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Existence of multiple positive solutions for nonhomogeneous elliptic problems in $\mathbb{R}^N$

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ABSTRACT

In this paper, we study the multiplicity of positive solutions for the nonhomogeneous elliptic problem: $-\Delta u+\lambda u=f(x)u^{p-1}+\mu h(x)$ in $\mathbb{R}^N$. We will show how the shape of the graph of $f(x)$ affects the number of positive solutions.

1. Introduction

In this paper, we study the multiplicity of positive solutions for the following nonhomogeneous elliptic problem:

$$\begin{cases} -\Delta u+\lambda u=f(x)u^{p-1}+\mu h(x) & \text{in } \mathbb{R}^N, \\ u>0 & \text{in } \mathbb{R}^N, \\ u\in H^1(\mathbb{R}^N), \end{cases} \tag{$E_{\lambda,\mu}$}$$

where $2<p<\frac{2N}{N-2}$ $(N\geq 3)$, the parameters $\lambda,\mu>0$, $f\in C(\mathbb{R}^N)$ and $h\in L^2(\mathbb{R}^N)\setminus\{0\}$ with $h\geq 0$.

Under the assumption $\mu\neq 0$, our equation $(E_{\lambda,\mu})$ can be regarded as a perturbation problem of the following semilinear elliptic equation:

$$\begin{cases} -\Delta u+\lambda u=f(x)u^{p-1} & \text{in } \mathbb{R}^N, \\ u>0 & \text{in } \mathbb{R}^N, \\ u\in H^1(\mathbb{R}^N). \end{cases} \tag{1}$$

It is known that the existence of positive solutions of Eq. (1) is affected by the shape of the graph of $f(x)$. This has been the focus of a great deal of research by several authors (see [1–7] etc.). Furthermore, if f is a positive constant, then Eq. (1) has a unique radially symmetric positive solution (see [8,9]).

When $f\equiv 1$, h is the exponential decay and μ small, Hirano [10] and Zhu [11] showed that Eq. $(E_{\lambda,\mu})$ admits at least two positive solutions. Generalizations of the results of [10,11] were made by Cao and Zhou [12], Jeanjean [13] and Adachi and Tanaka [14,15]. In [15] Adachi and Tanaka showed the existence of at least four positive solutions of Eq. $(E_{\lambda,\mu})$ under the following assumptions:

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(f1) $f(x) \to 1$ as $|x| \to \infty$,
(f2) $f(x) \in (0,1]$ for all $x \in \mathbb{R}^N$ and $f(x) \not\equiv 1$,
(f3) there exist $\delta > 0$ and $C > 0$ such that

$$f(x) - 1 \geq -C \exp\left(-(2+\delta)|x|\right) \quad \text{for all } x \in \mathbb{R}^N,$$

and μ sufficiently small. In [12–14], the general equations

$$\begin{cases} -\Delta u + \lambda u = g(x,u) + \mu h(x) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right) \end{cases}$$

were studied, where g satisfies some suitable conditions and $h \in H^{-1}\left(\mathbb{R}^N\right) \setminus \{0\}$ is non-negative, and the existence of at least two positive solutions when μ sufficiently small was proved.

The main purpose of this paper is to use the shape of the graph of $f(x)$ to prove the multiplicity of positive solutions for Eq. $(E_{\lambda,\mu})$. First, we consider the following assumptions:

($\overline{\text{f3}}$) there exist $\delta > 0$ and $C > 0$ such that

$$f(x) - 1 \geq -C \exp\left(-(1+\delta)|x|\right) \quad \text{for all } x \in \mathbb{R}^N;$$

(f4) there exist some points $x^1, x^2, \ldots, x^k$ in $\mathbb{R}^N$ such that $f\left(x^i\right)$ are strict maximums on $\mathbb{R}^N$ with

$$1 = f\left(x^i\right) \equiv \max\left\{f(x) \mid x \in \mathbb{R}^N\right\} \quad \text{for all } i = 1, 2, \ldots, k.$$

Then we have the following results.

Theorem 1.1. *Suppose that the function f satisfies the conditions* (f1), (f2)*and* ($\overline{\text{f3}}$). *Then for each $h \in L^2\left(\mathbb{R}^N\right) \setminus \{0\}$ with $h \geq 0$ there exists a positive number Λ_* such that for any $\lambda, \mu > 0$ with $\mu\lambda^{\frac{N}{4} - \frac{p-1}{p-2}} < \Lambda_*$, Eq. $(E_{\lambda,\mu})$ has at least four positive solutions.*

Theorem 1.2. *Suppose that the function f satisfies the conditions* (f1), (f2), ($\overline{\text{f3}}$) *and* (f4). *Then for each $h \in L^2\left(\mathbb{R}^N\right) \setminus \{0\}$ with $h \geq 0$ there exist positive numbers $\overline{\Lambda}_*$ and λ_0 such that for any $\lambda > \lambda_0$ and $\mu > 0$ with $\mu\lambda^{\frac{N}{4} - \frac{p-1}{p-2}} < \overline{\Lambda}_*$, Eq. $(E_{\lambda,\mu})$ has at least $k+3$ positive solutions.*

This paper is organized as follows. In Section 2, we prove the existence of four positive solutions. In Section 3, we prove the existence of other k positive solutions. Based on this result, we can prove Theorem 1.2.

2. Existence of four solutions

By the change of variables $\eta = \frac{1}{\sqrt{\lambda}}$, $v(x) = \eta^{2/(p-2)} u(\eta x)$, Eq. $(E_{\lambda,\mu})$ is transformed to

$$\begin{cases} -\Delta v + v = f_\eta v^{p-1} + \mu \eta^{\frac{2(p-1)}{p-2}} h_\eta & \text{in } \mathbb{R}^N, \\ v > 0 & \text{in } \mathbb{R}^N, \\ v \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{2}$$

where $f_\eta = f(\eta x)$ and $h_\eta = h(\eta x)$. Associated with Eq. (2), we consider the following minimization problem: for $\eta > 0$, $\mu \geq 0$ and $u \in H^1\left(\mathbb{R}^N\right)$ define

$$I_{\eta,\mu}(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_{\mathbb{R}^N} f_\eta u_+^p dx - \mu \eta^{\frac{2(p-1)}{p-2}} \int_{\mathbb{R}^N} h_\eta u dx;$$
$$\mathbf{M}_{\eta,\mu} = \left\{u \in H^1\left(\mathbb{R}^N\right) \setminus \{0\} \mid \left\langle I'_{\eta,\mu}(u), u\right\rangle = 0\right\};$$
$$\alpha_{\eta,\mu} = \inf\left\{I_{\eta,\mu}(u) \mid u \in \mathbf{M}_{\eta,\mu}\right\}$$

where $\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\right)^{1/2}$ is a standard norm in $H^1\left(\mathbb{R}^N\right)$ and $u_+ = \max\{0, u\}$. It is well known that the solutions of Eq. (2) are the critical points of the energy functional $I_{\eta,\mu}$ (see [16]).

Now, we study the break down of the (PS)-condition for $I_{\eta,\mu}$. First, we introduce the following elliptic equation:

$$\begin{cases} -\Delta u + u = u^{p-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right). \end{cases} \tag{3}$$

We define the energy functional $I^{\infty} : H^1\left(\mathbb{R}^N\right) \to \mathbb{R}$ as follows

$$I^{\infty}(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_{\mathbb{R}^N} u_+^p dx.$$

Using the results of Berestycki and Lions [8] and Kwong [9], Eq. (3) has a unique radially symmetric positive solution $w(x)$ such that

$$\alpha^{\infty} = I^{\infty}(w) = \inf\left\{I^{\infty}(u) \mid \text{for any positive solution } u \text{ of Eq. (3)}\right\}$$

and for any $\delta > 0$ and $x \in \mathbb{R}^N$,

$$w(x) \le C\exp\left(-(1-\delta)|x|\right) \text{ and } |\nabla w(x)| \le C\exp\left(-(1-\delta)|x|\right)$$

for some $C > 0$. Moreover, the unique solution $w(x)$ of Eq. (3) plays an important role in describing the asymptotic behavior of a (PS)-sequence for $I_{\eta,\mu}$.

Proposition 2.1. *Let $\{u_n\}$ be a (PS)-sequence in $H^1(\mathbb{R}^N)$ for $I_{\eta,\mu}$. Then there exist a subsequence $\{u_n\}$, an integer $m \in \mathbb{N}\cup\{0\}$, m sequences $\left\{x_n^1\right\}, \left\{x_n^2\right\}, \ldots, \left\{x_n^m\right\} \subset \mathbb{R}^N$ and a critical point $u_0 \in H^1(\mathbb{R}^N)$ of $I_{\eta,\mu}$ such that*

$$\begin{aligned}
&\left|x_n^i\right| \to \infty \quad \text{for } 1 \le i \le m,\\
&\left|x_n^i - x_n^j\right| \to \infty \quad \text{for } 1 \le i, j \le m \text{ and } i \ne j,\\
&u_n \rightharpoonup u_0 \quad \text{weakly in } H^1(\mathbb{R}^N),\\
&u_n = u_0 + \sum_{i=1}^m w(\cdot - x_n^i) + o(1) \quad \text{strongly in } H^1(\mathbb{R}^N),\\
&I_{\eta,\mu}(u_n) = I_{\eta,\mu}(u_0) + mI^{\infty}(w) + o(1).
\end{aligned}$$

Proof. This is a standard result. See Lions [6,7] and Benci and Cerami [17] for analogous arguments. □

2.1. Existence of a local minimum

Define

$$\psi_{\eta,\mu}(u) = \left\langle I'_{\eta,\mu}(u), u\right\rangle = \|u\|_{H^1}^2 - \int_{\mathbb{R}^N} f_\eta u_+^p dx - \mu\eta^{\frac{2(p-1)}{p-2}}\int_{\mathbb{R}^N} h_\eta u dx.$$

Clearly, for $u \in \mathbf{M}_{f_\eta,h_\eta}$, we have

$$\left\langle \psi'_{\eta,\mu}(u), u\right\rangle = \|u\|_{H^1}^2 - (p-1)\int_{\mathbb{R}^N} f_\eta u_+^p dx \tag{4}$$

$$= (2-p)\|u\|_{H^1}^2 - (1-p)\mu\eta^{\frac{2(p-1)}{p-2}}\int_{\mathbb{R}^N} h_\eta u dx. \tag{5}$$

Let

$$\Lambda_0 = \frac{p-2}{p-1}\left(\frac{S_p^{p/2}}{p-1}\right)^{\frac{1}{p-2}}\|h\|_{L^2}^{-1}.$$

Then we have the following result.

Lemma 2.2. *For each $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \Lambda_0$, we have*

$$\left\langle \psi'_{\eta,\mu}(u), u\right\rangle \ne 0 \quad \text{for all } u \in \mathbf{M}_{\eta,\mu}.$$

Proof. Our proof is almost the same as that in [18, Lemma 2.3]. Suppose the contrary. Then there exist $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \Lambda_0$ such that

$$\left\langle \psi'_{\eta,\mu}(u), u\right\rangle = 0.$$

Then, for $u \in \mathbf{M}_{\eta,\mu}$ with $\left\langle \psi'_{\eta,\mu}(u), u\right\rangle = 0$, by (5) and the Hölder and Sobolev inequalities we have

$$\|u\|_{H^1}^2 = \mu\eta^{\frac{2(p-1)}{p-2}}\frac{p-1}{p-2}\int_{\mathbb{R}^N} h_\eta u dx \le \mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\frac{p-1}{p-2}\|h\|_{L^2}\|u\|_{H^1}$$

and so

$$\|u\|_{H^1} \leq \mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} \frac{p-1}{p-2} \|h\|_{L^2}.$$

Similarly, using (4), (f2)and the Sobolev inequality we have

$$\frac{1}{p-1} \|u\|_{H^1}^2 = \int_{\mathbb{R}^N} f_\eta u_+^p dx \leq S_p^{\frac{-p}{2}} \|u\|_{H^1}^p,$$

which implies

$$\|u\|_{H^1} \geq \left(\frac{S_p^{p/2}}{p-1}\right)^{\frac{1}{p-2}} \quad \text{for all } \mu \geq 0.$$

Hence, we must have

$$\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} \geq \frac{p-2}{p-1}\left(\frac{S_p^{p/2}}{p-1}\right)^{\frac{1}{p-2}} \|h\|_{L^2}^{-1} = \Lambda_0$$

which is a contradiction. This completes the proof. □

By Lemma 2.2, we may write $\mathbf{M}_{\eta,\mu} = \mathbf{M}_{\eta,\mu}^+ \cup \mathbf{M}_{\eta,\mu}^-$, where

$$\mathbf{M}_{\eta,\mu}^+ = \left\{u \in \mathbf{M}_{\eta,\mu} \,\middle|\, \|u\|_{H^1}^2 - (p-1)\int_{\mathbb{R}^N} f_\eta |u_+|^p dx > 0\right\};$$

$$\mathbf{M}_{\eta,\mu}^- = \left\{u \in \mathbf{M}_{\eta,\mu} \,\middle|\, \|u\|_{H^1}^2 - (p-1)\int_{\mathbb{R}^N} f_\eta |u_+|^p dx < 0\right\}$$

and define

$$\alpha_{\eta,\mu}^+ = \inf_{u\in\mathbf{M}_{\eta,\mu}^+} I_{\eta,\mu}(u) \quad \text{and} \quad \alpha_{\eta,\mu}^- = \inf_{u\in\mathbf{M}_{\eta,\mu}^-} I_{\eta,\mu}(u).$$

Then we have the following result.

Theorem 2.3. *We have the following.*

(i) $\alpha_{\eta,\mu}^+ < 0$ *for all* $\eta, \mu > 0$ *with* $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \Lambda_0$.

(ii) *If* $\eta, \mu > 0$ *with* $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \frac{\Lambda_0}{2}$, *then* $\alpha_{\eta,\mu}^- > c_0$ *for some* $c_0 > 0$.

In particular, for each $\eta, \mu > 0$ *with* $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \frac{\Lambda_0}{2}$, *Eq.* (2) *has a unique positive solution* $u_{\eta,\mu}^1 \in \mathbf{M}_{\eta,\mu}^+$ *such that* $I_{\eta,\mu}\left(u_{\eta,\mu}^1\right) = \alpha_{\eta,\mu}^+ = \alpha_{\eta,\mu}$.

Proof. Our proof is almost the same as that in [14, Lemma 1.4] and [19, Theorem 3.1]. □

2.2. Existence of two solutions

First, we establish the decay estimate for solutions of Eq. (2).

Lemma 2.4. *Let* $u_0 \in H^1\left(\mathbb{R}^N\right)$ *be a positive solution of Eq.* $(E_{\lambda,\mu})$. *Then* $v_0(x) = \eta^{2/(p-2)}u_0(\eta x)$ *is a positive solution of Eq.* (2) *and*

$$v_0(x) \geq C\eta^{\frac{6-p}{2(p-2)}} \exp\left(-(1+\varepsilon)\eta|x|\right), \quad \text{for all } |x| \geq R_0 \text{ for some } C > 0. \tag{6}$$

Proof. Our proof is almost the same as that in [20,21]. □

For $c > 0$, we define

$$I_{\eta,0}^c(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_{\mathbb{R}^N} cf_\eta |u_+|^p dx;$$

$$\mathbf{M}_{\eta,0}^c = \left\{u \in H^1\left(\mathbb{R}^N\right)\setminus\{0\} \mid \left\langle\left(I_{\eta,0}^c\right)'(u), u\right\rangle = 0\right\};$$

$$\mathbf{M}_{\eta,0} = \left\{u \in H^1\left(\mathbb{R}^N\right)\setminus\{0\} \mid \left\langle I_{\eta,0}'(u), u\right\rangle = 0\right\}.$$

Note that $I_{\eta,0}=I^c_{\eta,0}$ for $c=1$, and for each $u\in\mathbf{M}^-_{\eta,\mu}$ there is a unique $t^1=t^1(u)>0$ such that $t^1u\in\mathbf{M}_{\eta,0}$. Then we have the following results.

Lemma 2.5. *Suppose that $\eta,\mu>0$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}<\frac{\Lambda_0}{2}$. Then for each $u\in\mathbf{M}^-_{\eta,\mu}$, we have the following.*

(i) *There is a unique $t^c(u)>0$ such that $t^c(u)u\in\mathbf{M}^c_{\eta,0}$ and*

$$\max_{t\geq 0} I^c_{\eta,0}(tu)=I^c_{\eta,0}\left(t^c(u)u\right)=\left(\frac{1}{2}-\frac{1}{p}\right)c^{\frac{-2}{p-2}}\left[\frac{\|u\|^p_{H^1}}{\int_{\mathbb{R}^N}f_\eta|u_+|^p\,dx}\right]^{\frac{2}{p-2}}.$$

(ii) *For $\sigma\in\left(0,\mu^{-1}\eta^{\frac{N}{2}-\frac{2(p-1)}{p-2}}\right)$,*

$$I_{\eta,\mu}(u)\geq\left(1-\sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)^{\frac{p}{p-2}}I_{\eta,0}\left(t^1u\right)-\frac{\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}$$

and

$$I_{\eta,\mu}(u)\leq\left(1+\sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)^{\frac{p}{p-2}}I_{\eta,0}\left(t^1u\right)+\frac{\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}.$$

Proof. (i) Similar to the proof of Lemma 7.1 in Wu [19].

(ii) For each $u\in\mathbf{M}^-_{\eta,\mu}$, let $c=1/\left(1-\sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)$, $t^c=t^c(u)>0$ and $t^1=t^1(u)>0$ such that $t^cu\in\mathbf{M}^c_{\eta,0}$ and $t^1u\in\mathbf{M}_{\eta,0}$. For $\sigma\in(0,1)$, we have

$$\left|\int_{\mathbb{R}^N}h_\eta t^cudx\right|\leq\eta^{-\frac{N}{2}}\left\|t^cu\right\|_{H^1}\|h\|_{L^2}\leq\frac{\sigma\eta^{-\frac{N}{2}}}{2}\left\|t^cu\right\|^2_{H^1}+\frac{\eta^{-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}.$$

Then by part (i) and $\frac{2(p-1)}{p-2}-\frac{N}{2}>0$,

$$\begin{aligned}\sup_{t\geq 0}I_{\eta,\mu}(tu)&\geq I_{\eta,\mu}\left(t^cu\right)\geq\frac{1}{2}\left\|t^cu\right\|^2_{H^1}-\frac{1}{p}\int_{\mathbb{R}^N}f_\eta\left(t^cu_+\right)dx-\mu\eta^{\frac{2(p-1)}{p-2}}\left(\frac{\sigma\eta^{-\frac{N}{2}}}{2}\left\|t^cu\right\|^2_{H^1}+\frac{\eta^{-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}\right)\\&=\frac{1}{2c}\left\|t^cu\right\|^2_{H^1}-\frac{1}{p}\int_{\mathbb{R}^N}f_\eta\left(t^cu_+\right)^pdx-\frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}\\&=\frac{1}{c}I^c_{\eta,0}\left(t^cu\right)-\frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}\\&=\left(1-\sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)^{\frac{p}{p-2}}\left(\frac{1}{2}-\frac{1}{p}\right)\left[\frac{\|u\|^p_{H^1}}{\int_{\mathbb{R}^N}f_\eta|u_+|^p\,dx}\right]^{\frac{2}{p-2}}-\frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}\\&=\left(1-\sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)^{\frac{p}{p-2}}I_{f_\eta,0}\left(t^1u\right)-\frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}.\end{aligned}$$

Moreover, by Theorem 2.3 and [20, Lemma 2.4],

$$\sup_{t\geq 0}I_{\eta,\mu}(tu)=I_{\eta,\mu}(u).$$

Thus,

$$I_{\eta,\mu}(u)\geq\left(1-\sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)^{\frac{p}{p-2}}I_{f_\eta,0}\left(t^1u\right)-\frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}.$$

Moreover,

$$I_{f_\eta,h_\eta}(tu)\leq\frac{\left(1+\sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)}{2}\|tu\|^2_{H^1}-\frac{1}{p}\int_{\mathbb{R}^N}f_\eta|tu_+|^p\,dx+\frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}$$

and so

$$I_{f_\eta,h_\eta}(u)\leq\left(1+\sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)^{\frac{p}{p-2}}I_{f_\eta,0}\left(t^1u\right)+\frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}.$$

This completes the proof. □

Let $w(x)$ be a unique radially symmetric positive solution of Eq. (3) and $e \in S^{N-1} = \left\{x \in \mathbb{R}^N \mid |x| = 1\right\}$. We denote

$$w_l(x) = w(x + le), \quad l \in (0, \infty).$$

Then we have the following results.

Lemma 2.6. *Suppose that $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}} < \frac{\Lambda_0}{2}$. Then*

(i) *there exists $t_0 > 0$ such that*

$$I_{\eta,\mu}\left(u^1_{\eta,\mu} + tw_l\right) < I_{\eta,\mu}\left(u^1_{\eta,\mu}\right) \quad \text{for all } t \geq t_0 \text{ and } e \in S^{N-1};$$

(ii) *there exists $l_1 > 0$ such that for $l > l_1$,*

$$\sup_{t\geq 0} I_{\eta,\mu}\left(u^1_{\eta,\mu} + tw_l\right) < I_{\eta,\mu}\left(u^1_{\eta,\mu}\right) + \alpha^\infty = \alpha_{\eta,\mu} + \alpha^\infty,$$

where $u^1_{\eta,\mu}$ is the local minimum in Theorem 2.3.

Proof. (i) Since $u^1_{\eta,\mu}$ is a positive solution of Eq. (2), using the fact that $\int_{\mathbb{R}^N} \nabla u_{\eta,\mu} \nabla w_l dx = -\int_{\mathbb{R}^N} w_l \Delta u_{\eta,\mu} dx$ we have

$$\begin{aligned} I_{\eta,\mu}\left(u^1_{\eta,\mu} + tw_l\right) &\leq I_{\eta,\mu}\left(u^1_{\eta,\mu}\right) + \frac{t^2}{2}\|w_l\|^2_{H^1} + t\int_{\mathbb{R}^N} f_\eta w_l \left(u^1_{\eta,\mu}\right)^{p-1} dx - \frac{t^p}{p}\int_{\mathbb{R}^N} f_\eta w_l^p dx \\ &\leq I_{\eta,\mu}\left(u^1_{\eta,\mu}\right) + \frac{t^2}{2}\|w_l\|^2_{H^1} + t\int_{\mathbb{R}^N} f_\eta w_l \left(u^1_{\eta,\mu}\right)^{p-1} dx - \frac{t^p}{p}\int_{B^N(0;1)} w^p dx. \end{aligned}$$

Since $p > 2$ and $w > 0$ in $\mathbb{R}^N$, we can choose $t_0 > 0$ large enough such that (i) holds.

(ii) Since $I_{\eta,\mu}$ is continuous in $H^1\left(\mathbb{R}^N\right)$, there exists $t_1 > 0$ such that for $l > 0$,

$$I_{\eta,\mu}\left(u^1_{\eta,\mu} + tw_l\right) < I_{\eta,\mu}\left(u^1_{\eta,\mu}\right) + \alpha^\infty \quad \text{for all } t < t_1 \text{ and } e \in S^{N-1}.$$

Using part (i) we know that for $l > 0$,

$$\sup_{t\geq t_0} I_{\eta,\mu}\left(u^1_{\eta,\mu} + tw_l\right) < I_{\eta,\mu}\left(u^1_{\eta,\mu}\right) + \alpha^\infty \quad \text{for all } e \in S^{N-1}.$$

Thus, we only need to show that there exists $l_1 > 0$ such that for $l > l_1$,

$$\sup_{t_1\leq t\leq t_0} I_{\eta,\mu}\left(u^1_{\eta,\mu} + tw_l\right) < I_{\eta,\mu}\left(u^1_{\eta,\mu}\right) + \alpha^\infty \quad \text{for all } e \in S^{N-1}.$$

By Brown and Zhang [22] and Willem [23], we know that

$$\sup_{t>0} I^\infty(tw) = I^\infty(w) = \alpha^\infty. \tag{7}$$

For $l > 0$ and $t_1 \leq t \leq t_0$,

$$\begin{aligned} &I_{\eta,\mu}\left(u^1_{\eta,\mu} + tw_l\right) \\ &\quad\leq I_{\eta,\mu}\left(u^1_\eta\right) + I^\infty(tw) + \frac{1}{p}\int_{\mathbb{R}^N}\left(1 - f_\eta\right) t^p w_l^p dx - \frac{1}{p}\int_{B^N(le;1)} f_\eta \int_0^{t_1 w_l} \left(u^1_{\eta,\mu} + s\right)^{p-1} - \left(u^1_{\eta,\mu}\right)^{p-1} - s^{p-1} ds dx \\ &\quad\leq I_{\eta,\mu}\left(u^1_{\eta,\mu}\right) + I^\infty(tw) + (I) - (II). \end{aligned}$$

Using (7), we have

$$\sup_{t_1\leq t\leq t_0} I_{\eta,\mu}\left(u^1_{\eta,\mu} + tw_l\right) \leq I_{\eta,\mu}\left(u^1_{\eta,\mu}\right) + I^\infty(w) + (I) - (II).$$

We recall the fact that for some $c > 0$

$$w(|x|)|x|^{\frac{N-1}{2}} \exp(|x|) \to c \quad \text{as } |x| \to \infty.$$

(See [1,2,9,24]). In particular, there exists a constant $C_0 > 0$ such that

$$w(x) \leq C_0 \exp(-|x|) \quad \text{for all } x \in \mathbb{R}^N.$$

Then by the Taylor expansion,

$$\begin{aligned} \int_0^{t_1 w_l} \left(u^1_{\eta,\mu} + s\right)^{p-1} - \left(u^1_{\eta,\mu}\right)^{p-1} - s^{p-1} ds &\geq \int_0^{t_1 w_l} (p-1)s^{p-2} u^1_{\eta,\mu} - \left(u^1_{\eta,\mu}\right)^{p-1} ds \\ &= \left[(t_1 w_l)^{p-2} - \left(u^1_{\eta,\mu}\right)^{p-2}\right] t_1 w_l u^1_{\eta,\mu}. \end{aligned} \tag{8}$$

Since $w_l > 0$ in $\mathbb{R}^N$, there exists a number $c_1 > 0$ such that

$$w_l \geq c_1 \text{ in } B^N(le; 1). \tag{9}$$

Since $u^1_{\eta,\mu}(x) \to 0$ as $|x| \to \infty$, it follows from the definition of w_l that for l large enough,

$$t_1 w_l > u^1_{\eta,\mu} \text{ in } B^N(le; 1).$$

Thus, there exist $c_2 > 0$ and $l_0 > 0$ such that for $l > l_0$,

$$(t_1 w_l)^{p-2} - \left(u^1_{\eta,\mu}\right)^{p-2} > c_2. \tag{10}$$

Then combining (8)–(10) we have

$$\int_{B^N(le;1)} f_\eta \int_0^{t_1 w_l} \left(u^1_{\eta,\mu} + s\right)^{p-1} - \left(u^1_{\eta,\mu}\right)^{p-1} - s^{p-1} ds dx \geq f_{\min} c_1 c_2 \int_{B^N(le;1)} u^1_{\eta,\mu} dx,$$

where $f_{\min} = \min\left\{f(x) \mid x \in \mathbb{R}^N\right\}$. By (6), we find that for any $\varepsilon > 0$, there exists $C_1 > 0$ such that

$$\int_{B^N(le;1)} \int_0^{t_1 w_l} \left(u^1_{\eta,\mu} + s\right)^{p-1} - \left(u^1_{\eta,\mu}\right)^{p-1} - s^{p-1} ds dx \geq C_1 \eta^{\frac{6-p}{2(p-2)}} \exp\left(-\eta(1+\varepsilon) l\right). \tag{11}$$

We also have from condition (f3)

$$\begin{aligned}(I) &\leq \int_{\mathbb{R}^N} C \exp\left(-\eta(1+\delta)|x|\right) C_0 \exp\left(-p|x+le|\right) \\ &\leq C' \exp\left(-\min\{\eta(1+\delta), p\} l\right).\end{aligned}$$

Since (11) holds for any $\varepsilon > 0$, choosing positive numbers δ, ε such that $\varepsilon < \delta$, we can find some l_1 large enough such that

$$(II) > (I) \quad \text{for all } l \geq l_1. \tag{12}$$

This completes the proof of (ii). $\square$

In the following, we use an idea of Adachi and Tanaka [14]. For $c \in \mathbb{R}$, we denote

$$\left[I_{\eta,\mu} \leq c\right] = \left\{u \in \mathbf{M}^-_{\eta,\mu} \mid I_{\eta,\mu}(u) \leq c\right\}.$$

We then try to show for a sufficiently small $\sigma > 0$, we have

$$\operatorname{cat}\left(\left[I_{\eta,\mu} \leq \alpha_{\eta,\mu} + \alpha^\infty - \sigma\right]\right) \geq 2.$$

Let

$$A_1 = \left\{u \in H^1\left(\mathbb{R}^N\right) \setminus \{0\} \middle| \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) > 1\right\} \cup \{0\};$$

$$A_2 = \left\{u \in H^1\left(\mathbb{R}^N\right) \setminus \{0\} \middle| \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) < 1\right\}.$$

Then we have the following results.

Lemma 2.7. *For each $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}} < \frac{\Lambda_0}{2}$, we have the following.*

(i) $H^1\left(\mathbb{R}^N\right) \setminus \mathbf{M}^-_{\eta,\mu} = A_1 \cup A_2$.
(ii) $\mathbf{M}^+_{\eta,\mu} \subset A_1$.
(iii) *There exist $t_* > 1$ and $l_2 \geq l_1$ such that $u^1_{\eta,\mu} + t_* w_l \in A_2$ for all $l \geq l_2$, where l_1 is defined as in Lemma 2.6.*
(iv) *For each $l \geq l_2$ there exists $s_l \in (0, 1)$ such that $u^1_{\eta,\mu} + s_l t_* w_l \in \mathbf{M}^-_{\eta,\mu}$ and*

$$s_l t_* = 1 + o(1) \quad \text{as } \mu \to 0.$$

(v) $\alpha^-_{\eta,\mu} < \alpha_{\eta,\mu} + \alpha^\infty$.

Proof. The proof is essentially the same as that in Tarantello [18, Lemma 3.1] (or see [20,19]). $\square$

For $l \geq l_2$, we define a map $F_l : S^{N-1} \to H^1\left(\mathbb{R}^N\right)$ by

$$F_l(e)(x) = u^1_{\eta,\mu}(x) + s_l t_* w_l(x) \quad \text{for } e \in S^{N-1}.$$

Then we have the following result.

Lemma 2.8. *There exists a sequence* $\{\sigma_l\} \subset \mathbb{R}^+$ *such that*

$$F_l\left(S^{N-1}\right) \subset \left[I_{\eta,\mu} \leq \alpha_{\eta,\mu} + \alpha^\infty - \sigma_l\right].$$

Proof. By Lemmas 2.6 and 2.7(iv), for each $l \geq l_2$ we have $u^1_{\eta,\mu} + s_l t_* w_l \in \mathbf{M}^-_{\eta,\mu}$ and

$$\sup_{t\geq 0} I_{\eta,\mu}\left(u^1_{\eta,\mu} + t w_l\right) < \alpha_{\eta,\mu} + \alpha^\infty \quad \text{uniformly in } e \in S^{N-1}$$

since $F_l\left(S^{N-1}\right)$ is compact. Thus, $I_{\eta,\mu}\left(u^1_{\eta,\mu} + s_l t_* w_l\right) \leq \alpha_{\eta,\mu} + \alpha^\infty - \sigma_l$, so that the conclusion holds. □

The following lemma is a key lemma to prove our main result.

Lemma 2.9. *There exists* $\delta_0 > 0$ *such that if* $u \in \mathbf{M}_{\eta,0}$ *and* $I_{\eta,0}(u) \leq \alpha^\infty + \delta_0$, *then*

$$\int_{\mathbb{R}^N} \frac{x}{|x|}\left(|\nabla u|^2 + u^2\right) dx \neq 0.$$

Proof. Suppose the contrary. Then there exists a sequence $\{u_n\}$ in $\mathbf{M}_{\eta,0}$ such that $I_{\eta,0}(u_n) = \alpha^\infty + o(1)$ and

$$\int_{\mathbb{R}^N} \frac{x}{|x|}\left(|\nabla u_n|^2 + u_n^2\right) dx = 0.$$

By Wang and Wu [25, Lemma 7], $\{u_n\}$ is a $(\mathrm{PS})_{\alpha^\infty}$-sequence in $H^1\left(\mathbb{R}^N\right)$ for $I_{\eta,0}$. It follows from Proposition 2.1 and Bahri and Lions [2] that there exist a subsequence $\{u_n\}$ and a sequence $\{x_n\} \subset \mathbb{R}^N$ such that $u_n \rightharpoonup 0$ weakly in $H^1\left(\mathbb{R}^N\right)$, $|x_n| \to \infty$ as $n \to \infty$ and

$$u_n(x) = w(x - x_n) + o(1) \quad \text{strongly in } H^1\left(\mathbb{R}^N\right).$$

Assume $\frac{x_n}{|x_n|} \to e$ as $n \to \infty$, where $e \in S^{N-1}$. Then by the Lebesgue dominated theorem, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \frac{x}{|x|}\left(|\nabla u_n|^2 + u_n^2\right) dx = \int_{\mathbb{R}^N} \frac{x + x_n}{|x + x_n|}\left(|\nabla w|^2 + w^2\right) dx + o(1) \\ &= \left(\frac{2p}{p-2}\right) e\alpha^\infty + o(1), \end{aligned}$$

which is a contradiction. □

Lemma 2.10. *There exists a positive number* $\Lambda_1 \leq \frac{\Lambda_0}{2}$ *such that for any* $\eta, \mu > 0$ *with* $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}} < \Lambda_1$, *we have*

$$\int_{\mathbb{R}^N} \frac{x}{|x|}\left(|\nabla u|^2 + u^2\right) dx \neq 0$$

for all $u \in \left[I_{\eta,\mu} < \alpha_{\eta,\mu} + \alpha^\infty\right]$.

Proof. For $u \in \left[I_{\eta,\mu} < \alpha_{\eta,\mu} + \alpha^\infty\right]$, there exists $t^1 > 0$ such that $t^1 u \in \mathbf{M}_{\eta,0}$. By Lemma 2.5(ii), we have for any $\sigma \in \left(0, \mu^{-1}\eta^{\frac{N}{2} - \frac{2(p-1)}{p-2}}\right)$

$$I_{\eta,0}\left(t^1 u\right) \leq \left(1 - \sigma\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}}\right)^{-\frac{p}{p-2}} \left(I_{\eta,\mu}(u) + \frac{\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}\right). \tag{13}$$

Since $\alpha_{\eta,\mu} < 0$, we have $\left[I_{\eta,\mu} < \alpha_{\eta,\mu} + \alpha^\infty\right] \subset \left[I_{\eta,\mu} < \alpha^\infty\right]$. Thus, by (13),

$$I_{\eta,0}\left(t^1 u\right) \leq \left(1 - \sigma\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}}\right)^{-\frac{p}{p-2}} \left(\alpha^\infty + \frac{\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}\right)$$

for all $u \in \left[I_{\eta,\mu} < \alpha_{\eta,\mu} + \alpha^\infty\right]$. Since $\sigma \in \left(0, \mu^{-1}\eta^{\frac{N}{2} - \frac{2(p-1)}{p-2}}\right)$ is arbitrary, we have for each $\delta_0 > 0$ there exists a positive number $\Lambda_1 \leq \frac{\Lambda_0}{2}$ such that for any $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}} < \Lambda_1$, we have

$$I_{f_\eta,0}\left(t^1 u\right) < \alpha^\infty + \delta_0. \tag{14}$$

Since $t^1 u \in \mathbf{M}_{\eta,0}$ and $t^1 > 0$, by Lemma 2.9 and (14),

$$\int_{\mathbb{R}^N} \frac{x}{|x|} \left(\left|\nabla \left(t^1 u\right)\right|^2 + \left(t^1 u\right)^2 \right) dx \neq 0,$$

which implies

$$\int_{\mathbb{R}^N} \frac{x}{|x|} \left(|\nabla u|^2 + u^2 \right) dx \neq 0$$

for all $u \in \left[I_{\eta,\mu} < \alpha_{\eta,\mu} + \alpha^\infty\right]$. □

From Lemma 2.10, we define

$$G : \left[I_{\eta,\mu} < \alpha_{\eta,\mu} + \alpha^\infty\right] \to S^{N-1}$$

by

$$G(u) = \int_{\mathbb{R}^N} \frac{x}{|x|} \left(|\nabla u|^2 + |u|^2 \right) dx \Big/ \left| \int_{\mathbb{R}^N} \frac{x}{|x|} \left(|\nabla u|^2 + |u|^2 \right) dx \right|.$$

Then we have the following results.

Lemma 2.11. *There exist positive numbers $\Lambda_2 \leq \Lambda_1$ and $l_3 > l_2$ such that for any $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}} < \Lambda_2$ and $l \in (l_3, \infty)$, the map*

$$G \circ F_l : S^{N-1} \to S^{N-1}$$

is homotopic to the identity.

Proof. Let $\Theta = \left\{ u \in H^1\left(\mathbb{R}^N\right) \setminus \{0\} \mid \int_{\mathbb{R}^N} \frac{x}{|x|} \left(|\nabla u|^2 + |u|^2 \right) dx \neq 0 \right\}$. By Lemma 2.7(iv), for $\theta \in [0, 1/2)$

$$(1 - 2\theta) F_l(e) + 2\theta w(x + le) = w(x + le) + o(1) \quad \text{in } H^1\left(\mathbb{R}^N\right) \text{ as } \|h\|_{L^2} \to 0.$$

By an argument similar to that in Lemma 2.9, there exist positive numbers $\Lambda_2 \leq \Lambda_1$ and $l_3 > l_2$ such that for any $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}} < \Lambda_2$ and $l \in (l_3, \infty)$

$$(1 - 2\theta) F_l(e) + 2\theta w(x + le) \in \Theta \quad \text{for all } e \in S^{N-1} \text{ and } \theta \in [0, 1/2)$$

and

$$w\left(x + \frac{l}{2(1-\theta)} e\right) \in \Theta \quad \text{for all } e \in S^{N-1} \text{ and } \theta \in [1/2, 1).$$

Thus, we can define

$$\zeta_l(\theta, e) : [0, 1] \times S^{N-1} \to S^{N-1}$$

by

$$\zeta_l(\theta, e) = \begin{cases} G\left((1 - 2\theta) F_l(e) + 2\theta w(x + le)\right) & \text{for } \theta \in [0, 1/2); \\ G\left(w\left(x + \frac{l}{2(1-\theta)} e\right)\right) & \text{for } \theta \in [1/2, 1); \\ e & \text{for } \theta = 1. \end{cases}$$

Then $\zeta_l(0, e) = G(F_l(e)) = G(F_l(e))$ and $\zeta_l(1, e) = e$. First, we claim that $\lim_{\theta \to 1^-} \zeta_l(\theta, e) = e$ and $\lim_{\theta \to \frac{1}{2}^-} \zeta_l(\theta, e) = \overline{G}(w(x + le))$.

(a) $\lim_{\theta \to 1^-} \zeta_l(\theta, e) = e$: since

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{x}{|x|} \left(\left| \nabla \left[w\left(x + \frac{l}{2(1-\theta)} e\right) \right] \right|^2 + \left[w\left(x + \frac{l}{2(1-\theta)} e\right) \right]^2 \right) dx \\ &= \int_{\mathbb{R}^N} \frac{x - \frac{l}{2(1-\theta)} e}{\left| x - \frac{l}{2(1-\theta)} e \right|} \left(|\nabla [w(x)]|^2 + [w(x)]^2 \right) dx + o(1) \\ &= \left(\frac{2p}{p-2} \right) \alpha^\infty e + o(1) \quad \text{as } \theta \to 1^-, \end{aligned}$$

then $\lim_{\theta \to 1^-} \zeta_l(\theta, e) = e$.

(b) $\lim_{\theta \to \frac{1}{2}^-} \zeta_l(\theta, e) = G(w(x + le))$: since $G \in C\left(\Theta, S^{N-1}\right)$, we obtain $\lim_{\theta \to \frac{1}{2}^-} \zeta_l(\theta, l) = G(w(x + le))$.

Thus, $\zeta_l(\theta, e) \in C\left([0,1] \times S^{N-1}, S^{N-1}\right)$ and

$$\zeta_l(0, e) = G(F_l(e)) \quad \text{for all } e \in S^{N-1},$$
$$\zeta_l(1, e) = e \quad \text{for all } e \in S^{N-1},$$

provided $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}} < \Lambda_2$ and $l \in (l_3, \infty)$. This completes the proof. $\square$

Theorem 2.12. *For each $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}} < \Lambda_2$ and $l \in (l_3, \infty)$, the energy functional $I_{\eta,\mu}(u)$ has at least two critical points in*

$$\left[I_{\eta,\mu} < \alpha_{\eta,\mu} + \alpha^\infty\right].$$

Proof. Applying Lemma 2.11 and [14, Lemma 2.5], we have for each $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}} < \Lambda_2$ and $l \in (l_3, \infty)$,

$$\operatorname{cat}\left(\left[I_{\eta,\mu} \le \alpha_{\eta,\mu} + \alpha^\infty - \sigma_l\right]\right) \ge 2.$$

By Proposition 2.1, Lemma 2.7(v) and [26, Theorem 2.3] $I_{\eta,\mu}(u)$ has at least two critical points in $\left[I_{\eta,\mu} < \alpha_{\eta,\mu} + \alpha^\infty\right]$. $\square$

2.3. Proof of Theorem 1.1

Since $\alpha^-_{\eta,\mu} > 0$ for all $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}} < \Lambda_2$, we define

$$K_{\eta,\mu} = \sup_{t\ge 0} I_{\eta,\mu}(tu) = I_{\eta,\mu}\left(t^- u\right) > 0,$$

where $t^- u \in \mathbf{M}^-_{\eta,\mu}$. We observe that if $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}}$ is sufficiently small, Bahri–Li's minimax argument [1] also works for $K_{\eta,\mu}$. Let

$$\Gamma = \left\{\gamma \in C(B_r(0), \Sigma) \,|\, \gamma|_{\partial B_r(0)} = w(x-y) / \|w(x-y)\|_{H^1}\right\} \quad \text{for large } r = |y|,$$

where $\Sigma = \left\{u \in H^1\left(\mathbb{R}^N\right) | u \ge 0 \text{ and } \|u\|_{H^1} = 1\right\}$. Then we define

$$\theta_{\eta,\mu} = \inf_{\gamma\in\Gamma} \sup_{y\in\mathbb{R}^N} K_{\eta,\mu}(\gamma(y)).$$

By Lemma 2.5(ii), for $\sigma \in \left(0, \mu^{-1}\eta^{\frac{N}{2} - \frac{2(p-1)}{p-2}}\right)$, we have

$$\theta_{\eta,\mu} \ge \left(1 - \sigma\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}}\right)^{\frac{p}{p-2}} \theta_{\eta,0} - \frac{\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}}}{2\sigma} \|h\|^2_{L^2} \tag{15}$$

and

$$\theta_{\eta,\mu} \le \left(1 + \sigma\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}}\right)^{\frac{p}{p-2}} \theta_{\eta,0} + \frac{\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}}}{2\sigma} \|h\|^2_{L^2}. \tag{16}$$

Lemma 2.13. $\alpha^\infty < \theta_{\eta,0} < 2\alpha^\infty$.

Proof. Since Bahri and Li [1] (or see Bahri and Lions [2]) proved that equation

$$\begin{cases} -\Delta v + v = f_\eta v^{p-1} & \text{in } \mathbb{R}^N, \\ 0 < v \in H^1\left(\mathbb{R}^N\right) \end{cases} \tag{17}$$

admits at least one positive solution u_0 and $I_{\eta,0}(u_0) = \theta_{\eta,0} < 2\alpha^\infty$. Moreover, Eq. (17) does not have a ground state solution. Thus, $\alpha^\infty < \theta_{\eta,0} < 2\alpha^\infty$. $\square$

Theorem 2.14. *There exists a positive number $\Lambda_* \le \Lambda_2$ such that for any $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}} < \Lambda_*$, we have*

$$\alpha_{\eta,\mu} + \alpha^\infty < \theta_{\eta,\mu} < \alpha^-_{\eta,\mu} + \alpha^\infty.$$

Furthermore, there exists a positive solution $u^4_{\eta,\mu}$ of Eq. (2) such that $I_{\eta,\mu}\left(u^4_{\eta,\mu}\right) = \theta_{\eta,\mu}$.

Proof. By Lemma 2.5(ii), we also have that $\sigma \in \left(0, \mu^{-1}\eta^{\frac{N}{2} - \frac{2(p-1)}{p-2}}\right)$,

$$\alpha^-_{\eta,\mu} \ge \left(1 - \sigma\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}}\right)^{\frac{p}{p-2}} \alpha^\infty - \frac{\mu\eta^{\frac{2(p-1)}{p-2} - \frac{N}{2}}}{2\sigma} \|h\|^2_{L^2}$$

and

$$\alpha^-_{\eta,\mu} \le \left(1+\sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)^{\frac{p}{p-2}} \alpha^\infty + \frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma} \|h\|^2_{L^2}.$$

For any $\epsilon > 0$ there exists a positive number $\Lambda_{*,1} \le \Lambda_2$ such that for any $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \Lambda_{*,1}$, we have

$$\alpha^\infty - \epsilon < \alpha^-_{\eta,\mu} < \alpha^\infty + \epsilon.$$

Thus,

$$2\alpha^\infty - \epsilon < \alpha^-_{\eta,\mu} + \alpha^\infty < 2\alpha^\infty + \epsilon.$$

Applying (15) and (16), for any $\varepsilon > 0$ there exists a positive number $\Lambda_{*,2} \le \Lambda_2$ such that for any $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \Lambda_{*,2}$, we have

$$\theta_{\eta,0} - \varepsilon < \theta_{\eta,\mu} < \theta_{\eta,0} + \varepsilon.$$

Fix small numbers $0 < \delta < \left(2\alpha^\infty - \theta_{\eta,0}\right)/2$, since $\alpha^\infty < \theta_{\eta,0} < 2\alpha^\infty$, choosing an $\varepsilon > 0$ such that for any $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \Lambda_* = \min\left\{\Lambda_{*,1}, \Lambda_{*,2}\right\}$, we obtain

$$\alpha_{\eta,\mu} + \alpha^\infty < \alpha^\infty < \theta_{\eta,\mu} < 2\alpha^\infty - \epsilon < \alpha^-_{\eta,\mu} + \alpha^\infty.$$

Therefore, by Proposition 2.1, we obtain that there exists a positive solution $u^4_{\eta,\mu}$ of Eq. (2) such that $I_{\eta,\mu}\left(u^4_\eta\right) = \theta_{\eta,\mu}$. □

We can now complete the proof of Theorem 1.1. By Theorems 2.3, 2.12 and 2.14 we have that for any $\eta, \mu > 0$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \Lambda_*$, Eq. (2) has four positive solutions $u^1_{\eta,\mu}, u^2_{\eta,\mu}, u^3_{\eta,\mu}, u^4_{\eta,\mu}$ such that $I_{\eta,\mu}\left(u^1_{\eta,\mu}\right) = \alpha_{\eta,\mu} < 0$ and

$$0 < \alpha^-_{\eta,\mu} \le I_{\eta,\mu}\left(u^i_{\eta,\mu}\right) < \alpha_{\eta,\mu} + \alpha^\infty \quad \text{for } i = 2, 3;$$

$$\alpha_{\eta,\mu} + \alpha^\infty < I_{\eta,\mu}\left(u^4_{\eta,\mu}\right) < \alpha_{\eta,\mu} + 2\alpha^\infty.$$

Thus, $u^1_{\eta,\mu}, u^2_{\eta,\mu}, u^3_{\eta,\mu}$ and $u^4_{\eta,\mu}$ are different. Let $U^i_{\lambda,\mu}(x) = \lambda^{\frac{1}{p-2}} u^i_{\eta,\mu}\left(\sqrt{\lambda}x\right)$. We obtain that $U^1_{\lambda,\mu}, U^2_{\lambda,\mu}, U^3_{\lambda,\mu}$ and $U^4_{\lambda,\mu}$ are positive solutions of Eq. $(E_{\lambda,\mu})$, which implies that for any $\lambda, \mu > 0$ with $\mu\lambda^{\frac{N}{4}-\frac{p-1}{p-2}} < \Lambda_*$, Eq. $(E_{\lambda,\mu})$ has at least four positive solutions.

3. Existence of k Solutions

First, we use the graph of the coefficient f to find some Palais–Smale sequences which are used to prove Theorem 1.1. For $a > 0$, let $C_a\left(x^i\right)$ denote the hypercube $\Pi^N_{j=1}\left(x^i_j - a, x^i_j - a\right)$ centered at $x^i = \left(x^i_1, x^i_2, \ldots, x^i_N\right)$ for $i = 1, 2, \ldots, k$. Let $\overline{C_a\left(x^i\right)}$ and $\partial C_a\left(x^i\right)$ denote the closure and the boundary of $C_a\left(x^i\right)$, respectively. By the conditions (f1)–(f2), we can choose numbers $K, l > 0$ such that $C_l\left(x^i\right)$ are disjoint, $f(x) < f\left(x^i\right)$ for $x \in \partial C_l\left(x^i\right)$ for all $i = 1, 2, \ldots, k$ and $\bigcup^k_{i=1} C_l\left(x^i\right) \subset \prod^N_{i=1}(-K, K)$.

Define $\phi_\eta \in C(\mathbb{R}, \mathbb{R})$, $g_\eta \in C\left(H^1\left(\mathbb{R}^N\right), \mathbb{R}^N\right)$ by

$$\phi_\eta(t) = \begin{cases} \dfrac{2K}{\eta} & t > \dfrac{2K}{\eta}, \\ t & -\dfrac{2K}{\eta} \le t \le \dfrac{2K}{\eta}, \\ -\dfrac{2K}{\eta} & t < -\dfrac{2K}{\eta}, \end{cases}$$

$$g^j_\eta(u) = \frac{\int_{\mathbb{R}^N} \phi_\eta\left(x_j\right)|u|^p}{\int_{\mathbb{R}^N}|u|^p} \quad \text{for } j = 1, 2, \ldots, N$$

and

$$g_\eta(u) = \left(g^1_\eta(u), g^2_\eta(u), \ldots, g^N_\eta(u)\right).$$

Let $C^i_{l/\eta} \equiv C_{l/\eta}\left(\frac{x^i}{\eta}\right)$,

$$N^i_{\eta,\mu} = \left\{u \in \mathbf{M}^-_{\eta,\mu} \mid u \ge 0 \text{ and } g_\eta(u) \in C^i_{l/\eta}\right\},$$

$$\partial N^i_{\eta,\mu} = \left\{u \in \mathbf{M}^-_{\eta,\mu} \mid u \ge 0 \text{ and } g_\eta(u) \in \partial C^i_{l/\eta}\right\}$$

for $i = 1, 2, \ldots, k$. It is easy to verify that $N^i_{\eta,\mu}$ and $\partial N^i_{\eta,\mu}$ are non-empty sets for all $i = 1, 2, \ldots, k$. For $i = 1, 2, \ldots, k$, consider the minimization problems in $N^i_{\eta,\mu}$ and $\partial N^i_{\eta,\mu}$ for $I_{\eta,\mu}$,

$$\gamma^i_{\eta,\mu} = \inf_{u\in N^i_{\eta,\mu}} I_{\eta,\mu}(u) \quad \text{and} \quad \widetilde{\gamma}^i_{\eta,\mu} = \inf_{u\in \partial N^i_{\eta,\mu}} I_{\eta,\mu}(u).$$

Then we have the following result, whose proof can be found in Cao and Noussair [4].

Theorem 3.1. *There exists a positive number $\eta_0 = \eta_0(f, N, p)$ such that if $\eta < \eta_0$, then*

$$\alpha^\infty < \gamma^i_{\eta,0} < \min\left\{2\alpha^\infty, \widetilde{\gamma}^i_{\eta,0}\right\} \quad \textit{for all } i = 1, 2, \ldots, k.$$

By Lemma 2.5(ii), for each $u \in \mathbf{M}^-_{\eta,\mu}$ there is a unique $t_u > 0$ such that $t_u u \in \mathbf{M}^-_{\eta,0}$ and for $\sigma \in \left(0, \mu^{-1}\eta^{\frac{N}{2}-\frac{2(p-1)}{p-2}}\right)$,

$$I_{\eta,\mu}(u) \geq \left(1 - \sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)^{\frac{p}{p-2}} I_{\eta,0}(t_u u) - \frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2} \tag{18}$$

and

$$I_{\eta,\mu}(u) \leq \left(1 + \sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)^{\frac{p}{p-2}} I_{\eta,0}(t_u u) + \frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}. \tag{19}$$

Furthermore, if $u \in N^i_{\eta,\mu}$ (or $\partial N^i_{\eta,\mu}$), then $t_u u \in N^i_{\eta,0}$ (or $\partial N^i_{\eta,0}$).

Let $\eta_0 > 0$ be as in Theorem 3.1. Then for any $\eta < \eta_0$ we have the following results.

Lemma 3.2. *Let $\Lambda_* > 0$ be as in Theorem 3.1. Then for each $\delta > 0$ there exists a positive number $\overline{\Lambda} \leq \Lambda_*$ such that for any $\mu > 0$ and $\eta \in (0, \eta_0)$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \overline{\Lambda}$, we have*

$$\widetilde{\gamma}^i_{\eta,0} - \delta < \widetilde{\gamma}^i_{\eta,\mu} < \widetilde{\gamma}^i_{\eta,0} + \delta.$$

Proof. By (18) and (19), we have that for $\sigma \in \left(0, \mu^{-1}\eta^{\frac{N}{2}-\frac{2(p-1)}{p-2}}\right)$

$$\widetilde{\gamma}^i_{\eta,\mu} \geq \left(1 - \sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)^{\frac{p}{p-2}} \widetilde{\gamma}^i_{\eta,0} - \frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}$$

and

$$\widetilde{\gamma}^i_{\eta,\mu} \leq \left(1 + \sigma\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}\right)^{\frac{p}{p-2}} \widetilde{\gamma}^i_{\eta,0} + \frac{\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}}{2\sigma}\|h\|^2_{L^2}.$$

Then for each $\delta > 0$ there exists a positive number $\overline{\Lambda} \leq \Lambda_*$ such that for any $\mu > 0$ and $\eta \in (0, \eta_0)$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \overline{\Lambda}$, we have

$$\widetilde{\gamma}^i_{\eta,0} - \delta < \widetilde{\gamma}^i_{\eta,\mu} < \widetilde{\gamma}^i_{\eta,0} + \delta.$$

This completes the proof. □

Lemma 3.3. *Let $\overline{\Lambda} > 0$ be as in Lemma 3.2. Then there exists a positive number $\overline{\Lambda}_* \leq \overline{\Lambda}$ such that for any $\mu > 0$ and $\eta \in (0, \eta_0)$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \overline{\Lambda}_*$, we have*

$$\alpha_{\eta,\mu} + \alpha^\infty < \gamma^i_{\eta,\mu} < \min\left\{\alpha^-_{\eta,\mu} + \alpha^\infty, \widetilde{\gamma}^i_{\eta,\mu}\right\} \quad \textit{for all } i = 1, 2, \ldots, k.$$

Proof. Similar to the argument in the proof of Lemma 3.2. For each $\epsilon > 0$ there exists a positive number $\widetilde{\Lambda}_0 \leq \overline{\Lambda}$ such that if $\mu > 0$ and $\eta \in (0, \eta_0)$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \widetilde{\Lambda}_0$, then

$$\alpha^\infty - \epsilon < \alpha^-_{\eta,\mu} < \alpha^\infty + \epsilon.$$

Thus,

$$2\alpha^\infty - \epsilon < \alpha^-_{\eta,\mu} + \alpha^\infty < 2\alpha^\infty + \epsilon.$$

Applying (18) and (19), for any $\varepsilon > 0$ there exists a positive number $\widetilde{\Lambda}_i \leq d_6$ such that if $\mu > 0$ and $\eta \in (0, \eta_0)$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}} < \widetilde{\Lambda}_i$, then

$$\gamma^i_{\eta,0} - \varepsilon < \gamma^i_{\eta,\mu} < \gamma^i_{\eta,0} + \varepsilon \quad \text{for } i = 1, 2, , \ldots, k.$$

Fix small numbers

$$0<\delta<\min_{1\leq i\leq k}\left\{\widetilde{\gamma}_{\eta,0}^{i}-\gamma_{\eta,0}^{i}\right\}\Big/2$$

and

$$0<\epsilon<\min_{1\leq i\leq k}\left\{\min\left\{2\alpha^{\infty},\widetilde{\gamma}_{\eta,0}^{i}\right\}-\gamma_{\eta,0}^{i}\right\}\Big/2,$$

since

$$\alpha^{\infty}<\gamma_{\eta,0}^{i}<\min\left\{2\alpha^{\infty},\widetilde{\gamma}_{\eta,0}^{i}\right\}\quad\text{for all } i=1,2,\ldots,k,$$

choosing an $\varepsilon>0$ such that for $\mu>0$ and $\eta\in(0,\eta_0)$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}<\overline{\Lambda}_*=\min_{0\leq j\leq k}\left\{\widetilde{\Lambda}_j\right\}$, we obtain

$$\alpha_{\eta,\mu}+\alpha^{\infty}<\alpha^{\infty}<\gamma_{\eta,\mu}^{i}<2\alpha^{\infty}-\epsilon<\alpha_{\eta,\mu}^{-}+\alpha^{\infty}$$

and

$$\gamma_{\eta,\mu}^{i}<\widetilde{\gamma}_{\eta,0}^{i}-\delta<\widetilde{\gamma}_{\eta,\mu}^{i},$$

which implies

$$\alpha_{\eta,\mu}+\alpha^{\infty}<\gamma_{\eta,\mu}^{i}<\min\left\{\alpha_{\eta,\mu}^{-}+\alpha^{\infty},\widetilde{\gamma}_{\eta,\mu}^{i}\right\}$$

for all $i=1,2,\ldots,k$. This completes the proof. □

Furthermore, by the Ekeland variational principle [27], we may prove the following result (or see [4]).

Proposition 3.4. *For any $\mu>0$ and $\eta\in(0,\eta_0)$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}<\overline{\Lambda}_*$, there exists a sequence $\{u_n\}\subset N_{\eta,\mu}^{i}$ such that*

$$I_{\eta,\mu}(u_n)=\gamma_{\eta,\mu}^{i}+o(1)\quad\text{and}\quad I_{\eta,\mu}'(u_n)=o(1)\quad\text{in } H^{-1}\left(\mathbb{R}^N\right)$$

for all $i=1,2,\ldots,k$.

Theorem 3.5. *For any $\mu>0$ and $\eta\in(0,\eta_0)$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}<\overline{\Lambda}_*$, Eq. (2) has k positive solutions $v_{\eta,\mu}^{1},v_{\eta,\mu}^{2},\ldots,v_{\eta,\mu}^{k}$ such that $v_{\eta,\mu}^{i}\in N_{\eta,\mu}^{i}$ and $I_{\eta,\mu}\left(v_{\eta,\mu}^{i}\right)=\gamma_{\eta,\mu}^{i}$ for all $i=1,2,\ldots,k$.*

Proof. Fix an $i\in\{1,2,\ldots,k\}$. Assume that $\left\{u_n^i\right\}\subset N_{\eta,\mu}^{i}$ is a sequence satisfying

$$I_{\eta,\mu}(u_n)=\gamma_{\eta,\mu}^{i}+o(1)\quad\text{and}\quad I_{\eta,\mu}'(u_n)=o(1)\quad\text{in } H^{-1}\left(\mathbb{R}^N\right).$$

Since $\alpha_{\eta,\mu}+\alpha^{\infty}<\gamma_{\eta,\mu}^{i}<\alpha_{\eta,\mu}^{-}+\alpha^{\infty}$. By Proposition 2.1, $I_{\eta,\mu}$ satisfies the $(\mathrm{PS})_{\beta}$-condition for $\beta=\gamma_{\eta,\mu}^{i}$ and there exists a critical point $v_{\eta,\mu}^{i}\in\overline{N}_{\eta,\mu}^{i}$ corresponding to $\gamma_{\eta,\mu}^{i}$ such that $v_{\eta,\mu}^{i}$ is a non-zero solution of Eq. (2). Moreover, by the maximum principle, $v_{\eta,\mu}^{i}>0$ in $\mathbb{R}^N$, since $g_{\eta}^{i}\left(v_{\eta,\mu}^{i}\right)\in\overline{C_{l/\eta}^{i}}$. Therefore, $v_{\eta,\mu}^{i}$ are different. □

We can now complete the proof of Theorem 1.2. By Theorems 2.3, 2.12 and 3.5 we have that for any $\mu>0$ and $\eta\in(0,\eta_0)$ with $\mu\eta^{\frac{2(p-1)}{p-2}-\frac{N}{2}}<\overline{\Lambda}_*$, Eq. (2) has $k+3$ positive solutions $u_{\eta,\mu}^{1},u_{\eta,\mu}^{2},u_{\eta,\mu}^{3},v_{\eta,\mu}^{1},v_{\eta,\mu}^{2},\ldots,v_{\eta,\mu}^{k}$ such that $I_{\eta,\mu}\left(u_{\eta}^{1}\right)=\alpha_{\eta,\mu}<0$ and

$$0<\alpha_{\eta,\mu}^{-}\leq I_{\eta,\mu}\left(u_{\eta,\mu}^{i}\right)<\alpha_{\eta,\mu}+\alpha^{\infty}\quad\text{for } i=2,3;$$

$$\alpha_{\eta,\mu}+\alpha^{\infty}<I_{\eta,\mu}\left(v_{\eta,\mu}^{i}\right)<\alpha_{\eta,\mu}+2\alpha^{\infty}\quad\text{for } i=1,2,\ldots,k.$$

Thus, $u_{\eta,\mu}^{1},u_{\eta,\mu}^{2},u_{\eta,\mu}^{3},v_{\eta,\mu}^{1},\ldots,v_{\eta,\mu}^{k}$ are different. Letting $\lambda_0=\eta_0^{-2}$, $U_{\lambda,\mu}^{i}(x)=\lambda^{\frac{1}{p-2}}u_{\eta,\mu}^{i}\left(\sqrt{\lambda}x\right)$ and $V_{\lambda,\mu}^{i}(x)=\lambda^{\frac{1}{p-2}}v_{\eta,\mu}^{i}\left(\sqrt{\lambda}x\right)$, we obtain that $U_{\lambda,\mu}^{i}$ and $V_{\lambda,\mu}^{i}$ are positive solutions of Eq. $(E_{\lambda,\mu})$, which implies that for any $\lambda>\lambda_0$ and $\mu>0$ with $\mu\lambda^{\frac{N}{4}-\frac{p-1}{p-2}}<\overline{\Lambda}_*$, Eq. $(E_{\lambda,\mu})$ has at least $k+3$ positive solutions.

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