

## MIXED BRIGHTNESS-INTEGRALS OF CONVEX BODIES

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ABSTRACT. The mixed width-integrals of convex bodies are defined by E. Lutwak. In this paper, the mixed brightness-integrals of convex bodies are defined. An inequality is established for the mixed brightness-integrals analogous to the Fenchel-Aleksandrov inequality for the mixed volumes. An isoperimetric inequality (involving the mixed brightness-integrals) is presented which generalizes an inequality recently obtained by Chakerian and Heil. Strengthened version of this general inequality is obtained by introducing indexed mixed brightness-integrals.

### 1. Introduction and main results

The setting for this paper will be the  $n$ -dimensional Euclidean space,  $\mathbb{R}^n$ . Let  $\mathcal{K}^n$  denote the set of convex bodies (compact, convex subset with non-empty interiors) and  $\mathcal{K}_o^n$  denote the subspace of  $\mathcal{K}^n$  consisting of all convex bodies that contain the origin in their interiors. Let  $\mathcal{S}_o^n$  denote the set of star bodies about the origin (star-shaped, continuous radial function) in  $\mathbb{R}^n$ . The unit  $n$ -ball and its surface will be denoted by  $U$  and  $S^{n-1}$ , respectively. The volume of the  $n$ -ball,  $U$ , will be denoted by  $\omega_n$ .

Lutwak introduced the notion of the mixed width-integrals of convex bodies in [8, p. 250]: For  $k \in \mathcal{K}^n$  and  $u \in S^{n-1}$ ,  $b(K, u)$  is half the width of  $K$  in the direction  $u$ . Mixed width-integrals  $A(K_1, K_2, \dots, K_n)$  of  $K_1, K_2, \dots, K_n \in \mathcal{K}^n$  was defined by

$$(1.1) \quad A(K_1, K_2, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} b(K_1, u)b(K_2, u) \cdots b(K_n, u)dS(u).$$

More in general, for a real number  $p \neq 0$ , the mixed width-integrals of order  $p$ ,  $A_p(K_1, K_2, \dots, K_n)$ , of  $K_1, K_2, \dots, K_n \in \mathcal{K}^n$  was also defined by Lutwak [8,

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p. 251],

$$(1.2) \quad A_p(K_1, K_2, \dots, K_n) = \omega_n \left[ \frac{1}{n\omega_n} \int_{S^{n-1}} b(K_1, u)^p \cdots b(K_n, u)^p dS(u) \right]^{\frac{1}{p}}.$$

And the properties of the mixed width-integrals of convex bodies were listed, such as positive, continuous, translation invariant, monotone under set inclusion, and homogeneous of degree one in each variable.

After that, the mixed chord-integrals of star bodies are defined by Fenghong Lu in [5]. For  $L \in \mathcal{S}_o^n$  and  $u \in S^{n-1}$ , let

$$(1.3) \quad d(L, u) = \frac{1}{2}\rho(L, u) + \frac{1}{2}\rho(L, -u)$$

denote half the chord of  $L$  in the direction  $u$ . The mixed chord-integral,  $B(L_1, \dots, L_n)$ , of  $L_1, \dots, L_n \in \mathcal{S}_o^n$  is defined by

$$(1.4) \quad B(L_1, \dots, L_n) = \frac{1}{n} \int_{S^{n-1}} d(L_1, u) \cdots d(L_n, u) dS(u).$$

Lutwak established some inequalities for mixed width-integrals in [6, 8]:

**Theorem A.** *If  $K_1, \dots, K_n \in \mathcal{K}^n$  and  $1 < m \leq n$ , then*

$$(1.5) \quad A^m(K_1, \dots, K_n) \leq \prod_{i=0}^{m-1} A(K_1, \dots, K_{n-m}, K_{n-i}, \dots, K_{n-i}),$$

*with equality if and only if  $K_{n-m+1}, K_{n-m+2}, \dots, K_n$  are all of similar width.*

**Theorem B.** *If  $K_1, \dots, K_n \in \mathcal{K}^n$ , then*

$$(1.6) \quad V(K_1) \cdots V(K_n) \leq A^n(K_1 \cdots K_n),$$

*with equality if and only if  $K_1, K_2, \dots, K_n$  are  $n$ -ball.*

Strengthened versions of inequality (1.6) are obtained by introducing indexed mixed width-integrals.

**Theorem C.** *If  $K_1, \dots, K_n \in \mathcal{K}^n$ ,  $p \neq 0$  and  $-1 \leq p \leq \infty$ , then*

$$(1.7) \quad V(K_1) \cdots V(K_n) \leq A_p^n(K_1 \cdots K_n),$$

*with equality if and only if  $K_1, K_2, \dots, K_n$  are  $n$ -ball.*

In this paper, the mixed brightness-integrals of convex bodies are defined. Half the brightness is defined by

$$(1.8) \quad \delta(K, u) = \frac{1}{2}h(\Pi K, u).$$

The mixed brightness-integral  $D(K_1, \dots, K_n)$  of  $K_1, \dots, K_n \in \mathcal{K}^n$  is defined by

$$(1.9) \quad D(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \delta(K_1, u) \cdots \delta(K_n, u) dS(u).$$

Further, some inequalities for the mixed brightness-integrals analogous to the Fenchel-Aleksandrov inequality for the mixed volumes are established. And we obtain strengthened versions of the general inequality established by Chakerian. We mainly obtain the following results:

**Theorem 1.** *If  $K_1, \dots, K_n \in \mathcal{K}^n$  and  $1 < m \leq n$ , then*

$$(1.10) \quad D^m(K_1, \dots, K_n) \leq \prod_{i=0}^{m-1} D(K_1, \dots, K_{n-m}, K_{n-i}, \dots, K_{n-i}),$$

*with equality if and only if  $K_{n-m+1}, K_{n-m+2}, \dots, K_n$  are all of similar brightness.*

A strengthened version of inequality (1.10) is obtained:

**Theorem 2.** *If  $K_1, \dots, K_n \in \mathcal{K}^n$  and  $1 < m \leq n$ , then for  $p > 0$*

$$(1.11) \quad D_p^m(K_1, \dots, K_n) \leq \prod_{i=0}^{m-1} D_p(K_1, \dots, K_{n-m}, K_{n-i}, \dots, K_{n-i}),$$

*with equality if and only if  $K_{n-m+1}, K_{n-m+2}, \dots, K_n$  are all of similar brightness. For  $p < 0$ , inequality (1.11) is reversed.*

Theorem 1 and Theorem 2 are just analogs of the Fenchel-Aleksandrov inequality for the mixed volumes.

For  $K \in \mathcal{K}^n$ , and  $u \in S^{n-1}$ , let  $K^u$  denote the image of the orthogonal projection of  $K$  onto  $\xi_u$ , the  $(n-1)$ -dimensional subspace of  $\mathbb{R}^n$  that is orthogonal to  $u$ .  $v(K_1^u, \dots, K_n^u)$  denote the mixed volume of  $K_1^u, \dots, K_n^u$  and  $v(K^u)$  denote the volume of  $K^u$ .

**Theorem 3.** *If  $K_1, \dots, K_n \in \mathcal{K}^n$ , and  $u \in S^{n-1}$ , then*

$$(1.12) \quad D^n(K_1, \dots, K_n) \leq v(K_1^u) \cdots v(K_n^u),$$

*with equality if and only if  $K_1, K_2, \dots, K_n$  are all of similar brightness.*

A strengthened versions of inequality (1.12) is obtained:

**Theorem 4.** *If  $K_1, \dots, K_n \in \mathcal{K}^n$ ,  $u \in S^{n-1}$  and  $-\infty \leq p \leq 1$ , then*

$$(1.13) \quad D_p^n(K_1, \dots, K_n) \leq v(K_1^u) \cdots v(K_n^u),$$

*with equality if and only if  $K_1, K_2, \dots, K_n$  are all of similar brightness and have constant joint brightness.*

## 2. Preliminaries

### 2.1. Support function and radial function

Let  $h(K, u)$  denote the support function (restricted to the unit sphere) of  $K \in \mathcal{K}^n$ ; i.e., for  $u \in S^{n-1}$ ,

$$(2.1) \quad h(K, u) = \max\{u \cdot x : x \in K\},$$

where  $u \cdot x$  denote the usual inner product of  $x$  and  $u$  in  $\mathbb{R}^n$ .

For  $K_1, K_2, \dots, K_n \in \mathcal{K}^n$  and  $\lambda_1, \dots, \lambda_n \geq 0$ , the Minkowski linear combination  $\lambda_1 K_1 + \dots + \lambda_n K_n \in \mathcal{K}^n$  is defined by

$$(2.2) \quad \lambda_1 K_1 + \dots + \lambda_n K_n = \{\lambda_1 x_1 + \dots + \lambda_n x_n \in \mathcal{K}^n : x_i \in K_i, 1 \leq i \leq n\}.$$

It is trivial to verify that

$$(2.3) \quad h(\lambda_1 K_1 + \dots + \lambda_n K_n, u) = \lambda_1 h(K_1, u) + \dots + \lambda_n h(K_n, u).$$

The support function  $h(K, u)$  is a sublinear function, which satisfies

$$(2.4) \quad h(K, \lambda u) = \lambda h(K, u), \quad h(K, u + v) \leq h(K, u) + h(K, v)$$

for  $\lambda \geq 0$ .

It is very clear from the definition that  $K \subset L$  if and only if

$$(2.5) \quad h(K, u) \leq h(L, u).$$

A compact convex set  $K$  is centered if and only if

$$(2.6) \quad h(K, u) = h(K, -u).$$

The group of nonsingular linear transformations is denoted by  $GL(n)$ . Let  $\phi \in GL(n)$ , the transpose and inverse are denoted by  $\phi^t$  and  $\phi^{-1}$ . Then

$$(2.7) \quad h(\phi K, u) = h(K, \phi^t u) = \|\phi^t u\| h\left(K, \frac{\phi^t u}{\|\phi^t u\|}\right)$$

for all  $u \in S^{n-1}$ .

The radial function  $\rho(K, u)$  of the convex body  $K$  is

$$(2.8) \quad \rho(K, u) = \sup\{\lambda > 0 : \lambda u \in K\}, \quad u \in S^{n-1}.$$

**2.2. Brightness and mixed brightness**

For convex bodies  $K_1, \dots, K_{n-1} \in \mathcal{K}^n$  and a direction  $u \in S^{n-1}$ , the mixed brightness of  $K_1, \dots, K_{n-1}$  in the direction  $u$ ,  $\sigma(K_1, \dots, K_{n-1}; u)$ , is defined by

$$(2.9) \quad \sigma(K_1, \dots, K_{n-1}; u) = nV(K_1, \dots, K_{n-1}, \langle u \rangle),$$

where  $V(K_1, \dots, K_{n-1}, \langle u \rangle)$  denote the mixed volume of  $K_1, \dots, K_{n-1}, \langle u \rangle$  and  $\langle u \rangle$  denote the closed line segment.

Since  $h(\langle u \rangle, \bar{u}) = \frac{1}{2}|u \cdot \bar{u}|$ , we obtain

$$(2.10) \quad \sigma(K_1, \dots, K_{n-1}; u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot \bar{u}| d(K_1, \dots, K_{n-1}; \bar{u}).$$

For  $K \in \mathcal{K}^n$ , and  $u \in S^{n-1}$ , the mixed brightness of  $K_1, \dots, K_{n-1}$  in the direction  $u$  can be written as

$$(2.11) \quad \sigma(K_1, \dots, K_{n-1}; u) = v(K_1^u, \dots, K_{n-1}^u),$$

If  $K_1 = \dots = K_{n-i-1} = K$  and  $K_{n-i} = \dots = K_{n-1} = \bar{K}$ , then the mixed brightness  $\sigma(K_1, \dots, K_{n-1}; u)$  is written as  $\sigma_i(K, \bar{K}; u)$ . If  $i = 0$ , then  $\sigma(K, u)$  is called the brightness of  $K$  in the direction  $u$ . From (2.11) we have

$$(2.12) \quad \sigma_i(K, \bar{K}; u) = u(K^u, \bar{K}^u), \quad \sigma(K, u) = v(K^u).$$

**2.3. Projection and mixed projection bodies**

The projection body,  $\Pi K$ , of the body  $K \in \mathcal{K}^n$  is defined as the convex figure whose support function is given, for  $u \in S^{n-1}$ , by

$$(2.13) \quad h(\Pi K, u) = v(K^u).$$

From (2.10), (2.12), and (2.13), we can see that the homogeneous extension of degree 1 of  $h(\Pi K, u)$  is a convex function and hence  $\Pi K$  is a convex figure. From (2.13), it is easy to see that a projection body is always centered (symmetric about the origin), and if  $K$  has interior points, then  $\Pi K$  will have interior points as well.

If  $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ , then the mixed projection body of  $K_1, \dots, K_{n-1}$  is denoted by  $\Pi(K_1, \dots, K_{n-1})$ , and defined by

$$(2.14) \quad h(\Pi(K_1, \dots, K_{n-1}), u) = \sigma(K_1, \dots, K_{n-1}; u).$$

It is easy to see that the mixed projection body,  $\Pi(K_1, \dots, K_{n-1})$ , must be a convex body that is symmetric with respect to the origin from (2.10) and (2.14).

The following is a list of the basic properties of the mixed projection operator.

The projection operator is multilinear with respect to Minkowski linear combinations; i.e., if  $K_1, K'_1, K_2, \dots, K_{n-1} \in \mathcal{K}^n$  and  $\lambda, \lambda' \geq 0$ , then

$$(2.15) \quad \begin{aligned} &\Pi(\lambda K_1 + \lambda' K'_1, K_2, \dots, K_{n-1}) \\ &= \lambda \Pi(K_1, K_2, \dots, K_{n-1}) + \lambda' \Pi(K'_1, K_2, \dots, K_{n-1}). \end{aligned}$$

If  $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ , and  $\phi \in GL(n)$ , then

$$(2.16) \quad \Pi(\phi K_1, \dots, \phi K_{n-1}) = |\det \phi| \phi^{-t}(\Pi(K_1, \dots, K_{n-1})).$$

The mixed projection operator is monotone nondecreasing with respect to set inclusion (by seeing [10, p. 907]); i.e., if  $K_i, L_i \in \mathcal{K}^n$ , and  $K_i \subset L_i$ ,  $1 \leq i \leq n - 1$ , then

$$(2.17) \quad \Pi(K_1, \dots, K_{n-1}) \subset \Pi(L_1, \dots, L_{n-1}).$$

From the corresponding properties of the  $(n - 1)$ -dimensional mixed volumes and (2.9) or (2.11), it follows that the mixed projection bodies  $\Pi(K_1, \dots, K_{n-1})$  is symmetric in its argument, and for  $x_1, \dots, x_n \in \mathbb{R}^n$ , we have

$$(2.18) \quad \Pi(x_1 + K_1, \dots, x_n + K_n) = \Pi(K_1, \dots, K_n).$$

**2.4. Mixed volumes**

For  $K_1, \dots, K_n \in \mathcal{K}^n$ , and  $u \in S^{n-1}$ , then the following equation relates mixed volumes  $V(K_1, \dots, K_n)$  and mixed area measures  $S(K_1, \dots, K_{n-1}, u)$ :

$$(2.19) \quad V(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} h(K_n, u) dS(K_1, \dots, K_{n-1}, u).$$

**3. Mixed brightness-integrals of convex bodies**

**3.1. Half the brightness**

**Definition 1.** If  $K \in \mathcal{K}^n$  and  $u \in S^{n-1}$ , we let

$$\delta(K, u) = \frac{1}{2} h(\Pi K, u),$$

i.e.,  $\delta(K, u)$  denotes half the brightness of  $K$  in the direction  $u$ . Convex bodies  $K_1, \dots, K_n$  are said to have similar brightness if there exist constants  $\lambda_1, \lambda_2, \dots, \lambda_n > 0$  such that  $\lambda_1 \delta(K_1, u) = \dots = \lambda_n \delta(K_n, u)$  for all  $u \in S^{n-1}$ ; they are said to have constant joint brightness if the product  $\delta(K_1, u) \cdots \delta(K_n, u)$  is constant for all  $u \in S^{n-1}$ . For reference see Gardner [3] and schneider [11].

**3.2. Mixed brightness-integral**

**Definition 2.** Following Lutwak, we define the mixed brightness-integrals of convex bodies:

For  $K_1, K_2, \dots, K_n \in \mathcal{K}^n$ , the mixed brightness-integral

$$D(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \delta(K_1, u) \cdots \delta(K_n, u) dS(u).$$

By this definition,  $D$  is a map

$$D : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_n \rightarrow \mathbb{R}.$$

**3.3. The properties of mixed brightness-integrals**

We list some of its elementary properties.

- (1) (*Positively homogeneous*) If  $K_1, \dots, K_n \in \mathcal{K}^n$  and  $\lambda_1, \dots, \lambda_n > 0$ , then

$$D(\lambda_1 K_1, \dots, \lambda_n K_n) = \lambda_1 \cdots \lambda_n D(K_1, \dots, K_n).$$

- (2) (*Continuity*) The mixed brightness-integrals  $D(K_1, \dots, K_n)$  is a continuous function of  $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ .
- (3) (*Monotonicity for set inclusion*) If  $K_i, L_i \in \mathcal{K}^n$ ,  $K_i \subset L_i$  and  $1 \leq i \leq n$ , then

$$D(K_1, \dots, K_n) \leq D(L_1, \dots, L_n),$$

with equality if and only if  $K_i = L_i$  for  $1 \leq i \leq n$ .

- (4) (*Nonnegativity*) For  $K_1, \dots, K_n \in \mathcal{K}^n$ ,  $D(K_1, \dots, K_n) \geq 0$ .

(5) (*Invariance under individual translation*) If  $x \in \mathbb{R}^n$ , then

$$D(K_1 + x, K_2, \dots, K_n) = D(K_1, \dots, K_n).$$

(6) (*Invariance under linear transformation*) If  $K_1, \dots, K_n \in \mathcal{K}^n$ , and  $\phi \in GL(n)$ , then

$$D(\phi K_1, \dots, \phi K_n) = D(K_1, \dots, K_n).$$

*Proof.* (1) From (2.3) and (2.15), we can get

$$\delta(\lambda_i K_i, u) = \frac{1}{2} h(\Pi \lambda_i K_i, u) = \frac{1}{2} h(\lambda_i \Pi K_i, u) = \frac{1}{2} \lambda_i h(\Pi K_i, u) = \lambda_i \delta(K_i, u)$$

for  $1 \leq i \leq n$ . Then, from the definition, we can obtain,

$$\begin{aligned} D(\lambda_1 K_1, \dots, \lambda_n K_n) &= \frac{1}{n} \int_{S^{n-1}} \delta(\lambda_1 K_1, u) \cdots \delta(\lambda_n K_n, u) dS(u) \\ &= \lambda_1 \cdots \lambda_n \frac{1}{n} \int_{S^{n-1}} \delta(K_1, u) \cdots \delta(K_n, u) dS(u) \\ &= \lambda_1 \cdots \lambda_n D(K_1, \dots, K_n). \end{aligned}$$

(2) The polar coordinate formula for mixed volume of bodies  $K_1, \dots, K_n$  in  $\mathbb{R}^n$  is

$$\tilde{V}(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) d(u).$$

From the continuity of the mixed volume and Minkowski addition, we can see the support function is continuous. Hence, the mixed brightness-integral is a continuous function.

(3) For  $K_i, L_i \in \mathcal{K}^n$ ,  $K_i \subset L_i$  and  $1 \leq i \leq n$ , from (2.17) and (2.5), we have

$$\Pi K_i \subset \Pi L_i,$$

then

$$h(\Pi K_i, u) \leq h(\Pi L_i, u),$$

hence,

$$D(K_1, \dots, K_n) \leq D(L_1, \dots, L_n),$$

with equality if and only if  $K_i = L_i$  for  $1 \leq i \leq n$ .

(4) The mixed projection body,  $\Pi(K_1, \dots, K_n)$ , is a convex body that is symmetric with respect to the origin, then  $h(\Pi K_i, u) > 0$  for  $1 \leq i \leq n$ . From the definition,

$$D(K_1, \dots, K_n) > 0.$$

In particular, if any  $K_i$  is a single point, then  $\Pi K_i$  is the origin. In this case,  $h(\Pi K_i) = 0$ , then

$$D(K_1, \dots, K_n) = 0.$$

Hence,

$$D(K_1, \dots, K_n) \geq 0.$$

(5) From (2.18), we have

$$\delta(K_1 + x, u) = \frac{1}{2}h(\Pi(K_1 + x), u) = \frac{1}{2}h(\Pi(K_1), u) = \delta(K_1, u),$$

hence,

$$D(K_1 + x, K_2, \dots, K_n) = D(K_1, \dots, K_n).$$

(6) From (2.16) and (2.7), we can get

$$\begin{aligned} \delta(\phi K, u) &= \frac{1}{2}h(\Pi(\phi K), u) \\ &= \frac{1}{2}h(|\det \phi| \phi^{-t}(\Pi K), u) \\ &= \frac{1}{2}|\det \phi| h(\phi^{-t}(\Pi K), u) \\ &= \frac{1}{2}|\det \phi| h(\Pi K, \phi^{-1}u) \\ &= \frac{1}{2}|\det \phi| \|\phi^{-1}u\| h\left(\Pi K, \frac{\phi^{-1}u}{\|\phi^{-1}u\|}\right) \\ &= \frac{1}{2}h(\Pi K, u) \\ &= \delta(\phi K, u'), \end{aligned}$$

where  $u' \in S^{n-1}$ .

Hence,

$$D(\phi K_1, \dots, \phi K_n) = D(K_1, \dots, K_n). \quad \square$$

### 3.4. Mixed brightness-integral of order $p$

Just as the width-integral  $B_i(K)$  [6] of  $K \in \mathcal{K}^n$ , are defined to be the special mixed width-integral

$$A(\underbrace{K, \dots, K}_{n-i}, \underbrace{U, \dots, U}_i),$$

the brightness-integral  $C_i(K)$  of  $K \in \mathcal{K}^n$ , can be defined as the special mixed brightness-integral

$$D(\underbrace{K, \dots, K}_{n-i}, \underbrace{U, \dots, U}_i).$$

Now we generalize the notion of the mixed brightness-integral of convex bodies: For  $K_1, \dots, K_n \in \mathcal{K}^n$ , and a real number  $p \neq 0$ , the mixed brightness-integral of order  $p$ ,  $D_p(K_1, \dots, K_n)$  of  $K_1, \dots, K_n$  is defined by

$$D_p(K_1, K_2, \dots, K_n) = \omega_n \left[ \frac{1}{n\omega_n} \int_{S^{n-1}} \delta(K_1, u)^p \cdots \delta(K_n, u)^p dS(u) \right]^{\frac{1}{p}}.$$

Specially  $p = 1$ , this definition is just Definition 1.



**4. Inequalities for the mixed-brightness integrals**

In order to prove the conclusions in the introduction, we require the following simply extension of Hölder' s inequality.

**Lemma 1.** *If  $f_0, f_1, \dots, f_m$  are (strictly) positive continuous functions defined on  $S^{n-1}$  and  $\lambda_1, \dots, \lambda_m$  are positive constants the sum of whose reciprocals is unity, then*

$$\int_{S^{n-1}} f_0(u)f_1(u)\cdots f_m(u)dS(u) \leq \prod_{i=1}^m \left[ \int_{S^{n-1}} f_0(u)f_i^{\lambda_i}(u)dS(u) \right]^{\frac{1}{\lambda_i}},$$

with equality if and only if there exist positive constants  $\alpha_1, \alpha_2, \dots, \alpha_m$  such that  $\alpha_1 f_1^{\lambda_1}(u) = \dots = \alpha_m f_m^{\lambda_m}(u)$  for all  $u \in S^{n-1}$ .

*Proof of Theorem 1.* For  $K_1, \dots, K_n \in \mathcal{K}^n$ , let

$$\begin{aligned} \lambda_i &= m(1 \leq i \leq m), \\ f_0 &= \delta(K_1, u) \cdots \delta(K_{n-m}, u) \quad (f_0 = 1 \text{ if } m = n), \\ f_i &= \delta(K_{n-i+1}, u) \quad (1 \leq i \leq m). \end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned} &\int_{S^{n-1}} \delta(K_1, u) \cdots \delta(K_n, u) dS(u) \\ &\leq \prod_{i=1}^m \left[ \int_{S^{n-1}} \delta(K_1, u) \cdots \delta(K_{n-m}, u) \delta(K_{n-i+1}, u)^m dS(u) \right]^{\frac{1}{m}}, \end{aligned}$$

with equality if and only if  $K_{n-m+1}, K_{n-m+2}, \dots, K_n$  are all of similar brightness, i.e.,

$$D^m(K_1, \dots, K_n) \leq \prod_{i=0}^{m-1} D(K_1, \dots, K_{n-m}, K_{n-i}, \dots, K_{n-i}),$$

with equality if and only if  $K_{n-m+1}, K_{n-m+2}, \dots, K_n$  are all of similar brightness. □

*Proof of Theorem 2.* For  $K_1, \dots, K_n \in \mathcal{K}^n$ , let

$$\begin{aligned} \lambda_i &= m(1 \leq i \leq m), \\ f_0 &= \delta^p(K_1, u) \cdots \delta^p(K_{n-m}, u) \quad (f_0 = 1 \text{ if } m = n), \\ f_i &= \delta^p(K_{n-i+1}, u) \quad (1 \leq i \leq m). \end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned} &\int_{S^{n-1}} \delta^p(K_1, u) \cdots \delta^p(K_n, u) dS(u) \\ &\leq \prod_{i=1}^m \left[ \int_{S^{n-1}} \delta^p(K_1, u) \cdots \delta^p(K_{n-m}, u) \delta^{pm}(K_{n-i+1}, u) dS(u) \right]^{\frac{1}{m}}, \end{aligned}$$

with equality if and only if  $K_{n-m+1}, K_{n-m+2}, \dots, K_n$  are all of similar brightness.

For  $p > 0$ , we get

$$\begin{aligned} & \omega_n \left[ \frac{1}{n\omega_n} \int_{S^{n-1}} \delta^p(K_1, u) \cdots \delta^p(K_n, u) dS(u) \right]^{\frac{1}{p}} \\ & \leq \omega_n \prod_{i=1}^m \left[ \frac{1}{n\omega_n} \int_{S^{n-1}} \delta^p(K_1, u) \cdots \delta^p(K_{n-m}, u) \delta^{pm}(K_{n-i+1}, u) dS(u) \right]^{\frac{1}{pm}}, \end{aligned}$$

with equality if and only if  $K_{n-m+1}, K_{n-m+2}, \dots, K_n$  are all of similar brightness. For  $p < 0$ , inequality above is reversed.

Thus we obtain the conclusion. □

*Proof of Theorem 3.* For  $K_1, \dots, K_n \in \mathcal{K}^n$ , from (2.12), (2.19) and Lemma 1, we can get

$$\begin{aligned} & D(K_1, \dots, K_n) \\ & = \frac{1}{n} \int_{S^{n-1}} \delta(K_1, u) \cdots \delta(K_n, u) dS(u) \\ & \leq n^{-\frac{1}{n}} \left[ \int_{S^{n-1}} \delta^n(K_1, u) dS(u) \right]^{\frac{1}{n}} \cdots \left[ \int_{S^{n-1}} \delta^n(K_n, u) dS(u) \right]^{\frac{1}{n}} \\ & = n^{-\frac{1}{n}} \left\{ \int_{S^{n-1}} \left[ \frac{1}{2} h(\Pi K_1, u) \right]^n dS(u) \right\}^{\frac{1}{n}} \cdots \left\{ \int_{S^{n-1}} \left[ \frac{1}{2} h(\Pi K_n, u) \right]^n dS(u) \right\}^{\frac{1}{n}} \\ & = v^{\frac{1}{n}}(K_1^u) \cdots v^{\frac{1}{n}}(K_n^u), \end{aligned}$$

with equality if and only if  $K_1, \dots, K_n$  are all of similar brightness.

Thus we get

$$D^n(K_1, \dots, K_n) \leq v(K_1^u) \cdots v(K_n^u). \quad \square$$

For  $p$  equal to  $-\infty, 0$  or  $\infty$ , we respectively define the mixed brightness-integral of order  $p$  by

$$D_p(K_1, \dots, K_n) = \lim_{s \rightarrow p} D_s(K_1, \dots, K_n).$$

As a direct consequence of Jensen's inequality [4] we have:

**Proposition 1.** *If  $K_1, \dots, K_n \in \mathcal{K}^n$  and  $-\infty \leq p \leq q \leq \infty$ , then*

$$D_p(K_1, \dots, K_n) \leq D_q(K_1, \dots, K_n),$$

*with equality if and only if  $K_1, \dots, K_n$  have constant joint brightness.*

*Proof of Theorem 4.* For  $K_1, \dots, K_n \in \mathcal{K}^n$  and  $-\infty \leq p \leq 1$ . By combining Theorem 3 with Proposition 1 we obtain

$$\begin{aligned} D_p^n(K_1, \dots, K_n) & \leq D_1^n(K_1, \dots, K_n) \\ & \leq v(K_1^u) \cdots v(K_n^u). \end{aligned}$$

In view of the equality conditions of Theorem 1 with Proposition 1, equality holds if and only if  $K_1, K_2, \dots, K_n$  are all of similar brightness and have constant joint brightness. Thus we obtain the conclusion.  $\square$

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