpha}},
\end{aligned}
$$

where $P_{k,i,j}(x_2,\cdots,x_N)$'s are rational functions.

Let (k,i,j) be an arbitrary triple with $2\leq k,i,j\leq N$. Let $(z_2^\alpha,\cdots,z_N^\alpha)$ be an arbitrary point on V. By directly calculating, we get

$$P_{k,i,j}(\log|z_2^\alpha|,\cdots,\log|z_N^\alpha|) = \lim_{\varepsilon\to 0} Q_{k,i,j}(\log|\varepsilon|,\log|z_2^\alpha|,\cdots,\log|z_N^\alpha|).$$

On the other hand,

$$d(\Psi|_{V_\varepsilon}) = (d\Psi)|_{V_\varepsilon} \equiv 0|_{V_\varepsilon} \equiv 0.$$

Thus $Q_{k,i,j}(\log|\varepsilon|,\log|z_2^\alpha|,\cdots,\log|z_N^\alpha|) = 0$ for any sufficiently small ε. Therefore,

$$P_{k,i,j}(\log|w_2^\alpha|,\cdots,\log|w_N^\alpha|) \equiv 0 \quad \text{on } V$$

and so $d\mathrm{Res}_{D_1}(\partial\overline{\partial}\log\Phi_{g,\Gamma}) = 0$.

□

4.1. **Some lemmas on local volume functions.** For any polynomial T in $\mathbb{R}[x_1,\cdots,x_N]$, let $\deg_i T$ be the degree of T with respect to x_i. For example, we write $T = a_l x_1^l + a_{l-1}x_1^{l-1} + \cdots + a_0$ with $a_l\neq 0$ where each a_i is a polynomial in $\mathbb{R}[x_2,\cdots,x_N]$, then $\deg_1(T) = l$.

Lemma 4.3. *Let F be an arbitrary local volume function. We have :*

1. That

$$\deg_k T_{i,j}^\alpha \begin{cases} = 2\deg_k F^\alpha - 2, & i=j=k \\ \leq 2\deg_k F^\alpha - 1, & i=k,\ j\neq k \text{ or } i\neq k,\ j=k \\ \leq 2\deg_k F^\alpha, & i\neq k,\ j\neq k \end{cases}$$

and

$$\deg_k \det(T_{i,j}^\alpha)_{1\leq i,j\leq N} \leq 2N\deg_k F^\alpha - 2.$$

2. That $\deg_i F^\alpha \geq 1$ for all $i=1,\cdots N$.

Proof. The (1) is obvious. We just prove the (2). Otherwise, $\deg_k F^\alpha = 0$ for some k. Then $T_{k,j}^\alpha$ and $T_{j,k}^\alpha$ are zero polynomials for all integers $j\in[1,N]$, so that $\det(T_{i,j}^\alpha)_{1\leq i,j\leq N}$ is a zero polynomials. But on U_α^*,

$$\det(T_{i,j}^\alpha)(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|) = (-1)^{\frac{g(g+1)}{2}}(F^\alpha)^{(g+1)(g-1)}(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|)\neq 0$$

by the theorem 3.2. It is a contradiction. □

For convenience, we now allow any function to take $\pm\infty$ value. Let

$$\nu_0 = -\log 0^+ := \lim_{r\to 0+} \log \frac{1}{r}.$$

We have the following reasonable definitions and rules :

$$\begin{aligned} 0\times\nu_0 &:= 0, & (\nu_0)^0 &= 1, \\ \alpha\nu_0 &:= \lim_{r\to 0+} \log(\frac{1}{r})^\alpha \ (\alpha\in\mathbb{R}), & \alpha\nu_0+\beta\nu_0 &:= (\alpha+\beta)\nu_0 \ (\alpha,\beta\in\mathbb{R}), \\ (\nu_0)^\alpha &:= \lim_{r\to 0+} (\log\frac{1}{r})^\alpha \ (\alpha\in\mathbb{R}_{>0}), & (\nu_0)^{-\alpha} &:= \frac{1}{(\nu_0)^\alpha} \ (\alpha\in\mathbb{R}_{>0}), \\ \nu_0^\alpha\times\nu_0^\beta &:= \nu_0^{\alpha+\beta} \ (\alpha,\beta\in\mathbb{R}). \end{aligned}$$

In this paper, the addition of $\alpha\nu_0^a$ and $\beta\nu_0^b$ is formally written as $\alpha\nu_0^a+\beta\nu_0^b$ for any two nonzero different real numbers a,b(we particularly prohibit to use the rule that $\alpha\nu_0^a+\beta\nu_0^b=\alpha\nu_0^a$ if $a>b$); we set the rule of multiplication by $(\sum_{i=1}^n \alpha_i\nu_0^{a_i})(\sum_{j=1}^m \beta_j\nu_0^{b_j}) := \sum_{1\leq i\leq n, 1\leq j\leq m} \alpha_i\beta_j\nu_0^{a_i+b_j}$. We also allow that any coefficient of a matrix take $+\infty$ value. A $n\times n$ real matrix M is said to be a **∞-positive** if M can be written as $M = M_1+(\nu_0-c)M_2$ for some positive $n\times n$ matrix M_1, some semi-positive $n\times n$ matrix M_2 and some non-negative real number c. Certainly, M_1, M_2 and c are not unique for any ∞-positive matrix M.

Lemma 4.4. *The determine of a $n\times n$ ∞-positive matrix $M=M_1+(\nu_0-c)M_2$ is never zero. Moreover,* $\det(M) = \sum_{i=0}^{\mathrm{rank}(M_2)} c_i\nu_0^i$ *with some finite number $c_{\mathrm{rank}(M_2)}>0$.*

Lemma 4.5. *Let n,g be positive integers such that $n\geq g(g+1)/2$. Let B be an open set in $(\mathbb{C}^n,(z_1,\cdots,z_n))$ containing $(0,\cdots,0)$ and let $B^* := B\backslash\bigcup_{i=1}^n B_i$ where $B_i := \{z_i=0\}$.*

*Let $M(x_1,\cdots,x_n)$ be a $g\times g$ **logarithmical positive matrix function** on B^* with n real variables $x_1,\cdots,x_n$, i.e., $M(x_1,\cdots,x_n) = \sum_{i=1}^n x_jE_j$ for n nonzero semi-positive symmetric real $g\times g$ matrices $E_1,\cdots,E_n$ and $M(-\log|w_1|,\cdots,-\log|w_n|)$ is a positive matrix at every point $(w_1,\cdots,w_n)$ in B^*.*

Let $S(x_1,\cdots,x_n) := \det(M(x_1,\cdots,x_n))$. For any integer $i\in[1,n]$, we have :

1. $M(-\log|w_1|,\cdots,-\log|w_n|)$ is a ∞-positive matrix at any point $(w_1,\cdots,w_n)\in B$.
2. $\deg_i S = \mathrm{rank}E_i$, and

$$S(x_1,\cdots,x_n) = S_i(x_1,\cdots\widehat{x_i}\cdots,x_n)x_i^{\deg_i S} + \text{ terms with lower degree of } x_i$$

where $S_i(x_1,\cdots\widehat{x_i}\cdots,x_n)$ *is a homogenous polynomial of degree* $(g-\deg_i S)$ *with* $n-1$ *variables* $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n$.

3. *There exists a positive number* α *and a* $(g-\mathrm{rank}E_i)\times(g-\mathrm{rank}E_i)$ *logarithmical positive matrix function* $M_i(x_1,\cdots\widehat{x_i}\cdots,x_n)$ *on* $B_i^*:=B_i\setminus\bigcup_{m\neq i}(B_i\cap B_m)$ *such that*

$$S_i(x_1,\cdots\widehat{x_i}\cdots,x_n)=\alpha\det(M^{(i)}(x_1,\cdots\widehat{x_i}\cdots,x_n)).$$

Moreover, $M^{(i)}(-\log|w_1|,\cdots\widehat{-\log|w_i|}\cdots,-\log|w_n|)$ *is a* ∞*-positive matrix at any point* $(w_1,\cdots,w_n)\in B$. *In particular,* $S_i(-\log|w_1|,\cdots\widehat{-\log|w_i|}\cdots,-\log|w_n|)$ *is never zero at any point* $(w_1,\cdots,w_n)\in B$.

4. *Let* $i\in[1,n]$ *be an integer and let* $B_i^*:=B_i\setminus\bigcup_{j\neq i}B_j$. *Let* $Q\in\mathbb{R}[x_1,\cdots,x_n]$ *be a homogenous polynomial with* $\deg_i Q\leq\deg_i S$ *such that*

$$Q=Q_i(x_1,\cdots\widehat{x_i}\cdots,x_n)x_i^{\deg_i Q}+\text{ terms with lower degree of } x_i\ .$$

where $Q_i(x_1,\cdots\widehat{x_i}\cdots,x_n)$ *is a homogenous polynomial with* $n-1$ *variables* $x_1,\cdots,x_{i-1}$, $x_{i+1},\cdots,x_n$. *Define a function* $A(z_1,\cdots,z_n):=(Q/S)(-\log|z_1|,\cdots,-\log|z_n|)$ *on* B^*.

a) *That* $J(w_1,\cdots,\underbrace{0}_{i},\cdots,w_n):=\lim_{t_i\to 0}(Q/S)(-\log|w_1|,\cdots,-\log|t_i|,\cdots,-\log|w_n|)$ *exists as a finite real number for any point* $(w_1,\cdots,\underbrace{0}_{i},\cdots,w_n)\in B_i^*$;

b) *the function* A *can be extended to a continuous function* $\widetilde{A}$ *on* $B^*\cup B_i^*$, *where*

$$\begin{aligned}&\widetilde{A}(w_1,\cdots,w_n)\\ :=\ &\begin{cases}A(w_1,\cdots,w_n) & (w_1,\cdots,w_n)\in B^*,\\ J(w_1,\cdots,w_{i-1},0,w_{i+1},\cdots,w_n) & (w_1,\cdots,w_{i-1},0,w_{i+1},\cdots,w_n)\in B_i^*;\end{cases}\end{aligned}$$

c) $\mathrm{Res}_i(A):=\widetilde{A}|_{B_i^*}$ *is a smooth function on* B_i^*.

Proof. 1. It is sufficient to show that $M(-\log|w_1|,\cdots,-\log|w_n|)$ is a ∞-positive matrix at any point $(w_1,\cdots,w_n)\in\bigcup_{i=1}^n\{w_i=0\}$.

Let Λ be a subset of $\{1,\cdots,n\}$.

Let $(w_1,\cdots,w_n)$ be a point in $\bigcup_{i=1}^n\{w_i=0\}$ such that $\begin{cases}w_i=0, & i\in\Lambda\\ w_i\neq 0, & i\notin\Lambda\end{cases}$. We define a system of points $\{(w_1(\epsilon),\cdots,w_n(\epsilon))\}_{\epsilon\in\mathbb{R}_+}$ given by $\begin{cases}w_i(\epsilon):=\epsilon, & i\in\Lambda\\ w_i(\epsilon):=w_i, & i\notin\Lambda\end{cases}$. It is easy to check that there is a positive $\epsilon_0<1$ such that $(w_1(\epsilon),\cdots,w_n(\epsilon))\in B^*$ for $\forall\epsilon\in(0,\epsilon_0]$.

We always have

$$\begin{aligned} M(-\log|w_1(r)|,\cdots,-\log|w_n(r)|) &= M(-\log|w_1(\epsilon_0)|,\cdots,-\log|w_n(\epsilon_0)|) \\ &\quad +(-\log r+\log\varepsilon_0)(\sum_{i\in\Lambda}E_i) \end{aligned}$$

for any sufficient small real positive number r. Since $M(-\log|w_1(\epsilon_0)|,\cdots,-\log|w_n(\epsilon_0)|)$ is a positive matrix as $(w_1(\epsilon_0),\cdots,w_n(\epsilon_0))\in B^*$ and $\sum_{i\in\Lambda}E_i$ is a semi-positive matrix, we obtain $M(-\log|w_1|,\cdots,-\log|w_n|)$ is a ∞-positive matrix and

$$\begin{aligned} & M(-\log|w_1|,\cdots,-\log|w_n|) \\ = & M(-\log|w_1(\epsilon_0)|,\cdots,-\log|w_n(\epsilon_0)|)+(\nu_0-\log\frac{1}{\epsilon_0})(\sum_{i\in\Lambda}E_i). \end{aligned}$$

2. We prove the second and the third statements in this lemma together. By the symmetry, it is sufficient to prove all statements in case of $i=1$.

   Since E_1 is semi-positive symmetric and nonzero, E_1 is diagonalized by an orthogonal matrix O such that

$$O^TE_1O=\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_g \end{pmatrix}$$

with

$$\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{k_1}>0=\lambda_{k_1+1}=\lambda_{k_1+2}=\cdots$$

where $k_1=\mathrm{rank}E_1$.

   Then, we have $O^TMO=x_1\mathrm{diag}[\lambda_1,\cdots,\lambda_{k_1},0,\cdots,0]+\sum_{i=2}^n x_iO^TE_iO$ and

$$\begin{aligned} S &= \det\big(x_1\mathrm{diag}[\lambda_1,\cdots,\lambda_{k_1},0,\cdots,0]+\sum_{i=2}^n x_iO^TE_iO\big) \\ &= P_{k_1}(x_2,\cdots,x_n)(\prod_{i=1}^{k_1}\lambda_i)x_1^{k_1}+\sum_{i=1}^{k_1}P_{k_1-i}(x_2,\cdots,x_n)a_{k_1-i}(\lambda_1,\cdots,\lambda_{k_1})x_1^{k_1-i}, \end{aligned}$$

where each $P_i(x_2,\cdots,x_n)$ is a homogenous polynomial of degree $g-i$ and each $a_i(y_1,\cdots,y_{k_1})\in\mathbb{R}[y_1,\cdots,y_{k_1}]$ is a homogenous polynomial of degree i.

   For any $2\leq j\leq n$, we define $E_j^{(1)}$ to be the $(g-k_1,g-k_1)$ matrix by deleting rows $1,\cdots,k_1$ and columns $1,\cdots,k_1$ of the matrix O^TE_jO. So all $E_i^{(1)}$ are semi-positive. Let $M^{(1)}(x_2,\cdots,x_n):=\sum\limits_{j=2}^n x_jE_j^{(1)}$. Then, $M^{(1)}(x_2,\cdots,x_n)$ is the $(g-$

$k_1, g-k_1)$ matrix by deleting rows $1, \cdots, k_1$ and columns $1, \cdots, k_1$ of the matrix $O^T M(x_1, \cdots, x_n) O$, and $P_{k_1}(x_2, \cdots, x_n) = \det(M^{(1)}(x_2, \cdots, x_n))$.

At any point $(w_1, \cdots, w_n) \in B$, $M^{(1)}(-\log|w_2|, \cdots, -\log|w_n|)$ is a ∞-positive $(g-k_1, g-k_1)$ matrix since $O^T M(-\log|w_1|, \cdots, -\log|w_n|)O$ is a ∞-positive (g,g) matrix by (1) of this lemma. Let $(0, w_2^{'}, \cdots, w_n^{'}) \in B_1^*$ be an arbitrary point. The matrix $O^T M(-\log|\epsilon|, -\log|w_1^{'}| \cdots, -\log|w_n^{'}|)O$ is positive as $(\epsilon, w_2, \cdots, w_n) \in B^*$ for any nonzero sufficiently small real number ϵ, and so $M^{(1)}(-\log|w_2^{'}|, \cdots, -\log|w_n^{'}|)$ is a positive $(g - \mathrm{rank}E_i) \times (g - \mathrm{rank}E_i)$ matrix. Thus $M^{(1)}(x_2, \cdots, x_n)$ is a $(g - \mathrm{rank}E_i) \times (g - \mathrm{rank}E_i)$ logarithmical positive matrix function on B_1^*. Therefore, $P_{k_1}(-\log|w_2|, \cdots, -\log|w_n|)$ is non zero at any point $(w_1, \cdots, w_n) \in B$ by the statement (1) of this lemma. In particular, we have that $P_{k_1}(x_2, \cdots, x_n)$ is a nonzero polynomial and $\deg_1 S = \mathrm{rank}E_1$.

3. See the proof of the statement(2).
4. Let $(w_1, \cdots, \underbrace{0}_{i}, \cdots, w_n) \in B_i^*$ be an arbitrary point. Then $(w_1, \cdots, t_i, \cdots, w_n)$ is in B^* for any $t_i \in \mathbb{C}^*$ with sufficiently small $|t_i|$. We have that

$$\begin{aligned} A(w_1, \cdots, t_i \cdots, w_n) &= \frac{(-\log|t_i|)^{\deg_i S}}{S(-\log|w_1|, \cdots, -\log|t_i| \cdots, -\log|w_n|)} \\ &\quad \times \frac{Q(-\log|w_1|, \cdots, -\log|t_i| \cdots, -\log|w_n|}{(-\log|t_i|)^{\deg_i S}} \end{aligned}$$

for any $t_i \in \mathbb{C}^*$ near zero point. Since both

$$\lim_{t_i \to 0} \frac{(-\log|t_i|)^{\deg_i S}}{S(-\log|w_1|, \cdots, -\log|t_i| \cdots, -\log|w_n|)} < \infty$$

and

$$\lim_{t_i \to 0} \frac{Q(-\log|w_1|, \cdots, -\log|t_i| \cdots, -\log|w_n|}{(-\log|t_i|)^{\deg_i S}} < \infty$$

exist, $J(z_1, \cdots, \underbrace{0}_{i}, \cdots, z_n)$ is well-defined on B_i^*. It is easy to check that for any $p \in B_i^*$, there is a neighborhood V_p of p in $B^* \cup B_i^*$ such that $\widetilde{A}(z_1, \cdots, z_n)$ is continues on V_p. Moreover, we have

$$\begin{aligned} &\mathrm{Res}_i(A)(w_1, \cdots, \underbrace{0}_{i}, \cdots, w_n) \\ = &\begin{cases} 0, & \deg_i Q < \deg_i S; \\ \frac{Q_i}{S_i}(-\log|w_1|, \cdots \widehat{-\log|w_i|} \cdots, -\log|w_n|), & \deg_i Q = \deg_i S. \end{cases} \end{aligned}$$

Thus, $\mathrm{Res}_i(A)$ is a smooth function on B_i^*.

$\square$

4.2. **Behaviors of logarithmical canonical line bundles.** Let $\widetilde{t}_{i,j} = (-1)^{i+j}\det(\widetilde{T}_{i,j})$ be the (i,j)-th cofactor of $(T^{\alpha}_{i,j})$ where $\widetilde{T}_{i,j}$ is a $(N-1)\times(N-1)$ the matrix that from deleting row i and column j of the $N\times N$ matrix $(T^{\alpha}_{i,j})$.

Let $\widetilde{K}_{i,j}$ be the $(N-1)\times(N-1)$ the matrix that from deleting row i and column j of the $N\times N$ matrix $(K_{i,j})$. Let $\widetilde{k}_{i,j} = (-1)^{i+j}\det(\widetilde{K}_{i,j})$ be the (i,j)-th cofactor of the matrix $(K_{i,j})$.

For all pair (i,j) with $1\le i,j\le N$, we define $(N-1,N-1)$ simple forms

$$\nu_{i,j} = \begin{cases} dw^{\alpha}_i \wedge d\overline{w^{\alpha}_j} \bigwedge\limits_{k\neq i,k\neq j}^{1\le k\le N} dw^{\alpha}_k\wedge d\overline{w^{\alpha}_k}, & i\neq j;\\ \bigwedge\limits_{k\neq i}^{1\le k\le N} dw^{\alpha}_k\wedge d\overline{w^{\alpha}_k}, & i=j. \end{cases}$$

By 4.0.2, we obtain $(\partial\overline{\partial}\log\Phi_{\alpha})^{N-1} = (N-1)!(\sum_{i=1}^{N}\widetilde{k}_{i,i}\nu_{ii} - \sum_{i\neq j}^{1\le i,j\le N}\widetilde{k}_{j,i}\nu_{i,j})$.

Lemma 4.6. *Let $\widetilde{h}$ be an arbitrary smooth Hermitian metric on the logarithmical cotangent bundle $[K_{\overline{\mathcal{A}}_{g,\Gamma}}+D_\infty]$. Then, there holds that*

$$\int_{\overline{\mathcal{A}}_{g,\Gamma}} c_1([K_{\overline{\mathcal{A}}_{g,\Gamma}}+D_\infty],\widetilde{h})^{N-k}\wedge\eta = \int_{\mathcal{A}_{g,\Gamma}} c_1(\mathcal{O}_{\mathcal{A}_{g,\Gamma}}(K_{\mathcal{A}_{g,\Gamma}}),h_B)^{N-k}\wedge\eta$$

for any d-closed smooth $2k$-form$(1\le k\le N)$ η on $\overline{\mathcal{A}}_{g,\Gamma}$.

Proof. According to the lemma 4.1, the statement can be obtained directly by Mumford's argument in Theorem 1.4 of [25] and Kollár argument of 5.18 in [20]. ☐

Theorem 4.7. *Let $\Gamma\subset \mathrm{Sp}(g,\mathbb{Z})$ be a neat arithmetic subgroup and $\Sigma_{\mathfrak{F}_0}$ a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$ regular with respect to Γ. Let $\overline{\mathcal{A}}_{g,\Gamma}$ be the toroidal compactification of $\mathcal{A}_{g,\Gamma}:=\mathfrak{H}_g/\Gamma$ constructed by $\Sigma_{\mathfrak{F}_0}$.*

Assume that the boundary divisor $D_\infty := \overline{\mathcal{A}}_{g,\Gamma}\setminus\mathcal{A}_{g,\Gamma}$ is simple normal crossing. Let D_i be an arbitrary irreducible component of D_∞. The intersection number

$$D_i\cdot(K_{\overline{\mathcal{A}}_{g,\Gamma}}+D_\infty)^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}-1} = 0$$

if one of the following conditions is satisfied : (i) $g=2$, (ii) D_i is constructed from an edge ρ_i in $\Sigma_{\mathfrak{F}_{\min}}$ for some minimal cusp $\mathfrak{F}_{\min}$ such that $\mathrm{Int}(\rho_i)\subset C(\mathfrak{F}_{\min})$.

Proof. Let $||\cdot||_i$ be an arbitrary Hermitian metric on the line bundle $[D_i]$ and s_i the global section of $[D_i]$ defining D_i. By the lemma 4.6, we have :

(4.7.1)

$$(K_{\overline{\mathcal{A}}_{g,\Gamma}}+D_\infty)^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}-1}\cdot D_i = \lim_{\epsilon\to 0}\int_{\overline{\mathcal{A}}_{g,\Gamma}\setminus D_\infty(\epsilon)} (\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Phi_{g,\Gamma})^{N-1}\wedge c_1([D_i],||\cdot||_i)$$

where $D_\infty(\epsilon)$ is a tube of radius ϵ around D_∞. Thus

$$
\begin{aligned}
& (K_{\overline{\mathcal{A}}_{g,\Gamma}} + D_\infty)^{\dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma} - 1} \cdot D_i \\
= & \lim_{\epsilon \to 0} -\left(\frac{\sqrt{-1}}{2\pi}\right)^N \int_{\overline{\mathcal{A}}_{g,\Gamma} \setminus D_\infty(\epsilon)} (\partial\overline{\partial} \log \Phi_{g,\Gamma})^{N-1} \wedge \partial\overline{\partial} \log ||s_i||_i \\
= & \lim_{\epsilon \to 0} \frac{1}{2}\left(\frac{\sqrt{-1}}{2\pi}\right)^N \int_{\overline{\mathcal{A}}_{g,\Gamma} \setminus D_\infty(\epsilon)} (\partial\overline{\partial} \log \Phi_{g,\Gamma})^{N-1} \wedge d(\partial - \overline{\partial}) \log ||s_i||_i \\
= & -\lim_{\epsilon \to 0} \frac{\sqrt{-1}}{4\pi} C_{N-1} \int_{\partial D_\infty(\epsilon)} (\partial\overline{\partial} \log \Phi_{g,\Gamma})^{N-1} \wedge (\partial - \overline{\partial}) \log ||s_i||_i.
\end{aligned}
$$

Let $(U_\alpha, (w_1^\alpha, \cdots, w_N^\alpha))$ be a local chart as in 4.0.1 such that $D_i \cap U_\alpha = \{w_i^\alpha = 0\}$. We write $||s_i||_i = h_\alpha(w)|w_i^\alpha|^2$ on U_α. Then,

$$
\begin{aligned}
& \int_{\partial D_\infty(\epsilon) \cap U_\alpha} C_{N-1} (\partial\overline{\partial} \log \Phi_\alpha)^{N-1} \wedge (\partial - \overline{\partial}) \log(h_\alpha(w)|w_i^\alpha|^2) \\
= & \int_{\partial D_\infty(\epsilon) \cap U_\alpha} C_{N-1} (\partial\overline{\partial} \log \Phi_\alpha)^{N-1} \wedge \left\{2\sqrt{-1}\mathrm{Im}(\partial \log |w_i^\alpha|) + (\partial - \overline{\partial}) \log h_\alpha(w)\right\} \\
= & \; 2\sqrt{-1}\mathrm{Im}\big(C_{N-1} \int_{\partial D_\infty(\epsilon) \cap U_\alpha} \partial \log |w_i^\alpha| \wedge (\partial\overline{\partial} \log \Phi_\alpha)^{N-1}\big) \\
& + C_{N-1} \int_{\partial D_\infty(\epsilon) \cap U_\alpha} (\partial\overline{\partial} \log \Phi_\alpha)^{N-1} \wedge (\partial - \overline{\partial}) \log h_\alpha(w).
\end{aligned}
$$

Using similar calculation as Proposition 1.2 in [25], we get

$$
\lim_{\epsilon \to 0} \int_{\partial D_\infty(\epsilon) \cap U_\alpha} (\partial\overline{\partial} \log \Phi_\alpha)^{N-1} \wedge (\partial - \overline{\partial}) \log h_\alpha(w) = 0. \tag{4.7.2}
$$

On $U_\alpha^* = U_\alpha \setminus D_\infty$, we have

$$
\begin{aligned}
& \frac{1}{(N-1)!} \partial \log |w_i^\alpha| \wedge (\partial\overline{\partial} \log \Phi_\alpha)^{N-1} \\
= & \; \widetilde{k}_{i,i} \frac{dw_i^\alpha}{w_i^\alpha} \wedge \bigwedge_{1 \le m \le N, m \ne i} dw_m^\alpha \wedge d\overline{w_m^\alpha} \\
& + \sum_{1 \le m \le N, m \ne i} \widetilde{k}_{i,m} \frac{dw_m^\alpha}{w_i^\alpha} \wedge \bigwedge_{1 \le l \le N, l \ne m} dw_l^\alpha \wedge d\overline{w_l^\alpha}
\end{aligned}
$$

Let $T_j(\epsilon)$ be the tube neighborhood of $\{w_j = 0\}$ in U_α for $j = 1, \cdots, N$. Due to $dw_k^\alpha \wedge \overline{dw_k^\alpha} = 0$ on $\partial T_k(\epsilon) := \{|w_k^\alpha| = \epsilon\}$, we get

$$\begin{aligned}
&\int_{\partial D_\infty(\epsilon)\cap U_\alpha} \partial \log |w_i^\alpha| \wedge (\partial\overline{\partial} \log \Phi_\alpha)^{N-1} \\
= &\sum_{j=1}^{N} \int_{\partial T_j(\epsilon)\cap \partial D_\infty(\epsilon)} \partial \log |w_i^\alpha| \wedge (\partial\overline{\partial} \log \Phi_\alpha)^{N-1} \\
= &(N-1)! \int_{\partial T_i(\epsilon)\cap \partial D_\infty(\epsilon)} \frac{\widetilde{k}_{i,i}}{w_i^\alpha} dw_i^\alpha \wedge \bigwedge_{1\le m\le N, m\ne i} dw_m^\alpha \wedge \overline{dw_m^\alpha} \\
&+(N-1)! \sum_{1\le j\le N, j\ne i} \int_{\partial T_j(\epsilon)\cap \partial D_\infty(\epsilon)} \frac{\widetilde{k}_{i,j}}{w_i^\alpha} dw_j^\alpha \wedge \bigwedge_{1\le l\le N, l\ne j} dw_l^\alpha \wedge \overline{dw_l^\alpha}.
\end{aligned}$$

Since $\sqrt{-1}\partial\overline{\partial}\log \Phi_\alpha$ has Poincaré growth on boundary, the above integral is bounded uniformly on ϵ.

For any integer $j \in [1, N]$ with $j \neq i$, we have

$$\begin{aligned}
\frac{\widetilde{k}_{i,j}}{w_i^\alpha} = &(-1)^{N-1} (\frac{g+1}{4})^{N-1} \\
&\times \frac{\widetilde{t}_{i,j}(\log|w_1^\alpha|, \cdots, \log|w_N^\alpha|)}{w_j^\alpha (\prod_{1\le l\le N, l\ne j} |w_l^\alpha|^2)(F^\alpha)^{2(N-1)}(\log|w_1^\alpha|, \cdots, \log|w_N^\alpha|)} \quad \text{on } \partial T_j(\epsilon)\cap \partial D_\infty(\epsilon)
\end{aligned}$$

where $\widetilde{t}_{i,j} = (-1)^{i+j}\det(\widetilde{T}_{i,j})$ is the (i,j)-th cofactor of the matrix $(T_{l,m})_{1\le l,m\le N}$.

The lemma 4.3 says

$$\deg_j \widetilde{t}_{i,j} \le 2(N-1)\deg_j F - 1 \ \forall j \neq i,$$

we then get

$$\lim_{\epsilon\to 0} \int_{\partial T_j(\epsilon)\cap \partial D_\infty(\epsilon)} \frac{\widetilde{k}_{i,j}}{w_i^\alpha} dw_j^\alpha \wedge \bigwedge_{1\le l\le N, l\ne j} dw_l^\alpha \wedge \overline{dw_l^\alpha} = 0 \quad \forall j \neq i. \tag{4.7.3}$$

by using the generalized Cauchy integral formula and the Poincaré growth of $(\partial\overline{\partial}\log\Phi_\alpha)^{N-1}$.

Also, we have

$$\begin{aligned}
\frac{\widetilde{k}_{i,i}}{w_i^\alpha} = &(-1)^{N-1} (\frac{g+1}{4})^{N-1} \\
&\times \frac{\widetilde{t}_{i,i}(\log|w_1^\alpha|, \cdots, \log|w_N^\alpha|)}{w_i^\alpha (\prod_{1\le l\le N, l\ne i} |w_l^\alpha|^2)(F^\alpha)^{2(N-1)}(\log|w_1^\alpha|, \cdots, \log|w_N^\alpha|)} \quad \text{on } \partial T_i(\epsilon)\cap \partial D_\infty(\epsilon)
\end{aligned}$$

where $\widetilde{t}_{i,i} = \det(\widetilde{T}_{i,i})$ is the (i,i)-th cofactor of the matrix $(T_{l,m})_{1\le l,m\le N}$.

Let $d_i = \deg_i F^\alpha[x_1, \cdots, x_N]$. We can write

$$F^\alpha = P(x_1, \cdots, \widehat{x_i}, \cdots, x_N) x_i^{d_i} + \cdots,$$

where P is a homogenous polynomial in $\mathbb{R}[x_1, \cdots, \widehat{x_i}, \cdots, x_N]$ of degree $g - d_i$. For any integers l, m in $[1, N] \setminus \{i\}$, we define $A_{l,m} := PP_{lm} - P_l P_m$, where $P_{lm} := \frac{\partial^2 P}{\partial x_l \partial x_m}$, $P_l := \frac{\partial P}{\partial x_l}$. We get a $(N-1) \times (N-1)$ matrix $(A_{l,m})_{l,m \in [1,N] \setminus \{i\}}$. Then, the coefficient of the term $x_i^{2(N-1)d_i}$ in $\widetilde{t}_{i,i}$ is just $\det(A_{l,m})$. Again using the generalized Cauchy integral formula and the Poincaré growth of $(\partial\overline{\partial} \log \Phi_\alpha)^{N-1}$, we have

$$\begin{aligned}
& \lim_{\epsilon \to 0} \frac{1}{2\pi\sqrt{-1}} \int_{\partial T_i(\epsilon) \cap \partial D_\infty(\epsilon)} \frac{\widetilde{k}_{i,i}}{w_i^\alpha} dw_i^\alpha \wedge \bigwedge_{1 \leq m \leq N, m \neq i} dw_m^\alpha \wedge d\overline{w_m^\alpha} \\
= \ & (-1)^{N-1} (\frac{g+1}{4})^{N-1} \\
& \times \int_{\{w_i^\alpha = 0\}} \frac{\det(A_{l,m})(\log|w_1^\alpha|, \cdots, \widehat{\log|w_i^\alpha|}, \cdots, \log|w_N^\alpha|) \bigwedge_{1 \leq m \leq N, m \neq i} dw_m^\alpha \wedge d\overline{w_m^\alpha}}{(\prod_{1 \leq l \leq N, l \neq i} |w_l^\alpha|^2) P^{2(N-1)}(\log|w_1^\alpha|, \cdots, \widehat{\log|w_i^\alpha|}, \cdots, , \log|w_N^\alpha|)}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
& \lim_{\epsilon \to 0} \frac{1}{2} (\frac{\sqrt{-1}}{2\pi})^N \int_{\partial D_\infty(\epsilon) \cap U_\alpha} (\partial\overline{\partial} \log \Phi_{g,\Gamma})^{N-1} \wedge (\partial - \overline{\partial}) \log ||s_i||_i \\
= \ & \frac{\sqrt{-1}}{4\pi} \times \lim_{\epsilon \to 0} \int_{\partial D_\infty(\epsilon) \cap U_\alpha} (\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log \Phi_\alpha)^{N-1} \wedge (\partial - \overline{\partial}) \log(h_\alpha(w)|w_i^\alpha|^2) \\
= \ & \frac{-1}{2\pi} \mathrm{Im}\big(\lim_{\epsilon \to 0} \int_{\partial D_\infty(\epsilon) \cap U_\alpha} \partial \log|w_i^\alpha| \wedge (\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log \Phi_\alpha)^{N-1} \big) \\
= \ & \frac{-(N-1)!}{2\pi} \mathrm{Im}\big(\lim_{\epsilon \to 0} (\frac{\sqrt{-1}}{2\pi})^{N-1} \int_{\partial D_\infty(\epsilon) \cap T_i(\epsilon)} \frac{\widetilde{k}_{i,i}}{w_i^\alpha} dw_i^\alpha \wedge \bigwedge_{1 \leq m \leq N, m \neq i} dw_m^\alpha \wedge d\overline{w_m^\alpha} \big) \\
= \ & (-1)^N (N-1)! (\frac{g+1}{4})^{N-1} \\
& \times \int_{\{w_i^\alpha = 0\}} \frac{\det(PP_{lm} - P_l P_m)}{P^{2(N-1)}} (\log|w_1^\alpha|, \cdots, \widehat{,i,}, \cdots, \log|w_N^\alpha|) \frac{\bigwedge_{m \neq i}^{1 \leq m \leq N} (\frac{\sqrt{-1}}{2\pi} dw_m^\alpha \wedge d\overline{w_m^\alpha})}{\prod_{1 \leq l \leq N, l \neq i} |w_l^\alpha|^2}.
\end{aligned}$$

In all conditions(i),(ii), the polynomial P never takes zero value and $\det(PP_{lm} - P_l P_m)$ is always zero by the lemma 4.5. $\square$

4.3. **Intersection theory for infinity divisor boundaries and non ampleness of logarithmical canonical bundles.** Let d be an integer with $1 \leq d \leq N-1$. Let $D_1, \cdots, D_d$ be d irreducible components of the boundary divisor D_∞. For each integer $i \in [1, d]$, let $|| \cdot ||_i$ be an arbitrary Hermitian metric on the line bundle $[D_i]$ and let s_i be a global section of $[D_i]$ defining D_i.

Now we study the intersection number $(K_{\overline{\mathcal{A}}_{g,\Gamma}} + D_\infty)^{N-d} \cdot D_1 \cdots D_d$. By the lemma 4.6, we have :

$$
\begin{aligned}
& (K_{\overline{\mathcal{A}}_{g,\Gamma}} + D_\infty)^{N-d} \cdot D_1 \cdots D_d \\
= \ & \lim_{\epsilon \to 0} \int_{\overline{\mathcal{A}}_{g,\Gamma} - D_\infty(\epsilon)} (\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log \Phi_{g,\Gamma})^{N-d} \wedge \bigwedge_{i=1}^{d} c_1([D_i], || \cdot ||_i) \\
= \ & \lim_{\epsilon \to 0} -C_{N-d+1} \int_{\overline{\mathcal{A}}_{g,\Gamma} - D_\infty(\epsilon)} (\partial\overline{\partial} \log \Phi_{g,\Gamma})^{N-d} \wedge (\bigwedge_{i=2}^{d} c_1([D_i], || \cdot ||_i)) \wedge \partial\overline{\partial} \log ||s_1||_1 \\
= \ & \lim_{\epsilon \to 0} \frac{C_{N-d+1}}{2} \int_{\overline{\mathcal{A}}_{g,\Gamma} - D_\infty(\epsilon)} (\partial\overline{\partial} \log \Phi_{g,\Gamma})^{N-d} \wedge (\bigwedge_{i=2}^{d} c_1([D_i], || \cdot ||_i)) \wedge d(\partial - \overline{\partial}) \log ||s_1||_1 \\
= \ & -\lim_{\epsilon \to 0} \frac{\sqrt{-1}}{4\pi} C_{N-d} \int_{\partial D_\infty(\epsilon)} (\partial\overline{\partial} \log \Phi_{g,\Gamma})^{N-d} \wedge (\bigwedge_{i=2}^{d} c_1([D_i], || \cdot ||_i)) \wedge (\partial - \overline{\partial}) \log ||s_1||_1.
\end{aligned}
$$

Let $(U_\alpha, (w_1^\alpha, \cdots, w_N^\alpha))$ be a local coordinate chart as in 4.0.1 such that $D_1 \cap U_\alpha = \{w_1^\alpha = 0\}$ and $U_\alpha^* = U_\alpha \setminus D_\infty$. We write $||s_1||_1 = h_\alpha(w)|w_1^\alpha|^2$ on U_α. Since ω_{can} has Poincaré growth on D_∞, we get

$$
\begin{aligned}
& \lim_{\epsilon \to 0} \frac{\sqrt{-1}}{4\pi} C_{N-d} \int_{\partial D_\infty(\epsilon) \cap U_\alpha} (\partial\overline{\partial} \log \Phi_{g,\Gamma})^{N-d} \wedge (\bigwedge_{i=2}^{d} c_1([D_i], || \cdot ||_i)) \wedge (\partial - \overline{\partial}) \log ||s_1||_1. \\
= \ & \frac{-1}{2\pi} \mathrm{Im}\big(C_{N-d} \lim_{\epsilon \to 0} \int_{\partial D_\infty(\epsilon) \cap U_\alpha} \partial \log |w_1^\alpha| \wedge (\partial\overline{\partial} \log \Phi_\alpha)^{N-d} \wedge (\bigwedge_{i=2}^{d} c_1([D_i], || \cdot ||_i))\big).
\end{aligned}
$$

We now use $[i_1, \cdots, i_l]$ to mean a l-tuple $(i_1, \cdots, i_l)$ with $1 \leq i_1 < i_2 < \cdots < i_l \leq N$, and we say $[i_1, \cdots, i_l] = [j_1, \cdots, j_l]$ if and only if $i_k = j_k \ \forall k = 1, \cdots, l$. For any $[i_1, i_2, \cdots, i_l]$, let $j_1, \cdots, j_{N-l}$ be integers in $\{1, \cdots, N\} \setminus \{i_1, \cdots, i_l\}$ satisfying $1 \leq j_1 < j_2 < \cdots < j_{N-l} \leq N$, define $[i_1, \cdots, i_l]^\circ := [j_1, \cdots, j_{N-l}]$. So $[i_1, \cdots, i_l]^\circ = [j_1, \cdots, j_l]^\circ$ if and only if $[i_1, \cdots, i_l] = [j_1, \cdots, j_l]$. For any l-tuple $[i_i, \cdots, i_l]$, we define a simple (l, l)-form on U_α

$$
\nu_{[j_i, \cdots, j_l]}^{[i_i, \cdots, i_l]} := (dw_{i_1}^\alpha \wedge \cdots \wedge dw_{i_l}^\alpha) \bigwedge (d\overline{w_{j_1}^\alpha} \wedge \cdots \wedge d\overline{w_{j_l}^\alpha}).
$$

For a $N\times N$ matrix $A=(A_{ij})$, we use $A^{[i_i,\cdots,i_d]}_{[j_i,\cdots,j_d]}$ to mean a $(N-d,N-d)$ matrix by deleting rows $i_1,\cdots,i_d$ and column $j_1,\cdots,j_d$ of the matrix $A=(A_{i,j})$, and we define

$$\widetilde{A}^{[i_i,\cdots,i_d]}_{[j_i,\cdots,j_d]}=(-1)^{\sum_{k=1}^{d}(i_k+j_k)}\det(A^{[i_i,\cdots,i_d]}_{[j_i,\cdots,j_d]}).$$

Consider the $N\times N$ matrix $K=(K_{i,j})$ related to the equality 4.0.2, we have

$$\begin{aligned}&\frac{1}{(N-d)!}(\partial\overline{\partial}\log\Phi_\alpha)^{N-d}\\ =&\sum_{[i_1,\cdots,i_d]}\widetilde{K}^{[i_1,\cdots,i_d]}_{[i_1,\cdots,i_d]}\bigwedge_{1\leq l\leq N-d}^{[j_1,\cdots,j_{N-d}]=[i_1,\cdots,i_d]^\circ}(dw^\alpha_{j_l}\wedge d\overline{w^\alpha_{j_l}})+\sum_{[i_1,\cdots,i_d],[j_1,\cdots,j_d]}^{[i_1,\cdots,i_d]\neq[j_1,\cdots,j_d]}\pm\widetilde{K}^{[i_1,\cdots,i_d]}_{[j_1,\cdots,j_d]}\nu^{[i_1,\cdots,i_d]^\circ}_{[j_1,\cdots,j_d]^\circ}\end{aligned}$$

and we get the following equality on $\partial D_\infty(\epsilon)\cap U_\alpha$:

$$\begin{aligned}\frac{\partial\log|w^\alpha_1|\wedge(\partial\overline{\partial}\log\Phi_\alpha)^{N-d}}{(N-d)!}=&\underbrace{\sum_{[1,i_2,\cdots,i_d]}\frac{\widetilde{K}^{[1,i_2,\cdots,i_d]}_{[1,i_2\cdots,i_d]}}{2w^\alpha_1}dw^\alpha_1\wedge\bigwedge_{1\leq l\leq N-d}^{[j_1,\cdots,j_{N-d}]=[1,i_2,\cdots,i_d]^\circ}(dw^\alpha_{j_l}\wedge d\overline{w^\alpha_{j_l}})}_{I}\\ &+\underbrace{\sum_{[1,i_2,\cdots,i_d],[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]\neq[1,j_2,\cdots,j_d]}\pm\frac{\widetilde{K}^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]}}{2w^\alpha_1}dw^\alpha_1\wedge\nu^{[1,i_2,\cdots,i_d]^\circ}_{[1,j_2,\cdots,j_d]^\circ}}_{II}\\ &+\underbrace{\sum_{[1,i_2,\cdots,i_d],[j_1,\cdots,j_d]}^{j_1\neq1}\pm\frac{\widetilde{K}^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]}}{2w^\alpha_1}dw^\alpha_1\wedge\nu^{[1,i_2,\cdots,i_d]^\circ}_{[j_1,\cdots,j_d]^\circ}}_{III}\end{aligned}$$

Here I, II and III are smooth forms on $\partial D_\infty(\epsilon)\cap U_\alpha$. There is

$$\partial D_\infty(\epsilon)\cap U_\alpha=\bigcup_{j=1}^{N}\partial D_\infty(\epsilon)\cap\partial T_j(\epsilon),$$

where each $T_j(\epsilon)$ is the tube neighborhood of $\{w_j=0\}$ in U_α with radius ε. Certainly, it is not necessary that $D_j\cap U_\alpha=\{w_k=0\}$ for some k if $j\geq 2$.

Lemma 4.8. *Let η be an arbitrary smooth $(d-1,d-1)$-form on $\overline{\mathcal{A}}_{g,\Gamma}$. We have*

$$\lim_{\epsilon\to0}\int_{\partial D_\infty(\epsilon)\cap\partial T_k(\epsilon)}I\wedge\eta=0$$

for any integer $k\in[2,N]$.

Proof. Since $dw_k^\alpha \wedge d\overline{w_k^\alpha} = 0$ on $\partial T_k(\epsilon)(=\{|w_k^\alpha| = \epsilon\})$, we get

$$I = \sum_{[1,i_2,\cdots,i_d],k\in\{i_2,\cdots,i_d\}} \theta([1,i_2,\cdots i_d]) \quad \text{on } \partial T_k(\epsilon),$$

where

$$\theta([1,i_2,\cdots i_d]) := \widetilde{K}^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]} \frac{dw_1^\alpha}{2w_1^\alpha} \wedge \bigwedge_{l=1}^{N-d} (dw_{j_l}^\alpha \wedge d\overline{w_{j_l}^\alpha}) \quad \text{on } \partial T_k(\epsilon)$$

with $[j_1,\cdots,j_{N-d}] = [1,\cdots,i_l]^\circ$. On the coordinate chart $(U_\alpha,(w_1^\alpha,\cdots,w_N^\alpha))$, we write $\eta = \sum_\beta c_\beta\eta_\beta$ where each c_β is a smooth function on U_α and each η_β is a simple $(d-1,d-1)$ form given by the wedge product of some dw_l^α's and some $d\overline{w_m^\alpha}$'s with coefficient 1. It is sufficient to prove the equality

$$\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap\partial T_k(\epsilon)} \theta([1,i_2,\cdots i_d])\wedge c_\beta\eta_\beta = 0$$

for any d-tuple $[1,i_2,\cdots i_d]$ satisfying $k\in\{i_2,\cdots,i_d\}$, any simple $(d-1,d-1)$ form η_β with coefficient 1 and any smooth function c_β on U_α.

Let η_β be an arbitrary simple $(d-1,d-1)$ form with coefficient 1 and c_β an arbitrary smooth function on U_α. We may require that η_β does not contain the factor dw_1^α and contains neither a factor like $dw_{j_l}^\alpha$ nor a factor like $d\overline{w_{j_m}^\alpha}$(or else $\theta([1,i_2,\cdots i_d])\wedge\eta_\beta = 0$). Since $\theta\wedge\eta_\beta$ is a simple $(N,N-1)$-form with coefficient $\frac{\widetilde{K}^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]}}{2w_1^\alpha}$, η_β must contain dw_k^α. We also require that η_β does not contain the factor $d\overline{w_k^\alpha}$(or else $\eta_\beta = 0$ on $\partial T_k(\epsilon)$). Then, η_β contains the factor $d\overline{w_1^\alpha}$ and

$$\begin{aligned}
&\theta([1,i_2,\cdots i_d])\wedge\eta_\beta \\
= \;& \pm\widetilde{K}^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]} dw_k^\alpha \wedge \frac{dw_1^\alpha\wedge d\overline{w_1^\alpha}}{2w_1^\alpha} \wedge \bigwedge_{j=2}^{k-1}(dw_j^\alpha\wedge d\overline{w_j^\alpha}) \wedge \bigwedge_{j=k+1}^{N}(dw_j^\alpha\wedge d\overline{w_j^\alpha}) \quad \text{on } \partial T_k(\epsilon).
\end{aligned}$$

Since $(\partial\overline{\partial}\log\Phi_\alpha)^{N-d}$ is a $(N-d,N-d)$ form of Poincaré growth, there is a small neighborhood V_0 of the origin point $0\in U_\alpha$ such that

$$|\widetilde{K}^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]}| \le \frac{C}{\prod_{m=1}^{N-d}|w_{j_m}^\alpha|^2(\log|w_{j_m}^\alpha|)^2} \text{ on } V_0\cap U_\alpha^*$$

where C is a constant depending on V_0. It is well-known that there is

$$\lim_{\varepsilon\to 0^+}\varepsilon\Big(\int_\varepsilon^A \frac{dr}{r(\log r)^2}\Big)^n = 0$$

for any real numbers $A, n > 0$. Therefore, we have

$$\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap\partial T_k(\epsilon)} \theta([1, i_2, \cdots i_d]) \wedge c_\beta\eta_\beta = 0.$$

□

Lemma 4.9. *Let η be an arbitrary smooth $(d-1, d-1)$-form on $\overline{\mathcal{A}}_{g,\Gamma}$. We have*

$$\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap\partial T_k(\epsilon)} II \wedge \eta = 0$$

for any integer $k \in [2, N]$.

Proof. We can write $II = \sum_{[1,i_2,\cdots,i_d],[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]\neq[1,j_2,\cdots,j_d]} \pm\theta^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]}$, where

$$\theta^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]} := \widetilde{K}^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]}\frac{dw_1^\alpha}{2w_1^\alpha}\wedge \nu^{[1,i_2,\cdots,i_d]^\circ}_{[1,j_2,\cdots,j_d]^\circ}.$$

Since $dw_k^\alpha \wedge \overline{dw_k^\alpha} = 0$ on $\partial T_k(\epsilon)$, it is sufficient to show

$$\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap\partial T_k(\epsilon)} \theta^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]} \wedge c_\beta\eta_\beta = 0$$

for any simple $(d-1, d-1)$ form η_β with coefficient 1, any smooth function c_β on U_α and any two d-tuples $[1, i_2, \cdots, i_d], [1, j_2, \cdots, j_d]$ satisfying

$$(!)\quad [1, i_2, \cdots, i_d] \neq [1, j_2, \cdots, j_d] \quad\text{and}\quad k \in \{i_2, \cdots, i_d\}\bigcup\{j_2, \cdots, j_d\}.$$

Let η_β be an arbitrary simple $(d-1, d-1)$ form with coefficient 1 and c_β an arbitrary smooth function on U_α. Let $[1, i_2, \cdots, i_d], [1, j_2, \cdots, j_d]$ be two arbitrary d-tuples satisfying the condition (!), and let $[i_{d+1}, \cdots, i_N] = [1, i_2, \cdots, i_d]^\circ$ and $[j_{d+1}, \cdots, j_N] = [1, j_2, \cdots, j_d]^\circ$. So $i_{d+1} \geq 2, j_{d+1} \geq 2$ and $\theta^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]} = \widetilde{K}^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]}\frac{dw_1^\alpha}{2w_1^\alpha}\wedge\nu^{[i_{d+1},\cdots,i_N]}_{[j_{d+1},\cdots,j_N]}$.

We can require that the simple form η_β contains no factor in the set

$$\{dw_1^\alpha\} \cup \{dw^\alpha_{i_{d+1}}, \cdots, dw^\alpha_{i_N}\} \cup \{\overline{dw_1^\alpha}, \overline{dw^\alpha_{j_{d+2}}}, \cdots, \overline{dw^\alpha_{j_N}}\}$$

(or else $\theta^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]} \wedge \eta_\beta = 0$). There are three cases :

- $k \in \{i_2, \cdots, i_d\} \cap \{j_2, \cdots, j_d\}$: Then $k \notin \{i_{d+1}, \cdots, i_N\} \cup \{j_{d+1}, \cdots, j_N\}$. Suppose that $\theta^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]} \wedge \eta_\beta$ is a nonzero $(N, N-1)$ simple form. Then η_β contains the factor dw_k^α by that $k \notin \{i_{d+1}, \cdots, i_N\}$. We may require that η_β does not contain the factor $\overline{dw_k^\alpha}$(otherwise $\eta = 0$ on $\partial T_k(\epsilon)$), and then η_β contains the factor $\overline{dw_1^\alpha}$. We have that

$$\theta^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]} \wedge \eta_\beta = \pm\widetilde{K}^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]} dw_k^\alpha \wedge \frac{dw_1^\alpha\wedge\overline{dw_1^\alpha}}{2w_1^\alpha}\wedge(\bigwedge_{l=2}^{k-1} dw_l^\alpha\wedge\overline{dw_l^\alpha})\wedge(\bigwedge_{k+1}^{N} dw_l^\alpha\wedge\overline{dw_l^\alpha}).$$

by that $k \notin \{j_{d+1}, \cdots, j_N\}$. Since that $k \notin \{i_{d+1}, \cdots, i_N\} \cup \{j_{d+1}, \cdots, j_N\}$, we obtain

$$\lim_{\epsilon\to 0} \int_{\partial D_\infty(\epsilon)\cap\partial T_k(\epsilon)} \theta^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]} \wedge c_\beta \eta_\beta = 0$$

by the Poincaré growth of the form $(\partial\overline{\partial}\log\Phi_\alpha)^{N-d}$.

- $k \in \{i_2, \cdots, i_d\}$ but $k \notin \{j_2, \cdots, j_d\}$: Then, $k \in \{j_{d+1}, \cdots, j_N\}$, and so
$$\theta^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]} \wedge \eta_\beta = 0 \quad \text{on} \quad \partial T_k(\epsilon).$$
- $k \in \{j_2, \cdots, j_d\} \setminus \{i_2, \cdots, i_d\}$: Then $k \in \{i_{d+1}, \cdots, i_N\}$ but $k \notin \{j_{d+1}, \cdots, j_N\}$. We require that η_β does not have the factor $d\overline{w^\alpha_k}$(or else $\theta^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]} \wedge \eta_\beta = 0$ on $\partial T_k(\epsilon)$ by $k \in \{i_{d+1}, \cdots, i_N\}$), and so η_β has the factor $d\overline{w^\alpha_1}$ as $j_{d+1} \geq 2$. Thus, we get

$$\begin{aligned}
&\theta^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]} \wedge \eta_\beta \\
= \;& \pm \frac{(-g-1)^{N-d}\det(T^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]})}{4^{N-d}(\prod^{i_l\neq k}_{d+1\leq l\leq N} w^\alpha_{i_l})(\prod^{N}_{l=d+1}\overline{w_{j_l}})(F^\alpha)^{2(N-d)}(\log|w^\alpha_1|, \cdots, \log|w^\alpha_N|)} \\
&\times \frac{dw^\alpha_k}{w^\alpha_k} \wedge \frac{dw^\alpha_1 \wedge d\overline{w^\alpha_1}}{2w^\alpha_1} \wedge (\bigwedge^{k-1}_{i=2} dw^\alpha_i d \wedge \overline{w^\alpha_i}) \wedge (\bigwedge^{N}_{i=k+1} dw^\alpha_i \wedge d\overline{w^\alpha_i}) \text{ on } \partial T_k(\epsilon).
\end{aligned}$$

Since $k \in \{j_{d+1}, \cdots, j_N\}$ but $k \notin \{i_{d+1}, \cdots, i_N\}$, we have

$$\deg_k \det(T^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]}) \leq 2(N-d)\deg_k F^\alpha - 1.$$

by the lemma 4.3. Therefore, we obtain

$$\lim_{\epsilon\to 0} \int_{\partial D_\infty(\epsilon)\cap\partial T_k(\epsilon)} \theta^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]} \wedge c_\beta \eta_\beta = \int_{\{w^\alpha_k=0\}} 0 = 0.$$

by using the generalized Cauchy integral formula and using the Poincaré growth of $(\partial\overline{\partial}\log\Phi_\alpha)^{N-d}$.

□

Lemma 4.10. *Let η be an arbitrary smooth $(d-1, d-1)$-form on $\overline{\mathcal{A}}_{g,\Gamma}$. We have that*

$$\lim_{\epsilon\to 0} \int_{\partial D_\infty(\epsilon)\cap\partial T_k(\epsilon)} III \wedge \eta = 0$$

for any integer $k \in [1, N]$.

Proof. Since $dw_k^\alpha \wedge d\overline{w_k^\alpha} = 0$ on $\partial T_k(\epsilon)$, $III = \sum_{[1,i_2,\cdots,i_d],[j_1,\cdots,j_d],j_1\neq 1}^{k\in\{1,i_2,\cdots,i_d\}\cup\{j_1,\cdots,j_d\}} \pm\theta_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]}$ on $\partial T_k(\epsilon)$ where

$$\theta_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]} := \widetilde{K}_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]} \frac{dw_1^\alpha}{2w_1^\alpha} \wedge \nu_{[j_1,\cdots,j_d]^\circ}^{[1,\cdots,i_d]^\circ}.$$

As $j_1 \neq 1$, we must have $[j_1, \cdots, j_d]^\circ = [1, \cdots]$ and so

$$III = 0 \text{ on } \partial T_1(\epsilon).$$

Now we suppose that $k > 1$. It is sufficient to show

$$\lim_{\epsilon\to 0} \int_{\partial D_\infty(\epsilon)\cap\partial T_k(\epsilon)} \theta_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]} \wedge c_\beta \eta_\beta = 0$$

for any simple $(d-1, d-1)$ form η_β with coefficient 1, any smooth function c_β on U_α and any two d-tuples $[1, i_2, \cdots, i_d], [j_1, \cdots, j_d]$ satisfying

$$(!!) \quad j_1 \neq 1 \text{ and } k \in \{1, i_2, \cdots, i_d\} \cup \{j_1, \cdots, j_d\}.$$

Let η_β be an arbitrary simple $(d-1, d-1)$ form with coefficient 1 and c_β an arbitrary smooth function on U_α. Let $[1, i_2, \cdots, i_d], [j_1, \cdots, j_d]$ be two arbitrary d-tuples satisfying the condition (!!), and let $[i_{d+1}, \cdots, i_N] = [1, i_2, \cdots, i_d]^\circ$ and $[1, j_{d+2}, \cdots, j_N] = [j_1, \cdots, j_d]^\circ$. Then, we get

$$\theta_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]} = \widetilde{K}_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]} \frac{dw_1^\alpha \wedge d\overline{w_1^\alpha}}{2w_1^\alpha} \wedge \bigwedge_{l=d+1}^{N} dw_{i_l}^\alpha \wedge \bigwedge_{l=d+2}^{N} d\overline{w_{j_l}^\alpha}.$$

We can require that the simple form η_β contains no factor in the set

$$\{dw_1^\alpha\} \cup \{dw_{i_{d+1}}^\alpha, \cdots, dw_{i_N}^\alpha\} \cup \{d\overline{w_1^\alpha}, d\overline{w_{j_{d+2}}^\alpha}, \cdots, d\overline{w_{j_N}^\alpha}\}$$

(or else $\theta_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]} \wedge \eta_\beta = 0$). There are three cases :

- $k \in \{1, i_2, \cdots, i_d\} \cap \{j_1, \cdots, j_d\}$: So $k \notin \{i_{d+1}, \cdots, i_N\} \cup \{1, j_{d+2}, \cdots, j_N\}$.

  Suppose that $\theta_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]} \wedge \eta_\beta$ is a nonzero $(N, N-1)$ simple form with coefficient $\frac{\widetilde{K}_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]}}{2w_1^\alpha}$. Then η_β contains the factor dw_k^α by $k \notin \{i_{d+1}, \cdots, i_N\}$. We may require η_β does not contain the factor $d\overline{w_k^\alpha}$(otherwise, $\eta = 0$ on $\partial T_k(\epsilon)$). We have

$$\theta_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]} \wedge \eta_\beta = \pm\widetilde{K}_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]} dw_k^\alpha \wedge \frac{dw_1^\alpha \wedge d\overline{w_1^\alpha}}{2w_1^\alpha} \wedge (\bigwedge_{l=2}^{k-1} dw_l^\alpha \wedge d\overline{w_l^\alpha}) \wedge (\bigwedge_{k+1}^{N} dw_l^\alpha \wedge d\overline{w_l^\alpha})$$

  by that $k \notin \{1, j_{d+2}, \cdots, j_N\}$. Then,

$$\lim_{\epsilon\to 0} \int_{\partial D_\infty(\epsilon)\cap\partial T_k(\epsilon)} \theta_{[j_1,\cdots,j_d]}^{[1,i_2,\cdots,i_d]} \wedge c_\beta \eta_\beta = 0$$

by that $k \notin \{i_{d+1}, \cdots, i_N\} \cup \{1, j_{d+2}, \cdots, j_N\}$ and the Poincaré growth of the form $(\partial\overline{\partial}\log\Phi_\alpha)^{N-d}$.

- $k \in \{1, i_2, \cdots, i_d\}$ bur $k \notin \{j_1, \cdots, j_d\}$: So $k \in \{1, j_{d+2}, \cdots, j_N\}$, and $i_{d+1} > 1$. Then,

$$\theta^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]} \wedge \eta_\beta = 0 \quad \text{on} \quad \partial T_k(\epsilon).$$

- $k \in \{j_1, \cdots, j_d\}$ but $k \notin \{1, i_2, \cdots, i_d\}$: So $k \in \{i_{d+1}, \cdots, i_N\}$ but $k \notin \{1, j_{d+2}, \cdots, j_N\}$ and $i_{d+1} > 1$. We require that η_β has no factor $d\overline{w^\alpha_k}$(otherwise, $\theta^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]} \wedge \eta_\beta = 0$ on $\partial T_k(\epsilon)$). Then, we have

$$\begin{aligned} & \theta^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]} \wedge \eta_\beta \\ = \ & \pm \frac{(-g-1)^{N-d}\det(T^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]})}{4^{N-d}(\prod^{i_l\neq k}_{d+1\leq l\leq N} w^\alpha_{i_l})(\prod^{N}_{l=d+1}\overline{w_{j_l}})(F^\alpha)^{2(N-d)}(\log|w^\alpha_1|,\cdots,\log|w^\alpha_N|)} \\ & \times \frac{dw^\alpha_k}{w^\alpha_k} \wedge \frac{dw^\alpha_1 \wedge d\overline{w^\alpha_1}}{2w^\alpha_1} \wedge (\bigwedge^{k-1}_{i=2} dw^\alpha_i \wedge d\overline{w^\alpha_i}) \wedge (\bigwedge^{N}_{i=k+1} dw^\alpha_i \wedge d\overline{w^\alpha_i}) \text{ on } \partial T_k(\epsilon). \end{aligned}$$

The lemma 4.3 says that

$$\deg_k \det(T^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]}) \leq 2(N-d)\deg_k F^\alpha - 1.$$

Therefore, we obtain

$$\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap\partial T_k(\epsilon)} \theta^{[1,i_2,\cdots,i_d]}_{[j_1,\cdots,j_d]} \wedge c_\beta\eta_\beta = 0.$$

by the generalized Cauchy integral formula and the Poincaré growth of $(\partial\overline{\partial}\log\Phi_\alpha)^{N-d}$.

□

By the lemma 4.5, we can write

$$F^\alpha(x_1,\cdots,x_N) = S^{\alpha,1}(x_2,\cdots,x_N)x_1^{\deg_1 F^\alpha} + \text{ terms of lower degree of } x_1\ ,$$

where S_1 is a homogenous polynomial in $\mathbb{R}[x_1,\cdots,\widehat{x_i},\cdots,x_N]$ of degree $(g-\deg_1 F^\alpha)$. For any integers l, m in $[1, N]$, we define a $N\times N$ matrix $A^\alpha(1) := (A^\alpha(1)_{l,m})$ by setting

$$A^\alpha(1)_{l,m} := S^{\alpha,1}\frac{\partial^2 S^{\alpha,1}}{\partial x_l\partial x_m} - \frac{\partial S^{\alpha,1}}{\partial x_l}\frac{\partial S^{\alpha,1}}{\partial x_m}. \tag{4.10.1}$$

Here is a direct consequence of the lemma 4.3:

Lemma 4.11. *The coefficient of term* $x_1^{2(N-d)\deg_1 F^\alpha}$ *in* $\det(T^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]})$ *is* $\det(P^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]})$ *where* $P := A^\alpha(1)$.

We write the real $(N-d, N-d)$-form

$$C_{N-d}(\partial\overline{\partial}\log\Phi_\alpha)^{N-d} = (N-d)! \sum_{[i_1,\cdots,i_d],[j_1,\cdots,j_d]} \varsigma^{[i_1,\cdots,i_d]}_{[j_1,\cdots,j_d]} \tag{4.11.1}$$

with $\varsigma^{[i_1,\cdots,i_d]}_{[j_1,\cdots,j_d]} = \overline{\varsigma^{[j_1,\cdots,j_d]}_{[i_1,\cdots,i_d]}}$. Each term in $C_{N-d}(\partial\overline{\partial}\log\Phi_\alpha)^{N-d}$ can be written

$$\varsigma^{[i_1,\cdots,i_d]}_{[j_1,\cdots,j_d]} = c^{[i_1,\cdots,i_d]}_{[j_1,\cdots,j_d]} \det(K^{[i_1,\cdots,i_d]}_{[j_1,\cdots,j_d]}) \bigwedge_{l=d+1}^{N} (dw^\alpha_{i_l} \wedge d\overline{w^\alpha_{j_l}}), \tag{4.11.2}$$

such that $c^{[i_1,\cdots,i_d]}_{[j_1,\cdots,j_d]}$ is the sign $\pm$ depending on $([i_1,\cdots,i_d],[j_1,\cdots,j_d])$, and

$$c^{[i_1,\cdots,i_d]}_{[j_1,\cdots,j_d]} = c^{[j_1,\cdots,j_d]}_{[i_1,\cdots,i_d]}, \quad c^{[i_1,\cdots,i_d]}_{[i_1,\cdots,i_d]} = 1.$$

Lemma 4.12. *Let $P := A^\alpha(1)$ be a $N \times N$ matrix of polynomials given in 4.10.1. Given a d-tuple $[\alpha_1,\cdots,\alpha_d]$ where $1 \leq \alpha_1,\cdots,\alpha_d \leq N$, we always use $N-d$ tuple $[\alpha_{d+1},\cdots,\alpha_N]$ to represent $[\alpha_1,\cdots,\alpha_d]^\circ$, where $\{\alpha_{d+1},\cdots,\alpha_N\} = \{1,\cdots,N\} \setminus \{\alpha_1,\cdots,\alpha_d\}$.*

For any two d-tuples $[1,i_2,\cdots,i_d],[1,j_2,\cdots,j_d]$, we define a rational function

$$\xi^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]}(x_2,\cdots,x_N) = \frac{\det(P^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]})(x_2,\cdots,x_N)}{(S^{\alpha,1})^{2(N-d)}(x_2,\cdots,x_N)}, \tag{4.12.1}$$

and we define a real $(N-d, N-d)$ form

$$\delta^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]} := \frac{C_{N-d}}{2}((\prod_{l=d+1}^{N} w^\alpha_{i_l}\overline{w^\alpha_{j_l}}) \bigwedge_{l=d+1}^{N}(dw^\alpha_{i_l} \wedge d\overline{w^\alpha_{j_l}}) + (\prod_{l=d+1}^{N} w^\alpha_{j_l}\overline{w^\alpha_{i_l}}) \bigwedge_{l=d+1}^{N}(dw^\alpha_{j_l} \wedge d\overline{w^\alpha_{i_l}})).$$

Let η be an arbitrary real smooth $(d-1,d-1)$ form on $\overline{\mathcal{A}}_{g,\Gamma}$. We have :

$$\begin{aligned}
& \lim_{\epsilon\to 0} \frac{\sqrt{-1}}{4\pi} C_{N-d} \int_{\partial D_\infty(\epsilon)\cap U_\alpha} (\partial\overline{\partial}\log\Phi_{g,\Gamma})^{N-d} \wedge \eta \wedge (\partial - \overline{\partial})\log||s_1||_1. \\
= \; & (-1)^{N-d+1}(\frac{g+1}{4})^{N-d}(N-d)! \\
& \times\{ \sum_{[1,i_2,\cdots,i_d]} \int_{\{w_1=0\}} \xi^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]}(\log|w^\alpha_2|,\cdots,\log|w^\alpha_N|) \frac{\delta^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]} \wedge \eta}{\prod_{l=d+1}^{N} |w^\alpha_{i_l}|^4} \\
& + \sum_{[1,i_2,\cdots,i_d],[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]\neq[1,j_2,\cdots,j_d]} c^{[i_1,\cdots,i_d]}_{[j_1,\cdots,j_d]} \int_{\{w_1=0\}} \xi^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]}(\log|w^\alpha_2|,\cdots,\log|w^\alpha_N|) \frac{\delta^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]} \wedge \eta}{\prod_{l=d+1}^{N} |w^\alpha_{i_l} w^\alpha_{j_l}|^2} \}
\end{aligned}$$

where each $c^{[i_1,\cdots,i_d]}_{[j_1,\cdots,j_d]}$ *is a sign defined in 4.11.1-4.11.2.*

Proof. For any two d-tuples $[1,i_2,\cdots,i_d],[1,j_2,\cdots,j_d]$, we have

$$\begin{aligned}
\frac{dw_1^\alpha}{2w_1^\alpha}\wedge\varsigma^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]} &= C_{N-d}\det(K^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]})\frac{dw_1^\alpha}{2w_1^\alpha}\wedge\bigwedge_{l=d+1}^{N}(dw^\alpha_{i_l}\wedge d\overline{w^\alpha_{i_l}})\\
\frac{dw_1^\alpha}{2w_1^\alpha}\wedge\varsigma^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]} &= c^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]}C_{N-d}\det(K^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]})\frac{dw_1^\alpha}{2w_1^\alpha}\wedge\bigwedge_{l=d+1}^{N}(dw^\alpha_{i_l}\wedge d\overline{w^\alpha_{j_l}}).
\end{aligned}$$

Since $C_{N-d}(\partial\overline{\partial}\log\Phi_\alpha)^{N-d}$ has Poincaré growth on D_∞, we get

$$|\det(K^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]})|<\frac{C}{\prod_{l=d+1}^{N}(|w^\alpha_{i_l}||\log|w^\alpha_{i_l}|)(|w^\alpha_{j_l}||\log|w^\alpha_{j_l}|)}\quad\text{on } V_0\cap U^*_\alpha,$$

where V_0 is a sufficient small neighborhood of the origin point $(0,\cdots,0)$ in U_α and C is a constant depending on V_0. Thus the $|\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap U_\alpha}\theta([1,i_2,\cdots,i_d])\wedge\eta|$ is finite. Moreover, we get that

$$\det(K^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]})=\frac{(-g-1)^{N-d}}{4^{N-d}}\frac{\det(T^{[1,i_2,\cdots,i_d]}_{[1,j_2,\cdots,j_d]})(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|)}{(\prod_{l=d+1}^{N}w^\alpha_{i_l}\overline{w^\alpha_{j_l}})(F^\alpha)^{2(N-d)}(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|)}\ \text{on } \partial T_1(\epsilon)$$

Using the generalized Cauchy integral formula, we obtain that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap\partial T_1(\epsilon)}\frac{dw_1^\alpha}{2w_1^\alpha}\wedge\varsigma^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]}\wedge\eta\\
=&\ \ 2\pi\sqrt{-1}\frac{(-g-1)^{N-d}}{4^{N-d}}\int_{\{w_1=0\}}\xi^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]}(\log|w_2^\alpha|,\cdots,\log|w_N^\alpha|)\frac{C_{N-d}\bigwedge_{l=d+1}^{N}(dw^\alpha_{i_l}\wedge d\overline{w^\alpha_{i_l}})\wedge\eta}{\prod_{l=d+1}^{N}|w^\alpha_{i_l}|^2}\\
=&\ \ 2\pi\sqrt{-1}\frac{(-g-1)^{N-d}}{4^{N-d}}\int_{\{w_1=0\}}\xi^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]}(\log|w_2^\alpha|,\cdots,\log|w_N^\alpha|)\frac{\delta^{[1,i_2,\cdots,i_d]}_{[1,i_2,\cdots,i_d]}\wedge\eta}{\prod_{l=d+1}^{N}|w^\alpha_{i_l}|^4}
\end{aligned}$$

and that if $[1,i_2,\cdots,i_d]\neq[1,j_2,\cdots,j_d]$ then

$$\frac{1}{2}\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap\partial T_1(\epsilon)}\frac{dw_1^\alpha}{2w_1^\alpha}\wedge(\varsigma_{[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]}+\varsigma_{[1,i_2,\cdots,i_d]}^{[1,j_2,\cdots,j_d]})\wedge\eta$$
$$= c_{[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]}2\pi\sqrt{-1}\frac{(-g-1)^{N-d}}{4^{N-d}}\int_{\{w_1=0\}}\xi_{[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]}(\log|w_2^\alpha|,\cdots,\log|w_N^\alpha|)\frac{\delta_{[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]}\wedge\eta}{\prod_{l=d+1}^{N}|w_{i_l}^\alpha w_{j_l}^\alpha|^2}$$

We then have that

$$\frac{C_{N-d}}{(n-d)!}\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap U_\alpha}\partial\log|w_1^\alpha|\wedge(\partial\overline{\partial}\log\Phi_\alpha)^{N-d}\wedge\eta$$
$$= C_{N-d}\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap\partial T_1(\epsilon)}I\wedge\eta+C_{N-d}\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap\partial T_1(\epsilon)}II\wedge\eta$$
$$= \sum_{[1,i_2,\cdots,i_d]}\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap\partial T_1(\epsilon)}\frac{dw_1^\alpha}{2w_1^\alpha}\wedge\varsigma_{[1,i_2,\cdots,i_d]}^{[1,i_2,\cdots,i_d]}\wedge\eta$$
$$+\sum_{[1,i_2,\cdots,i_d],[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]\neq[1,j_2,\cdots,j_d]}\frac{1}{2}\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap\partial T_1(\epsilon)}\frac{dw_1^\alpha}{2w_1^\alpha}\wedge(\varsigma_{[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]}+\varsigma_{[1,i_2,\cdots,i_d]}^{[1,j_2,\cdots,j_d]})\wedge\eta.$$

Finally, we obtain that

$$\lim_{\epsilon\to 0}\frac{\sqrt{-1}}{4\pi}C_{N-d}\int_{\partial D_\infty(\epsilon)\cap U_\alpha}(\partial\overline{\partial}\log\Phi_{g,\Gamma})^{N-d}\wedge\eta\wedge(\partial-\overline{\partial})\log||s_1||_1.$$
$$= \frac{-1}{2\pi}\mathrm{Im}\big(C_{N-d}\lim_{\epsilon\to 0}\int_{\partial D_\infty(\epsilon)\cap U_\alpha}\partial\log|w_1^\alpha|\wedge(\partial\overline{\partial}\log\Phi_\alpha)^{N-d}\wedge\eta\big).$$
$$= (-1)^{N-d+1}(\frac{g+1}{4})^{N-d}(N-d)!$$
$$\times\{\sum_{[1,i_2,\cdots,i_d]}\int_{\{w_1=0\}}\xi_{[1,i_2,\cdots,i_d]}^{[1,i_2,\cdots,i_d]}(\log|w_2^\alpha|,\cdots,\log|w_N^\alpha|)\frac{\delta_{[1,i_2,\cdots,i_d]}^{[1,i_2,\cdots,i_d]}\wedge\eta}{\prod_{l=d+1}^{N}|w_{i_l}^\alpha|^4}$$
$$+\sum_{[1,i_2,\cdots,i_d],[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]\neq[1,j_2,\cdots,j_d]}c_{[j_1,\cdots,j_d]}^{[i_1,\cdots,i_d]}\int_{\{w_1=0\}}\xi_{[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]}(\log|w_2^\alpha|,\cdots,\log|w_N^\alpha|)\frac{\delta_{[1,j_2,\cdots,j_d]}^{[1,i_2,\cdots,i_d]}\wedge\eta}{\prod_{l=d+1}^{N}|w_{i_l}^\alpha w_{j_l}^\alpha|^2}\}.$$

□

We have the following generalization of the theorem 4.7.

Theorem 4.13. *Let $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ be a neat arithmetic subgroup and $\Sigma_{\mathfrak{F}_0}$ a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$ regular with respect to Γ. Let $\overline{\mathcal{A}}_{g,\Gamma}$ be the toroidal compactification of $\mathcal{A}_{g,\Gamma} := \mathfrak{H}_g/\Gamma$ constructed by $\Sigma_{\mathfrak{F}_0}$.*

Assume that the boundary divisor $D_\infty := \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ is simple normal crossing. Let d be an integer with $1 \leq d \leq \dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma} - 1$. Let $D_1, \cdots, D_d$ be d irreducible components of the boundary divisor D_∞. For each D_i, let $||\cdot||_i$ be an arbitrary Hermitian metric on the line bundle $[D_i]$.

There is

$$\begin{aligned} &(K_{\overline{\mathcal{A}}_{g,\Gamma}} + D_\infty)^{\dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma} - d} \cdot D_1 \cdots D_d \\ = &\int_{D_i} \mathrm{Res}_{D_i}\Big(\Big(\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log \Phi_{g,\Gamma}\Big)^{\dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma} - d}\Big) \wedge \Big(\bigwedge_{1\leq j\leq d, j\neq i} c_1([D_j], ||\cdot||_j)\Big) \end{aligned}$$

for any integer $i \in [1, \dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma}]$, where the operator Res_{D_i} is defined in 4.2.1. Moreover, the intersection number

$$(K_{\overline{\mathcal{A}}_{g,\Gamma}} + D_\infty)^{\dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma} - d} \cdot D_1 \cdots D_d = 0$$

if one of the following conditions is satisfied : (i) $g = 2$; (ii) there is a $D_i \in \{D_1, \cdots, D_d\}$ constructed from an edge ρ_i in $\Sigma_{\mathfrak{F}_{\min}}$ for some minimal cusp $\mathfrak{F}_{\min}$ such that $\mathrm{Int}(\rho) \subset C(\mathfrak{F}_{\min})$.

Remark. Let $\rho = \mathbb{R}_{\geq 0}E$ be an edge in $\Sigma_{\mathfrak{F}_0}$ where E is a semi-positive $g \times g$ matrix with rational coefficients. By the lemma 2.15, if $\mathrm{rank}(E) \neq g$(i.e, $\mathrm{Int}(\rho) \subset C(\mathfrak{F}_0)$) then $\mathrm{rank}(E) = 1$.

Proof. Using the proposition 4.2, we can get the integral formula for $i = 1$ immediately by putting $\eta := \bigwedge_{j=2}^{d} c_1([D_j], ||\cdot||_j)$ into the lemmas 4.8, 4.9, 4.10, 4.12. □

The theorem 2.20 and the corollary 2.24 obviously imply the following fact.

Claim * :*Let $D_1, \cdots, D_d$ be $d(\leq \dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma})$ different irreducible components of D_∞. We have that the set $\mathfrak{D} := \bigcap_{i=1}^{d} D_i$ is not empty if and only if there exists a minimal cusp $\mathfrak{F}_{\min}$ and a top-dimensional cone $\sigma_{\max}$ in the polyhedral decomposition $\Sigma_{\mathfrak{F}_{\min}}$ of the convex cone $C(\mathfrak{F}_{\min})$ such that each D_i corresponds one-one to an edge of $\sigma_{\max}$. Moreover, if $\mathfrak{D} \neq \emptyset$ then every local chart U_α in 4.0.1 satisfying $U_\alpha \cap \mathfrak{D} \neq \emptyset$ is constructed by a top-dimensional cone $\sigma_{\max}$ in $\Sigma_{\mathfrak{F}_{\min}}$ for some minimal cusp $\mathfrak{F}_{\min}$.*

For any two different irreducible components D_α, D_β of $D_\infty = \bigcup_i D_i$, we define

$$(D_\alpha \cap D_\beta)_\infty := \bigcup_{i\neq\alpha,\beta} D_\alpha \cap D_\beta \cap D_i \quad\text{and}\quad (D_\alpha \cap D_\beta)^* := (D_\alpha \cap D_\beta) \setminus (D_\alpha \cap D_\beta)_\infty.$$

Let p be an arbitrary positive integer. By the lemma 4.5, we can define a form $\mathrm{Res}_{D_i\cap D_j}(\mathrm{Res}_{D_i}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p))$ on $(D_i\cap D_j)^*$ for any two different irreducible components D_i, D_j of D_∞ with $D_i\cap D_j\neq\emptyset$. Using similar arguments in the proposition 4.2, we get that $\mathrm{Res}_{D_i\cap D_j}(\mathrm{Res}_{D_i}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p))$ is a closed smooth form on $(D_i\cap D_j)^*$ having Poincaré growth on $(D_i\cap D_j)_\infty$.

Lemma 4.14. *Let p be a positive integer. Let D_i, D_j be two different irreducible components of D_∞ such that $D_i\cap D_j\neq\emptyset$. Both* $\mathrm{Res}_{D_i\cap D_j}(\mathrm{Res}_{D_i}((\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Phi_{g,\Gamma})^p))$ *and* $\mathrm{Res}_{D_i\cap D_j}(\mathrm{Res}_{D_j}((\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Phi_{g,\Gamma})^p))$ *are positive closed currents on $D_i\cap D_j$ such that*

$$\mathrm{Res}_{D_i\cap D_j}(\mathrm{Res}_{D_i}((\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)) = \mathrm{Res}_{D_i\cap D_j}(\mathrm{Res}_{D_j}((\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)).$$

Proof. Let $i=1, j=2$ for convenience. There exists a minimal cusp $\mathfrak{F}_{\min}$ and a top-dimensional cone $\sigma_{\max}$ in the polyhedral decomposition $\Sigma_{\mathfrak{F}_{\min}} := \{\sigma_\alpha\}_\alpha$ of the convex cone $C(\mathfrak{F}_{\min})$ such that D_1, D_2 correspond to two different edges $\rho_1 := \mathbb{R}_{\geq 0}E_1, \rho_2 := \mathbb{R}_{\geq 0}E_2$ of $\sigma_{\max}$. Let $(U_\alpha, (w_1^\alpha,\cdots,w_N^\alpha))$ be a local chart as 4.0.1 such that $\{w_i^\alpha = 0\} = U_\alpha\cap D_i$ for $i=1,2$. In this chat,we write the form $\Phi_{g,\Gamma}$ on $U_\alpha^* := U_\alpha\setminus D_\infty$ as

$$\Phi_\alpha = \frac{(\frac{\sqrt{-1}}{2})^N 2^{\frac{g(g-1)}{2}} \bigwedge_{1\leq i\leq N} dw_i^\alpha\wedge d\overline{w_i^\alpha}}{(\prod_{1\leq i\leq N}|w_i^\alpha|^2)(F^\alpha(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|))^{g+1}}.$$

For convenience, let $\mathfrak{F}_{\min}$ be the standard minimal cusp $\mathfrak{F}_0$. Then, E_1 and E_2 are semi-positive $g\times g$ symmetric real matrices. By the lemma 2.15, we have

$$\mathrm{rank}(E_i) = 1 \text{ or } g \quad \text{for } i=1,2.$$

Then, we need to check the following three cases.

- $\mathrm{rank}E_1 = g$ or $\mathrm{rank}E_2 = g$: Let $\mathrm{rank}E_1 = g$. Then we have
$$\mathrm{Res}_{D_1}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p) = 0 \quad\text{and}\quad \mathrm{Res}_{D_1\cap D_2}(\mathrm{Res}_{D_2}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)) = 0 \quad\text{on } U_\alpha^*$$
since $F^\alpha(x_1,x_2,\cdots,x_N) = \mathrm{Constant}\cdot x_1^g +$ terms of lower dgeree of x_1.
- $\mathrm{rank}E_1 = \mathrm{rank}E_2 = 1$ and there is an orthogonal $g\times g$ matrix O such that
$$O^T E_1 O = \mathrm{diag}[\lambda_{1,1},0,\cdots,0](\lambda_{1,1}>0) \text{ and } O^T E_2 O = \mathrm{diag}[\lambda_{2,1},0,\cdots,0](\lambda_{2,1}>0):$$
Then, the homogenous polynomial F^α has the form
$$F^\alpha(x_1,x_2,\cdots,x_N) = S(x_3,\cdots,x_N)\cdot(\lambda_{1,1}x_1+\lambda_{2,1}x_2) + \text{ terms without } x_1 \text{ and } x_2,$$
and so $\mathrm{Res}_{D_1\cap D_2}(\mathrm{Res}_{D_1}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)) = \mathrm{Res}_{D_1\cap D_2}(\mathrm{Res}_{D_2}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)) = 0$ on U_α^*.

- Otherwise, the homogenous polynomial F^α has the form

$$F^\alpha(x_1, x_2, \cdots, x_N) = S(x_3, \cdots, x_N) \cdot x_1 x_2 + \text{ terms without } x_1 \text{ and } x_2,$$

and so $\mathrm{Res}_{D_1\cap D_2}(\mathrm{Res}_{D_1}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p)) = \mathrm{Res}_{D_1\cap D_2}(\mathrm{Res}_{D_2}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^p))$ on U^*_α.

□

We observe that these $\mathrm{Res}_{D_i\cap D_j}(\mathrm{Res}_{D_i}(\partial\overline{\partial}\log\Phi_{g,\Gamma}))$'s$(i \neq j)$ and $\mathrm{Res}_{D_i}(\partial\overline{\partial}\log\Phi_{g,\Gamma}))$'s all have similar type as that of $\partial\overline{\partial}\log\Phi_{g,\Gamma}$.

Theorem 4.15. *Let $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ be a neat arithmetic subgroup and $\Sigma_{\mathfrak{F}_0}$ a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$ regular with respect to Γ. Let $\overline{\mathcal{A}}_{g,\Gamma}$ be the toroidal compactification of $\mathcal{A}_{g,\Gamma} := \mathfrak{H}_g/\Gamma$ constructed by $\Sigma_{\mathfrak{F}_0}$.*

Assume that the boundary divisor $D_\infty := \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ is simple normal crossing. Let d be an integer with $1 \leq d \leq \dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma} - 1$ and let $D_1, \cdots, D_d$ be any d different irreducible components of the boundary divisor D_∞ such that $\bigcap\limits_{k=1}^{d} D_k \neq \emptyset$. We have :

1. *Let $i_1, \cdots, i_d$ be d positive integers. If $d \geq g-1$ and $\dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma} - \sum\limits_{k=1}^{d} i_k \geq 2$(or if $d \geq g$ and $\dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma} - \sum\limits_{k=1}^{d} i_k = 1$) then the intersection number*

$$D_1^{i_1} \cdots D_d^{i_d} \cdot (K_{\overline{\mathcal{A}}_{g,\Gamma}} + D_\infty)^{\dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma} - \sum_{k=1}^{d} i_k} = 0.$$

2. *The divisor $K_{\overline{\mathcal{A}}_{g,\Gamma}} + D_\infty$ is not ample on $\overline{\mathcal{A}}_{g,\Gamma}$.*
3. *Define $D^{(1)} := D_1, D^{(2)} := D^{(1)} \cap D_2, \cdots, D^{(d)} := D^{(d-1)} \cap D_d$. If $d < g-1$ then there is*

$$\begin{aligned}&(K_{\overline{\mathcal{A}}_{g,\Gamma}} + D_\infty)^{\dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma} - d} \cdot D_1 \cdots D_d \\ = &\int_{\bigcap\limits_{k=1}^{d} D_k} \mathrm{Res}_{D^{(d)}}(\mathrm{Res}_{D^{(d-1)}} \cdots (\mathrm{Res}_{D^{(1)}}((\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Phi_{g,\Gamma})^{\dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma} - d})) \cdots)\end{aligned}$$

where the $\mathrm{Res}_{D^{(k)}}(\cdots)$ $k = 1, \cdots, d$ are current on $D^{(k)}$ defined recursively by the lemma 4.5.

Proof. The statement (2) is a direct consequence of the statement (1). For each i, let $||\cdot||_i$ be an arbitrary Hermitian metric on the line bundle $[D_i]$. We have that

$$\begin{aligned}
&\int_{D_1} \mathrm{Res}_{D_1}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}-d}) \wedge (\bigwedge_{l=2}^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}} c_1([D_l],||\cdot||_l)) \\
= &\int_{D_1\cap D_2} \mathrm{Res}_{D_1\cap D_2}(\mathrm{Res}_{D_1}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}-d})) \wedge (\bigwedge_{l=3}^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}} c_1([D_l],||\cdot||_l))
\end{aligned}$$

by the estimates in the lemmas 4.8-4.12. Therefore, we can finish proving the statements (1) and (3) by recursion since all local volume functions have only degree g. $\square$

Example 4.16. Suppose that d different irreducible components $D_1,\cdots .D_d$ of $D_\infty := \overline{\mathcal{A}}_{g,\Gamma}\setminus\mathcal{A}_{g,\Gamma}$ has $\bigcap\limits_{l=1}^{d} D_l \neq \emptyset$. Let $(U_\alpha,(w_1^\alpha,\cdots,w_N^\alpha))(N=\frac{g(g+1)}{2})$ be a local chart as 4.0.1 such that $\{w_i^\alpha=0\}=U_\alpha\cap D_i$ for all $i=1,\cdots,d$. The following is an algorithm processor to produce $\chi_\alpha := \mathrm{Res}_{D^{(d)}}(\cdots(\mathrm{Res}_{D^{(1)}}((\partial\overline{\partial}\log\Phi_{g,\Gamma})^{N-d})))|_{U_\alpha\cap\bigcap\limits_{l=1}^{d} D_l}$.

00: Begin
10: In put the local volume form on $U_\alpha^* := U_\alpha\setminus D_\infty$:

$$\Phi_\alpha = \frac{(\frac{\sqrt{-1}}{2})^N 2^{\frac{g(g-1)}{2}}\mathrm{vol}_\Gamma(\sigma_{\max})^2 \bigwedge\limits_{1\le i\le N} dw_i^\alpha\wedge d\overline{w_i^\alpha}}{(\prod_{1\le i\le N}|w_i^\alpha|^2)(F^\alpha(\log|w_1^\alpha|,\cdots,\log|w_N^\alpha|))^{g+1}},$$

where $F^\alpha\in\mathbb{R}[x_1,\cdots,x_N]$ of degree g is the local volume function.
20: Let $k=0$ and let $S_0(x_1,\cdots,x_N) := F^\alpha(x_1,\cdots,x_N)$.
30: Let $f(x_1,\cdots,x_N) := S_k(x_{k+1},\cdots,x_N)$ and $q=k+1$.
40: Let $k=q$ and let $n := \deg_k f(x_1,\cdots,x_N)$. Write

$$f = S_k(x_{k+1},\cdots,x_N)x_k^n + \text{terms of lower degree of } x_k.$$

50: Let $P=(P_{lm})_{1\le l,m\le N}$ be a $N\times N$ matrix given by $P_{lm}=S_k\frac{\partial^2 S_k}{\partial x_l\partial x_m}-\frac{\partial S_k}{\partial x_l}\frac{\partial S_k}{\partial x_m}$.
60: Let $g_k(x_{k+1},\cdots x_N) := \det(P^{[1,\cdots,d]}_{[1,\cdots,d]})$ (Thus, $\deg S_k\le g-k$ and $\deg g_k\le 2(N-d)(\deg S_k-1)\le 2(N-d)(g-k-1)$; if $\deg S_k=0$ then $g_k\equiv 0$; if $\deg S_k=1$ and $N-d\ge 2$ then $g_k\equiv 0$).
70: If $k<d$ then goto **30**.

80: Output

$$\chi_\alpha = (-1)^{N-d}(\frac{g+1}{4})^{N-d}(N-d)! \times \frac{g_d(\log|w^\alpha_{d+1}|,\cdots,\log|w^\alpha_N|)}{S_d(\log|w^\alpha_{d+1}|,\cdots,\log|w^\alpha_N|)^{2(N-d)}} \frac{\bigwedge_{i=d+1}^{N} dw_i^\alpha \wedge d\overline{w_i^\alpha}}{(\prod_{i=d+1}^{N}|w_i^\alpha|)^2}.$$

90: End.

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