

A STRONG MULTIPLICITY ONE THEOREM FOR SL_2

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ABSTRACT. It is known that multiplicity one property holds for SL_2 , while the strong multiplicity one property fails. However, in this paper, we show that if we require further that a pair of cuspidal representations π and π' of SL_2 have the same local components at archimedean places and the places above 2, and they are generic with respect to the same additive character, then they also satisfy the strong multiplicity one property. The proof is based on a local converse theorem for SL_2 .

1. INTRODUCTION

Let F be a number field, and $\mathbb{A} = \mathbb{A}_F$ be its ring of adèles. Let G be a linear reductive algebraic group defined over F . The study of the space of automorphic forms $L^2(G(F)\backslash G(\mathbb{A}))$ has been a central topic in the Langlands program and representation theory. Let $L_0^2(G(F)\backslash G(\mathbb{A}))$ be the subspace of cuspidal representations. Suppose π is an irreducible automorphic representation of $G(\mathbb{A})$. It is known that π occurs discretely with finite multiplicity m_π in $L_0^2(G(F)\backslash G(\mathbb{A}))$.

The multiplicities m_π are important in the study of automorphic forms and number theory. By the work of Piatetski-Shapiro and Jacquet and Shalika [JaSh] the group $G = GL_n$ and its inner forms have the property of multiplicity one, that is, $m_\pi \leq 1$ for any π . This is also true for SL_2 by the famous work of D.Ramakrishnan [Ra]. But in general, the multiplicity one property fails, for example [B, GaGJ, Li, LL] to list a few.

In the case of GL_n , a stronger theorem, called the strong multiplicity one, holds. It says that for two cuspidal representations π_1, π_2 , if they have isomorphic local components almost everywhere, then they coincide in the space of cusp forms (not only isomorphic). It follows from the results in [LL] that SL_2 does not have this strong multiplicity one property. The multiplicity one property is already rare, and the strong multiplicity one is even more rare. To the authors' knowledge, the examples other than GL_n in this direction are the strong multiplicity one theorems for $U(2, 1)$ ([GePS3, Ba2]) and GSp_4 ([So]) and the rigidity theorem for $SO(2n + 1)$ ([JS, Theorem 5.3]).

The main purpose of this paper is to prove a weaker version of strong multiplicity one result for $Sp_2 = SL_2$. Although we know the strong multiplicity one does not hold in general for a pair of cuspidal representations π_1, π_2 of $SL_2(\mathbb{A})$, if we require that both π_1, π_2 are generic with respect to the same additive character ψ of \mathbb{A} , then we can show that they also satisfy the strong multiplicity one property.

The reason for the failure of the strong multiplicity one for SL_2 is the existence of L-packets. According to the local conjecture of Gan-Gross-Prasad, [GaGP, Conjecture 17.3], there is at most one ψ -generic representation in each L-packet. For SL_2 , the result is known by the local discussion in [LL]. In this paper, we prove a local converse theorem for $SL_2(F)$ when F is a p -adic field such that its residue characteristic is not 2, which will reprove this result and confirm a local converse conjecture of Jiang, see [Jng, Conjecture 3.7] and [JngN, Conjecture 6.3]. This also implies our version of strong multiplicity one easily.

We now give some details of our results. In [GePS2], Gelbart and Piatetski-Shapiro constructed some Rankin-Selberg integrals to study L -functions on the group $G_n \times GL(n)$, at least when G_n is a classical group with split rank n . In particular, in Method C in that paper, if π is a globally generic cuspidal representation of $Sp_{2n}(\mathbb{A})$, τ is a cuspidal representation of $GL_n(\mathbb{A})$, consider the

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$$I(s, \phi, E) = \int_{\mathrm{Sp}_{2n}(F) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \phi(g) \theta(g) E(g, s) dg$$

where ϕ belongs to the space of π , $E(g, s)$ is a genuine Eisenstein series on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ built from the representation induced from the representation τ on $\mathrm{GL}_n(\mathbb{A})$ twisted by $|\det|^s$, and $\theta(g)$ is some theta series on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. Note that the product $\theta(g)E(g, s)$ is well-defined on Sp_{2n} . The global integral is shown to be Eulerian. The local functional equations and unramified calculations were also carried out in [GePS2] by Gelbart and Piatetski-Shapiro. Although we will only consider the easiest case when $n = 1$ of Gelbart and Piatetski-Shapiro's construction, we remark here that Ginzburg, Rallis and Soudry generalized the above construction to $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$, for any k , in [GIRS1, GIRS2].

We study more details of Gelbart and Piatetski-Shapiro's local integral

$$\Psi(W_v, \phi_v, f_{s,v}) = \int_{N(F_v) \backslash \mathrm{SL}_2(F_v)} W_v(h) (\omega_{\psi_v^{-1}}(h) \phi_v)(1) f_{s,v}(h) dh$$

(for the unexplained notations, see sections below) when v is finite. These local zeta integrals satisfy certain functional equations, which come from the intertwining operators on induced representations and certain uniqueness statements. These functional equations can then be used to define local gamma factors $\gamma(s, \pi_v, \eta_v, \psi_v)$, where π_v is a generic representation of $\mathrm{SL}_2(F_v)$, η_v is a character of F_v^\times , and ψ_v is a nontrivial additive character. The main local result of this paper can be formulated as follows.

Theorem 1.1 (Theorem 3.10, local converse theorem and stability of γ). *Suppose that the residue characteristic of the p -adic field F is not 2 and ψ is a nontrivial additive character of F . Let (π, V_π) and $(\pi', V_{\pi'})$ be two ψ -generic representations of $\mathrm{SL}_2(F)$ with the same central character.*

- (1) *If $\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi)$ for all quasi-characters η of F^\times , then $\pi \cong \pi'$.*
- (2) *There is an integer $l = l(\pi, \pi')$ such that if η is a quasi-character of F^\times with conductor $\mathrm{cond}(\eta) > l$, then*

$$\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi).$$

The proof of this result follows closely [Ba1, Ba2, Zh1] and Howe vectors play an important role. With the help of this result, combined with a nonvanishing result on archimedean local integrals proved in Lemma 4.8, we follow the argument in Theorem 7.2.13 in [Ba2], or Theorem 2 in [Ca], to prove the main global result of this paper.

Theorem 1.2 (Theorem 4.7, Strong Multiplicity One for SL_2). *Let ψ be a nontrivial additive character of $F \backslash \mathbb{A}$ and let $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ be two irreducible cuspidal automorphic representations of $\mathrm{SL}_2(\mathbb{A})$ with the same central character. Suppose that π and π' are both ψ -generic. Let S be a finite set of **finite** places such that no place in S is above 2. If $\pi_v \cong \pi'_v$ for all $v \notin S$, then $\pi = \pi'$.*

We remark here the restriction on residue characteristic comes from Lemma 3.3. It is expected that this restriction can be removed.

Besides the above, we also in this paper include a discussion of relations between global genericity and local genericity. An irreducible cuspidal automorphic representation (π, V_π) is called globally ψ -generic if for some $\phi \in V_\pi$, the integral

$$\int_{N(F) \backslash N(\mathbb{A})} \phi(ug) \psi^{-1}(u) du \neq 0$$

for some $g \in \mathrm{SL}_2(\mathbb{A})$. The representation π is called locally ψ -generic if each of its local component is generic for the corresponding local components of ψ . It is easy to see that if π is globally ψ -generic, then π is also locally ψ -generic. It is a conjecture that on a reductive algebraic group G , the converse is also true. This conjecture is closely related to the Ramanujan conjecture. See [Sh] for more detailed discussions. In this paper, we confirm this conjecture for SL_2 .

Theorem 1.3 (Theorem 4.3). *Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ and $\psi = \otimes_v \psi_v$ be a nontrivial additive character of $F \backslash \mathbb{A}$. Then π is ψ -generic if and only if each π_v is ψ_v -generic.*

In [GeRS, Proposition 2.5], Gelbart, Rogawski and Soudry proved similar results for $U(1, 1)$ and for endoscopic cuspidal automorphic representations of $U(2, 1)$. From the discussions given in [GeRS], Theorem 1.3 follows from the results of [LL] directly. Here, we include this result because we adopt a local argument (see Proposition 2.1) which is different from that given in [LL]. Hopefully, this local argument can be extended to more general groups.

As explained above, there is essentially nothing new in this paper. All the results and proofs should be known to the experts. Our task here is simply to try to write down the details and to check everything works out as expected.

This paper is organized as follows. In section 2, we collect basic results about the local zeta integrals which will be needed. In section 3, we study the Howe vectors and use them to prove the local converse theorem and stability of local gamma factors. In section 4, we prove the main global results.

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Notations: Let F be a field. In $SL_2(F)$, we consider the following subgroups. Let B be the upper triangular subgroup. Let $B = TN$ be the Levi decomposition, where T is the diagonal torus and N is the upper triangular unipotent. Denote

$$t(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \in T, \text{ for } a \in F^\times, \text{ and } n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in N, \text{ for } b \in F.$$

Let \bar{N} be the lower triangular unipotent and denote

$$\bar{n}(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}.$$

Let $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

Denote by St the natural inclusion $SO_3(\mathbb{C}) \rightarrow GL_3(\mathbb{C})$ and view it as the ‘‘standard’’ representation of ${}^L SL_2 = SO_3(\mathbb{C})$.

2. THE LOCAL ZETA INTEGRAL

2.1. The genericity of representations of $SL_2(F)$. In this section, let F be a local field, and ψ be a nontrivial additive character of F , which is also viewed as a character of $N(F)$. For $\kappa \in F^\times$, $g \in SL_2(F)$, we define

$$g^\kappa = \begin{pmatrix} \kappa & \\ & 1 \end{pmatrix} g \begin{pmatrix} \kappa^{-1} & \\ & 1 \end{pmatrix}.$$

Explicitly,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}^\kappa = \begin{pmatrix} x & \kappa y \\ \kappa^{-1} z & w \end{pmatrix}.$$

Note that if $\kappa \in F^{\times, 2}$, say $\kappa = a^2$, then $g^\kappa = t(a)gt(a)^{-1}$, i.e., $g \mapsto g^\kappa$ is an inner automorphism on $SL_2(F)$. Let (π, V_π) be an infinite dimensional irreducible smooth representation of $SL_2(F)$. We consider the representation $(\pi^\kappa, V_{\pi^\kappa})$ defined by

$$V_{\pi^\kappa} = V_\pi, \text{ and } \pi^\kappa(g) = \pi(g^\kappa).$$

Let ψ_κ be the character of F defined by $\psi_\kappa(b) = \psi(\kappa b)$. If (π, V_π) is ψ -generic with a nonzero ψ Whittaker functional $\Lambda : V_\pi \rightarrow \mathbb{C}$, one verifies that

$$\Lambda(\pi^\kappa(n)v) = \Lambda(\pi(n^\kappa)v) = \psi_\kappa(n)\Lambda(v)$$

for all $n \in N(F)$, $v \in V_{\pi^\kappa} = V_\pi$. Hence $(\pi^\kappa, V_{\pi^\kappa})$ is ψ_κ -generic.

Proposition 2.1. *If π is both ψ - and ψ_κ -generic, then $\pi \cong \pi^\kappa$.*

Proof. If F is non-archimedean, the result is proved in a more general setting in [Zh1, Theorem 1.1].

If F is archimedean, the case $F = \mathbb{C}$ is easy, as every κ has a square root in \mathbb{C} . Now consider $F = \mathbb{R}$, and we will work with the category of smooth representations of moderate growth of finite length. The Whittaker functional is an exact functor from this category to the category of vector spaces by [CMH, Theorem 8.2].

We first consider the case when $\pi = \text{Ind}_B^G(\chi)$ for some quasi-character χ of F^\times . For $f \in I(\chi)$, consider the function f^κ on $\text{SL}_2(F)$ defined by $f^\kappa(g) = f(g^{\kappa^{-1}})$. It is clear that $f^\kappa \in I(\chi)^\kappa$ and the map $f \mapsto f^\kappa$ defines an isomorphism $I(\chi) \rightarrow I(\chi)^\kappa$.

By results in Chapter 2 of [Vo], if π is not a fully induced representation, then it can be embedded into a principal series $I(\chi)$. This $I(\chi)$ has two irreducible infinite dimensional subrepresentations, and use π' to denote the other one. The quotient of $I(\chi)$ by the sum of π and π' , denoted by π'' , is finite dimensional, i.e., we have a short exact sequence

$$0 \rightarrow \pi \oplus \pi' \rightarrow I(\chi) \rightarrow \pi'' \rightarrow 0$$

First by Theorem 6.1 in [CMH], we know that the dimension of Whittaker functionals on $I(\chi)$ is one dimensional for either ψ or ψ_κ . Note that π'' cannot be generic as it is finite dimensional. Since the Whittaker functor is exact, it follows that the dimension of Whittaker functionals on $\pi \oplus \pi'$ is also one dimensional for either ψ or ψ_κ . By the assumption π is both ψ - and ψ_κ -generic, thus π' is neither ψ nor ψ_κ generic.

Now since π is ψ -generic, π^κ is ψ_κ -generic. Hence the image of π under the isomorphism $I(\chi) \rightarrow I(\chi)^\kappa$ given by $f \mapsto f^\kappa$ is again ψ_κ -generic, and hence it has to be ψ -generic and isomorphic to π , which finishes the proof. \square

2.2. Weil representations of $\widetilde{\text{SL}}_2$. Let $\widetilde{\text{SL}}_2$ be the metaplectic double cover of SL_2 . Then we have an exact sequence

$$0 \rightarrow \mu_2 \rightarrow \widetilde{\text{SL}}_2 \rightarrow \text{SL}_2 \rightarrow 0,$$

where $\mu_2 = \{\pm 1\}$.

The product on $\widetilde{\text{SL}}_2(F)$ is given by

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, \zeta_1 \zeta_2 c(g_1, g_2)),$$

where $c : \text{SL}_2(F) \times \text{SL}_2(F) \rightarrow \{\pm 1\}$ is defined by Hilbert symbols as

$$c(g_1, g_2) = (\mathbf{x}(g_1), \mathbf{x}(g_2))_F (-\mathbf{x}(g_1)\mathbf{x}(g_2), \mathbf{x}(g_1 g_2))_F,$$

where

$$\mathbf{x} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c, & c \neq 0, \\ d, & c = 0. \end{cases}$$

For these formulas for the Kubota cocycle, see section 3 in [Ku].

For a subgroup A of $\text{SL}_2(F)$, we denote \tilde{A} the preimage of A in $\widetilde{\text{SL}}_2(F)$, which is a subgroup of $\widetilde{\text{SL}}_2(F)$. It is easy to see that $\tilde{N}(F) = N(F) \times \mu_2$ and $\tilde{N}(F) = \tilde{N}(F) \times \mu_2$. For an element $g \in \text{SL}_2(F)$, we sometimes write $(g, 1) \in \widetilde{\text{SL}}_2(F)$ as g by abuse of notation.

A representation π of $\widetilde{\text{SL}}_2(F)$ is called genuine if $\pi(\zeta g) = \zeta \pi(g)$ for all $g \in \widetilde{\text{SL}}_2(F)$ and $\zeta \in \mu_2$. Let ψ be an additive character of F . Then there is Weil representation ω_ψ of $\widetilde{\text{SL}}_2(F)$ on the space $\mathcal{S}(F)$ of Schwartz-Bruhat functions on F . For $f \in \mathcal{S}(F)$, we have the well-know formulas:

$$\begin{aligned} \left(\omega_\psi \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right) f(x) &= \gamma(\psi) \hat{f}(x), \\ \left(\omega_\psi \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \right) f(x) &= \psi(bx^2) f(x), b \in F \\ \left(\omega_\psi \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) f(x) &= |a|^{1/2} \frac{\gamma(\psi)}{\gamma(\psi_a)} f(ax), a \in F^\times, \end{aligned}$$

and

$$\omega_\psi(\zeta) f(x) = \zeta f(x), \zeta \in \mu_2.$$

Here $\hat{f}(x) = \int_F f(y)\psi(2xy)dy$, where dy is normalized so that $\hat{f}(x) = f(-x)$, $\gamma(\psi)$ is the Weil index and $\psi_a(x) = \psi(ax)$.

Let \tilde{T} be the inverse image of $T = \left\{ t(a) := \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, a \in F^\times \right\} \subset \mathrm{SL}_2(F)$ in $\widetilde{\mathrm{SL}}_2(F)$. The product in \tilde{T} is given by the Hilbert symbol, i.e.,

$$(t(a), \zeta_1)(t(b), \zeta_2) = (t(ab), \zeta_1 \zeta_2(a, b)_F),$$

where $(a, b)_F$ is the Hilbert symbol. The function

$$\mu_\psi(a) = \frac{\gamma(\psi)}{\gamma(\psi_a)}$$

satisfies

$$\mu_\psi(a)\mu_\psi(b) = \mu_\psi(ab)(a, b)_F,$$

and thus defines a genuine character of \tilde{T} .

The representation ω_ψ is not irreducible, and we have $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$, where ω_ψ^+ (resp. ω_ψ^-) is the subrepresentation on even (resp. odd) functions in $\mathcal{S}(F)$. All the above facts can be found in section 1 in [GePS1].

2.3. The local zeta integral. Let $\mu_\psi(a) = \frac{\gamma(\psi)}{\gamma(\psi_a)}$, which is viewed as a character of \tilde{T} . Let η be a character of F^\times and η_s denotes the character $\eta|\cdot|^s$ of F^\times , and consider the induced representation $I(s, \eta, \psi) = \mathrm{Ind}_{\widetilde{\mathrm{B}}(F)}^{\widetilde{\mathrm{SL}}_2(F)}(\eta_{s-1/2}\mu_\psi)$.

Let (π, V) be a ψ -generic representation of $\mathrm{SL}_2(F)$ with its Whittaker model $\mathcal{W}(\pi, \psi)$. Choose $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F)$ and $f_s \in I(s, \eta, \psi^{-1})$, note that $(\omega_{\psi^{-1}}(h)\phi)(1)f_s(h)$ is well-defined as a function on $\mathrm{SL}_2(F)$, and consider the integral

$$\Psi(W, \phi, f_s) = \int_{N(F)\backslash\mathrm{SL}_2(F)} W(h)(\omega_{\psi^{-1}}(h)\phi)(1)f_s(h)dh.$$

By results in section 5 and section 12 in [GePS2], the above integral is absolutely convergent when $\mathrm{Re}(s)$ is large enough and has a meromorphic continuation to the whole plane.

Remark: In [GePS2], Gelbart and Piatetski-Shapiro constructed a global zeta integral for $\mathrm{Sp}_{2n} \times \mathrm{GL}_n$, showed that it is Eulerian and sketched a proof of the local functional equation. This is Method C in [GePS2]. The above integral is the simplest case of the Gelbart and Piatetski-Shapiro integral, namely, when $n = 1$.

2.4. Local functional equation. The trilinear form $(W, \phi, f_s) \mapsto \Psi(W, \phi, f_s)$ defines an element in

$$\mathrm{Hom}_{\mathrm{SL}_2}(\pi \otimes \omega_{\psi^{-1}} \otimes I(s, \eta, \psi^{-1}), \mathbb{C}),$$

which has dimension at most one. The proof of this fact is given in [GePS2, §11] and also can be deduced by the uniqueness of Fourier-Jacobi model for SL_2 , see [Su]. Let

$$M_s : I(s, \eta, \psi^{-1}) \rightarrow I(1-s, \eta^{-1}, \psi^{-1})$$

be the standard intertwining operator, i.e.,

$$M_s(f_s)(g) = \int_N f_s(wng)dn.$$

By the uniqueness of the above Hom space, we then get the following

Proposition 2.2. *There is a meromorphic function $\gamma(s, \pi, \eta, \psi)$ such that*

$$\Psi(W, \phi, M_s(f_s)) = \gamma(s, \pi, \eta, \psi)\Psi(W, \phi, f_s),$$

for all $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F)$ and $f_s \in I(s, \eta, \psi^{-1})$.

2.5. Unramified calculation. The unramified calculation of Method C is in fact not included in [GePS2], but it can be simply done in the SL_2 -case.

Suppose everything is unramified. Suppose that the residue characteristic is not 2 so that μ_ψ is unramified. See [Sz, Page 2188]. Suppose the representation (π, V) has a Satake parameter a , which means that π is the unramified component $\mathrm{Ind}_{B(F)}^{\mathrm{SL}_2(F)}(\nu)$ for an unramified character ν and $a = \nu(p_F)$, where p_F is some prime element of F . Let

$$b_k = \mathrm{diag}(p_F^k, p_F^{-k}),$$

and W be the spherical Whittaker functional normalized by $W(e) = 1$. Then $W(b_k) = 0$ for $k < 0$, and

$$W(b_k) = \frac{q^{-k}}{a-1}(a^{k+1} - a^{-k}),$$

by the general Casselman-Shalika formula [CS, Theorem 5.4]. For $k \geq 0$, we have

$$(\omega_{\psi^{-1}}(b_k)\phi)(1) = \mu_{\psi^{-1}}(p_F^k)|p_F^k|^{1/2},$$

where ϕ is the characteristic function of ring of integers \mathcal{O}_F . On the other hand, let f_s be the standard spherical section of $I(s, \eta, \psi^{-1})$ normalized by $f_s(1) = 1$. Then we have

$$f_s(b_k) = \eta(p_F^k)|p_F^k|^{s+1/2}\mu_{\psi^{-1}}(p_F^k).$$

Since $\mu_{\psi^{-1}}(p_F^k)\mu_{\psi^{-1}}(p_F^k) = (p_F^k, p_F^k)_F = (p_F, -1)_F^k$, we have

$$\begin{aligned} \Psi(W, \phi, f_s) &= \int_{F^\times} \int_K W(t(a)k)\omega_{\psi^{-1}}(t(ak)\phi)(1)f_s(t(a)k)|a|^{-2}dkda \\ &= \int_{F^\times} W(t(a))\omega_{\psi^{-1}}(t(a))\phi(1)f_s(t(a))|a|^{-2}da \\ &= \sum_{k \geq 0} W(b_k)(\omega_{\psi^{-1}}(b_k)\phi)(1)f_s(b_k)|p_F^k|^{-2} \\ &= \frac{1}{a-1} \sum_{k \geq 0} (a^{k+1} - a^{-k})(p_F, -1)^k \eta(p_F)^k q_F^{-ks} \\ &= \frac{1+c}{(1-ac)(1-a^{-1}c)} = \frac{1-c^2}{(1-c)(1-ac)(1-a^{-1}c)} \\ &= \frac{L(s, \pi, St \otimes \eta\chi)}{L(2s, \eta^2)}, \end{aligned}$$

where $c = (p_F, -1)\eta(p_F)q_F^{-s}$, and $\chi(a) = (a, -1)_F$. Recall that St is the standard representation of ${}^L\mathrm{SL}_2 = \mathrm{SO}_3(\mathbb{C})$.

Remark: Let $M_s : I(s, \eta, \psi^{-1}) \rightarrow I(1-s, \eta^{-1}, \psi^{-1})$ be the standard intertwining operator, i.e.,

$$M_s(f_s)(g) = \int_N f_s(wng)dn.$$

From the calculation of the local coefficients for $\widetilde{\mathrm{SL}}_2(F)$ given in [Sz, Theorem 1.1], one can check that

$$M_s(f_s) = \frac{L(2s-1, \eta^2)}{L(2s, \eta^2)} f_{1-s},$$

where f_s (resp. f_{1-s}) is the standard spherical section in $I(s, \eta, \psi^{-1})$ (resp. $I(1-s, \eta^{-1}, \psi^{-1})$). Thus the factor $L(2s, \eta^2)$ appearing in the above unramified calculation will play the role of the normalizing factor of a global intertwining operator or Eisenstein series.

3. HOWE VECTORS AND THE LOCAL CONVERSE THEOREM

In this section, we assume the residue characteristic of F is not 2. We will follow Baruch's method, [Ba1] and [Ba2], to give a proof of local converse theorem for generic representations of $\mathrm{SL}_2(F)$.

3.1. Howe vectors. Let ψ be an unramified character. For a positive integer m , let $K_m = (1 + M_{2 \times 2}(\mathcal{P}_F^m)) \cap SL_2(F)$ where \mathcal{P}_F denotes the maximal ideal in \mathcal{O}_F . Define a character τ_m of K_m by

$$\tau_m(k) = \psi(p_F^{-2m} k_{12}).$$

for $k = (k_{ij}) \in K_m$. It is easy to see that τ_m is indeed a character on K_m .

Let $d_m = \text{diag}(p_F^{-m}, p_F^m)$. Consider the subgroup $J_m = d_m K_m d_m^{-1}$. Then

$$J_m = \begin{pmatrix} 1 + \mathcal{P}_F^m & \mathcal{P}_F^{-m} \\ \mathcal{P}_F^{3m} & 1 + \mathcal{P}_F^m \end{pmatrix} \cap SL_2(F).$$

Define $\psi_m(j) = \tau_m(d_m^{-1} j d_m)$ for $j \in J_m$. For a subgroup $H \subset SL_2(F)$, denote $H_m = H \cap J_m$. It is easy to check that $\psi_m|_{N_m} = \psi|_{N_m}$.

Let π be an irreducible smooth ψ -generic representation of $SL_2(F)$, and let $v \in V_\pi$ be a vector such that $W_v(1) = 1$. For $m \geq 1$, as in [Ba1] and [Ba2], we consider

$$(3.1) \quad v_m = \frac{1}{\text{Vol}(N_m)} \int_{N_m} \psi(n)^{-1} \pi(n) v dn.$$

Let $L \geq 1$ be an integer such that v is fixed by K_L . Following E.M. Baruch, we call $\{v_m\}_{m \geq L}$ **Howe vectors**.

Lemma 3.1. *We have*

- (1) $W_{v_m}(1) = 1$.
- (2) If $m \geq L$, $\pi(j)v_m = \psi_m(j)v_m$ for all $j \in J_m$.
- (3) If $k \leq m$, then

$$v_m = \frac{1}{\text{Vol}(N_m)} \int_{N_m} \psi(u)^{-1} \pi(u) v_k du.$$

The proof of this lemma is the same as the proof in the $U(2, 1)$ case, which is given in [Ba2, Lemma 5.2].

Lemma 3.2. *Let $m \geq L$ and $t = t(a)$ for $a \in F^\times$. Then*

- (1) if $W_{v_m}(t) \neq 0$, we have

$$a^2 \in 1 + \mathcal{P}_F^m;$$

- (2) if $W_{v_m}(tw) \neq 0$, we have

$$a^2 \in \mathcal{P}^{-3m}.$$

Proof. (1) Take $x \in \mathcal{P}^{-m}$. We then have $n(x) \in N_m \subset J_m$. From the relation

$$tn(x) = n(a^2 x)t,$$

and (2) of Lemma 3.1, we have

$$\psi(x)W_{v_m}(t) = \psi(a^2 x)W_{v_m}(t).$$

If $W_{v_m}(t) \neq 0$, we get $\psi(x) = \psi(a^2 x)$ for all $x \in \mathcal{P}^{-m}$. Since ψ is unramified, we get $a^2 \in 1 + \mathcal{P}^m$.

(2) For $x \in \mathcal{P}^{3m}$, we have $\bar{n}(x) \in \bar{N}_m$. From the relation $tw\bar{n}(x) = n(-a^2 x)tw$ and Lemma 3.1 (2), we get

$$W_{v_m}(tw) = \psi(-a^2 x)W_{v_m}(tw).$$

Thus if $W_{v_m}(tw) \neq 0$, we get $\psi(-a^2 x) = 1$ for all $x \in \mathcal{P}^{3m}$. Thus $a^2 \in \mathcal{P}^{-3m}$. \square

Lemma 3.3. *For $m \geq 1$, consider the square map $^2 : 1 + \mathcal{P}^m \rightarrow 1 + \mathcal{P}^m$, $a \mapsto a^2$ is well-defined and surjective.*

This lemma requires the residue field of F is not of characteristic 2 which we assume throughout this section.

Proof. For $x \in \mathcal{P}^m$, it is clear that $(1+x)^2 = 1+2x+x^2 \in 1+\mathcal{P}^m$. Thus the square map is well-defined. On the other hand, we take $u \in 1+\mathcal{P}^m$ and consider the equation $f(X) := X^2 - u = 0$. We have $f'(X) = 2X$. Since $q^{-m} = |1-u| = |f(1)| < |f'(1)|^2 = |2|^2 = 1$, by Newton's Lemma, see, for example, [Lg, Proposition 2, Chapter II], there is a root $a \in \mathcal{O}_F$ of $f(X)$ such that

$$|a-1| \leq \frac{|f(1)|}{|f'(1)|^2} = |1-u| = q^{-m}.$$

Thus we get a root $a \in 1+\mathcal{P}^m$ of $f(X)$. This completes the proof. \square

Let $Z = \{\pm 1\}$, and identify Z with the center of $\mathrm{SL}_2(F)$. Denote ω_π the central character of π .

Corollary 3.4. *Let $m \geq L$. Then we have*

$$W_{v_m}(t(a)) = \begin{cases} \omega_\pi(z), & \text{if } a = za', \text{ for some } z \in Z, a' \in 1+\mathcal{P}^m, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose that $W_{v_m}(t(a)) \neq 0$. Then by Lemma 3.2, we have $a^2 \in 1+\mathcal{P}^m$. By Lemma 3.3, there exists an $a' \in 1+\mathcal{P}^m$ such that $a^2 = (a')^2$. Thus $a = za'$ for some $z \in Z$. Since $a' \in 1+\mathcal{P}^m$, we get $t(a') \in J_m$. The assertion follows from Lemma 3.1. \square

From now on, we fix two ψ -generic representations (π, V_π) and $(\pi', V_{\pi'})$ with the same central characters. Fix v, v' such that $W_v(1) = 1 = W_{v'}(1)$. Let L be an integer such that both v and v' are fixed by K_L . For $m \geq 1$, we consider the Howe vectors v_m and v'_m .

By Corollary 3.4 and the fact that $\omega_\pi = \omega_{\pi'}$, we get

Corollary 3.5. *For $m \geq L$, we have $W_{v_m}(g) = W_{v'_m}(g)$ for all $g \in B$.*

Lemma 3.6 (Baruch). *If $m \geq 4L$ and $n \in N - N_m$, we have*

$$W_{v_m}(twn) = W_{v'_m}(twn),$$

for all $t \in T$.

Proof. This is a special case of [Ba1, Lemma 6.2.2]. A similar result for $\mathrm{U}(2,1)$ is given in [Ba2, Proposition 5.7]. We just remark that the proof of this lemma depends on Corollary 3.5, and hence require the residue characteristic of F is not 2. \square

3.2. Induced representations. Note that $\bar{N}(F)$ and $N(F)$ split in $\widetilde{\mathrm{SL}}_2(F)$. Moreover, for $g_1 \in N$ and $g \in \bar{N}$ we have $c(g_1, g_2) = 1$. In fact, if $g_1 = n(y)$ and $g_2 = \bar{n}(x)$ with $x \neq 0$, we have $\mathbf{x}(g_1) = 1$ and $\mathbf{x}(g_2) = x$, and thus

$$c(g_1, g_2) = (1, x)_F(-x, x)_F = 1.$$

This shows that $N(F) \cdot \bar{N}(F) \subset \mathrm{SL}_2(F)$, where $\mathrm{SL}_2(F)$ denotes the subset of $\widetilde{\mathrm{SL}}_2(F)$ which consists of elements of the form $(g, 1)$ for $g \in \mathrm{SL}_2(F)$.

Let X be an open compact subgroup of $N(F)$. For $x \in X$ and $i > 0$, we consider the set $A(x, i) = \{\bar{n} \in \bar{N}(F) : \bar{n}x \in B \cdot \bar{N}_i\}$.

Lemma 3.7. (1) *For any positive integer c , there exists an integer $i_1 = i_1(X, c)$ such that for all $i \geq i_1$, $x \in X$ and $\bar{n} \in A(x, i)$, we have*

$$\bar{n}x = nt(a)\bar{n}_0$$

with $n \in N, \bar{n}_0 \in \bar{N}_i$ and $a \in 1+\mathcal{P}^c$.

(2) *There exists an integer $i_0 = i_0(X)$ such that for all $i \geq i_0$, we have $A(x, i) = \bar{N}_i$ for all $i \geq i_1$.*

Proof. By abuse of notation, for $x \in X$, we write $x = n(x)$. Since X is compact, there is a constant C such that $|x| < C$ for all $n(x) \in X \subset N$.

For $n(x) \in X, \bar{n}(y) \in A(x, i)$, we have $\bar{n}(y)n(x) \in B \cdot \bar{N}_i$. Thus we can assume that

$$\bar{n}(y)n(x) = \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \bar{n}(\bar{y})$$

for $a \in F^\times, b \in F$ and $\bar{y} \in \mathcal{P}^{3i}$. Rewrite the above expression as

$$\bar{n}(-y) \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} = n(x)\bar{n}(-\bar{y}),$$

or

$$\begin{pmatrix} a & b \\ -ay & a^{-1} - by \end{pmatrix} = \begin{pmatrix} 1 - x\bar{y} & x \\ -\bar{y} & 1 \end{pmatrix}.$$

Thus we get

$$a = 1 - x\bar{y} \text{ and } ay = \bar{y}.$$

Since $|x| < C$ and $\bar{y} \in \mathcal{P}^{3i}$, it is clear that for any positive integer c , we can choose $i_1(X, c)$ such that $a = 1 - x\bar{y} \in 1 + \mathcal{P}^c$ for all $n(x) \in X$ and $\bar{n}(y) \in A(x, i)$. This proves (1).

If we take $i_0(X) = i_1(X, 1)$, we get $a \in 1 + \mathcal{P} \subset \mathcal{O}^\times$ for $i \geq i_0$. From $ay = \bar{y}$, we get $y \in \mathcal{P}^{3i}$. Thus we get that for $i \geq i_0(X)$, we have $\bar{n}(y) \in \bar{N}_i$, i.e., $A(x, i) \subset \bar{N}_i$.

The other direction can be checked similarly if i is large. We omit the details. \square

Given a positive integer i and a complex number $s \in \mathbb{C}$, we consider the following function f_s^i on $\widetilde{SL}_2(F)$:

$$f_s^i(\tilde{g}) = \begin{cases} \zeta \mu_{\psi^{-1}}(a) \eta_{s+1/2}(a), & \text{if } \tilde{g} = \left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \zeta \right) \bar{n}(x), \text{ with } a \in F^\times, b \in F, \zeta \in \mu_2, x \in \mathcal{P}^{3i}, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.8. (1) *There exists an integer $i_2(\eta)$ such that for all $i \geq i_2(\eta)$, f_s^i defines a section in $I(s, \eta, \psi^{-1})$.*

(2) *Let X be an open compact subset of N . Then there exists an integer $I(X, \eta) \geq i_2(\eta)$ such that for all $i \geq I(X, \eta)$, we have*

$$\tilde{f}_s^i(wx) = \text{vol}(\bar{N}_i) = q_F^{-3i},$$

for all $x \in X$, where $\tilde{f}_s^i = M_s(f_s^i)$.

$$\text{Here } w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Proof. (1) From the definition, it is clear that

$$f_s^i \left(\left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \zeta \right) \tilde{g} \right) = \zeta \mu_{\psi^{-1}}(a) \eta_{s+1/2}(a) f_s^i(\tilde{g}),$$

for $a \in F^\times, b \in F, \zeta \in \mu_2$, and $\tilde{g} \in \widetilde{SL}_2(F)$. It suffices to show that for i large, there is an open compact subgroup $\tilde{H}_i \subset \widetilde{SL}_2(F)$ such that $f_s^i(\tilde{g}\tilde{h}) = f_s^i(\tilde{g})$ for all $\tilde{g} \in \widetilde{SL}_2(F)$, and $\tilde{h} \in \tilde{H}_i$.

If ψ is unramified and the residue characteristic is not 2 as we assumed, the character $\mu_{\psi^{-1}}$ is trivial on \mathcal{O}_F^\times , see, for example, [Sz, Page 2188].

Let c be a positive integer such that η is trivial on $1 + \mathcal{P}^c$. Let $i_2(\eta) = \max\{c, i_0(N \cap K_c), i_1(N \cap K_c, c)\}$. For $i \geq i_2(\eta)$, we take $\tilde{H}_i = K_{4i} = 1 + M_2(\mathcal{P}^{4i})$. Note that K_{4i} splits, and thus can be viewed as a subgroup of \widetilde{SL}_2 . We now check that for $i \geq i_2(\eta)$, we have $f_s^i(\tilde{g}h) = f_s^i(\tilde{g})$ for all $\tilde{g} \in \widetilde{SL}_2$ and $h \in K_{4i}$. We have the decomposition $K_{4i} = (N \cap K_{4i})(T \cap K_{4i})(\bar{N} \cap K_{4i})$. For $h \in \bar{N} \cap K_{4i} \subset \bar{N}_i$, it is clear that $f_s^i(\tilde{g}h) = f_s^i(\tilde{g})$ by the definition of f_s^i . Now we take $h \in T \cap K_{4i}$. Write $h = t(a_0)$, with $a_0 \in 1 + \mathcal{P}^{4i}$. We have $\bar{n}(x)h = h\bar{n}(a_0^{-2}x)$. It is clear that $x \in \mathcal{P}^{3i}$ if and only if $a_0^{-2}x \in \mathcal{P}^{3i}$. On the other hand, for any $a \in F^\times, b \in F$, we have

$$c \left(\left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, t(a_0) \right) \right) = (a, a_0) = 1,$$

since $a_0 \in 1 + \mathcal{P}_F^{4i} \subset F^{\times, 2}$, by Lemma 3.3. Thus we get

$$\left(\left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \zeta \right) \bar{n}(x)h \right) = \left(\left(\begin{pmatrix} aa_0 & ba_0^{-1} \\ & a^{-1}a_0^{-1} \end{pmatrix}, \zeta \right) \bar{n}(a_0^{-2}x) \right).$$

By definition of f_s^i , if $x \in \mathcal{P}^{3i}$, for $g = \left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \zeta \right) \bar{n}(x)$, we get

$$f_s^i(gh) = \mu_{\psi^{-1}}(a_0 a) \eta_{s+1/2}(a a_0) = \mu_{\psi^{-1}}(a) \eta_{s+1/2}(a) = f_s^i(g),$$

by the assumption on i .

Finally, we consider $h \in N \cap K_{4i} \subset N \cap K_c$. By assumption on i , we get

$$A(h, i) = A(h^{-1}, i) = \bar{N}_i.$$

In particular, for $\bar{n} \in \bar{N}_i$, we have $\bar{n}h \in B \cdot \bar{N}_i$ and $\bar{n}h^{-1} \in B \cdot \bar{N}_i$. Now it is clear that $\tilde{g} \in \tilde{B} \cdot \bar{N}_i$ if and only if $\tilde{g}h \in \tilde{B} \cdot \bar{N}_i$. Thus $f_s^i(\tilde{g}) = 0$ if and only if $f_s^i(\tilde{g}h) = 0$. Moreover, for $\bar{n} \in \bar{N}_i$, we have

$$\bar{n}h = \begin{pmatrix} a_0 & b_0 \\ & a_0^{-1} \end{pmatrix} \bar{n}_0,$$

for $a_0 \in 1 + \mathcal{P}^c$, $b_0 \in F$ and $\bar{n}_0 \in \bar{N}_i$. Thus for $\tilde{g} = \left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \zeta \right) \bar{n}$ with $\bar{n} \in \bar{N}_i$, we get

$$\tilde{g}h = \left(\begin{pmatrix} a a_0 & a b_0 + a_0^{-1} b \\ & a_0^{-1} a^{-1} \end{pmatrix}, \zeta \right) \bar{n}_0.$$

Here we used the fact that $a_0 \in 1 + \mathcal{P}^c$ is a square, and thus

$$c \left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \begin{pmatrix} a_0 & b_0 \\ & a_0^{-1} \end{pmatrix} \right) = 1.$$

Since $\mu_{\psi^{-1}}(a_0) = 1$, $(a, a_0) = 1$ and $\eta_{s+1/2}(a_0) = 1$, we get

$$f_s^i(\tilde{g}h) = f_s^i(\tilde{g}).$$

This finishes the proof of (1).

(2) As in the proof of (1), let c be a positive integer such that η is trivial on $1 + \mathcal{P}^c$. Take $I(X, \eta) = \max \{i_1(X, c), i_0(X)\}$. We have

$$\tilde{f}_s^i(wx) = \int_N f_s^i(w^{-1}nwx) dn.$$

By the definition of f_s^i , $f_s^i(w^{-1}nwx) \neq 0$ if and only if $w^{-1}nwx \in B\bar{N}_i$, if and only if $w^{-1}nw \in A(x, i) = \bar{N}_i$ for all $i \geq I(X)$ and $x \in X$. On the other hand, if $w^{-1}nw \in A(x, i)$, we have

$$w^{-1}nwx = \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \bar{n}_0,$$

with $a \in 1 + \mathcal{P}^c$. Thus

$$f_s^i(w^{-1}nwx) = \eta_{s+1/2}(a) \mu_{\psi^{-1}}(a) = 1.$$

Now the assertion is clear. \square

3.3. The local converse theorem.

Lemma 3.9. *Let ϕ^m be the characteristic function of $1 + \mathcal{P}^m$. Then*

- (1) for $n \in N_m$, we have $\omega_{\psi^{-1}}(n)\phi^m = \psi^{-1}(n)\phi^m$;
- (2) for $\bar{n} \in \bar{N}_m$, we have $\omega_{\psi^{-1}}(\bar{n})\phi^m = \phi^m$.

Proof. (1) For $n = n(b) \in N_m$, we have $b \in \mathcal{P}^{-m}$. For $x \in 1 + \mathcal{P}^m$, we have $bx^2 - b \in \mathcal{O}_F$ and thus

$$\omega_{\psi^{-1}}(n)\phi^m(x) = \psi^{-1}(bx^2)\phi^m(x) = \psi^{-1}(b)\phi^m(x).$$

For $x \notin 1 + \mathcal{P}^m$, we have $\omega_{\psi^{-1}}(n)\phi^m(x) = \psi^{-1}(bx^2)\phi^m(x) = 0$. The first assertion follows.

(2) For $\bar{n} \in \bar{N}_m$, we can write $\bar{n} = w^{-1}n(b)w$ with $b \in \mathcal{P}^{3m}$. Let $\phi' = \omega_{\psi^{-1}}(w)\phi^m$. We have

$$\begin{aligned} \phi'(x) &= \gamma(\psi^{-1}) \int_F \phi^m(y) \psi^{-1}(2xy) dy \\ &= \gamma(\psi^{-1}) \psi^{-1}(2x) \int_{\mathcal{P}^m} \psi^{-1}(2xz) dz \\ &= \gamma(\psi^{-1}) \psi^{-1}(2x) \text{vol}(\mathcal{P}^m) \text{Char}(\mathcal{P}^{-m})(x) \end{aligned}$$

where $\mathrm{Char}(\mathcal{P}^{-m})$ denotes the characteristic function of the set \mathcal{P}^{-m} . It is clear that $\omega_{\psi^{-1}}(n(b))\phi' = \phi'$. Thus we have

$$\omega_{\psi^{-1}}(\bar{n})\phi^m = \omega_{\psi^{-1}}(w^{-1}n(b))\phi' = \omega_{\psi^{-1}}(w^{-1})\phi' = \omega_{\psi^{-1}}(w^{-1})\omega_{\psi^{-1}}(w)\phi^m = \phi^m.$$

This completes the proof. \square

Given a quasi-character η of F^\times , recall that we have defined a local gamma factor $\gamma(s, \pi, \eta, \psi)$ in Proposition 2.2.

Theorem 3.10. *Suppose that the residue characteristic of F is not 2 and ψ is a nontrivial additive character of F . Let (π, V_π) and $(\pi', V_{\pi'})$ be two ψ -generic representations of $\mathrm{SL}_2(F)$ with the same central character.*

- (1) *If $\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi)$ for all quasi-characters η of F^\times , then $\pi \cong \pi'$.*
- (2) *There is an integer $l = l(\pi, \pi')$ such that if η is quasi-character of F^\times with conductor $\mathrm{cond}(\eta) > l$, then*

$$\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi).$$

Remark: Theorem 3.10 can be viewed one example of a general local converse conjecture for classical groups, see [Jng, Conjecture 3.7] or [JngN, Conjecture 6.3].

Proof. We will first treat the case when ψ is unramified, and prove the general case at the end. We fix the notations $v \in V_\pi, v' \in V_{\pi'}$ and L as before.

Let η be a quasi-character of F^\times . We take an integer $m \geq \max\{6L, \mathrm{cond}(\eta)\}$. We consider the Howe vectors v_m and v'_m . We also take an integer $i \geq \max\{i_2(\eta), I(N_m, \eta), m\}$. In particular we have a section $f_s^i \in I(s, \eta, \psi)$. Let $W_m = W_{v_m}$ or $W_{v'_m}$. We compute the integral $\Psi(W_m, \phi^m, f_s^i)$ on the open dense subset $T\bar{N}(F) = N(F) \setminus N(F)T\bar{N}(F)$ of $N(F) \setminus \mathrm{SL}_2(F)$. For $g = nt(a)\bar{n}$, we can take the quotient measure as $dg = |a|^{-2}d\bar{n}da$. By the definition of f_s^i , we get

$$\begin{aligned} \Psi(W_m, \phi^m, f_s^i) &= \int_{T \times \bar{N}(F)} W_m(t(a)\bar{n})(\omega_{\psi^{-1}}(t(a)\bar{n})\phi^m)(1)f_s^i(t(a)\bar{n})|a|^{-2}d\bar{n}da \\ &= \int_{T \times \bar{N}_i} W_m(t(a)\bar{n})\mu_{\psi^{-1}}(a)|a|^{1/2}\omega_{\psi^{-1}}(\bar{n})\phi^m(a)\mu_{\psi^{-1}}(a)\eta_{s+1/2}(a)|a|^{-2}d\bar{n}da \\ &= \int_{T \times \bar{N}_i} W_m(t(a)\bar{n})\omega_{\psi^{-1}}(\bar{n})\phi^m(a)\chi(a)\eta_{s-1}(a)d\bar{n}da, \end{aligned}$$

where $\chi(a) = \mu_{\psi^{-1}}(a)\mu_{\psi^{-1}}(a) = (a, -1)_F$. Since $i \geq m$, we get $\bar{N}_i \subset \bar{N}_m$. By Lemma 3.1, and Lemma 3.9, we get $W_m(t(a)\bar{n}) = W_m(t(a))$ and $\omega_{\psi^{-1}}(\bar{n})\phi^m = \phi^m$. Thus we get

$$\Psi(W_m, \phi^m, f_s^i) = q^{-3i} \int_{F^\times} W_m(t(a))\phi^m(a)\chi(a)\eta_{s-1}(a)da.$$

Since $\phi^m = \mathrm{Char}(1 + \mathcal{P}^m)$ and, for $a \in 1 + \mathcal{P}^m$, we have $W_m(t(a)) = 1$ by Lemma 3.1, we get

$$\Psi(W_m, \phi^m, f_s^i) = q^{-3i} \int_{1 + \mathcal{P}^m} \chi(a)\eta(a)da.$$

Since $\chi(a) = 1$ for $a \in 1 + \mathcal{P}^m$, and $m \geq \mathrm{cond}(\eta)$ by assumption, we get

$$\Psi(W_m, \phi^m, f_s^i) = q^{-3i-m}.$$

The above calculation works for both W_{v_m} and $W_{v'_m}$. Thus we have

$$(3.2) \quad \Psi(W_{v_m}, \phi^m, f_s^i) = \Psi(W_{v'_m}, \phi^m, f_s^i) = q^{-3i-m}.$$

Next, we compute the other side of the local functional equation of $\Psi(W_m, \phi^m, \tilde{f}_s^i)$ on the open dense subset $N(F) \setminus N(F)T_wN(F) \subset N(F) \setminus \mathrm{SL}_2(F)$, where $\tilde{f}_s^i = M_s(f_s^i)$. We have

$$\begin{aligned} \Psi(W_m, \phi^m, \tilde{f}_s^i) &= \int_{T \times N(F)} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\tilde{f}_s^i(t(a)wn)|a|^{-2}dnda \\ &= \int_{T \times N_m} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\tilde{f}_s^i(t(a)wn)|a|^{-2}dnda \\ &\quad + \int_{T \times (N(F) - N_m)} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\tilde{f}_s^i(t(a)wn)|a|^{-2}dnda. \end{aligned}$$

By Lemma 3.6, we get $W_{v_m}(t(a)wn) = W_{v'_m}(t(a)wn)$ for all $n \in N(F) - N_m$. Thus

$$\begin{aligned} &\Psi(W_{v_m}, \phi^m, \tilde{f}_s^i) - \Psi(W_{v'_m}, \phi^m, \tilde{f}_s^i) \\ &= \int_{T \times N_m} (W_{v_m}(t(a)wn) - W_{v'_m}(t(a)wn))(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\tilde{f}_s^i(t(a)wn)|a|^{-2}dnda. \end{aligned}$$

Since $i \geq I(N_m, \eta)$, we get

$$\tilde{f}_s^i(t(a)wn) = \mu_{\psi^{-1}}(a)\eta_{3/2-s}^{-1}(a)q_F^{-3i},$$

by Lemma 3.8. On the other hand, by Lemma 3.1 and Lemma 3.9, for $n \in N_m$, we get

$$W_m(t(a)wn) = \psi(n)W_m(t(a)w), \quad (\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1) = \psi^{-1}(n)(\omega_{\psi^{-1}}(t(a)w)\phi^m)(1).$$

Thus

$$\begin{aligned} (3.3) \quad &\Psi(W_{v_m}, \phi^m, \tilde{f}_s^i) - \Psi(W_{v'_m}, \phi^m, \tilde{f}_s^i) \\ &= q_F^{-3i+m} \int_T (W_{v_m}(t(a)w) - W_{v'_m}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)\chi(a)\eta^{-1}(a)|a|^{-s}da. \end{aligned}$$

By (3.2), (3.3) and the local functional equation, we get

$$\begin{aligned} (3.4) \quad &q^{-2m}(\gamma(s, \pi, \eta, \psi) - \gamma(s, \pi', \eta, \psi)) \\ &= \int_{F^\times} (W_{v_m}(t(a)w) - W_{v'_m}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)\chi(a)\eta^{-1}(a)|a|^{-s}da. \end{aligned}$$

Let $k = 4L$. Since $m \geq 6L > k$, by Lemma 3.1 and Lemma 3.6, we get

$$\begin{aligned} W_{v_m}(t(a)w) - W_{v'_m}(t(a)w) &= \frac{1}{\mathrm{vol}(N_m)} \int_{N_m} (W_{v_k}(t(a)wn) - W_{v'_k}(t(a)wn))\psi^{-1}(n)dn \\ &= \frac{1}{\mathrm{vol}(N_m)} \int_{N_k} (W_{v_k}(t(a)wn) - W_{v'_k}(t(a)wn))\psi^{-1}(n)dn \\ &= \frac{\mathrm{vol}(N_k)}{\mathrm{vol}(N_m)} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)) \\ &= q^{k-m} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)). \end{aligned}$$

Now we can rewrite (3.4) as

$$\begin{aligned} (3.5) \quad &q^{-m-k}(\gamma(s, \pi, \eta, \psi) - \gamma(s, \pi', \eta, \psi)) \\ &= \int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)\chi(a)\eta^{-1}(a)|a|^{-s}da. \end{aligned}$$

By Lemma 3.2, if $a \notin \mathcal{P}^{-6L}$, i.e., $a^2 \notin \mathcal{P}^{-3k}$, we get $W_{v_k}(t(a)w) = 0 = W_{v'_k}(t(a)w)$. Thus the integral on the right side in the above formula (3.5) can be taken over \mathcal{P}^{-6L} . For $a \in \mathcal{P}^{-6L}$ and $m \geq 6L$ (as we assumed), by the calculation given in the proof of Lemma 3.9, we have

$$\begin{aligned} (\omega_{\psi^{-1}}(w)\phi^m)(a) &= \gamma(\psi^{-1})\psi^{-1}(2a)\mathrm{vol}(\mathcal{P}^m)\mathrm{Char}(\mathcal{P}^{-m})(a) \\ &= \gamma(\psi^{-1})\psi^{-1}(2a)q^{-m}. \end{aligned}$$

Plugging this into (3.5), we get

$$(3.6) \quad \begin{aligned} & q^{-k} \gamma(\psi^{-1})^{-1} (\gamma(s, \pi, \eta, \psi) - \gamma(s, \pi', \eta, \psi)) \\ &= \int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)) \psi^{-1}(2a) \chi(a) \eta^{-1}(a) |a|^{-s} da. \end{aligned}$$

Now we can prove our theorem. We consider (1) first. Suppose that $\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi)$ for all quasi-characters η of F^\times , then we get

$$\int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)) \psi^{-1}(2a) \chi(a) \eta^{-1}(a) |a|^{-s} da = 0$$

for all quasi-characters η .

We rewrite the equality as

$$\begin{aligned} 0 &= \int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)) \psi^{-1}(2a) \chi(a) \eta^{-1}(a) |a|^{-s} da \\ &= \sum_{m=-\infty}^{\infty} \int_{|a|=q^m} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)) \psi^{-1}(2a) \chi(a) \eta^{-1}(a) da q^{-ms}. \end{aligned}$$

It follows that all the coefficients in the above Laurent series in q^s have to be zero. So

$$\int_{|a|=q^m} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)) \psi^{-1}(2a) \chi(a) \eta^{-1}(a) da = 0$$

for all quasi-characters η .

Since the set $\{a \in F^\times : |a| = q^m\}$ is compact open in F^\times , the left side of the above equality can be viewed as Mellin transform of a compactly supported function on F^\times . By the inverse Mellin transform, we get

$$(W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)) \psi^{-1}(2a) = 0,$$

or

$$W_{v_k}(t(a)w) = W_{v'_k}(t(a)w).$$

By Lemma 3.1, Lemma 3.6, Corollary 3.5 and the Bruhat decomposition $SL_2(F) = B \cup BwB$, we get

$$W_{v_k}(g) = W_{v'_k}(g),$$

for all $g \in SL_2(F)$. By the uniqueness of Whittaker model, we get $\pi \cong \pi'$. This proves (1).

Next, we consider (2). Let $l = l(\pi, \pi')$ be an integer such that $l \geq 6L$ and

$$W_{v_k}(t(a_0a)w) = W_{v_k}(t(a)w), \text{ and } W_{v'_k}(t(a_0a)w) = W_{v'_k}(t(a)w),$$

for all $a_0 \in 1 + \mathcal{P}^l$ and all $a \in \mathcal{P}^{-6L}$. Such an l exists because the functions $a \mapsto W_{v_k}(t(a)w)$, $a \mapsto W_{v'_k}(t(a)w)$ on $\mathcal{P}^{-6L} \subset F^\times$ are continuous. Note that $k = 4L$ and L only depends on the choices of v and v' . On the other hand, for $a \in \mathcal{P}^{-6L}$, it is easy to see that

$$\psi^{-1}(2a_0a) = \psi^{-1}(2a), \text{ for all } a_0 \in 1 + \mathcal{P}^l,$$

since $l \geq 6L$. It is also clear that $\chi(a_0a) = \chi(a)$ for all $a_0 \in 1 + \mathcal{P}^l$. Since the character χ is unramified. As we noted before, the integrand of the right side integral of (3.6) has support in \mathcal{P}^{-6L} . Let η be a quasi-character of F^\times with $\text{cond}(\eta) > l$. Then it is clear that the integral of the right side of (3.6) vanishes. Thus we get

$$\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi).$$

This finishes the proof when ψ is unramified.

Now let us consider the general case when ψ is ramified. The proof is essentially the same as the unramified case, and we will indicate the necessary changes in the above proof. If ψ has conductor c , i.e., $\psi(\mathcal{P}_F^c) = 1$ but $\psi(\mathcal{P}_F^{c-1}) \neq 1$, we define $d_m = \text{diag}(p_F^{-2m+c}, 1) \in GL_2(F)$ and $J_m = d_m K_m d_m^{-1}$. Then

$$J_m = \left(\begin{array}{cc} 1 + \mathcal{P}_F^m & \mathcal{P}_F^{-m+c} \\ \mathcal{P}_F^{3m-c} & 1 + \mathcal{P}_F^m \end{array} \right) \cap SL_2(F).$$

For $j = (j_{il})_{1 \leq i, l \leq 2} \in J_m$, we define $\psi_m(j) = \psi(j_{12})$. It is clear that ψ_m defines a character of J_m . Given a ψ -generic representation (π, V) of $\mathrm{SL}_2(F)$, and a vector $v \in V$, we define v_m in the same way as before, i.e., by (3.1). In this case, we fix an integer L such that $L \geq c$ and v is fixed by K_L . We call $\{v_m\}_{m \geq L}$ the Howe vectors. We note that in the proof of Lemma 3.8, we used that ψ is unramified to make sure $\mu_{\psi^{-1}}$ is trivial on \mathcal{O}_F^\times . If ψ is ramified, by continuity, $\mu_{\psi^{-1}}$ is trivial on $1 + \mathcal{P}_F^i$ for i large. This is all what we need in the proof of Lemma 3.8 to extend it to the ramified case. Now one can check easily that all of the above proofs go through and we get the theorem in general. \square

4. A STRONG MULTIPLICITY ONE THEOREM

Let F be a number field and \mathbb{A} be its adèle ring.

4.1. Global genericity. In this subsection, we discuss the relation between global genericity and local genericity. Let φ be a cusp form on $\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A})$. Since the group $N(F) \backslash N(\mathbb{A})$ is compact and abelian, we have the Fourier expansion

$$\varphi(g) = \sum_{\psi \in \widehat{N(F) \backslash N(\mathbb{A})}} W_\varphi^\psi(g),$$

where

$$W_\varphi^\psi(g) = \int_{N(F) \backslash N(\mathbb{A})} \varphi(n g) \psi^{-1}(n) dn.$$

Since φ is a cusp form, we get $W_\varphi^{\psi_0} \equiv 0$, where ψ_0 is the trivial character of $F \backslash \mathbb{A}$. Thus we get

$$\varphi(g) = \sum_{\substack{\psi \in \widehat{N(F) \backslash N(\mathbb{A})} \\ \psi \neq \psi_0}} W_\varphi^\psi(g).$$

Fix a nontrivial additive character ψ of $N(F) \backslash N(\mathbb{A})$. Then

$$\widehat{(N(F) \backslash N(\mathbb{A})) \setminus \{0\}} = \{\psi_\kappa : \kappa \in F^\times\},$$

where $\psi_\kappa(a) = \psi(\kappa a)$, $a \in \mathbb{A}$. If $\kappa \in F^{\times, 2}$, say $\kappa = a^2$, we have

$$W_\varphi^{\psi_\kappa}(g) = W_\varphi^\psi(t(a)g).$$

Thus we get

$$\varphi(g) = \sum_{\kappa \in F^\times / F^{\times, 2}} \sum_{a \in F^\times} W_\varphi^{\psi_\kappa}(t(a)g).$$

Corollary 4.1. *If φ is a nonzero cusp form, there exists $\kappa \in F^\times$ such that*

$$W_\varphi^{\psi_\kappa} \neq 0.$$

Let (π, V_π) be a cuspidal automorphic representation of $\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A})$. We say π is ψ_κ -generic, if there exists $\varphi \in V_\pi$ such that

$$W_\varphi^{\psi_\kappa} \neq 0.$$

Corollary 4.2. *Let π be a cuspidal automorphic representation of $\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A})$ and ψ be a nontrivial additive character of $F \backslash \mathbb{A}$. Then there exists $\kappa \in F^\times$ such that π is ψ_κ -generic.*

Theorem 4.3. *Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ and $\psi = \otimes_v \psi_v$ be a nontrivial additive character of $F \backslash \mathbb{A}$. Then π is ψ -generic if and only if each π_v is ψ_v -generic.*

Proof. A similar result is proved for $\mathrm{U}(1, 1)$ by Gelbart, Rogawski and Soudry in [GeRS, Proposition 2.5].

One direction is trivial. Now we assume each π_v is ψ_v -generic.

We assume π is ψ_κ -generic for some $\kappa \in F^\times$, i.e. there exists $\varphi \in V_\pi$ such that

$$W_\varphi^{\psi_\kappa}(g) = \int_{N(F) \backslash N(\mathbb{A})} \varphi(n g) \psi_\kappa^{-1}(n) dn \neq 0.$$

Then it is clear that π_v is also $\psi_{\kappa,v}$ -generic, where $\psi_{\kappa,v}(a) = \psi_v(\kappa a)$. By Proposition 2.1, we get $\pi_v \cong \pi_v^\kappa$.

For $\varphi \in V_\pi$, consider the function $\varphi^\kappa(g) = \varphi(g^\kappa)$, where $g^\kappa = \text{diag}(\kappa, 1)g\text{diag}(\kappa^{-1}, 1)$. Then

$$\begin{aligned} \int_{N(F)\backslash N(\mathbb{A})} \varphi^\kappa(n g) dn &= \int_{N(F)\backslash N(\mathbb{A})} \varphi((n g)^\kappa) dn \\ &= \int_{N(F)\backslash N(\mathbb{A})} \varphi(n^\kappa g^\kappa) dn = \int_{N(F)\backslash N(\mathbb{A})} \varphi(n g^\kappa) dn = 0 \end{aligned}$$

hence φ^κ is also a cusp form. Let V_π^κ be the space which consists functions of the form φ^κ for all $\varphi \in V_\pi$. Let π^κ denote the cuspidal automorphic representation of $SL_2(\mathbb{A})$ on V_π^κ .

Claim: $(\pi^\kappa)_v = \pi_v^\kappa$.

proof of the claim: Let $\Lambda : V_\pi \rightarrow \mathbb{C}$ be a nonzero ψ_κ -Whittaker functional for π , and let Λ_v be nonzero $(\psi_\kappa)_v$ -Whittaker functional on V_{π_v} satisfying that, if $\varphi = \otimes_v \varphi_v$ is a pure tensor, then

$$\Lambda(\pi(g)\varphi) = \prod_v \Lambda_v(\pi_v(g_v)\varphi_v)$$

Note that Λ is in fact given by

$$\Lambda(\varphi) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(n)\psi_\kappa^{-1}(n)dn$$

Then the ψ_{κ^2} -Whittaker functional of π^κ is given by

$$\int_{N(F)\backslash N(\mathbb{A})} \varphi^\kappa(n)\psi_{\kappa^2}^{-1}(n)dn$$

This means that if $W_\varphi(g)$ is a ψ_κ -Whittaker function of π , then $W_{\varphi^\kappa}(g) = W_\varphi(g^\kappa)$ is a ψ_{κ^2} -Whittaker function of π^κ .

Hence with $\varphi = \otimes_v \varphi_v$ a pure tensor, we have $W_\varphi(g) = \prod_v W_{\varphi_v}(g_v)$, and $\{W_{\varphi_v}(g_v)\}$ is the Whittaker model of π_v . While $W_{\varphi^\kappa}(g) = W_\varphi(g^\kappa) = \prod_v W_{\varphi_v}(g_v^\kappa)$, and $\{W_{\varphi_v}(g_v^\kappa)\}$ is the Whittaker model of $(\pi^\kappa)_v$.

Now $W_v(g_v) \rightarrow W_v(g_v^\kappa)$ gives an isomorphism between π_v^κ and $(\pi^\kappa)_v$, which proves the claim.

Now let us continue the proof of the theorem. By the claim we then have $\pi_v \cong (\pi^\kappa)_v$, or $\pi \cong \pi^\kappa$. By the multiplicity one theorem for SL_2 of [Ra], we get $\pi = \pi^\kappa$. Since π is ψ_κ -generic, we get that π^κ is ψ_{κ^2} -generic, and hence ψ -generic. Since $\pi = \pi^\kappa$, the theorem follows. \square

4.2. Eisenstein series on $\widetilde{SL}_2(\mathbb{A})$. Let $\widetilde{SL}_2(\mathbb{A})$ be the double cover of $SL_2(\mathbb{A})$. It is well-known that $SL_2(F)$ splits over the projection $\widetilde{SL}_2(\mathbb{A}) \rightarrow SL_2(\mathbb{A})$. Let μ_ψ be the genuine character of $T(F)\backslash \widetilde{T}(\mathbb{A})$ whose local components are μ_{ψ_v} as given in §2.

Let η be a quasi-character of $F^\times \backslash \mathbb{A}^\times$, and $s \in \mathbb{C}$. We consider the induced representation

$$I(s, \chi, \psi) = \text{Ind}_{\widetilde{B}(\mathbb{A})}^{\widetilde{SL}_2(\mathbb{A})}(\mu_\psi \eta_{s-1/2}).$$

For $f_s \in I(s, \eta, \psi)$, we consider the Eisenstein series $E(g, f_s)$ on $\widetilde{SL}_2(\mathbb{A})$:

$$E(g, f_s) = \sum_{B(F)\backslash SL_2(F)} f_s(\gamma g), g \in \widetilde{SL}_2(\mathbb{A}).$$

The above sum is absolutely convergent when $\text{Re}(s) \gg 0$, and can be meromorphically continued to the whole s -plane.

There is an intertwining operator $M_s : I(s, \eta, \psi) \rightarrow I(1-s, \eta^{-1}, \psi)$:

$$M_s(f_s)(g) = \int_{N(F)\backslash N(\mathbb{A})} f_s(wng)dn.$$

The above integral is absolutely convergent for $\text{Re}(s) \gg 0$ and defines a meromorphic function of $s \in \mathbb{C}$.

Proposition 4.4. (1) *If $\eta^2 \neq 1$, then the Eisenstein series $E(g, f_s)$ is holomorphic for all s . If $\eta^2 = 1$, the only possible poles of $E(g, f_s)$ are at $s = 0$ and $s = 1$. Moreover, the order of the poles are at most 1.*

(2) We have the functional equation

$$E(g, f_s) = E(g, M_s(f_s)), \text{ and } M_s(\eta) \circ M_{1-s}(\eta^{-1}) = 1.$$

See [GaQT, Proposition 6.1] for example.

4.3. The global zeta integral. Let ψ be a nontrivial additive character of $F \setminus \mathbb{A}$. Then there is a global Weil representation ω_ψ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ on $\mathcal{S}(\mathbb{A})$. For $\phi \in \mathcal{S}(\mathbb{A})$, we consider the theta series

$$\theta_\psi(\phi)(g) = \sum_{x \in F} (\omega_\psi(g)\phi)(x).$$

It is well-known that θ_ψ defines an automorphic form on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.

Let (π, V_π) be a ψ -generic cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$. For $\varphi \in V_\pi, \phi \in \mathcal{S}(\mathbb{A})$ and $f_s \in I(s, \eta, \psi^{-1})$, consider the integral

$$(4.1) \quad Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s)) = \int_{\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A})} \varphi(g) \theta_{\psi^{-1}}(\phi)(g) E(g, f_s) dg.$$

Proposition 4.5 (Gelbart-Piatetski-Shapiro, [GePS2, Theorem 4.C]). *For $\mathrm{Re}(s) \gg 0$, the integral $Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s))$ is absolutely convergent, and*

$$Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s)) = \int_{N(\mathbb{A}) \backslash \mathrm{SL}_2(\mathbb{A})} W_\varphi^\psi(g) (\omega_{\psi^{-1}}(g)\phi)(1) f_s(g) dg,$$

where $W_\varphi^\psi(g) = \int_{N(F) \backslash N(\mathbb{A})} \varphi(ng) \psi^{-1}(n) dn$ is the ψ -th Whittaker coefficient of φ .

Corollary 4.6. *We take $\phi = \otimes_v \phi_v, f_s = \otimes_v f_{s,v}$ to be pure tensors. Let S be a finite set of places such that for all $v \notin S$, v is finite and $\pi_v, \psi_v, f_{s,v}$ are unramified. Then for $\mathrm{Re}(s) \gg 0$, we have*

$$Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s)) = \prod_{v \in S} \Psi(W_v, \phi_v, f_{s,v}) \frac{L^S(s, \pi, \mathrm{St} \otimes (\chi\eta))}{L^S(2s, \eta^2)},$$

where χ is the character of $F^\times \setminus \mathbb{A}^\times$ defined by

$$\chi((a_v)) = \prod_v (a_v, -1)_{F_v}, (a_v)_v \in \mathbb{A}^\times.$$

Moreover, we have the following functional equation

$$Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s)) = Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, M_s(f_s)))$$

This follows from Proposition 4.2, the unramified calculation and the functional equation of Eisenstein series in Proposition 4.4 directly.

4.4. A strong multiplicity one theorem. With the above preparation, we are now ready to prove the main global result of this paper.

Theorem 4.7. *Let $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ be two irreducible cuspidal automorphic representations of $\mathrm{SL}_2(\mathbb{A})$ with the same central character. Suppose that π and π' are both ψ -generic. Let S be a finite set of **finite** places such that no place in S is above 2. If $\pi_v \cong \pi'_v$ for all $v \notin S$, then $\pi = \pi'$.*

Proof. For $\phi = \otimes_v \phi_v \in V_\pi$, consider $Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s))$. By Proposition 4.5, we have

$$\begin{aligned} Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s)) &= \prod_v \Psi(W_v, \phi_v, f_{s,v}) \\ &= \prod_v \int_{N(F_v) \backslash \mathrm{SL}_2(F_v)} W_v(h) (\omega_{\psi^{-1}}(h)\phi_v)(1) f_{s,v}(h) dh. \end{aligned}$$

By the functional equation in Corollary 4.6, we have

$$\begin{aligned} &\prod_v \int_{N(F_v) \backslash \mathrm{SL}_2(F_v)} W_v(h) (\omega_{\psi^{-1}}(h)\phi_v)(1) f_{s,v}(h) dh \\ &= \prod_v \int_{N(F_v) \backslash \mathrm{SL}_2(F_v)} W_v(h) (\omega_{\psi^{-1}}(h)\phi_v)(1) M_s(f_{s,v})(h) dh. \end{aligned}$$

At the archimedean places, by Lemma 4.8 below, we can choose the data so that the archimedean local integrals are nonzero. At the finite places, by the local functional equation discussed in section 1.4, there are local gamma factors $\gamma(s, \pi_v, \eta_v, \psi_v)$ such that

$$\int_{N(F_v)\backslash SL_2(F_v)} W_v(h)(\omega_{\psi^{-1}}(h)\phi_v)(1)M_s(f_{s,v})(h)dh = \gamma(s, \pi_v, \eta_v, \psi_v) \int_{N(F_v)\backslash SL_2(F_v)} W_v(h)(\omega_{\psi^{-1}}(h)\phi_v)(1)f_{s,v}(h)dh.$$

Hence we have

$$1 = \prod_{v<\infty} \gamma(s, \pi_v, \eta_v, \psi_v) \prod_{v|\infty} \frac{\Psi(W_v, \phi_v, M_s(f_{s,v}))}{\Psi(W_v, \phi_v, f_{s,v})}.$$

Similarly, we have for π' the identity

$$1 = \prod_{v<\infty} \gamma(s, \pi'_v, \eta_v, \psi_v) \prod_{v|\infty} \frac{\Psi(W'_v, \phi_v, M_s(f_{s,v}))}{\Psi(W'_v, \phi_v, f_{s,v})}.$$

By assumptions that $\pi_v \cong \pi'_v$ for all $v \notin S$ with S does not have archimedean places, we then have

$$\prod_{v \in S} \gamma(s, \pi_v, \eta_v, \psi_v) = \prod_{v \in S} \gamma(s, \pi'_v, \eta_v, \psi_v).$$

Fix $v_0 \in S$. By [JL, Lemma 12.5], given an arbitrary character η_{v_0} , we can find a character η of \mathbb{A}^\times which restricts to v_0 is η_{v_0} and has arbitrary high conductor at the other places of S . By Theorem 3.10 (2), we conclude that

$$\gamma(s, \pi_{v_0}, \eta_{v_0}, \psi_{v_0}) = \gamma(s, \pi'_{v_0}, \eta_{v_0}, \psi_{v_0}),$$

for all characters η_{v_0} . Thus by Theorem 3.10 (1), we conclude that $\pi_{v_0} \cong \pi'_{v_0}$. This applies also to the other places of S . Thus we proved that $\pi_v \cong \pi'_v$ for all places v . Now the theorem follows from the multiplicity one theorem for SL_2 of [Ra]. \square

Remark: We expect that the restriction about residue characteristics on the finite set S in Theorem 4.7 can be removed.

Finally, we prove a nonvanishing result on the archimedean local zeta integrals which is used in the above proof. We formulate and prove the result both for the p -adic and the archimedean cases simultaneously.

Lemma 4.8. *Let F be a local field, ψ be a nontrivial additive character of F , η be a quasi-character of F^\times and π be a ψ -generic representation of $SL_2(F)$. Then there exists $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F)$ and $f_s \in \text{Ind}_{\widetilde{B}}^{\widetilde{SL}_2(F)}$ such that*

$$\Psi(W, \phi, f_s) = \int_{N(F)\backslash SL_2(F)} W(h)(\omega_{\psi^{-1}}\phi)(h)f_s(h) \neq 0.$$

Proof. We note that the Bruhat cell $\Omega = N(F)TwN(F)$ is open dense in $SL_2(F)$. Thus the above integral is reduced to

$$\Psi(W, \phi, f_s) = \int_{TN(F)} W(\omega_t(a)n(u))(\omega_{\psi^{-1}}(wt(a)n(u))\phi)(1)f_s(wt(a)n(u))\Delta(a)dadu,$$

where $\Delta(a) = |a|^{-2}$.

Using the formulas for the Weil representation $\omega_{\psi^{-1}}$, we find

$$(\omega_{\psi^{-1}}(\omega_2 t(a)n(u))\phi)(x) = |a|^{1/2} \frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})} \int_F \psi(ua^2y^2)\phi(ay)\psi(2xy)dy = |a|^{1/2} \frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})} \hat{\Phi}_{a,u}(x),$$

where $\hat{\Phi}_{a,u}(x) = \psi(ua^2x^2)\phi(ax)$ which is again an Schwartz function on F and depends continuously on a and u .

We first explain the set $\{(g, 1) : g \in N(F)TWN(F)\}$, still denoted as Ω , is open in $\widetilde{\mathrm{SL}}_2(F)$. Note that there is a double covering map $p : \widetilde{\mathrm{SL}}_2(F) \rightarrow \mathrm{SL}_2(F)$. For any $(g, 1) \in \Omega$, its projection of p is g . As p is a covering map, there exists an open neighborhood U_g of g contained in $N(F)TWN(F)$, such that $p^{-1}(U_g)$ is a disjoint union of two open subsets of $\widetilde{\mathrm{SL}}_2(F)$ each is homeomorphic to U_g by p . Then one component of $p^{-1}(U_g)$ is an open neighborhood of $(g, 1)$ in Ω , which shows that Ω is open in $\widetilde{\mathrm{SL}}_2(F)$.

Now define $f_s \in I(s, \eta, \psi^{-1})$ on the set $\{(g, 1) : g \in \mathrm{SL}_2(F)\}$ by

$$f_s(g) = \begin{cases} \delta(b)^{1/2}(\eta_{s-1/2}\mu_{\psi^{-1}})(b)f_2(u) & \text{if } g = bwn(u) \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

where $b \in B(F) = TN(F)$, $u \in F$, and f_2 is compactly supported to be determined later. Then we extend the definition of f_2 to the set $\{(g, -1) : g \in \mathrm{SL}_2(F)\}$ to make it genuine, i.e., $f_s(g, -1) = -1f_s(g, 1)$.

Then the integral Ψ can be reduced further to

$$\Psi(W, \phi, f_s) = \int_{TN(F)} W(wau)|a|^{1/2} \frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})} \hat{\Phi}_{a,u}(1) \delta(a)^{1/2} (\eta_{s-1/2}\mu_{\psi^{-1}})(a) f_2(u) \Delta(a) da du. \quad \dots (*)$$

Case 1: F is p -adic.

Consider the Howe vector W_{v_m} . By Corollary 3.4, taking m large enough, W_{v_m} can have arbitrarily small compact open support around 1 when restricted to T . Then $W_{w.v_m}(t(a^{-1})w)$ has small compact open support around $a = 1$.

First choose ϕ so that $\hat{\Phi}_{a,u}(1) \neq 0$ when $a = 1, u = 0$. Then choose m so that $W_{w.v_m}(wt(a)) = W_{w.v_m}(t(a^{-1})w)$ has small compact support around 1 and all the other data involving a in the integral (*) are nonzero constants. For this $W_{w.v_m}$, consider $W_{w.v_m}(wt(a)u)$ with $u \in N$. When u is close to 1 enough, we have $W_{w.v_m}(wt(a)u) = W_{w.v_m}(wt(a))$ for all a in that small compact support around 1. Then take f_2 with support u close to 1 satisfying the above. With these choices of $W_{w.v_m}(g), f_2, \phi$, the integral (*) is nonzero.

Case 2: F is archimedean.

We will concentrate on the case $F = \mathbb{R}$. The case $F = \mathbb{C}$ is similar as we have the same formulas for the Weil representation by Proposition 1.3 in [JL]. We begin with the formulas

$$\Psi(W, \phi, f_s) = \int_{TN(F)} W(wau)|a|^{1/2} \frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})} \hat{\Phi}_{a,u}(1) \delta(a)^{1/2} (\eta_{s-1/2}\mu_{\psi^{-1}})(a) f_2(u) \Delta(a) da du. \quad \dots (**)$$

where $\Phi_{a,u}(x) = \psi(ua^2x^2)\phi(ax)$ is again a Schwartz function as ϕ is, and it depends on a and u continuously. Since the Fourier transform is an isometry of the Schwartz space, we can choose ϕ so that the Fourier transform $\hat{\Phi}_{a,u}(1) > 0$ when $a = 1, u = 0$ and depends on a and u continuously.

Now let (π, V) be an irreducible generic smooth representation of $\mathrm{SL}_2(\mathbb{R})$ of moderate growth. Realize π as a quotient of a smooth principal series $I(\chi, s)$, i.e.,

$$0 \rightarrow V' \rightarrow I(\chi, s) \rightarrow V \rightarrow 0$$

Let $\lambda : V \rightarrow \mathbb{C}$ be the unique nonzero continuous Whittaker functional on V . Then the composition

$$\Lambda : I(\chi, s) \rightarrow V \xrightarrow{\lambda} \mathbb{C}$$

gives the unique nonzero continuous Whittaker functional on $I(\chi, s)$ up to a scalar. It follows that the two spaces $\{\lambda(\pi(g)v) : g \in \mathrm{SL}_2(F), v \in V\}$ and $\{\Lambda(R(g).f) : g \in \mathrm{SL}_2(F), f \in I(\chi, s)\}$ are the same, although the first is the Whittaker model of π , while the later may not be a Whittaker model of $I(\chi, s)$.

The Whittaker functional on $I(\chi, s)$ is given by the following

$$\Lambda(f) = \int_{N(F)} f(wu)\psi^{-1}(u)du,$$

when s is in some right half plane, and its continuation gives Whittaker functionals for all $I(\chi, s)$. Also when f has support inside $\Omega = N(F)TwN(F)$, the above integral always converges for any s , and gives the Whittaker functional.

Now for such f , one computes that for $a = t(a) \in T$

$$\begin{aligned} \Lambda(I(a).f) &= \int_{N(F)} f(wua)\psi^{-1}(u)du = \chi'(a) \int_F f(wu)\psi^{-1}(a^2u)du \\ &= \chi'(a) \int_F f_1(u)\psi^{-1}(a^2u)du = \chi'(a)\hat{f}_1(a^2), \end{aligned}$$

where f_1 is the restriction of f to wN which can be chosen to be a Schwartz function, \hat{f}_1 is its Fourier transform, and χ' is a certain character. Again, as the Fourier transform gives an isometry of Schwartz functions, we can always choose f so that its Whittaker function $W_f(a)$ has arbitrarily small compact support around 1. By a right translation by w , we show that one can always choose f so that $W_{w.f}(aw)$ has small compact support around 1.

In order to prove the proposition, note that we have chosen Φ . Let

$$R(a, u) = |a|^{1/2} \frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})} \hat{\Phi}_{a,u}(1) \delta(a)^{1/2} (\eta_{s-1/2} \mu_{\psi^{-1}})(a) \Delta(a).$$

Then $R(a, u)$ is a continuous function of a and u and $R(1, 0) \neq 0$. This means that there exist neighborhoods U_1 of $a = 1$ and U_2 of $u = 0$, such that $R(a, u) > R(1, 0)/2 > 0$ for all $a \in U_1, u \in U_2$.

Now choose f so that $W_{w.f}(aw)$ has small compact support in a neighborhood V_1 of 1 with $V_1 \subset U_1$, and $W_{w.f}(w) > 0$. For this Whittaker function, since $W_{w.f}(awu)$ is continuous on u , we can choose f_2 so that it is positively supported in a neighborhood V_2 of 0 such that

- (1). $V_2 \subset U_2$;
- (2). $W_{w.f}(awu) > W_{w.f}(w)/2 > 0$ for all $u \in V_2$.

Then (**) becomes

$$\int W_{w.f}(awu)R(a, u)f_2(u)dadu > \frac{W_{w.f}(w)}{2} \frac{R(1, 0)}{2} \int_{V_1} \int_{V_2} f_2(u)dadu > 0,$$

which proves the non vanishing. □

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