A STRONG MULTIPLICITY ONE THEOREM FOR SL_2

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Abstract. It is known that multiplicity one property holds for SL_2, while the strong multiplicity one property fails. However, in this paper, we show that if we require further that a pair of cuspidal representations π and π' of SL_2 have the same local components at archimedean places and the places above 2, and they are generic with respect to the same additive character, then they also satisfy the strong multiplicity one property. The proof is based on a local converse theorem for SL_2.

Keywords: strong multiplicity one theorem, local converse theorem, Howe vectors

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1. Introduction

Let F be a number field, and $\mathbb{A} = \mathbb{A}_F$ be its ring of adeles. Let G be a linear reductive algebraic group defined over F. The study of space of automorphic forms $L^2(G(F)\backslash G(\mathbb{A}))$ has been a central topic in Langlands program and representation theory. Let $L^2_0(G(F)\backslash G_(\mathbb{A}))$ be the subspace of cuspidal representations. Suppose π is an irreducible automorphic representation of $G(\mathbb{A})$. It is known that π occurs discretely with finite multiplicity m_π in $L^2_0(G(F)\backslash G(\mathbb{A}))$.

The multiplicities m_π are important in the study of automorphic forms and number theory. By the work of Piatetski-Shapiro and Jacquet and Shalika ([JaSh]) the group $G = \mathrm{GL}_n$ has property of multiplicity one, that is, $m_\pi \le 1$ for any π. This is also true for SL_2 by the famous work of D.Ramakrishnan ([Ra]). But in general, the multiplicity one property fails, for example [B, GaGJ, Li, LL] to list a few.

In the case of GL_n, a stronger theorem, called the strong multiplicity one, holds. It says that two cuspidal representations π_1, π_2, if they have isomorphic local components almost everywhere, then they coincide in the space of cusp forms(not only isomorphic). It follows from the results in [LL] that SL_2 doesn't have this strong multiplicity one property. Multiplicity one property is already very few, and the strong multiplicity one is even rare. To the authors' knowledge, the only example other than GL_n in this direction is the rigidity theorem for $\mathrm{SO}(2n+1)$, Theorem 5.3 of [JS], which is established by D.Jiang and D.Soudry.

The main purpose of this paper is to prove a weaker version of strong multiplicity one result for $\mathrm{Sp}_2 = \mathrm{SL}_2$. Although we know the strong multiplicity one doesn't hold in general for a pair of cuspidal representations π_1, π_2 of $\mathrm{SL}_2(\mathbb{A})$, but if we require that both π_1, π_2 are generic with respect to the same additive character ψ of $\mathbb{A}$, then we can show that they also satisfy the strong multiplicity one property.

The reason for the failure of the strong multiplicity one for SL_2 is the existence of L-packets. According the local conjecture of Gan-Gross-Prasad, Conjecture 17.3 of [GaGP], there is at most one ψ-generic representation in each L-packet. For SL_2, the result is known by the local discussion in [LL]. In this paper, we prove a local converse theorem for $\mathrm{SL}_2(F)$ when F is a p-adic field such that its residue characteristic is not 2, which will reprove this result and confirm a local converse conjecture in [Jng]. This also implies our version of strong multiplicity one easily.

We now give some details of our results. In [GePS2], Gelbart and Piatetski-Shapiro constructed some Rankin-Selberg integrals to study L-functions on the group $G_n \times \mathrm{GL}(n)$, at least when G_n has split rank n. In particular, in Method C in that paper, if π is a globally generic cuspidal representation of $\mathrm{Sp}_{2n}(\mathbb{A})$, τ is cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$, consider the global Shimura type zeta-integral

$$I(s,\phi,E)=\int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})}\phi(g)\theta(g)E(g,s)$$

where ϕ belongs to the space of π, $E(g,s)$ is a genuine Eisenstein series on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ built from the representation induced from $\mathrm{GL}_n(\mathbb{A})$ by τ twisted by $|\det|^s$, and $\theta(g)$ is some theta series on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. Note that the product $\theta(g)E(g,s)$ is well-defined on Sp_{2n}. The global integral is shown to be Eulerian. The functional equations and unramified calculations were also carried out in [GePS2] by Gelbart and Piatetski-Shapiro. Although we will only consider the easiest case when $n=1$ of Gelbart and Piatetski-Shapiro's construction, we remark here that Ginzburg, Rallis and Soudry generalized the above construction to $\mathrm{Sp}_{2n}\times\mathrm{GL}_k$, for any k, in [GiRS].

We study in more details of Gelbart and Piatetski-Shapiro's local integral

$$\Psi(W_v,\phi_v,f_{s,v})=\int_{N(F_v)\backslash \mathrm{SL}_2(F_v)}W_v(h)(\omega_{\psi_v^{-1}}(h)\phi_v)(1)f_{s,v}(h)dh$$

(for the unexplained notations, see sections below) when v is finite. These local zeta integrals satisfy certain functional equations, which come from the intertwining operators on induced representation and certain uniqueness statements. These functional equations can then be used to define local gamma factors $\gamma(s,\pi_v,\eta_v,\psi_v)$, where π_v is a generic representation of $\mathrm{SL}_2(F_v)$, η_v is a character of $F_v^\times$, and ψ_v is a nontrivial additive character. The main local result of this paper can be formulated as follows.

Theorem 1.1 (Theorem 3.10, local converse theorem and stability of γ)**.** *Suppose that the residue characteristic of a p-adic field F is not 2 and ψ is an unramified character of F. Let (π,V_π) and $(\pi',V_{\pi'})$ be two ψ-generic representations of $\mathrm{SL}_2(F)$ with the same central character.*

1. *If $\gamma(s,\pi,\eta,\psi)=\gamma(s,\pi',\eta,\psi)$ for all quasi-characters η of $F^\times$, then $\pi\cong\pi'$.*
2. *There is an integer $l=l(\pi,\pi')$ such that if η is a quasi-character of $F^\times$ with conductor $\mathrm{cond}(\eta)>l$, then*
$$\gamma(s,\pi,\eta,\psi)=\gamma(s,\pi',\eta,\psi).$$

The proof of this result follows closely to [Ba1, Ba2, Zh], and Howe vectors play an important role. With the help of this result, combining with a nonvanishing result on archimedean local integrals proved in Lemma 4.9, we follow the argument in Theorem 7.2.13 in [Ba2], or Theorem 2 in [Ca], to prove the main global result of this paper.

Theorem 1.2 (Theorem 4.8, Strong Multiplicity One for SL_2)**.** *Let $\pi=\otimes\pi_v$ and $\pi'=\otimes\pi'_v$ be two irreducible cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ with the same central character. Suppose that π and π' are both ψ-generic. Let S be a finite set of* ***finite*** *places such that no place in S is above 2. If $\pi_v\cong\pi'_v$ for all $v\notin S$, then $\pi=\pi'$.*

We remark here the restriction on residue characteristic comes from Lemma 3.3. It is expected that this restriction can be removed.

Besides the above, we also in this paper include a discussion of relations between globally genericness and locally genericness. An irreducible cuspidal automorphic representation (π,V_π) is called globally generic if for some $\phi\in V_\pi$, the integral

$$\int_{N(F)\backslash N(\mathbb{A})}\phi(ug)\psi^{-1}(u)du\neq 0$$

for some $g\in\mathrm{SL}_2(\mathbb{A})$. The representation π is called locally generic if each of its local component is generic. It is easy to see that if π is globally generic, then π is also locally generic. It is a conjecture that on reductive algebraic group G, the converse is also true. This conjecture is closely related to Ramanujan conjecture. See [Sh] for more detailed discussions. In this paper, we confirm this conjecture for SL_2.

Theorem 1.3 (Theorem 4.3)**.** *Let $\pi=\otimes_v\pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ and $\psi=\otimes\psi_v$ be a nontrivial additive character of $F\setminus\mathbb{A}$. Then π is ψ-generic if and only if each π_v is ψ_v-generic.*

In [GeRS] Proposition 2.5, Gelbart, Rogawski and Soudry proved similar results for $\mathrm{U}(1,1)$ and for endoscopic cuspidal automorphic representation of $\mathrm{U}(2,1)$. From the discussions given in [GeRS], Theorem 1.3 follows from the results of [LL] directly. Here, we insist to include this result because we adopt a local argument (see Proposition 2.1) which is different from that given in [LL]. Hopefully, this local argument can be extended to more general groups.

As explained above, there is essentially nothing new in this paper. All the results and proofs should be known to the experts. Our task here is simply to try to write down the details and to check everything works out as expected.

This paper is organized as follows. In section 2, we collect basic results about the local zeta integrals which will be needed. In section 3, we study the Howe vectors and use it to prove the local converse theorem and stability of local gamma factors. In section 4, we prove the main global results.

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Notations: Let F be a field. In $\mathrm{SL}_2(F)$, we consider the following subgroups. Let B be the upper triangular subgroup. Let $B = TN$ be the Levi decomposition, where T is the torus and N is the upper triangular unipotent. Denote

$$t(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \in T, \text{ for } a \in F^\times n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in N, \text{ for } b \in F.$$

Let $\bar{N}$ be the lower triangular unipotent and denote

$$\bar{n}(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}.$$

Let $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

Denote St the natural inclusion $\mathrm{SO}_3(\mathbb{C}) \to \mathrm{GL}_3(\mathbb{C})$ and view it as the "standard" representation of ${}^L\mathrm{SL}_2 = \mathrm{SO}_3(\mathbb{C})$.

2. The local zeta integral

2.1. **The genericness of representations of** $\mathrm{SL}_2(F)$**.** In this section, let F be a local field, and ψ be a nontrivial additive character of F, which is also viewed as a character of $N(F)$. For $\kappa \in F^\times$, $g \in \mathrm{SL}_2(F)$, we define

$$g^\kappa = \begin{pmatrix} \kappa & \\ & 1 \end{pmatrix} g \begin{pmatrix} \kappa^{-1} & \\ & 1 \end{pmatrix}.$$

Explicitly,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}^\kappa = \begin{pmatrix} x & \kappa y \\ \kappa^{-1} z & w \end{pmatrix}.$$

Note that if $\kappa \in F^{\times,2}$, say $\kappa = a^2$, then $g^\kappa = t(a) g t(a)^{-1}$, i.e., $g \to g^\kappa$ is an inner automorphism on $\mathrm{SL}_2(F)$. Let (π, V_π) be an infinite dimensional irreducible smooth representation of $\mathrm{SL}_2(F)$, we consider the representation $(\pi^\kappa, V_{\pi^\kappa})$ defined by

$$V_{\pi^\kappa} = V_\pi, \text{ and } \pi^\kappa(g) = \pi(g^\kappa).$$

Let ψ_κ be the character of F defined by $\psi_\kappa(b) = \psi(\kappa b)$. If (π, V_π) is ψ-generic with a nonzero ψ Whittaker functional $\Lambda : V_\pi \to \mathbb{C}$, one verifies that

$$\Lambda(\pi^\kappa(n)v) = \Lambda(\pi(n^\kappa)v) = \psi_\kappa(n)\Lambda(v)$$

for all $n \in N(F), v \in V_{\pi^\kappa} = V_\pi$. Hence $(\pi^\kappa, V_{\pi^\kappa})$ is ψ_κ-generic.

Proposition 2.1. *If π is both ψ- and ψ_κ-generic, then $\pi \cong \pi^\kappa$.*

Proof. If F is non-archimedean, we proved the same result in the $\mathrm{U}(1,1)$-case in [Zh], Theorem 2.12. The proof in the SL_2-case is the same.

If F is archimedean, the case $F = \mathbb{C}$ is easy, as every κ has a square root in $\mathbb{C}$. Now consider $F = \mathbb{R}$, and we will work with the category of smooth representations of moderate growth of finite length. The Whittaker functional is an exact functor from this category to the category of vector spaces by Theorem 8.2 in [CMH].

We first consider the case when $\pi = \mathrm{Ind}_B^G(\chi)$ for some quasi-character χ of $F^\times$. For $f \in I(\chi)$, consider the function f^κ on $\mathrm{SL}_2(F)$ defined by $f^\kappa(g) = f(g^{\kappa^{-1}})$. It is clear that $f^\kappa \in I(\chi)^\kappa$ and the map $f \mapsto f^\kappa$ defines an isomorphism $I(\chi) \to I(\chi)^\kappa$.

By results in Chapter 2 of [Vo], if π is not a fully induced representation, then it can be embedded into a principal series $I(\chi)$. This $I(\chi)$ has two irreducible infinite dimensional subrepresentations, and use π' to denote the other one. The quotient of $I(\chi)$ by the sum of π and π', denoted by π'', is finite dimensional, i.e., we have an exact short sequence

$$0 \to \pi \oplus \pi' \to I(\chi) \to \pi'' \to 0$$

First by Theorem 6.1 in [CMH], we know that the dimension of Whittaker functionals on $I(\chi)$ is one dimensional for either ψ or ψ_κ. Note that π'' cannot be generic as it is finite dimensional. Since the Whittaker functor is exact, it follows that the dimensional of Whittaker functionals on $\pi \oplus \pi'$ is also one dimensional for either ψ or ψ_κ. By the assumption π is both ψ and ψ_κ generic, thus π' is neither ψ nor ψ_κ generic.

Now since π is ψ generic, then π^κ is ψ_κ generic. Hence the image of π under the isomorphism $f \to f^\kappa$ between $I(\chi) \to I(\chi)^\kappa$ is again ψ_κ generic, hence it has to be ψ generic and isomorphic to π, which finishes the proof. □

2.2. **Weil representations of $\widetilde{\mathrm{SL}}_2$.** Let $\widetilde{\mathrm{SL}}_2$ be the metaplectic double cover of SL_2. Then we have an exact sequence

$$0 \to \mu_2 \to \widetilde{\mathrm{SL}}_2 \to \mathrm{SL}_2 \to 0,$$

where $\mu_2 = \{\pm 1\}$.

The product on $\widetilde{\mathrm{SL}}_2(F)$ is given by

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, \zeta_1 \zeta_2 c(g_1, g_2)),$$

where $c : \mathrm{SL}_2(F) \times \mathrm{SL}_2(F) \to \{\pm 1\}$ is defined by

$$c(g_1, g_2) = (\mathbf{x}(g_1), \mathbf{x}(g_2))_F (-\mathbf{x}(g_1)\mathbf{x}(g_2), \mathbf{x}(g_1 g_2))_F,$$

where

$$\mathbf{x}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c, & c \neq 0, \\ d, & c = 0. \end{cases}$$

For these formulas, see [Sz] for example.

For a subgroup A of $\mathrm{SL}_2(F)$, we denote $\tilde{A}$ the preimage of A in $\widetilde{\mathrm{SL}}_2(F)$, which is a subgroup of $\widetilde{\mathrm{SL}}_2(F)$. It is easy to see that $\widetilde{N}(F) = N(F) \times \mu_2$ and $\widetilde{\bar{N}}(F) = \bar{N}(F) \times \mu_2$. For an element $g \in \mathrm{SL}_2(F)$, we sometimes write $(g, 1) \in \widetilde{\mathrm{SL}}_2(F)$ as g by abuse of notation.

A representation π of $\widetilde{\mathrm{SL}}_2(F)$ is called genuine if $\pi(\zeta g) = \zeta \pi(g)$ for all $g \in \widetilde{\mathrm{SL}}_2(F)$ and $\zeta \in \mu_2$. Let ψ be an additive character of F, there is Weil representation ω_ψ of $\widetilde{\mathrm{SL}}_2(F)$ on $\mathcal{S}(F)$, the Bruhat-Schwartz functions on F. For $f \in \mathcal{S}(F)$, we have the familiar formulas:

$$\left(\omega_\psi \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) f(x) = \gamma(\psi) \hat{f}(x),$$

$$\left(\omega_\psi \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right) f(x) = \psi(bx^2) f(x), b \in F$$

$$\left(\omega_\psi \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\right) f(x) = |a|^{1/2} \frac{\gamma(\psi)}{\gamma(\psi_a)} f(ax), a \in F^\times,$$

and

$$\omega_\psi(\zeta) f(x) = \zeta f(x), \zeta \in \mu_2.$$

Here $\hat{f}(x) = \int_F f(y)\psi(2xy)dy$, where dy is normalized so that $(\hat{f})\hat{}(x) = f(-x)$, $\gamma(\psi)$ is the Weil index and $\psi_a(x) = \psi(ax)$.

Let $\widetilde{T}$ be the inverse image of $T = \left\{t(a) := \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, a \in F^\times \right\} \subset \mathrm{SL}_2(F)$ in $\widetilde{\mathrm{SL}}_2(F)$. The product in $\widetilde{T}$ is given by the Hilbert symbol, i.e.,

$$(t(a), \zeta_1)(t(b), \zeta_2) = (t(ab), \zeta_1\zeta_2(a,b)_F),$$

where $(a,b)_F$ is the Hilbert symbol. The function

$$\mu_\psi(a) = \frac{\gamma(\psi)}{\gamma(\psi_a)}$$

satisfies

$$\mu_\psi(a)\mu_\psi(b) = \mu_\psi(ab)(a,b),$$

and thus defines a genuine character of $\widetilde{T}$.

The representation ω_ψ is not irreducible, and we have $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$, where ω_ψ^+ (resp. ω_ψ^-) is the subrepresentation on even (resp. odd) functions in $\mathcal{S}(F)$. All the above facts can be found in section 1 in [GePS1] for example.

2.3. **The local zeta integral.** Let $\mu_\psi(a) = \frac{\gamma(\psi)}{\gamma(\psi_a)}$, which is viewed as a character of $\widetilde{T}$. Let η be a character of $F^\times$, and we consider the induced representation $I(s,\eta,\psi) = \mathrm{Ind}_{\widetilde{B}(F)}^{\widetilde{\mathrm{SL}}_2(F)}(\eta_{s-1/2}\mu_\psi)$.

Let (π, V) be a ψ-generic representation of $\mathrm{SL}_2(F)$ with its Whittaker model $\mathcal{W}(\pi,\psi)$. Choose $W \in \mathcal{W}(\pi,\psi)$ $\phi \in \mathcal{S}(F)$ and $f_s \in I(s,\eta,\psi^{-1})$, note that $(\omega_{\psi^{-1}}(h)\phi)(1)f_s(h)$ is well-defined as a function on $\mathrm{SL}_2(F)$, and we consider the integral

$$\Psi(W,\phi,f_s) = \int_{N(F)\backslash \mathrm{SL}_2(F)} W(h)(\omega_{\psi^{-1}}(h)\phi)(1)f_s(h)dh.$$

By results in section 5 and section 12 in [GePS2], the above integral is absolutely convergent when $Re(s)$ is large enough and has a meromorphic continuation to the whole plane.

Remark: In [GePS2], Gelbart and Piatetski-Shapiro constructed a global zeta integral for $\mathrm{Sp}_{2n} \times \mathrm{GL}_n$, showed that it is Eulerian and sketched a proof of the local functional equation. This is the so-called method C in [GePS2]. The above integral is the easiest case of the Gelbart and Piatetski-Shapiro integral, namely, when $n = 1$.

2.4. **local functional equation.** The trilinear form $(W,\phi,f_s) \mapsto \Psi(W,\phi,f_s)$ defines an element in

$$\mathrm{Hom}_{\mathrm{SL}_2}(\pi \times \omega_{\psi^{-1}} \otimes I(s,\eta,\psi^{-1}), \mathbb{C}),$$

which has dimension at most one. The proof of this fact is given in [GePS2](section 11) and also can be deduced by the uniqueness of Fourier-Jacobi model for SL_2, see [Su]. Let

$$M_s : I(s,\eta,\psi^{-1}) \to I(1-s,\eta^{-1},\psi^{-1})$$

be the standard intertwining operator, i.e.,

$$M_s(f_s)(g) = \int_N f_s(wng)dn.$$

By the uniqueness of the above Hom space, we then get the following

Proposition 2.2. *There is a meromorphic function $\gamma(s,\pi,\eta,\psi)$ such that*

$$\Psi(W,\phi,M_s(f_s)) = \gamma(s,\pi,\eta,\psi)\Psi(W,\phi,f_s),$$

for all $W \in \mathcal{W}(\pi,\psi)$, $\phi \in \mathcal{S}(F)$ and $f_s \in I(s,\eta,\psi^{-1})$.

2.5. **Unramified calculation.** The unramified calculation of Method C is in fact not included in [GePS2], but it is really easy to do that in our super easy case.

Suppose everything is unramified. Suppose that the residue characteristic is not 2 so that μ_ψ is unramified, see [Sz]. Suppose the representation (π, V) has Satake parameter a, which means that π is the unramified component $\mathrm{Ind}_{B(F)}^{\mathrm{SL}_2(F)}(\nu)$ for an unramified character ν and $a = \nu(p_F)$, where p_F is some prime element of F. Let

$$b_k = \mathrm{diag}(p_F^k, p_F^{-k}),$$

and W be the spherical Whittaker functional normalized by $W(e) = 1$. Then $W(b_k) = 0$ for $k < 0$, and

$$W(b_k) = \frac{q^{-k}}{a-1}(a^{k+1} - a^{-k}),$$

by the general Casselman-Shalika formula. For $k \geq 0$, we have

$$(\omega_{\psi^{-1}}(b_k)\phi)(1) = \mu_{\psi^{-1}}(p_F^k)|p_F^k|^{1/2},$$

where ϕ is the characteristic function of $\mathcal{O}_F$. On the other hand, let f_s be the standard spherical section of $I(s, \eta, \psi^{-1})$ normalized by $f_s(1) = 1$. Then we have

$$f_s(b_k) = \eta(p_F^k)|p_F^k|^{s+1/2}\mu_{\psi^{-1}}(p_F^k).$$

Since $\mu_{\psi^{-1}}(p_F^k)\mu_{\psi^{-1}}(p_F^k) = (p_F^k, p_F^k)_F = (p_F, -1)_F^k$, We have

$$\begin{aligned}
\Psi(W, \phi, f_s) &= \int_{F^\times}\int_K W(t(a)k)\omega_{\psi^{-1}}(t(ak)\phi)(1)f_s(t(a)k)|a|^{-2}dkda \\
&= \int_{F^\times} W(t(a))\omega_{\psi^{-1}}(t(a))\phi(1)f_s(t(a))|a|^{-2}da \\
&= \sum_{k\geq 0} W(b_k)(\omega_{\psi^{-1}}(b_k)\phi)(1)f_s(b_k)|p_F^k|^{-2} \\
&= \frac{1}{a-1}\sum_{k\geq 0}(a^{k+1} - a^{-k})(p_F, -1)^k\eta(p_F)^k q_F^{-ks} \\
&= \frac{1+c}{(1-ac)(1-a^{-1}c)} = \frac{1-c^2}{(1-c)(1-ac)(1-a^{-1}c)} \\
&= \frac{L(s, \pi, St \otimes \eta\chi)}{L(2s, \eta^2)},
\end{aligned}$$

where $c = (p_F, -1)\eta(p)q_F^{-s}$, and $\chi(a) = (a, -1)_F$. Recall that St is the standard representation of ${}^L\mathrm{SL}_2 = \mathrm{SO}_3(\mathbb{C})$.

Remark: Let $M_s : I(s, \eta, \psi^{-1}) \to I(1-s, \eta^{-1}, \psi^{-1})$ be the standard intertwining operator, i.e.,

$$M_s(f_s)(g) = \int_N f_s(wng)dn.$$

From the calculation given in [Sz], it is easy to see that

$$M_s(f_s) = \frac{L(2s-1, \eta^2)}{L(2s, \eta^2)}f_{1-s},$$

where f_s (resp. f_{1-s}) is the standard spherical section in $I(s, \eta, \psi)$ (resp. $I(1-s, \eta^{-1}, \psi)$). Thus the factor $L(2s, \eta^2)$ appeared in the above unramified calculation will play the role of the normalizing factor of a global intertwining operator or Eisenstein series.

3. Howe vectors and the local converse theorem

In this section, we assume the residue characteristic of F is not 2. We will follow Baruch's method, [Ba1] and [Ba2], to give a proof of local converse theorem for generic representations of $\mathrm{SL}_2(F)$.

3.1. **Howe vectors.** Let ψ be an unramified character. For a positive integer m, let $K_m = (1 + M_{2\times 2}(\mathcal{P}_F^m)) \cap \mathrm{SL}_2(F)$ where $\mathcal{P}_F$ denotes the maximal ideal in $\mathcal{O}_F$. Define a character τ_m of K_m by

$$\tau_m(k) = \psi(p_F^{-2m} k_{12}).$$

for $k = (k_{ij}) \in K_m$. It is easy to see that τ_m is indeed a character on K_m.

Let $d_m = \mathrm{diag}(p_F^{-m}, p_F^m)$. Consider the subgroup $J_m = d_m K_m d_m^{-1}$. Then

$$J_m = \begin{pmatrix} 1+\mathcal{P}_F^m & \mathcal{P}_F^{-m} \\ \mathcal{P}^{3m} & 1+\mathcal{P}_F^m \end{pmatrix}.$$

Define $\psi_m(j) = \tau_m(d_m^{-1} j d_m)$ for $j \in J_m$. For a subgroup $H \subset \mathrm{SL}_2(F)$, denote $H_m = H \cap J_m$. It is easy to check that $\psi_m|_{N_m} = \psi|_{N_m}$.

Let π be an irreducible smooth ψ-generic representation of $\mathrm{SL}_2(F)$, let $v \in V_\pi$ be a vector such that $W_v(1) = 1$. For $m \geq 1$, as in [Ba1] and [Ba2], we consider

$$v_m = \frac{1}{\mathrm{Vol}(N_m)} \int_{N_m} \psi(u)^{-1} \pi(n) v dn. \tag{3.1}$$

Let $L \geq 1$ be an integer such that v is fixed by K_L.

Lemma 3.1. *We have*

1. $W_{v_m}(1) = 1$.
2. *If* $m \geq L$, $\pi(j) v_m = \psi_m(j) v_m$ *for all* $j \in J_m$.
3. *If* $k \leq m$, *then*
$$v_m = \frac{1}{\mathrm{Vol}(N_m)} \int_{N_m} \psi(u)^{-1} \pi(u) v_k du.$$

The proof of this lemma is the same as the proof in the $\mathrm{U}(2,1)$ case, which is given in [Ba2].

Lemma 3.2. *Let* $m \geq L$ *and* $t = t(a)$ *for* $a \in F^\times$. *Then*

1. *if* $W_{v_m}(t) \neq 0$, *we have*
$$a^2 \in 1 + \mathcal{P}_F^m;$$
2. *if* $W_{v_m}(tw) \neq 0$, *we have*
$$a^2 \in \mathcal{P}^{-3m}.$$

Proof. (1) Take $x \in \mathcal{P}^{-m}$, we then have $n(x) \in N_m \subset J_m$. From the relation

$$tn(x) = n(a^2 x) t,$$

and the above lemma, we have

$$\psi(x) W_{v_m}(t) = \psi(a^2 x) W_{v_m}(t).$$

If $W_{v_m}(t) \neq 0$, we get $\psi(x) = \psi(a^2)x$ for all $x \in \mathcal{P}^{-m}$. Since ψ is unramified, we get $a^2 \in 1 + \mathcal{P}^m$.

(2) For $x \in \mathcal{P}^{3m}$, we have $\bar{n}(x) \in \bar{N}_m$. From the relation $tw\bar{n}(x) = n(-a^2 x) tw$ and the above lemma, we get

$$W_{v_m}(tw) = \psi(-a^2 x) W_{v_m}(tw).$$

Thus if $W_{v_m}(tw) \neq 0$, we get $\psi(-a^2 x) = 1$ for all $x \in \mathcal{P}^{3m}$. Thus $a^2 \in \mathcal{P}^{-3m}$. □

Lemma 3.3. *For* $m \geq 1$, *consider the square map* ${}^2 : 1 + \mathcal{P}^m \to 1 + \mathcal{P}^m$, $a \mapsto a^2$ *is well-defined and surjective.*

This lemma requires the residue field of F is not of characteristic 2 which we assumed throughout this section.

Proof. For $x \in \mathcal{P}^m$, it is clear that $(1+x)^2 = 1 + 2x + x^2 \in 1 + \mathcal{P}^m$. Thus the square map is well-defined. On the other hand, we take $u \in 1 + \mathcal{P}^m$ and consider the equation $f(X) := X^2 - u = 0$. We have $f'(X) = 2X$. Since $q^{-m} = |1-u| = |f(1)| < |f'(1)|^2 = |2|^2 = 1$, by Newton's Lemma, Proposition 2, Chapter II of [Lg], there is root $a \in \mathcal{O}_F$ of $f(X)$ such that

$$|a-1| \leq \frac{|f(1)|}{|f'(1)|^2} = |1-u| = q^{-m}.$$

Thus we get a root $a \in 1 + \mathcal{P}^m$ of $f(X)$. This completes the proof. □

Let $Z=\{\pm 1\}$, and identify Z with the center of $\mathrm{SL}_2(F)$. Use ω_π to denote the central character of π.

Corollary 3.4. *Let $m\geq L$, then we have*

$$W_{v_m}(t(a))=\begin{cases}\omega_\pi(z), & \text{if } a=za', \text{ for some } z\in Z, a'\in 1+\mathcal{P}^m,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Suppose that $W_{v_m}(t(a))\neq 0$, then by Lemma 3.2, we have $a^2\in 1+\mathcal{P}^m$. By Lemma 3.3, there exists an $a'\in 1+\mathcal{P}^m$ such that $a^2=(a')^2$. Thus $a=za'$ for some $z\in Z$. Since $a'\in 1+\mathcal{P}^m$, we get $t(a')\in J_m$. The assertion follows from Lemma 3.1. □

From now on, we fix two ψ-generic representations (π, V_π) and $(\pi', V_{\pi'})$ with the same central characters. Fix v, v' such that $W_v(1)=1=W_{v'}(1)$. Let L be an integer such that both v and v' are fixed by K_L. For $m\geq 1$, we consider the Howe vectors v_m and v'_m.

By Corollary 3.4 and the fact that $\omega_\pi=\omega_{\pi'}$, we get

Corollary 3.5. *For $m\geq L$, we have $W_{v_m}(g)=W_{v'_m}(g)$ for all $g\in B$.*

Lemma 3.6 (Baruch)**.** *If $m\geq 4L$ and $n\in N-N_m$, we have*

$$W_{v_m}(twn)=W_{v'_m}(twn),$$

for all $t\in T$.

Proof. It's completely the same as Proposition 3.4 in [Zh], which is also a special case of Lemma 6.2.2 of [Ba1]. Similar type result for $\mathrm{U}(2,1)$ can be found in [Ba2]. We just remark that the proof of this lemma depends on Corollary 3.5, and hence require the residue characteristic of F is not 2. □

3.2. **Induced representations.** Note that $\bar{N}(F)$ and $N(F)$ splits in $\widetilde{\mathrm{SL}}_2(F)$. Moreover, for $g_1\in N$ and $g\in\bar{N}$ we have $c(g_1,g_2)=1$. In fact, if $g_1=n(y)$ and $g_2=\bar{n}(x)$ with $x\neq 0$, we have $\mathbf{x}(g_1)=1$ and $\mathbf{x}(g_2)=x$, and thus

$$c(g_1,g_2)=(1,x)_F(-x,x)_F=1.$$

This shows that $N(F)\cdot\bar{N}(F)\subset\mathrm{SL}_2(F)$, where $\mathrm{SL}_2(F)$ denotes the subset of $\widetilde{\mathrm{SL}}_2(F)$ which consists elements of the form $(g,1)$ for $g\in\mathrm{SL}_2(F)$.

Let X be an open compact subgroup of $N(F)$. For $x\in X$ and $i>0$, we consider the set $A(x,i)=\left\{\bar{n}\in\bar{N}(F):\bar{n}x\in B\cdot\bar{N}_i\right\}$.

Lemma 3.7. (1) *For any positive integer c, there exists an integer $i_1=i_1(X,c)$ such that for all $i\geq i_1$, $x\in X$ and $\bar{n}\in A(x,i)$, we have*

$$\bar{n}x=nt(a)\bar{n}_0$$

with $n\in N, \bar{n}_0\in\bar{N}_i$ and $a\in 1+\mathcal{P}^c$.

(2) *There exists an integer $i_0=i_0(X)$ such that for all $i\geq i_0$, we have $A(x,i)=\bar{N}_i$ for all $i\geq i_1$.*

Proof. By abuse of notation, for $x\in X$, we write $x=n(x)$. Since X is compact, there is a constant C such that $|x|<C$ for all $n(x)\in X\subset N$.

For $n(x)\in X, \bar{n}(y)\in A(x,i)$, we have $\bar{n}(y)n(x)\in B\cdot\bar{N}_i$, thus we can assume that

$$\bar{n}(y)n(x)=\begin{pmatrix}a & b\\ & a^{-1}\end{pmatrix}\bar{n}(\bar{y})$$

for $a\in F^\times, b\in F$ and $\bar{y}\in\mathcal{P}^{3i}$. Rewrite the above expression as

$$\bar{n}(-y)\begin{pmatrix}a & b\\ & a^{-1}\end{pmatrix}=n(x)\bar{n}(-\bar{y}),$$

or

$$\begin{pmatrix}a & b\\ -ay & a^{-1}-by\end{pmatrix}=\begin{pmatrix}1-x\bar{y} & x\\ -\bar{y} & 1\end{pmatrix}.$$

Thus we get

$$a=1-x\bar{y}, ay=\bar{y}.$$

Since $|x| < C$ and $\bar{y} \in \mathcal{P}^{3i}$, it is clear that for any positive integer c, we can choose $i_1(X, c)$ such that $a = 1 - x\bar{y} \in 1 + \mathcal{P}^c$ for all $n(x) \in X$ and $\bar{n}(y) \in A(x, i)$. This proves (1).

If we take $i_0(X) = i_1(X, 1)$, we get $a \in 1 + \mathcal{P} \subset \mathcal{O}^\times$ for $i \geq i_0$. From $ay = \bar{y}$, we get $y \in \mathcal{P}^{3i}$. Thus we get that for $i \geq i_0(X)$, we have $\bar{n}(y) \in \bar{N}_i$, i.e., $A(x, i) \subset \bar{N}_i$.

The other direction can be checked similarly if i is large. We omit the details. □

Given a positive integer i and a complex number $s \in \mathbb{C}$, we consider the following function f_s^i on $\widetilde{\mathrm{SL}}_2(F)$:

$$f_s^i((g,\zeta)) = \begin{cases} \zeta\mu_{\psi^{-1}}(a)\eta_{s+1/2}(a), & \text{if } g = \left(\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}, \zeta\right)\bar{n}(x), \text{ with } a \in F^\times, b \in F, \zeta \in \mu_2, x \in \mathcal{P}^{3i}, \\ 0, & \text{otherwise.}\end{cases}$$

Lemma 3.8. (1) *There exists an integer $i_2(\eta)$ such that for all $i \geq i_2(\eta)$, f_s^i defines a section in $I(s, \eta, \psi^{-1})$.*

(2) *Let X be an open compact subset of N, then there exists an integer $I(X, \eta) \geq i_2(\eta)$ such that for all $i \geq I(X, \eta)$, we have*

$$\tilde{f}_s^i(wx) = \mathrm{vol}(\bar{N}_i) = q_F^{-3i},$$

for all $x \in X$, where $\tilde{f}_s^i = M_s(f_s^i)$.

Here $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

Proof. (1) From the definition, it is clear that

$$f_s^i\left(\left(\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}, \zeta\right)\tilde{g}\right) = \zeta\mu_{\psi^{-1}}(a)\eta_{s+1/2}(a)f_s^i(\tilde{g}),$$

for $a \in F^\times, b \in F, \zeta \in \mu_2$, and $\tilde{g} \in \widetilde{\mathrm{SL}}_2(F)$. It suffices to show that for i large, there is an open compact subgroup $\widetilde{H}_i \subset \widetilde{\mathrm{SL}}_2(F)$ such that $f_s^i(\tilde{g}\tilde{h}) = f_s^i(\tilde{g})$ for all $\tilde{g} \in \widetilde{\mathrm{SL}}_2(F)$, and $\tilde{h} \in \widetilde{H}_i$.

If ψ is unramified and the residue characteristic is not 2 as we assumed, the character $\mu_{\psi^{-1}}$ is trivial on $\mathcal{O}_F^\times$, see [Sz] for example.

Let c be a positive integer such that η is trivial on $1+\mathcal{P}^c$. Let $i_2(\eta) = \{c, i_0(N \cap K_c), i_1(N \cap K_c, c)\}$. For $i \geq i_2(\eta)$, we take $\widetilde{H}_i = K_{4i} = 1 + M_2(\mathcal{P}^{4i})$. Note that K_{4i} splits, and thus can be viewed as a subgroup of $\widetilde{\mathrm{SL}}_2$. We now check that for $i \geq i_2(\eta)$, we have $f_s^i(\tilde{g}h) = f_s(\tilde{g})$ for all $\tilde{g} \in \widetilde{\mathrm{SL}}_2$ and $h \in K_{4i}$. We have the Iwahori decomposition $K_{4i} = (N \cap K_{4i})(T \cap K_{4i})(\bar{N} \cap K_{4i})$. For $h \in \bar{N} \cap K_{4i} \subset \bar{N}_i$, it is clear that $f_s^i(\tilde{g}h) = f_s^i(\tilde{g})$ by the definition of f_s^i. Now we take $h \in T \cap K_{4i}$. Write $h = t(a_0)$, with $a_0 \in 1 + \mathcal{P}^{4i}$. We have $\bar{n}(x)h = h\bar{n}(a_0^{-2}x)$. It is clear that $x \in \mathcal{P}^{3i}$ if and only if $a_0^{-2}x \in \mathcal{P}^{3i}$. On the other hand, for any $a \in F^\times, b \in F$, we have

$$c\left(\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}, t(a_0)\right) = (a, a_0) = 1,$$

since $a_0 \in 1 + \mathcal{P}_F^{4i} \subset F^{\times,2}$, by Lemma 3.3. Thus we get

$$\left(\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}, \zeta\right)\bar{n}(x)h = \left(\begin{pmatrix} aa_0 & ba_0^{-1} \\ & a^{-1}a_0^{-1}\end{pmatrix}, \zeta\right)\bar{n}(a_0^{-2}x).$$

By the definition of f_s^i, if $x \in \mathcal{P}^{3i}$, for $g = \left(\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}, \zeta\right)\bar{n}(x)$ we get

$$f_s^i(gh) = \mu_{\psi^{-1}}(a_0a)\eta_{s+1/2}(aa_0) = \mu_{\psi^{-1}}(a)\eta_{s+1/2}(a) = f_s^i(g),$$

by the assumption on i.

Finally, we consider $h \in N \cap K_{4i} \subset N \cap K_c$. By assumption on i, we get

$$A(h, i) = A(h^{-1}, i) = \bar{N}_i.$$

In particular, for $\bar{n} \in \bar{N}_i$, we have $\bar{n}h \in B \cdot \bar{N}_i$ and $\bar{n}h^{-1} \in B \cdot \bar{N}_i$. Now it is clear that $\tilde{g} \in \widetilde{B} \cdot \bar{N}_i$ if and only if $\tilde{g}h \in \widetilde{B} \cdot \bar{N}_i$. Thus $f_s^i(\tilde{g}) = 0$ if and only if $f_s^i(\tilde{g}h) = 0$. Moreover, for $\bar{n} \in \bar{N}_i$, we have

$$\bar{n}h = \begin{pmatrix} a_0 & b_0 \\ & a_0^{-1} \end{pmatrix} \bar{n}_0,$$

for $a_0 \in 1 + \mathcal{P}^c$, $b_0 \in F$ and $\bar{n}_0 \in \bar{N}_i$. Thus for $\tilde{g} = \left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \zeta \right) \bar{n}$ with $\bar{n} \in \bar{N}_i$, we get

$$\tilde{g}h = \left(\begin{pmatrix} aa_0 & ab_0 + a_0^{-1}b \\ & a_0^{-1}a^{-1} \end{pmatrix}, \zeta \right) \bar{n}_0.$$

Here we used the fact that $a_0 \in 1 + \mathcal{P}^c$ is a square, and thus

$$c\left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}, \begin{pmatrix} a_0 & b_0 \\ & a_0^{-1} \end{pmatrix} \right) = 1.$$

Since $\mu_{\psi^{-1}}(a_0) = 1$, $(a, a_0) = 1$ and $\eta_{s+1/2}(a_0) = 1$, we get

$$f_s^i(\tilde{g}h) = f_s^i(g).$$

This finishes the proof of (1).

(2) As in the proof of (1), let c be a positive integer such that η is trivial on $1 + \mathcal{P}^c$. Take $I(X, \eta) = \max\{i_1(X, c), i_0(X)\}$. We have

$$\tilde{f}_s^i(wx) = \int_N f_s^i(w^{-1}nwx)dn.$$

By the definition of f_s^i, $f_s^i(w^{-1}nwx) \neq 0$ if and only if $w^{-1}nwx \in B\bar{N}_i$, if and only if $w^{-1}nw \in A(x, i) = \bar{N}_i$ for all $i \geq I(X)$, and $x \in X$. On the other hand, if $w^{-1}nw \in A(x, i)$, we have

$$w^{-1}nwx = \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \bar{n}_0,$$

with $a \in 1 + \mathcal{P}^c$. Thus

$$f_s^i(w^{-1}nwx) = \eta_{s+1/2}(a)\mu_{\psi^{-1}}(a) = 1.$$

Now the assertion is clear. □

3.3. **The local converse theorem.**

Lemma 3.9. *Let ϕ^m be the characteristic function of $1 + \mathcal{P}^m$. Then*

(1) *for $n \in N_m$, we have $\omega_{\psi^{-1}}(n)\phi^m = \psi^{-1}(n)\phi^m$;*
(2) *for $\bar{n} \in \bar{N}_m$, we have $\omega_{\psi^{-1}}(\bar{n})\phi^m = \phi^m$.*

Proof. (1) For $n = n(b) \in N_m$, we have $b \in \mathcal{P}^{-m}$. For $x \in 1 + \mathcal{P}^m$, we have $bx^2 - b \in \mathcal{O}_F$ and thus

$$\omega_{\psi^{-1}}(n)\phi^m(x) = \psi^{-1}(bx^2)\phi^m(x) = \psi^{-1}(b)\phi^m(x).$$

For $x \notin 1 + \mathcal{P}^m$, we have $\omega_{\psi^{-1}}(n)\phi^m(x) = \psi^{-1}(bx^2)\phi^m(x) = 0$. The first assertion follows.

(2) For $\bar{n} \in \bar{N}_m$, we can write $\bar{n} = w^{-1}n(b)w$ with $b \in \mathcal{P}^{3m}$. Let $\phi' = \omega_{\psi^{-1}}(w)\phi^m$. We have

$$\begin{aligned}
\phi'(x) &= \gamma(\psi^{-1}) \int_F \phi^m(y)\psi^{-1}(2xy)dy \\
&= \gamma(\psi^{-1})\psi^{-1}(2x) \int_{\mathcal{P}^m} \psi^{-1}(2xz)dz \\
&= \gamma(\psi^{-1})\psi^{-1}(2x)\mathrm{vol}(\mathcal{P}^m)\mathrm{Char}(\mathcal{P}^{-m})(x)
\end{aligned}$$

where $\mathrm{Char}(\mathcal{P}^{-m})$ denotes the characteristic function of the set $\mathcal{P}^{-m}$. It is clear that $\omega_{\psi^{-1}}(n(b))\phi' = \phi'$. Thus we have

$$\omega_{\psi^{-1}}(\bar{n})\phi^m = \omega_{\psi^{-1}}(w^{-1}n(b))\phi' = \omega_{\psi^{-1}}(w^{-1})\phi' = \omega_{\psi^{-1}}(w^{-1})\omega_{\psi^{-1}}(w)\phi^m = \phi^m.$$

This completes the proof. □

Theorem 3.10. *Suppose that the residue characteristic of F is not 2 and ψ is an unramified character of F. Let (π, V_π) and $(\pi', V_{\pi'})$ be two ψ-generic representations of $\mathrm{SL}_2(F)$ with the same central character.*

1. *If $\gamma(s,\pi,\eta,\psi) = \gamma(s,\pi',\eta,\psi)$ for all quasi-characters η of $F^\times$, then $\pi \cong \pi'$.*
2. *There is an integer $l = l(\pi,\pi')$ such that if η is quasi-character of $F^\times$ with conductor $\mathrm{cond}(\eta) > l$, then*
$$\gamma(s,\pi,\eta,\psi) = \gamma(s,\pi',\eta,\psi).$$

Remark: D. Jiang conjectured the local converse theorem for any reductive group G, in [Jng], Conjecture 3.7. Our theorem can be viewed one example of that general conjecture.

Proof. We fix the notations $v \in V_\pi, v' \in V_{\pi'}$ and L as before.

Let η be a quasi-character of $F^\times$. We take an integer $m \geq \{6L, \mathrm{cond}(\eta)\}$. We consider the Howe vectors v_m and v'_m. We also take an integer $i \geq \{i_2(\eta), I(N_m,\eta), m\}$. In particular we have a section $f_s^i \in I(s,\eta,\psi)$. Let $W_m = W_{v_m}$ or $W_{v'_m}$. We compute the integral $\Psi(W_m,\phi^m,f_s^i)$ on the open dense subset $T\bar{N}(F) = N(F)\setminus N(F)T\bar{N}(F)$ of $N(F)\setminus \mathrm{SL}_2(F)$. For $g = nt(a)\bar{n}$, we can take the quotient measure as $dg = |a|^{-2}d\bar{n}da$. By the definition of f_s^i, we get

$$\begin{aligned}\Psi(W_m,\phi^m,f_s^i) &= \int_{T\times\bar{N}(F)} W_m(t(a)\bar{n})(\omega_{\psi^{-1}}(t(a)\bar{n})\phi^m)(1)f_s^i(t(a)\bar{n})|a|^{-2}d\bar{n}da\\ &= \int_{T\times\bar{N}_i} W_m(t(a)\bar{n})\mu_{\psi^{-1}}(a)|a|^{1/2}\omega_{\psi^{-1}}(\bar{n})\phi^m(a)\mu_{\psi^{-1}}(a)\eta_{s+1/2}(a)|a|^{-2}d\bar{n}da\\ &= \int_{T\times\bar{N}_i} W_m(t(a)\bar{n})\omega_{\psi^{-1}}(\bar{n})\phi^m(a)\chi(a)\eta_{s-1}(a)d\bar{n}da,\end{aligned}$$

where $\chi(a) = \mu_{\psi^{-1}}(a)\mu_{\psi^{-1}}(a) = (a,-1)_F$. Since $i \geq m$, we get $\bar{N}_i \subset \bar{N}_m$. By Lemma 3.1, and Lemma 3.9, we get $W_m(t(a)\bar{n}) = W_m(t(a))$ and $\omega_{\psi^{-1}}(\bar{n})\phi^m = \phi^m$. Thus we get

$$\Psi(W_m,\phi^m,f_s^i) = q^{-3i}\int_{F^\times} W_m(t(a))\phi^m(a)\chi(a)\eta_{s-1}(a)da.$$

Since $\phi^m = \mathrm{Char}(1+\mathcal{P}^m)$, and for $a \in 1+\mathcal{P}^m$, we have $W_m(t(a)) = 1$ by Lemma 3.1, we get

$$\Psi(W_m,\phi^m,f_s^i) = q^{-3i}\int_{1+\mathcal{P}^m} \chi(a)\eta(a)da.$$

Since $\chi(a) = 1$ for $a \in 1+\mathcal{P}^m$, and $m \geq \mathrm{cond}(\eta)$ by assumption, we get

$$\Psi(W_m,\phi^m,f_s^i) = q^{-3i-m}.$$

The above calculation works for both W_{v_m} and $W_{v'_m}$, thus we have

$$\Psi(W_{v_m},\phi^m,f_s^i) = \Psi(W_{v'_m},\phi^m,f_s^i) = q^{-3i-m}. \tag{3.2}$$

Next, we compute the other side local zeta integral $\Psi(W_m,\phi^m,\tilde{f}_s^i)$ on the open dense subset $N(F)\setminus N(F)TwN(F) \subset N(F)\setminus\mathrm{SL}_2(F)$, where $\tilde{f}_s^i = M_s(f_s^i)$. We have

$$\begin{aligned}\Psi(W_m,\phi^m,\tilde{f}_s^i) &= \int_{T\times N(F)} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\tilde{f}_s^i(t(a)wn)|a|^{-2}dnda\\ &= \int_{T\times N_m} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\tilde{f}_s^i(t(a)wn)|a|^{-2}dnda\\ &+ \int_{T\times(N(F)-N_m)} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\tilde{f}_s^i(t(a)wn)|a|^{-2}dnda.\end{aligned}$$

By Lemma 3.6, we get $W_{v_m}(t(a)wn) = W_{v'_m}(t(a)wn)$ for all $n \in N(F)-N_m$. Thus

$$\begin{aligned}&\Psi(W_{v_m},\phi^m,\tilde{f}_s^i) - \Psi(W_{v'_m},\phi^m,\tilde{f}_s^i)\\ =&\int_{T\times N_m}(W_{v_m}(t(a)wn) - W_{v'_m}(t(a)wn))(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)\tilde{f}_s^i(t(a)wn)|a|^{-2}dnda.\end{aligned}$$

Since $i \geq I(N_m, \eta)$, we get

$$\tilde{f}_s^i(t(a)wn) = \mu_{\psi^{-1}}(a)\eta_{3/2-s}^{-1}(a)q_F^{-3i},$$

by Lemma 3.8. On the other hand, by Lemma 3.1 and Lemma 3.9, for $n \in N_m$, we get

$$W_m(t(a)wn) = \psi(n)W_m(t(a)w), (\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1) = \psi^{-1}(n)(\omega_{\psi^{-1}}(t(a)w)\phi^m)(1).$$

Thus

$$\begin{aligned}(3.3)\qquad &\Psi(W_{v_m}, \phi^m, \tilde{f}_s^i) - \Psi(W_{v'_m}, \phi^m, \tilde{f}_s^i)\\ =&q_F^{-3i+m}\int_T (W_{v_m}(t(a)w) - W_{v'_m}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)\chi(a)\eta^{-1}(a)|a|^{-s}da.\end{aligned}$$

By (3.2), (3.3) and the local functional equation, we get

$$\begin{aligned}(3.4)\qquad &q^{-2m}(\gamma(s,\pi,\eta,\psi) - \gamma(s,\pi',\eta,\psi))\\ =&\int_{F^\times}(W_{v_m}(t(a)w) - W_{v'_m}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)\chi(a)\eta^{-1}(a)|a|^{-s}da.\end{aligned}$$

Let $k = 4L$. Since $m \geq 6L > k$, by Lemma 3.1 and Lemma 3.6, we get

$$\begin{aligned}W_{v_m}(t(a)w) - W_{v'_m}(t(a)w) =&\frac{1}{\mathrm{vol}(N_m)}\int_{N_m}(W_{v_k}(t(a)wn) - W_{v'_k}(t(a)wn))\psi^{-1}(n)dn\\ =&\frac{1}{\mathrm{vol}(N_m)}\int_{N_k}(W_{v_k}(t(a)wn) - W_{v'_k}(t(a)wn))\psi^{-1}(n)dn\\ =&\frac{\mathrm{vol}(N_k)}{\mathrm{vol}(N_m)}(W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\\ =&q^{k-m}(W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)).\end{aligned}$$

Now we can rewrite (3.4) as

$$\begin{aligned}(3.5)\qquad &q^{-m-k}(\gamma(s,\pi,\eta,\psi) - \gamma(s,\pi',\eta,\psi))\\ =&\int_{F^\times}(W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)\chi(a)\eta^{-1}(a)|a|^{-s}da.\end{aligned}$$

By Lemma 3.2, if $a \notin \mathcal{P}^{-6L}$, i.e., $a^2 \notin \mathcal{P}^{-3k}$, we get $W_{v_k}(t(a)w) = 0 = W_{v'_k}(t(a)w)$. Thus the integral on the right side in the above formula (3.5) can be taken over $\mathcal{P}^{-6L}$. For $a \in \mathcal{P}^{-6L}$ and $m \geq 6L$ (as we assumed), by the calculation given in the proof of Lemma 3.9, we have

$$\begin{aligned}(\omega_{\psi^{-1}}(w)\phi^m)(a) &= \gamma(\psi^{-1})\psi^{-1}(2a)\mathrm{vol}(\mathcal{P}^m)\mathrm{Char}(\mathcal{P}^{-m})(a)\\ &= \gamma(\psi^{-1})\psi^{-1}(2a)q^{-m}.\end{aligned}$$

Plugging this into (3.5), we get

$$\begin{aligned}(3.6)\qquad &q^{-k}\gamma(\psi^{-1})^{-1}(\gamma(s,\pi,\eta,\psi) - \gamma(s,\pi',\eta,\psi))\\ =&\int_{F^\times}(W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)|a|^{-s}da.\end{aligned}$$

Now we can prove our theorem. We consider (1) first. Suppose that $\gamma(s,\pi,\eta,\psi) = \gamma(s,\pi',\eta,\psi)$ for all quasi-characters η of $F^\times$, we get

$$\int_{F^\times}(W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)|a|^{-s}da = 0$$

for all quasi-characters η. By Mellin inversion, we get

$$(W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a) \equiv 0,$$

or

$$W_{v_k}(t(a)w) \equiv W_{v'_k}(t(a)w).$$

By Lemma 3.1, Lemma 3.6, Corollary 3.5 and the Iwasawa decomposition $\mathrm{SL}_2 = B \cup BwB$, we get

$$W_{v_k}(g) = W_{v'_k}(g),$$

for all $g \in \mathrm{SL}_2(F)$. By the uniqueness of Whittaker model, we get $\pi \cong \pi'$. This proves (1).

Next, we consider (2). Let $l = l(\pi, \pi')$ be an integer such that $l \geq 6L$ and

$$W_{v_k}(t(a_0a)w) = W_{v_k}(t(a)w), \text{ and } W_{v'_k}(t(a_0a)w) = W_{v'_k}(t(a)w),$$

for all $a_0 \in 1 + \mathcal{P}^l$ and all $a \in \mathcal{P}^{-6L}$. Such an l exists because the functions $a \mapsto W_{v_k}(t(a)w)$, $a \mapsto W_{v'_k}(t(a)w)$ on $\mathcal{P}^{-6L} \subset F^\times$ are continuous. Note that $k = 4L$ and L only depends on the choice of v and v'. On the other hand, for $a \in \mathcal{P}^{-6L}$, it is easy to see that

$$\psi^{-1}(2a_0a) = \psi^{-1}(2a), \text{ for all } a_0 \in 1 + \mathcal{P}^l,$$

since $l \geq 6L$. It is also clear that $\chi(a_0a) = \chi(a)$ for all $a_0 \in 1 + \mathcal{P}^l$. In fact, the character χ is unramified. As we noticed before, the integrand of the right side integral of (3.6) has support in $\mathcal{P}^{-6L}$. Let η be a quasi-character of $F^\times$ with $\mathrm{cond}(\eta) > l$, now it is clear that integral of the right side of (3.6) vanishes, thus we get

$$\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi).$$

This finishes the proof. □

4. A strong multiplicity one theorem

4.1. **Global genericness.** In this subsection, we discuss the relation between globally generic and locally generic. Let F be a number field and $\mathbb{A}$ be its adele. Let φ be a cusp form on $\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})$. Since the group $N(F) \setminus N(\mathbb{A})$ is compact and abelian, we have the Fourier expansion

$$\varphi(g) = \sum_{\psi \in \widehat{N(F)\backslash N(\mathbb{A})}} W_\varphi^\psi(g),$$

where

$$W_\varphi^\psi(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\psi^{-1}(n)dg.$$

Since φ is a cusp form, we get $\varphi_0 \equiv 0$, thus we get

$$\varphi(g) = \sum_{\substack{\psi \in \widehat{N(F)\backslash N(\mathbb{A})} \\ \psi \neq 0}} W_\varphi^\psi(g).$$

Fix a nontrivial additive character ψ of $N(F) \setminus N(\mathbb{A})$, then

$$\left\{\widehat{N(F) \setminus N(\mathbb{A})}\right\} \setminus \{0\} = \left\{\psi_\kappa, \kappa \in F^\times\right\},$$

where $\psi_\kappa(a) = \psi(\kappa a), a \in \mathbb{A}$. If $\kappa \in F^{\times,2}$, say $\kappa = a^2$, we have

$$W_\varphi^{\psi_\kappa}(g) = W_\varphi^\psi(t(a)g).$$

Thus we get

$$\varphi(g) = \sum_{\kappa \in F^\times/F^{\times,2}} \sum_{a \in F^\times} W_\varphi^{\psi_\kappa}(t(a)g).$$

Corollary 4.1. *If φ is a nonzero cusp form, there exists $\kappa \in F^\times$ such that*

$$W_\varphi^{\psi_\kappa} \neq 0.$$

Let (π, V_π) be a cuspidal automorphic representation of $\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})$. We call π is ψ_κ-generic, if there exists $\varphi \in V_\pi$ such that

$$W_\varphi^{\psi_\kappa} \not\equiv 0.$$

Corollary 4.2. *Let π be a cuspidal automorphic representation of $\mathrm{SL}_2(F) \setminus \mathrm{SL}_2(\mathbb{A})$ and ψ be a nontrivial additive character of $F \setminus \mathbb{A}$. Then there exists $\kappa \in F^\times$, such that π is ψ_κ-generic.*

Theorem 4.3. *Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ and $\psi = \otimes\psi_v$ be a nontrivial additive character of $F \setminus \mathbb{A}$. Then π is ψ-generic if and only if each π_v is ψ_v-generic.*

Proof. A similar result is proved for $\mathrm{U}(1,1)$ by Gelbart, Rogawski and Soudry, in Proposition 2.5, [GeRS].

One direction is trivial. Now we assume each π_v is ψ_v-generic.

We assume π is ψ_κ-generic for some $\kappa \in F^\times$, i.e., there exists $\varphi \in V_\pi$ such that

$$W_\varphi^{\psi_\kappa}(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\psi_\kappa^{-1}(n)dn \neq 0.$$

Then it is clear that π_v is also $\psi_{\kappa,v} - generic$, where $\psi_{\kappa,v}(a) = \psi_v(\kappa a)$. By the theorem in the first section, we get $\pi_v \cong \pi_v^\kappa$.

For $\varphi \in V_\pi$, consider the function $\varphi^\kappa(g) = \varphi(g^\kappa)$, where $g^\kappa = \mathrm{diag}(\kappa,1)g\mathrm{diag}(\kappa^{-1},1)$. Then

$$\int_{N(F)\backslash N(\mathbb{A})} \varphi^\kappa(ng)dn = \int_{N(F)\backslash N(\mathbb{A})} \varphi((ng)^\kappa)dn$$

$$= \int_{N(F)\backslash N(\mathbb{A})} \varphi(n^\kappa g^\kappa)dn = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng^\kappa)dn = 0$$

hence φ^κ is also a cusp form. Let V_π^κ be the space which consists functions of the form φ^κ for all $\varphi \in V_\pi$. Let π^κ denote the cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ on V_π^κ.

Claim: $(\pi^\kappa)_v = \pi_v^\kappa$.

proof of the claim: Let $\Lambda : V_\pi \to \mathbb{C}$ be a nonzero ψ_κ-Whittaker functional of π, and let Λ_v be nonzero $(\psi_\kappa)_v$-Whittaker functional on V_{π_v} satisfying that, if $\varphi = \otimes_v \varphi_v$ is a pure tensor, then

$$\Lambda(\pi(g)\varphi) = \prod_v \Lambda_v(\pi_v(g_v)\varphi_v)$$

Note that Λ is in fact given by

$$\Lambda(\varphi) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(n)\psi_\kappa^{-1}(n)dn$$

Then the ψ_{κ^2}-Whittaker functional of π^κ is given by

$$\int_{N(F)\backslash N(\mathbb{A})} \varphi^\kappa(n)\psi_{\kappa^2}^{-1}(n)dn$$

This means that if $W_\varphi(g)$ is a ψ_κ-Whittaker function of π, then $W_{\varphi^\kappa}(g) = W_\varphi(g^\kappa)$ is a ψ_{κ^2}-Whittaker function of π^κ.

Hence with $\varphi = \otimes_v \varphi_v$ a pure tensor, we have $W_\varphi(g) = \prod_v W_{\varphi_v}(g_v)$, and $\{W_{\varphi_v}(g_v)\}$ is the Whittaker model of π_v. While $W_{\varphi^\kappa}(g) = W_\varphi(g^\kappa) = \prod_v W_{\varphi_v}(g_v^\kappa)$, and $\{W_{\varphi_v}(g_v^\kappa)\}$ is the Whittaker model of $(\pi^\kappa)_v$.

Now $W_v(g_v) \to W_v(g_v^\kappa)$ gives an isomorphism between π_v^k and $(\pi^\kappa)_v$, which proves the claim.

Now let's continue the proof of the theorem, by the claim we then have $\pi_v \cong (\pi^\kappa)_v$, or $\pi \cong \pi^\kappa$. By multiplicity one theorem for SL_2, see [Ra], we get $\pi = \pi^\kappa$. It is clear that π^κ is ψ_{κ^2}-generic, and hence ψ-generic. The theorem follows. □

4.2. **Eisenstein series on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.** Now let F be a number field, and $\mathbb{A}$ be its adele ring. Let $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ be the double cover of $\mathrm{SL}_2(\mathbb{A})$. It is well-known that the projection $\widetilde{\mathrm{SL}}_2(\mathbb{A}) \to \mathrm{SL}_2(\mathbb{A})$ factors through $\mathrm{SL}_2(F)$. Let μ_ψ be the genuine character of $T(F) \setminus \widetilde{T}(\mathbb{A})$ whose local components are μ_{ψ_v} given in §2.

Let η be a quasi-character of $F^\times \setminus \mathbb{A}^\times$, and $s \in \mathbb{C}$, we consider the induced representation

$$I(s,\chi,\psi) = \mathrm{Ind}_{\widetilde{B}(\mathbb{A})}^{\widetilde{\mathrm{SL}}_2(\mathbb{A})}(\mu_\psi \eta_{s-1/2}).$$

For $f_s \in I(s,\eta,\psi)$, we consider the Eisenstein series $E(g,f_s)$ on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$:

$$E(g,f_s) = \sum_{B(F)\backslash \mathrm{SL}_2(F)} f_s(\gamma g), g \in \mathrm{SL}_2(\mathbb{A}).$$

The above sum is absolutely convergent when $\mathrm{Re}(s) >> 0$, and can be meromorphically continued to the whole s-plane.

There is an intertwining operator $M_s : I(s,\eta,\psi) \to I(1-s,\eta^{-1},\psi)$:

$$M_s(f_s)(g) = \int_{N(F)\backslash N(\mathbb{A})} f_s(wng)dn.$$

The above integral is absolutely convergent for $\mathrm{Re}(s) >> 0$ and defines a meromorphic function of $s \in \mathbb{C}$.

Proposition 4.4. (1) *If $\eta^2 \neq 1$, then the Eisenstein series $E(g,f_s)$ is holomorphic for all s. If $\eta^2 = 1$, the only possible poles of $E(g,f_s)$ are at $s=0$ and $s=1$. Moreover, the order of the poles are at most 1.*

(2) *We have the functional equation*

$$E(g,f_s) = E(g,M_s(f_s)), \text{ and } M_s(\eta)\circ M_{1-s}(\eta^{-1}) = 1.$$

See Proposition 6.1 of [GaQT] for example.

4.3. **The global zeta integral.** Let ψ be a nontrivial additive character of $F\setminus\mathbb{A}$. Then there is a global Weil representation representation ω_ψ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ on $\mathcal{S}(\mathbb{A})$. For $\phi\in\mathcal{S}(\mathbb{A})$, we consider the theta series

$$\theta_\psi(\phi)(g) = \sum_{x\in F}(\omega_\psi(g)\phi)(x).$$

It is well-known that θ_ψ defines an automorphic form on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.

Let (π, V_π) be a ψ-generic cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$. For $\varphi\in V_\pi, \phi\in\mathcal{S}(\mathbb{A})$ and $f_s\in I(s,\eta,\psi^{-1})$, we consider the integral

$$Z(\varphi,\theta_{\psi^{-1}}(\phi),E(\cdot,f_s)) = \int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})} \varphi(g)\theta_{\psi^{-1}}(\phi)(g)E(g,f_s)dg. \tag{4.1}$$

Proposition 4.5 (Theorem 4.C of [GePS2])**.** *For $\mathrm{Re}(s) >> 0$, the integral $Z(\varphi,\theta_{\psi^{-1}}(\phi),E(\cdot,f_s))$ is absolutely convergent, and*

$$Z(\varphi,\theta_{\psi^{-1}}(\phi),E(\cdot,f_s)) = \int_{N(\mathbb{A})\backslash\mathrm{SL}_2(\mathbb{A})} W_\varphi^\psi(g)(\omega_{\psi^{-1}}(g))\phi(1)f_s(g)dg,$$

where $W_\varphi^\psi(g) = \int_{N(F)\backslash N(\mathbb{A})}\varphi(ng)\psi^{-1}(n)dn$ is the ψ-th Whittaker coefficient of φ.

Corollary 4.6. *We take $\phi=\otimes_v\phi_v, f_s=\otimes f_{s,v}$ to be pure tensors. Let S be a finite set of places such that for all $v\notin S$, then v is finite and $\pi_v,\psi_v,f_{s,v}$ are unramified, then for $\mathrm{Re}(s)>>0$, we have*

$$Z(\varphi,\theta_{\psi^{-1}}(\phi),E(\cdot,f_s)) = \prod_{v\in S}\Psi(W_v,\phi_v,f_{s,v})\frac{L^S(s,\pi,St\otimes(\chi\eta))}{L^S(2s,\eta^2)},$$

where χ is the character of $F^\times\setminus\mathbb{A}^\times$ defined by

$$\chi((a_v)) = \prod_v (a_v,-1)_{F_v}, (a_v)_v\in\mathbb{A}^\times.$$

Moreover, we have the following functional equation

$$Z(\varphi,\theta_{\psi^{-1}}(\phi),E(\cdot,f_s)) = Z(\varphi,\theta_{\psi^{-1}}(\phi),E(\cdot,M_s(f_s)))$$

This follows from Proposition 4.2, the unramified calculation and the functional equation of Eisenstein series in Proposition 4.4 directly.

Corollary 4.7. (1) *The partial L-function $L^S(s,\pi,St\otimes\chi\eta)$ can be extended to a meromorphic of s.*

(2) *If $\eta^2\neq 1$, then $L^S(s,\pi,St\otimes\chi\eta)$ is holomorphic for $\mathrm{Re}(s)>1/2$.*

(3) *If $\eta^2=1$, then on the region $\mathrm{Re}(s)>1/2$, the only possible pole of $L^S(s,\pi,St\otimes\chi\eta)$ is at $s=1$. Moreover, the order of the pole of $L^S(s,\pi,St\otimes(\chi\eta))$ at $s=1$ is at most 1.*

Proof. By Proposition 4.4 and Corollary 4.6, it suffices to show that, for each place v, and for any fixed point $s\in\mathbb{C}$ we can choose datum $(W_v,\phi_v,f_{s,v})$ such that $\Psi(W_v,\phi_v,f_{s,v})\neq 0$. If v is non-archimedean, this is proved in the proof of Theorem 3.10, see Eq.(3.2). We will prove the general case later, see Lemma 4.9. □

4.4. **A strong multiplicity one theorem.** With the above preparation, we are now ready to prove the main global result of this paper.

Theorem 4.8. *Let $\pi = \otimes\pi_v$ and $\pi' = \otimes\pi'_v$ be two irreducible cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ with the same central character. Suppose that π and π' are both ψ-generic. Let S be a finite set of* ***finite*** *places such that no place in S is above 2. If $\pi_v \cong \pi'_v$ for all $v \notin S$, then $\pi = \pi'$.*

Proof. For $\phi = \otimes_v\phi_v \in V_\pi$, consider $Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s))$. By Proposition 4.5, we have

$$Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s)) = \prod_v \Psi(W_v, \phi_v, f_{s,v})$$

$$= \prod_v \int_{N(F_v)\backslash \mathrm{SL}_2(F_v)} W_v(h)(\omega_{\psi^{-1}}(h)\phi_v)(1)f_{s,v}(h)dh$$

By functional equation in Corollary 4.6, we have

$$\prod_v \int_{N(F_v)\backslash \mathrm{SL}_2(F_v)} W_v(h)(\omega_{\psi^{-1}}(h)\phi_v)(1)f_{s,v}(h)dh$$

$$= \prod_v \int_{N(F_v)\backslash \mathrm{SL}_2(F_v)} W_v(h)(\omega_{\psi^{-1}}(h)\phi_v)(1)M_s(f_{s,v})(h)dh$$

On archimedean places, by Lemma 4.9 below, we can choose datum so that the archimedean local integrals are nonzero. While on the finite places, by local functional equation discussed in section 1.4, there are local gamma factors $\gamma(s, \pi_v, \eta_v, \psi_v)$ such that

$$\int_{N(F_v)\backslash \mathrm{SL}_2(F_v)} W_v(h)(\omega_{\psi^{-1}}(h)\phi_v)(1)M_s(f_{s,v})(h)dh =$$

$$\gamma(s, \pi_v, \eta_v, \psi_v) \int_{N(F_v)\backslash \mathrm{SL}_2(F_v)} W_v(h)(\omega_{\psi^{-1}}(h)\phi_v)(1)f_{s,v}(h)dh$$

Hence we have

$$1 = \prod_{v<\infty} \gamma(s, \pi_v, \eta_v, \psi_v) \prod_{v|\infty} \frac{\Psi(W_v, \phi_v, M_s(f_{s,v}))}{\Psi(W_v, \phi_v, f_{s,v})}$$

Similarly, we have for π' the identity

$$1 = \prod_{v<\infty} \gamma(s, \pi'_v, \eta_v, \psi_v) \prod_{v|\infty} \frac{\Psi(W'_v, \phi_v, M_s(f_{s,v}))}{\Psi(W'_v, \phi_v, f_{s,v})}$$

By assumptions that $\pi_v \cong \pi'_v$ for all $v \notin S$ with S doesn't have archimedean places, we then have

$$\prod_{v\in S} \gamma(s, \pi_v, \eta_v, \psi_v) = \prod_{v\in S} \gamma(s, \pi'_v, \eta_v, \psi_v)$$

Fix $v_0 \in S$, by Lemma 12.5 in [JL], given an arbitrary character η_{v_0}, we can find a character η of $\mathbb{A}^\times$ which restricts to v_0 is η_{v_0} and has arbitrary high conductor at other places of S. By Theorem 3.10 (2), we conclude that

$$\gamma(s, \pi_{v_0}, \eta_{v_0}, \psi_{v_0}) = \gamma(s, \pi'_{v_0}, \eta_{v_0}, \psi_{v_0})$$

for all characters η_{v_0}. Thus by Theorem 3.10 (1), we conclude that $\pi_{v_0} \cong \pi'_{v_0}$. This applies also to other places of S. Thus we proved that $\pi_v \cong \pi'_v$ for all places v. Now the theorem follows from the multiplicity one theorem for SL_2, [Ra]. □

Remark: We expect that the restriction on the finite set S in Theorem 4.8 can be removed.

Finally, we prove a nonvanishing result on archimedean local zeta integrals which is used in the above proof. We formulate and prove the result both for p-adic and archimedean cases simultaneously.

Lemma 4.9. *Let F be a local field, ψ be a nontrivial additive character of F, η be a quasi-character of $F^\times$ and π be a ψ-generic representation of $\mathrm{SL}_2(F)$. Then there exists $W \in \mathcal{W}(\pi,\psi), \phi \in \mathcal{S}(F)$ and $f_s \in \mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{SL}}_2(F)}$ such that*

$$\Psi(W,\phi,f_s) = \int_{N(F)\backslash \mathrm{SL}_2(F)} W(h)(\omega_{\psi^{-1}}\phi)(h)f_s(h) \neq 0.$$

Proof. We note that the Bruhat cell $\Omega = N(F)TwN(F)$ is open dense in $\mathrm{SL}_2(F)$. Thus the above integral is reduced to

$$\Psi(W,\phi,f_s) = \int_{TN(F)} W(\omega_2 t(a)n(u))(\omega_{\psi^{-1}}(wt(a)n(u))\phi)(1)f_s(wt(a)n(u))\Delta(a)dadu$$

where $\Delta(a)$ is certain Jacobian.

Use the formulas for the Weil representation $\omega_{\psi^{-1}}$, we find

$$(\omega_{\psi^{-1}}(\omega_2 t(a)n(u))\phi)(x) = |a|^{1/2}\frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})}\int_F \psi(ua^2y^2)\phi(ay)\psi(2xy)dy = |a|^{1/2}\frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})}\hat{\Phi}_{a,u}(x)$$

where $\Phi_{a,u}(x) = \psi(ua^2x^2)\phi(ax)$ which is again an Schwartz function on F and depends continuously on a,u.

Note that $\Omega = N(F)TwN(F)$ is open in $\widetilde{\mathrm{SL}}_2(F)$. Now define $f_s \in I(s,\eta,\psi^{-1})$ on the set $\{(g,1) : g \in \mathrm{SL}_2(F)\}$ by

$$f_s(g) = \begin{cases} \delta(b)^{1/2}(\eta_{s-1/2}\mu_{\psi^{-1}})(b)f_2(u) & \text{if } \ g = bwn(u) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

where $b \in B(F) = TN(F), u \in F$, and f_2 is compactly supported to be determined later. Then we extend the definition of f_2 to the set $\{(g,-1) : g \in \mathrm{SL}_2(F)\}$ to make it genuine, i.e., $f_s(g,-1) = -1f_s(g,1)$.

Then the integral Ψ can be reduced further to

$$\Psi(W,\phi,f_s) = \int_{TN(F)} W(wau)|a|^{1/2}\frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})}\hat{\Phi}_{a,u}(1)\delta(a)^{1/2}(\eta_{s-1/2}\mu_{\psi^{-1}})(a)f_2(u)\Delta(a)dadu \qquad \cdots(*)$$

Case 1: F is p-adic.

Consider Howe vector W_{v_m}. By Corollary 3.4, taking m large enough, W_{v_m} can have arbitrarily small compact open support around 1 when restricted to T. Then $W_{w.v_m}(t(a^{-1})w)$ has small compact open support around $a = 1$.

First choose ϕ so that $\hat{\Phi}_{a,u}(1) \neq 0$ when $a = 1, u = 0$. Then choose m so that $W_{\omega_2.v_m}(wt(a)) = W_{w.v_m}(t(a^{-1})w)$ has small compact support around 1 and all other datum involving a in the integral (*) are nonzero constants. For this $W_{w.v_m}$, consider $W_{w.v_m}(\omega_2 t(a)u)$ with $u \in N$. When u is close to 1 enough, we have $W_{w.v_m}(wt(a)u) = W_{w.v_m}(wt(a))$ for all a in that small compact support around 1. Then take f_2 with support u close to 1 satisfying the above. With these choices of $W_{w.v_m}(g), f_2, \phi$, the integral $(*)$ is nonzero.

Case 2: F is archimedean.

We will concentrate on the case $F = \mathbb{R}$. The case $F = \mathbb{C}$ is similar as we have the same formulas for Weil representations by Proposition 1.3 in [JL]. We begin with the formulas

$$\Psi(W,\phi,f_s) = \int_{TN(F)} W(\omega_2 au)|a|^{1/2}\frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})}\hat{\Phi}_{a,u}(1)\delta(a)^{1/2}(\eta_{s-1/2}\mu_{\psi^{-1}})(a)f_2(u)\Delta(a)dadu \qquad \cdots(**)$$

where $\Phi_{a,u}(x) = \psi(ua^2x^2)\phi(ax)$ is again a Schwartz function as ϕ is, and it depends on a,u continuously. Since the Fourier transform is again an isometry of Schwartz function space, we can choose

ϕ so that the Fourier transform $\hat{\Phi}_{a,u}(1) > 0$ when $a = 1, u = 0$ and depends on a, u continuously.

Now let (π, V) be an irreducible generic smooth representation of $\mathrm{SL}_2(\mathbb{R})$ of moderate growth. Realize π as a quotient of principal series smooth $I(\chi, s)$, i.e.,

$$0 \to V' \to I(\chi, s) \to V \to 0$$

Let $\lambda : V \to \mathbb{C}$ be the unique nonzero continuous Whittaker functional on V, then the composition

$$\Lambda : I(\chi, s) \to V \xrightarrow{\lambda} \mathbb{C}$$

gives the unique nonzero continuous Whittaker functional on $I(\chi, s)$ up to a scalar. It follows that the two spaces $\{\lambda(\pi(g)v) : g \in \mathrm{SL}_2(F), v \in V\}$ and $\{\Lambda(R(g).f) : g \in \mathrm{SL}_2(F), f \in I(\chi, s)\}$ are the same, although the first is the Whittaker model of π, while the later may not be a Whittaker model of $I(\chi, s)$.

The Whittaker functional on $I(\chi, s)$ is given by the following

$$\Lambda(f) = \int_{N(F)} f(\omega_2 u)\psi^{-1}(u)du$$

when s is in some right half plane, and its continuation gives Whittaker functional for all $I(\chi, s)$. Also when f has support inside $\Omega = N(F)T\omega_2 N(F)$, the above integral always converges for any s, and gives the Whittaker functional.

Now for such f, one compute that for $a = t(a) \in T$

$$\Lambda(I(a).f) = \int_{N(F)} f(\omega_2 ua)\psi^{-1}(u)du = \chi'(a) \int_F f(\omega_2 u)\psi^{-1}(a^2 u)du$$

$$= \chi'(a) \int_F f_1(u)\psi^{-1}(a^2 u)du = \chi'(a)\hat{f}_1(a^2)$$

where f_1 is the restriction of f to $\omega_2 N$ which can be chosen to be a Schwartz function, $\hat{f}_1$ is its Fourier transform, and χ' is certain character. Again, as Fourier transform gives an isometry of Schwartz functions, we can always choose f so that its Whittaker function $W_f(a)$ has arbitrary small compact support around 1. By a right translation by ω_2, we shows that one can always choose f so that $W_{\omega_2.f}(a\omega_2)$ has small compact support around 1.

In order to prove the proposition, note that we have chosen Φ. Let

$$R(a, u) = |a|^{1/2} \frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})} \hat{\Phi}_{a,u}(1)\delta(a)^{1/2}(\eta_{s-1/2}\mu_{\psi^{-1}})(a)\Delta(a)$$

Then $R(a, u)$ is a continuous function of a, u and $R(1, 0) \neq 0$. This means that there exist neighborhoods U_1 of $a = 1$ and U_2 of $u = 0$, such that $R(a, u) > R(1, 0)/2 > 0$ for all $a \in U_1, u \in U_2$.

Now choose f so that $W_{\omega_2.f}(a\omega_2)$ has small compact support in a neighborhood V_1 of 1 with $V_1 \subset U_1$, and $W_{\omega_2.f}(\omega_2) > 0$. For this Whittaker function, since $W_{\omega_2.f}(a\omega_2)u$ is continuous on u, we can choose f_2 so that it is positively supported in a neighborhood V_2 of 0 such that
(1). $V_2 \subset U_2$;
(2). $W_{\omega_2.f}(a\omega_2 u) > W_{\omega_2.f}(\omega_2)/2 > 0$ for all $u \in V_2$.

Then $(**)$ becomes

$$\int W_{\omega_2.f}(a\omega_2 u)R(a, u)f_2(u)dadu > \frac{W_{\omega_2.f}(\omega_2)}{2} \frac{R(1, 0)}{2} \int_{V_1} \int_{V_2} f_2(u)dadu > 0$$

which proves the non vanishing.

□

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