INTRODUCTION TO CYCLIC COVERS IN ALGEBRAIC GEOMETRY

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Contents

1. Main result

For any integer $n \geq 1$, let $G_n =< \sigma >$ be the cyclic group of order n and let $\xi_n \in \mu_n := \{x \in \mathbb{C} \,|\, x^n = 1\}$ be a primitive n-th root of unity. Then G_n is isomorphic to the multiplicative group μ_n by $\sigma \mapsto \xi_n$ for any integer $n \geq 1$.

For any non-singular complex variety Y with an invertible sheaf L and an effective divisor $D = \sum_{j=1}^{r} \nu_j D_j$ satisfying $L^n = \mathcal{O}_Y(D)$ for a positive integer n, we obtain a cyclic cover $\pi : Z \to Y$ by taking the n-th root out of D as follows : Let

$$\mathcal{F} := \mathcal{O}_Y \oplus L^{-1} \oplus L^{-2} \oplus \cdots \oplus L^{-(n-1)}$$

and let s be a section of L^n defining the divisor D. The $\mathcal{F}$ becomes an $\mathcal{O}_Y$-algebra generated by L^{-1} due to the following laws :

- $L^{-a} \otimes L^{-b} = L^{-(a+b)}$;

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- for any positive integer m expressed as $m = ln + k, 0 \leq k \leq n-1$, one fixes an $\mathcal{O}_X$-homomorphism $L^{-m} \longrightarrow L^{-k}\, h \mapsto hs^l$.

Then we get a finite and flat morphism $p : \mathrm{Spec}(\mathcal{F}) \to Y$ of degree n ramified at D. The composite morphism $\pi := p \circ \mathrm{Nor}$ of the morphism p and the normalization $\mathrm{Nor} : Z \to \mathrm{Spec}(\mathcal{F})$ is called the **cyclic cover over Y obtained by taking the n-th root out of D**.

There is a well-known result as follows.

Theorem 1.1 (see [EV],[K85]). *Let X be a non-singular complex variety, let $D = \sum_{j=1}^{r} \nu_j D_j$ be an effective divisor and let L be an invertible sheaf with $L^{N_0} = \mathcal{O}_X(D)$ for a positive integer N_0.*

Let $\pi : Z \to X$ be the cyclic cover obtained by taking the N_0-th root out of D and let G be the cyclic group $< \sigma >$ of order N_0. We have :

1.
$$\pi_* \mathcal{O}_Z = \bigoplus_{k=0}^{N_0-1} L^{(k)^{-1}} \text{ for } L^{(k)} := L^k \otimes \mathcal{O}_X(-[\frac{kD}{N_0}]) \tag{1.1.1}$$

   where $[\frac{kD}{N_0}]$ denotes the integral part of the $\mathbb{Q}$-divisor $\frac{kD}{N_0}$. So that the cover π is a finite and flat morphism.
2. *Z is normal, the singularities of Z are lying over the singularities of D_{red}.*
3. *The cyclic group G acts on Z. One can choose a primitive N_0-th root of unit ξ such that the sheaf $L^{(k)}$ in 1) is the sheaf of eigenvectors for σ in $\pi_* \mathcal{O}_Z$ with eigenvalue ξ^k.*
4. *X is irreducible if $L^{(k)} \neq \mathcal{O}_X$ for $k = 1, \cdots, N_0 - 1$. In particular this holds true if the common factor of the integers $N_0, \nu_1, \cdots, \nu_r$ has $\gcd(N_0, \nu_1, \cdots, \nu_r) = 1$.*
5. *Assume that $D = \sum_{j=1}^{r} \nu_j D_j$ is an effective normal crossing divisor. Define*

$$D^{(j)} := \{\frac{jD}{N_0}\}_{\mathrm{red}} = \sum_{\{\frac{i\nu_j}{N_0}\} \neq 0} D_j \;\; 0 \leq i \leq N_0 - 1,$$

   where $\{\frac{kD}{N_0}\} := \frac{kD}{N_0} - [\frac{kD}{N_0}]$ is the fractional part of $\frac{kD}{N_0}$. For any $p \geq 0$, there is

$$\pi_* \widehat{\Omega_Z^p} \cong \bigoplus_{i=0}^{N_0-1} \Omega_X^p(\log D^{(i)}) \otimes \mathcal{L}^{-(i)} \tag{1.1.2}$$

   where $\widehat{\Omega_Z^p}$ is the reflexive hull of Ω_Z^p. In particular, $\Omega_X^p = (\pi_* \widehat{\Omega_Z^p})^G$.

6. *Assume that X is a proper scheme and that $D = \sum_{j=1}^{r} D_j$ is an effective normal crossing reduced divisor. There is*

$$(\pi_* \mathcal{T}_Z)^G = \mathcal{T}_X(-\log D). \tag{1.1.3}$$

2. Preliminary on commutative Algebra

As begining, we investigate the local structure of cyclic cover.

Let R be both a finitely generated $\mathbb{C}$-algebra and an integral normal ring, let K be the fraction field of R.

Lemma 2.1. *Let $n \geq 1$ be an integer. Let f be an arbitrary nonzero element in R. We have :*

a) Any irreducible factor of $T^n - f$ in $K[T]$ with coefficient 1 of the first term is of form $T^{n'} - f' \in R[T]$ satisfying $n'|n$ and $f = (f')^{n/n'}$.

b) Let $d|n$ be the maximal positive integer such that $T^d - f$ has a root K, say g. Then $T^{n/d} - g$ is irreducible in $K[T]$ and $g \in R$.

c) Assume that $f = uf_1^{\nu_1} \cdots f_m^{\nu_m}$ is an element in R where f_i's are prime elements in R and ν_i's are positive integers. Let d be an integer. Then $T^d - f$ has a root in K if and only if $d|\gcd(\nu_1, \cdots, \nu_m)$ and there is a unit $u_0 \in R$ satisfying $u_0^d = u$.

Proof. a) Let $\phi(T) = T^m + a_{m-1}T^{m-1} + a_{m-2}T^{m-2} + \cdots + a_0 \in K[T]$ be an irreducible factor of $T^n - f$. Since R is $\mathbb{C}$-algebra, $\phi(T)$ is separable and it has m different roots $\alpha_1, \cdots, \alpha_m$ in an algebraic closure $\overline{K}$. Because every root of $\phi(T)$ is a root of $T^n - f$, all α_i's are integral over R, so that the coefficients of $\phi(T)$ are integral over R. Since R is integrally closed, we have $\phi(T) \in R[T]$.

Let $y = \alpha_1$ be a root of $g(T)$. We have

$$T^n - f = (T - y)(T - \xi y) \cdots (T - \xi^{n-1} y) = (T - y_0)(T - y_1) \cdots (T - y_{n-1})$$

where ξ is a primitive n-th root of unity. Then $\phi(T)$ is the minimal polynomial $\min(K, y)$ of y in $K[T]$, we write as

$$\phi(T) = \prod_{j=1}^{m} (T - y_{i_j}) \text{ with } y_{i_1} = y.$$

Thus $a_0 = y^m \xi^l$ for some integer l, then $f' := y^m \in R$. Hence $\phi(T) | T^m - y^m$ in $K[T]$, we must have $\phi(T) = T^m - f'$.

For any $\sigma \in \mu_n = \{x \in \mathbb{C} \mid x^n = 1\}$, we define $\phi^\sigma(T) := \prod_{j=1}^{m}(T - \sigma y_{i_j})$. We have that each

$$\phi^\sigma(T) = \prod_{j=1}^{m}(T - \sigma y_{i_j}) = T^m + \sigma a_{m-1}T^{m-1} + \sigma^2 a_{m-2}T^{m-2} + \cdots + \sigma^m a_0$$

is also an irreducible factor of $T^n - f$ in $K[T]$ of degree m, so that $\phi^\sigma(T) \in R[T]$ is the minimal polynomial $\min(K, \sigma y)$ of σy in $K[T]$, and we have either $\phi(T) = \phi^\sigma(T)$ or that $\phi(T)$ has no common factor with $\phi^\sigma(T)$ in $\overline{K}[T]$. We list all different elements in $\{\phi^\sigma(T) \mid \sigma \in \mu_n\}$ as $\phi_1(T) = \phi(T), \phi_2(T), \cdots, \phi_k(T)$. Hence, we have

$$T^n - f = \phi_1(T)\phi_2(T)\cdots\phi_k(T),$$

so that $n = mk$. Therefore, we must have $\phi(T) = T^m - f'$ with $(f')^{n/m} = f$.

b) It is a consequence of (a).

c) If $d|\gcd(\nu_1, \cdots, \nu_m)$ and $u_0^d = u$ for some unit $u_0 \in R$, then $T^d - f \in R[T]$ has a root $u_0 f_1^{\nu_1/d} \cdots f_m^{\nu_m/d} \in R$.

Suppose that there is $g \in K$ with $g^d = f$. The $T - g$ is an irreducible factor of $T^d - f$ in $K[T]$, thus $g \in R$ by (a). We write $g_0 := g$. Then we have

$$g_0^d = u f_1^{\nu_1} \cdots f_n^{\nu_m} b_0^d \in (f_1).$$

We then have $g_0 = f_1 g_1$, and get $f_1^d g_1^d = u f_1^{\nu_1} \cdots f_n^{\nu_m}$. Since f_1 is a prime element, it is impossible that $d > \nu_1$. So that $\nu_1 - d \geq 0$ and we have $g_1^d = u f_1^{\nu_1 - d} \cdots f_n^{\nu_m}$. If $\nu_1 - d \geq 1$, we have $g_1^d = u f_1^{\nu_1 - d} \cdots f_n^{\nu_m} \in (f_1)$. We deal with g_1 similarly like g_0, and so on. We finally get that $\nu_1 = dk$ for some integer k. By symmetry, we have $d|\nu_i$ for other ν_i's. Then $d|\gcd(n, \nu_1, \cdots, \nu_m)$ and $(\frac{g}{f_1^{\nu_1/d} \cdots f_n^{\nu_m/d}})^d = u$. Again by (a), we have $u_0 := \frac{g}{f_1^{\nu_1/d} \cdots f_n^{\nu_m/d}}$ is a unit in R.

□

Corollary 2.2. *Let $T^n - f$ be a polynomial in $R[T]$. We have :*

a) The $T^n - f$ is irreducible in $K[T]$ if and only if it is irreducible in $R[T]$.

b) If $T^n - f$ is irreducible in $R[T]$ then it is a prime element in $R[T]$.

For a polynomial $T^n - f (n \geq 1)$ with nonzero $f \in R$, let $d|n$ be the maximal positive integer satisfying $T^d - f$ has a root $g \in K$, we have $T^n - f = \prod_{i-1}^{d}(T^{n'} - \zeta_d^i g)$ where ζ_d is a primitive d-th root of unit and each $T^{n'} - \zeta_d^i g$ is a prime element in $R[T]$ for $n' := n/d$; the quotient ring $K[T]/(T^n - f)$ can be regarded as a finite K-linear space

$$K[T]/(T^n - f) = \bigoplus_{i=0}^{n-1} K t^i$$

where $t := T \mod (T^n - f)$ can be regarded as a root of $T^n - f$ in $\overline{K}$, that t is also integral over R and the ring $R[T]/(T^n - f)$ is integral if and only if $T^n - f$ is irreducible, the group $G_d =< \sigma >$ has a natural action on $K[T]/(T^n - f)$ by $\sigma : T \mapsto \zeta_n T$ and $\sigma|_K = \mathrm{id}_K$, the μ_n-invariant part of $K[T]/(T^n - f)$ is K.

Lemma 2.3. *Let f be an nonzero element in the domain R. Let n be a positive integer and $d|n$ the maximal positive integer such that $T^d - f$ has a root in K and $n' := n/d$. For any $\zeta \in \mu_d := \{x \in \mathbb{C} \,|\, x^n = 1\}$, define a polynomial*

$$P_\zeta = \frac{1}{\prod\limits_{\zeta' \in \mu_d \setminus \zeta} (\zeta g - \zeta' g)} \frac{T^n - f}{T^{n'} - \zeta g}$$

where $g \in K$ is a root of $T^d - f$.

1. *In the quotient ring $\frac{K[T]}{(T^n - f)}$, these P_ζ's are idempotent,i.e. $P_\zeta \cdot P_{\zeta'} = 0\, \zeta \neq \zeta', P_\zeta^2 = P_\zeta$ with $\sum\limits_{\zeta \in \mu_d} P_\zeta = 1$. Moreover, the cyclic group $G_n =< \sigma >$ acts transitively on the set $\{P_\zeta\}_{\zeta \in \mu_d}$ with $(P_\zeta)^\sigma = P_{\xi_n^{-n'}\zeta}\, \forall \zeta \in \mu_d$.*
2. *Let K_ζ be the field $\frac{K[T]}{(T^{n'} - \zeta g)}$ for each $\zeta \in \mu_d$. Any $\xi \in \mu_n$ defines an isomorphism $K_\zeta \xrightarrow{\cong} K_{\xi^{n'}\zeta}$ of fields by sending T to ξT. Moreover, there is a natural G_n-action on the algebra $\prod\limits_{\zeta \in \mu_d} K_\zeta$.*
3. *There is a G_n-equivariant isomorphism of K-algebras*

$$P : \prod_{\zeta \in \mu_d} \frac{K[T]}{(T^{n'} - \zeta g)} \xrightarrow{\cong} \frac{K[T]}{(T^n - f)}, (\alpha_\zeta)_\zeta \mapsto \sum_\zeta \alpha_\zeta P_\zeta. \tag{2.3.1}$$

Corollary 2.4. *Let $T^n - f$ be a polynomial in $R[T]$ with degree $n \geq 1$. Let $A = R[T]/(T^n - f)$ be a ring and $S := \{a \in A \,|\, r \text{ is not zero divisor}\}$ a multiplicative system in A. Then $S^{-1}A = K[T]/(T^n - f)$.*

Proof. Let $h(T) \in R[T]$ such that it's image $\overline{h}$ is not a zero-divisor in A. Consider the K-isomorphism 2.3.1, we get that $P^{-1}(\overline{h}) = (\alpha_\zeta)_\zeta$ with $\alpha_\zeta := h(T) \mod (T - \zeta g)$ in $K[T]\, \forall \zeta \in \mu_d$. Thus that $\overline{h}$ is not a zero-divisor in R if and only if for each $\zeta \in \mu_d$ there is a $\eta_\zeta(T) \in K[T]$ such that $\eta_\zeta(T) h(T) = 1 \mod (T^{n'} - \zeta g)$ in $K[T]$ We get $\overline{\eta}\overline{h} = 1$ in $K[T]/(T^n - f)$, where $\overline{\eta}$ is the projection image of the polynomial $\eta(T) := \sum_{\zeta \in \mu_d} \eta_\zeta P_\zeta$. Thus, we have that any non zero divisor in A is an unit in $\frac{K[T]}{(T^n - f)}$. Hence,

$$S^{-1}A = S^{-1}\left(\frac{K[T]}{(T^n - f)}\right) = \frac{K[T]}{(T^n - f)}.$$

□

Consider the ring homomorphism $R \to K[T]/(T^n - f)$ for a polynomial $T^n - f \in R[T]$ of degree $n \geq 1$. Let $\bar{R}$ be the integral closure of R in $K[T]/(T^n - f)$.

Lemma 2.5. *Let f be an arbitrary nonzero element in R. If a polynomial $T^n - f \in R[T]$ with degree $n \geq 1$ is irreducible in $R[T]$ then $\bar{R}$ is the normalization of the domain $\frac{R[T]}{(T^n-f)}$ in the field extension $\frac{K[T]}{(T^n-f)}$ of K.*

Therefore, by Corollary 2.4 the $\bar{R}$ is also the normalization of $\frac{R[T]}{(T^n-f)}$ even though $T^n - f$ is not irreducible since R is a normal integral ring.

Lemma 2.6. *Let g be a nontrivial prime element in R. Then the principal prime ideal (g) is of height one.*

Proof. We claim that any nonzero prime idea $\mathfrak{p}$ of R has an irreducible element : Suppose the claim is not true. Let a_0 be any nonzero element in $\mathfrak{p}$. Then $a = bc$ such that both b and c are not unit. Then one of b, c is in $\mathfrak{p}$, say a_1. The a_1 is also irre4ducible and $(a_0) \subsetneq (a_1) \subset \mathfrak{p}$. We do same steps for a_1 as dealing with a_0 and so on. Thus we have an strictly increasing sequence of ideals

$$(a_0) \subsetneq (a_1) \subsetneq (a_2) \subsetneq \cdots \subsetneq \mathfrak{p},$$

which contradicts that R is a Noetherian ring.

Let $\mathfrak{p}$ be any nontrivial prime ideal of R such that $0 \subsetneq \mathfrak{p} \subset (g)$. Let $a \in \mathfrak{p}$ be an irreducible element. We have $a = gc$ due to $a \in (g)$, but c is a unit since a is irreducible. Hence $\mathfrak{p} = (g)$. □

For any height one prime idea $\mathfrak{p}$ of R, the local ring $R_{\mathfrak{p}}$ is a DVR, thus we can define $\mathrm{div}(x)$ for any $x \in K$.

Theorem 2.7. *Assume a finitely generated $\mathbb{C}$-algebra R is integral and normal. Let K be the fraction field of R. Let $f = u f_1^{\nu_1} \cdots f_l^{\nu_l}$ ($\forall \nu_i \geq 1$) be an element in R such that f_i's are different prime elements in R and u is a unit in R. Let $n \geq 1$ be an integer and $d|n$ the maximal positive integer such that $T^d - f$ has a root in K, say g, and denote by $n' := n/d$.*

1. *With respect to the injective ring homomorphism $R \to \frac{K[T]}{(T^n-f)}$, the integral closure of R is*

$$\bar{R} = \bigoplus_{i=0}^{n-1} t^i f_1^{-[\frac{\nu_1 i}{n}]} \cdots f_l^{-[\frac{\nu_l i}{n}]} R \tag{2.7.1}$$

   where $t := T \mod (T^n - f)$. Moreover, the cyclic group $\mathbb{Z}_n =< \sigma >$ has a natural action on $\bar{R}$ by defining $\sigma : t \mapsto \xi_n t, \sigma|_R = \mathrm{id}_R$, and the G_n-invariant part of $\bar{R}$ is

$$\bar{R}^{\mathbb{Z}_n} = R.$$

2. *Let $\bar{R}_\zeta$ be normalization of an integral ring $R_\zeta := \frac{R[T]}{(T^{n'}-\zeta g)}$ in $\frac{K[T]}{(T^{n'}-\zeta g)}$ for each $\zeta \in \mu_d$. Then*

$$\bar{R}_\zeta = \bigoplus_{i=0}^{n'-1} t_\zeta^i f_1^{-[\frac{\nu_1 i}{n}]} \cdots f_l^{-[\frac{\nu_l i}{n}]} R \tag{2.7.2}$$

where $t_\zeta := T \mod (T^n - \zeta g)$. *Moreover,* $R \to \bar{R}_\zeta$ *is Galois, with Galois group* $\mathbb{Z}_{n'}$.

3. *The group* $\mathbb{Z}_n$ *has a natural action on the product* R*-algebra* $\prod_{\zeta\in\mu_d} \bar{R}_\zeta$. *The group permutes the factors of the decomposition, and there is a* G_n*-equivariant isomorphism of* R*-algebras*

$$\prod_{\zeta\in\mu_d} \bar{R}_\zeta \longrightarrow \bar{R},\ (\alpha_\zeta)_\zeta \mapsto \sum_\zeta \alpha_\zeta P_\zeta.$$

Proof. For any $x \in K$, it is easy to verify the following conditions are equivalent :

i. $x^n f^i \in R$,

ii. $n\mathrm{div}(x) + \sum_{j=1}^{l} i\nu_j \mathrm{div}(f_j) \geq 0$(we use $\mathrm{div}(u) = 0$).

iii. $\mathrm{div}(x f_1^{[\frac{\nu_1 i}{n}]} \cdots f_l^{[\frac{\nu_l i}{n}]}) + \sum_{j=1}^{l} (\frac{i\nu_j}{n} - [\frac{\nu_1 i}{n}])[f_j] \geq 0$(we use that (f_i)'s are prime ideal of height one and so that $\mathrm{div}(f_j)$ is the Weil divisor $[f_j] := \mathrm{Zero}(f_j)$),

iv. $\mathrm{div}(x f_1^{[\frac{\nu_1 i}{n}]} \cdots f_l^{[\frac{\nu_l i}{n}]}) \geq 0$(we use that $\mathrm{div}(x f_1^{[\frac{\nu_1 i}{n}]} \cdots f_l^{[\frac{\nu_l i}{n}]})$ has integral coefficients and f_i's are prime elements),

v. $x f_1^{[\frac{\nu_1 i}{n}]} \cdots f_l^{[\frac{\nu_l i}{n}]} \in R$.

1. Thus there is

$$\{x \in K; x^n f^i \in R\} = f_1^{-[\frac{\nu_1 i}{n}]} \cdots f_l^{-[\frac{\nu_l i}{n}]} R.$$

We now show

$$\bar{R} = \bigoplus_{i=0}^{n-1} \{x \in K; x^n f^i \in R\} t^i.$$

Let $\alpha = \sum_{i=0}^{n-1} x_i t^i$ $(x_i \in K)$ be an element in $\frac{K[T]}{(T^n - f)}$. We claim that $\alpha \in \bar{R}$ if and only if $x_i t^i \in \bar{R}$ for every i. Suppose $\alpha \in \bar{R}$. We have $\alpha^\zeta = \sum_{i=0}^{n-1} \zeta^i x_i t^i \in \bar{R}$, for every $\zeta \in \mu_n$. Let $\mu_n = \{\zeta_1, \zeta_2, \ldots, \zeta_n\}$. The Vandermonde determinant

$$\det \begin{bmatrix} \zeta_1^0 & \zeta_1^1 & \cdots & \zeta_1^{n-1} \\ \zeta_2^0 & \zeta_1^1 & \cdots & \zeta_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_n^0 & \zeta_n^1 & \cdots & \zeta_n^{n-1} \end{bmatrix} = \prod_{1\leq i<j\leq n} (\zeta_j - \zeta_i)$$

is a unit in R, so that each $x_i t^i$ is a combination of the α^ζ's with coefficients in R by Cramer's rule. Therefore $x_i t^i \in \bar{R}$. The converse is obvious.

Let $x \in K$ and $0 \leq i < n$. We claim that $xt^i \in \bar{R}$ if and only if $x^n f^i \in R$. Indeed, we have that $xt^i \in \bar{R}$ if and only if $y = (xt^i)^n \in \bar{R}$. Due to

$$y = (xt^i)^n = x^n (t^n)^i = x^n f^i \in K,$$

We have that $y \in \bar{R}$ if and only if $y \in K \cap \bar{R} = R$ since R is integrally closed in K.

2. By the statement (1), the integral closure of R in $\frac{K[T]}{(T^{n'} - \zeta g)}$ is the integrally closed domain $\bar{R}_\zeta = \bigoplus_{j=0}^{n'-1} \{x \in K; x^{n'}(\zeta g)^j \in R\} t_\zeta^j$. Since R is integrally closed in K, $x^{n'}(\zeta g)^j \in R$ if and only if $(x^{n'}(\zeta g)^j)^d \in R$. That is $x^n f^j \in R$. Therefore

$$\bar{R}_\zeta = \bigoplus_{j=0}^{n'-1} \{x \in K; x^n f^j \in R\} t_\zeta^j.$$

3. In the ring $\frac{K[T]}{(T^n - f)}$, we have that $f(\sum_\zeta \alpha_\zeta P_\zeta) = \sum_\zeta f(\alpha_\zeta) P_\zeta$ for any polynomial $f \in R[T]$. Hence that $\sum_\zeta \alpha_\zeta P_\zeta$ is integral over R if and only if each $\alpha_\zeta \in \frac{K[T]}{(T^{n'} - \zeta g)}$ is integral over $\frac{R[T]}{(T^{n'} - \zeta g)}$. The product decomposition of Lemma 2.3 induces an isomorphism of R-algebras $\prod_{\zeta \in \mu_d} \bar{R}_\zeta \xrightarrow{\cong} \bar{R}$.

□

3. Proof of Theorem 1.1

Now we investigate the local structure of cyclic cover.

Lemma 3.1. *Let $U := \mathrm{Spec} R$ be an irreducible complex normal affine variety of dimension m and $f = u f_1^{\nu_1} \cdots f_e^{\nu_e} (\forall \nu_i \geq 0)$ an element in R such that f_i's are prime elements in R and u is a unit in R. Let $D = \sum_{i=1}^{e} \nu_i D_i$ is zero locus of f, i.e., $D_i = \mathrm{Zero}(f_i) \forall i$.*

Denote by $U[\sqrt[n]{f}] := \{(p, y) \in U \times \mathbb{A}^1 \mid y^n = f(p)\}$ for an integer $n \geq 2$.

a) Let p_0 be a smooth closed point of U. Then $U[\sqrt[n]{f}]$ is singular at a point $(p_0, *)$ if and only if $p_0 \in \mathrm{Sing}(D)$. Moreover, $U[\sqrt[n]{f}]$ is smooth if and only if U is smooth and D is reduced with only one smooth component.

b) Assume that U is smooth. If $D = \nu_1 D_1$ and $f = u f_1^{\nu_1} (\nu_1 \geq 1)$ then the normalization $\overline{U}[\sqrt[n]{f}]$ of $U[\sqrt[n]{f}]$ has only singularities over $\mathrm{Sing}(D_1)$.

Proof. The proof of (a) is directly follows from Jocobi's criterion : Let $(Z_1, \cdots Z_m)$ be an regular parameter near an open neighborhood V of p_0. Let

$$F(Z_0, Z_1, \cdots Z_m) = Z_0^n - f(Z_1, \cdots, Z_m).$$

The point $q := (p_0, y_0)$ in $U[\sqrt[n]{f}]$ is singular if and only if the F satisfies

$$\begin{aligned} F(q) &= 0 \\ \frac{\partial F}{\partial Z_0}(q) = \frac{\partial F}{\partial Z_1}(q) = \cdots = \frac{\partial F}{\partial Z_m}(q) &= 0. \end{aligned}$$

Thus we prove the (a).

Now $R = A(U)$ is an regular algebra. The problem (b) is now reduced to show that the $\overline{U}[\sqrt[n]{f}]$ is smooth if the D_1 is a smooth divisor. In case of $\nu = 0, 1$ the (b) is obvious by (a.) With lost of generality, we assume that $\gcd(n, \nu) = 1$. Let R be the ring corresponding to the U. By Theorem 2.7, We get $\overline{U}[\sqrt[n]{f}] = \mathrm{Spec}(\bar{R})$ such that

$$\bar{R} = \bigoplus_{i=0}^{n-1} t^i f_1^{-[\frac{\nu_1 i}{n}]} R$$

where $t := T \mod (T^n - u f_1^{\nu_1})$.

Since $\gcd(n, \nu) = 1$, we have an ring-isomorphism $\frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{\times \nu_1} \frac{\mathbb{Z}}{n\mathbb{Z}}$, and so we can get $a \in \{0, \cdots n-1\}$ with $a\nu_1 = 1 + ln$ for some $l \in \mathbb{Z}$. Denote by $g = t^a f_1^{-[\frac{a\nu_1}{n}]}$. We get that

$$g^n = u^a f_1 \;\text{ and }\; t = u^{-l} g^{\nu_1}.$$

Thus

$$\bigoplus_{i=0}^{n-1} g^i R \cong \frac{R[W]}{(W^n - u^a f_1)}$$

is an R-subalgebra of $\bar{R}$. Moreover, we have a tower of inclusions of R-algebras

$$R \xrightarrow{\subset} \frac{R[T]}{(T^n - u f^{\nu_1})} \xrightarrow[T \mapsto u^{-l} g^{\nu_1}]{\subset} \bigoplus_{i=0}^{n-1} g^i R \xrightarrow{\subset} \bar{R}.$$

such that the integral domains $\frac{R[T]}{(T^n - u f^{\nu_1})}$, $\bigoplus_{i=0}^{n-1} g^i R$ and $\bar{R}$ all have same quotient field.

Since the algebra $\frac{R[W]}{(W^n - u^a f_1)}$ is regular by (a) and the algebra $\bar{R}$ is normal, we have to get

$$\bar{R} \cong \frac{R[W]}{(W^n - u^a f_1)}.$$

Therefore the $\overline{U}[\sqrt[n]{f}]$ is smooth. □

We only need to prove the statements 5) and 6) of Theorem 1.1.

Let $d = \dim_{\mathbb{C}} X$ and let ξ be a primitive root of $x^{N_0} - 1$. Since π is étale outside of D, there is

$$\pi_* \widehat{\Omega^p_Z}|_{X-D} \cong (\Omega^p_X \otimes \pi_* \mathcal{O}_Z)|_{X-D}$$

and the isomorphism 1.1.2 holds over $X - D$. On the other hand, the $\pi_* \widehat{\Omega^p_Z}$ is a reflexive sheaf on X by the following lemma 3.2, thus it is sufficient to check the assertion 1.1.2 only in codimension one on X.

Lemma 3.2 ([H80])**.** *Let $\mathcal{F}$ be a coherent sheaf on a normal integral scheme X. The followin two conditions are equivalent:*

i. *$\mathcal{F}$ is reflexive;*

ii. *$\mathcal{F}$ is torsion-free, and is normal,i.e.,for every open set $U \subset X$ and every closed subset $B \subset U$ of codimension 2, the restriction map $\mathcal{F}(U) \to \mathcal{F}(U - Y)$ bijective.*

Without lost of generality, we may assume that the D_{red} is smooth, moreover there is a local regular parameter $(x_1, \cdots, x_d)$ of X such that

$$X = \operatorname{Spec}(\mathbb{C}[x_1, \cdots, x_d]),\ D = (x_1^a).$$

Then $X' = \operatorname{Spec}(\mathbb{C}[x_1, \cdots, x_d, t]/(T^{N_0} - ux_1^a))$ where u is a unit in the domain $\mathbb{C}[x_1, \cdots, x_d]$ and Z is normalization of X'.

Let $R := \mathbb{C}[x_1, x_2, \cdots, x_d]$ a normal integral ring. Let $A := \frac{R[T]}{(T^{N_0} - ux_1^a)}$ and B the normalization of A. As Ω^1_B is generated as a B-module, by the image $d : B \to \Omega^1_B$. We know as that

$$\wedge^p \Omega^1_B = \{b_0 db_1 \wedge db_2 \cdots \wedge db_p \mid b_i \in B\, i = 1, \cdots, N_0 - 1\}.$$

Let $t := T \mod (T^{N_0} - ux_1^a)$. By Proposition 2.7 we have that $B = \bigoplus_{i=0}^{N_0-1} q_i R$ as an R-module and that the group $G =< \sigma >$ has a natural action on $\wedge^p \Omega^1_B$, where $q_i := t^i x_1^{-[\frac{ai}{N_0}]}$ $0 \le i \le N_0 - 1$. Since

$$q_i q_j = q_i := t^{i+j} x_1^{-[\frac{a(i+j)}{N_0}]} x_1^{[\frac{a(i+j)}{N_0}] - [\frac{ai}{N_0}] - [\frac{aj}{N_0}]}\ \forall i, j \ge 0,$$

we have that $q_i q_j \in q_{i+j} R$ if $i + j < N_0$ and that $q_i q_j \in q_{i+j-n} R$ if $i + j \ge N_0$. Let ξ be the primitive N_0-th root of unit as we choose in the statement 2). We then have an decomposition of $\wedge^p \Omega^1_B = \bigoplus_{i=1}^{N_0-1} Q_i$ such that each Q_i is the R-submodule of eigenvectors for σ in $\wedge^p \Omega^1_B$ with eigenvalue ξ^i. We now find out these Q_i's. We have that

$$\begin{aligned} \frac{dt}{t} &= \frac{a}{N_0} \frac{dx_1}{x_1} \\ dq_i &= \{\frac{ai}{N_0}\} q_i \frac{dx_1}{x_1} \end{aligned}$$

On the other hand, each $b \in B$ can be written uniquely as $b = \sum_{i=1}^{N_0-1} q_i r_i$ with each $r_i \in R$ and so $db = \sum_{i=1}^{N_0-1}(q_i dr_i + r_i dq_i)$. For each $0 \le i \le N_0 - 1$, the elements

$$t^i x_1^{-[\frac{ia}{N_0}]} dy_{e_1} \wedge \cdots \wedge dy_{e_p},\ e_1 < e_2 < \cdots < e_p < d \text{ for } e_1 \neq 1,$$

with

$$\begin{cases} t^i x_1^{-[\frac{ia}{N_0}]} \frac{dx_1}{x_1} \wedge dx_{e_2} \wedge \cdots \wedge dy_{e_p},\ 1 < e_2 < \cdots < e_p < d & \text{if } \{\frac{ai}{N_0}\} \neq 0 \\ t^i x_1^{-[\frac{ia}{N_0}]} dx_1 \wedge dx_{e_2} \wedge \cdots \wedge dx_{e_p},\ 1 < e_2 < \cdots < e_p < d & \text{otherwise} \end{cases}$$

becomes an R-basic of the module Q_i. In particular, G-invariant submodule of $\wedge^p \Omega^1_B$ has $Q_0 = \wedge^p \Omega^1_R$. Therefore, we get the formula 1.1.2 and $\Omega^p_X = (\pi_* \widehat{\Omega^p_Z})^G$.

Now we are going to prove the statement 6). For the normal varieties X and Z, by definition, the tangent sheaf

$$\mathcal{T}_Z := \mathcal{H}om_{\mathcal{O}_Z}(\Omega^1_Z, \mathcal{O}_Z)$$

is a reflexive sheaf which is composed of derivations from $\mathcal{O}_Z$ to $\mathcal{O}_Z$, and $\mathcal{T}_X(-\log D^{(1)})$ is the subsheaf of $\mathcal{T}_X$ of consisting of derivations which send the ideal sheaf $\mathcal{I}$ of $D^{(1)}$ into itself. Since X is smooth, we get(see [D72]) that

$$\mathcal{T}_X(-\log D^{(1)}) = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X(\log D^{(1)}), \mathcal{O}_X).$$

Suppose X is proper and D is reduced. Then $D = D^{(i)}$ for all $1 \le i \le N_0 - 1$ and by 1) Z is also a proper scheme. Thus the dualizing sheaf $\omega_Z = \pi^! \omega_X$ has a natural $\mathcal{O}_Z$-module(see Proposition 5.68. [KM]). Because π is an affine morphism, we can define an equivalent functor $\sim$ of categories between the category of quasi-coherent $\pi_*\mathcal{O}_Z$-modules and the category of quasi-coherent $\mathcal{O}_Z$-module(see Ex. II. 5.17(e) [Hart]). Since $\pi_*\mathcal{O}_Z$ is locally free, we have

$$\pi^! \omega_X := \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_Z, \omega_X)^{\sim} = (\mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_Z, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \omega_X)^{\sim} = \pi^! \mathcal{O}_X \otimes_{\mathcal{O}_Z} \pi^* \omega_X.$$

Thus the relative dualizing sheaf $\omega_{Z/X} = \pi^! \mathcal{O}_X$.

We claim that

$$\omega_{Z/X} = \pi^* L^{(N_0-1)} = \pi^*(L^{N_0-1}).$$

We can take a local regular parameter $(x_1, \cdots, x_d)$ of X such that

$$X = \mathrm{Spec}(\mathbb{C}[x_1, \cdots, x_d]),\ D = (x_1 \cdots x_k).$$

Thus we get that

$$X' = \mathrm{Spec}(\mathbb{C}[x_1, \cdots, x_d, t]/(T^{N_0} - x_1 \cdots x_k))$$

and Z is the normalization of X'. Let

$$R := \mathbb{C}[x_1, x_2, \cdots, x_d] \text{ and } A := \frac{R[T]}{(T^{N_0} - x_1 \cdots x_k)},$$

and let B be the normalization of A. By Proposition 2.7 we have that

$$B = \bigoplus_{i=0}^{N_0-1} p_i R$$

as an R-module where $\{p_i := t^i\}_{i=0,\cdots N_0-1}$ is an R-basis of B. Let

$$\eta_i \in B^* := \mathrm{Hom}_R(B, R)$$

be dual of q_i, i.e., $\eta_i(p_i) = 1, \eta_i(p_j) = 0 \forall j \neq i$. The B-module structure of $\mathrm{Hom}_R(B, R)$ is clear : $b \cdot \phi := \phi_b$ where $\phi_b \in \mathrm{Hom}_R(B, R)$is given by $\phi_b(b_1) := \phi(bb_1) \forall b_1 \in B$. We observe that

$$p_i p_j = q_{i+j} \text{ if } i + j < N_0 \text{ and } p_i p_j \in p_{i+j-n} R \text{ if } i + j \geq N_0.$$

So that

$$p_i \cdot \eta_{N_0-1} = \eta_{N_0-1-i}$$

for all $i = 0, \cdots N_0 - 1$. As an B-module, the $\mathrm{Hom}_R(B, R)$ must be a module of rank one generated by η_{N_0-1}. Hence $\pi^! \mathcal{O}_X = \pi^*(L^{(N_0-1)})$.

Let $\mathrm{Sing}(D)$ be singularities of D and $X_0 := X \setminus \mathrm{Sing}(D)$. We know codimension of $\mathrm{Sing}(D)$ in X is more than two and $\pi^{-1}(X_0)$ is smooth open sub-scheme of Z. Using the duality theory for finite morphism(cf. Theorem 5.67 [KM]), on X_0 we get

$$\begin{aligned}
\pi_* \mathcal{T}_Z &= \pi_* \mathcal{H}om_{\mathcal{O}_Z}(\Omega^1_Z, \mathcal{O}_Z) \\
&= \pi_* \mathcal{H}om_{\mathcal{O}_Z}(\omega_{Z/X} \otimes_{\mathcal{O}_Z} \Omega^1_Z, \omega_{Z/X}) \\
&= \pi_* \mathcal{H}om_{\mathcal{O}_Z}(\pi^*(L^{(N_0-1)}) \otimes_{\mathcal{O}_Z} \Omega^1_Z, \pi^! \mathcal{O}_X) \\
&= \mathcal{H}om_{\mathcal{O}_X}(L^{(N_0-1)} \otimes_{\mathcal{O}_X} \pi_* \Omega^1_Z, \mathcal{O}_X) \\
&= \mathcal{H}om_{\mathcal{O}_X}(L^{(N_0-1)} \otimes_{\mathcal{O}_X} (\bigoplus_{i=0}^{N_0-1} \Omega^1_X(\log D^{(i)}) \otimes \mathcal{L}^{-(i)}), \mathcal{O}_X) \\
&= (L^{-(N_0-1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X) \oplus \bigoplus_{i=1}^{N_0-1} (L^{(i-N_0+1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X(-\log D)).
\end{aligned}$$

Since both $\pi_* \mathcal{T}_Z$ and $(L^{-(N_0-1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X) \oplus \bigoplus_{i=1}^{N_0-1} (L^{(i-N_0+1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X(-\log D))$ are reflexive sheaves on X, there is

$$\pi_* \mathcal{T}_Z = L^{-(N_0-1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X) \oplus \bigoplus_{i=1}^{N_0-1} (L^{(i-N_0+1)} \otimes_{\mathcal{O}_X} \mathcal{T}_X(-\log D)) \text{ on } X.$$

Hence $(\pi_* \mathcal{T}_Z)^G = \mathcal{T}_X(-\log D)$ on X.

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