TRIPLE PRODUCT FORMULA AND THE SUBCONVEXITY BOUND OF TRIPLE PRODUCT L-FUNCTION IN LEVEL ASPECT

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Abstract. In this paper we derived a nice general formula for the local integrals of triple product formula whenever one of the representations has sufficiently higher level than the other two. As an application we generalized Venkatesh and Woodbury's work on the subconvexity bound of triple product L-function in level aspect, allowing joint ramifications, higher ramifications, general unitary central characters and general special values of local epsilon factors.

1. introduction

1.1. **Triple product formula.** Let $\mathbb{F}$ be a number field. Let π_i, $i = 1, 2, 3$ be three irreducible unitary cuspidal automorphic representations, such that the product of their central characters is trivial:

$$\prod_i w_{\pi_i} = 1. \tag{1.1}$$

Let $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$. Then one can define the triple product L-function $L(\Pi, s)$ associated to them. It was first studied in [6] by Garrett in classical languages, where explicit integral representation was given. In particular the triple product L-function has analytic continuation and functional equation. Later on Shapiro and Rallis in [19] reformulated his work in adelic languages.

In this paper we shall study the following integral representing the special value of triple product $L-$function (see Section 2.2 for more details):

$$|\int\limits_{Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A})} f_1(g)f_2(g)f_3(g)dg|^2 = \frac{\zeta_{\mathbb{F}}^2(2)L(\Pi, 1/2)}{8L(\Pi, Ad, 1)} \prod_v I_v^0(f_{1,v}, f_{2,v}, f_{3,v}), \tag{1.2}$$

Here $f_i \in \pi_i^{\mathbb{D}}$ for a specific quaternion algebra $\mathbb{D}$, and $\pi_i^{\mathbb{D}}$ is the image of π_i under Jacquet-Langlands correspondence. The local integral I_v^0 is defined as

$$I_v^0(f_{1,v}, f_{2,v}, f_{3,v}) = \frac{L_v(\Pi_v, Ad, 1)}{\zeta_v^2(2)L_v(\Pi_v, 1/2)} I_v(f_{1,v}, f_{2,v}, f_{3,v}), \tag{1.3}$$

where

$$I_v(f_{1,v}, f_{2,v}, f_{3,v}) = \int\limits_{\mathbb{F}_v^*\backslash\mathbb{D}^*(\mathbb{F}_v)} \prod_{i=1}^{3} < \pi_i^{\mathbb{D}}(g) f_{i,v}, f_{i,v} > dg. \tag{1.4}$$

Here $< \cdot, \cdot >$ is a bilinear and $\mathbb{D}^*(\mathbb{F}_v)-$ invariant unitary pairing for $\pi_{i,v}^{\mathbb{D}}$. At unramified places, $I_v^0 = 1$.

This is a special version of Ichino's formula in [11]. Its local factors are, however, not explicit, and can not be used directly for applications. Later on, there are several works computing the local factors with ramifications. The work by Watson in [23], Nelson in [17] and Woodbury in [24] covered all possible situations with square-free levels. The work of Nelson-Pitale-Saha in [18], however, is the only work that deal with higher ramifications, essentially with the assumption that $\pi_1 = \pi_2$ (and correspondingly $f_1 = f_2$) and π_3 is unramified.

Such results are already very useful. The local computations by Woodbury together with the work of Venkatesh in [22] proved the subconvexity bound of Triple product $L-$function for square-free levels. The local computations by Nelson-Pitale-Saha was used to prove the mass equidistribution on modular curve $Y_0(1)$. It is natural to expect that computing the local integral for more general situations would also generalize these applications.

We remark here that the work in [18] is based on Lemma (3.4.2) of [16], which relates I_v to the local Rankin-Selberg integral. But this method can't be generalized to the case when, for example, all the representations are supercuspidal, which is necessary for our consideration.

Before I give the main result for local calculations in this paper, let me first give a very special and explicit corollary, which is already new.

For any holomorphic new form f of weight k, suppose that it has Fourier expansion

$$f(z) = \sum_{n>0} \frac{a_f(n)}{n^{\frac{1-k}{2}}} e^{2\pi i n z}. \tag{1.5}$$

For two modular forms f_1, f_2 of same weight k and level N, define their Petersson inner product

$$< f_1, f_2 >_{Pet} = \frac{1}{[\mathrm{SL}_2 : \Gamma_0(N)]} \int\limits_{\Gamma_0(N)\backslash\mathbb{H}} f_1(z)\overline{f}_2(z) y^k \frac{dxdy}{y^2}. \tag{1.6}$$

Corollary 1.1. *Let f_i be three holomorphic new forms of weight k_i, such that $k_1 + k_2 = k_3$. For simplicity assume that f_1 f_2 are level 1, and f_3 is of level p^n for $n > 1$. Let $a_{f_i}(n)$ be the $n-$th Fourier coefficient for f_i as above. Then*

$$\frac{| < f_1 f_2(p^n z), f_3 >_{Pet} |^2}{\prod_{i=1}^{3} < f_i, f_i >_{Pet}} = \frac{L(f_1 \otimes f_2 \otimes f_3, 1/2)}{8 \prod L(f_i, Ad, 1)} C_\infty C_p, \tag{1.7}$$

where

$$C_\infty = \begin{cases} 2, & \text{if } k_i \neq 0 \text{ for all } i, \\ 1, & \text{otherwise.} \end{cases} \tag{1.8}$$

$$C_p = \frac{\prod L_p(f_i, Ad, 1)}{\zeta_p^2(2)} \frac{((\sqrt{p} + 1/\sqrt{p})^2 - a_{f_1}(p)^2)((\sqrt{p} + 1/\sqrt{p})^2 - a_{f_2}(p)^2)}{(p+1)^3 p^{n-1}}. \tag{1.9}$$

Note that it is necessary to use $f_2(p^n z)$ as a test function since if we simply use new forms, the integral will be zero trivially for the level reason. Scaling by p^n is exactly the classical way to produce an old form, and in this case guarantees that the integral becomes nonzero.

In general for any non-archimedean place v, let q be the size of the corresponding residue field. We have the following result for the local integral.

Theorem 1.2. *Let c_i be the level of π_i at v, that is, there exists a unique up to constant new form in π_i which is invariant by $K_1(\varpi_v^{c_i})$ (see (2.3)). Assume that $c_3 \geq 2\max\{c_1, c_2, 1\}$. Then the local integral*

$$I_v = \frac{(1-A)(1-B)}{(q+1)q^{c_3-1}}. \tag{1.10}$$

Here the constants A, B can be explicitly given in terms of π_1, π_2 respectively, and are bounded and away from 1. In particular the local integral is nonzero.

Note that the test vectors are normalized and chosen similarly as in the corollary above. The explicit values of A and B will also be given. See Theorem 4.1 for more details.

Remark 1.3. The assumption that $c_3 \geq 2\max\{c_1, c_2, 1\}$ essentially means that one of the levels is sufficiently larger than the other two. We put 1 here to avoid the squarefree level case, which is already well studied before.

If we further assume the central characters to be trivial, the assumption can be further improved to be that $c_3 > \max\{c_1, c_2, 1\}$.

The local integral becomes more complicated when, for example, $c_2 = c_3 > c_1$. In a subsequent work in [10], we shall give an upper bound for the local integral in this case, and use this to prove mass equidistribution on $Y_0(N)$. In this paper, we shall use the theorem above to prove the subconvexity bound of triple product $L-$function in level aspect.

The tool developed in this paper can also be used to compute other local integrals, like Rankin-Selberg integral. We expect to see more applications of this tool in future research.

1.2. **Subconvexity bound.** We shall give the application of the above theorem to the subconvexity bound for the special value of triple product L-function $L(\Pi, 1/2)$. This topic can be approached in different ways. For example Bernstein and Reznikov in [1] gave the subconvexity bound in the eigenvalue aspect. What we seek is the subconvexity bound in the level aspect. In particular, we will fix π_1 and π_2, let π_3 vary with finite conductor N. The finite conductor of Π is roughly N^4. By the general principle for convexity bound, one can get the trivial bound

$$L(\Pi, 1/2) << N^{1+\epsilon}. \tag{1.11}$$

We'd like to establish the subconvexity bound

$$L(\Pi, 1/2) << N^{1-\delta}. \tag{1.12}$$

as $N \to \infty$.

The idea comes from Venkatesh's work in [22]. One starts with the integral representation of the special value of triple product L-function (1.2). Suppose that the cusp forms and their local components are properly normalized. The idea is to give first an upper bound for the global integral on the left-hand side of (1.2). Then a lower bound for the local integrals I_v^0 will result in an upper bound for $L(\Pi, 1/2)$, which turn out to be a subconvexity bound in the level aspect. The local integrals basically gives the main term as in the trivial bound, and the power saving comes from the global upper bound. In particular, assume that π_3 is of prime conductor $\mathfrak{p}$. Venkatesh's work together with Woodbury's work on local integrals in [24] proved the following:

$$L(\Pi, 1/2) << N(\mathfrak{p})^{1-1/12}. \tag{1.13}$$

Their result, however, is based on the following conditions:

(1) π_i essentially have disjoint ramifications and π_3 has square-free finite conductor $\mathfrak{p}$.
(2) All the central characters are trivial.
(3) The special values of local epsilon factors $\epsilon_v(\Pi, 1/2) = 1$ for all places.
(4) The infinity component of π_3 is bounded.

In this paper, we will remove the first three conditions and prove a similar subconvexity bound. So we will allow high ramifications and joint ramifications, and general unitary central characters.

The third condition is related to Prasad's thesis work on local trilinear forms, and turns out to be free to remove. This is because, as we will see later, all key calculations will be done on the GL_2 side. We still assume the last condition to control the contribution from archimedean places.

We shall prove the following main theorem:

Theorem 1.4. *Let $\pi_i, i = 1,2,3$ be three unitary cuspidal automorphic representations of GL_2, such that*

$$\prod_i w_{\pi_i} = 1. \tag{1.14}$$

Fix π_1 and π_2, and let π_3 vary with changing finite conductor $\mathcal{N}$ and $N = Nm(\mathcal{N})$. Suppose that the infinity component of π_3 is still bounded. Then

$$L(\pi_1 \otimes \pi_2 \otimes \pi_3, 1/2) \ll N^{1-1/12}, \text{ as } N \to \infty. \tag{1.15}$$

The proof follows the same strategy. In particular we will generalized the power saving upper bound for the global integral, and the lower bound for local integrals directly comes from Theorem 1.2.

We remark here that the proof for the global upper bound can be refined to slightly more general setting, that is, to allow the conductors of π_1 and π_2 also to change, but they have to be relatively small compared to N. In this way, one can obtain a hybrid type subconvexity bound in level aspect. See Remark 3.6 for more details.

1.3. **Organizations of this paper.** In Section 2 we will review necessary tools and results, as well as derive some new results which will be used in this paper. In particular we will discuss the Whittaker functional and matrix coefficient for highly ramified representations of GL_2, which is the key ingredient for the local integral of Triple product formula. In Section 3, we basically imitate Venkatesh's proof and get a power saving upper bound for the global integral in more general setting. We will use amplification method and reduce the problem to a bound for global matrix coefficient. In Section 4 we will prove Theorem 1.2, or Theorem 4.1. Despite its generality, its proof seems quite concise considering all possible combinations of cases. In Section 5 we will finish the proof of Theorem 1.4 by combining the results from the previous two sections. In the appendix we will prove the bound for the global matrix coefficient which is used in the proof in Section 3.

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2. Notations and preliminary results

2.1. **Basic Notations and facts.** Let $\mathbb{F}$ denote a number field. Let G be a reductive algebraic $\mathbb{F}-$group. In this paper we will focus on G being GL_2 or $\mathbb{D}^*$, where $\mathbb{D}$ is a quaternion algebra. Let $X = Z_G(\mathbb{A})G(\mathbb{F})\backslash G(\mathbb{A})$. Let $L^2(X)$ be the space of square integrable functions on X, and $< \cdot, \cdot >$ be the natural pairing on it given by

$$< f_1, f_2 >= \int_X f_1(g)\overline{f_2(g)}dg. \tag{2.1}$$

Any unitary cuspidal automorphic representation can be naturally embedded into $L^2(X)$ with the compatible unitary pairings.

Let $\mathbb{F}_v$ be the corresponding local field of $\mathbb{F}$ at a place v. Let K_v denote the standard maximal compact subgroup of $G(\mathbb{F}_v)$, and

$$K = \prod_v K_v. \tag{2.2}$$

When v is a finite place, let ϖ_v denote a uniformizer of $\mathbb{F}_v$ and O_v denote the ring of integers at v. Let $q^{-1} = |\varpi_v|_v$. At places where $G(\mathbb{F}_v) \simeq \mathrm{GL}_2(\mathbb{F}_v)$, define for an integer $c > 0$:

$$K_1(\varpi_v^c) = \{k \in K_v | \ k \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod (\varpi_v^c)\}. \tag{2.3}$$

We will call a function spherical if it is right invariant under K_v. For unramified representations, there is a unique up to constant spherical element. We will call a local representation to be of level c if there exists a unique up to constant element which is $K_1(\varpi_v^c)-$invariant. Such an element is then called a new form.

Now we record some basic facts about integrals on $\mathrm{GL}_2(\mathbb{F}_v)$ when v is finite.

Lemma 2.1. *For every positive integer c,*

$$GL_2(\mathbb{F}_v) = \coprod_{0\leq i\leq c} B \begin{pmatrix} 1 & 0 \\ \varpi_v^i & 1 \end{pmatrix} K_1(\varpi_v^c).$$

Here B is the Borel subgroup of GL_2.

We normalize the Haar measure on $\mathrm{GL}_2(\mathbb{F}_v)$ such that K_v has volume 1. Then we have the following easy result (see, for example, [9, Appendix A]).

Lemma 2.2. *Locally let f be a $K_1(\varpi_v^c)-$invariant function, on which the center acts trivially. Then*

$$\int_{F_v^*\backslash GL_2(\mathbb{F}_v)} f(g)dg = \sum_{0\leq i\leq c} A_i \int_{\mathbb{F}_v^*\backslash B(\mathbb{F}_v)} f(b\begin{pmatrix} 1 & 0 \\ \varpi_v^i & 1 \end{pmatrix})db. \tag{2.4}$$

Here db is the left Haar measure on $\mathbb{F}_v^*\backslash B(\mathbb{F}_v)$, and

$$A_0 = \frac{q}{q+1}, \quad A_c = \frac{1}{(q+1)q^{c-1}}, \quad \textit{and } A_i = \frac{q-1}{(q+1)q^i} \textit{ for } 0 < i < c.$$

2.2. **Integral representation of special values of Triple product $L-$function.** The story begins with Prasad's thesis work. For the triple product L-function $L(\Pi, s)$, there exist local epsilon factors $\epsilon_v(\Pi_v, \psi_v, s)$ and global epsilon factor $\epsilon(\Pi, s) = \prod_v \epsilon(\Pi_v, \psi_v, s)$, such that,

$$L(\Pi, 1-s) = \epsilon(\Pi, s)L(\check{\Pi}, s). \tag{2.5}$$

With the assumption that $\prod_i w_{\pi_i} = 1$, we have

$$\Pi \cong \check{\Pi}.$$

The special values of local epsilon factors $\epsilon_v(\Pi_v, \psi_v, 1/2)$ are actually independent of ψ_v and always take value ± 1. For simplicity, we will write

$$\epsilon_v(\Pi_v, 1/2) = \epsilon_v(\Pi_v, \psi_v, 1/2).$$

For any place v, there is a unique (up to isomorphism) division algebra $\mathbb{D}_v$. Then Prasad proved in [20] the following theorem about the dimension of the space of local trilinear forms:

Theorem 2.3. (1) $\dim \mathrm{Hom}_{GL_2(\mathbb{F}_v)}(\Pi_v, \mathbb{C}) \leq 1$, *with the equality if and only if* $\epsilon_v(\Pi_v, 1/2) = 1$.

(2) $\dim \mathrm{Hom}_{\mathbb{D}_v}(\Pi_v^{\mathbb{D}_v}, \mathbb{C}) \leq 1$, *with the equality if and only if* $\epsilon_v(\Pi_v, 1/2) = -1$.
Here $\Pi_v^{\mathbb{D}_v}$ *is the image of* Π_v *under Jacquet-Langlands correspondence.*

This motivated the following result which is conjectured by Jacquet and later on proved by Harris and Kudla in [7] and [8]:

Theorem 2.4.

$$\{L(\Pi, 1/2) \neq 0\} \Longleftrightarrow \left\{ \begin{array}{c} \text{there exist } \mathbb{D} \text{ and } f_i \in \pi_i^{\mathbb{D}} \text{ s.t.} \\ \int\limits_{Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A})} f_1(g)f_2(g)f_3(g)dg \neq 0 \end{array} \right\}$$

This result hints that

$$\int\limits_{Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A})} f_1(g)f_2(g)f_3(g)dg$$

could be a potential integral representation of special value of triple product L-function. Later on there are a lot of work on explicitly relating both sides. In particular one can see Ichino's work in [11]. We only need a special version here (as in the introduction).

$$|\int\limits_{Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A})} f_1(g)f_2(g)f_3(g)dg|^2 = \frac{\zeta_{\mathbb{F}}^2(2)L(\Pi, 1/2)}{8L(\Pi, Ad, 1)} \prod_v I_v^0(f_{1,v}, f_{2,v}, f_{3,v}), \tag{2.6}$$

where

$$I_v^0(f_{1,v}, f_{2,v}, f_{3,v}) = \frac{L_v(\Pi_v, Ad, 1)}{\zeta_v^2(2)L_v(\Pi_v, 1/2)} I_v(f_{1,v}, f_{2,v}, f_{3,v}), \tag{2.7}$$

and

$$I_v(f_{1,v}, f_{2,v}, f_{3,v}) = \int\limits_{\mathbb{F}_v^*\backslash\mathbb{D}^*(\mathbb{F}_v)} \prod_{i=1}^{3} < \pi_i^{\mathbb{D}}(g)f_{i,v}, f_{i,v} > dg. \tag{2.8}$$

2.3. **Hecke operators.** For most of this subsection we refer to [22]. Let f be a function on a group G and σ a compactly supported measure on G. Define the convolution of f with σ by

$$f * \sigma(x) = \int\limits_g f(xg)d\sigma(g). \tag{2.9}$$

If σ_1 and σ_2 are two compactly supported measures on G, we define the convolution $\sigma_1 * \sigma_2$ to be the pushforward to G of $\sigma_1 \times \sigma_2$ on $G \times G$, under the multiplication map

$$(g_1, g_2) \in G \times G \mapsto g_1 g_2. \tag{2.10}$$

Then one has the following compatibility relation

$$(f * \sigma_2) * \sigma_1 = f * (\sigma_1 * \sigma_2). \tag{2.11}$$

Now we introduce the Hecke operators in this language. At a non-archimedean place v, let $\mathfrak{l}$ be the maximal prime ideal and $r \geq 0$ be an integer. Define the measure $\mu_{\mathfrak{l}^r}^*$ on $\mathrm{GL}_2(\mathbb{F}_v)$ to be the restriction of Haar measure to the set

$$K_v \begin{pmatrix} \varpi_v^r & 0 \\ 0 & 1 \end{pmatrix} K_v,$$

so that the total mass of $\mu^*_{\mathfrak{l}^r}$ is $\begin{cases}(q+1)q^{r-1}, & \text{if } r \geq 1;\\ 1, & \text{if } r = 0.\end{cases}$

Define

$$\mu_{\mathfrak{l}^r} = \frac{1}{q^{r/2}} \sum_{0\leq k\leq r/2} \mu^*_{\mathfrak{l}^{r-2k}}. \tag{2.12}$$

Via the natural inclusion of $\mathrm{GL}_2(\mathbb{F}_v)$ in $\mathrm{GL}_2(\mathbb{A}_f)$ where $\mathbb{A}_f$ is the ring of finite adeles, we can regard $\mu_{\mathfrak{l}^r}$ as a compactly supported measure on $\mathrm{GL}_2(\mathbb{A}_f)$. If $\mathfrak{n}$ is an integral ideal $\prod_v \mathfrak{l}_v^{r_v}$, define

$$\mu_{\mathfrak{n}} = \prod_v \mu_{\mathfrak{l}_v^{r_v}}. \tag{2.13}$$

Convolution by $\mu_{\mathfrak{n}}$ can be thought of as $\mathfrak{n}-$th Hecke operator.

For functions on which the center acts trivially, convolution with $\mu_{\mathfrak{n}}$ is a self-dual operator, that is,

$$\int_X f_1 \cdot (f_2 * \mu_{\mathfrak{n}}) dg = \int_X (f_1 * \mu_{\mathfrak{n}}) \cdot f_2 dg. \tag{2.14}$$

One can also check for such functions that

$$\int_{G(\mathbb{A})} f(xg) d\mu_{\mathfrak{n}}(g) = \int_{G(\mathbb{A})} f(xg^{-1}) d\mu_{\mathfrak{n}}(g). \tag{2.15}$$

We have the following nice lemma about compositions of Hecke operators:

Lemma 2.5. *Let $\mathfrak{n}$, $\mathfrak{m}$ be ideals. Let h be a function on $G(\mathbb{A})$ that is spherical at all places $v|\mathfrak{n}\mathfrak{m}$, and the center acts on h trivially. Then*

$$\int_{G(\mathbb{A})} h(xg) d(\mu_{\mathfrak{n}} * \mu_{\mathfrak{m}})(g) = \sum_{\mathfrak{d}|(\mathfrak{n},\mathfrak{m})} \int_{G(\mathbb{A})} h(xg) d\mu_{\mathfrak{n}\mathfrak{m}\mathfrak{d}^{-2}}(g). \tag{2.16}$$

We will also need to consider functions on which the center acts by a fixed non-trivial unitary character w. From now on we will focus only on operators of form $\mu_{\mathfrak{l}}$ or $\mu_{\mathfrak{l}^2}$. Let $\check{\mu}_{\mathfrak{l}}$ be the dual of $\mu_{\mathfrak{l}}$ in the sense of (2.14). Then one can easily check that

$$\check{\mu}_{\mathfrak{l}} = w(\varpi_v^{-1})\mu_{\mathfrak{l}} = \frac{w(\varpi_v^{-1})}{\sqrt{q}} \mu^*_{\mathfrak{l}}. \tag{2.17}$$

Similarly let $\check{\mu}_{\mathfrak{l}^2}$ be the dual of $\mu_{\mathfrak{l}^2}$. Then

$$\check{\mu}_{\mathfrak{l}^2} = \frac{1}{q}(w(\varpi_v^{-2})\mu^*_{\mathfrak{l}^2} + \mu^*_{\mathfrak{o}}). \tag{2.18}$$

When acting on spherical functions, $\mu^*_{\mathfrak{l}}$ and $\mu^*_{\mathfrak{l}^2}$ are related as follows:

$$\mu^*_{\mathfrak{l}} * \mu^*_{\mathfrak{l}} = \mu^*_{\mathfrak{l}^2} + (q+1)w(\varpi_v)\mu^*_{\mathfrak{o}}. \tag{2.19}$$

Given a spherical function, let $\check{\lambda}_{\mathfrak{l}}$ and $\check{\lambda}_{\mathfrak{l}^2}$ be the eigenvalues of $\check{\mu}_{\mathfrak{l}}$ and $\check{\mu}_{\mathfrak{l}^2}$ acting on it. Putting (2.17), (2.18) and (2.19) together, we have

$$\check{\lambda}_{\mathfrak{l}^2} = \check{\lambda}_{\mathfrak{l}}^2 + (q^{-1} - \frac{q+1}{qw(\varpi_v)}). \tag{2.20}$$

Note that

$$|q^{-1} - \frac{q+1}{qw(\varpi_v)}| \geq 1.$$

Then one can easily check that,

Lemma 2.6.

$$|\check{\lambda}_{t2}| + |\check{\lambda}_{t1}| \geq 1.$$

2.4. **Bounds for matrix coefficient.** If π is an irreducible unitary cuspidal automorphic representation, then its local component at v is also a unitary representation. At a non-archimedean place, it can be classified into one of the following four types:

(1) supercuspidal representation;
(2) $\pi(\chi_1, \chi_2)$ where χ_i are unitary characters;
(3) special representation $\sigma(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$ where χ is unitary;
(4) $\pi(\chi|\cdot|^{\tau}, \chi|\cdot|^{-\tau})$, where χ is unitary and $0 < \tau < 1/2$.

The first three types are tempered representations. The generalized Ramanujan Conjecture implies that only tempered representations can be the local component of a unitary cuspidal automorphic representation. What is known is a bound α towards Ramanujan Conjecture. This means if type (4) ever happens, then $\tau < \alpha$. The smaller α is, the closer we are to the Ramanujan Conjecture for GL_2. For our purpose, any $\alpha < 1/4$ would be enough to get a subconvexity bound. The current record is $\alpha = 7/64$. See [14], [2].

Using the bound towards Ramanujan Conjecture, one can bound the matrix coefficient for the local component of a unitary cuspidal automorphic representation.

Locally for $f_1 \in \pi_v \cong \pi(\chi_1, \chi_2)$, $f_2 \in \check{\pi}_v \cong \pi(\chi_1^{-1}, \chi_2^{-1})$ in the standard model for the induced representations, we can define the pairing by

$$< f_1, f_2 >= \int_{K_v} f_1(k) f_2(k) dk. \tag{2.21}$$

We can define the matrix coefficient of π_v associated to f_1, f_2 as

$$\Phi(g) =< \pi_v(g) f_1, f_2 > . \tag{2.22}$$

See later subsections for the alternative definition and the definition when the representation is supercuspidal.

We first record here the matrix coefficient for spherical elements. (See for example, [3].) For simplicity, let χ_i denote $\chi_i(\varpi_v)$ in the following formula if we don't specify which element the characters are taking.

Lemma 2.7. *Let $\pi = \pi(\chi_1, \chi_2)$ be an unramified unitary representation of GL_2. Let Φ be the matrix coefficient associated to the spherical element in π such that $\Phi(1) = 1$. Then it's bi-K−invariant and*

$$\Phi(\begin{pmatrix} \varpi_v^n & 0 \\ 0 & 1 \end{pmatrix}) = \frac{q^{-n/2}}{1+q^{-1}} \frac{\chi_1^n(\chi_1 - \chi_2 q^{-1}) - \chi_2^n(\chi_2 - \chi_1 q^{-1})}{\chi_1 - \chi_2}. \tag{2.23}$$

Now we state the result for the bound of local matrix coefficient for general elements.

Lemma 2.8. *Let π_v be the local component of a unitary cuspidal automorhpic representation of GL_2 at a finite place v. Let f_1, f_2 be two K_v−finite elements in π_v, stabilized respectively by compact open subgroups $K_{1,v}$ and $K_{2,v}$. Then for any $x \in \mathbb{F}_v$ and $\epsilon > 0$,*

$$|<\pi(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix})f_1, f_2>| \ll_{\epsilon,\mathbb{F}} [K_v : K_{1,v}]^{1/2}[K_v : K_{2,v}]^{1/2} q^{(\alpha-1/2+\epsilon)|v(x)|}\|f_1\|_v\|f_2\|_v. \tag{2.24}$$

Proof. It follows from, for example, Lemma 9.1 of [22]. Here we briefly describe how to prove this result for induced representations at non-archimedean places. For spherical elements, one can use Lemma 2.7 above to check the inequality directly. More specifically if $|v(x)| = n$ and f_i's are spherical and normalized, then

(2.25)

$$\begin{aligned} |<\pi(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix})f_1, f_2>| &= |\frac{q^{-n/2}}{1+q^{-1}} \frac{\chi_1^n(\chi_1 - \chi_2 q^{-1}) - \chi_2^n(\chi_2 - \chi_1 q^{-1})}{\chi_1 - \chi_2}| \\ &= |\frac{q^{-n/2}}{1+q^{-1}}((\chi_1^n + \chi_1^{n-1}\chi_2 + \cdots + \chi_2^n) - q^{-1}\chi_1\chi_2(\chi_1^{n-2} + \chi_1^{n-3}\chi_2 + \cdots + \chi_2^{n-2}))| \\ &\leq (n+1)q^{(\alpha-1/2)n}. \end{aligned}$$

The coefficient $(n+1)$ will be essentially bounded by $q^{\epsilon n}$ for any $\epsilon > 0$, and the implicit constant can be taken to be 1 when q is large enough. When f_1, f_2 are not spherical, one can use the trick as in [5] to reduce the inequality to the spherical case. □

Remark 2.9. This proof actually allow one to control the implicit constant. In particular one can take a product of the local inequalities and get a global inequality.

Now we give a bound for the global matrix coefficient. Let $\mathbb{D}$ be a global quaternion algebra. Let ρ denote the right regular representation of $\mathbb{D}^*(\mathbb{A})$ on $L^2(Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A}))$. Let F_1, $F_2 \in L^2(Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A}))$ be two rapidly decreasing and K−finite automorphic forms which don't have 1-dim components in their spectrum decompositions. Let S be a finite set of non-archimedean places. We assume that $\mathbb{D}$ is locally the matrix algebra at the places in S. Let $K_S = \prod_{v\in S} K_v$ and $K_{i,S} = \prod_{v\in S} K_{i,v}$, where $K_{i,v}$ stabilizes the local component of F_i at v. Let $\mathcal{N} = \prod_v \varpi_v^{e_v}$ for $e_v \geq 0$, and $N = \mathrm{Nm}(\mathcal{N})$. Define the matrix

$$a([\mathcal{N}]) = \prod_v \begin{pmatrix} \varpi^{-e_v} & 0 \\ 0 & 1 \end{pmatrix},$$

which can be naturally thought of as an element of $\mathbb{D}^*(\mathbb{A})$.

Proposition 2.10. *With the setting as above, we have*

(2.26)

$$|\int\limits_{Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A})} F_1(g)\rho(a([\mathcal{N}]))F_2(g)dg| \ll_{\epsilon,\mathbb{F}} [K_S : K_{1,S}]^{1/2}[K_S : K_{2,S}]^{1/2} N^{\alpha-1/2+\epsilon}\|F_1\|_{L^2}\|F_2\|_{L^2}.$$

We will prove this proposition in the appendix. Now the question is, for any given automorphic form $F \in L^2(Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A}))$, how can we separate out the 1-dimensional components. Suppose that in general the center acts on F by a unitary central character w. Then its 1-dimensional components can be given by the following projection:

$$F \mapsto \mathcal{P}F(x) = \sum_{\chi^2=w} \chi(x) \int_X f(y)\overline{\chi(y)}dy, \tag{2.27}$$

where χ is a unitary Hecke character, and $\chi(x) = \chi(\det x)$. Then the remaining part $F - \mathcal{P}F$ doesn't have any 1-dimensional components.

2.5. **Whittaker model for induced representations.** Here we recall some basic results about the Whittaker model for induced representations at a finite place. This and the next subsection are purely local, so we will suppress the subscript v for all notations.

Fix an additive character ψ. Without loss of generality, we will always assume ψ is unramified. Let π be a local irreducible (generic) representation of G. Then there is a unique realization of π in the space of functions W on G such that

$$W(\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}g) = \psi(n)W(g). \tag{2.28}$$

Locally for an induced representation of GL_2, one can compute its Whittaker functional by the following formula:

$$W(g) = \int\limits_{m\in\mathbb{F}} \varphi(\omega\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}g)\psi(-m)dm, \tag{2.29}$$

where φ is an element of π in the model of induced representation and ω is the matrix $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

When π is unitary, one can define a unitary pairing on π using the Whittaker model:

$$< W_1, W_2 >= \int_{\mathbb{F}^*} W_1(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix})\overline{W_2(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix})}d^*\alpha. \tag{2.30}$$

To get the Whittaker functional explicitly using (2.29), the first step is to write

$$\omega\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix} = \begin{pmatrix}\varpi^i & 1\\ -\alpha - m\varpi^i & -m\end{pmatrix}$$

in form of $B(\mathbb{F})\begin{pmatrix}1 & 0\\ \varpi^k & 1\end{pmatrix}K_1(\varpi^c)$ for $0 \le i, k \le c$. Note that if $k = c$, then $\begin{pmatrix}1 & 0\\ \varpi^k & 1\end{pmatrix}$ is absorbed into $K_1(\varpi^c)$. Same for i.

We record the following results from [9].

Lemma 2.11. (1) *Suppose $k = 0$.*

(1i) *If $i = 0$, we need $m \notin \alpha(-1 + \varpi O_F)$ for $\begin{pmatrix}\varpi^i & 1\\ -\alpha - m\varpi^i & -m\end{pmatrix} \in B\begin{pmatrix}1 & 0\\ \varpi^k & 1\end{pmatrix}K_1(\varpi^c)$;*

(1ii) *If $i > 0$, we need $v(m) \ge v(\alpha)$.*

Under above conditions we can write $\begin{pmatrix}\varpi^i & 1\\ -\alpha - m\varpi^i & -m\end{pmatrix}$ as

$$\begin{pmatrix}-\frac{\alpha}{\alpha+m\varpi^i} & \varpi^i + \frac{\alpha}{\alpha+m\varpi^i}\\ 0 & -\alpha - m\varpi^i\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1 + \frac{m}{\alpha+m\varpi^i}\\ 0 & 1\end{pmatrix}.$$

(2) *Suppose $k = c$.*

(2i) *If $i < c$, we need $m \in \alpha\varpi^{-i}(-1 + \varpi^{c-i}O_F)$;*

(2ii) *If $i = c$, we need $v(m) \le v(\alpha) - c$.*

Under above conditions, we can write $\begin{pmatrix}\varpi^i & 1\\ -\alpha - m\varpi^i & -m\end{pmatrix}$ *as*

$$\begin{pmatrix}-\frac{\alpha}{m} & 1\\ 0 & -m\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{\alpha}{m}+\varpi^i & 1\end{pmatrix}.$$

(3) *Suppose* $0 < k < c$.

(3i) *If* $i < k$, *we need* $m \in \alpha\varpi^{-i}(-1+\varpi^{k-i}O_F^*)$;

(3ii) *If* $i > k$, *we need* $v(m) = v(\alpha) - k$;

(3iii) *If* $i = k$, *we need* $v(m) \le v(\alpha) - k$ *but* $m \notin \alpha\varpi^{-k}(-1+\varpi O_F)$.

Under above conditions we can write $\begin{pmatrix}\varpi^i & 1\\ -\alpha - m\varpi^i & -m\end{pmatrix}$ *as*

$$\begin{pmatrix}-\frac{\alpha\varpi^k}{\alpha+m\varpi^i} & 1\\ 0 & -m\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^k & 1\end{pmatrix}\begin{pmatrix}\frac{\alpha+m\varpi^i}{m\varpi^k} & 0\\ 0 & 1\end{pmatrix}.$$

Now let π be a unitary induced representation $\pi(\mu_1,\mu_2)$, where μ_1 and μ_2 are both ramified of level k_1 and k_2. Let $c = k_1 + k_2$ be the level of π. Then by the classical results in [4], there exists a new form in the model of induced representation, which is right $K_1(\varpi^c)-$invariant and supported on

$$B\begin{pmatrix}1 & 0\\ \varpi^{k_2} & 1\end{pmatrix}K_1(\varpi^c),$$

where B is the Borel subgroup.

We shall consider the Whittaker function W associated to this new form. Let

$$C = \int_{u\in O_F^*} \mu_1(-\varpi^{k_2})\mu_2(-\varpi^{-k_2}u)\psi(-\varpi^{-k_2}u)du. \tag{2.31}$$

Denote

$$W^{(i)}(\alpha) = W(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}). \tag{2.32}$$

Then the next lemma follows from (2.29) and (3) of the above lemma.

Lemma 2.12. (i) *If* $i < k_2$, *then*

$$W^{(i)}(\alpha) = C^{-1}\int_{u\in O_F^*} \mu_1(-\frac{\varpi^i}{u})\mu_2(\alpha\varpi^{-i}(1-\varpi^{k_2-i}u))\psi(\alpha\varpi^{-i}(1-\varpi^{k_2-i}u))q^{i-k_2-v(\alpha)/2}du. \tag{2.33}$$

Its integral in α *for fixed* $v(\alpha)$ *is always 0.*

(ii) *If* $k_2 < i \le c$, *then*

$$W^{(i)}(\alpha) = C^{-1}\int_{u\in O_F^*} \mu_1(-\frac{\varpi^{k_2}}{1+u\varpi^{i-k_2}})\mu_2(-\varpi^{-k_2}\alpha u)q^{-v(\alpha)/2}\psi(-\varpi^{-k_2}\alpha u)du. \tag{2.34}$$

Its integral in α *is zero unless* $i = c$ *or* $c-1$. *In particular*

$$W^{(c)}(\alpha) = \begin{cases}1, & \text{if } v(\alpha) = 0;\\ 0, & \text{otherwise.}\end{cases}. \tag{2.35}$$

When $i < c$,

$$\int\limits_{v(\alpha)\ fixed} W^{(i)}(\alpha)d^*\alpha = \begin{cases} -\frac{1}{q-1}, & \text{if } i = c-1 \text{ and } v(\alpha) = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{2.36}$$

(iii) *If $i = k_2$,*

$$W^{(k_2)}(\alpha) = C^{-1} \int\limits_{v(u)\leq -k_2, u\notin \varpi^{-k_2}(-1+\varpi O_F)} \mu_1(-\frac{\varpi^{k_2}}{1+u\varpi^{k_2}})\mu_2(-\alpha u)|\frac{\varpi^{k_2}}{\alpha u(1+u\varpi^{k_2})}|^{1/2}\psi(-\alpha u)q^{-v(\alpha)}du. \tag{2.37}$$

The integral of $W^{(k_2)}$ in α for fixed $v(\alpha)$ is always zero if either $k_1 > 1$ or $v(\alpha) \neq 0$. When $k_1 = 1 = c - i$ and $v(\alpha) = 0$, its integral against 1 is the same as expected from (2.36).

We shall also consider the case when $\pi \cong \pi(\mu_1, \mu_2)$, where μ_1 is unramified and μ_2 is ramified of level k. Then the level of π is k. In this case the new form is right $K_1(\varpi^k)-$invariant and supported on $BK_1(\varpi^k)$. Then by (2) of Lemma 2.11, we have

Lemma 2.13. (1) *When $i = k$,*

$$\begin{aligned} W^{(k)}(\alpha) &= \int\limits_{v(m)\leq v(\alpha)-k} \mu_1(-\frac{\alpha}{m})\mu_2(-m)\psi(-m)q^{-\frac{1}{2}v(\alpha)+v(m)}dm \\ &= \begin{cases} q^{-\frac{1}{2}v(\alpha)-k}\mu_1^{v(\alpha)+k}(\varpi) \int\limits_{v(m)=-k} \mu_2(-m)\psi(-m)dm, & \text{if } v(\alpha)\geq 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.38}$$

(2) *When $i < k$,*

$$W^{(i)}(\alpha) = \mu_1^i(\varpi) \int\limits_{u\in O_{\mathbb{F}}} \mu_2(\alpha\varpi^{-i}(1-\varpi^{k-i}u))\psi(\alpha\varpi^{-i}(1-\varpi^{k-i}u))q^{-\frac{1}{2}v(\alpha)-k+i}du. \tag{2.39}$$

Remark 2.14. In this lemma, the Whittaker functional is not normalized. But this turns out to be enough.

2.6. **Kirillov model for supercuspidal representations.** Now let's consider supercuspidal representations. For the fixed additive character ψ, the Kirillov model of π is a unique realization on the space of Schwartz functions $S(\mathbb{F}^*)$ such that

$$\pi(\begin{pmatrix} a_1 & m \\ 0 & a_2 \end{pmatrix})\varphi(x) = w_\pi(a_2)\psi(ma_2^{-1}x)\varphi(a_1a_2^{-1}x), \tag{2.40}$$

where w_π is the central character for π. Let W_φ be the Whittaker function associated to φ. Then they are related by

$$\varphi(\alpha) = W_\varphi(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}),$$

$$W_\varphi(g) = \pi(g)\varphi(1).$$

When π is unitary, one can define the $G-$invariant unitary pairing on Kirillov model by

$$< f_1, f_2 >= \int\limits_{\mathbb{F}^*} f_1(x)\overline{f_2}(x)d^*x. \tag{2.41}$$

By Bruhat decomposition, one just has to know the action of $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to understand the whole group action.

Define

$$\mathbf{1}_{\nu,n}(x) = \begin{cases} \nu(u), & \text{if } x = u\varpi^n \text{ for } u \in O_F^*; \\ 0, & \text{otherwise.} \end{cases}$$

Roughly speaking, it's the character ν supported at $v(x) = n$. We can then describe the action of $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbf{1}_{\nu,n}$ explicitly according to [13]:

$$\pi(\omega)\mathbf{1}_{\nu,n} = C_{\nu w_0^{-1}} z_0^{-n} \mathbf{1}_{\nu^{-1}w_0, -n+n_{\nu^{-1}}}. \tag{2.42}$$

Here $z_0 = w(\varpi)$ and $w_0 = w_\pi|_{O_F^*}$. n_ν is an integer decided by the representation π and the character ν (and independent of n). It's well-known that $n_\nu \leq -2$ for any ν. When we pick ν to be the trivial character, the number $-n_1$ is actually the level of this supercuspidal representation. Denote $c = -n_1$. The local new form is simply $\mathbf{1}_{1,0}$.

The relation $\omega^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ implies

$$n_\nu = n_{\nu^{-1}w_0^{-1}}, \quad C_\nu C_{\nu^{-1}w_0^{-1}} = w_0(-1) z_0^{n_\nu}. \tag{2.43}$$

It is essentially proved in [9, Proposition B.3] that

Proposition 2.15. *Suppose that $c = -n_1 \geq 2$ is the level of a supercuspidal representation π whose central character is unramified or level 1. If $p \neq 2$ and ν is a level i character, then we have*

$$n_\nu = \min\{-c, -2i\}.$$

When $p = 2$ or the central character of π is highly ramified, we have the same statement, except when $c \geq 4$ is an even integer and $i = c/2$. In that case, we only claim $n_\nu \geq -c$.

Remark 2.16. In [9, Proposition B.3], we didn't prove the result for highly ramified central characters. But one can use the same method. If we denote the level of the central character by $c(w_\pi)$, the key observation is that

$$c(w_\pi) \leq c/2.$$

Definition 2.17. We will say a function $f(x)$ consists of level i components, or simply of level i, if we can write

$$f(x) = \sum_{\nu,n} a_{\nu,n} \mathbf{1}_{\nu,n},$$

where $a_{\nu,n} \neq 0$ only if ν is of level i. Equivalently,

$$\int_{v(x)=n} f(x)\nu(x) d^*x \neq 0$$

only if ν is a level i character.

Let ψ be an unramified additive character. By abuse of notation, we will say $\psi(\varpi^{-i}x)$ is of level i at $v(x) = 0$. But it should be understood that when $i = 1$, $\psi(\varpi^{-1}x)$ at $v(x) = 0$ actually consists of level 1 and level 0 components.

As a direct corollary to the above proposition, we have the following result about the Whittaker functional for supercuspidal representations:

Corollary 2.18. (1) $W^{(c)}(\alpha) = \mathbf{1}_{1,0}$.

(2) *In general for* $0 \le i < c$, $W^{(i)}(\alpha)$ *is supported only at* $v(\alpha) = \min\{0, 2i - c\}$*, consisting of level* $c - i$ *components and also level 0 components when* $i = c - 1$.

(3) *The exception happens when* $p = 2$ *or the central character is highly ramified, and* $c \ge 4$ *is an even number and* $i = c/2$*. In that case,* $W^{(c/2)}$ *is supported at* $v(\alpha) \ge 0$*, consisting of level* $c/2$ *components.*

Let's see how the results above can be applied to the matrix coefficient of a supercuspidal representation in general. Let

$$\Phi(g) =< \pi(g)F, F >= \int_{\mathbb{F}^*} \pi(g)F(x)\overline{F(x)}d^*x,$$

where $F = \mathbf{1}_{1,k} \in S(\mathbb{F}^*)$ for an integer $k \ge 0$, and π is supercuspidal of level c. One can easily check that F is $K_1(\varpi^{c+k})-$invariant, so $\Phi(g)$ is bi-$K_1(\varpi^{c+k})-$invariant. But we will only make use of the right $K_1(\varpi^{c+k})-$invariance now.

By Lemma 2.1, to understand $\Phi(g)$, it will be enough to understand $\Phi(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ for $0 \le i \le c + k$.

Proposition 2.19. (i) *For* $c + k - 1 \le i \le c + k$, $\Phi(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ *is supported on* $v(a) = 0$ *and* $v(m) \ge -k - 1$*. On the support, we have*

$$\Phi(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}) = \begin{cases} 1, & \text{if } v(m) \ge -k \text{ and } i = c + k; \\ -\frac{1}{q-1}, & \text{if } v(m) = -k - 1 \text{ and } i = c + k; \\ -\frac{1}{q-1}, & \text{if } v(m) \ge -k \text{ and } i = c + k - 1. \end{cases} \tag{2.44}$$

When $v(m) = -k - 1$ *and* $i = c + k - 1$, $\Phi(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c+k-1} & 1 \end{pmatrix})$ *consists of level 1 and 0 components as a function in a (or m), and*

$$\int_{v(m)=-k-1} \Phi(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c+k-1} & 1 \end{pmatrix})dm = \frac{1}{q-1}q^k. \tag{2.45}$$

(ii) *For* $0 \le i < c+k-1$, $i \ne c/2+k$, $\Phi(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ *is supported on* $v(a) = \min\{0, 2i-c-2k\}$, $v(m) = i - c - 2k$*. It is of level* $c + k - i$ *as a function in a.*

(iii) *When* $i = c/2 + k$*, the conclusion in (ii) still holds except when* $p = 2$ *or the central character is highly ramified, and* $c \ge 4$ *is an even number. In that case,* $\Phi(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ *is supported on* $v(a) \ge 0$, $v(m) = i - c - 2k = -c/2 - k$*. It is of level* $c/2$ *in a.*

Proof. By definition,

$$\Phi(g) = \int_{v(x)=k} \pi(g)F(x)d^*x. \tag{2.46}$$

To get a non-zero value for $\Phi(g)$, we just need a level 0 component supported at $v(x) = k$ for $\pi(g)F(x)$. We shall prove (ii) first. Let $0 \le i < c + k - 1$, $i \ne c/2 + k$. According to Proposition

2.15,

$$\pi(\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix})\mathbf{1}_{1,k}(x) = \pi(-\omega\begin{pmatrix}1 & -\varpi^i\\ 0 & 1\end{pmatrix}\omega)\mathbf{1}_{1,k}(x)$$

is supported at $v(x) = \min\{k, 2i - c - k\}$, being a linear combination of all level $c + k - i$ characters. To see this, assume for simplicity that the central character is trivial. Then

$$\pi(\omega)\mathbf{1}_{1,k}(x) = C_1\mathbf{1}_{1,-k-c}(x). \tag{2.47}$$

The action of $\begin{pmatrix}1 & -\varpi^i\\ 0 & 1\end{pmatrix}$ will then give a linear combination of all level $c+k-i$ characters supported at $v(x) = -c - k$, according to the classical results about Gauss sum. Then Proposition 2.15 will imply the claim above.

Now by definition,

$$\pi(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix})\mathbf{1}_{1,k}(x) = \psi(mx)\pi(\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix})\mathbf{1}_{1,k}(ax), \tag{2.48}$$

$$\Phi(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}) = \int\limits_{v(x)=k} \psi(mx)\pi(\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix})\mathbf{1}_{1,k}(ax)d^*x. \tag{2.49}$$

One can see that we need

$$v(a) = \min\{0, 2i - c - 2k\}$$

to change the support of $\pi(\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix})\mathbf{1}_{1,k}(ax)$ to $v(x) = k$.

When $0 \leq i < c + k - 1$, we need $\psi(mx)$ also to be of level $c + k - i$ at $v(x) = k$ to get level 0 components from the product. So it's supported at

$$v(m) = i - c - 2k.$$

It's clear now that $\Phi(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix})$ as a function of a or m is of level $c + k - i$. So (ii) is proved.

(iii) can be proved using the same method.

When one uses the same method for (i), there will be two differences which are worth noting. The first difference is that when $i = c + k - 1$, $\pi(\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix})\mathbf{1}_{1,k}(x)$ is a linear combination of level 1 and also level 0 components. The second difference is that $\psi(mx)$ has level 0 component at $v(x) = k$ when $v(m) \geq -k - 1$.

Now we will prove (2.45) and leave (2.44) to the readers, as the latter is actually much easier to check.

So suppose $i = c + k - 1$, $v(a) = 0$ and $v(m) = -k - 1$. Then $\pi(\begin{pmatrix}1 & 0\\ \varpi^{c+k-1} & 1\end{pmatrix})\mathbf{1}_{1,k}(ax)$ and $\psi(mx)$ will both be linear combinations of level 1 and level 0 characters.

$$\int_{v(m)=-k-1} \Phi(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c+k-1} & 1 \end{pmatrix})dm = \int_{v(m)=-k-1}\int_{v(x)=k} \psi(mx)\pi(\begin{pmatrix} 1 & 0 \\ \varpi^{c+k-1} & 1 \end{pmatrix})\mathbf{1}_{1,k}(ax)d^*xdm$$
$$= \int_{v(x)=k}\int_{v(m)=-k-1} \psi(mx)\pi(\begin{pmatrix} 1 & 0 \\ \varpi^{c+k-1} & 1 \end{pmatrix})\mathbf{1}_{1,k}(ax)dmd^*x$$
$$= -q^k \int_{v(x)=k} \pi(\begin{pmatrix} 1 & 0 \\ \varpi^{c+k-1} & 1 \end{pmatrix})\mathbf{1}_{1,k}(ax)d^*x$$

The last step is to see that the level 0 component of $\pi(\begin{pmatrix} 1 & 0 \\ \varpi^{c+k-1} & 1 \end{pmatrix})\mathbf{1}_{1,k}$ is $-\frac{1}{q-1}\mathbf{1}_{1,k}$. □

3. Upper bound for the global period integral

From now on we take $G = \mathbb{D}^*$ as decided in Theorem 2.4. Denote $X = Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A})$. Let $\pi_i, i = 1, 2, 3$ be three unitary automorphic cuspidal representations of GL_2. Let $\pi_i^{\mathbb{D}}$ be the image of π_i under Jacquet-Langlands correspondence. They are naturally embedded in $L^2(X)$. We will fix π_1 and π_2 and let π_3 have varying finite conductor, but with bounded components at infinity.

Definition 3.1. At a finite place v, let c_i denote the levels of π_i at v. Let

$$S = \{v \text{ finite } | c_3 \geq 2\max\{c_1, c_2\} \text{ at } v,\ c_3 \neq 0\}.$$

Let

$$\mathcal{N} = \prod_{v\in S} \varpi_v^{c_3-c_2}, \qquad N = \mathrm{Nm}(\mathcal{N}) = \prod_{v} |\varpi_v|^{-(c_3-c_2)}.$$

Remark 3.2. Note that we don't take $\mathcal{N}$ here to be exactly the conductor of π_3. But their difference is controlled by the conductors of π_1 and π_2 which are fixed. In particular this difference is negligible when we consider the asymptotic behavior.

We claim without proof here that for $v \in S$, the local epsilon factor $\epsilon_v(\Pi_v, 1/2) = 1$, so $\mathbb{D}$ is the matrix algebra at these places. (We will prove this claim in Corollary 5.2.) For this reason, the following definition makes sense:

Definition 3.3. For $\mathcal{N}$ defined as above, let

$$a_v([\mathcal{N}]) = \begin{cases} \begin{pmatrix} \varpi_v^{-(c_3-c_2)} & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } v \in S; \\ 1, & \text{otherwise.} \end{cases}$$

and

$$a([\mathcal{N}]) = \prod_v a_v([\mathcal{N}]).$$

$a([\mathcal{N}])$ can be thought of as an element of $\mathbb{D}^*(\mathbb{A})$.

Take cusp forms $f_i \in \pi_i^{\mathbb{D}}, i = 1, 2, 3$. We want to bound the global period integral

$$\mathbb{I}(f_1, \rho(a([\mathcal{N}]))f_2, f_3) = \int_X f_1(x)f_2(xa([\mathcal{N}]))f_3(x)dx. \tag{3.1}$$

Let's first specify our choices of local components for f_i's.

(i) At the places when all three representations are unramified, we will just choose local components to be spherical;
(ii) For places in S, we will always pick new forms for all local components;
(iii) For the remaining places, we will pick proper new forms or old forms to guarantee that the local integral $I_v^0 \geq \delta$ for some $\delta > 0$. In particular the local components of f_1 and f_2 can be chosen from a finite set of test vectors.
(iv) f_i's are globally normalized with respect to (2.1), and locally normalized with respect to (2.30).

Remark 3.4. Note the level of π_3 is controlled by the levels of π_1 and π_2 for places outside S. (iii) was essentially proven in Lemma 6.3 and Lemma 6.4 of [24]. We briefly describe why it's true here. If we fix the level of π_3, the parametrization of all possible representations with fixed central character is compact. Theorem 2.4 guarantees that the local integral I_v^0 is not zero with a proper choice of test vectors. So I_v^0 is bounded away from zero in an open neighborhood of the parametrization by continuity, and (iii) is true because of compactness.

Now we can state our result on the upper bound of global period integral:

Proposition 3.5. *Let $\pi_i, i = 1, 2, 3$ be three unitary automorphic cuspidal representations with π_1 and π_2 fixed. Let $f_i \in \pi_i, i = 1, 2, 3$ be cusp forms with local components specified as above. Then*

$$\mathbb{I}(f_1, \rho(a([\mathcal{N}]))f_2, f_3) = \int_X f_1(x) f_2(xa([\mathcal{N}])) f_3(x) dx \ll N^{-\delta}. \tag{3.2}$$

Here δ can be taken to be any positive number less than $-\frac{(\alpha-1/2)(2\alpha-1/2)}{4\alpha-3}$, where α is a bound towards Ramanujan Conjecture. In particular when we take $\alpha = 7/64$, $-\frac{(\alpha-1/2)(2\alpha-1/2)}{4\alpha-3} > \frac{1}{24}$.

Proof. We will basically generalize the proof in [22]. First we specify a signed measure σ on $G(\mathbb{A})$ we are going to use. We will take $\sigma = \sum_{\mathfrak{n}} a_{\mathfrak{n}} \mu_{\mathfrak{n}}$, where $\mu_{\mathfrak{n}}$ is the measure associated to $\mathfrak{n}-$th Hecke operator as defined in Section 2.3. We will choose the sequence of complex numbers $a_{\mathfrak{n}}$ as follows:

Let b be a fixed small positive real number to be chosen. For every finite place v, let $\mathfrak{l}$ be the maximal prime ideal there. Let T be the set of prime ideals $\mathfrak{l}$ such that $\mathrm{Nm}(\mathfrak{l}) \in [N^b, 2N^b]$ and π_i's are locally unramified. In particular by the choice of local component (i), f_i's are spherical at these places. As f_1 and f_2 have fixed conductor, and primes involved in the conductor of π_3 are asymptotically less than N^ϵ for any $\epsilon > 0$ as $N \to \infty$, T will essentially contain all such primes with norm in $[N^b, 2N^b]$. More specifically by the distribution of primes, we have

$$N^{b-\epsilon} \ll |T| \ll N^{b+\epsilon}.$$

For $z \in \mathbb{C}$ we put $\mathrm{sign}(z) = z/|z|$ for $z \neq 0$ and $\mathrm{sign}(0) = 1$. Put

$$a_{\mathfrak{n}} = \begin{cases} \overline{\mathrm{sign}(\check{\lambda}_{\mathfrak{n}})}, & \text{if } \mathfrak{n} \in T \text{ or } \mathfrak{n} = \mathfrak{l}^2, \mathfrak{l} \in T; \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Here $\check{\lambda}_{\mathfrak{n}}$ is the eigenvalue of the Hecke operator $\check{\mu}_{\mathfrak{n}}$ acting on f_3, as the local component of f_3 at this place is spherical. Then by the definition above, one can easily verify the following inequalities, which we will make use of later:

$$|\sum_{\mathfrak{n}} a_{\mathfrak{n}} \check{\lambda}_{\mathfrak{n}}| \gg_{\epsilon, \mathbb{F}} N^{b-\epsilon}. \tag{3.4}$$

$$\sum_{\mathfrak{n}} \mathrm{Nm}(\mathfrak{n})^{1/2+\epsilon} |a_{\mathfrak{n}}| \ll_\epsilon N^{2b+\epsilon}. \tag{3.5}$$

$$(3.6)\qquad \sum_{\mathfrak{n},\mathfrak{m}}\sum_{\mathfrak{d}|(\mathfrak{n},\mathfrak{m})}(\mathrm{Nm}(\frac{\mathfrak{n}\mathfrak{m}}{\mathfrak{d}^2}))^{2\alpha-1/2}|a_{\mathfrak{n}}||a_{\mathfrak{m}}| \ll N^{(4\alpha+1)b}$$

The first equality follows from Lemma 2.6. The second and the third inequalities are more direct to check. α in the last inequality is a bound towards Ramanujan conjecture, and we need the fact that one can take $\alpha < 1/4$.

Now for the measure σ defined as above, we have $f_3 * \check{\sigma} = \lambda f_3$, where

$$(3.7)\qquad \lambda = \sum_{\mathfrak{n}} a_{\mathfrak{n}}\check{\lambda}_{\mathfrak{n}}.$$

Let $\Psi(x) = f_1(x)f_2(xa([\mathcal{N}])) \in C^\infty(X)$. Then

$$\begin{aligned}(3.8)\qquad \lambda\mathbb{I} &= \int_X \Psi(x)(f_3 * \check{\sigma})(x)dx = \int_X (\Psi * \sigma)(x)f_3(x)dx \leq (\int_X |\Psi * \sigma|^2 dx)^{1/2}\\
&= (\int_X\int_{g,g'\in G(\mathbb{A})}(\rho(g)\Psi)\overline{(\rho(g')\Psi)}d\sigma(g)d\sigma(g')dx)^{1/2}\\
&= (\int_X\int_{g,g'\in G(\mathbb{A})} f_1(xg)f_2(xa([\mathcal{N}])g)\overline{f_1(xg')f_2(xa([\mathcal{N}])g')}d\sigma(g)d\sigma(g')dx)^{1/2}\\
&= (\int_X\int_{g,g'\in G(\mathbb{A})} f_1(xg)f_2(xga([\mathcal{N}]))\overline{f_1(xg')f_2(xg'a([\mathcal{N}]))}d\sigma(g)d\sigma(g')dx)^{1/2}.\end{aligned}$$

Note that the support of σ commmutes with $a([\mathcal{N}])$ according to our choice of σ. So the last equality above is valid. Now we want to change the order of the integral, separate the constant part and use Proposition 2.10 to bound the difference. In particular, let $h_i(x) = f_i(xg)\overline{f_i(xg')}$, $i = 1, 2$, so the center acts trivially on $h_i(x)$. So we have

$$\begin{aligned}(3.9)\qquad &|\int_X h_1(x)h_2(xa(\mathcal{N}))dx - \sum_{\chi^2=1}\chi(\mathcal{N})\int_X h_1(x)\chi(x)dx\int_X h_2(x)\chi(x)dx|\\
&= | < h_1, \rho(a(\mathcal{N}))h_2 > - < \mathcal{P}h_1, \rho(a(\mathcal{N}))\mathcal{P}h_2 > |\\
&= | < h_1 - \mathcal{P}h_1, \rho(a(\mathcal{N}))(h_2 - \mathcal{P}h_2) >\\
&\ll N^{\alpha-1/2+\epsilon}||h_1||_{L^2}||h_2||_{L^2}\\
&\ll N^{\alpha-1/2+\epsilon}\end{aligned}$$

The implicit constant depends on the compact open subgroups that stabilize f_1 and f_2 at places in S, and is thus bounded. In the last inequality we have used $||h_i||_{L^2} \leq ||f_i||^2_{L^4}$, which is finite and bounded because f_i's are normalized cusp forms chosen from a finite fixed collection for $i = 1, 2$.

Combining (3.8) and (3.9), we have

(3.10)

$$|\lambda\mathbb{I}|^2 \ll N^{\alpha-1/2+\epsilon}||\sigma||^2 + \sum_{\chi^2=1}\int_{g,g'} | < \rho(g^{-1}g')f_1, f_1\otimes\chi >< \rho(g^{-1}g')f_2, f_2\otimes\chi > |d|\sigma|(g)d|\sigma|(g'),$$

where $|\sigma| = \sum_{\mathfrak{n}}|a_{\mathfrak{n}}|\mu_{\mathfrak{n}}$ is the total variation measure associated to σ, $||\sigma|| = |\sigma|(X)$ is the total variation of σ.

Note that if we consider $| < \rho(g^{-1}g')f_1, f_1\otimes\chi >< \rho(g^{-1}g')f_2, f_2\otimes\chi > |$ as a function of g or g', the center acts on it trivially as the central characters are unitary. So Hecke operators $\mu_{\mathfrak{n}}$ act on it nicely. In particular, define $\sigma^{(2)} = |\sigma| * |\sigma|$. Then we can rewrite the above result as

$$|\lambda \mathbb{I}|^2 \ll N^{\alpha-1/2+\epsilon}\|\sigma\|^2 + \sum_{\chi^2=1} \int_g | < \rho(g)f_1, f_1 \otimes \chi >< \rho(g)f_2, f_2 \otimes \chi > |d\sigma^{(2)}(g). \tag{3.11}$$

According to Lemma 2.5, we have the following formula for spherical local components on which the center acts trivially:

$$\sigma^{(2)} = \sum_{\mathfrak{n},\mathfrak{n}} |a_{\mathfrak{n}}||a_{\mathfrak{m}}| \sum_{\mathfrak{d}|(\mathfrak{n},\mathfrak{m})} \mu_{\mathfrak{n}\mathfrak{m}\mathfrak{d}^{-2}}. \tag{3.12}$$

According to Lemma 2.8, one can easily prove that for spherical functions:

$$\int_{g\in G(\mathbb{A})} | < \rho(g)f_1, f_1 \otimes \chi >< \rho(g)f_2, f_2 \otimes \chi > |d\mu_{\mathfrak{n}}(g) \ll_\epsilon \mathrm{Nm}(\mathfrak{n})^{2\alpha-1/2+\epsilon}. \tag{3.13}$$

Moreover, for fixed $g \in \mathrm{Supp}(\mu_{\mathfrak{n}})$, the inner product $< \rho(g)f_1, f_1 \otimes \chi >$ is nonvanishing only if χ is unramified whenever f_1 is unramified and the place does not divide $\mathfrak{n}$. The number of such quadratic characters is $O_\epsilon(\mathrm{Nm}(\mathfrak{n})^\epsilon N^\epsilon)$, where the implicit constant is allowed to depend on the base field $\mathbb{F}$. Thus

$$\sum_{\chi^2=1} \int_{g\in G(\mathbb{A})} | < \rho(g)f_1, f_1 \otimes \chi >< \rho(g)f_2, f_2 \otimes \chi > |d\mu_{\mathfrak{n}}(g) \ll_\epsilon \mathrm{Nm}(\mathfrak{n})^{2\alpha-1/2+\epsilon} N^\epsilon. \tag{3.14}$$

One can also check that

$$\|\sigma\| \ll_\epsilon \sum_{\mathfrak{n}} \mathrm{Nm}(\mathfrak{n})^{1/2+\epsilon}|a_{\mathfrak{n}}|. \tag{3.15}$$

Now combine formulae (3.12), (3.14) and (3.15) into (3.11), we have

$$|\mathbb{I}| \ll N^\epsilon \frac{((\sum_{\mathfrak{n}} \mathrm{Nm}(\mathfrak{n})^{1/2+\epsilon}|a_{\mathfrak{n}}|)^2 N^{\alpha-1/2+\epsilon} + \sum_{\mathfrak{n},\mathfrak{m}} \sum_{\mathfrak{d}|(\mathfrak{n},\mathfrak{m})} (\mathrm{Nm}(\frac{\mathfrak{n}\mathfrak{m}}{\mathfrak{d}^2}))^{2\alpha-1/2+\epsilon}|a_{\mathfrak{n}}||a_{\mathfrak{m}}|)^{1/2}}{|\sum_{\mathfrak{n}} a_{\mathfrak{n}}\check{\lambda}_{\mathfrak{n}}|}. \tag{3.16}$$

According to the inequalities (3.4), (3.5) and (3.6), we have

$$|\mathbb{I}| \ll N^\epsilon \frac{(N^{4b+\alpha-1/2} + N^{(4\alpha+1)b})^{1/2}}{N^b}. \tag{3.17}$$

Now choose $b = \frac{\alpha-1/2}{4\alpha-3} > 0$ as we can pick $\alpha < 1/4$. Then the above inequality becomes

$$|\mathbb{I}| \ll N^{\frac{(\alpha-1/2)(2\alpha-1/2)}{4\alpha-3}+\epsilon}. \tag{3.18}$$

Again $\frac{(\alpha-1/2)(2\alpha-1/2)}{4\alpha-3} < 0$. When we pick $\alpha = 7/64$,

$$\frac{(\alpha-1/2)(2\alpha-1/2)}{4\alpha-3} = -\frac{225}{5248} < -\frac{1}{24}. \tag{3.19}$$

□

Remark 3.6. (1) The roles of f_1 and f_2 are interchangeable. One can also, for example, assume $\mathcal{N} = \mathcal{N}_1\mathcal{N}_2$ with $\mathcal{N}_1$ and $\mathcal{N}_2$ relatively prime, and get a similar inequality

$$\int_X f_1(xa([\mathcal{N}_1]))f_2(xa([\mathcal{N}_2]))f_3(x)dx \ll N^{-\delta}. \tag{3.20}$$

(2) It's also possible to refine the proof and allow the conductors N_1, N_2 of f_1 f_2 to change at the same time, under somewhat restrictive condition that essentially $N_1^a, N_2^a|N$ for a large enough integer a ($a = 9$ should be enough). Such condition is used to control the implied constant in, for example, (3.9). One would also need this condition to control the L^4 norm using the normalized L^2 norm and a proper bound of L^∞ norm in level aspect.

Then one can also get a hybrid type subconvexity bound of triple product L-function in level aspect under the same condition.

4. Local integral for the triple product L-function

In this section, we shall compute the local integral for the triple product $L-$function at finite places. As we will work purely locally, let's suppress subscript v in this section.

Let $\pi_i, i = 1, 2, 3$ be three local irreducible unitary representations of GL_2. Let $f_i \in \pi_i$ be the normalized new forms for places $v \in S$ according to our choice in the last section. Let

$$\Phi_i(g) =< \pi_i(g)f_i, f_i > \text{ for } i = 1, 3 \quad \Phi_2(g) =< \pi_2(ga_v([\mathcal{N}]))f_2, \pi_2(a_v([\mathcal{N}]))f_2 >,$$

where $a_v([\mathcal{N}])$ is as in Definition 3.3.

We will compute in this section the following integral

$$I_v(f_1, \pi_2(a_v([\mathcal{N}]))f_2, f_3) = \int_{\mathbb{F}^*\backslash \mathrm{GL}_2(\mathbb{F})} \Phi_1(g)\Phi_2(g)\Phi_3(g)dg. \tag{4.1}$$

We will assume that $c_3 \geq 2\max\{c_1, c_2, 1\}$. The difference between this assumption and the condition for the set of places S is the case $c_1 = c_2 = 0$ and $c_3 = 1$. But this case was already considered in [24]. In general the exact value of the matrix coefficient is very difficult to write out explicitly, and so is the local integral (4.1). But with the assumption $c_3 \geq 2\max\{c_1, c_2, 1\}$, the computations turn out to be very nice and simple.

We will consider all possible local irreducible unitary representations which can be classified into the following three types:

Type 1. π is either supercuspidal, or of form $\pi(\mu_1, \mu_2)$ where μ_i is ramified of level $k_i > 0$ for $i = 1, 2$;
Type 2. π is unramified or special unramified;
Type 3. π is of form $\pi(\mu_1, \mu_2)$ where μ_1 is unramified and μ_2 is ramified of level k.

Note we don't have to consider the case when μ_1 is ramified and μ_2 is unramified, as $\pi(\mu_1, \mu_2) \cong \pi(\mu_2, \mu_1)$. Also when π is of form $\sigma(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$ where χ is ramified, we can pick the new form similarly as in the second case of Type 1. So this case won't be considered as a different case.

Theorem 4.1. *Let π_1, π_2, π_3 be three local irreducible unitary representations of GL_2, with levels satisfying $c_3 \geq 2\max\{c_1, c_2, 1\}$. Let $f_i \in \pi_i$ be normalized local new forms. Then the local integral*

$$I_v(f_1, \pi_2(a_v([\mathcal{N}]))f_2, f_3) = \int_{\mathbb{F}^*\backslash GL_2(\mathbb{F})} \Phi_1(g)\Phi_2(g)\Phi_3(g)dg = \frac{(1-A)(1-B)}{(q+1)q^{c_3-1}}, \tag{4.2}$$

where

$$A = \Phi_1(\begin{pmatrix} 1 & \varpi^{-1} \\ 0 & 1 \end{pmatrix}) \text{ and } B = \Phi_2(\begin{pmatrix} 1 & 0 \\ \varpi^{c_3-1} & 1 \end{pmatrix}).$$

More specifically we have the following tables of values of A and B for all three types of irreducible unitary representations

π_1	*Type 1*	*unramified of form* $\pi(\chi_1, \chi_2)$	*special unramified*	*Type 3*
A	$-\frac{1}{q-1}$	$\frac{1}{q+1}(\frac{\chi_1}{\chi_2} + \frac{\chi_2}{\chi_1} + 1 - q^{-1})$	$-q^{-1}$	*0*

π_2	*Type 1*	*unramified of form* $\pi(\eta_1,\eta_2)$	*special unramified*	*Type 3*
B	$-\frac{1}{q-1}$	$\frac{1}{q+1}(\frac{\eta_1}{\eta_2}+\frac{\eta_2}{\eta_1}+1-q^{-1})$	$-q^{-1}$	*0*

4.1. **General strategy.** As all the matrix coefficients will be right $K_1(\varpi^{c_3})-$invariant, it's natural to break the integral on $\mathbb{F}^*\backslash \mathrm{GL}_2(\mathbb{F})$ into integrals on the sets of the form

$$\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} K_1(\varpi^{c_3})$$

for $0 \leq i \leq c_3$. So one would like to know the values of Φ_i on matrices of the form $\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}$.

As we have assumed that $c_3 \geq 2\max\{c_1, c_2, 1\}$, π_3 will always be of Type 1. In particular π_3 is either supercuspidal, or of form $\pi(\mu_1, \mu_2)$ where μ_1 and μ_2 are ramified of same level $c_3/2$. This is because we have assumed that the product of central characters is always trivial.

Lemma 4.2. *Let π_3 be a supercuspidal representation or $\pi(\mu_1,\mu_2)$ where μ_1 μ_2 are both of level $c_3/2$.*

(1) *As a function in a, $\Phi_3(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ contains level 0 components only when $i = c_3$ or $i = c_3 - 1$, $v(a) = 0$ and $v(m) \geq -1$. Then we have the following special values and integral formula:*

$$\Phi_3(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}) = \begin{cases} 1, & \text{if } v(m) \geq 0 \text{ and } i = c_3; \\ -\frac{1}{q-1}, & \text{if } v(m) = -1 \text{ and } i = c_3; \\ -\frac{1}{q-1}, & \text{if } v(m) \geq 0 \text{ and } i = c_3 - 1. \end{cases} \tag{4.3}$$

When $v(a) = 0$, $v(m) = -1$ and $i = c_3 - 1$,

$$\int\limits_{v(m)=-1} \Phi_3(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c_3-1} & 1 \end{pmatrix})dm = \frac{1}{q-1}. \tag{4.4}$$

(2) *When $i \geq c_3/2$, $\Phi_3(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ is supported at $v(m) \geq -c_3/2$ and consists of level$\leq$ $c_3/2$ components in a.*

(3) *When $0 \leq i < c_3/2$, $\Phi_3(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ is supported at $v(a) = 2i - c_3$ and $v(m) = i - c_3$. As a function in a, it consists of level $c_3 - i$ components.*

Proof. When π_3 is supercuspidal, the above results follow directly from (actually is weaker than) Proposition 2.19. When π_3 is of the form $\pi(\mu_1, \mu_2)$ where μ_i are both of level $c_3/2$, the claims basically follow from Lemma 2.12.

By definition

$$\begin{aligned} \Phi_3(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}) &= \int \psi(m\alpha)W^{(i)}(a\alpha)\overline{W^{(c_3)}(\alpha)}d^*\alpha \\ &= \int\limits_{v(\alpha)=0} \psi(m\alpha)W^{(i)}(a\alpha)d^*\alpha. \end{aligned} \tag{4.5}$$

To find the level 0 component of $\Phi_3(\begin{pmatrix} a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \varpi^i & 1\end{pmatrix})$ in a is equivalent to find the level 0 component in $W^{(i)}(a\alpha)$ at $v(\alpha)=0$, which only occurs when $i=c_3$ or $i=c_3-1$ and $v(a\alpha)=0$ from Lemma 2.12. Then we further need $v(a)=0$ and $v(m)\geq -1$ for the integral in α to be nonzero.

The special values and the special integral just follow from (ii) of Lemma 2.12. As an example, we will prove (4.4). When $v(a)=0$ and $i=c_3-1$,

$$\begin{aligned}(4.6)\qquad &\int\limits_{v(m)=-1}\Phi_3(\begin{pmatrix} a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \varpi^{c_3-1} & 1\end{pmatrix})dm=\int\limits_{v(\alpha)=0}\int\limits_{v(m)=-1}\psi(m\alpha)W^{(c_3-1)}(a\alpha)dmd^*\alpha\\ &=-\int\limits_{v(\alpha)=0}W^{(c_3-1)}(a\alpha)d^*\alpha=-\int\limits_{v(\alpha)=0}W^{(c_3-1)}(\alpha)d^*\alpha=\frac{1}{q-1}.\end{aligned}$$

Now to prove (2), suppose $v(m)<-c_3/2$ in (4.5). Then $\psi(m\alpha)$ is of level $\geq c_3/2+1$ in α. But one can check explicitly from (ii) and (iii) of Lemma 2.12 that $W^{(i)}$ is of level $\leq c_3/2$ in α. For example, when $i=c_3/2$,

$$\begin{aligned}(4.7)\qquad &W^{(\frac{c_3}{2})}(a\alpha)\\ &=C^{-1}\int\limits_{v(u)\leq -\frac{c_3}{2},u\notin\varpi^{-\frac{c_3}{2}}(-1+\varpi O_F)}\mu_1(-\frac{\varpi^{\frac{c_3}{2}}}{1+u\varpi^{\frac{c_3}{2}}})\mu_2(-a\alpha u)|\frac{\varpi^{\frac{c_3}{2}}}{a\alpha u(1+u\varpi^{\frac{c_3}{2}})}|^{1/2}\psi(-a\alpha u)q^{-v(a\alpha)}du.\end{aligned}$$

As functions in u, $\mu_1(-\frac{\varpi^{\frac{c_3}{2}}}{1+u\varpi^{\frac{c_3}{2}}})\mu_2(-a\alpha u)$ is of level $\leq c_3/2$ on the domain, $\psi(a\alpha u)$ is of level $-v(a\alpha u)$. Then we need $v(a\alpha u)\geq -c_3/2$ for the integral in u ever to be nonzero. As a result, the level of $W^{(i)}(a\alpha)$ in α (and also in a) is $\leq c_3/2$. Then (4.5) has to be zero as it's the integral of product of level$\geq c_3/2+1$ components with level$\leq c_3/2$ components. One can also see from this argument that the level of Φ_3 in a is $\leq c_3/2$.

(3) follows from (i) of Lemma 2.12: when $i<c_3/2$,
(4.8)

$$W^{(i)}(a\alpha)=C^{-1}\int\limits_{u\in O_F^*}\mu_1(-\frac{\varpi^i}{u})\mu_2(a\alpha\varpi^{-i}(1-\varpi^{\frac{c_3}{2}-i}u))q^{v(a\alpha)/2-i}\psi(a\alpha\varpi^{-i}(1-\varpi^{\frac{c_3}{2}-i}u))q^{2i-\frac{c_3}{2}-v(a\alpha)}du.$$

As functions in u, $\mu_1(-\frac{\varpi^i}{u})$ is of level $c_3/2$, $\mu_2(a\alpha\varpi^{-i}(1-\varpi^{\frac{c_3}{2}-i}u))$ is of level $\leq i<c_3/2$, $\psi(a\alpha\varpi^{-i}(1-\varpi^{\frac{c_3}{2}-i}u))$ is of level $2i-c_3/2-v(a\alpha)$. As $v(\alpha)=0$ in (4.5), $W^{(i)}(a\alpha)$ is not zero only when $v(a)=2i-c_3$. We will assume this for the remaining discussions.

As functions in α, $\mu_2(a\alpha\varpi^{-i}(1-\varpi^{\frac{c_3}{2}-i}u))$ is of level $c_3/2$, and $\psi(a\alpha\varpi^{-i}(1-\varpi^{\frac{c_3}{2}-i}u))$ is now of level $i-v(a\alpha)=c_3-i>c_3/2$. Then $W^{(i)}(a\alpha)$ as a function in α is of level c_3-i. Thus for the integral in (4.5) to be vanishing, we require $v(m)=i-c_3$.

In this argument, one can easily see that $\Phi_3(\begin{pmatrix} a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \varpi^i & 1\end{pmatrix})$ as a function in a for $i<c_3/2$ is of level c_3-i. □

Now we can explain the strategy to prove Theorem 4.1. As we mentioned earlier, we will add up the integrals on the double cosets of the form

$$\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} K_1(\varpi^{c_3})$$

for $0 \leq i \leq c_3$. We will show that the nonzero contributions will only come from $i = c_3$ and $i = c_3 - 1$, where we know special values or integrals for Φ_3. In particular, we will prove the following two claims about Φ_1 and Φ_2 for various types of representations:

Claim 1. (1) When $i \geq c_3/2$, $\Phi_1(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ is of level 0 in a for fixed valuations. In particular we have the following special values:

$$\Phi_1(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}) = \begin{cases} 1, & \text{if } v(a) = 0 \text{ and } v(m) \geq 0 \\ A, & \text{if } v(a) = 0 \text{ and } v(m) = -1. \end{cases}$$

(2) When $i < c_3/2$, $v(a) = 2i - c_3$ and $v(m) = i - c_3$, $\Phi_1(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ as a function in a is of level $\leq c_1 < c_3 - i$.

Remark 4.3. This claim should be clear by intuition. When $i \geq c_3/2 \geq c_1$, $\Phi_1(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}) = \Phi_1(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix})$. Its special value at $v(a) = 0$ and $v(m) \geq 0$ is just a matter of normalization.

Claim 2. (1) For $i \geq c_3/2$, and $v(m) \geq -c_3/2$, $\Phi_2(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ is of level 0 as a function in a and independent of m. When $i = c_3$, $v(a) = 0$ and $v(m) \geq -c_3/2$,

$$\Phi_2(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}) = 1. \tag{4.9}$$

When $i = c_3 - 1$, $v(a) = 0$ and $v(m) \geq -c_3/2$,

$$\Phi_2(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c_3-1} & 1 \end{pmatrix}) = B. \tag{4.10}$$

(2) For $i < c_3/2$, $v(a) = 2i - c_3$ and $v(m) = i - c_3$, $\Phi_2(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ is of level $\leq c_2 < c_3 - i$ as a function in a.

Remark 4.4. Again the special value in the case $i = c_3$, $v(a) = 0$ and $v(m) \geq -c_3/2$ is just a matter of normalization.

Now suppose that these two claims are always true for all three types of representations. When $i < c_3 - 1$, the level in a of Φ_1 and Φ_2 is strictly less than that of Φ_3. So indeed the only nonzero contribution to the final integral will come from $i = c_3$ and $i = c_3 - 1$. One can then have the following table of values:

$v(a)$ is always 0	Φ_1	Φ_2	Φ_3
$i=c_3, v(m)\geq 0$	1	1	1
$i=c_3, v(m)=-1$	A	1	$-\frac{1}{q-1}$
$i=c_3-1, v(m)\geq 0$	1	B	$-\frac{1}{q-1}$
$i=c_3-1, v(m)=-1$	A	B	satisfying (4.4)

Then by Lemma 2.2, one can easily compute that

$$\begin{aligned}(4.11)\qquad &\int\limits_{\mathbb{F}^*\backslash \mathrm{GL}_2(\mathbb{F})} \Phi_1(g)\Phi_2(g)\Phi_3(g)dg\\ =&\frac{1}{(q+1)q^{c_3-1}}[1+(q-1)A(-\frac{1}{q-1})]+\frac{q-1}{(q+1)q^{c_3-1}}[B(-\frac{1}{q-1})+AB\frac{1}{q-1}]\\ =&\frac{(1-A)(1-B)}{(q+1)q^{c_3-1}}.\end{aligned}$$

So the theorem will be proved if we can verify Claim 1 and Claim 2, and also figure out A and B for various types of representations. We will do this in the remaining of this section. First let's give the formulae for Φ_1 and Φ_2 more explicitly in terms of Whittaker functionals.

Let W_i be the corresponding Whittaker functionals for the normalized new forms f_i, and

$$W_i^{(j)}(\alpha)=W_i(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^j & 1\end{pmatrix})$$

as in Section 2.5.

Φ_1 is right $K_1(\varpi^{c_1})-$invariant, and thus automatically $K_1(\varpi^{c_3})-$invariant. When $i\geq c_1$,

$$(4.12)\qquad \Phi_1(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix})=\int\limits_{\alpha}\psi(m\alpha)W_1^{(c_1)}(a\alpha)\overline{W_1^{(c_1)}(\alpha)}d^*\alpha.$$

When $i<c_1$,

$$(4.13)\qquad \Phi_1(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix})=\int\limits_{\alpha}\psi(m\alpha)W_1^{(i)}(a\alpha)\overline{W_1^{(c_1)}(\alpha)}d^*\alpha.$$

Now for Φ_2, by definition ,

(4.14)

$$\Phi_2(g)=<\pi_2(g\begin{pmatrix}\varpi^{-c_3+c_2} & 0\\ 0 & 1\end{pmatrix})f_2,\pi_2(\begin{pmatrix}\varpi^{-c_3+c_2} & 0\\ 0 & 1\end{pmatrix})f_2>=<\pi_2(\begin{pmatrix}\varpi^{c_3-c_2} & 0\\ 0 & 1\end{pmatrix}g\begin{pmatrix}\varpi^{-c_3+c_2} & 0\\ 0 & 1\end{pmatrix})f_2,f_2>.$$

It is again right $K_1(\varpi^{c_3})-$invariant as

$$\begin{pmatrix}\varpi^{c_3-c_2} & 0\\ 0 & 1\end{pmatrix}K_1(\varpi^{c_3})\begin{pmatrix}\varpi^{-c_3+c_2} & 0\\ 0 & 1\end{pmatrix}\subset K_1(\varpi^{c_2}),$$

and f_2 is $K_1(\varpi^{c_2})-$invariant.

Now if $i\geq c_3-c_2$,

$$\begin{pmatrix}\varpi^{c_3-c_2} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}\begin{pmatrix}\varpi^{-c_3+c_2} & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}a & m\varpi^{c_3-c_2}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^{i-c_3+c_2} & 1\end{pmatrix},$$

and

$$\Phi_2(\begin{pmatrix} a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \varpi^i & 1\end{pmatrix}) = \int \psi(m\varpi^{c_3-c_2}\alpha)W_2^{(i-c_3+c_2)}(a\alpha)\overline{W_2^{(c_2)}(\alpha)}d^*\alpha. \tag{4.15}$$

If $i < c_3 - c_2$,

$$\begin{pmatrix} \varpi^{c_3-c_2} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \varpi^i & 1\end{pmatrix}\begin{pmatrix} \varpi^{-c_3+c_2} & 0\\ 0 & 1\end{pmatrix}$$
$$= \varpi^{i-c_3+c_2}\begin{pmatrix} a\varpi^{-2i+2c_3-2c_2} & a(\varpi^{-i+c_3-c_2} - \varpi^{-2i+2c_3-2c_2}) + m\varpi^{c_3-c_2}\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & -1+\varpi^{-i+c_3-c_2}\\ 0 & 1\end{pmatrix},$$

and

(4.16)

$$\Phi_2(\begin{pmatrix} a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \varpi^i & 1\end{pmatrix})$$
$$= w_{\pi_2}(\varpi^{i-c_3+c_2})\int \psi((a(\varpi^{-i+c_3-c_2} - \varpi^{-2i+2c_3-2c_2}) + m\varpi^{c_3-c_2})\alpha)W_2^{(0)}(a\varpi^{-2i+2c_3-2c_2}\alpha)\overline{W_2^{(c_2)}(\alpha)}d^*\alpha.$$

We can actually reduce many parts of Claim 1 and Claim 2 to the following simple lemma:

Lemma 4.5. *Let π be a unitary local representation of GL_2. Let W be a Whittaker functional associated to a new form in π. Then $W(\begin{pmatrix} \alpha & 0\\ 0 & 1\end{pmatrix})$ is of level 0 in α and supported on $v(\alpha) \geq 0$.*

Proof. The claim follows directly from Lemma 2.12, Lemma 2.13 and Corollary 2.18 for Type 1 and Type 3. It's also well know for unramified representations. For special unramified representations $\pi \cong \sigma(\mu|\cdot|^{1/2}, \mu|\cdot|^{-1/2})$ where μ is unramified, one can see, for example, [9]. There I proved that the Whittaker functional associated to a new form satisfies the following formula:

$$W(\begin{pmatrix} \alpha & 0\\ 0 & 1\end{pmatrix}) = \begin{cases} \mu(\alpha)q^{-v(\alpha)}, & \text{if } v(\alpha) \geq 0;\\ 0, & \text{otherwise.}\end{cases}$$

□

Now we shall prove Claim 1 and Claim 2 except part (2) of Claim 1 and explicit computations for A and B. We start with part (1) of Claim 1. Let $i \geq c_1$, then

$$\Phi_1(\begin{pmatrix} a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \varpi^i & 1\end{pmatrix}) = \int \psi(m\alpha)W_1^{(c_1)}(a\alpha)\overline{W_1^{(c_1)}(\alpha)}d^*\alpha. \tag{4.17}$$

Both of $W_1^{(c_1)}(a\alpha)$ and $\overline{W_1^{(c_1)}(\alpha)}$ are of level 0 in α because of the lemma above. Then the result should be of level 0 in a. One can make a change of variable to see that A is well-defined.

Now we consider part (1) of Claim 2. If $v(m) \geq -c_3/2$, then $v(m\varpi^{c_3-c_2}\alpha) \geq 0$ for $v(\alpha) \geq 0$. So when $i \geq c_3 - c_2$,

$$\begin{aligned}\Phi_2(\begin{pmatrix} a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \varpi^i & 1\end{pmatrix}) &= \int_{v(\alpha)\geq 0} \psi(m\varpi^{c_3-c_2}\alpha)W_2^{(i-c_3+c_2)}(a\alpha)\overline{W_2^{(c_2)}(\alpha)}d^*\alpha\\ &= \int_{v(\alpha)\geq 0} W_2^{(i-c_3+c_2)}(a\alpha)\overline{W_2^{(c_2)}(\alpha)}d^*\alpha.\end{aligned} \tag{4.18}$$

This is to find level 0 components of $W_2^{(i-c_3+c_2)}$, and should be of level 0 in a. It's also clearly independent of m. That's why B is well-defined.

Similarly when $c_3/2 \le i < c_3 - c_2$,

(4.19)

$$\begin{aligned}
&\Phi_2(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}) \\
=&w_{\pi_2}(\varpi^{i-c_3+c_2}) \int_{v(\alpha)\ge 0} \psi((a(\varpi^{-i+c_3-c_2} - \varpi^{-2i+2c_3-2c_2}) + m\varpi^{c_3-c_2})\alpha) W_2^{(0)}(a\varpi^{-2i+2c_3-2c_2}\alpha)\overline{W_2^{(c_2)}(\alpha)}d^*\alpha \\
=&w_{\pi_2}(\varpi^{i-c_3+c_2}) \int_{v(\alpha)\ge 0} \psi(a(\varpi^{-i+c_3-c_2} - \varpi^{-2i+2c_3-2c_2})\alpha) W_2^{(0)}(a\varpi^{-2i+2c_3-2c_2}\alpha)\overline{W_2^{(c_2)}(\alpha)}d^*\alpha.
\end{aligned}$$

Again this integral is to find level 0 components of $\psi(a(\varpi^{-i+c_3-c_2}-\varpi^{-2i+2c_3-2c_2})\alpha)W_2^{(0)}(a\varpi^{-2i+2c_3-2c_2}\alpha)$ and should be of level 0 in a and independent of m.

One can also prove (2) of Claim 2 without referring to any specific type of representation. When $i < c_3/2$, $v(a) = 2i - c_3$ and $v(m) = i - c_3$, we know

$$\begin{aligned}
\Phi_2(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}) = w_{\pi_2}(\varpi^{i-c_3+c_2}) \int_{v(\alpha)\ge 0} &\psi(m\varpi^{c_3-c_2}\alpha)\psi((\varpi^{-i+c_3-c_2} - \varpi^{-2i+2c_3-2c_2})a\alpha) \\
&W_2^{(0)}(\varpi^{-2i+2c_3-2c_2}a\alpha)\overline{W^{(c_2)}(\alpha)}d^*\alpha.
\end{aligned} \tag{4.20}$$

By the previous lemma, $\overline{W^{(c_2)}(\alpha)}$ is of level 0 in α, and $v(\alpha) \ge 0$ in the integral. So $v(m\varpi^{c_3-c_2}\alpha) \ge i - c_3 + c_3 - c_2 = i - c_2 \ge -c_2$, which means $\psi(m\varpi^{c_3-c_2}\alpha)$ is of level$\le c_2$ in α. Now note that if $\psi((\varpi^{-i+c_3-c_2} - \varpi^{-2i+2c_3-2c_2})a\alpha)W_2^{(0)}(\varpi^{-2i+2c_3-2c_2}a\alpha)$ has a component of level j in α, then this component is also of level j in a. As only those components of level$\le c_2$ will be detected by the integral, $\Phi_2(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix})$ can only be of level$\le c_2$ in a. Another way to argue this is just to do a change of variable for the integral.

As a result, we will only need to verify part (2) of Claim 1 and compute A and B explicitly for various types of unitary representations.

4.2. **Type 1 occuring.** First let's consider supercuspidal representations. When π_1 is supercuspidal, part (2) of Claim 1 follows directly from Proposition 2.19 for $k = 0$ there and one can also easily see that $A = -\frac{1}{q-1}$.

When π_2 is supercuspidal, take $k = c_3 - c_2$ in Proposition 2.19. Note $\pi_2(\begin{pmatrix} \varpi^{-c_3+c_2} & 0 \\ 0 & 1 \end{pmatrix})\mathbf{1}_{1,0} = \mathbf{1}_{1,c_3-c_2}$, and thus

$$\Phi_2(g) =< \pi_2(g)\mathbf{1}_{1,c_3-c_2}, \mathbf{1}_{1,c_3-c_2} > .$$

So again we can apply Proposition 2.19 and get $B = -\frac{1}{q-1}$.

Now suppose π_1 is of form $\pi(\chi_1, \chi_2)$, where χ_1 and χ_2 are both ramified. For any m with $v(m) = -1$,

$$\Phi_1(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}) = \int_{v(\alpha)=0} \psi(m\alpha)d^*\alpha = -\frac{1}{q-1}. \tag{4.21}$$

This is the value of A. When $i < c_1$, the proof of (2) of Claim 1 is actually similar to the proof of Lemma 4.2. We will leave this to the readers.

Now let π_2 be of form $\pi(\chi_1, \chi_2)$, where χ_1 and χ_2 are both ramified. By formula (4.18), we have the following for $v(a) = 0$, $i = c_3 - 1$ and $v(m) \geq -c_3/2$:

$$\Phi_2(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}) = \int\limits_{v(\alpha)=0} W_2^{(c_2-1)}(\alpha) d^*\alpha = -\frac{1}{q-1}. \tag{4.22}$$

In the last equality we have used Lemma 2.12. This gives the value B.

4.3. **Type 2 occuring.** In this subsection we will consider unramified and special unramified representations. We first recall the existing work of matrix coefficients for these representations. For unramified representations, just recall Lemma 2.7.

For special unramified representations, let $\sigma_n = \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}$, and $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Lemma 4.6. *Let $\pi = \sigma(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$ be a special unramified unitary representation of GL_2. It has a normalized $K_1(\varpi)-$invariant new form. The associated matrix coefficient Φ for this new form is bi-$K_1(\varpi)-$invariant and can be given in the following table for double $K_1(\varpi)-$cosets:*

g	1	ω	σ_n	$\omega\sigma_n$	$\sigma_n\omega$	$\omega\sigma_n\omega$
$\Phi(g)$	1	$-q^{-1}$	$\chi^n q^{-n}$	$-\chi^n q^{1-n}$	$-\chi^n q^{-1-n}$	$\chi^n q^{-n}$

In this table $n \geq 1$.

This result is due to [24].

Now we consider unramified representations. Part (2) of Claim 1 is actually automatic in this case as Φ_1 is $K-$invariant. Let's figure out A and B values using Lemma 2.7. Let Φ be the matrix coefficient as defined in Lemma 2.7. For $v(m) = -1$,

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/m & 1 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & 1/m \end{pmatrix}\begin{pmatrix} 1/m & 1 \\ -1 & 0 \end{pmatrix}.$$

So we have

$$\begin{aligned} \Phi_1(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}) &= \Phi_1(\begin{pmatrix} m & 0 \\ 0 & 1/m \end{pmatrix}) = w_{\pi_1}^{-1}\Phi(\sigma_2) \\ &= \frac{1}{\chi_1\chi_2}\left(\frac{q^{-1}}{1+q^{-1}}\frac{\chi_1^2(\chi_1 - \chi_2 q^{-1}) - \chi_2^2(\chi_2 - \chi_1 q^{-1})}{\chi_1 - \chi_2}\right) \\ &= \frac{1}{q+1}\left(\frac{\chi_1}{\chi_2} + \frac{\chi_2}{\chi_1} + 1 - q^{-1}\right). \end{aligned} \tag{4.23}$$

This is the value A.

On the other hand,

$$\begin{pmatrix} \varpi^{c_3} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c_3-1} & 1 \end{pmatrix}\begin{pmatrix} \varpi^{-c_3} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varpi^{-1} & 1 \end{pmatrix} = -\omega\begin{pmatrix} 1 & -\varpi^{-1} \\ 0 & 1 \end{pmatrix}\omega.$$

So we can similarly show that $B = \frac{1}{q+1}(\frac{\eta_1}{\eta_2} + \frac{\eta_2}{\eta_1} + 1 - q^{-1})$ if $\pi_2 \cong \pi(\eta_1, \eta_2)$.

Now let π_1 be a special unramified representation of form $\sigma(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$, and let Φ be the matrix coefficient as given in Lemma 4.6. When $v(m) = -1$,

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 1/m & 1 \end{pmatrix}\omega\begin{pmatrix} 1/m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/m & 1 \end{pmatrix}.$$

So

$$A = \Phi(\omega \begin{pmatrix} 1/m & 0 \\ 0 & m \end{pmatrix}) = w_{\pi_1}^{-1}\Phi(\omega\sigma_2) = \frac{1}{\chi^2}(-\chi^2 q^{-1}) = -q^{-1}.$$

To check (2) of Claim 1, we just need to show that when $i = 0$ and $v(m) = v(a) = -c_3$, $\Phi_1(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix})$ is at most level 1. If $v(a+m) > v(m)$, we have

$$\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \omega \begin{pmatrix} a/m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} 1 & -m/a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{a+m}{m} & 1 \end{pmatrix},$$

so

$$\Phi_1(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}) = \Phi(\omega \begin{pmatrix} a/m & 0 \\ 0 & m \end{pmatrix}) = w_{\pi_1}^{-c_3}\Phi(\omega\sigma_{c_3}). \tag{4.24}$$

If $v(a+m) = v(m)$, we have

$$\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ \frac{1}{a+m} & 1 \end{pmatrix}\omega\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}\omega\begin{pmatrix} \frac{a+m}{m} & 1 \\ 0 & \frac{a}{a+m} \end{pmatrix},$$

so

$$\Phi_1(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}) = \Phi(\omega\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}\omega) = w_{\pi_1}^{-c_3}\Phi(\omega\sigma_{c_3}\omega). \tag{4.25}$$

Put together, one can conclude that $\Phi_1(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix})$ is at most level 1 in a when $v(a) = v(m) = -c_3$.

Now we compute the value B for special unramified representations. Since

$$\begin{aligned} \begin{pmatrix} \varpi^{c_3-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c_3-1} & 1 \end{pmatrix}\begin{pmatrix} \varpi^{-c_3+1} & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &= -\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{4.26}$$

we have

$$B = \Phi_2(\begin{pmatrix} \varpi^{c_3-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c_3-1} & 1 \end{pmatrix}\begin{pmatrix} \varpi^{-c_3+1} & 0 \\ 0 & 1 \end{pmatrix}) = \Phi(\omega) = -q^{-1}. \tag{4.27}$$

4.4. **Type 3 occuring.** In this subsection, we consider the representations of form $\pi(\chi_1,\chi_2)$, where χ_1 is unramified and χ_2 is of level k. The results will basically follow from Lemma 2.13. Let's first check part (2) of Claim 1. By (4.13), when $i < k$,

$$\Phi_1(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}) = \int\limits_{v(\alpha)\geq 0} \psi(m\alpha)W^{(i)}(a\alpha)\overline{W^{(k)}(\alpha)}d^*\alpha, \tag{4.28}$$

Here

$$W^{(k)}(\alpha) = \begin{cases} q^{-\frac{1}{2}v(\alpha)-k}\chi_1^{v(\alpha)+k}(\varpi)\int\limits_{v(m)=-k}\chi_2(-m)\psi(-m)dm, & \text{if } v(\alpha)\geq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{4.29}$$

and

$$(4.30) \qquad W^{(i)}(a\alpha) = \chi_1^i(\varpi) \int_{u\in O_{\mathbb{F}}} \chi_2(a\alpha\varpi^{-i}(1-\varpi^{k-i}u))\psi(a\alpha\varpi^{-i}(1-\varpi^{k-i}u))q^{-\frac{1}{2}v(a\alpha)-k+i}du.$$

They are not normalized, but it turns out that this is enough.

For fixed $v(u)\geq 0$, $\chi_2(a\alpha\varpi^{-i}(1-\varpi^{k-i}u))$ is of level $\leq i-v(u)$ in u, $\psi(a\alpha\varpi^{-i}(1-\varpi^{k-i}u))$ is additive of level $2i-k-v(a\alpha)-v(u)$ in u. For (4.30) to be nonzero, we need $2i-k-v(a\alpha)-v(u)\leq i-v(u)$, that is $v(a\alpha)\geq i-k$. This is because if $2i-k-v(a\alpha)-v(u) > i-v(u)$, then the integral will be automatically zero for fixed $v(u)<i$, and

(4.31)

$$\int_{v(u)\geq i} \chi_2(a\alpha\varpi^{-i}(1-\varpi^{k-i}u))\psi(a\alpha\varpi^{-i}(1-\varpi^{k-i}u))du = \chi_2(a\alpha\varpi^{-i}) \int_{v(u)\geq i} \psi(a\alpha\varpi^{-i}(1-\varpi^{k-i}u))du = 0,$$

as $v(a\alpha\varpi^{k-2i})<-i$.

Then as functions in a, $\chi_2(a\alpha\varpi^{-i}(1-\varpi^{k-i}u))$ is of level k in a, and $\psi(a\alpha\varpi^{-i}(1-\varpi^{k-i}u))$ is of level $i-v(a\alpha)\leq k$ in a. In particular, $W^{(i)}(a\alpha)$ if of level$\leq k$ in a. So (2) of Claim 1 is verified for this case.

Now for $v(m)=-1$,

$$(4.32) \qquad \Phi_1(\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}) = \int_{v(\alpha)\geq 0} \psi(m\alpha)W^{(k)}(\alpha)\overline{W^{(k)}(\alpha)}d^*\alpha$$
$$= q^{-2k}|\int_{v(m)=-k} \chi_2(-m)\psi(-m)dm|^2 \int_{v(\alpha)\geq 0} \psi(m\alpha)q^{-v(\alpha)}d^*\alpha.$$

Here we have used the fact that for a representation of Type 3 to be unitary, both χ_1 and χ_2 have to be unitary. Up to a nonzero constant, $\int_{v(\alpha)\geq 0} \psi(m\alpha)q^{-v(\alpha)}d^*\alpha$ is just

$$\int_{v(\alpha)\geq 0} \psi(m\alpha)d\alpha,$$

which is zero when $v(m)=-1$. So $A=0$.

Now let π_2 be of Type 3. By (4.15),

$$(4.33) \qquad \Phi_2(\begin{pmatrix}1 & 0\\ \varpi^{k-1} & 1\end{pmatrix}) = \int_{v(\alpha)\geq 0} W^{(k-1)}(\alpha)\overline{W^{(k)}(\alpha)}d^*\alpha.$$

Here

$$(4.34) \qquad W^{(k-1)}(\alpha) = \chi_1^{k-1}(\varpi) \int_{u\in O_{\mathbb{F}}} \chi_2(\alpha\varpi^{-k+1}(1-\varpi u))\psi(\alpha\varpi^{-k+1}(1-\varpi u))q^{-\frac{1}{2}v(\alpha)-1}du.$$

As functions in α, $\chi_2(\alpha\varpi^{-k+1}(1-\varpi u))$ is of level k, but $\psi(\alpha\varpi^{-k+1}(1-\varpi u))$ is of level $k-1-v(\alpha)\leq k-1$. Then the integral in (4.33) has to be zero as $W^{(k-1)}(\alpha)$ doesn't have any level 0 components. So $B=0$.

5. Conclusion

Using Theorem 4.1, we get the following lower bound for local integrals:

Proposition 5.1.

$$\prod_{v\in S} I_v^0(f_{1,v},\pi_{2,v}(a_v([\mathcal{N}]))f_{2,v},f_{3,v}) \gg N^{-1-\epsilon} \tag{5.1}$$

Proof. The case when $\pi_{1,v}$ and $\pi_{2,v}$ are unramified and $\pi_{3,v}$ is special unramified representation was considered in [24]. The remaining situations are covered by Theorem 4.1, as

$$I_v = \frac{(1-A)(1-B)}{(q+1)q^{c_3-1}} = \frac{(1-A)(1-B)}{(1+q^{-1})}\frac{1}{q^{c_3}}. \tag{5.2}$$

The factor $\frac{1}{q^{c_3}}$ here will contribute to the main part N^{-1}. The other factors, including the normalizing L-factors, are uniformly bounded and can be absorbed into $N^{-\epsilon}$ when taking the product. This is to say if we have, for example, local inequalities

$$I_v^0 \geq C\frac{1}{q^{c_3}},$$

then we are safe to take a product and claim that

$$\prod_{v\in S} I_v^0 \gg N^{-1-\epsilon}.$$

This is because C will be finally strictly greater than $\frac{1}{q^{c_3\epsilon}}$, when either $q\to+\infty$ or $c_3\to+\infty$.

So in particular we don't have to worry about factors like $\frac{1}{1+q^{-1}}$ and $\frac{1}{\zeta_v(2)} = 1-q^{-2}$. We check here that A and B should be bounded away from 1. This is clear from Theorem 4.1 for Types 1 and 3 and also special unramified representations. For unramified representations, we have

$$|1-A| = \frac{|q+q^{-1}-\frac{\chi_1}{\chi_2}-\frac{\chi_2}{\chi_1}|}{q+1} \geq \frac{q+q^{-1}-(|\frac{\chi_1}{\chi_2}|+|\frac{\chi_2}{\chi_1}|)}{q+1}.$$

This is clearly bounded below if the representation is tempered. When it's not tempered, we need to use a bound α towards Ramanujan Conjecture. So

$$|1-A| \geq \frac{q+q^{-1}-(q^{2\alpha}+q^{-2\alpha})}{q+1}$$

for $\alpha < 1/4$, and is clearly bounded below. □

Corollary 5.2. *For $v\in S$ where S is as in Definition 3.1, we always have*

$$\epsilon_v(\Pi_v, 1/2) = 1.$$

Proof. The claim just follows from Prasad's work and that the local integrals in Theorem 4.1 are nonzero. □

Now we can easily prove the main theorem:

Proof of Theorem 1.4. By Ichino's formula,

$$|\mathbb{I}(f_1,\rho(a[\mathcal{N}])f_2,f_3)|^2 = \frac{\zeta_{\mathbb{F}}^2(2)L(\Pi,1/2)}{8L(\Pi,Ad,1)}\prod_v I_v^0(f_{1,v},\pi_{2,v}(a_v([\mathcal{N}]))f_{2,v},f_{3,v}). \tag{5.3}$$

According to [12],

$$L(\Pi, Ad, 1) \ll C(\Pi)^{\epsilon}, \tag{5.4}$$

where the implicit constant depends continuously on the Langlands parameter of the infinity component. In particular when π_1 π_2 are fixed and π_3 has bounded infinity component,

$$L(\Pi, Ad, 1) \ll N^{\epsilon}. \tag{5.5}$$

By Proposition 3.5 and Remark 3.2,

$$|\mathbb{I}(f_1, \rho(a[\mathcal{N}])f_2, f_3)|^2 \ll N^{2\frac{(\alpha-1/2)(2\alpha-1/2)}{4\alpha-3}+\epsilon}. \tag{5.6}$$

By Proposition 5.1 and Remark 3.4,

$$\prod_v I_v^0(f_{1,v}, \pi_{2,v}(a_v([\mathcal{N}]))f_{2,v}, f_{3,v}) \gg N^{-1-\epsilon}. \tag{5.7}$$

Then one can prove the theorem by combining (5.3), (5.5), (5.6) and (5.7) and take $\alpha = \frac{7}{64}$. □

Appendix A. Bound of global matrix coefficient

In the appendix we will prove Proposition 2.10, which we record here again. Let $\mathbb{D}$ be a global quaternion algebra. Let ρ denote the right regular representation of $\mathbb{D}^*(\mathbb{A})$ on $L^2(Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A}))$. Let $F_1, F_2 \in L^2(Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A}))$ be two rapidly decreasing and $K-$finite automorphic forms which don't have 1-dim components in their spectrum decompositions. Implicitly the center acts on F_i trivially. Let S be a finite set of non-archimedean places. We assume that $\mathbb{D}$ is locally the matrix algebra at the places in S. Let $K_S = \prod_{v\in S} K_v$ and

$$K_{i,S} = \prod_{v\in S} K_{i,v} \quad K_i = \prod_{v \text{ finite}} K_{i,v},$$

where $K_{i,v}$ stabilizes the local component of F_i at v. Let $\mathcal{N} = \prod_v \varpi_v^{e_v}$ for $e_v \geq 0$, and $N = \mathrm{Nm}(\mathcal{N})$. Define the matrix

$$a([\mathcal{N}]) = \prod_v \begin{pmatrix} \varpi^{-e_v} & 0 \\ 0 & 1 \end{pmatrix},$$

which can be naturally thought of as an element of $\mathbb{D}^*(\mathbb{A})$.

Proposition A.1. *With the setting as above, we have*

(A.1)
$$|\int_{Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A})} F_1(g)\rho(a([\mathcal{N}]))F_2(g)dg| \ll_{\epsilon,\mathbb{F}} [K_S : K_{1,S}]^{1/2}[K_S : K_{2,S}]^{1/2}N^{\alpha-1/2+\epsilon}\|F_1\|_{L^2}\|F_2\|_{L^2}.$$

The case when $\mathbb{D}$ is a division algebra is easier to prove. This is basically because there is no continuous spectrum for $L^2(\mathbb{Z}_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A}))$. First of all, for an irreducible unitary cuspidal representation, the global unitary pairing is equal to the product of local pairings. In the case when the automorphic forms are from a single cuspidal representation, one just need to take a product of local bounds in Lemma 2.8 according to Remark 2.9. In general we consider the spectrum decomposition for F_i:

$$F_i = \sum_{\pi} \sum_{f\in\mathcal{B}(\pi)} < F_i, f > f,$$

where $\mathcal{B}(\pi)$ is an orthonormal basis under the unitary pairing for a cuspidal automorphic representation π. If F_i is invariant under K_i, then its cuspidal component in π:

$$\sum_{f\in\mathcal{B}(\pi)} < F_i, f > f$$

is also invariant under K_i. This is true because of Plancherel Theorem. As a result, one can apply the argument for the previous case for each such component, and then use Cauchy-Schwarz inequality.

When $\mathbb{D}$ is the matrix algebra, one can argue similarly if F_1, F_2 have only cuspidal spectrums. But in general, they can have continuous spectrums. By intuition the continuous spectrums shouldn't mess things up because they are related to Eisenstein series defined by unitary characters, and they look like tempered representations locally.

To argue more strictly, let's first recall some results.

A.1. **The spectrums of $L^2(\mathbb{Z}_{\mathbb{A}}\mathbf{GL}_2(\mathbb{F})\backslash\mathbf{GL}_2(\mathbb{A}))$ and the Plancherel formula.** For a more detailed reference of this subsection, see Section 2.2 in [16].

Let $\mathcal{X}$ denote the set of pairs (M, σ), where M is a $\mathbb{F}-$Levi subgroup of a $\mathbb{F}-$parabolic subgroup (containing a maximally $\mathbb{F}-$split torus T) of G, and σ is an irreducible subrepresentation of the space of functions on $M(\mathbb{F})\backslash M(\mathbb{A})$ such that any $f \in \sigma$ is square-integrable in the sense that

$$\|f\|_\sigma^2 =< f, f >_\sigma= \int\limits_{\mathbb{Z}_M M(\mathbb{F})\backslash M(\mathbb{A})} |f|^2 < \infty. \tag{A.2}$$

For $G = \mathrm{GL}_2$ there are two cases. When M is the whole group GL_2, then σ is just a cuspidal representation. When M is the torus T, then σ is actually a unitary character on the torus.

We can equip $\mathcal{X}$ with a measure in the following way: we write

$$\mathcal{X} = \bigsqcup_M \mathcal{X}_M, \tag{A.3}$$

indexed by levis containing T. We require that for any continuous assignment of $\chi \in \mathcal{X}_M$ to f_χ in the underlying space of χ,

$$\int\limits_{\mathbb{Z}_M M(\mathbb{F})\backslash M(\mathbb{A})} |\int\limits_\chi f_\chi d\chi|^2 = \int\limits_\chi \|f_\chi\|_\sigma^2 d\chi. \tag{A.4}$$

This uniquely specifies a measure $d\chi$ on $\mathcal{X}_M$, and so also on $\mathcal{X}$.

(M, σ) is said to be equivalent to (M', σ') if there exists ω in the normalizer of T with $Ad(\omega)M = M'$ and $Ad(\omega)\sigma = \sigma'$. There is a natural quotient measure on $\mathcal{X}/\sim$.

For $\chi = (M, \sigma) \in \mathcal{X}$, we donte by $\mathcal{I}(\chi)$ the unitary induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\sigma$, where P is any parabolic subgroup containing M. One can define a unitary pairing on $\mathcal{I}(\chi)$ by

$$< f_1, f_2 >_{\mathrm{Eis}}= \int\limits_K < f_1(k), f_2(k) >_\sigma dk, \tag{A.5}$$

where K is equipped with Haar probability measure. When $M = T$ and σ is just a unitary character of T, this pairing is just

$$< f_1, f_2 >_{\mathrm{Eis}}= \int\limits_K f_1(k)\overline{f_2(k)}dk, \tag{A.6}$$

which is directly a product of local integrals. With this pairing, one can talk about orthonormal basis for $\mathcal{I}(\chi)$. We will denote such an orthonormal basis by $\mathcal{B}(\chi)$.

For any element $\varphi \in \mathcal{I}(\chi)$, one can define the corresponding Eisenstein series by just averaging over $P(\mathbb{F})\backslash G(\mathbb{F})$ and analytic continuation. We will denote the corresponding Eisenstein series by $E_{\chi,\varphi}$. When $M = G$, $E_{\chi,\varphi} = \varphi$.

For rapidly decreasing functions on $Z_{\mathbb{A}}G(\mathbb{F})\backslash G(\mathbb{A})$, we have the following formulae

$$F = \int_{\chi\in\mathcal{X}/\sim} \sum_{\varphi\in\mathcal{B}(\chi)} < F, E_{\chi,\varphi} > E_{\chi,\varphi} d\chi, \tag{A.7}$$

$$< F_1, F_2 >= \int_{\chi\in\mathcal{X}/\sim} \sum_{\varphi\in\mathcal{B}(\chi)} < F_1, E_{\chi,\varphi} > \overline{< F_2, E_{\chi,\varphi} >} d\chi. \tag{A.8}$$

A.2. **Proof continued.** Now we shall finish the proof of Proposition 2.10. As we've already proved the proposition for the cuspidal part and we can use Cauchy-Schwartz inequality to piece together, we assume from now on that F_1 and F_2 have only continuous spectrums.

By (A.8),

$$\begin{aligned} < F_1, \rho(a([\mathcal{N}]))F_2 > &= \int_{\chi\in\mathcal{X}_T} \sum_{\varphi\in\mathcal{B}(\chi)} < F_1, E_{\chi,\varphi} > \overline{< \rho(a([\mathcal{N}]))F_2, E_{\chi,\varphi} >} d\chi \\ &= \int_{\chi\in\mathcal{X}_T} \sum_{\varphi\in\mathcal{B}(\chi)} < F_1, E_{\chi,\varphi} > \overline{< F_2, \rho(a([\mathcal{N}])^{-1})E_{\chi,\varphi} >} d\chi. \end{aligned} \tag{A.9}$$

Note that

$$\rho(a([\mathcal{N}])^{-1})E_{\chi,\varphi} = E_{\chi,\rho(a([\mathcal{N}])^{-1})\varphi},$$

where $\rho(a([\mathcal{N}])^{-1})\varphi$ is still in $\mathcal{I}(\chi)$ and $K-$finite. In particular, we can decompose $\rho(a([\mathcal{N}])^{-1})\varphi$ using the orthonormal basis $\mathcal{B}(\chi)$:

$$\rho(a([\mathcal{N}])^{-1})\varphi = \sum_{\varphi'\in\mathcal{B}(\chi)} < \rho(a([\mathcal{N}])^{-1})\varphi, \varphi' >_{\text{Eis}} \varphi'. \tag{A.10}$$

Note this is a purely local argument, and the sum on the right hand side is just a finite sum if we pick the basis properly. Correspondingly,

$$\rho(a([\mathcal{N}])^{-1})E_{\chi,\varphi} = \sum_{\varphi'\in\mathcal{B}(\chi)} < \rho(a([\mathcal{N}])^{-1})\varphi, \varphi' >_{\text{Eis}} E_{\chi,\varphi'}. \tag{A.11}$$

Now the part associated to χ in (A.9) becomes

$$\begin{aligned} &\sum_{\varphi\in\mathcal{B}(\chi)} < F_1, E_{\chi,\varphi} > \overline{< F_2, \rho(a([\mathcal{N}])^{-1})E_{\chi,\varphi} >} \\ &= \sum_{\varphi,\varphi'} < F_1, E_{\chi,\varphi} >< \rho(a([\mathcal{N}])^{-1})\varphi, \varphi' >_{\text{Eis}} \overline{< F_2, E_{\chi,\varphi'} >} \\ &=< \sum_{\varphi} < F_1, E_{\chi,\varphi} > \rho(a([\mathcal{N}])^{-1})\varphi, \sum_{\varphi'} < F_2, E_{\chi,\varphi'} > \varphi' >_{\text{Eis}} \\ &=< \sum_{\varphi} < F_1, E_{\chi,\varphi} > \varphi, \rho(a([\mathcal{N}])) \sum_{\varphi'} < F_2, E_{\chi,\varphi'} > \varphi' >_{\text{Eis}} . \end{aligned} \tag{A.12}$$

For each $\sum_{\varphi\in\mathcal{B}(\chi)} < F_i, E_{\chi,\varphi} > \varphi$ in the expression above we have the following lemma:

Lemma A.2. *If F_i is K_i-invariant, then for any cuspidal datum χ associated to T,*

$$\sum_{\varphi\in\mathcal{B}(\chi)} < F_i, E_{\chi,\varphi} > \varphi \tag{A.13}$$

is also K_i- invariant.

Proof. First of all (A.13) is independent of the choices of the basis. In particular one can pick an orthonormal basis for $\mathcal{I}(\chi)^{K_i}$ first, then extend it to an orthonormal basis for $\mathcal{I}(\chi)$. To prove the lemma, it is enough to show for this basis that if $\varphi \in \mathcal{B}(\chi) - \mathcal{B}(\mathcal{I}(\chi)^{K_i})$, then $< F_i, E_{\chi,\varphi} >= 0$.

By definition and the standard unfolding technique,

$$\begin{aligned} < F_i, E_{\chi,\varphi} > &= \int_{g\in Z_{\mathbb{A}}\mathrm{GL}_2(\mathbb{F})\backslash \mathrm{GL}_2(\mathbb{A})} F_i(g)\overline{E_{\chi,\varphi}(g)}dg \\ &= \int_{g\in Z_{\mathbb{A}}B(\mathbb{F})\backslash \mathrm{GL}_2(\mathbb{A})} F_i(g)\overline{\varphi(g)}dg \\ &= \int_{g\in Z_{\mathbb{A}}B(\mathbb{F})\backslash \mathrm{GL}_2(\mathbb{A})/K_i} F_i(g) \int_{k\in g^{-1}Z_{\mathbb{A}}B(\mathbb{F})g\cap K_i\backslash K_i} \overline{\varphi(gk)}dkdg. \end{aligned} \tag{A.14}$$

Note that if $< F_i, E_{\chi,\varphi} >$ is ever going to be nonzero, both F and φ have to be invariant under $Z_{\mathbb{A}} \cap K_i$. If $a \in g^{-1}Z_{\mathbb{A}}B(\mathbb{F})g \cap K_i$, then there exists $b \in Z_{\mathbb{A}}B(\mathbb{F}) \cap gK_ig^{-1}$ such that

$$ga = bg. \tag{A.15}$$

It's easily seen that $Z_{\mathbb{A}}B(\mathbb{F}) \cap gK_ig^{-1} = Z_{\mathbb{A}} \cap gK_ig^{-1} = Z_{\mathbb{A}} \cap K_i$, thus

$$\varphi(gak) = \varphi(bgk) = \varphi(gk). \tag{A.16}$$

So $\int_{k\in K_i} \overline{\varphi(gk)}dk$ will be a nontrivial multiple of $\int_{k\in g^{-1}Z_{\mathbb{A}}B(\mathbb{F})g\cap K_i\backslash K_i} \overline{\varphi(gk)}dk$. They are simultaneously zero or nonzero.

Then we'd like to see for every g, whether the following integral is zero or not:

$$\int_{K_i} \overline{\varphi(gk)}dk = \prod_v \int_{K_{i,v}} \overline{\varphi_v(g_vk_v)}dk_v. \tag{A.17}$$

Now fix g. For every v, consider the double coset decomposition

$$\mathrm{GL}_2(\mathbb{F}_v) = \coprod_{a_i} Ba_iK_{i,v},$$

where $\{a_i\}$ is the set of double coset representatives. Locally we can write $g_v = b_va_{g(v)}k'_v$, where $a_{g(v)} \in \{a_i\}$. Note for almost all places, $K_{i,v} = K_v$ and $g_v \in K_v$, so b_v and $a_{g(v)}$ will be trivial there. Then (A.17) is zero if and only if

$$\prod_v \int_{K_{i,v}} \overline{\varphi_v(a_{g(v)}k_v)}dk_v \tag{A.18}$$

is zero.

But for every fixed g this integral is the same (up to a nonzero constant) as the pairing $<\cdot,\cdot>_{\mathrm{Eis}}$ in $\mathcal{I}(\chi)$ between φ and another element $\varphi' \in \mathcal{I}(\chi)$ whose local component at v is only supported on $Ba_{g(v)}K_{i,v}$. This element φ' is clearly in $\mathcal{I}(\chi)^{K_i}$, so by the choice of φ,

$$<\varphi',\varphi>_{\mathrm{Eis}}=0. \tag{A.19}$$

So (A.17) is zero and $<F_i, E_{\chi,\varphi}>=0$ for $\varphi\in\mathcal{B}(\chi)-\mathcal{B}(\mathcal{I}(\chi)^{K_i})$. □

Recall the pairing $<\cdot,\cdot>_{\mathrm{Eis}}$ is directly a product of local pairings, and for the local pairing, we can use Lemma 2.8 to bound the local matrix coefficient. Note the local components of $\mathcal{I}(\chi)$ are always tempered. By taking a product and using the result that $\sum_{\varphi\in\mathcal{B}(\chi)} <F_i, E_{\chi,\varphi}>\varphi$ is K_i-invariant, we can get

$$\begin{aligned}
&\sum_{\varphi\in\mathcal{B}(\chi)} <F_1, E_{\chi,\varphi}>\overline{<F_2,\rho(a([\mathcal{N}])^{-1})E_{\chi,\varphi}>}\\
&\ll_{\epsilon,\mathbb{F}} [K_S:K_{1,S}]^{1/2}[K_S:K_{2,S}]^{1/2}N^{-1/2+\epsilon}\|\sum_{\varphi}<F_i,E_{\chi,\varphi}>\varphi\|_{\mathrm{Eis}}\|\sum_{\varphi}<F_2,E_{\chi,\varphi}>\varphi\|_{\mathrm{Eis}}\\
&\leq [K_S:K_{1,S}]^{1/2}[K_S:K_{2,S}]^{1/2}N^{\alpha-1/2+\epsilon}(\sum_{\varphi}|<F_1,E_{\chi,\varphi}>|^2)^{1/2}(\sum_{\varphi}|<F_2,E_{\chi,\varphi}>|^2)^{1/2},
\end{aligned} \tag{A.20}$$

for any bound α towards Ramanujan Conjecture. Finially when we do the integral in cuspidal datum χ, just apply Cauchy-Schwartz inequality.

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