INTRODUCTION TO CYCLIC COVERS IN ALGEBRAIC GEOMETRY

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1. MAIN RESULT

For any integer $n \geq 1$, let $G_n = \langle \sigma \rangle$ be the cyclic group of order $n$ and let $\xi_n \in \mu_n := \{x \in \mathbb{C} | x^n = 1\}$ be a primitive $n$-th root of unity. Then $G_n$ is isomorphic to the multiplicative group $\mu_n$ by $\sigma \mapsto \xi_n$ for any integer $n \geq 1$.

For any non-singular complex variety $Y$ with an invertible sheaf $L$ and an effective divisor $D = \sum_{j=1}^{r} \nu_j D_j$ satisfying $L^n = \mathcal{O}_Y(D)$ for a positive integer $n$, we obtain a cyclic cover $\pi: Z \rightarrow Y$ by taking the $n$-th root out of $D$ as follows: Let

$$\mathcal{F} := \mathcal{O}_Y \oplus L^{-1} \oplus L^{-2} \oplus \cdots \oplus L^{-(n-1)}$$

and let $s$ be a section of $L^n$ defining the divisor $D$. The $\mathcal{F}$ becomes an $\mathcal{O}_Y$-algebra generated by $L^{-1}$ due to the following laws:

- $L^{-a} \otimes L^{-b} = L^{-(a+b)}$;

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• for any positive integer \( m \) expressed as \( m = ln + k, 0 \leq k \leq n - 1 \), one fixes an \( \mathcal{O}_X \)-homomorphism \( L^{-m} \to L^{-k} h \mapsto hs^j \).

Then we get a finite and flat morphism \( p : \text{Spec}(\mathcal{F}) \to Y \) of degree \( n \) ramified at \( D \). The composite morphism \( \pi := p \circ \text{Nor} \) of the morphism \( p \) and the normalization \( \text{Nor} : Z \to \text{Spec}(\mathcal{F}) \) is called the cyclic cover over \( Y \) obtained by taking the \( n \)-th root out of \( D \).

There is a well-known result as follows.

**Theorem 1.1** (see [EV], [K85]). Let \( X \) be a non-singular complex variety, let \( D = \sum_{j=1}^r \nu_jD_j \) be an effective divisor and let \( L \) be an invertible sheaf with \( L^{N_0} = \mathcal{O}_X(D) \) for a positive integer \( N_0 \).

Let \( \pi : Z \to X \) be the cyclic cover obtained by taking the \( N_0 \)-th root out of \( D \) and let \( G \) be the cyclic group \( < \sigma > \) of order \( N_0 \).

We have:

1. \begin{equation}
\pi_*\mathcal{O}_Z = \bigoplus_{k=0}^{N_0-1} L^{(k)-1} \text{ for } L^{(k)} := L^k \otimes \mathcal{O}_X(-\lfloor \frac{kD}{N_0} \rfloor)
\end{equation}

where \( \lfloor \frac{kD}{N_0} \rfloor \) denotes the integral part of the \( \mathbb{Q} \)-divisor \( \frac{kD}{N_0} \). So that the cover \( \pi \) is a finite and flat morphism.

2. \( Z \) is normal, the singularities of \( Z \) are lying over the singularities of \( D_{\text{red}} \).

3. The cyclic group \( G \) acts on \( Z \). One can choose a primitive \( N_0 \)-th root of unit \( \xi \) such that the sheaf \( L^{(k)} \) in 1) is the sheaf of eigenvectors for \( \sigma \) in \( \pi_*\mathcal{O}_Z \) with eigenvalue \( \xi^k \).

4. \( X \) is irreducible if \( L^{(k)} \neq \mathcal{O}_X \) for \( k = 1, \cdots, N_0 - 1 \). In particular this holds true if the common factor of the integers \( N_0, \nu_1, \cdots, \nu_r \) has \( \gcd(N_0, \nu_1, \cdots, \nu_r) = 1 \).

5. Assume that \( D = \sum_{j=1}^r \nu_jD_j \) is an effective normal crossing divisor. Define

\[
D^{(j)} := \{ \lfloor \frac{jD}{N_0} \rfloor_{\text{red}} = \sum_{\lfloor \frac{\nu_j}{N_0} \rfloor \neq 0} D_j 0 \leq i \leq N_0 - 1,
\]

where \( \{ \lfloor \frac{kD}{N_0} \rfloor \} := \frac{kD}{N_0} - \lfloor \frac{kD}{N_0} \rfloor \) is the fractional part of \( \frac{kD}{N_0} \). For any \( p \geq 0 \), there is

\begin{equation}
\pi_*\Omega^p_Z \simeq \bigoplus_{i=0}^{N_0-1} \Omega^p_X(\log D^{(i)}) \otimes L^{-i}
\end{equation}

where \( \Omega^p_Z \) is the reflexive hull of \( \Omega^p_Z \). In particular, \( \Omega^p_X = (\pi_*\Omega^p_Z)^G \).
6. Assume that $X$ is a proper scheme and that $D = \sum_{j=1}^{r} D_j$ is an effective normal crossing reduced divisor. There is

(1.1.3) \[(\pi_* T_Z)^G = T_X(-\log D).\]

2. Preliminary on commutative Algebra

As beginning, we investigate the local structure of cyclic cover. Let $R$ be both a finitely generated $\mathbb{C}$-algebra and an integral normal ring, let $K$ be the fraction field of $R$.

Lemma 2.1. Let $n \geq 1$ be an integer. Let $f$ be an arbitrary nonzero element in $R$. We have:

a) Any irreducible factor of $T^n - f$ in $K[T]$ with coefficient 1 of the first term is of form $T^{n'} - f' \in R[T]$ satisfying $n'|n$ and $f = (f')^{n/n'}$.

b) Let $d | n$ be the maximal positive integer such that $T^d - f$ has a root in $K$, say $g$. Then $T^{n/d} - g$ is irreducible in $K[T]$ and $g \in R$.

c) Assume that $f = uf_1^{\nu_1} \cdots f_m^{\nu_m}$ is an element in $R$ where $f_i$'s are prime elements in $R$ and $\nu_i$'s are positive integers. Let $d$ be an integer. Then $T^d - f$ has a root in $K$ if and only if $d | \gcd(\nu_1, \cdots, \nu_m)$ and there is a unit $u_0 \in R$ satisfying $u_0^d = u$.

Proof. a) Let $\phi(T) = T^n + a_{m-1}T^{n-1} + a_{m-2}T^{n-2} + \cdots + a_0 \in K[T]$ be an irreducible factor of $T^n - f$. Since $R$ is $\mathbb{C}$-algebra, $\phi(T)$ is separable and it has $m$ different roots $\alpha_1, \cdots, \alpha_m$ in an algebraic closure $\overline{K}$. Because every root of $\phi(T)$ is a root of $T^n - f$, all $\alpha_i$'s are integral over $R$, so that the coefficients of $\phi(T)$ are integral over $R$. Since $R$ is integrally closed, we have $\phi(T) \in R[T]$.

Let $y = \alpha_1$ be a root of $g(T)$. We have

$T^n - f = (T - y)(T - \xi y) \cdots (T - \xi^{n-1} y) = (T - y_0)(T - y_1) \cdots (T - y_{n-1})$

where $\xi$ is a primitive $n$-th root of unity. Then $\phi(T)$ is the minimal polynomial $\min(K, y)$ of $y$ in $K[T]$, we write as

$\phi(T) = \prod_{j=1}^{m} (T - y_j)$ with $y_1 = y$.

Thus $a_0 = y^m \xi^l$ for some integer $l$, then $f' := y^m \in R$. Hence $\phi(T)|T^m - y^m$ in $K[T]$, we must have $\phi(T) = T^m - f'$. 
For any $\sigma \in \mu_n = \{x \in \mathbb{C} \mid x^n = 1\}$, we define $\phi^\sigma(T) := \prod_{j=1}^{m}(T - \sigma y_{i_j})$. We have that each

$$\phi^\sigma(T) = \prod_{j=1}^{m}(T - \sigma y_{i_j}) = T^m + \sigma a_{m-1}T^{m-1} + \sigma^2 a_{m-2}T^{m-2} + \cdots + \sigma^m a_0$$

is also an irreducible factor of $T^n - f$ in $K[T]$ of degree $m$, so that $\phi^\sigma(T) \in R[T]$ is the minimal polynomial $\min(K, \sigma y)$ of $\sigma y$ in $K[T]$, and we have either $\phi(T) = \phi^\sigma(T)$ or that $\phi(T)$ has no common factor with $\phi^\sigma(T)$ in $\overline{K}[T]$. We list all different elements in $\{\phi^\sigma(T) \mid \sigma \in \mu_n\}$ as $\phi_1(T) = \phi(T), \phi_2(T), \cdots, \phi_k(T)$. Hence, we have

$$T^n - f = \phi_1(T)\phi_2(T) \cdots \phi_k(T),$$

so that $n = mk$. Therefore, we must have $\phi(T) = T^m - f'$ with $(f')^{n/m} = f$.

b) It is a consequence of (a).

c) If $d|\gcd(\nu_1, \cdots, \nu_m)$ and $u_0^d = u$ for some unit $u_0 \in R$, then $T^d - f \in R[T]$ has a root $u_0\nu_1/d \cdots \nu_m/d \in R$.

Suppose that there is $g \in K$ with $g^d = f$. The $T - g$ is an irreducible factor of $T^d - f$ in $K[T]$, thus $g \in R$ by (a). We write $g_0 := g$. Then we have

$$g_0^d = u_0\nu_1/d \cdots \nu_m/d \in (f_1).$$

We then have $g_0 = f_1g_1$, and get $f_1^dg_1^d = u_0\nu_1/d \cdots \nu_m/d$. Since $f_1$ is a prime element, it is impossible that $d > \nu_1$. So that $\nu_1 - d \geq 0$ and we have $g_1^d = u_0\nu_1/d \cdots \nu_m/d$. If $\nu_1 - d \geq 1$, we have $g_1^d = u_0\nu_1/d \cdots \nu_m/d \in (f_1)$. We deal with $g_1$ similarly like $g_0$, and so on. We finally get that $\nu_1 = dk$ for some integer $k$. By symmetry, we have $d|\nu_i$ for other $\nu_i$’s. Then we get $d|\nu_1, \cdots, \nu_m$ and $(\nu_1/d, \cdots, \nu_m/d)^d = u$. Again by (a), we have $u_0 := u_0\nu_1/d \cdots \nu_m/d \nu_i/d$ is a unit in $R$.

\[ \square \]

**Corollary 2.2.** Let $T^n - f$ be a polynomial in $R[T]$. We have:

a) The $T^n - f$ is irreducible in $K[T]$ if and only if it is irreducible in $R[T]$.

b) If $T^n - f$ is irreducible in $R[T]$ then it is a prime element in $R[T]$.

For a polynomial $T^n - f(n \geq 1)$ with nonzero $f \in R$, let $d|n$ be the maximal positive integer satisfying $T^d - f$ has a root $g \in K$, we have $T^n - f = \prod_{i=-1}^{d-1}(T^{n'} - \zeta_d^i g)$ where $\zeta_d$ is a primitive $d$-th root of unit and each $T^{n'} - \zeta_d^i g$ is a prime element in $R[T]$ for $n' := n/d$; the quotient ring $K[T]/(T^n - f)$ can be regarded as a finite $K$-linear space

$$K[T]/(T^n - f) = \bigoplus_{i=0}^{n-1} Kt^i$$
where $t := T \mod (T^n - f)$ can be regarded as a root of $T^n - f$ in $\overline{K}$, that $t$ is also integral over $R$ and the ring $R[T] / (T^n - f)$ is integral if and only if $T^n - f$ is irreducible, the group $G_d = < \sigma >$ has a natural action on $K[T] / (T^n - f)$ by $\sigma : T \mapsto \zeta_n T$ and $\sigma|_K = \text{id}_K$, the $\mu_n$-invariant part of $K[T] / (T^n - f)$ is $K$.

**Lemma 2.3.** Let $f$ be an nonzero element in the domain $R$. Let $\eta$ be a positive integer and $\eta \not| n$ the maximal positive integer such that $T^\eta - f$ has a root in $K$ and $n' := n/d$. For any $\zeta \in \mu_d := \{ x \in \mathbb{C} \mid x^n = 1 \}$, define a polynomial

$$P_\zeta = \frac{1}{\prod_{\zeta' \in \mu_d} (\zeta' - \zeta)} \frac{T^n - f}{T^{n'} - \zeta}$$

where $g \in K$ is a root of $T^d - f$.

1. In the quotient ring $\frac{K[T]}{(T^n - f)}$, these $P_\zeta$’s are idempotent, i.e. $P_\zeta \cdot P_\zeta' = 0 \neq \zeta' = P_\zeta \cdot P_\zeta$, with $\sum_{\zeta \in \mu_d} P_\zeta = 1$. Moreover, the cyclic group $G_n = < \sigma >$ acts transitively on the set $\{ P_\zeta \}_{\zeta \in \mu_d}$ with $(P_\zeta)^\sigma = P_{\zeta^{-n'}} \forall \zeta \in \mu_d$.

2. Let $K_\zeta$ be the field $\frac{K[T]}{(T^n - \zeta)}$ for each $\zeta \in \mu_d$. Any $\zeta \in \mu_n$ defines an isomorphism $K_\zeta \xrightarrow{\cong} K_{\zeta'}$ of fields by sending $T$ to $\zeta T$. Moreover, there is a natural $G_n$-action on the algebra $\prod_{\zeta \in \mu_n} K_\zeta$.

3. There is a $G_n$-equivariant isomorphism of $K$-algebras

$$P : \prod_{\zeta \in \mu_d} \frac{K[T]}{(T^{n'} - \zeta)} \cong \frac{K[T]}{(T^n - f)} : (\alpha_\zeta) \mapsto \sum_{\zeta} \alpha_\zeta P_\zeta.$$

**Corollary 2.4.** Let $T^n - f$ be a polynomial in $R[T]$ with degree $n \geq 1$. Let $A = R[T] / (T^n - f)$ be a ring and $S := \{ a \in A \mid a$ is not zero divisor $\}$ a multiplicative system in $A$. Then $S^{-1}A = K[T] / (T^n - f)$.

**Proof.** Let $h(T) \in R[T]$ such that it’s image $\overline{h}$ is not a zero-divisor in $A$. Consider the $K$-isomorphism 2.3.1, we get that $P^{-1}(\overline{h}) = (\alpha_\zeta)$ with $\alpha_\zeta := h(T) \mod (T - \zeta g)$ in $K[T] \forall \zeta \in \mu_d$. Thus that $\overline{h}$ is not a zero-divisor in $R$ if and only if for each $\zeta \in \mu_d$ there is a $\eta_\zeta(T) \in K[T]$ such that $\eta_\zeta(T)h(T) = 1 \mod (T^{n'} - \zeta g)$ in $K[T]$. We get $\overline{\eta h} = 1$ in $K[T] / (T^n - f)$, where $\overline{\eta}$ is the projection image of the polynomial $\eta(T) := \sum_{\zeta \in \mu_d} \eta_\zeta P_\zeta$. Thus, we have that any non zero divisor in $A$ is an unit in $K[T] / (T^n - f)$. Hence,

$$S^{-1}A = S^{-1}(\frac{K[T]}{(T^n - f)}) = \frac{K[T]}{(T^n - f)}.$$

Consider the ring homomorphism $R \rightarrow K[T] / (T^n - f)$ for a polynomial $T^n - f \in R[T]$ of degree $n \geq 1$. Let $\overline{R}$ be the integral closure of $R$ in $K[T] / (T^n - f)$.
Lemma 2.5. Let $f$ be an arbitrary nonzero element in $R$. If a polynomial $T^n - f \in R[T]$ with degree $n \geq 1$ is irreducible in $R[T]$ then $\tilde{R}$ is the normalization of the domain $\frac{R[T]}{(T^n - f)}$ in the field extension $\frac{K[T]}{(T^n - f)}$ of $K$.

Therefore, by Corollary 2.4 the $\tilde{R}$ is also the normalization of $\frac{R[T]}{(T^n - f)}$ even though $T^n - f$ is not irreducible since $R$ is a normal integral ring.

Lemma 2.6. Let $g$ be a nontrivial prime element in $R$. Then the principal prime ideal $(g)$ is of height one.

Proof. We claim that any nonzero prime ideal $\mathfrak{p}$ of $R$ has an irreducible element: Suppose the claim is not true. Let $a_0$ be any nonzero element in $\mathfrak{p}$. Then $a = bc$ such that both $b$ and $c$ are not unit. Then one of $b, c$ is in $\mathfrak{p}$, say $a_1$. The $a_1$ is also irreducible and $(a_0) \subsetneq (a_1) \subsetneq \mathfrak{p}$. We do same steps for $a_1$ as dealing with $a_0$ and so on. Thus we have an strictly increasing sequence of ideals

$$(a_0) \subsetneq (a_1) \subsetneq (a_2) \subsetneq \cdots \subsetneq \mathfrak{p},$$

which contradicts that $R$ is a Noetherian ring.

Let $\mathfrak{p}$ be any nontrivial prime ideal of $R$ such that $0 \subsetneq \mathfrak{p} \subset (g)$. Let $a \in \mathfrak{p}$ be an irreducible element. We have $a = gc$ due to $a \in (g)$, but $c$ is a unit since $a$ is irreducible. Hence $\mathfrak{p} = (g)$.

For any height one prime idea $\mathfrak{p}$ of $R$, the local ring $R_\mathfrak{p}$ is a DVR, thus we can define $\text{div}(x)$ for any $x \in K$.

Theorem 2.7. Assume a finitely generated $\mathbb{C}$-algebra $R$ is integral and normal. Let $K$ be the fraction field of $R$. Let $f = u f_1^{\nu_1} \cdots f_l^{\nu_l} (\forall \nu_i \geq 1)$ be an element in $R$ such that $f_i$’s are different prime elements in $R$ and $u$ is a unit in $R$. Let $n \geq 1$ be an integer and $d | n$ the maximal positive integer such that $T^d - f$ has a root in $K$, say $g$, and denote by $n' := n/d$.

1. With respect to the injective ring homomorphism $R \to \frac{K[T]}{(T^n - f)}$, the integral closure of $R$ is

$$\tilde{R} = \bigoplus_{i=0}^{n-1} t^i f_1^{[-\frac{n'}{d}]} \cdots f_l^{[-\frac{n'}{d}]} R$$

where $t := T \mod (T^n - f)$. Moreover, the cyclic group $\mathbb{Z}_n = \langle \sigma \rangle$ has a natural action on $\tilde{R}$ by defining $\sigma : t \mapsto \xi_n t, \sigma|_R = \text{id}_R$, and the $G_n$-invariant part of $\tilde{R}$ is

$$\tilde{R}^{G_n} = R.$$

2. Let $\tilde{R}_\zeta$ be normalization of an integral ring $R_\zeta := \frac{R[T]}{(T^n - \zeta g)}$ in $\frac{K[T]}{(T^n - \zeta g)}$ for each $\zeta \in \mu_d$. Then

$$\tilde{R}_\zeta = \bigoplus_{i=0}^{n'-1} t^i f_1^{[-\frac{n'}{d}]} \cdots f_l^{[-\frac{n'}{d}]} R$$
where \( t_{\zeta} := T \mod (T^n - \zeta g) \). Moreover, \( R \to \bar{R}_{\zeta} \) is Galois, with Galois group \( \mathbb{Z}_n' \).

3. The group \( \mathbb{Z}_n \) has a natural action on the product \( R \)-algebra \( \prod_{\zeta \in \mu_d} \bar{R}_{\zeta} \). The group permutes the factors of the decomposition, and there is a \( G_n \)-equivariant isomorphism of \( R \)-algebras

\[
\prod_{\zeta \in \mu_d} \bar{R}_{\zeta} \longrightarrow \bar{R}, \quad (\alpha_{\zeta})_{\zeta} \mapsto \sum_{\zeta} \alpha_{\zeta} P_{\zeta}.
\]

**Proof.** For any \( x \in K \), it is easy to verify the following conditions are equivalent:

i. \( x^n f_i \in R \),

ii. \( n \text{div}(x) + \sum_{j=1}^l i\nu_j \text{div}(f_j) \geq 0 \) (we use \( \text{div}(u) = 0 \)).

iii. \( \text{div}(x f_1^{\frac{\nu_1}{n}} \cdots f_l^{\frac{\nu_l}{n}}) + \sum_{j=1}^l \left( \frac{\nu_j}{n} - \frac{[\nu_j]}{n} \right) [f_j] \geq 0 \) (we use that \( (f_i)'s \) are prime ideal of height one and so that \( \text{div}(f_j) \) is the Weil divisor \( [f_j] := \text{Zero}(f_j) \)),

iv. \( \text{div}(x f_1^{\frac{\nu_1}{n}} \cdots f_l^{\frac{\nu_l}{n}}) \geq 0 \) (we use that \( \text{div}(x f_1^{\frac{\nu_1}{n}} \cdots f_l^{\frac{\nu_l}{n}}) \) has integral coefficients and \( f_i's \) are prime elements),

v. \( x f_1^{\frac{\nu_1}{n}} \cdots f_l^{\frac{\nu_l}{n}} \in R \).

1. Thus there is

\[
\{ x \in K; x^n f_i \in R \} = f_1^{-\frac{\nu_1}{n}} \cdots f_l^{-\frac{\nu_l}{n}} R.
\]

We now show

\[
\bar{R} = \bigoplus_{i=0}^{n-1} \{ x \in K; x^n f_i \in R \} t^i.
\]

Let \( \alpha = \sum_{i=0}^{n-1} x_i t^i \) \((x_i \in K)\) be an element in \( \frac{K[T]}{(T^n - f)} \). We claim that \( \alpha \in \bar{R} \) if and only if \( x_i t^i \in \bar{R} \) for every \( i \). Suppose \( \alpha \in \bar{R} \). We have \( \alpha^\zeta = \sum_{i=0}^{n-1} \zeta^i x_i t^i \in \bar{R} \), for every \( \zeta \in \mu_n \). Let \( \mu_n = \{ \zeta_1, \zeta_2, \ldots, \zeta_n \} \). The Vandermonde determinant

\[
\begin{vmatrix}
\zeta_1^0 & \zeta_1^1 & \cdots & \zeta_1^{n-1} \\
\zeta_2^0 & \zeta_2^1 & \cdots & \zeta_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_n^0 & \zeta_n^1 & \cdots & \zeta_n^{n-1}
\end{vmatrix} = \prod_{1 \leq i < j \leq n} (\zeta_j - \zeta_i)
\]

is a unit in \( R \), so that each \( x_i t^i \) is a combination of the \( \alpha^\zeta \)'s with coefficients in \( R \) by Cramer’s rule. Therefore \( x_i t^i \in \bar{R} \). The converse is obvious.
Let $x \in K$ and $0 \leq i < n$. We claim that $xt^i \in \bar{R}$ if and only if $x^n f^i \in R$. Indeed, we have that $xt^i \in \bar{R}$ if and only if $y = (xt^i)^n \in \bar{R}$. Due to

$$y = (xt^i)^n = x^n(t^n)^i = x^n f^i \in K,$$

We have that $y \in \bar{R}$ and only if $y \in K \cap \bar{R} = R$ since $R$ is integrally closed in $K$.

2. By the statement (1), the integral closure of $R$ in $\frac{K[T]}{(T^n - \zeta g)}$ is the integrally closed domain $\tilde{R}_\zeta = \bigoplus_{j=0}^{n' - 1} \{x \in K; x^n(\zeta g)^j \in R\} t^j_\zeta$. Since $R$ is integrally closed in $K$, $x^n(\zeta g)^j \in R$ if and only if $(x^n(\zeta g)^j)^d \in R$. That is $x^n f^j \in R$. Therefore

$$\tilde{R}_\zeta = \bigoplus_{j=0}^{n' - 1} \{x \in K; x^n f^j \in R\} t^j_\zeta.$$

3. In the ring $\frac{K[T]}{(T^n - \zeta g)}$, we have that $f(\sum \zeta \alpha_i P_\zeta) = \sum \zeta f(\alpha_i)P_\zeta$ for any polynomial $f \in R[T]$. Hence $\sum \zeta \alpha_i P_\zeta$ is integral over $R$ if and only if each $\alpha_i \in \frac{K[T]}{(T^n - \zeta g)}$ is integral over $\frac{R[T]}{(T^n - \zeta g)}$. The product decomposition of Lemma 2.3 induces an isomorphism of $R$-algebras $\prod_{\zeta \in \mu_d} \tilde{R}_\zeta \cong \bar{R}$.

3. **Proof of Theorem 1.1**

Now we investigate the local structure of cyclic cover.

**Lemma 3.1.** Let $U := \text{Spec} R$ be an irreducible complex normal affine variety of dimension $m$ and $f = u f_1^{\nu_1} \cdots f_e^{\nu_e}$ ($\forall \nu_i \geq 0$) an element in $R$ such that $f_i$’s are prime elements in $R$ and $u$ is a unit in $R$. Let $D = \sum_{i=1}^e \nu_i D_i$ is zero locus of $f$, i.e., $D_i = \text{Zero}(f_i) \forall i$.

Denote by $U[\sqrt{n}] := \{(p, y) \in U \times \mathbb{A}^1 \mid y^n = f(p)\}$ for an integer $n \geq 2$.

- **a)** Let $p_0$ be a smooth closed point of $U$. Then $U[\sqrt{n}]$ is singular at a point $(p_0, *)$ if and only if $p_0 \in \text{Sing}(D)$. Moreover, $U[\sqrt{n}]$ is smooth if and only if $U$ is smooth and $D$ is reduced with only one smooth component.

- **b)** Assume that $U$ is smooth. If $D = \nu_1 D_1$ and $f = u f_1^{\nu_1}$ ($\nu_1 \geq 1$) then the normalization $\bar{U}[\sqrt{n}]$ of $U[\sqrt{n}]$ has only singularities over $\text{Sing}(D_1)$.

**Proof.** The proof of (a) is directly follows from Jacobi’s criterion: Let $(Z_1, \cdots, Z_m)$ be an regular parameter near an open neighborhood $V$ of $p_0$. Let $F(Z_0, Z_1, \cdots, Z_m) = Z_0^n - f(Z_1, \cdots, Z_m)$.
The point \( q := (p_0, y_0) \) in \( U[\sqrt[\nu]{f}] \) is singular if and only if the \( F \) satisfies

\[
\frac{\partial F}{\partial Z_0}(q) = \frac{\partial F}{\partial Z_1}(q) = \cdots = \frac{\partial F}{\partial Z_m}(q) = 0.
\]

Thus we prove the (a).

Now \( R = A(U) \) is an regular algebra. The problem (b) is now reduced to show that the \( \overline{U}\left[\sqrt[\nu]{f}\right] \) is smooth if the \( D_1 \) is a smooth divisor. In case of \( \nu = 0,1 \) the (b) is obvious by (a.) With lost of generality, we assume that \( \gcd(n, \nu) = 1 \). Let \( R \) be the ring corresponding to the \( U \). By Theorem 2.7, We get \( \overline{U}\left[\sqrt[\nu]{f}\right] = \text{Spec}(\overline{R}) \) such that

\[
\overline{R} = \bigoplus_{i=0}^{n-1} t^i f_1^{-\left[\frac{\nu_1}{n}\right]} R
\]

where \( t := T \mod (T^n - uf_1^{\nu_1}) \).

Since \( \gcd(n, \nu) = 1 \), we have an ring-isomorphism \( \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{\nu_1} \mathbb{Z} \), and so we can get \( a \in \{0, \ldots, n-1\} \) with \( av_1 = 1 + ln \) for some \( l \in \mathbb{Z} \). Denote by \( g = t^a f_1^{-\left[\frac{\nu_1}{n}\right]} \). We get that

\[
g^n = u^a f_1 \quad \text{and} \quad t = u^{-l} g^{\nu_1}.
\]

Thus

\[
\bigoplus_{i=0}^{n-1} g^i R \cong \frac{R[W]}{(W^n - u^a f_1)}
\]

is an \( R \)-subalgebra of \( \overline{R} \). Moreover, we have a tower of inclusions of \( R \)-algebras

\[
R \subset \frac{R[T]}{(T^n - uf_1^{\nu_1})} \subset \bigoplus_{i=0}^{n-1} g^i R \subset \overline{R}.
\]

such that the integral domains \( \frac{R[T]}{(T^n - uf_1^{\nu_1})} \), \( \bigoplus_{i=0}^{n-1} g^i R \) and \( \overline{R} \) all have same quotient field.

Since the algebra \( \frac{R[W]}{(W^n - u^a f_1)} \) is regular by (a) and the algebra \( \overline{R} \) is normal, we have to get

\[
\overline{R} \cong \frac{R[W]}{(W^n - u^a f_1)}.
\]

Therefore the \( \overline{U}\left[\sqrt[\nu]{f}\right] \) is smooth. \( \square \)

We only need to prove the statements 5) and 6) of Theorem 1.1.
Let \( d = \dim \mathbb{C} X \) and let \( \xi \) be a primitive root of \( x^{N_0} - 1 \). Since \( \pi \) is étale outside of \( D \), there is

\[
\pi_! \Omega^p_2|_{X - D} \cong (\Omega^p_X \otimes \pi_* \mathcal{O}_Z)|_{X - D}
\]

and the isomorphism 1.1.2 holds over \( X - D \). On the other hand, the \( \pi_* \Omega^p_2 \) is a reflexive sheaf on \( X \) by the following lemma 3.2, thus it is sufficient to check the assertion 1.1.2 only in codimension one on \( X \).

**Lemma 3.2 ([H80]).** Let \( \mathcal{F} \) be a coherent sheaf on a normal integral scheme \( X \). The following two conditions are equivalent:

i. \( \mathcal{F} \) is reflexive;

ii. \( \mathcal{F} \) is torsion-free, and is normal, i.e., for every open set \( U \subset X \) and every closed subset \( B \subset U \) of codimension 2, the restriction map \( \mathcal{F}(U) \to \mathcal{F}(U - Y) \) bijective.

Without lost of generality, we may assume that the \( D_{\text{red}} \) is smooth, moreover there is a local regular parameter \( (x_1, \cdots, x_d) \) of \( X \) such that

\[
X = \text{Spec}(\mathbb{C}[x_1, \cdots, x_d]), \quad D = (x_i^0).
\]

Then \( X' = \text{Spec}(\mathbb{C}[x_1, \cdots, x_d, t]/(T^{N_0} - ux_i^0)) \) where \( u \) is a unit in the domain \( \mathbb{C}[x_1, \cdots, x_d] \) and \( Z \) is normalization of \( X' \).

Let \( R := \mathbb{C}[x_1, x_2, \cdots, x_d] \) a normal integral ring. Let \( A := \frac{R[T]}{(T^{N_0} - ux^0_i)} \) and \( B \) the normalization of \( A \). As \( \Omega^1_B \) is generated as a \( B \)-module, by the image \( d : B \to \Omega^1_B \).

We know as that

\[
\wedge^p \Omega^1_B \equiv \{ b_0 db_1 \wedge db_2 \cdots \wedge db_p \mid b_i \in B i = 1, \cdots, N_0 - 1 \}.
\]

Let \( t := T \mod (T^{N_0} - ux^0_i) \). By Proposition 2.7 we have that \( B = \bigoplus_{i=0}^{N_0-1} q_i R \) as an \( R \)-module and that the group \( G = < \sigma > \) has a natural action on \( \wedge^p \Omega^1_B \), where \( q_i := t^i x_1^{-\frac{a^i}{N_0}} \), \( 0 \leq i \leq N_0 - 1 \). Since

\[
q_i q_j = q_i := i + j x_1^{-\frac{a(i+j)}{N_0}} x_1^{-\frac{a(i+j)}{N_0}} x_1^{-\frac{a(j)}{N_0}} \forall i, j \geq 0,
\]

we have that \( q_i q_j \in q_{i+j} R \) if \( i + j < N_0 \) and that \( q_i q_j \in q_{i+j-n} R \) if \( i + j \geq N_0 \). Let \( \xi \) be the primitive \( N_0 \)-th root of unit as we choose in the statement 2). We then have an decomposition of \( \wedge^p \Omega^1_B = \bigoplus_{i=1}^{N_0-1} Q_i \) such that each \( Q_i \) is the \( R \)-submodule of eigenvectors for \( \sigma \) in \( \wedge^p \Omega^1_B \) with eigenvalue \( \xi^i \). We now find out these \( Q_i \)'s. We have that

\[
\frac{dt}{t} = \frac{a dx_1}{N_0 x_1},
\]

\[
dq_i = \left\{ \frac{ai}{N_0} \right\} q_i \frac{dx_1}{x_1}.
\]
On the other hand, each $b \in B$ can be written uniquely as $b = \sum_{i=1}^{N_0-1} q_i r_i$ with each $r_i \in R$ and so $db = \sum_{i=1}^{N_0-1} (q_i dr_i + r_i dq_i)$. For each $0 \leq i \leq N_0 - 1$, the elements

$$t^i x_1^{-N_0/n_0} dy_{e_1} \wedge \cdots \wedge dy_{e_p}, \quad e_1 < e_2 < \cdots < e_p < d \quad \text{for} \quad e_1 \neq 1,$$

becomes an $R$-basic of the module $Q_i$. In particular, $G$-invariant submodule of $\wedge^p \Omega^1_B$ has $Q_0 = \wedge^p \Omega^1_B$. Therefore, we get the formula 1.1.2 and $\Omega^1_Z = (\pi_* \Omega^1_B)^G$.

Now we are going to prove the statement 6). For the normal varieties $X$ and $Z$, by definition, the tangent sheaf

$$\mathcal{T}_Z := \mathcal{H}om_{\mathcal{O}_Z}(\Omega^1_Z, \mathcal{O}_Z)$$

is a reflexive sheaf which is composed of derivations from $\mathcal{O}_Z$ to $\mathcal{O}_Z$, and $\mathcal{T}_X(-\log D^{(1)})$ is the subsheaf of $\mathcal{T}_X$ of consisting of derivations which send the ideal sheaf $\mathcal{I}$ of $D^{(1)}$ into itself. Since $X$ is smooth, we get (see [D72]) that

$$\mathcal{T}_X(-\log D^{(1)}) = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X(\log D^{(1)}), \mathcal{O}_X).$$

Suppose $X$ is proper and $D$ is reduced. Then $D = D^{(i)}$ for all $1 \leq i \leq N_0 - 1$ and by 1) $Z$ is also a proper scheme. Thus the dualizing sheaf $\omega_Z = \pi^* \omega_X$ has a natural $\mathcal{O}_Z$-module (see Proposition 5.68. [KM]). Because $\pi$ is an affine morphism, we can define an equivalent functor $\sim$ of categories between the category of quasi-coherent $\pi_* \mathcal{O}_Z$-modules and the category of quasi-coherent $\mathcal{O}_Z$-module (see Ex. II. 5.17(e) [Hart]). Since $\pi_* \mathcal{O}_Z$ is locally free, we have

$$\pi^! \omega_X := \mathcal{H}om_{\mathcal{O}_X}(\pi_* \mathcal{O}_Z, \omega_X)^\sim = (\mathcal{H}om_{\mathcal{O}_X}(\pi_* \mathcal{O}_Z, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \omega_X)^\sim = \pi^! \mathcal{O}_X \otimes_{\mathcal{O}_Z} \pi^* \omega_X.$$

Thus the relative dualizing sheaf $\omega_{Z/X} = \pi^! \mathcal{O}_X$.

We claim that

$$\omega_{Z/X} = \pi^* L^{(N_0-1)} = \pi^*(L^{N_0-1}).$$

We can take a local regular parameter $(x_1, \cdots, x_d)$ of $X$ such that

$$X = \text{Spec}(\mathbb{C}[x_1, \cdots, x_d]), \quad D = (x_1 \cdots x_k).$$

Thus we get that

$$X' = \text{Spec}(\mathbb{C}[x_1, \cdots, x_d, t]/(T^{N_0} - x_1 \cdots x_k))$$
and $Z$ is the normalization of $X'$. Let
\[ R := \mathbb{C}[x_1, x_2, \ldots, x_d] \quad \text{and} \quad A := \frac{R[T]}{(T^{N_0} - x_1 \cdots x_k)}, \]
and let $B$ be the normalization of $A$. By Proposition 2.7 we have that
\[ B = \bigoplus_{i=0}^{N_0-1} p_i R \]
as an $R$-module where $\{p_i := t^i\}_{i=0 \ldots N_0-1}$ is an $R$-basis of $B$. Let
\[ \eta_i \in B^* := \text{Hom}_R(B, R) \]
be dual of $q_i$, i.e., $\eta_i(p_i) = 1$, $\eta_i(p_j) = 0 \forall j \neq i$. The $B$-module structure of $\text{Hom}_R(B, R)$ is clear: $b \cdot \phi := \phi_b$ where $\phi_b \in \text{Hom}_R(B, R)$ is given by $\phi_b(b_1) := \phi(bb_1) \forall b_1 \in B$. We observe that
\[ p_ip_j = q_{i+j} \text{ if } i + j < N_0 \quad \text{and} \quad p_ip_j \in p_{i+j-n}R \text{ if } i + j \geq N_0. \]
So that
\[ p_i \cdot \eta_{N_0-1-i} = \eta_{N_0-1-i} \]
for all $i = 0, \ldots, N_0-1$. As an $B$-module, the $\text{Hom}_R(B, R)$ must be a module of rank one generated by $\eta_{N_0-1}$. Hence $\pi^*\mathcal{O}_X = \pi^*(L^{(N_0-1)})$.

Let $\text{Sing}(D)$ be singularities of $D$ and $X_0 := X \setminus \text{Sing}(D)$. We know codimension of $\text{Sing}(D)$ in $X$ is more than two and $\pi^{-1}(X_0)$ is smooth open sub-scheme of $Z$. Using the duality theory for finite morphism(cf. Theorem 5.67 [KM]), on $X_0$ we get
\[ \pi_* \mathcal{T}_Z = \pi_* \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_Z, \mathcal{O}_Z) \]
\[ = \pi_* \mathcal{H}om_{\mathcal{O}_X}(\omega_{Z/X} \otimes_{\mathcal{O}_X} \Omega^1_Z, \omega_{Z/X}) \]
\[ = \pi_* \mathcal{H}om_{\mathcal{O}_X}(\pi^*(L^{(N_0-1)}) \otimes_{\mathcal{O}_X} \Omega^1_Z, \pi^*\mathcal{O}_X) \]
\[ = \mathcal{H}om_{\mathcal{O}_X}(L^{(N_0-1)} \otimes_{\mathcal{O}_X} \pi_* \Omega^1_Z, \mathcal{O}_X) \]
\[ = \mathcal{H}om_{\mathcal{O}_X}(L^{(N_0-1)} \otimes_{\mathcal{O}_X} (\bigoplus_{i=0}^{N_0-1} \Omega^1_X((\log D)^{i}) \otimes L^{-i}), \mathcal{O}_X) \]
\[ = (L^{-(N_0-1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X) \oplus \bigoplus_{i=1}^{N_0-1} (L^{(i-N_0+1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X(-\log D)). \]

Since both $\pi_* \mathcal{T}_Z$ and $L^{-(N_0-1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X \oplus \bigoplus_{i=1}^{N_0-1} (L^{(i-N_0+1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X(-\log D))$ are reflexive sheaves on $X$, there is
\[ \pi_* \mathcal{T}_Z = L^{-(N_0-1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X \oplus \bigoplus_{i=1}^{N_0-1} (L^{(i-N_0+1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X(-\log D)) \text{ on } X. \]
Hence $(\pi_* \mathcal{T}_Z)^G = \mathcal{T}_X(-\log D)$ on $X$. 
REFERENCES


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