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Multiple positive solutions for a Dirichlet problem involving critical Sobolev exponent

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ABSTRACT

In this paper, we study the decomposition of the Nehari manifold via the combination of concave and convex nonlinearities. Furthermore, we use this result and the Ljusternik–Schnirelmann category to prove that the existence of multiple positive solutions for a Dirichlet problem involving critical Sobolev exponent.

1. Introduction

In this paper, we study the multiplicity of positive solutions for the following semilinear elliptic problem:

$$\begin{cases} -\Delta u = f_\lambda(x)|u|^{q-2}u + g(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{E_λ}$$

where $1 < q < 2 < p \leqslant 2^* = \frac{2N}{N-2}$ $(N \geqslant 3)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, the parameter $\lambda > 0$ and the weight functions $f_\lambda = \lambda f_+ + f_-$ $(f_\pm = \pm\max\{f, 0\})$, g are continuous on $\overline{\Omega}$ which satisfy the following condition:

(Q) there exist a non-empty closed set $M = \{x \in \overline{\Omega} \mid g(x) = \max_{x\in\overline{\Omega}} g(x) \equiv 1\}$ and $\rho > N-2$ such that

$$M \subset \{x \in \Omega \mid f(x) > 0\}$$

and

$$g(z) - g(x) = O\left(|x-z|^\rho\right) \quad \text{as } x \to z \text{ and uniformly in } z \in M.$$

Remark 1.1. Let $M_r = \{x \in \mathbb{R}^N \mid \mathrm{dist}(x, M) < r\}$ for $r > 0$. Then by the condition (Q), we may assume that there exist two positive constants C_0 and r_0 such that

$$f(x), g(x) > 0 \quad \text{for all } x \in M_{r_0} \subset \Omega$$

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and

$$g(z) - g(x) \leqslant C_0|x-z|^\rho \quad \text{for all } x \in B^N(z, r_0) \text{ and for all } z \in M.$$

That the existence and multiplicity of positive solutions of Eq. (E_λ) may be influenced by the concave and convex nonlinearities. These issues have been the focus of a great deal of research in recent years. When the weight functions $f_\lambda \equiv \lambda$ and $g \equiv 1$, Ambrosetti, Brezis and Cerami [1] has proved that there exists $\lambda_0 > 0$ such that Eq. (E_λ) admits at least two positive solutions for $\lambda \in (0, \lambda_0)$, a positive solution for $\lambda = \lambda_0$ and no positive solution for $\lambda > \lambda_0$. Actually, Adimurthy, Pacella and Yadava [2], Ouyang and Shi [20] and Tang [22] proved that there exists $\lambda_0 > 0$ such that Eq. (E_λ) in unit ball $B^N(0;1)$ has exactly two positive solutions for all $\lambda \in (0, \lambda_0)$, has exactly one positive solution for $\lambda = \lambda_0$ and no positive solution for all $\lambda > \lambda_0$. When the weight functions f, g are sign-changing, Brown and Wu [6,7], de Figueiredo, Gossez and Ubilla [14], Hsu and Lin [15] and Wu [25,26] they proved that Eq. (E_λ) has at least two positive solutions when λ is sufficiently small. Furthermore, if $p = 2^*$ and $f, g \in C(\overline{\Omega})$ satisfy the following conditions:

- $(D1)$ $B^N(2\delta_0)\backslash\overline{B^N(\delta_0)} \subset \Omega$ for some $\delta_0 > 0$, where $B^N(r) = \{x \in \mathbb{R}^N \mid |x| < r\}$;
- $(D2)$ $f = f_+ - f_-$, where $f^\pm : \overline{\Omega} \to \mathbb{R}$ are continuous functions and there exists a domain $B^N(2\delta_0)\backslash\overline{B^N(\delta_0)} \subset \Theta \subset \Omega$ of class C^1 such that for all $x \in \Theta$, $f_-(x) = 0$, $f_+(x) > 0$ and for all $x \in \Omega\backslash\overline{\Theta}$, $f_-(x) \geqslant 0$, $f_+(x) = 0$;
- $(D3)$ the N-ball $B^N(2\delta_0) \subset \Omega$ such that for all $x \in B^N(\delta_0)$, $0 < g(x) < 1$ and for all $x \in \overline{\Omega}\backslash B^N(2\delta_0)$, $0 \leqslant g(x) < 1$;
- $(D4)$ $g(x) = 0$ for all $x \in B^N(2\delta_0)\backslash\overline{B^N(\delta_0)}$,

then Wu [27] proved that Eq. (E_λ) has at least three positive solutions when λ is sufficiently small.

The main purpose of this paper is to consider the relation between the number of solutions and the topology of the global maxima set of weight function g. Our main result improves a recent multiplicity result of [15,25,27]. The following theorem is our main result.

Theorem 1.1. *Suppose that $p = 2^*$. Then for each $\delta < r_0$ there exists $\Lambda_\delta > 0$ such that for $\lambda < \Lambda_\delta$, Eq. (E_λ) has at least $cat_{M_\delta}(M) + 1$ positive solutions.*

Hereafter *cat* is the Ljusternik–Schnirelmann category (see e.g. [17]).

In the following sections, we proceed to proof Theorem 1.1. We use the variational methods to find positive solutions of Eq. (E_λ). Associated with Eq. (E_λ), we consider the energy functional J_λ in $H_0^1(\Omega)$,

$$J_\lambda(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{q}\int_\Omega f_\lambda |u|^q\,dx - \frac{1}{2^*}\int_\Omega g|u|^{2^*}\,dx$$

where $\|u\|_{H^1} = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$ is the standard norm in $H_0^1(\Omega)$. It is well known that the solutions of Eq. (E_λ) are the critical points of the energy functional J_λ in $H_0^1(\Omega)$.

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we discuss some concentration behavior. In Section 4, we prove Theorem 1.1.

2. Notations and preliminaries

Throughout this paper, we denote by S the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ which is given by

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{(\int_{\mathbb{R}^N} |u|^{2^*}\,dx)^{2/2^*}} > 0. \tag{2.1}$$

It is well known that S is independent of $\Omega \subset \mathbb{R}^N$ in the sense that if

$$S(\Omega) = \inf_{u \in H_0^1(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{(\int_\Omega |u|^{2^*}\,dx)^{2/2^*}} > 0,$$

then $S(\Omega) = S(\mathbb{R}^N) = S$, and the function

$$u_\varepsilon(x) = \frac{\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}, \quad \varepsilon > 0 \text{ and } x \in \mathbb{R}^N,$$

is an extremal function for the minimum problem (2.1). Moreover, for each $\varepsilon > 0$,

$$v_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}} \tag{2.2}$$

which is a positive solution of critical problem:

$$-\Delta u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N \tag{2.3}$$

with $\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2\,dx = \int_{\mathbb{R}^N} |v_\varepsilon|^{2^*}\,dx = S^{\frac{N}{2}}$.

We define the Palais–Smale (simply by (PS)) sequences, (PS)-values, and (PS)-conditions in $H_0^1(\Omega)$ for J_λ as follows.

Definition 2.1.

(i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_λ if $J_\lambda(u_n) = \beta + o(1)$; and $J'_\lambda(u_n) = o(1)$; strongly in $H^{-1}(\Omega)$ as $n \to \infty$.
(ii) J_λ satisfies the $(\mathrm{PS})_\beta$-condition in $H_0^1(\Omega)$ if every $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_λ contains a convergent subsequence.

As the energy functional J_λ is not bounded below on $H_0^1(\Omega)$, it is useful to consider the functional on the Nehari manifold

$$\mathbf{N}_\lambda = \bigl\{u \in H_0^1(\Omega)\backslash\{0\} \bigm| \langle J'_\lambda(u), u\rangle = 0\bigr\}.$$

Thus, $u \in \mathbf{N}_\lambda$ if and only if

$$\|u\|_{H^1}^2 - \int_\Omega f_\lambda |u|^q\,dx - \int_\Omega g|u|^{2^*}\,dx = 0.$$

Moreover, we have the following results.

Lemma 2.2. *The energy functional J_λ is coercive and bounded below on $\mathbf{N}_\lambda$.*

Proof. If $u \in \mathbf{N}_\lambda$, then by the Young and Sobolev inequalities

$$J_\lambda(u) = \frac{1}{N}\|u\|_{H^1}^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\int_\Omega f_\lambda |u|^q\,dx \tag{2.4}$$

$$\geqslant \frac{1}{N}\|u\|_{H^1}^2 - \lambda\frac{2^*-q}{q2^*}\|f_+\|_{L^{q^*}} S^{-\frac{q}{2}}\|u\|_{H^1}^q \tag{2.5}$$

$$\geqslant \frac{1}{N}\|u\|_{H^1}^2 - \frac{1}{N}\|u\|_{H^1}^2 - \overline{D}\lambda^{2/(2-q)}$$

$$= -D_0\lambda^{2/(2-q)}, \tag{2.6}$$

where $q^* = \frac{2^*}{2^*-q}$ and D is a positive constant depending on q, N, S and $\|f_+\|_{L^{q^*}}$. Thus, J_λ is coercive and bounded below on $\mathbf{N}_\lambda$. $\square$

Define

$$\psi_\lambda(u) = \langle J'_\lambda(u), u\rangle = \|u\|_{H^1}^2 - \int_\Omega f_\lambda |u|^q\,dx - \int_\Omega g|u|^{2^*}\,dx.$$

Then for $u \in \mathbf{N}_\lambda$,

$$\langle \psi'_\lambda(u), u\rangle = 2\|u\|_{H^1}^2 - q\int_\Omega f_\lambda |u|^q\,dx - 2^*\int_\Omega g|u|^{2^*}\,dx$$

$$= -\frac{4}{N-2}\|u\|_{H^1}^2 - (q-2^*)\int_\Omega f_\lambda |u|^q\,dx \tag{2.7}$$

$$= (2-q)\|u\|_{H^1}^2 - (2^*-q)\int_\Omega g|u|^{2^*}\,dx. \tag{2.8}$$

Similarly to the method used in [26], we split $\mathbf{N}_\lambda$ into three parts:

$$\mathbf{N}_\lambda^+ = \bigl\{u \in \mathbf{N}_\lambda \bigm| \langle \psi'_\lambda(u), u\rangle > 0\bigr\}; \qquad \mathbf{N}_\lambda^0 = \bigl\{u \in \mathbf{N}_\lambda \bigm| \langle \psi'_\lambda(u), u\rangle = 0\bigr\}; \qquad \mathbf{N}_\lambda^- = \bigl\{u \in \mathbf{N}_\lambda \bigm| \langle \psi'_\lambda(u), u\rangle < 0\bigr\}.$$

We now derive some basic properties of $\mathbf{N}_\lambda^+$, $\mathbf{N}_\lambda^0$ and $\mathbf{N}_\lambda^-$.

Lemma 2.3. *Suppose that u_0 is a local minimizer for J_λ on $\mathbf{N}_\lambda$ and that $u_0 \notin \mathbf{N}_\lambda^0$. Then $J'_\lambda(u_0) = 0$ in $H^{-1}(\Omega)$. Furthermore, if u_0 is a non-trivial nonnegative function in Ω, then u_0 is a positive solution of Eq. (E_λ).*

Proof. Similarly the argument in [8, Theorem 2.3] (or see [3]), we have $J'_\lambda(u_0) = 0$ in $H^{-1}(\Omega)$. This implies u_0 is a weak solution of Eq. (E_λ). Now, if u_0 is a non-trivial nonnegative function in Ω, then by [13, Lemma 2.1], we have $u_0 \in L^\infty(\Omega)$. Moreover, we can apply the Harnack inequality due to [23] in order to get u_0 is positive in Ω. □

Lemma 2.4. *For each $\lambda > 0$ we have the following*:

(i) *For any $u \in \mathbf{N}_\lambda^+$, we have $\int_\Omega f_\lambda |u|^q\,dx > 0$.*
(ii) *For any $u \in \mathbf{N}_\lambda^0$, we have $\int_\Omega f_\lambda |u|^q\,dx > 0$ and $\int_\Omega g|u|^{2^*}\,dx > 0$.*
(iii) *For any $u \in \mathbf{N}_\lambda^-$, we have $\int_\Omega g|u|^{2^*}\,dx > 0$.*

Proof. The result now follows immediately from (2.7) and (2.8). □

Let $\Lambda_1 = \frac{S^{\frac{N(2-q)}{4}+\frac{q}{2}}}{\|f_+\|_{L^{q^*}}}(\frac{2-q}{2^*-q})^{\frac{(N-2)(2-q)}{4}}(\frac{2^*-2}{2^*-q})$. Then by an argument similar to that the proof Lemma 2.1 in [26], we have $\mathbf{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, \Lambda_1)$. Thus, we can write $\mathbf{N}_\lambda = \mathbf{N}_\lambda^+ \cup \mathbf{N}_\lambda^-$ and define

$$\alpha_\lambda^+ = \inf_{u\in\mathbf{N}_\lambda^+} J_\lambda(u) \quad \text{and} \quad \alpha_\lambda^- = \inf_{u\in\mathbf{N}_\lambda^-} J_\lambda(u).$$

Moreover, we have the following result, whose proof can be found in [27, Theorem 3.1].

Theorem 2.5. *We have the following*:

(i) *$\alpha_\lambda^+ < 0$ for all $\lambda \in (0, \Lambda_1)$.*
(ii) *If $\lambda < \Lambda_2 = \frac{q}{2}\Lambda_1$, then $\alpha_\lambda^- > c_0$ for some $c_0 > 0$.*

In particular, $\alpha_\lambda^+ = \inf_{u\in\mathbf{N}_\lambda} J_\lambda(u)$ for all $\lambda \in (0, \Lambda_2)$.

For each $u \in H_0^1(\Omega)\backslash\{0\}$ with $\int_\Omega g|u|^{2^*}\,dx > 0$, we write

$$t_{\max} = \left(\frac{(2-q)\|u\|_{H^1}^2}{(2^*-q)\int_\Omega g|u|^{2^*}\,dx}\right)^{\frac{N-2}{4}} > 0.$$

Then we have the following lemma.

Lemma 2.6. *For each $u \in H_0^1(\Omega)\backslash\{0\}$ we have the following*:

(i) *If $\int_\Omega f_\lambda|u|^q\,dx \leqslant 0$, then there is a unique $t^- = t^-(u) > t_{\max}$ such that $t^-u \in \mathbf{N}_\lambda^-$ and h_u is increasing on $(0, t^-)$ and decreasing on (t^-, ∞). Moreover,*

$$J_\lambda\big(t^-u\big) = \sup_{t\geqslant 0} J_\lambda(tu). \tag{2.9}$$

(ii) *If $\int_\Omega f_\lambda|u|^q\,dx > 0$, then there are unique $0 < t^+ = t^+(u) < t_{\max} < t^-$ such that $t^+u \in \mathbf{N}_\lambda^+$, $t^-u \in \mathbf{N}_\lambda^-$, h_u is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on (t^-, ∞). Moreover,*

$$J_\lambda\big(t^+u\big) = \inf_{0\leqslant t\leqslant t_{\max}} J_\lambda(tu); \qquad J_\lambda\big(t^-u\big) = \sup_{t\geqslant t^+} J_\lambda(tu). \tag{2.10}$$

Proof. The proof is almost the same as the in [26, Lemma 2.4]. □

For $c > 0$, we define

$$J_0^c(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{c}{2^*}\int_\Omega g|u|^{2^*}\,dx; \qquad \mathbf{N}_0^c = \big\{u \in H_0^1(\Omega)\backslash\{0\} \bigm| \big\langle\big(J_0^c\big)'(u), u\big\rangle = 0\big\}.$$

Note that $\mathbf{N}_0^c = \mathbf{N}_0 = \mathbf{N}_0^-$ for $c = 1$ and for each $u \in \mathbf{N}_\lambda^-$ there is a unique $t_u > 0$ such that $t_u u \in \mathbf{N}_0$. Moreover, we have the following result, whose proof can be found in [25, Lemma 5.2].

Lemma 2.7. *Let $q^* = \frac{2^*}{2^*-q}$. Then for each $u \in \mathbf{N}_\lambda^-$ we have the following*:

(i) *There is a unique $t^c(u) > 0$ such that $t^c(u)u \in \mathbf{N}_0^c$ and*

$$\sup_{t \geqslant 0} J_0^c(tu) = J_0^c\big(t^c(u)u\big) = \frac{1}{N}\left(\frac{\|u\|_{H^1}^{2^*}}{c\int_\Omega g|u|^{2^*}\,dx}\right)^{\frac{N-2}{2}}.$$

(ii)

$$J_\lambda(u) \geqslant (1-\lambda)^{\frac{N}{2}} J_0(t_u u) - \frac{\lambda(2-q)}{2q}\big(\|f_+\|_{L^{q^*}} S^{\frac{-q}{2}}\big)^{\frac{2}{2-q}}$$

and

$$J_\lambda(u) \leqslant (1+\lambda)^{\frac{N}{2}} J_0(t_u u) + \frac{\lambda(2-q)}{2q}\big(\|f_+\|_{L^{q^*}} S^{\frac{-q}{2}}\big)^{\frac{2}{2-q}}.$$

3. Concentration behavior

First, we consider the following critical problem:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{$\widehat{E}_0$}$$

Associated with Eq. $(\widehat{E}_0)$, we consider the energy functional J^∞ in $H_0^1(\Omega)$,

$$J^\infty(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx.$$

It is well known that

$$\inf_{u \in \mathbf{N}^\infty(\mathbb{R}^N)} J^\infty(u) = \inf_{u \in \mathbf{N}^\infty(\Omega)} J^\infty(u) = \frac{1}{N}S^{\frac{N}{2}} \quad \text{for all domain } \Omega \subset \mathbb{R}^N,$$

where

$$\mathbf{N}^\infty\big(\mathbb{R}^N\big) = \big\{u \in D^{1,2}\big(\mathbb{R}^N\big)\backslash\{0\} \,\big|\, \big\langle\big(J^\infty\big)'(u), u\big\rangle = 0\big\}$$

and

$$\mathbf{N}^\infty(\Omega) = \big\{u \in H_0^1(\Omega)\backslash\{0\} \,\big|\, \big\langle(u), u\big\rangle = 0\big\}$$

are the Nehari manifolds. Actually, $\inf_{u \in \mathbf{N}^\infty(\Omega)} J^\infty(u)$ is never attained on a domain $\Omega \subsetneqq \mathbb{R}^N$. Following the method of [4] and Remark 1.1, let $\eta \in C_0^\infty(\mathbb{R}^N)$ be a radially symmetric function such that $0 \leqslant \eta \leqslant 1$, $|\nabla\eta| \leqslant C$ and

$$\eta(x) = \begin{cases} 1, & |x| \leqslant r_0/2, \\ 0, & |x| \geqslant r_0. \end{cases}$$

For any $z \in M$, let

$$w_{\varepsilon,z}(x) = \eta(x-z)v_\varepsilon(x-z), \tag{3.1}$$

where $v_\varepsilon(x)$ as in (2.2). Then, by an argument similar to that the proof Lemma 4.2 in [16] (or see Struwe [21]), we have

$$\|w_{\varepsilon,z}\|_{H^1}^2 = S^{\frac{N}{2}} + O\big(\varepsilon^{N-2}\big) \quad \text{uniformly in } z \in M. \tag{3.2}$$

Moreover, we have the following results.

Lemma 3.1. *We have*

$$\inf_{u \in \mathbf{N}_0} J_0(u) = \inf_{u \in \mathbf{N}^\infty(\Omega)} J^\infty(u) = \frac{1}{N}S^{\frac{N}{2}}.$$

Furthermore, Eq. (E_0) does not admit any positive solution u_0 such that $J_0(u_0) = \frac{1}{N}S^{\frac{N}{2}}$.

Proof. Let $w_{\varepsilon,z}$ be as in (3.1) and define $\overline{g}:\mathbb{R}^N\to\mathbb{R}$ by

$$\overline{g}(x)=\begin{cases} g(x), & \text{if } x\in\overline{\Omega},\\ 0, & \text{if } x\in\overline{\Omega}^c,\end{cases}$$

as an extension of g. Then, by Lemma 2.6, there is a unique $t_0(w_{\varepsilon,z})>0$ such that $t_0(w_{\varepsilon,z})w_{\varepsilon,z}\in\mathbf{N}_0$ for all $\varepsilon>0$, that is

$$\left\|t_0(w_{\varepsilon,z})w_{\varepsilon,z}\right\|_{H^1}^2=\int_\Omega g\left|t_0(w_{\varepsilon,z})w_{\varepsilon,z}\right|^{2^*}dx$$

and so

$$\left[t_0(w_{\varepsilon,z})\right]^{4/(N-2)}=\frac{\int_\Omega g|w_{\varepsilon,z}|^{2^*}dx}{\|w_{\varepsilon,z}\|_{H^1}^2}. \tag{3.3}$$

By the definition of v_ε, we get that

$$\int_\Omega g|w_{\varepsilon,z}|^{2^*}dx=\int_{B^N(z,2r_0)} g(x)\left|\eta(x-z)v_\varepsilon(x-z)\right|^{2^*}dx=\int_{\mathbb{R}^N}\frac{[N(N-2)\varepsilon^2]^{N/2}\overline{g}(x+z)\eta^{2^*}(x)}{(\varepsilon^2+|x|^2)^N}dx.$$

Thus, by the condition (Q),

$$\begin{aligned}
0&\leqslant\frac{1}{[N(N-2)\varepsilon^2]^{N/2}}\left(\int_{\mathbb{R}^N}|v_\varepsilon|^{2^*}dx-\int_\Omega g|w_{\varepsilon,z}|^{2^*}dx\right)\\
&=\int_{\mathbb{R}^N\setminus B^N(0,r_0/2)}\frac{[1-\overline{g}(x+z)\eta^{2^*}(x)]}{(\varepsilon^2+|x|^2)^N}dx+\int_{B^N(0,r_0/2)}\frac{[1-\overline{g}(x+z)]}{(\varepsilon^2+|x|^2)^N}dx\\
&\leqslant\int_{\mathbb{R}^N\setminus B^N(0,r_0/2)}\frac{1}{|x|^{2N}}dx+C_0\int_{B^N(0,r_0/2)}\frac{|x|^\rho}{(\varepsilon^2+|x|^2)^N}dx\\
&\leqslant N\omega_N\int_{r_0/2}^\infty r^{-(N+1)}dr+\frac{C_0N\omega_N}{\varepsilon^2}\int_0^{r_0/2}r^{\rho-N+1}dr\\
&=\omega_N\left(\frac{r_0}{2}\right)^{-N}+\frac{C_0N\omega_N}{\varepsilon^2(\rho-(N-2))}\left(\frac{r_0}{2}\right)^{\rho-(N-2)}\\
&=C_1+\frac{C_2}{\varepsilon^2}\quad\text{for all } z\in M,
\end{aligned}\tag{3.4}$$

where ω_N is the volume of the unit ball $B^N(0,1)$ in $\mathbb{R}^N$. Then

$$\lim_{\varepsilon\to0}\int_\Omega g|w_{\varepsilon,z}|^{2^*}dx=S^{\frac{N}{2}}\quad\text{uniformly in } z\in M. \tag{3.5}$$

From the condition (Q) and the same argument of (3.5), we have

$$\lim_{\varepsilon\to0}\int_\Omega f_-|w_{\varepsilon,z}|^{2^*}dx=0\quad\text{uniformly in } z\in M. \tag{3.6}$$

By (3.2), (3.5) and (3.6),

$$\lim_{\varepsilon\to0}t_0(w_{\varepsilon,z})=1\quad\text{and}\quad\lim_{\varepsilon\to0}\|w_{\varepsilon,z}\|_{H^1}^2=S^{\frac{N}{2}}\quad\text{uniformly in } z\in M.$$

Thus,

$$\inf_{u\in\mathbf{N}_0}J_0(u)\leqslant J_0\big(t_0(w_{\varepsilon,z})w_{\varepsilon,z}\big)\to\frac{1}{N}S^{\frac{N}{2}}\quad\text{as }\varepsilon\to0$$

and so

$$\inf_{u\in\mathbf{N}_0}J_0(u)\leqslant\inf_{u\in\mathbf{N}^\infty(\Omega)}J^\infty(u)=\frac{1}{N}S^{\frac{N}{2}}.$$

Let $u \in \mathbf{N}_0$. Then, by Lemma 2.6(i), $J_0(u) = \sup_{t \geqslant 0} J_0(tu)$. Moreover, there is a unique $t_u > 0$ such that $t_u u \in \mathbf{N}^{\infty}(\Omega)$. Thus,

$$J_0(u) \geqslant J_0(t_u u) \geqslant J^{\infty}(t_u u) \geqslant \frac{1}{N} S^{\frac{N}{2}}.$$

This implies $\inf_{u \in \mathbf{N}_0} J_0(u) \geqslant \frac{1}{N} S^{\frac{N}{2}}$. Therefore,

$$\inf_{u \in \mathbf{N}_0} J_0(u) = \inf_{u \in \mathbf{N}^{\infty}(\Omega)} J^{\infty}(u) = \frac{1}{N} S^{\frac{N}{2}}.$$

Next, we will show that Eq. (E_0) does not admit any solution u_0 such that $J_0(u_0) = \inf_{u \in \mathbf{N}_0} J_0(u)$. Suppose the contrary. Then we can assume that there exists $u_0 \in \mathbf{N}_0$ such that $J_0(u_0) = \inf_{u \in \mathbf{N}_0} J_0(u)$. Since $J_0(u_0) = J_0(|u_0|)$ and $|u_0| \in \mathbf{N}_0$, by Lemma 2.3, we may assume that u_0 is a positive solution of Eq. (E_0). Moreover, by Lemma 2.6(i), $J_0(u_0) = \sup_{t \geqslant 0} J_0(tu_0)$. Thus, there is a unique $t_{u_0} > 0$ such that $t_{u_0} u_0 \in \mathbf{N}^{\infty}(\Omega)$ and so

$$\begin{aligned} \frac{1}{N} S^{\frac{N}{2}} &= \inf_{u \in \mathbf{N}_0} J_0(u) = J_0(u_0) \geqslant J_0(t_{u_0} u_0) \\ &= J^{\infty}(t_{u_0} u_0) + \frac{t_{u_0}^{2^*}}{2^*} \int_{\Omega} (1-g)|u_0|^{2^*} dx \\ &\geqslant \frac{1}{N} S^{\frac{N}{2}} + \frac{t_{u_0}^{2^*}}{2^*} \int_{\Omega} (1-g)|u_0|^{2^*} dx. \end{aligned}$$

This implies $\int_{\Omega} (1-g)|u_0|^{2^*} dx = 0$, which is a contradiction. This completes the proof. □

Lemma 3.2. *Suppose that $\{u_n\}$ is a minimizing sequence for J_0 in $\mathbf{N}_0$. Then*

(i) $\int_{\Omega} f_- |u_n|^q \, dx = o(1)$;
(ii) $\int_{\Omega} (1-g)|u_n|^{2^*} dx = o(1)$.

Furthermore, $\{u_n\}$ is a $(\mathrm{PS})_{\frac{1}{N} S^{\frac{N}{2}}}$-sequence for J^{∞} in $H_0^1(\Omega)$.

Proof. For each n, there is a unique $t_n > 0$ such that $t_n u_n \in \mathbf{N}^{\infty}(\Omega)$, that is

$$t_n^2 \int_{\Omega} |\nabla u_n|^2 \, dx = t_n^{2^*} \int_{\Omega} |u_n|^{2^*} dx.$$

Then, by Lemma 2.6(i),

$$\begin{aligned} J_0(u_n) \geqslant J_0(t_n u_n) &= J^{\infty}(t_n u_n) - \frac{t_n^q}{q} \int_{\Omega} f_- |u_n|^q \, dx + \frac{t_n^{2^*}}{2^*} \int_{\Omega} (1-g)|u_n|^{2^*} dx \\ &\geqslant \frac{1}{N} S^{\frac{N}{2}} - \frac{t_n^q}{q} \int_{\Omega} f_- |u_n|^q \, dx + \frac{t_n^{2^*}}{2^*} \int_{\Omega} (1-g)|u_n|^{2^*} dx. \end{aligned}$$

Since $J_0(u_n) = \frac{1}{N} S^{\frac{N}{2}} + o(1)$ from Lemma 3.1, we have

$$\frac{t_n^q}{q} \int_{\Omega} f_- |u_n|^q \, dx = o(1)$$

and

$$\frac{t_n^{2^*}}{2^*} \int_{\Omega} (1-g)|u_n|^{2^*} dx = o(1).$$

We will show that there exists $c_0 > 0$ such that $t_n > c_0$ for all n. Suppose the contrary. Then we may assume $t_n \to 0$ as $n \to \infty$. Since

$$J_0(u_n) = \frac{1}{N} S^{\frac{N}{2}} + o(1)$$

and

$$J_0(u_n) = \frac{1}{N} \|u_n\|_{H^1}^2 + o(1),$$

by Lemma 2.2, $\|u_n\|$ is uniformly bounded and so $\|t_n u_n\|_{H^1} \to 0$ or $J^\infty(t_n u_n) \to 0$ and this contradicts $J^\infty(t_n u_n) \geqslant \frac{1}{N} S^{\frac{N}{2}} > 0$. Thus,

$$\int_\Omega f_- |u_n|^q \, dx = o(1)$$

and

$$\int_\Omega (1-g)|u_n|^{2^*} \, dx = o(1),$$

this implies

$$\int_\Omega |\nabla u_n|^2 \, dx = \int_\Omega |u_n|^{2^*} \, dx + o(1)$$

and

$$J^\infty(u_n) = \frac{1}{N} S^{\frac{N}{2}} + o(1).$$

Similar to the method used in [24, Lemma 7], we have $\{u_n\}$ is a $(\mathrm{PS})_{\frac{1}{N} S^{\frac{N}{2}}}$-sequence for J^∞ in $H_0^1(\Omega)$. $\square$

For the positive numbers d, consider the filtration of the Nehari manifold $\mathbf{N}_0$ as follows:

$$\mathbf{N}_0(d) = \left\{ u \in \mathbf{N}_0 \;\middle|\; J_0(u) \leqslant \frac{1}{N} S^{\frac{N}{2}} + d \right\}.$$

Let $\Phi : H_0^1(\Omega) \to \mathbb{R}^N$ be a barycenter map defined by

$$\Phi(u) = \frac{\int_\Omega x|u|^{2^*} \, dx}{\int_\Omega |u|^{2^*} \, dx}. \tag{3.7}$$

Such a map has been constructed in Bartsch and Weth [5], Coron [10], Cerami and Passaseo [11] and Clapp and Weth [12], etc. Then we have the following result.

Lemma 3.3. *For each positive number $\delta < r_0$, there exists $d_\delta > 0$ such that*

$$\Phi(u) \in M_\delta \quad \text{for all } u \in \mathbf{N}_0(d_\delta).$$

Proof. Suppose the contrary. Then we can assume that there exist a sequence $\{u_n\} \in \mathbf{N}_0$ and $\delta_0 < r_0$ such that $J_0(u_n) = \frac{1}{N} S^{\frac{N}{2}} + o(1)$ and

$$\Phi(u_n) \notin M_{\delta_0} \quad \text{for all } n. \tag{3.8}$$

Then, by Lemma 3.2, we have $\{u_n\}$ is a $(\mathrm{PS})_{\frac{1}{N} S^{\frac{N}{2}}}$-sequence for J^∞ in $H_0^1(\Omega)$. It follows from Lemma 2.2 that there exist a subsequence $\{u_n\}$ and $u_0 \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. Since Ω is a bounded domain, by the concentration-compactness principle (see Lions [18,19] or Struwe [21, Theorem 3.1]), there exist two sequences $\{x_n\} \subset \Omega$, $\{R_n\} \subset \mathbb{R}^+$, $x_0 \in \overline{\Omega}$ and a positive solution $v_0 \in D^{1,2}(\mathbb{R}^N)$ of critical problem (2.3) with $J^\infty(v_0) = \frac{1}{N} S^{\frac{N}{2}}$ such that $x_n \to x_0$ and $R_n \to \infty$ as $n \to \infty$ and

$$\left\| u_n(x) - R_n^{\frac{N-2}{2}} v_0\big(R_n(x - x_n)\big) \right\|_{D^{1,2}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

Then

$$\begin{aligned}\Phi(u_n) &= \frac{\int_\Omega x|u_n|^{2^*}\,dx}{\int_\Omega |u_n|^{2^*}\,dx} = \frac{\int_\Omega x|R_n^{\frac{N-2}{2}} v_0(R_n(x-x_n))|^{2^*}\,dx}{\int_\Omega |R_n^{\frac{N-2}{2}} v_0(R_n(x-x_n))|^{2^*}\,dx} + o(1)\\ &= \frac{\int_\Omega (\frac{x}{R_n}+x_n)|v_0(x)|^{2^*}\,dx}{\int_\Omega |v_0(x)|^{2^*}\,dx} + o(1)\\ &= x_0 + o(1).\end{aligned}$$

Now we will show that $x_0 \in M_{\delta_0}$. Since

$$\begin{aligned}\int_\Omega g|u_n|^{2^*}\,dx &= \int_\Omega g(x)\left|R_n^{\frac{N-2}{2}} v_0\bigl(R_n(x-x_n)\bigr)\right|^{2^*}dx + o(1)\\ &= \int_\Omega g\left(\frac{x}{R_n}+x_n\right)\left|v_0(x)\right|^{2^*}dx + o(1)\\ &= g(x_0)S^{\frac{N}{2}} + o(1),\end{aligned}$$

which implies $g(x_0) = \max_{x\in\overline{\Omega}} g(x) \equiv 1$, and so

$$x_0 \in \left\{x\in\overline{\Omega} \;\middle|\; g(x) = \max_{x\in\overline{\Omega}} g(x) = 1\right\} = M, \tag{3.10}$$

which is a contradiction. This completes the proof. □

Now, we consider the filtration of the manifold $\mathbf{N}_\lambda^-$ as follows:

$$N_\lambda(c) = \left\{u \in \mathbf{N}_\lambda^- \;\middle|\; J_\lambda(u) \leqslant c\right\}.$$

Let $\overline{w}_{\varepsilon,z} = [N(N-2)\varepsilon^2]^{-(N-2)/4} w_{\varepsilon,j}$. Then we have the following results.

Lemma 3.4. *Let $\Lambda_2 > 0$ as in Theorem 2.5 and let $\varepsilon = \lambda^{2/(2-q)(N-2)}$. Then there exists $0 < \Lambda_* \leqslant \Lambda_2$ such that for $\lambda < \Lambda_*$,*

$$\sup_{t\geqslant 0} J_\lambda(t\overline{w}_{\varepsilon,z}) < c_\lambda = \frac{1}{N}S^{\frac{N}{2}} - \lambda^{2/(2-q)}D_0 \quad \text{uniformly in } z \in M,$$

where $\lambda^{2/(2-q)}D_0$ as in (2.6). Furthermore, there exists $t_z^- > 0$ such that $t_z^-\overline{w}_{\varepsilon,z} \in N_\lambda(c_\lambda)$ and $\Phi(t_z^-\overline{w}_{\varepsilon,z}) \in M_\delta$ for all $z \in M$.

Proof. By (3.4) and $\int_{\mathbb{R}^N} |v_\varepsilon|^{2^*}\,dx = S^{\frac{N}{2}} > 0$ for all $\varepsilon > 0$, we have

$$0 \leqslant 1 - S^{-\frac{N}{2}}\int_\Omega g|w_{\varepsilon,z}|^{2^*}\,dx \leqslant \left(C_1 + \frac{C_2}{\varepsilon^2}\right)S^{-\frac{N}{2}}\bigl[N(N-2)\varepsilon^2\bigr]^{N/2} \quad \text{for all } z \in M, \tag{3.11}$$

that is

$$1 - \left(C_1 + \frac{C_2}{\varepsilon^2}\right)S^{-\frac{N}{2}}\bigl[N(N-2)\varepsilon^2\bigr]^{N/2} \leqslant S^{-\frac{N}{2}}\int_\Omega g|w_{\varepsilon,z}|^{2^*}\,dx \leqslant 1 \quad \text{for all } z \in M. \tag{3.12}$$

Since $\varepsilon = \lambda^{2/(2-q)(N-2)}$ and $N \geqslant 3$, there exists a positive number Λ_3 such that

$$0 < 1 - \left(C_1 + \frac{C_2}{\varepsilon^2}\right)S^{-\frac{N}{2}}\bigl[N(N-2)\varepsilon^2\bigr]^{N/2} < 1 \quad \text{for all } \lambda \in (0, \Lambda_3),$$

then from (3.12) we can deduce that

$$\begin{aligned}1 - \left(C_1 + \frac{C_2}{\varepsilon^2}\right)S^{-\frac{N}{2}}\bigl[N(N-2)\varepsilon^2\bigr]^{N/2} &< \left(1 - \left(C_1 + \frac{C_2}{\varepsilon^2}\right)S^{-\frac{N}{2}}\bigl[N(N-2)\varepsilon^2\bigr]^{N/2}\right)^{2/2^*}\\ &\leqslant \left(S^{-\frac{N}{2}}\int_\Omega g|w_{\varepsilon,z}|^{2^*}\,dx\right)^{2/2^*} \leqslant 1 \quad \text{for all } z \in M,\end{aligned}$$

which yields that

$$S^{\frac{N-2}{2}} - \left(C_1 + \frac{C_2}{\varepsilon^2}\right) S^{-1}\left[N(N-2)\varepsilon^2\right]^{N/2} < \left(\int_\Omega g|w_{\varepsilon,z}|^{2^*}\,dx\right)^{2/2^*} \leqslant S^{\frac{N-2}{2}} \quad \text{for all } z \in M,$$

which implies that

$$\left(\int_\Omega g|w_{\varepsilon,z}|^{2^*}\,dx\right)^{2/2^*} = S^{\frac{N-2}{2}} + O\left(\varepsilon^{N-2}\right) \quad \text{uniformly in } z \in M. \tag{3.13}$$

Combining with (3.2) and (3.13), we obtain

$$\begin{aligned}
\Psi(\overline{w}_{\varepsilon,z}) &= \frac{\|\overline{w}_{\varepsilon,z}\|_{H^1}^2}{(\int_\Omega g|\overline{w}_{\varepsilon,z}|^{2^*}\,dx)^{2/2^*}} = \frac{\|[N(N-2)\varepsilon^2]^{-(N-2)/4} w_{\varepsilon,z}\|_{H^1}^2}{(\int_\Omega g|[N(N-2)\varepsilon^2]^{-(N-2)/4} w_{\varepsilon,z}|^{2^*}\,dx)^{2/2^*}} \\
&= \frac{\|w_{\varepsilon,z}\|_{H^1}^2}{(\int_\Omega g|w_{\varepsilon,z}|^{2^*}\,dx)^{2/2^*}} = \frac{S^{N/2} + O(\varepsilon^{N-2})}{S^{\frac{N-2}{2}} + O(\varepsilon^{N-2})} \quad \text{uniformly in } z \in M.
\end{aligned}$$

Hence

$$\begin{aligned}
\Psi(\overline{w}_{\varepsilon,z}) - S &= \frac{S^{N/2} + O(\varepsilon^{N-2})}{S^{\frac{N-2}{2}} + O(\varepsilon^{N-2})} - S = \frac{O(\varepsilon^{N-2})}{S^{\frac{N-2}{2}} + O(\varepsilon^{N-2})} \\
&= O\left(\varepsilon^{N-2}\right) \quad \text{uniformly in } z \in M.
\end{aligned} \tag{3.14}$$

Using the fact

$$\max_{t \geqslant 0}\left(\frac{t^2}{2}a - \frac{t^{2^*}}{2^*}b\right) = \frac{1}{N}\left(\frac{a}{b^{2/2^*}}\right)^{N/2} \quad \text{for all } a, b > 0,$$

we can deduce that

$$\sup_{t \geqslant 0} J_0(t\overline{w}_{\varepsilon,z}) = \frac{1}{N}\left(\Psi(\overline{w}_{\varepsilon,z})\right)^{N/2}.$$

Then, by (3.14), we conclude that

$$\sup_{t \geqslant 0} J_0(t\overline{w}_{\varepsilon,z}) = \frac{1}{N} S^{N/2} + O\left(\varepsilon^{N-2}\right) \quad \text{uniformly in } z \in M. \tag{3.15}$$

Now, we will show that exists $0 < \Lambda_* \leqslant \Lambda_2$ such that for $\lambda < \Lambda_*$,

$$\sup_{t \geqslant 0} J_\lambda(t\overline{w}_{\varepsilon,z}) < \frac{1}{N} S^{\frac{N}{2}} - \lambda^{2/(2-q)} D_0 \quad \text{for all } z \in M.$$

Let $\Lambda_4 \leqslant \min\{\Lambda_2, \Lambda_3\}$ be a positive number such that

$$\frac{1}{N} S^{\frac{N}{2}} - \lambda^{2/(2-q)} D_0 > 0 \quad \text{for all } \lambda \in (0, \Lambda_4). \tag{3.16}$$

Since

$$J_\lambda(t\overline{w}_{\varepsilon,z}) = \frac{t^2}{2}\|\overline{w}_{\varepsilon,z}\|_{H^1}^2 - \frac{t^q}{q}\int_\Omega f_\lambda|\overline{w}_{\varepsilon,z}|^q\,dx - \frac{t^*}{2^*}\int_\Omega g|\overline{w}_{\varepsilon,z}|^{2^*}\,dx$$

and

$$\int_\Omega f_\lambda|\overline{w}_{\varepsilon,z}|^q\,dx > 0 \quad \text{and} \quad \int_\Omega g|\overline{w}_{\varepsilon,z}|^{2^*}\,dx > 0, \tag{3.17}$$

we have

$$J_\lambda(t\overline{w}_{\varepsilon,z}) < \frac{t^2}{2}\|\overline{w}_{\varepsilon,z}\|_{H^1}^2 \quad \text{for all } t \geqslant 0 \text{ and } \lambda > 0.$$

Then, by (3.16), there exists $t_0 > 0$ such that

$$\sup_{0\leqslant t\leqslant t_0} J_\lambda(t\overline{w}_{\varepsilon,z}) < \frac{1}{N}S^{\frac{N}{2}} - \lambda^{2/(2-q)}D_0 \quad \text{for all } \lambda \in (0, \Lambda_4). \tag{3.18}$$

Now, we only need to show that

$$\sup_{t\geqslant t_0} J_\lambda(t\overline{w}_{\varepsilon,z}) < \frac{1}{N}S^{\frac{N}{2}} - \lambda^{2/(2-q)}D_0 \quad \text{for all } z \in M.$$

Since

$$\sup_{t\geqslant t_0} J_\lambda(t\overline{w}_{\varepsilon,z}) = \sup_{t\geqslant t_0}\left(J_0(t\overline{w}_{\varepsilon,z}) - \frac{t^q}{q}\int_\Omega f_\lambda|\overline{w}_{\varepsilon,z}|^q\,dx\right) \leqslant \frac{1}{N}S^{N/2} + O\left(\varepsilon^{N-2}\right) - \frac{\lambda t_0^q f_{\min}}{q}\int_{B^N(z,r_0)}|\overline{w}_{\varepsilon,z}|^q\,dx, \tag{3.19}$$

where

$$f_{\min} = \min\left\{f(x)\;\middle|\; x \in \overline{M}_{r_0}\right\} > 0.$$

Let $0 < \lambda \leqslant (r_0/2)^{(2-q)(N-2)/2}$. Then we have

$$0 < \varepsilon = \lambda^{2/(2-q)(N-2)} \leqslant r_0/2$$

and

$$\begin{aligned}\int_{B^N(z,r_0/2)}|\overline{w}_{\varepsilon,z}|^q &= \int_{B^N(z,r_0/2)}\frac{1}{(\varepsilon^2+|x-z|^2)^{q(N-2)/2}}\,dx = \int_{B^N(0,r_0/2)}\frac{1}{(\varepsilon^2+|x|^2)^{q(N-2)/2}}\,dx \\ &\geqslant \int_{B^N(0,r_0)}\frac{1}{r_0^{q(N-2)}}\,dx = D_1(N,q,r_0) \quad \text{for all } z \in M.\end{aligned} \tag{3.20}$$

Combining with (3.19) and (3.20), for $\varepsilon = \lambda^{2/(2-q)(N-2)}$ and $\lambda \in (0, (r_0/2)^{(2-q)(N-2)/2})$, we obtain

$$\sup_{t\geqslant t_0} J_\lambda(t\overline{w}_{\varepsilon,z}) \leqslant \frac{1}{N}S^{N/2} + O\left(\lambda^{2/(2-q)}\right) - \frac{t_0^q f_{\min}}{q}D_1(N,q,r_0)\lambda.$$

Thus, we can choose $0 < \Lambda_* \leqslant \min\{\Lambda_4, (r_0/2)^{(2-q)(N-2)/2}\}$ such that

$$O\left(\lambda^{2/(2-q)}\right) - \frac{t_0^q f_{\min}}{q}D_1(N,q,r_0)\lambda < -\lambda^{2/(2-q)}D_0 \quad \text{for all } \lambda \in (0, \Lambda_*)$$

and so

$$\sup_{t\geqslant t_0} J_\lambda(t\overline{w}_{\varepsilon,z}) < \frac{1}{N}S^{\frac{N}{2}} - \lambda^{2/(2-q)}D_0 \quad \text{for all } \lambda \in (0, \Lambda_*). \tag{3.21}$$

By (3.18) and (3.21), we can conclude that for all $\lambda < \Lambda_*$,

$$\sup_{t\geqslant 0} J_\lambda(t\overline{w}_{\varepsilon,z}) < \frac{1}{N}S^{\frac{N}{2}} - \lambda^{2/(2-q)}D_0 \quad \text{for all } z \in M.$$

Finally, we will show that there exists $t_z^- > 0$ such that $t_z^-\overline{w}_{\varepsilon,z} \in N_\lambda(c_\lambda)$ for all $z \in M$. By Lemma 2.6 and (3.17), there exists $t_z^- > 0$ such that $t_z^-\overline{w}_{\varepsilon,z} \in \mathbf{N}_\lambda^-$ and

$$J_\lambda\left(t_z^-\overline{w}_{\varepsilon,z}\right) < c_\lambda = \frac{1}{N}S^{\frac{N}{2}} - \lambda^{2/(2-q)}D_0 \quad \text{for all } z \in M$$

and so $t_z^-\overline{w}_{\varepsilon,z} \in N_\lambda(c_\lambda)$. Moreover, by definition of $\overline{w}_{\varepsilon,z}$, we have

$$\Phi\left(t_z^-\overline{w}_{\varepsilon,z}\right) = \Phi(\overline{w}_{\varepsilon,z}) \in M_\delta \quad \text{for all } z \in M. \tag{3.22}$$

This completes the proof. □

Lemma 3.5. *Let $\delta, d_\delta > 0$ be as in Lemma 3.3. Then there exists $0 < \Lambda_\delta \leqslant \Lambda_*$ such that for $\lambda < \Lambda_\delta$ we have*

$$\Phi(u) \in M_\delta \quad \textit{for all } u \in N_\lambda(c_\lambda).$$

Proof. For $u \in N_\lambda(c_\lambda)$. By Lemma 2.7, there is a unique $t_u > 0$ such that $t_u u \in \mathbf{N}_0$ and

$$J_0(t_u u) \leqslant (1-\lambda)^{-\frac{N}{2}}\left(J_\lambda(u) + \frac{\lambda(2-q)}{2q}\left(\|f_+\|_{L^{q^*}} S^{\frac{-q}{2}}\right)^{\frac{2}{2-q}}\right)$$
$$\leqslant (1-\lambda)^{-\frac{N}{2}}\left(\frac{1}{N}S^{\frac{N}{2}} - \lambda^{2/(2-q)}D_0 + \frac{\lambda(2-q)}{2q}\left(\|f_+\|_{L^{q^*}} S^{\frac{-q}{2}}\right)^{\frac{2}{2-q}}\right).$$

Then there exists $0 < \Lambda_\delta \leqslant \Lambda_*$ such that for $\lambda < \Lambda_\delta$,

$$J_0(t_u u) \leqslant \frac{1}{N}S^{\frac{N}{2}} + d_\delta \quad \text{for all } u \in N_\lambda(c_\lambda). \tag{3.23}$$

By Lemma 3.3, we have $t_u u \in \mathbf{N}_0(d_\delta)$ and

$$\Phi(u) = \frac{\int_\Omega x|t_u u|^{2^*}\,dx}{\int_\Omega |t_u u|^{2^*}\,dx} = \Phi(t_u u) \in M_\delta \quad \text{for all } u \in N_\lambda(c_\lambda). \tag{3.24}$$

This completes the proof. □

4. Proof of Theorem 1.1

First, we establish the existence of a local minimum for J_λ on $\mathbf{N}_\lambda^+$.

Theorem 4.1. *For each $\lambda < \Lambda_1$, the functional J_λ has a minimizer u_λ^+ in $\mathbf{N}_\lambda^+$ and it satisfies:*

(i) $J_\lambda(u_\lambda^+) = \alpha_\lambda^+ = \inf_{u\in\mathbf{N}_\lambda^+} J_\lambda(u)$;
(ii) u_λ^+ *is a positive solution of Eq.* (E_λ);
(iii) $J_\lambda(u_\lambda^+) \to 0$ *as* $\lambda \to 0$.

Proof. The proof is almost the same as the in [25, Theorem 3.4]. □

We need the following proposition provides a precise description for the Palais–Smale sequence of J_λ.

Proposition 4.2. *For each $\lambda < \Lambda_\delta$, the functional J_λ satisfies Palais–Smale condition on the sublevel $N_\lambda(c_\lambda) = \{u \in \mathbf{N}_\lambda^- \mid J_\lambda(u) \leqslant c_\lambda\}$.*

Proof. The proof is almost the same as the in [15, Lemma 5.2]. □

By Lemmas 3.3, 3.5 and Proposition 4.2, we can find $\Lambda_\delta > 0$ such that J_λ satisfies Palais–Smale condition on $N_\lambda(c_\lambda)$ and $\Phi(u) \in M_\delta$ for all $u \in N_\lambda(c_\lambda)$ for all $\lambda < \Lambda_\delta$. Let $F_\varepsilon(z) = t_z^- \overline{w}_{\varepsilon,z} \in N_\lambda(c_\lambda)$ as in Lemma 3.4. Then we have the following result.

Theorem 4.3. *Let $\delta, \Lambda_\delta > 0$ be as in Lemmas* 3.3, 3.5. *Then for each $\lambda < \Lambda_\delta$, J_λ has at least $cat_{M_\delta}(M)$ critical points on $N_{\lambda,+}(c_\lambda) = \{u \in N_\lambda(c_\lambda) \mid u \geqslant 0\}$.*

Proof. By Lemma 3.4, we can assume that for any such λ and for any $z \in M$ we have

$$J_\lambda\big(F_\varepsilon(z)\big) < c_\lambda = \frac{1}{N}S^{\frac{N}{2}} - \lambda^{2/(2-q)}D_0,$$

thus $F_\varepsilon(M) \subset N_\lambda(c_\lambda)$. Moreover, by Lemma 3.5, $\Phi(N_\lambda(c_\lambda)) \subset M_\delta$. Then, by (3.22), the map $\Phi \circ F$ is homotopic to the inclusion $j : M \to M_\delta$ in M_δ, for any $\lambda < \Lambda_\delta$. Thus, by an argument similar to that the proof of Theorem 1.1 in [9], J_λ has at least $cat_{M_\delta}(M)$ critical points on $N_{\lambda,+}(c_\lambda) = \{u \in N_\lambda(c_\lambda) \mid u \geqslant 0\}$. This completes the proof. □

Now, we begin to show the proof of Theorem 1.1: By Lemma 2.3 and Theorems 4.1, 4.3, Eq. (E_λ) has at least $cat_{M_\delta}(M) + 1$ positive solutions.

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