A WEAK KERNEL FORMULA FOR BESSEL FUNCTIONS

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Abstract. In this paper, we prove a weak kernel formula of Bessel functions attached to irreducible generic representations of p-adic $GL(n)$. As an application, we show that the Bessel function defined by Bessel distribution coincides with the Bessel function defined via uniqueness of Whittaker models on the open Bruhat cell.

1. Introduction

For a generic irreducible smooth representation π of $GL(n, F)$ with its contragredient $\tilde{\pi}$, where F is a p-adic field, there are two ways to attach Bessel functions to π. The first is via Bessel distribution $B_{l,l'}$, where l, l' are Whittaker functionals on π and $\tilde{\pi}$ respectively. Such distributions were used by Gelfand and Kazhdan ([14]), and Shalika ([18]) to prove uniqueness of Whittaker functionals. In [2], for more general quasi-split groups, E.Baruch showed that the restriction of $B_{l,l'}$ to the big open Bruhat cell is given by a locally constant function $j_0(g)$.

On the other hand, for Whitaker function $W \in \mathcal{W}(\pi)$, where $\mathcal{W}(\pi)$ denotes the Whittaker model of π, the integral

$$\int_{N_n} W(gu)\psi^{-1}(u)du$$

converges in the stable sense if g is in the big open cell, and thus defines a function $j_\pi(g)$ there such that

$$j_\pi(g)W(I) = \int_{N_n} W(gu)\psi^{-1}(u)du$$

because of uniqueness of Whittaker functionals. This Bessel function j_π was first defined in this way by David Soudry in [19] for $GL(2, F)$, and then was generalized by E.Baruch to $GL(n, F)$. For more details see [3, 5].

In the case $GL(3, F)$ (also $GL(2, F)$), E.Baruch in [1, 4] proved the Bessel function $j_\pi(g)$ is locally integrable on the whole group, and gives the Bessel distribution $B_{l,l'}$ on $GL(3, F)$, which implies that the above two functions j_0, j_π are the same.

These Bessel functions and Bessel distributions have many applications to the theory of automorphic forms, to list a few, for example see [6, 7, 8, 10, 11, 17]. Thus it is desirable to generalize E.Baruch's important results to more general $GL(n, F)$. The obstacle is the local integrability of $j_\pi(g)$. This is done by E.Baruch in [4] for $GL(3, F)$ case using Shalika germs, but seems to be very difficult in general.

In this paper, we generalize some above results to $GL(n, F)$. More precisely, we proved the following result.

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Theorem 1. If π is irreducible, smooth and generic, then we have

$$j_0(g) = j_\pi(g)$$

for all $g \in N_n\omega_n A_n N_n$, where N_n is the upper triangular unipotent subgroup, A_n is the subgroup of diagonal matrices, and ω_n is the longest element in the Weyl group.

Once we know local integrability of Bessel functions, we can also show $B_{l,l'}$ is given by j_π on the whole group following the method in [4].

We here essentially follow Baruch's approach, and an important ingredient in the proof is a kernel formula, which has its own interests and can be stated as follows.

Theorem 2(Theorem 4.2, Theorem 7.1). Assume either π is supercuspidal and W is any Whittaker function of π, or π is irreducible smooth and generic and W is the normalized Howe vectors with sufficiently large level(see section 5 for the definition of Howe vectors). For any $b\omega_n$, $b = diag(b_1, ..., b_n) \in A_n$, and any $W \in \mathcal{W}(\pi, \psi)$, we have

$$W(b\omega_n) =$$

$$\int j_\pi\left(b\omega_n \begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ & & & & 1 \end{pmatrix}^{-1}\right) W\begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ & & & & 1 \end{pmatrix}$$

$$|a_1|^{-(n-1)}da_1|a_2|^{-(n-2)}dx_{21}da_2\cdots|a_{n-1}|^{-1}dx_{n-1,1}\cdots dx_{n-1,n-2}da_{n-1}$$

where the right side is an iterated integral, a_i is integrated over $F^\times \subset F$ for $i = 1, ..., n-1$, x_{ij} is integrated over F for all relevant i, j, and all measures are additive self-dual Haar measures on F.

We remark that in the case of general generic representation π, this kernel formula is expected to be true for a wide class of Whittaker functions of π(though not all of them), but currently we are only able to prove it for Howe vectors which is sufficient for the purpose of this paper.

Such formula was first proved by David Soudry in [19] for generic irreducible representations of $GL(2, F)$, and then was generalized to $GL(3, F)$ by E.Baruch in [4]. Due to the lack of local integrability of j_π, we have to write the above integral as an iterated integral.

E.Baruch in [2] showed the existence of j_0 for smooth generic irreducible representations of quasi-split reductive groups over local fields of characteristic zero. Recently E.Lapid and Zhengyu Mao in [16] defined j_π using uniqueness of Whittaker functionals for split reductive groups. It is interesting to see if the results here can be generalized to these cases.

The paper is organized as follows. In section 2 and 3, we recall some results about Bessel functions $j_\pi(g)$ and Bessel distributions $B_{l,l'}$. Section 4 is to prove the weak kernel formula. Section 5 is devoted to prove some properties of Howe vectors, which will be needed later. In section 6 we show that these two Bessel functions are equivalent in the supercuspidal case. In the last section we generalize the results to generic case.

Notations.

Let F be a p-adic field with ring of integers $\mathcal{O}$, use $|\cdot|$ to denote the valuation on F. We will always fix a self-dual Haar measure on F. Let $K_n = GL(n, \mathcal{O})$ and $G_n = GL(n, F)$, π a generic irreducible smooth admissible representation of G_n, with its contragredient $\tilde{\pi}$. Let N_n be the maximal unipotent subgroup of upper triangular matrices. Let A_n be the group of diagonal matrices. Let

$\bar{N_n}$ be the transpose of N_n. $B_n = A_n N_n$, $\bar{B_n} = A_n \bar{N_n}$.

Let ψ be a nontrivial additive character of F with conductor exactly $\mathcal{O}$. We extend ψ to a character of N_n by $\psi(u) = \psi(\sum_{i=1}^{n-1} u_{i,i+1})$ if $u = (u_{ij}) \in N_n$, and still denote it as ψ.

Use l, l' to denote the Whittaker functionals on $\pi, \tilde{\pi}$ with respect to ψ and ψ^{-1}, respectively. Let $\mathcal{W} = \mathcal{W}(\pi, \psi)$, $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}(\tilde{\pi}, \psi^{-1})$ be the corresponding Whittaker models.

Let $\mathbb{W}$ be the Weyl group of G_n, and use ω_n to denote the longest Weyl element in $\mathbb{W}$, i.e. $\omega_n = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$.

2. Bessel functions

In this section, we review some results about Bessel functions $j_\pi(g)$ in [5]. For our purpose, we will restrict to supercuspidal representations, though most of the notions and results can be generalized to smooth irreducible generic representations. We refer to [5] for more details in general.

So let (π, V) be an irreducible supercuspidal representation of G_n. If ψ is a nondegenerate character of N_n, use $\mathcal{W} = \mathcal{W}(\pi, \psi)$ to denote the Whittaker model of π w.r.t. ψ. If $M > 0$ is a positive constant, let

$$A_n^M = \{a \in A_n : |\frac{a_i}{a_{i+1}}| < M \quad for \quad i = 1, 2, ..., n-1, if \quad a = diag(a_1, ..., a_n)\}$$

Note that as $M \to \infty$, A_n^M cover A_n.

We start with the following important result of Baruch.

Proposition 2.1. *For any $W \in \mathcal{W}(\pi, \psi)$, $M > 0$, the function on $A_n^M \times N_n$ defined by*

$$(a, u) \to W(a\omega_n u)$$

is compactly supported in N_n with support independent of $a \in A_n^M$. That is, if $W(a\omega_n u) \neq 0$, with $a \in A_n^M$, $u \in N_n$, then there exists a compact subset $U \subset N_n$, which is independent of a, such that $u \in U$.

Proof. This follows from Theorem 5.7 and Lemma 6.1 in [5]. □

The above result allows us to define Bessel functions for supercuspidal representations as follows. Take $W \in \mathcal{W}$. Consider the integrals for $g \in N_n A_n \omega_n N_n$

$$\int_{Y_i} W(gu)\psi^{-1}(u)du$$

where $Y_1 \subset Y_2 \subset ... \subset Y_i \subset Y_{i+1} \subset ...$ is an increasing filtration of N_n with compact open subgroups.

By Proposition 2.1, if Y_i is large enough, these integrals become stable. The stable limit is independent of the choices of sequence $\{Y_i\}$. Use $\int_{N_n}^*$ to denote this limit, which defines a nontrivial Whittaker functional on $\mathcal{W}$. Thus there exists a scalar $j_{\pi,\psi}(g)$ such that

$$\int_{N_n}^* W(gu)\psi^{-1}(u)du = j_{\pi,\psi}(g)W(I)$$

Definition 2.2. *The assignment $g \to j_\pi(g) = j_{\pi,\psi}$ defines a function on $N_n A_n \omega_n N_n$, which is called the Bessel function of π attached to ω_n.*

We extend j_π to G_n by putting $j_\pi(g)=0$ if $g\notin N_nA_n\omega_nN_n$, and still use j_π to denote it and call it the Bessel function of π.

- j_π is locally constant on $N_nA_n\omega_nN_n$.

- For any $u_1,u_2\in N_n$, any $g\in G_n$, we have $j_\pi(u_1gu_2)=\psi(u_1)\psi(u_2)j_\pi(g)$.

- One may also attach Bessel functions to other Weyl elements. For more details, see [5]. For Bessel functions defined in this way for split reductive groups, see [16].

For $W\in\mathcal{W}$, let $\widetilde{W}(g)=W(\omega_n\cdot {}^tg^{-1})$, then $\{\widetilde{W}:W\in\mathcal{W}\}$ is the Whittaker model for the contragredient $\widetilde{\pi}$ with respect to ψ^{-1}. By **Corollary 8.5** in [5], we have the following relation

$$j_{\tilde{\pi},\psi^{-1}}(g)=j_{\pi,\psi}(g^{-1}),\qquad g\in B_n\omega_nB_n \tag{2.1}$$

3. Bessel distributions

In this section, we collect some useful properties about Bessel distributions. Let π be an irreducible smooth generic representation of G_n. Let π^* and $\widetilde{\pi}^*$ denote the linear dual of π and $\widetilde{\pi}$ respectively. Let f be a locally constant function with compact support on G_n, take $l\in\pi^*$, $l'\in\widetilde{\pi}^*$. Define $\widetilde{\pi}(f)l'$ as

$$\widetilde{\pi}(f)l'=\int_{G_n}f(g)\widetilde{\pi}(g)l'dg$$

or equivalently, for any $\tilde{v}\in\widetilde{\pi}$,

$$\begin{aligned}<\widetilde{\pi}(f)l',\tilde{v}>&=\int_{G_n}f(g)<\widetilde{\pi}(g)l',\tilde{v}>dg\\&=\int_{G_n}f(g)<l',\widetilde{\pi}(g^{-1})(\tilde{v})>dg\end{aligned}$$

then $\widetilde{\pi}(f)l'$ is a smooth linear functional on $\widetilde{\pi}$, hence can be identified with a vector $v_{f,l'}\in\pi$.

Definition 3.1. *Define Bessel distribution $B_{l,l'}(f)$ as*

$$B_{l,l'}(f)=l(v_{f,l'})$$

- The definition of $B_{l,l'}$ depends on l,l' and Haar measure dg on G_n.

- When both l,l' are Whittaker functionals, such $B_{l,l'}$ is the Bessel distribution first studied by Gelfand and Kazhdan ([14]) for GL_n in p-adic case, and by Shalika ([18]) in archimedean case, by Baruch ([2]) for quasi-split groups in both non-archemedean and archemedean cases.

- It was shown in [2] that when both l,l' are Whittaker functionals with respect to ψ and ψ^{-1} respectively, $B_{l,l'}$ can be represented by a locally constant function in non-archimedean case, and by a real analytic function in archimedean case, when restricted to the open Bruhat cell. We will denote this function by $j_0(g)$, $g\in N_nA_n\omega_nN_n$.

- When l' is the Whittaker functional on $\widetilde{\pi}$ with respect to ψ^{-1}, for any $\hat{W}_{\widetilde{v}}\in\mathcal{W}(\widetilde{\pi},\psi^{-1})$, we have

$$<v_{f,l'},\widetilde{v}>=\int_{G_n}f(g)\hat{W}_{\widetilde{v}}(g^{-1})dg$$

From now on, we will always assume l, l' are nonzero Whittaker functionals. Fix l on π with respect to ψ, for $W \in \mathcal{W}(\pi,\psi)$, $\hat{W} \in \mathcal{W}(\tilde{\pi},\psi^{-1})$, if either $W\begin{pmatrix} h & \\ & 1\end{pmatrix}$ or $\hat{W}\begin{pmatrix} h & \\ & 1\end{pmatrix}$ is compactly supported mod N_{n-1}, by results in [9], we can normalize l', so that

$$< v, \tilde{v} >= \int_{N_{n-1}\backslash G_{n-1}} W_v\begin{pmatrix} h & \\ & 1\end{pmatrix} \hat{W}_{\tilde{v}}\begin{pmatrix} h & \\ & 1\end{pmatrix} dh$$

where the right side integral defines a P_n invariant pairing between π and $\tilde{\pi}$, here P_n is the so-called mirabolic subgroup of G_n.

Lemma 3.2. *With the above normalization, if $\hat{W}_{\tilde{v}}\begin{pmatrix} h & \\ & 1\end{pmatrix}$ is compactly supported mod N_{n-1}, we have for any $f \in C_c^\infty(G)$,*

$$\int_{G_n} f(g)\hat{W}_{\tilde{v}}(g^{-1})dg = \int_{N_{n-1}\backslash G_{n-1}} B_{l,l'}\left(L\begin{pmatrix} h & \\ & 1\end{pmatrix}.f\right)\hat{W}_{\tilde{v}}\begin{pmatrix} h & \\ & 1\end{pmatrix} dh$$

where L denotes the left action of G_n on f.

Proof. Let $v_{f,l'}$ be the vector in π as in Definition 3.1, so $B_{l,l'}(f) = l(v_{f,l'})$. For $\widetilde{v} \in \widetilde{\pi}$, denote $\underline{h} = \begin{pmatrix} h & \\ & 1\end{pmatrix}$, we have

$$< v_{L(\underline{h})f,l'}, \widetilde{v} >= \int_{G_n} f(\underline{h}^{-1}g) < l', \widetilde{\pi}(g^{-1})\widetilde{v} > dg$$

$$= \int_{G_n} f(g) < l', \widetilde{\pi}(g^{-1})\widetilde{\pi}(\underline{h}^{-1})\widetilde{v} > dg$$

$$=< v_{f,l'}, \widetilde{\pi}(\underline{h}^{-1})\widetilde{v} >=< \pi(\underline{h})v_{f,l'}, \widetilde{v} >$$

Thus $B_{l,l'}\left(L\begin{pmatrix} h & \\ & 1\end{pmatrix}.f\right) = l(v_{L(\underline{h})f,l'}) = l(\pi(\underline{h})v_{f,l'}) = W_{v_{f,l'}}(\underline{h})$.

It follows that the right side in lemma is $< v_{f,l'}, \widetilde{v} >$ by normalization, which equals the left side. □

4. Kernel formula

In this section, we will prove a weak kernel formula for Bessel functions attached to supercuspidal representations as in section 1, which is the first main result of this paper. The method of the proof follows that of Baruch in [4] by generalizing corresponding results there to $GL(n)$.

So through out of this section, π will be an irreducible supercuspidal representation of G_n. Let Y_i be the unipotent part of the parabolic subgroup of G_n associated to the partition (n-i+1,1,...,1), $1 \le i \le n$. Note that $Y_1 = \{I_n\}$, $Y_n = N_n$. By Proposition 2.1, we have the following lemma.

Lemma 4.1. *For any $b\omega_n$, $b = diag(b_1, ..., b_n) \in A_n$, then as a function of $u_i \in F^{n-i}$, the function*

$$\int_{Y_i} W\left(b\omega_n y_i \begin{pmatrix} I & u_i & \\ & 1 & \\ & & I_{i-1}\end{pmatrix}\right)\psi^{-1}(y_i)dy_i$$

is compactly supported, where $\psi(y_i)$ is the restriction of Whittaker character to $Y_i \subset N_n$.

Proof. By Proposition 2.1, if $b \in A_n^M$ for some constant $M > 0$, then the function $W(b\omega_n u)$ is compactly supported as a function $u \in N_n$, with support independent of $b \in A_n^M$. Then its restriction to

$$\{y_i \begin{pmatrix} I & u_i & \\ & 1 & \\ & & I_{i-1} \end{pmatrix} : y_i \in Y_i, u_i \in F^{n-i}\}$$

is again compactly supported. Now the lemma follows immediately. □

Theorem 4.2. *(weak kernel formula) For any $b\omega_n$, $b = diag(b_1, ..., b_n) \in A_n$, and any $W \in \mathcal{W}$, we have*

$$W(b\omega_n) =$$

$$\int j_\pi \left(b\omega_n \begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ & & & & 1 \end{pmatrix}^{-1} \right) W \begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ & & & & 1 \end{pmatrix}$$

$$|a_1|^{-(n-1)} da_1 |a_2|^{-(n-2)} dx_{21} da_2 \cdots |a_{n-1}|^{-1} dx_{n-1,1} \cdots dx_{n-1,n-2} da_{n-1}$$

where the right side is an iterated integral, a_i is integrated over $F^\times \subset F$ for $i = 1, ..., n-1$, x_{ij} is integrated over F for all relevant i, j, and all measures are additive self-dual Haar measures on F.

Proof. The proof is based an inductive argument. We begin with the proof of the following identity, for any $b\omega_n$,

$$\int_{Y_{n-1}} W(b\omega_n y_{n-1}) \psi(-y_{n-1}) dy_{n-1} =$$

$$\int_{F^\times} j_\pi \left(b\omega_n \begin{pmatrix} a_1^{-1} & \\ & I_{n-1} \end{pmatrix} \right) W \begin{pmatrix} a_1 & \\ & I_{n-1} \end{pmatrix} |a_1|^{-(n-1)} da_1 \tag{4.1}$$

where da_1 is the additive Haar measure on F. Note that because of Proposition 2.1, the left side integral is absolutely convergent. Because π is supercuspidal, W is compactly supported mod $N_n Z_n$, where Z_n is the center of G_n, then $W \begin{pmatrix} a_1 & \\ & I_{n-1} \end{pmatrix}$ is compactly supported in $F^\times$ as a function of a_1. Since j_π is locally constant on the big cell, $j_\pi \left(b\omega_n \begin{pmatrix} a_1^{-1} & \\ & I_{n-1} \end{pmatrix} \right)$ is also locally constant as a function of a_1, then the right side integral reduces to a finite sum, and hence is also absolutely convergent.

For this consider the following function $M_{n-1}(x) : F \to \mathbb{C}$ by

$$M_{n-1}(x) = \int_{Y_{n-1}} W \left(b\omega_n y_{n-1} \begin{pmatrix} 1 & x & \\ & 1 & \\ & & I_{n-2} \end{pmatrix} \right) \psi(-y_{n-1}) dy_{n-1}$$

By Lemma 4.1, this is a compactly supported function in x. Thus its Fourier transform $\widehat{M}_{n-1}(y)$ is also compactly supported, and we have Fourier inversion formula

$$M_{n-1}(x) = \int_F \widehat{M}_{n-1}(y) \psi(yx) dy = \int_{F^\times} \widehat{M}_{n-1}(y) \psi(yx) dy$$

where the last equality follows from the facts that dy is the additive Haar measure, and $F^\times$ is of full measure in F.

Put $x = 0$, we get

$$M_{n-1}(0) = \int_{F^\times} \widehat{M}_{n-1}(y)dy$$

Now we compute the Fourier coefficient $\widehat{M}_{n-1}(y)$ when $y = a_1 \neq 0$.

$$\begin{aligned}
&\widehat{M}_{n-1}(a_1) \\
= \quad & \int_F \int_{Y_{n-1}} W\left(b\omega_n y_{n-1}\begin{pmatrix}1 & x & \\ & 1 & \\ & & I_{n-2}\end{pmatrix}\right)\psi(-y_{n-1})\psi(-a_1x)dy_{n-1}dx \\
= \quad & \int_F \int_{Y_{n-1}} W\left(b\omega_n y_{n-1}\begin{pmatrix}1 & a_1^{-1}x & \\ & 1 & \\ & & I_{n-2}\end{pmatrix}\right)\psi(-y_{n-1})\psi(-x)|a_1|^{-1}dy_{n-1}dx \\
= \quad & \int_F \int_{Y_{n-1}} W\left(b\omega_n \begin{pmatrix}a_1^{-1} & \\ & I_{n-1}\end{pmatrix} y_{n-1}\begin{pmatrix}1 & x & \\ & 1 & \\ & & I_{n-2}\end{pmatrix}\begin{pmatrix}a_1 & \\ & I_{n-1}\end{pmatrix}\right)\psi(-y_{n-1})\psi(-x)|a_1|^{-(n-1)}dy_{n-1}dx
\end{aligned}$$

Put $y_n = y_{n-1}\begin{pmatrix}1 & x & \\ & 1 & \\ & & I_{n-2}\end{pmatrix} \in Y_n = N_n$, the above integral becomes

$$\begin{aligned}
= \quad & \int_{Y_n} W\left(b\omega_n \begin{pmatrix}a_1^{-1} & \\ & I_{n-1}\end{pmatrix} y_n \begin{pmatrix}a_1 & \\ & I_{n-1}\end{pmatrix}\right)\psi^{-1}(y_n)|a_1|^{-(n-1)}dy_n \\
= \quad & j_\pi\left(b\omega_n \begin{pmatrix}a_1^{-1} & \\ & I_{n-1}\end{pmatrix}\right) W\begin{pmatrix}a_1 & \\ & I_{n-1}\end{pmatrix}|a_1|^{-(n-1)}
\end{aligned}$$

where the last equality follows form the identity

$$\int_{Y_n}\left(\pi\begin{pmatrix}a_1 & \\ & I_{n-1}\end{pmatrix}.W\right)\left(b\omega_n\begin{pmatrix}a_1^{-1} & \\ & I_{n-1}\end{pmatrix}y_n\right)\psi^{-1}(y_n)dy_n =$$
$$j_\pi\left(b\omega_n\begin{pmatrix}a_1^{-1} & \\ & I_{n-1}\end{pmatrix}\right)\left(\pi\begin{pmatrix}a_1 & \\ & I_{n-1}\end{pmatrix}.W\right)(I)$$

which is the definition of j_π.

Now we get

$$\begin{aligned}
& \int_{Y_{n-1}} W\left(b\omega_n y_{n-1}\right)\psi(-y_{n-1})dy_{n-1} \\
= \quad & M_{n-1}(0) = \int_{F^\times}\widehat{M}_{n-1}(a_1)da_1 \\
= \quad & \int_{F^\times} j_\pi\left(b\omega_n\begin{pmatrix}a_1^{-1} & \\ & I_{n-1}\end{pmatrix}\right)W\begin{pmatrix}a_1 & \\ & I_{n-1}\end{pmatrix}|a_1|^{-(n-1)}da_1
\end{aligned}$$

which is exactly what we want to show in (4.1).

Now set $h_2 = \begin{pmatrix}1 & & \\ x_{21} & a_2 & \\ & & I_{n-2}\end{pmatrix}$ with $x_{21} \in F$, $a_2 \in F^\times$, we also use h_2 to denote the left upper corner 2×2 matrix. We first note the following identity of product of matrices

$$b\omega_n h_2^{-1} = \begin{pmatrix}I_{n-2} & & \\ & 1 & -b_{n-1}b_n^{-1}a_2^{-1}x_{21} \\ & & 1\end{pmatrix}\begin{pmatrix}I_{n-2} & & \\ & a_2^{-1} & \\ & & 1\end{pmatrix}b\omega_n$$

Hence

$$\begin{aligned} W_v\left(b\omega_n h_2^{-1} y_{n-1} h_2\right) &= W_v\left(\begin{pmatrix} I_{n-2} & & \\ & 1 & -b_{n-1}b_n^{-1}a_2^{-1}x_{21} \\ & & 1 \end{pmatrix}\begin{pmatrix} I_{n-2} & & \\ & a_2^{-1} & \\ & & 1 \end{pmatrix} b\omega_n y_{n-1} h_2\right) \\ &= \psi(-b_{n-1}b_n^{-1}a_2^{-1}x_{21}) W_{\pi(h_2)v}\left(\begin{pmatrix} I_{n-2} & & \\ & a_2^{-1} & \\ & & 1 \end{pmatrix} b\omega_n y_{n-1}\right) \end{aligned}$$

and

$$\begin{aligned} & j_\pi\left(b\omega_n h_2^{-1}\begin{pmatrix} a_1^{-1} & \\ & I_{n-1} \end{pmatrix}\right) \\ = & j_\pi\left(\begin{pmatrix} I_{n-2} & & \\ & 1 & -b_{n-1}b_n^{-1}a_2^{-1}x_{21} \\ & & 1 \end{pmatrix}\begin{pmatrix} I_{n-2} & & \\ & a_2^{-1} & \\ & & 1 \end{pmatrix} b\omega_n \begin{pmatrix} a_1^{-1} & \\ & I_{n-1} \end{pmatrix}\right) \\ = & \psi(-b_{n-1}b_n^{-1}a_2^{-1}x_{21}) j_\pi\left(\begin{pmatrix} I_{n-2} & & \\ & a_2^{-1} & \\ & & 1 \end{pmatrix} b\omega_n \begin{pmatrix} a_1^{-1} & \\ & I_{n-1} \end{pmatrix}\right) \end{aligned}$$

Now apply (4.1) to $b' = diag(b_1, ..., b_{n-2}, a_2^{-1}b_{n-1}, b_n)$, and $W = W_{\pi(h_2)v}$, we will get

$$\int_{Y_{n-1}} W_{\pi(h_2)v}\left(b'\omega_n y_{n-1}\right)\psi(-y_{n-1})dy_{n-1} =$$
$$\int_{F^\times} j_\pi\left(b'\omega_n\begin{pmatrix} a_1^{-1} & \\ & I_{n-1} \end{pmatrix}\right) W_{\pi(h_2)v}\begin{pmatrix} a_1 & \\ & I_{n-1} \end{pmatrix} |a_1|^{-(n-1)}da_1$$

Multiply by $\psi(-b_{n-1}b_n^{-1}a_2^{-1}x_{21})$ on both sides, and then

$$\int_{Y_{n-1}} W\left(b\omega_n h_2^{-1} y_{n-1} h_2\right)\psi(-y_{n-1})dy_{n-1} =$$

$$\int_{F^\times} j_\pi\left(b\omega_n h_2^{-1}\begin{pmatrix} a_1^{-1} & \\ & I_{n-1} \end{pmatrix}\right) W\left(\begin{pmatrix} a_1 & \\ & I_{n-1} \end{pmatrix} h_2\right)|a_1|^{-(n-1)}da_1 \tag{4.2}$$

Write $y_{n-1} = y_{n-2}\begin{pmatrix} I_2 & u_2 & \\ & 1 & \\ & & I_{n-3} \end{pmatrix}$ with u_2 a column vector in F^2, then the left side of (4.2) is

$$\int_{Y_{n-1}} W\left(b\omega_n y_{n-2}\begin{pmatrix} I_2 & h_2^{-1}.u_2 & \\ & 1 & \\ & & I_{n-3} \end{pmatrix}\right)\psi(-y_{n-2})\psi\begin{pmatrix} I_2 & -u_2 & \\ & 1 & \\ & & I_{n-3} \end{pmatrix}|a_2|^{(n-3)}dy_{n-1} \tag{4.3}$$

Now put

$$M_{n-2}(u_2) = \int_{Y_{n-2}} W\left(b\omega_n y_{n-2}\begin{pmatrix} I_2 & u_2 & \\ & 1 & \\ & & I_{n-3} \end{pmatrix}\right)\psi(-y_{n-2})dy_{n-2}$$

By Lemma 4.1, $M_{n-2}(u_2)$ is compactly supported, and its Fourier inversion formula is

$$M_{n-2}(u_2) = \int_{F^2}\widehat{M}_{n-2}(v_2)\psi(v_2u_2)dv_2 = \int_{F\times F^\times}\widehat{M}_{n-2}(x,y)\psi((x,y)u_2)dxdy$$

if we write $v_2 = (x, y)$ as a row vector, with $x \in F$, $y \in F^\times$, and the last equality follows from the facts that $dxdy$ are additive Haar measures and $F \times F^\times$ is of full measure in F^2.

Put $u_2 = 0$, we get

$$M_{n-2}(0) = \int_{F\times F^\times} \widehat{M}_{n-2}(x,y)dxdy \qquad (*)$$

Now we compute the Fourier coefficient $\widehat{M}_{n-2}(x,y)$ with $x = x_{21} \in F$, $y = a_2 \neq 0$, we have

$$\begin{aligned}
&\widehat{M}_{n-2}(x_{21},a_2)\\
=\ &\int_{F^2}\int_{Y_{n-2}} W\left(b\omega_n y_{n-2}\begin{pmatrix} I_2 & u_2 & \\ & 1 & \\ & & I_{n-3}\end{pmatrix}\right)\psi(-y_{n-2})\psi(-(x_{21},a_2)u_2)dy_{n-2}du_2\\
=\ &\int_{F^2}\int_{Y_{n-2}} W\left(b\omega_n y_{n-2}\begin{pmatrix} I_2 & u_2 & \\ & 1 & \\ & & I_{n-3}\end{pmatrix}\right)\psi(-y_{n-2})\psi\begin{pmatrix} I_2 & -h_2.u_2 & \\ & 1 & \\ & & I_{n-3}\end{pmatrix}dy_{n-2}du_2\\
=\ &\int W\left(b\omega_n y_{n-2}\begin{pmatrix} I_2 & h_2^{-1}.u_2 & \\ & 1 & \\ & & I_{n-3}\end{pmatrix}\right)\psi(-y_{n-2})\psi\begin{pmatrix} I_2 & -u_2 & \\ & 1 & \\ & & I_{n-3}\end{pmatrix}|a_2|^{-1}dy_{n-2}du_2
\end{aligned}$$

Thus by (4.3),

$$|a_2|^{-(n-2)}\int_{Y_{n-1}} W\left(b\omega_n h_2^{-1}y_{n-1}h_2\right)\psi(-y_{n-1})dy_{n-1} = \widehat{M}_{n-2}(x_{21},a_2)$$

and then by (4.2),

$$|a_2|^{-(n-2)}\int_{F^\times} j_\pi\left(b\omega_n h_2^{-1}\begin{pmatrix} a_1^{-1} & \\ & I_{n-1}\end{pmatrix}\right)W\left(\begin{pmatrix} a_1 & \\ & I_{n-1}\end{pmatrix}h_2\right)|a_1|^{-(n-1)}da_1 = \widehat{M}_{n-2}(x_{21},a_2)$$

Plug it into the Fourier inversion formula $(*)$, we find

$$\int_{Y_{n-2}} W\left(b\omega_n y_{n-2}\right)\psi(-y_{n-2})dy_{n-2} = M_{n-2}(0) = \int_{F\times F^\times}\int_{F^\times}$$

(4.4)

$$j_\pi\left(b\omega_n\begin{pmatrix} a_1 & & \\ x_{21} & a_2 & \\ & & I_{n-2}\end{pmatrix}^{-1}\right)W\left(b\omega_n\begin{pmatrix} a_1 & & \\ x_{21} & a_2 & \\ & & I_{n-2}\end{pmatrix}\right)|a_1|^{-(n-1)}da_1|a_2|^{-(n-2)}dx_{21}da_2$$

where we write

$$\begin{pmatrix} a_1 & \\ & I_{n-1}\end{pmatrix}h_2 = \begin{pmatrix} a_1 & & \\ x_{21} & a_2 & \\ & & I_{n-2}\end{pmatrix}.$$

Inductively, we will have

$$\int_{Y_2} W(b\omega_n y_2)\psi(-y_2)dy_2 =$$

$$\int j_\pi\left(b\omega_n\begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-2,1} & \cdots & x_{n-2,n-3} & a_{n-2} & \\ & & & & I_2\end{pmatrix}^{-1}\right)W\begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-2,1} & \cdots & x_{n-2,n-3} & a_{n-2} & \\ & & & & I_2\end{pmatrix}$$

$$|a_1|^{-(n-1)}da_1|a_2|^{-(n-2)}dx_{21}da_2\cdots|a_{n-2}|^{-2}dx_{n-2,1}\cdots dx_{n-2,n-3}da_{n-2} \tag{4.5}$$

where the right side is an iterated integral, a_i is integrated over $F^\times \subset F$ for $i = 1,...,n-2$, x_{ij} is integrated over F for all relevant i,j here, and all measures are additive self-dual Haar measures on F.

To prove the weak kernel formula, set

$$h_{n-1}=\begin{pmatrix} & I_{n-2} & & & \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ & & & & 1\end{pmatrix}$$

where $x_{n-1,i}\in F$, $i=1,2,...,n-2$, $a_{n-1}\in F^{\times}$. We also use h_{n-1} to denote the left upper corner matrix of size $(n-1)\times(n-1)$. Note that we have identity

$$b\omega_n h_{n-1}^{-1}=\begin{pmatrix}1 & & & & \\ & 1 & a_{n-1}^{-1}b_3^{-1}b_2x_{n-1,n-2} & \dots & a_{n-1}^{-1}b_n^{-1}b_2x_{n-1,1}\\ & & I_{n-2} & & \end{pmatrix}\begin{pmatrix}1 & & \\ & a_{n-1}^{-1} & \\ & & I_{n-2}\end{pmatrix}b\omega_n$$

Then we carry out the same argument as we derived (4.2) from (4.1), then by (4.5), we will have

$$\int_{Y_2}W(b\omega_n h_{n-1}^{-1}y_2h_{n-1})\psi(-y_2)dy_2=\int$$

$$j_\pi\left(b\omega_n h_{n-1}^{-1}\begin{pmatrix} a_1 & & & & & \\ x_{21} & a_2 & & & & \\ & & \ddots & & & \\ x_{n-2,1} & \cdots & x_{n-2,n-3} & a_{n-2} & & \\ & & & & & I_2\end{pmatrix}^{-1}\right)W\left(\begin{pmatrix} a_1 & & & & & \\ x_{21} & a_2 & & & & \\ & & \ddots & & & \\ x_{n-2,1} & \cdots & x_{n-2,n-3} & a_{n-2} & & \\ & & & & & I_2\end{pmatrix}h_{n-1}\right)$$

$$|a_1|^{-(n-1)}da_1|a_2|^{-(n-2)}dx_{21}da_2\cdots|a_{n-2}|^{-2}dx_{n-2,1}\cdots dx_{n-2,n-3}da_{n-2}\tag{4.6}$$

The left side integral is

$$\int_{F^{n-1}}W\left(b\omega_n\begin{pmatrix}I_{n-1} & h_{n-1}^{-1}.u_{n-1}\\ & 1\end{pmatrix}\right)\psi\begin{pmatrix}I_{n-1} & -u_{n-1}\\ & 1\end{pmatrix}du_{n-1}\tag{4.7}$$

Now let

$$M_1(u_{n-1})=W\left(b\omega_n\begin{pmatrix}I_{n-1} & u_{n-1}\\ & 1\end{pmatrix}\right)$$

which is a compactly supported function in column vector u_{n-1}. Its Fourier inversion formula is

$$\begin{aligned}M_1(u_{n-1}) &= \int_{F^{n-1}}\widehat{M}_1(v_{n-1})\psi(-v_{n-1}u_{n-1})dv_{n-1}\\ &= \int_{F^{n-2}\times F^{\times}}\widehat{M}_1(z_1,...,z_{n-1})\psi(-(z_1,...,z_{n-1})u_{n-1})dz_1...dz_{n-1}\end{aligned}$$

where we write $v_{n-1}=(z_1,...,z_{n-1})$, with $z_1,...,z_{n-2}\in F$, $z_{n-1}\in F^{\times}$, and the last equality follows from the facts that $dz_1...dz_{n-1}$ is the additive Haar measure and $F^{n-2}\times F^{\times}$ is of full measure in F^{n-1}.

Put $u_{n-1}=0$, we get

$$M_1(0)=\int_{F^{n-2}\times F^{\times}}\widehat{M}_1(z_1,...,z_{n-1})dz_1...dz_{n-1}\tag{$**$}$$

We compute the Fourier coefficient $\widehat{M}_1(z_1,...,z_{n-1})$, with $z_1=x_{n-1,1},...,z_{n-2}=x_{n-1,n-2}\in F, z_{n-1}=a_{n-1}\in F^{\times}$,

$$\begin{aligned}
&\widehat{M}_1(x_{n-1,1},...,x_{n-1,n-2},a_{n-1})\\
=&\int_{F^{n-1}} W\left(b\omega_n\begin{pmatrix} I_{n-1} & u_{n-1}\\ & 1\end{pmatrix}\right)\psi(-(x_{n-1,1},...,x_{n-1,n-2},a_{n-1})u_{n-1})du_{n-1}\\
=&\int_{F^{n-1}} W\left(b\omega_n\begin{pmatrix} I_{n-1} & u_{n-1}\\ & 1\end{pmatrix}\right)\psi\left(-\begin{pmatrix} I_{n-1} & h_{n-1}.u_{n-1}\\ & 1\end{pmatrix}\right)du_{n-1}\\
=&\int_{F^{n-1}} W\left(b\omega_n\begin{pmatrix} I_{n-1} & h_{n-1}^{-1}.u_{n-1}\\ & 1\end{pmatrix}\right)\psi\left(-\begin{pmatrix} I_{n-1} & u_{n-1}\\ & 1\end{pmatrix}\right)|a_{n-1}^{-1}|du_{n-1}
\end{aligned}$$

Thus by (4.7), we have

$$|a_{n-1}|^{-1}\int_{Y_2} W(b\omega_n h_{n-1}^{-1}y_2h_{n-1})\psi(-y_2)dy_2=\widehat{M}_1(x_{n-1,1},...,x_{n-1,n-2},a_{n-1})$$

Thus by (4.6),

$$\widehat{M}_1(x_{n-1,1},...,x_{n-1,n-2},a_{n-1})=|a_{n-1}|^{-1}\int$$

$$j_\pi\left(b\omega_n h_{n-1}^{-1}\begin{pmatrix} a_1 &&&&\\ x_{21} & a_2 &&&\\ && \ddots &&\\ x_{n-2,1} & \cdots & x_{n-2,n-3} & a_{n-2} &\\ &&&& I_2\end{pmatrix}^{-1}\right)W\left(\left(\begin{pmatrix} a_1 &&&&\\ x_{21} & a_2 &&&\\ && \ddots &&\\ x_{n-2,1} & \cdots & x_{n-2,n-3} & a_{n-2} &\\ &&&& I_2\end{pmatrix}h_{n-1}\right)\right)$$
$$|a_1|^{-(n-1)}da_1|a_2|^{-(n-2)}dx_{21}da_2\cdots|a_{n-2}|^{-2}dx_{n-2,1}\cdots dx_{n-2,n-3}da_{n-2}$$

Plug it into $(**)$, we finally find

$$W(b\omega_n)=$$
$$\int j_\pi\left(b\omega_n\begin{pmatrix} a_1 &&&&\\ x_{21} & a_2 &&&\\ && \ddots &&\\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} &\\ &&&& 1\end{pmatrix}^{-1}\right)W\begin{pmatrix} a_1 &&&&\\ x_{21} & a_2 &&&\\ && \ddots &&\\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} &\\ &&&& 1\end{pmatrix}$$
$$|a_1|^{-(n-1)}da_1|a_2|^{-(n-2)}dx_{21}da_2\cdots|a_{n-1}|^{-1}dx_{n-1,1}\cdots dx_{n-1,n-2}da_{n-1}$$

where the right side is an iterated integral, a_i is integrated over $F^\times\subset F$ for $i=1,...,n-1$, x_{ij} is integrated over F for all relevant i,j, and all measures are additive Haar measure on F. This finishes the proof.

□

• Since we don't have absolute convergence of right side integral, we have to write it as an iterated integral. If we know the local integrability of j_π, then using the same argument as in Lemma 5.3, [4], one can show that the right side integral is then absolutely convergent, and it equals

$$\int_{N_{n-1}\backslash G_{n-1}} j_\pi\left(y\begin{pmatrix} h^{-1} & \\ & 1\end{pmatrix}\right)W\begin{pmatrix} h & \\ & 1\end{pmatrix}dh$$

• The space of functions $\left\{W\begin{pmatrix} g & \\ & 1\end{pmatrix}: W\in\mathcal{W}\right\}$ is the Kirillov model of π. Theorem 4.2 gives the action of the longest Weyl element ω_n on this model in terms of Bessel functions. It thus follows that if we want to show two supercuspidal representations are equivalent, it suffices to show they have the same Bessel functions.

• In order to generalize the above argument to generic smooth irreducible representations of $GL(n,F)$, we need to know the space $\mathcal{W}^0$, as defined in section 5 of [5], is invariant under right translations by elements like

$$\begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ & & & & 1 \end{pmatrix}.$$

But this is not clear, and we plan to address this issue in future.

Corollary 4.3. *Let $\widetilde{W}(g) = W(\omega_n \cdot {}^t g^{-1}) \in \mathcal{W}(\widetilde{\pi}, \psi^{-1})$, then for any $b\omega_n$, $b \in A_n$, we have*

$$\widetilde{W}((b\omega_n)^{-1}) =$$
$$\int j_\pi \left(\begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ & & & & 1 \end{pmatrix} b\omega_n \right) \widetilde{W} \begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ & & & & 1 \end{pmatrix}$$
$$|a_1|^{-(n-1)} da_1 |a_2|^{-(n-2)} dx_{21} da_2 \cdots |a_{n-1}|^{-1} dx_{n-1,1} \cdots dx_{n-1,n-2} da_{n-1}$$

where the right side is an iterated integral, a_i is integrated over $F^\times \subset F$ for $i = 1, ..., n-1$, x_{ij} is integrated over F for all relevant i, j, and all measures are additive self-dual Haar measures on F.

Proof. Since $j_{\tilde{\pi},\psi^{-1}}(g) = j_{\pi,\psi}(g^{-1})$ for $g \in B_n\omega_n B_n$, apply the above theorem. □

5. Howe Vectors

In this section, we will discuss Howe vectors, which were introduced first by R.Howe. We will follow the exposition in [5] closely. Assume π is irreducible and generic.

For a positive integer m, let $K_n^m = I_n + M_n(\mathfrak{p}^m)$, here $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}$. Use ϖ to denote an uniformizer of F. Let

$$d = \begin{pmatrix} 1 & & & & \\ & \varpi^2 & & & \\ & & \varpi^4 & & \\ & & & \ddots & \\ & & & & \varpi^{2n-2} \end{pmatrix}$$

Put $J_m = d^m K_n^m d^{-m}$, $N_{n,m} = N_n \cap J_m$, $\bar{N}_{n,m} = \bar{N}_n \cap J_m$, $\bar{B}_{n,m} = \bar{B}_n \cap J_m$. Let $A_{n,m} = A_n \cap J_m$, then

$$J_m = \bar{N}_{n,m} A_{n,m} N_{n,m} = \bar{B}_{n,m} N_{n,m}$$

For $j \in J_m$, write $j = \bar{b}_j n_j$ with respect to the above decomposition, as in [5], define a character ψ_m on J_m by

$$\psi_m(j) = \psi(n_j)$$

Definition 5.1. *$W \in \mathcal{W}$ is called a Howe vector of π if for any m large enough, we have*

$$W(gj) = \psi_m(j) W(g) \tag{5.1}$$

for all $g \in G_n$, $j \in J_m$.

For each $W \in \mathcal{W}(\pi, \psi)$, let M be a positive constant such that $R(K_n^M).W = W$ where R denotes the action of right multiplication. For any $m > 3M$, put

$$W_m(g) = \int_{N_{n,m}} W(gu)\psi^{-1}(u)du$$

then by Lemma 7.1 in [5], we have

$$W_m(gj) = \psi_m(j)W_m(g), \forall j \in J_m, \quad \forall g \in G_n$$

This gives the existence of Howe vectors when m is large enough. The following lemma establishes its uniqueness in Kirillov model.

Theorem 5.2. *Assume $W \in \mathcal{W}$ satisfying (5.1). Let $h \in G_{n-1}$, if*

$$W\begin{pmatrix} h & \\ & 1 \end{pmatrix} \neq 0$$

then $h \in N_{n-1}\bar{B}_{n-1,m}$. Moreover

$$W\begin{pmatrix} h & \\ & 1 \end{pmatrix} = \psi(u)W(I)$$

if $h = u\bar{b}$, with $u \in N_{n-1}$, $\bar{b} \in \bar{B}_{n-1,m}$.

• Howe vectors were first introduced by R.Howe in an unpublished paper ([15]), in which he proved certain existence and uniqueness properties of such vectors based on Gelfand-Kazhdan method. We will below give an elementary proof of this theorem which calculates the Howe vectors in Kirillov models. This result also provides Howe vector as an candidate for the 'unramified' vector other than new vectors even in the 'ramified' representations.

Proof. We will use an inductive argument. Write

$$h = \begin{pmatrix} h_{11} & \cdots & h_{1,n-1} \\ & \cdots & \\ h_{n-1,1} & \cdots & h_{n-1,n-1} \end{pmatrix}$$

Take

$$u = \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & & u_1 \\ & 1 & 0\cdots & & u_2 \\ & & \cdots & & \\ & & & 1 & u_{n-1} \\ & & & & 1 \end{pmatrix} \in J_m$$

We have

$$\begin{aligned} \psi(u_{n-1})W\begin{pmatrix} h & \\ & 1 \end{pmatrix} &= W\left(\begin{pmatrix} h & \\ & 1 \end{pmatrix}\begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix}\right) \\ &= W\begin{pmatrix} h & h.u \\ & 1 \end{pmatrix} = W\left(\begin{pmatrix} I_{n-1} & h.u \\ & 1 \end{pmatrix}\begin{pmatrix} h & \\ & 1 \end{pmatrix}\right) \\ &= \psi(\sum_{i=1}^{n-1} h_{n-1,i}u_i)W\begin{pmatrix} h & \\ & 1 \end{pmatrix} \end{aligned}$$

Since $W\begin{pmatrix} h & \\ & 1 \end{pmatrix} \neq 0$, we get

$$\psi(\sum_{i=1}^{n-2} h_{n-1,i}u_i + (h_{n-1,n-1} - 1)u_{n-1}) = 1$$

Note that $u_i \in \mathfrak{p}^{(2i-2n+1)m}$, $i = 1, 2, ..., n-1$, it follows that $h_{n-1,i} \in \mathfrak{p}^{(2n-1-2i)m}$, $i = 1, 2, ..., n-2$, and $h_{n-1,n-1} \in 1 + \mathfrak{p}^m$. So we may write

$$h = \begin{pmatrix} I_{n-2} & y \\ & 1 \end{pmatrix} \begin{pmatrix} g & \\ x & a \end{pmatrix} = \begin{pmatrix} I_{n-2} & y \\ & 1 \end{pmatrix} \begin{pmatrix} g & \\ & 1 \end{pmatrix} \begin{pmatrix} I_{n-2} & \\ x & a \end{pmatrix}$$

with $x = (h_{n-1,1}, ..., h_{n-1,n-2})$, $a = h_{n-1,n-1}$, $y = h_{n-1,n-1}^{-1}\,{}^t(h_{1,n-1}, h_{2,n-1}, ..., h_{n-2,n-1})$, $g = h_{n-2} - y \cdot x$, where y is a column vector, x is a row vector, and

$$h_{n-2} = \begin{pmatrix} h_{11} & \cdots & h_{1,n-2} \\ & \cdots & \\ h_{n-2,1} & \cdots & h_{n-2,n-2} \end{pmatrix}$$

Since $j = \begin{pmatrix} I_{n-2} & \\ x & a \end{pmatrix} \in J_m$, and by the assumption on W, we get

$$\begin{aligned} W\begin{pmatrix} h & \\ & 1 \end{pmatrix} &= W\left(\begin{pmatrix} I_{n-2} & y & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} g & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_{n-2} & & \\ x & a & \\ & & 1 \end{pmatrix} \right) \\ &= \psi(y_{n-2})\psi_m(j) W \begin{pmatrix} g & & \\ & 1 & \\ & & 1 \end{pmatrix} \end{aligned}$$

Note that $\psi_m(j) = 1$ and it follows that

$$W \begin{pmatrix} g & & \\ & 1 & \\ & & 1 \end{pmatrix} \neq 0$$

and now we can argue inductively to get the result. □

• It follows from this lemma that $W(I) \neq 0$ for Howe vectors, we will normalize it so that $W(I) = 1$.

We apply the kernel formula to Howe vector to prove the following result.

Proposition 5.3. *Assume π is supercuspidal. Fix $b = diag(b_1, ..., b_n) \in A_n$, choose m large enough so that*

(1). $R(A_{n,m}).j_\pi(b\omega_n) = j_\pi(b\omega_n)$, and $L(A_{n,m}).j_\pi(b\omega_n) = j_\pi(b\omega_n)$;

(2). $\frac{b_{i-1}}{b_i} \in \mathfrak{p}^{-3m}, i = 3, ..., n$
Then

$$W(b\omega_n) = vol(\bar{B}_{n-1,m}) j_\pi(b\omega_n)$$

Proof. We first note that although we don't know whether the weak kernel formula is absolutely convergent, but when applying it to Howe vectors, by Theorem 5.2, Howe vectors have nice compact support modulo N_{n-1} in the Kirillov model, hence in this case the weak kernel formula is absolutely convergent. Write

$$x = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ & & \ddots & & \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & 1 & \\ & & & & 1 \end{pmatrix}$$

and

$$a = diag(a_1, a_2, ..., a_{n-1}, 1)$$

Apply the kernel formula to Howe vector W_m and by Theorem 5.2, we find

$$W_m(b\omega_n) = \int_{\bar{B}_{n-1,m}} j_\pi(b\omega_n(xa)^{-1}) da_1 dx_{21} da_2 \cdots dx_{n-1,1} \cdots dx_{n-1,n-2} da_{n-1}$$

Note that $b\omega_n a^{-1} x^{-1} a\omega_n b^{-1}$ is a upper triangular unipotent matrix and

$$\psi(b\omega_n a^{-1} x^{-1} a\omega_n b^{-1}) = \psi(-x_{21}\frac{a_1}{a_2}\frac{b_{n-1}}{b_n} - ... - x_{n-1,n-2}\frac{a_{n-2}}{a_{n-1}}\frac{b_2}{b_3})$$

then the above integral equals

$$\int_{\bar{B}_{n-1,m}} \psi(-x_{21}\frac{a_1}{a_2}\frac{b_{n-1}}{b_n} - ... - x_{n-1,n-2}\frac{a_{n-2}}{a_{n-1}}\frac{b_2}{b_3}) j_\pi(b\omega_n a^{-1}) da_1 dx_{21} da_2 \cdots dx_{n-1,1} \cdots dx_{n-1,n-2} da_{n-1}$$

$$= \int_{\bar{B}_{n-1,m}} j_\pi(b\omega_n a^{-1}) da_1 dx_{21} da_2 \cdots dx_{n-1,1} \cdots dx_{n-1,n-2} da_{n-1}$$

since $a_i \in A_{n,m}, i = 1, ..., n-1$, $\frac{b_{i-1}}{b_i} \in \mathfrak{p}^{-3m}, i = 3, ..., n$ by assumption (2), and $x_{i,i-1} \in \mathfrak{p}^{3m}, i = 2, ..., n-1$.

Now by assumption (1), $j_\pi(b\omega_n a^{-1}) = j_\pi(b\omega_n)$, and eventually we have

$$W(b\omega_n) = j_\pi(b\omega_n) \int_{\bar{B}_{n-1,m}} da_1 dx_{21} da_2 \cdots dx_{n-1,1} \cdots dx_{n-1,n-2} da_{n-1} = vol(\bar{B}_{n-1,m}) j_\pi(b\omega_n)$$

□

6. Bessel distributions and Bessel functions

In this section, we will show for supercuspidal representation π, the Bessel function $j_0(g)$ defined in section 3 via Bessel distribution, is equal to the Bessel function $j_\pi(g)$, defined in section 2 via uniqueness of Whittaker functional. We first review some results and constructions in [2], which will be useful for our purpose.

As in [2], use $L(\omega_n).f$ to denote the left translation by ω_n on f, for f a locally constant compactly supported function of G_n. Then this action induces an action on distributions, still denoted as $L(\omega_n)$. We now consider the distribution $J = L(\omega_n).B_{l,l'}$, where $B_{l,l'}$ is the Bessel distribution defined in section 3. An important result proved in [2] is that, the restriction of J to $\bar{N}_n A_n N_n$ is given by the locally constant function j_0, and the restriction of $B_{l,l'}$ to $N_n\omega_n A_n N_n$ is then given by $j_\pi = L(\omega_n).j_0$. We next describe the method used to prove this fact in section 3.3 of [2].

We first transform the distribution J on $Y = \bar{N}_n A_n N_n$ to a distribution σ_J on A_n using the constructions in [2]. For every $f \in C_c^\infty(Y)$, define $\beta_f \in C_c^\infty(A_n)$ by

$$\beta_f(a) = \int_{\bar{N}_n \times N_n} f(\bar{u}_1 a u_2)\psi(-\bar{u}_1)\psi(-u_2) du_1 du_2$$

where $\psi(\bar{u}_1) = \psi(\omega_n \bar{u}_1 \omega_n)$. Then by Proposition 1.12 in [18], the map sending f to β_f is a surjective map from $C_c^\infty(Y)$ onto $C_c^\infty(A_n)$, and there exists a unique distribution σ_J on A_n with

$$J(f) = \sigma_J(\beta_f)$$

Moreover, if the distribution σ_J on A_n is given by a locally constant function $\phi(a)$, then the distribution J on Y is given by the locally constant function $\psi(\bar{u}_1)\psi(u_2)\phi(a)\Delta^{-1}(a)$, where Δ satisfies $dg = \Delta(a) d\bar{u}_1 da du_2$ on Y.

To show σ_J is given by some locally constant $\phi(a)$, we need to introduce the following concept as in [2], specializing to our case.

Definition 6.1. *Let Θ be a distribution on A_n. Θ is said to be admissible if for any $a \in A_n$, there exists some compact open subgroup K (depending on a) of A_n, such that for every nontrivial character χ of K we have $\Theta(\chi_a) = 0$, where χ_a is the function defined on aK by $\chi_a(ak) = \chi(k)$, $k \in K$.*

We then have the following lemma.

Lemma 6.2. *A distribution Θ on A_n is admissible if and only if there exists a locally constant function θ on A_n such that*

$$\Theta(f) = \int_{A_n} \theta(a) f(a) da$$

for all $f \in C_c^\infty(A_n)$. Moreover, the value of $\theta(a)$ if given by $\frac{1}{vol(K(a))}\Theta(1_a)$, where $K(a)$ is any compact open subgroup of A_n satisfying Definition 6.1, and 1 denotes the trivial character of $K(a)$.

Proof. This is exactly Lemma 3.2 in [2] applied to our case. $\square$

In view of the above discussion, it suffices to show σ_J is admissible, and this is done in section 3.3 of [2]. Moreover fix $a = diag(a_1, ..., a_n) \in A_n$, choose m large enough. More precisely, let $M > 0$ be a positive constant as in Corollary 3.5 in [2], we then require m to satisfy that

(1). ψ is trivial on $\omega_n a \bar{N}_{n,m} a^{-1} \omega_n$ and $\omega_n a^{-1} \bar{N}_{n,m} a \omega_n$;

(2). $R(A_{n,m}).j_\pi(a\omega_n) = j_\pi(a\omega_n)$, and $L(A_{n,m}).j_\pi(a\omega_n) = j_\pi(a\omega_n)$.

(3). $m \geq M$.

As ψ has conductor exactly $\mathcal{O}$, if m is large (1) can then be satisfied. Because j_π is locally constant, for a given $a \in A_n$, (2) can be satisfied once m is large. Hence one can choose m large enough satisfying all the above (1),(2),(3).

Then $A_{n,m} = A_n \cap J_m$, which is a compact open subgroup of A_n, will satisfy Definition 6.1 by assumption (1) as shown in [2], and then by Lemma 6.2, $\phi(a) = \sigma_J(\frac{1}{vol(A_{n,m})}\chi_a)$ where χ_a is the characteristic function of $aA_{n,m}$.

Now let ϕ_1, ϕ_2 be a multiple of the characteristic function of $\bar{N}_n \cap K_n$, and $N_n \cap K_n$, respectively, with

$$\int_{\bar{N}_n} \phi_1(\bar{u}) \psi(-\omega_n \bar{u} \omega_n) d\bar{u} = 1$$

and

$$\int_{N_n} \phi_2(u) \psi(-u) du = 1$$

then $\Phi_a = \frac{1}{vol(A_{n,m})}\phi_1 \chi_a \phi_2$ is a locally constant compactly supported function on $\bar{N}_n A_n N_n$. Note that $\frac{1}{vol(A_{n,m})}\chi_a = \beta_{\Phi_a}$.

Hence $j_\pi(a) = \phi(a)\Delta^{-1}(a) = \sigma_J(\frac{1}{vol(A_{n,m})}\chi_a)\Delta^{-1}(a) = J(\Phi_a)\Delta^{-1}(a)$. Then

$$j_0(\omega_n a) = j_\pi(a) = J(\Phi_a)\Delta^{-1}(a) = B_{l,l'}(L(\omega_n)\Phi_a)\Delta^{-1}(a).$$

Note that $L(\omega_n).\Phi_a$ belongs to $C_c^\infty(N_n \omega_n A_n N_n)$, and hence can be viewed as an element in $C_c^\infty(G_n)$. Choose another positive integer m_1 large enough so that

(1). $L(\bar{b}).L(\omega_n).\Phi_a = L(\omega_n).\Phi_a$ for any $b \in \bar{B}_{n-1,m_1}$;

(2). $m_1 \geq m$;

(3). $\frac{a_i}{a_{i-1}} \in \mathfrak{p}^{-3m_1}, i = 3, ..., n$.

Since $L(\omega_n).\Phi_a \in C_c^\infty(G_n)$, hence it is bi-invariant under some open compact subgroup, then (1) is satisfied if m_1 is large. Thus we can choose m_1 large enough to satisfy all above (1),(2),(3).

Apply Lemma 3.2 to $f = L(\omega_n)\Phi_a$, $\hat{W} = \hat{W}_{m_1}$ the Howe vector, then we find by Theorem 5.2, the right hand side of Lemma 3.2 is

$$\begin{aligned}
&\int_{N_{n-1}\backslash G_{n-1}} B_{l,l'}\left(L\begin{pmatrix} h & \\ & 1\end{pmatrix}.L(\omega_n).\Phi_a\right)\hat{W}_{m_1}\begin{pmatrix} h & \\ & 1\end{pmatrix}dh \\
= \quad & B_{l,l'}(L(\omega_n)\Phi_a)vol(\bar{B}_{n-1,m_1})
\end{aligned}$$

While the left side integral of Lemma 3.2 is

$$\begin{aligned}
&\int f(g)\hat{W}_{m_1}(g^{-1})dg \\
= \quad & \frac{1}{vol(A_{n,m})}\int \phi_1(\omega_n u_1 \omega_n)\psi(-u_1)\phi_2(u_2)\psi(-u_2)du_1du_2\int_{A_{n,m}}\hat{W}_{m_1}(h^{-1}a^{-1}\omega_n)\Delta(ah)dh \\
= \quad & \frac{1}{vol(A_{n,m})}\Delta(a)\int_{A_{n,m}}\hat{W}_{m_1}(h^{-1}a^{-1}\omega_n)\Delta(h)dh
\end{aligned}$$

For any $h' \in A_{n,m_1} \subset A_{n,m}$, $h \in A_{n,m}$, since $j_{\tilde{\pi}}(g^{-1}) = j_\pi(g)$, we have

$$j_{\tilde{\pi}}(h'h^{-1}a^{-1}\omega_n) = j_\pi(\omega_n a h h'^{-1}) = j_\pi(\omega_n a h) = j_{\tilde{\pi}}(h^{-1}a^{-1}\omega_n)$$

$$j_{\tilde{\pi}}(h^{-1}a^{-1}\omega_n h') = j_\pi(h'^{-1}\omega_n a h) = j_\pi(\omega_n a h) = j_{\tilde{\pi}}(h^{-1}a^{-1}\omega_n)$$

and $\frac{h_i a_i}{h_{i-1}a_{i-1}} \in \mathfrak{p}^{-3m_1}, i = 3, ..., n$ if $h = diag(h_1, ..., h_n)$. Hence we can apply Proposition 5.3 to $\hat{W}_{m_1}(h^{-1}a^{-1}\omega_n)$ in the above integral and get

$$\begin{aligned}
&\int f(g)\hat{W}_{m_1}(g^{-1})dg \\
= \quad & \frac{1}{vol(A_{n,m})}\Delta(a)\int_{A_{n,m}} j_{\tilde{\pi}}(h^{-1}a^{-1}\omega_n)vol(\bar{B}_{n-1,m_1})\Delta(h)dh \\
= \quad & \frac{1}{vol(A_{n,m})}\Delta(a)\int_{A_{n,m}} j_\pi(\omega_n a h)vol(\bar{B}_{n-1,m_1})dh
\end{aligned}$$

where the last equality follows from the facts that $j_{\tilde{\pi}}(g^{-1}) = j_\pi(g)$ and $\Delta(h) = 1$ when restricted to $A_{n,m}$.

Now by assumption (2) on m, the above equals

$$\Delta(a)j_\pi(\omega_n a)vol(\bar{B}_{n-1,m_1})$$

which is the left hand side of Lemma 3.2.

Combining both sides of Lemma 3.2, we get

$$\Delta(a)j_\pi(\omega_n a)vol(\bar{B}_{n-1,m_1}) = B_{l,l'}(L(\omega_n)\Phi_a)vol(\bar{B}_{n-1,m_1})$$

Note that $B_{l,l'}(L(\omega_n)\Phi_a) = j_0(\omega_n a)\Delta(a)$, immediately we have

Theorem 6.3. *Assume π is supercuspidal. For all $g \in N_nA_n\omega_nN_n$, we have*

$$j_0(g) = j_\pi(g)$$

7. General Case

The aim of this section is to generalize **Theorem** 6.3 from supercuspidal case to general generic case. Now assume π is an irreducible admissible smooth generic representation of G_n with Whittaker model $\mathcal{W}(\pi,\psi)$. The point is to prove a weak kernel formula for Howe vectors of π. This weak kernel formula is expected to hold for a wider class of Whittaker functions, but currently we are only able to prove it for Howe vectors, which is sufficient for our purpose.

The proof is essentially the same as the proof of **Theorem** 4.2 with necessary modifications. Use W_m to denote the normalized Howe vector of level m of π as in section 5.

Theorem 7.1. *(weak kernel formula) For any $b\omega_n$, $b = diag(b_1, ..., b_n) \in A_n$, if m is large enough, we have*

$$W_m(b\omega_n) =$$

$$\int j_\pi\left(b\omega_n\begin{pmatrix} a_1 &&&& \\ x_{21} & a_2 &&& \\ && \ddots && \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ &&&& 1\end{pmatrix}^{-1}\right)W_m\begin{pmatrix} a_1 &&&& \\ x_{21} & a_2 &&& \\ && \ddots && \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ &&&& 1\end{pmatrix}$$

$$|a_1|^{-(n-1)}da_1|a_2|^{-(n-2)}dx_{21}da_2\cdots|a_{n-1}|^{-1}dx_{n-1,1}\cdots dx_{n-1,n-2}da_{n-1}$$

where the right side is an iterated integral, a_i is integrated over $F^\times \subset F$ for $i = 1, ..., n-1$, x_{ij} is integrated over F for all relevant i, j, and all measures are additive self-dual Haar measures on F.

We first note $W_m\begin{pmatrix} a_1 &&&& \\ x_{21} & a_2 &&& \\ && \ddots && \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ &&&& 1\end{pmatrix} \neq 0$ if and only if $\begin{pmatrix} a_1 &&&& \\ x_{21} & a_2 &&& \\ && \ddots && \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ &&&& 1\end{pmatrix}$ $\in \bar{B}_{n,m}$ by **Theorem** 5.2, in which case $W_m\begin{pmatrix} a_1 &&&& \\ x_{21} & a_2 &&& \\ && \ddots && \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ &&&& 1\end{pmatrix} = 1$. This is very important for our proof in this special case.

Introduce notations $h_i = \begin{pmatrix} & I_{i-1} &&& \\ x_{i,1} & ... & x_{i,i-1} & a_i & \\ &&&& I_{n-i}\end{pmatrix}$. We also use h_i to denote the left upper corner $i \times i$ matrix when there is no confusion. Note that $\begin{pmatrix} a_1 &&&& \\ x_{21} & a_2 &&& \\ && \ddots && \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} & \\ &&&& 1\end{pmatrix}$ $= h_1h_2...h_{n-1}$.

Proof. Now let

$$M_1(u_{n-1}) = W_m\left(b\omega_n\begin{pmatrix} I_{n-1} & u_{n-1} \\ & 1\end{pmatrix}\right)$$

which is a compactly supported function in column vector u_{n-1} by by Theorem 5.7 and Theorem 7.3 in [5] as $W_m \in \mathcal{W}^0$ if m is large enough. Its Fourier inversion formula is

$$\begin{aligned} M_1(u_{n-1}) &= \int_{F^{n-1}} \widehat{M_1}(v_{n-1})\psi(-v_{n-1}u_{n-1})dv_{n-1} \\ &= \int_{F^{n-2}\times F^\times} \widehat{M_1}(z_1,...,z_{n-1})\psi(-(z_1,...,z_{n-1})u_{n-1})dz_1...dz_{n-1}\end{aligned}$$

where we write $v_{n-1} = (z_1, ..., z_{n-1})$, with $z_1, ..., z_{n-2} \in F$, $z_{n-1} \in F^\times$, and the last equality follows from the facts that $dz_1...dz_{n-1}$ is the additive Haar measure and $F^{n-2}\times F^\times$ is of full measure in F^{n-1}.

Put $u_{n-1} = 0$, we get

$$M_1(0) = \int_{F^{n-2}\times F^\times} \widehat{M_1}(z_1,...,z_{n-1})dz_1...dz_{n-1} \tag{7.1}$$

By the same computations as in section 4, we find

$$|a_{n-1}|^{-1}\int_{Y_2} W(b\omega_n h_{n-1}^{-1}y_2h_{n-1})\psi(-y_2)dy_2 = \widehat{M_1}(x_{n-1,1},...,x_{n-1,n-2},a_{n-1})$$

So (7.1) becomes

$$W_m(b\omega_n) = \int_{F^{n-2}\times F^\times} \widehat{M_1}(x_{n-1,1},...,x_{n-1,n-2},a_{n-1})dx_{n-1,1}...dx_{n-1,n-2}da_{n-1}$$

$$= \int_{F^{n-2}\times F^\times} |a_{n-1}|^{-1}\int_{Y_2} W_m(b\omega_n h_{n-1}^{-1}y_2h_{n-1})\psi(-y_2)dy_2dx_{n-1,1}...dx_{n-1,n-2}da_{n-1} \tag{7.2}$$

Claim 1: As a function of h_{n-1}, $\int_{Y_2} W_m(b\omega_n h_{n-1}^{-1}y_2h_{n-1})\psi(-y_2)dy_2$ has support in $\bar{B}_{n,m}$.

Proof of Claim 1: The proof is similar to **Theorem** 5.2. Take

$$u = \begin{pmatrix} I_{n-1} & u \\ & 1\end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & u_1 \\ & 1 & 0\cdots & u_2 \\ & & \cdots & \\ & & 1 & u_{n-1} \\ & & & 1\end{pmatrix} \in J_m$$

then

$$\psi(u_{n-1})\int_{Y_2} W_m(b\omega_n h_{n-1}^{-1}y_2h_{n-1})\psi(-y_2)dy_2 = \int_{Y_2} W_m(b\omega_n h_{n-1}^{-1}y_2h_{n-1}u)\psi(-y_2)dy_2$$

$$= \int_{Y_2} W_m(b\omega_n h_{n-1}^{-1}y_2h_{n-1}uh_{n-1}^{-1}h_{n-1})\psi(-y_2)dy_2 = \int_{Y_2} W_m(b\omega_n h_{n-1}^{-1}y_2h_{n-1})\psi(-y_2)\psi(h_{n-1}uh_{n-1}^{-1})dy_2$$

where in the last equality we change variable $y_2h_{n-1}uh_{n-1}^{-1} \to y_2$.

Thus we find that if $\int_{Y_2} W_m(b\omega_n h_{n-1}^{-1}y_2h_{n-1})\psi(-y_2)dy_2 \neq 0$, then

$$\psi(u_{n-1}) = \psi(h_{n-1}uh_{n-1}^{-1})$$

As $u \in J_m$ is arbitrary, this forces $h_{n-1} \in \bar{B}_{n,m}$, which proves the claim.

□

Let's continue the proof of the theorem. Compare (7.2) with the desired formula in theorem and note the support of W_m and the claim, it suffices to show

$$\int_{Y_2} W_m(b\omega_n h_{n-1}^{-1} y_2 h_{n-1})\psi(-y_2)dy_2 = \int$$

$$j_\pi\left(b\omega_n h_{n-1}^{-1}\begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-2,1} & \cdots & x_{n-2,n-3} & a_{n-2} & \\ & & & & I_2\end{pmatrix}^{-1}\right) W_m\left(\left(\begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-2,1} & \cdots & x_{n-2,n-3} & a_{n-2} & \\ & & & & I_2\end{pmatrix} h_{n-1}\right)\right)$$

$$|a_1|^{-(n-1)}da_1|a_2|^{-(n-2)}dx_{21}da_2\cdots|a_{n-2}|^{-2}dx_{n-2,1}\cdots dx_{n-2,n-3}da_{n-2}$$

for $h_{n-1}\in \bar{B}_{n,m}$.

By properties of Howe vectors, this is equivalent to

$$\int_{Y_2} W_m(b\omega_n h_{n-1}^{-1} y_2)\psi(-y_2)dy_2 = \int$$

$$j_\pi\left(b\omega_n h_{n-1}^{-1}\begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-2,1} & \cdots & x_{n-2,n-3} & a_{n-2} & \\ & & & & I_2\end{pmatrix}^{-1}\right) W_m\begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-2,1} & \cdots & x_{n-2,n-3} & a_{n-2} & \\ & & & & I_2\end{pmatrix}$$

$$|a_1|^{-(n-1)}da_1|a_2|^{-(n-2)}dx_{21}da_2\cdots|a_{n-2}|^{-2}dx_{n-2,1}\cdots dx_{n-2,n-3}da_{n-2} \tag{7.3}$$

for $h_{n-1}\in \bar{B}_{n,m}$.

To prove (7.3), let

$$M_2(u_{n-2}) = \int_{Y_2} W_m\left(b\omega_n h_{n-1}^{-1} y_2\begin{pmatrix} I_{n-2} & u_{n-2} & \\ & 1 & \\ & & 1\end{pmatrix}\right)\psi(-y_2)dy_2$$

which is compactly supported function in column vector u_{n-2} by Theorem 5.7 and 7.3 in [5]. Its Fourier inversion formula is

$$\begin{aligned} M_2(u_{n-2}) &= \int_{F^{n-2}} \widehat{M_2}(v_{n-2})\psi(-v_{n-2}u_{n-2})dv_{n-2} \\ &= \int_{F^{n-3}\times F^\times} \widehat{M_2}(z_1,...,z_{n-2})\psi(-(z_1,...,z_{n-2})u_{n-2})dz_1...dz_{n-2}\end{aligned}$$

where we write $v_{n-2}=(z_1,...,z_{n-2})$, with $z_1,...,z_{n-3}\in F$, $z_{n-2}\in F^\times$, and the last equality follows from the facts that $dz_1...dz_{n-2}$ is the additive Haar measure and $F^{n-3}\times F^\times$ is of full measure in F^{n-2}.

Put $u_{n-2}=0$, we get

$$M_2(0) = \int_{F^{n-2}\times F^\times} \widehat{M_2}(z_1,...,z_{n-2})dz_1...dz_{n-2} \tag{7.4}$$

Similar computations as in section 4 shows that

$$\widehat{M_2}(x_{n-2,1},...,x_{n-2,n-3},a_{n-2}) = |deth_{n-2}|^{-2}$$

$$\int_{F^{n-2}}\int_{Y_2} W_m\left(b\omega_n h_{n-1}^{-1}h_{n-2}^{-1}y_2\begin{pmatrix} I_{n-2} & u_{n-2} & \\ & 1 & \\ & & 1\end{pmatrix}h_{n-2}\right)\psi(-y_2)\psi\left(-\begin{pmatrix} I_{n-2} & u_{n-2} & \\ & 1 & \\ & & 1\end{pmatrix}\right)dy_2du_{n-2}$$

$$= |deth_{n-2}|^{-2} \int_{Y_3} W_m(b\omega_n h_{n-1}^{-1} h_{n-2}^{-1} y_3 h_{n-2}) \psi(-y_3) dy_3$$

So (7.4) becomes

$$\int_{Y_2} W_m \left(b\omega_n h_{n-1}^{-1} y_2\right) \psi(-y_2) dy_2 =$$

$$\int_{F^{n-2}\times F^\times} |deth_{n-2}|^{-2} \int_{Y_3} W_m(b\omega_n h_{n-1}^{-1} h_{n-2}^{-1} y_3 h_{n-2}) \psi(-y_3) dy_3 dx_{n-2,1} ... da_{n-2} \tag{7.5}$$

Claim 2: As a function of h_{n-2}, $\int_{Y_3} W_m(b\omega_n h_{n-1}^{-1} h_{n-2}^{-1} y_3 h_{n-2})\psi(-y_3)dy_3$ has support in $\bar{B}_{n,m}$.

Proof of the Claim 2: Take

$$u = \begin{pmatrix} I_{n-2} & u_{n-2} & \\ & 1 & \\ & & 1 \end{pmatrix} \in J_m$$

then argue completely the same as the proof of Claim 1. The details will be omitted.

□

Compare (7.5) with (7.3) and note the support of W_m and Claim 2, then it suffices to show that

$$\int_{Y_3} W_m(b\omega_n h_{n-1}^{-1} h_{n-2}^{-1} y_3 h_{n-2}) \psi(-y_3) dy_3 = \int$$

$$j_\pi \left(b\omega_n h_{n-1}^{-1} h_{n-2}^{-1} \begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-3,1} & \cdots & x_{n-3,n-4} & a_{n-3} & \\ & & & & I_3 \end{pmatrix}^{-1} \right) W_m \left(\begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-3,1} & \cdots & x_{n-3,n-4} & a_{n-3} & \\ & & & & I_3 \end{pmatrix} h_{n-2} \right)$$

$$|a_1|^{-(n-1)} da_1 |a_2|^{-(n-2)} dx_{21} da_2 \cdots dx_{n-2,1} \cdots dx_{n-3,n-4} da_{n-3}$$

for $h_{n-2} \in \bar{B}_{n,m}$, which is equivalent to

$$\int_{Y_3} W_m(b\omega_n h_{n-1}^{-1} h_{n-2}^{-1} y_3) \psi(-y_3) dy_3 = \int$$

$$j_\pi \left(b\omega_n h_{n-1}^{-1} h_{n-2}^{-1} \begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-3,1} & \cdots & x_{n-3,n-4} & a_{n-3} & \\ & & & & I_3 \end{pmatrix}^{-1} \right) W_m \begin{pmatrix} a_1 & & & & \\ x_{21} & a_2 & & & \\ & & \ddots & & \\ x_{n-3,1} & \cdots & x_{n-3,n-4} & a_{n-3} & \\ & & & & I_3 \end{pmatrix}$$

$$|a_1|^{-(n-1)} da_1 |a_2|^{-(n-2)} dx_{21} da_2 \cdots dx_{n-2,1} \cdots dx_{n-3,n-4} da_{n-3} \tag{7.6}$$

To prove (7.6), inductively, it suffices to show that

$$\int_{Y_{n-1}} W_m \left(b\omega_n h_{n-1}^{-1} ... h_2^{-1} y_{n-1}\right) \psi(-y_{n-1}) dy_{n-1} =$$

$$\int_{F^\times} j_\pi \left(b\omega_n h_{n-1}^{-1} ... h_2^{-1} \begin{pmatrix} a_1^{-1} & \\ & I_{n-1} \end{pmatrix} \right) W_m \begin{pmatrix} a_1 & \\ & I_{n-1} \end{pmatrix} |a_1|^{-(n-1)} da_1$$

which can be proved completely in the same way as the proof of **Theorem** 4.2. Thus the proof of the theorem is finished.

□

Now use the same method as in section 6 , together with the above weak kernel formula for Howe vectors W_m with m large enough, we can show that for irreducible generic admissible representation π of G_n, the Bessel functions j_π defined via uniqueness of Whittaker models, coincide with the Bessel function j_0 defined via Bessel distributions, which generalize Theorem 6.3 to general generic representations. As the proof is completely the same as Theorem 6.3 we omit the details, and simply state the result as follows.

Theorem 7.2. *If π is an irreducible admissible smooth generic representation of G_n, then for any $g \in N_n\omega_n A_n N_n$, we have*

$$j_\pi(g) = j_0(g)$$

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