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Koszul duality of affine Kac–Moody algebras and cyclotomic rational double affine Hecke algebras ☆

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ABSTRACT

We give a proof of the parabolic/singular Koszul duality for the category O of affine Kac–Moody algebras. The main new tool is a relation between moment graphs and finite codimensional affine Schubert varieties. We apply this duality to q-Schur algebras and to cyclotomic rational double affine Hecke algebras. This yields a proof of a conjecture of Chuang–Miyachi relating the level-rank duality with the Ringel–Koszul duality of cyclotomic rational double affine Hecke algebras.

Contents

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1. Introduction

The purpose of this paper is to give a proof of the parabolic/singular Koszul duality for the category O of affine Kac–Moody algebras. The main motivation for this is the conjecture in [40] (proved in [36]) relating the parabolic affine category O and the category O of cyclotomic rational double affine Hecke algebras (CRDAHA for short). Using the present work, we deduce from this conjecture the main conjecture of [9] which claims that the category O of CRDAHA's is Koszul and that the Koszul equivalence is related to the level-rank duality on the Fock space.

There are several possible approaches to Koszul duality for affine Kac–Moody algebras. In [6], a geometric analogue of the composition of the Koszul and the Ringel duality is given, which involves Whittaker sheaves on the affine flag variety. Our principal motivation comes from representation theory of CRDAHA's. For this, we need to prove a Koszul duality for the category O itself rather than for its geometric analogues.

One difficulty of the Kac–Moody case comes from the fact that, at a positive level, the category O has no tilting modules, while at a negative level it has no projective modules. One way to overcome this is to use a different category of modules than the usual category O. For instance, one may use the Whittaker category as in [6], or a category of linear complexes as in [31]. To remedy this, we use a truncated version of the (affine parabolic) category O, more precisely we consider a Serre subcategory of O at negative levels and a Serre quotient of O at positive levels such that both have a finite number of simple objects. Note that this terminology may differ from the usual definition of a truncated category. Under truncation any singular block of an affine parabolic category O at a non-critical level yields a finite highest-weight category which contains both tilting and projective objects. We prove that these highest weight categories are Koszul and are Koszul dual to each other.

Note that the affine category O is related to two different types of geometry. In negative level it is related to the affine flag ind-scheme and to finite dimensional affine Schubert varieties. In positive level it is related to Kashiwara's affine flag manifold and to finite codimensional affine Schubert varieties. In negative level, a localization theorem (from the category O to perverse sheaves on the affine flag ind-scheme) has been worked out by Beilinson and Drinfeld in [4], and Frenkel and Gaitsgory in [17]. A difficulty in the proof of the main theorem comes from the absence of a localization theorem (from the category O to perverse sheaves on Kashiwara's affine flag manifold) at the positive level. To overcome this we use standard Koszul duality and Ringel duality to relate the positive and negative level.

Our general argument is similar to the one in [3,5]. For this, we need an affine analogue of the Soergel functor on the category O. We use the functor introduced by Fiebig in [13]. To define it, we must introduce the deformed category O, which is a highest weight category over a localization of a polynomial ring, and some category of sheaves over a moment graph.

By the work of Fiebig, sheaves over moment graphs give an algebraic analogue of equivariant perverse sheaves associated with finite dimensional affine Schubert varieties. An important new tool in our work is a relation between sheaves over some moment graph and equivariant perverse sheaves associated with finite codimensional affine Schubert varieties, see Appendix A. This relation is of independent interest.

Let us now explain the structure of the paper. Section 2 contains generalities on highest weight categories and standard Koszul duality. In Section 3 we introduce the affine parabolic category O and its truncated version. Section 4 contains some generalities on moment graphs and the relation with the deformed affine category O. The Section 5 is technical and contains the proof of the main theorem. Next, we apply the Koszul duality to CRDAHA and q-Schur algebras in Section 6.

The Kazhdan–Lusztig equivalence [27] implies that the module category of the q-Schur algebra is equivalent to a highest weight subcategory of the affine category O of GL_n at a negative level. Thus, our result implies that the q-Schur algebra is Morita equivalent to a Koszul algebra (and also to a standard Koszul algebra), see Section 6.6.[1] To our knowledge, this was not proved so far. There are different possible approaches for proving that the q-Schur algebra is Koszul. Some are completely algebraic, see e.g., [33]. Some

[1] After our paper was written, we received a copy of [9] where a similar result is obtained by different methods.

use analogues of the Bezrukavnikov–Mirkovic modular localization theorem, see e.g., [34]. Our approach has the advantage that it yields an explicit description of the Koszul dual of the q-Schur algebra.

Finally, we apply the Koszul duality of the category O to CRDAHA. More precisely, in [40] some higher analogue of the q-Schur algebra has been introduced. It is a highest-weight subcategory of the affine parabolic category O. Since the category is standard Koszul, these higher q-Schur algebras are also Koszul and their are Koszul dual to each other. Next, it was conjectured in [40] (and proved in [36]) that these higher q-Schur algebras are equivalent to the category O of the CRDAHA. Thus, our result also implies that the CRDAHA's are Koszul. Using this, we prove the level-rank conjecture for CRDAHA's in [9].

2. Preliminaries on Koszul rings and highest weight categories

2.1. Categories

For an object M of a category $\mathbf{C}$ let $\mathbf{1}_M$ be the identity endomorphism of M. Let $\mathbf{C}^{\mathrm{op}}$ be the category opposite to $\mathbf{C}$.

A functor of additive categories is always assumed to be additive. If $\mathbf{C}$ is an exact category then $\mathbf{C}^{\mathrm{op}}$ is equipped with the exact structure such that $0 \to M' \to M \to M'' \to 0$ is exact in $\mathbf{C}^{\mathrm{op}}$ if and only if $0 \to M'' \to M \to M' \to 0$ is exact in $\mathbf{C}$. An *exact functor* of exact categories is a functor which takes short exact sequences to short exact sequences. Unless specified otherwise, a functor will always be a covariant functor. A contravariant functor $F : \mathbf{C} \to \mathbf{C}'$ is exact if and only if the functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}'$ is exact.

Given an abelian category $\mathbf{C}$, let $\mathrm{Irr}(\mathbf{C})$ be the set of isomorphism classes of simple objects. Let $\mathrm{Proj}(\mathbf{C})$ and $\mathrm{Inj}(\mathbf{C})$ be the sets of isomorphism classes of indecomposable projective, injective objects respectively. For an object M of $\mathbf{C}$ we abbreviate $\mathrm{Ext}_{\mathbf{C}}(M) = \mathrm{Ext}_{\mathbf{C}}(M, M)$, where $\mathrm{Ext}_{\mathbf{C}}$ stands for the direct sum of all $\mathrm{Ext}^i_{\mathbf{C}}$'s.

Let R be a commutative, Noetherian, integral domain. An *R-category* is an additive category enriched over the tensor category of R-modules. Unless mentioned otherwise, a functor of R-categories is always assumed to be R-linear. A *hom-finite R-category* is an R-category whose Hom spaces are finitely generated over R.

An additive category is *Krull–Schmidt* if any object has a decomposition such that each summand is indecomposable with local endomorphism ring. A full additive subcategory of a Krull–Schmidt category is again Krull–Schmidt if every idempotent splits (i.e., if it is closed under direct summands). A hom-finite k-category over a field k is Krull–Schmidt if every idempotent splits (e.g., if it is abelian).

We call *finite abelian k-category* a k-category which is equivalent to the category of finite dimensional modules over a finite dimensional k-algebra. It is Krull–Schmidt.

For any abelian category $\mathbf{C}$, let $\mathbf{D}^b(\mathbf{C})$ be the corresponding bounded derived category.

2.2. Graded rings

For a ring A let A^{op} be the opposite ring. Let A-**Mod** be the category of left A-modules and let A-**mod** be the subcategory of the finitely generated ones. We abbreviate $\mathrm{Irr}(A) = \mathrm{Irr}(A\text{-}\mathbf{mod})$, $\mathrm{Proj}(A) = \mathrm{Proj}(A\text{-}\mathbf{mod})$ and $\mathrm{Inj}(A) = \mathrm{Inj}(A\text{-}\mathbf{mod})$.

By a graded ring $\bar{A}$ we'll always mean a $\mathbb{Z}$-graded ring. Let $\bar{A}$-**gmod** be the category of finitely generated graded $\bar{A}$-modules. We abbreviate $\mathbf{D}^b(\bar{A}) = \mathbf{D}^b(\bar{A}\text{-}\mathbf{gmod})$, $\mathrm{Irr}(\bar{A}) = \mathrm{Irr}(\bar{A}\text{-}\mathbf{gmod})$, $\mathrm{Inj}(\bar{A}) = \mathrm{Inj}(\bar{A}\text{-}\mathbf{gmod})$ and $\mathrm{Proj}(\bar{A}) = \mathrm{Proj}(\bar{A}\text{-}\mathbf{gmod})$. Given a graded $\bar{A}$-module M and an integer j, let $M\langle j\rangle$ be the graded $\bar{A}$-module obtained from M by shifting the grading by j, i.e., such that $(M\langle j\rangle)^i = M^{i-j}$. Given $M, N \in \bar{A}$-**gmod** let $\hom_{\bar{A}}(M, N)$ and $\mathrm{ext}_{\bar{A}}(M, N)$ be the morphisms and extensions in the category of graded modules.

We say that the ring $\bar{A}$ is *positively graded* if $\bar{A}^{<0} = 0$ and if $\bar{A}^0$ is a semisimple ring. A finite dimensional graded algebra $\bar{A}$ over a field k is positively graded if $\bar{A}^{<0} = 0$ and $\bar{A}^0$ is semisimple as an $\bar{A}$-module. Here $\bar{A}^0$ is identified with $\bar{A}/\bar{A}^{>0}$.

A graded module M is called *pure of weight* i if it is concentrated in degree $-i$, i.e., $M = M^{-i}$. Suppose $\bar{A}$ is a positively graded ring. Then any simple graded module is pure, and any pure graded module is semisimple.

Assume that k is a field and that $\bar{A}$ is a positively graded k-algebra. We say that $\bar{A}$ is *basic* if $\bar{A}^0$ isomorphic to a finite product of copies of k as a k-algebra.

Assume that $\bar{A}$ is finite dimensional. Let $\{1_x;\ x \in \mathrm{Irr}(\bar{A}^0)\}$ be a complete system of primitive orthogonal idempotents of $\bar{A}^0$. The *Hilbert polynomial* of $\bar{A}$ is the matrix $P(\bar{A}, t)$ with entries $\mathbb{N}[t]$ given by $P(\bar{A}, t)_{x,x'} = \sum_i t^i \dim(1_x \bar{A}^i 1_{x'})$ for each x, x'. Assume further that $\bar{A}$ is positively graded. We have canonical bijections $\mathrm{Irr}(A) = \mathrm{Irr}(\bar{A}^0) = \{\bar{A}^0 1_x\}$ such that x is the isomorphism class of $\bar{A}^0 1_x$. Since A-**mod** is Krull–Schmidt, there is a canonical bijection $\mathrm{top} : \mathrm{Proj}(A) \to \mathrm{Irr}(A)$, $P \mapsto \mathrm{top}(P)$. We have $\mathrm{top}(A 1_x) = \bar{A}^0 1_x = x$. The set $\{1_x;\ x \in \mathrm{Irr}(\bar{A}^0)\}$ is a complete system of primitive orthogonal idempotents of A.

Given a graded commutative, Noetherian, integral domain R, a *graded R-category* is an additive category enriched over the monoidal category of graded R-modules.

2.3. Koszul duality

Let k be a field and $\bar{A}$ be a graded k-algebra which is positively graded (in the sense of Section 2.2). Let A be the (non-graded) k-algebra underlying $\bar{A}$.

The *Koszul dual* of $\bar{A}$ is the graded k-algebra $E(\bar{A}) = \mathrm{Ext}_A(\bar{A}^0)$. Forgetting the grading of $E(\bar{A})$, we get a k-algebra $E(A)$. It is finite dimensional if A is finite dimensional and has finite global dimension.

A *linear projective resolution* of a graded $\bar{A}$-module M is a graded projective resolution $\ldots \to P_1 \to P_0 \to M \to 0$ such that, for each i, the projective graded $\bar{A}$-module P_i is finitely generated by a set of homogeneous elements of degree i.

We say that $\bar{A}$ is *Koszul* if each simple graded module which is pure of weight 0 has a linear projective resolution. If $\bar{A}$ is Koszul, then we say that A *has a Koszul grading*. If $\bar{A}$

is finite dimensional then this grading is unique up to isomorphism of graded k-algebras, see [5, cor. 2.5.2].

Assume that $\bar{A}$ is finite dimensional, has finite global dimension and is Koszul. Then $E(\bar{A})$ is Koszul and $E^2(\bar{A}) = \bar{A}$ canonically, see [5, thm. 1.2.5]. Put $\bar{A}^! = E(\bar{A})^{\mathrm{op}}$ and $A^! = E(A)^{\mathrm{op}}$. Note that $\bar{A}^!$ is also Koszul by [5, prop. 2.2.1].

For each $x \in \mathrm{Irr}(A) = \mathrm{Irr}(\bar{A}^0)$ the idempotent $1_x \in \bar{A}^0$ yields an idempotent $1^!_x = E(1_x) \in E(A)$ in the obvious way. The set $\{1^!_x;\ x \in \mathrm{Irr}(A)\}$ is a complete system of primitive orthogonal idempotents of $E(A)$.

By [5, thm. 2.12.5, 2.12.6], there is an equivalence of triangulated categories $E : \mathbf{D}^b(\bar{A}) \to \mathbf{D}^b(\bar{A}^!)$ such that $E(M\langle i\rangle) = E(M)[-i]\langle -i\rangle$ and $E(\bar{A}^0 1_x) = \bar{A}^! 1^!_x$ for each $x \in \mathrm{Irr}(A)$. We'll call it the *Koszul equivalence*. By forgetting the grading, we get a bijection $\mathrm{Irr}(A) \to \mathrm{Proj}(A^!)$, $x \mapsto A^! 1^!_x$. It induces a bijection $(\bullet)^! = \mathrm{top} \circ E : \mathrm{Irr}(A) \to \mathrm{Irr}(A^!)$, which will be called the *natural bijection* between $\mathrm{Irr}(A)$ and $\mathrm{Irr}(A^!)$.

Let $\mathbf{C}$ be a finite abelian k-category. We say that $\mathbf{C}$ *has a Koszul grading* if there is a projective generator P such that the ring $A = \mathrm{End}_{\mathbf{C}}(P)^{\mathrm{op}}$ has a Koszul grading $\bar{A}$. We may also simply say that $\mathbf{C}$ *is Koszul* and we abbreviate $\mathbf{C}^! = A^!$-**mod**.

The following lemmas are well-known.

Lemma 2.1. *Let P, A, $\mathbf{C}$ be as above. If $\bar{A}$ is a positively graded then $E(\bar{A}) = \mathrm{Ext}_{\mathbf{C}}(L)$, where $L = \mathrm{top}(P)$.*

Proof. The equivalence $\mathbf{C} \to A$-**mod**, $X \mapsto \mathrm{Hom}_{\mathbf{C}}(P, X)$ takes L to $\bar{A}/\bar{A}^{>0}$. Therefore $E(\bar{A}) = \mathrm{Ext}_{\mathbf{C}}(L)$. □

Lemma 2.2. *Let $\mathbf{C}$ be a finite abelian k-category with a Koszul grading. If $\mathbf{D}$ is a Serre subcategory and the inclusion $\mathbf{D} \subset \mathbf{C}$ induces injections on extensions, then $\mathbf{D}$ has also a Koszul grading.*

Proof. Since $\mathbf{C}$ is a Krull–Schmidt category, any simple object admits a projective cover. See Section 2.1 for details. Let P_L be the projective cover of $L \in \mathrm{Irr}(\mathbf{C})$. The set $\mathrm{Irr}(\mathbf{C})$ is finite and $P = \bigoplus_L P_L$ is a minimal projective generator. Set $A = \mathrm{End}_{\mathbf{C}}(P)^{\mathrm{op}}$.

Let A_I be the quotient of A by the two-sided ideal I generated by $\{\mathbf{1}_{P_L};\ L \notin \mathrm{Irr}(\mathbf{D})\}$. The pull-back by the ring homomorphism $A \to A_I$ identifies A_I-**mod** with the full subcategory of A-**mod** consisting of the modules killed by I.

An object $M \in \mathbf{C}$ belongs to $\mathbf{D}$ if and only if $\mathrm{Hom}_{\mathbf{C}}(P_L, M) = 0$ whenever $L \notin \mathrm{Irr}(\mathbf{D})$. Therefore, the equivalence $\mathrm{Hom}_{\mathbf{C}}(P, \bullet) : \mathbf{C} \to A$-**mod** identifies $\mathbf{D}$ with the full subcategory of A-**mod** consisting of the modules killed by I. We deduce that $\mathbf{D}$ is equivalent to A_I-**mod**.

Now, let $\bar{A}$ be the Koszul grading on A. Since $\bar{A}$ is positively graded, the idempotent $\mathbf{1}_{P_L}$ has degree 0. Thus I is a homogeneous ideal of $\bar{A}$. Hence $\bar{A}$ yields a grading $\bar{A}_I$ on A_I such that $\bar{A}_I^{<0} = 0$ and $\bar{A}_I^0$ is semi-simple as an $\bar{A}_I$-module. Thus $\bar{A}_I$ is also positively graded.

Fix a graded lift $\bar{L}$ of $L \in \mathrm{Irr}(A)$, see Section 2.6. The graded $\bar{A}$-module $\bar{L}$ is pure. Let d_L be its degree. By [5, prop. 2.1.3] we have $\mathrm{ext}^i_{\bar{A}}(\bar{L}, \bar{L}') = 0$ unless $i = d_{L'} - d_L$ for $L, L' \in \mathrm{Irr}(\mathbf{C})$. We must check that $\mathrm{ext}^i_{\bar{A}_I}(\bar{L}, \bar{L}') = 0$ unless $i = d_{L'} - d_L$ if $L, L' \in \mathrm{Irr}(\mathbf{D})$. This is obvious because the inclusion $\bar{A}_I$-**gmod** $\subset \bar{A}$-**gmod** induces injections on extensions. □

2.4. Highest weight categories

Let R be a commutative, Noetherian ring with 1 which is a local ring with residue field k.

Let $\mathbf{C}$ be an R-category which is equivalent to the category of finitely generated modules over a finite projective R-algebra A.

The category $\mathbf{C}$ is a *highest weight R-category* if it is equipped with a poset of isomorphism classes of objects $(\Delta(\mathbf{C}), \leqslant)$ called the standard objects satisfying the following conditions:

- the objects of $\Delta(\mathbf{C})$ are projective over R
- given $M \in \mathbf{C}$ such that $\mathrm{Hom}_{\mathbf{C}}(D, M) = 0$ for all $D \in \Delta(\mathbf{C})$, we have $M = 0$
- given $D \in \Delta(\mathbf{C})$, there is a projective object $P \in \mathbf{C}$ and a surjection $f : P \twoheadrightarrow D$ such that $\ker f$ has a (finite) filtration whose successive quotients are objects $D' \in \Delta$ with $D' > D$
- given $D \in \Delta$, we have $\mathrm{End}_{\mathbf{C}}(D) = R$
- given $D_1, D_2 \in \Delta$ with $\mathrm{Hom}_{\mathbf{C}}(D_1, D_2) \neq 0$, we have $D_1 \leq D_2$.

See [35, def. 4.11]. Note that since R is local, any finitely generated projective R-module is free, hence the set $\tilde{\Delta}$ in [35, def. 4.11] is the set of finite direct sums of objects in Δ. The partial order $\leqslant$ is called the *highest weight order* on $\mathbf{C}$. We write $\Delta(\mathbf{C}) = \{\Delta(\lambda)\}_{\lambda \in \Lambda}$ for Λ an indexing poset.

Lemma 2.3. *Let $\mathbf{C}$ be a highest weight R-category. Given $\lambda \in \Lambda$, there is a unique (up to isomorphism) indecomposable projective (resp. injective, tilting, costandard) object associate with λ, denoted by $P(\lambda)$ (rep. $I(\lambda)$, $T(\lambda)$, $\nabla(\lambda)$) such that*

(∇) $\mathrm{Hom}_{\mathbf{C}}(\Delta(\mu), \nabla(\lambda)) \simeq \delta_{\lambda\mu} R$ *and* $\mathrm{Ext}^1_{\mathbf{C}}(\Delta(\mu), \nabla(\lambda)) = 0$ *for all $\mu \in \Lambda$,*

(P) *there is a surjection $f : P(\lambda) \twoheadrightarrow \Delta(\lambda)$ such that* $\ker f$ *has a filtration whose successive quotients are $\Delta(\mu)$'s with $\mu > \lambda$,*

(I) *there is an injection $f : \nabla(\lambda) \hookrightarrow I(\lambda)$ such that* $\mathrm{coker} f$ *has a filtration whose successive quotients are $\nabla(\mu)$'s with $\mu > \lambda$,*

(T) *there is an injection $f : \Delta(\lambda) \hookrightarrow T(\lambda)$ and a surjection $g : T(\lambda) \twoheadrightarrow \nabla(\lambda)$ such that* $\mathrm{coker} f$ *(resp.* $\ker g$*) has a filtration whose successive quotients are $\Delta(\mu)$'s (resp. $\nabla(\mu)$'s) with $\mu < \lambda$.*

See e.g. [36, prop. 2.1]. The objects $\nabla(\lambda)$, $\Delta(\lambda)$, $P(\lambda)$, $I(\lambda)$ and $T(\lambda)$ are projective over R. We have $\mathrm{Proj}(\mathbf{C}) = \{P(\lambda)\}_{\lambda\in\Lambda}$, $\mathrm{Inj}(\mathbf{C}) = \{I(\lambda)\}_{\lambda\in\Lambda}$. The set $\mathrm{Tilt}(\mathbf{C}) = \{T(\lambda)\}_{\lambda\in\Lambda}$ is the set of isomorphism classes of indecomposable tilting objects in $\mathbf{C}$. Let $\nabla(\mathbf{C}) = \{\nabla(\lambda)\}_{\lambda\in\Lambda}$. Note that $\Delta(\lambda)$ has a unique simple quotient $L(\lambda)$. The set of isomorphism classes of simple objects in $\mathbf{C}$ is given by $\mathrm{Irr}(\mathbf{C}) = \{L(\lambda)\}_{\lambda\in\Lambda}$.

Let $\mathbf{C}^\Delta$, $\mathbf{C}^\nabla$ be the full subcategories of $\mathbf{C}$ consisting of the Δ-*filtered* and ∇-*filtered* objects, i.e., the objects having a finite filtration whose successive quotients are standard, costandard respectively. These categories are exact. Recall that a tilting object is by definition an object that is both Δ-filtered and ∇-filtered.

The opposite of $\mathbf{C}$ is a highest weight R-category such that $\Delta(\mathbf{C}^{\mathrm{op}}) = \nabla(\mathbf{C})$ with the opposite highest weight order.

Given a commutative local R-algebra S and an R-module M, we write $SM = M\otimes_R S$. Let $S\mathbf{C} = SA$-**mod**. We have the following, see e.g. [36, props. 2.1, 2.4, 2.5].

Proposition 2.4. *Let $\mathbf{C}$ be a highest weight R-category, and let S be a commutative local R-algebra with 1. For any $M, N \in \mathbf{C}$ the following holds:*

(a) if S is R-flat then $S\,\mathrm{Ext}^d_{\mathbf{C}}(M,N) = \mathrm{Ext}^d_{S\mathbf{C}}(SM,SN)$ for all $d \geqslant 0$,

(b) if either $M \in \mathbf{C}$ is projective or ($M \in \mathbf{C}^\Delta$ and $N \in \mathbf{C}^\nabla$), then we have $S\,\mathrm{Hom}_{\mathbf{C}}(M,N) = \mathrm{Hom}_{S\mathbf{C}}(SM,SN)$,

(c) if M is R-projective then M is projective in $\mathbf{C}$ (resp. M is tilting in $\mathbf{C}$, $M \in \mathbf{C}^\Delta$) if and only if kM is projective in $k\mathbf{C}$ (resp. kM is tilting in $k\mathbf{C}$, $kM \in k\mathbf{C}^\Delta$),

(d) if either (M is projective in $\mathbf{C}$ and N is R-projective) or ($M \in \mathbf{C}^\Delta$ and $N \in \mathbf{C}^\nabla$) then $\mathrm{Hom}_{\mathbf{C}}(M,N)$ is R-projective,

(e) the category $S\mathbf{C}$ is a highest weight S-category on the poset Λ with standard objects $S\Delta(\lambda)$ and costandard objects $S\nabla(\lambda)$. The projective, injective and tilting objects associated with λ are $SP(\lambda)$, $SI(\lambda)$ and $ST(\lambda)$. □

Remark 2.5. For any subset $\Sigma \subset \Lambda$, let $\mathbf{C}[\Sigma]$ be the Serre subcategory generated by all the $L(\lambda)$ with $\lambda \in \Sigma$ and let $\mathbf{C}(\Sigma)$ be the Serre quotient $\mathbf{C}/\mathbf{C}[\mathrm{Irr}(\mathbf{C}) \setminus \Sigma]$.

An *ideal* in the poset Λ is a subset of the form $I = \bigcup_{i\in I}\{\leqslant i\}$.

A *coideal* is the complement of an ideal, i.e., a subset of the form $J = \bigcup_{j\in J}\{\geqslant j\}$.

Now, assume that $\mathbf{C}$ is a highest weight category over a field k, and that I, J are respectively an ideal and a coideal of Λ. Then $\mathbf{C}[I]$, $\mathbf{C}(J)$ are highest weight categories and the inclusion $\mathbf{C}[I] \subset \mathbf{C}$ induces injections on extensions by [10, thm. 3.9], [11, prop. A.3.3].

2.5. Ringel duality

Let R be a commutative, Noetherian ring with 1 which is a local ring with residue field k.

Let $\mathbf{C}$ be a highest-weight R-category which is equivalent to A-**mod** for a finite projective R-algebra A.

We call $T = \bigoplus_{\lambda \in \Lambda} T(\lambda)$ the *characteristic tilting module*. Set $D(A) = \mathrm{End}_{\mathbf{C}}(T)$ and $A^\diamond = D(A)^{\mathrm{op}}$. The *Ringel dual* of A is the R-algebra $A^\diamond$, the Ringel dual of $\mathbf{C}$ is the category $\mathbf{C}^\diamond = A^\diamond$-**mod**.

The category $\mathbf{C}^\diamond$ is a highest-weight R-category on the poset Λ^{op}. We have an equivalence of triangulated categories $(\bullet)^\diamond : \mathbf{D}^b(\mathbf{C}) \to \mathbf{D}^b(\mathbf{C}^\diamond)$ called the *Ringel equivalence*. It restricts to an equivalence of exact categories $(\bullet)^\diamond : \mathbf{C}^\Delta \to (\mathbf{C}^\diamond)^\nabla$ such that $M \mapsto \mathrm{RHom}_{\mathbf{C}}(M, T)^*$. Here $(\bullet)^*$ is the dual as a k-vector space. We have $\Delta(\lambda)^\diamond = \nabla^\diamond(\lambda)$, $P(\lambda)^\diamond = T^\diamond(\lambda)$ and $T(\lambda)^\diamond = I^\diamond(\lambda)$, for each $\lambda \in \Lambda$, see [35, prop. 4.26].

The ring $(A^\diamond)^\diamond$ is Morita equivalent to A, see [35, prop. 4.26] and [11, sec. A.4].

Now, assume that $R = k$. For each primitive idempotent $e \in A$, there is a unique $\lambda \in \Lambda$ such that $Ae = P(\lambda)$. We define $e^\diamond \in A^\diamond$ to be the primitive idempotent such that $A^\diamond e^\diamond = P^\diamond(\lambda)$. The bijection $(\bullet)^\diamond : \mathrm{Irr}(\mathbf{C}) \to \mathrm{Irr}(\mathbf{C}^\diamond)$ given by $L(\lambda) \mapsto L^\diamond(\lambda)$ is called the *natural bijection* between $\mathrm{Irr}(\mathbf{C})$ and $\mathrm{Irr}(\mathbf{C}^\diamond)$.

The following is well-known, see, e.g., [11, prop. A.4.9].

Lemma 2.6. *A subset $I \subset \Lambda$ is an ideal if and only if it is a coideal in Λ^{op}. We have $\mathbf{C}[I]^\diamond = \mathbf{C}^\diamond(I)$, and the Ringel equivalence factors to an equivalence of categories $\mathbf{C}[I]^\Delta \to \mathbf{C}^\diamond(I)^\nabla$.*

2.6. Standard Koszul duality

Let k be a field and $\mathbf{C}$ be a highest weight k-category. Assume that $\mathbf{C}$ is equivalent to the category of finitely generated left modules over a finite dimensional k-algebra A.

Let $\bar{A}$ be a graded k-algebra which is isomorphic to A as a k-algebra. We call $\bar{A}$ a *graded lift* of A. A *graded lift* of an object $M \in \mathbf{C}$ is a graded $\bar{A}$-module which is isomorphic to M as an A-module.

Assume that $\bar{A}$ is positively graded (in the sense of Section 2.2). We have the following

Proposition 2.7. *Given $\lambda \in \Lambda$ there exists unique graded lifts $\bar{L}(\lambda)$, $\bar{P}(\lambda)$, $\bar{I}(\lambda)$, $\bar{\Delta}(\lambda)$, $\bar{\nabla}(\lambda)$, $\bar{T}(\lambda)$ such that*

- *$\bar{L}(\lambda)$ is pure of degree zero,*
- *the surjection $\bar{P}(\lambda) \twoheadrightarrow \bar{L}(\lambda)$ is homogeneous of degree zero,*
- *the injection $\bar{L}(\lambda) \hookrightarrow \bar{I}(\lambda)$ is homogeneous of degree zero,*
- *the surjection $f : \bar{P}(\lambda) \twoheadrightarrow \bar{\Delta}(\lambda)$ in (P) is homogeneous of degree zero,*
- *the injection $f : \bar{\nabla}(\lambda) \hookrightarrow \bar{I}(\lambda)$ in (I) is homogeneous of degree zero,*
- *both the injection $f : \bar{\Delta}(\lambda) \hookrightarrow \bar{T}(\lambda)$ and the surjection $g : \bar{T}(\lambda) \twoheadrightarrow \bar{\nabla}(\lambda)$ in (T) are homogeneous of degree zero.*

Proof. The existence of the graded lifts is proved in [30, cor. 4,5]. They are unique up to isomorphisms because, by [5, lem. 2.5.3], the graded lift of an indecomposable object of $k\mathbf{C}$ is unique up to a graded $\bar{A}$-module isomorphism and up to a shift of the grading. □

The gradings above will be called the *natural gradings*. In particular, let $\bar{T} = \bigoplus_{\lambda\in\Lambda}\bar{T}(\lambda)$. The grading on $\bar{T}$ induces a grading on the k-algebra $A^\diamond$ given by $\bar{A}^\diamond = \mathrm{End}_{\mathbf{C}}(\bar{T})^{\mathrm{op}}$. It is called the *natural grading*.

A chain complex of projective (resp. injective, tilting) modules $\ldots \to M_i \to M_{i-1} \to \ldots$ is called *linear* provided that for every $i \in \mathbb{Z}$ all indecomposable direct summands of $M_i\langle -i\rangle$ have the natural grading.

Following [2], we say that $\bar{A}$ is *standard Koszul* provided that all standard modules have linear projective resolutions and all costandard modules have linear injective coresolutions. By [2, thm. 1], a standard Koszul graded algebra is Koszul.

We will identify $\Lambda = \{L(\lambda)\}_{\lambda\in\Lambda} = \{1_x;\ x \in \mathrm{Irr}(A)\}$ and equip the latter with the partial order $\leqslant$. Assume that $\bar{A}$ is Koszul. By [2, thm. 3], the graded k-algebra $\bar{A}$ is standard Koszul if and only if $\bar{A}^!$ is quasi-hereditary relatively to the poset $\{1^!_x;\ x \in \mathrm{Irr}(A)\}$ such that $1^!_x \leqslant 1^!_y \Leftrightarrow 1_x \geqslant 1_y$.

Following [29], we say that $\bar{A}$ is *balanced* provided that all standard modules have linear tilting coresolutions and all costandard modules have linear tilting resolutions. By [30, thm. 7], the graded k-algebra $\bar{A}$ is balanced if and only if it is standard Koszul and if the graded k-algebra $D(\bar{A})$ is positive.

If $\bar{A}$ is balanced, then the following hold [29, thm. 1]:

- $\bar{A}$, $\bar{A}^\diamond$, $\bar{A}^!$ and $(\bar{A}^!)^\diamond$ are positively graded, quasi-hereditary, Koszul, standard Koszul and balanced,
- $(\bar{A}^!)^\diamond = (\bar{A}^\diamond)^!$ as graded quasi-hereditary k-algebras,
- the natural bijection $\mathrm{Irr}(A) \to \mathrm{Irr}(A^!)$ takes the highest weight order on $\mathrm{Irr}(A)$ to the opposite of the highest weight order on $\mathrm{Irr}(A^!)$.

Note that the notation $E(\bar{A})$ in [29] corresponds to our notation $\bar{A}^!$.

We'll say that $\mathbf{C}$ *is standard Koszul or balanced* if we can choose the graded k-algebra $\bar{A}$ in such a way that it is standard Koszul or balanced respectively.

Remark 2.8. Assume that the graded k-algebra $\bar{A}$ is standard Koszul. In particular, it is finite dimensional, Koszul and with finite global dimension. The Koszul equivalence is given by $E = \mathrm{RHom}_A(\bar{A}^0, \bullet)$, up to the grading. See [5, sec. 2], [37, sec. 3] for details. It takes costandard (resp. injective, simple) modules to standard (resp. simple, projective) ones, by [2, prop. 2.7], [5, thm. 2.12.5].

Remark 2.9. Assume that $\bar{A}$, $\bar{A}^\diamond$ are both positively graded (in the sense of Section 2.2). The functor $\mathrm{Hom}_{\mathbf{C}}(\bullet, \bar{T})^*$ takes the natural graded indecomposable tilting objects in $\bar{A}$-**gmod** to the natural graded indecomposable projective ones in $\bar{A}^\diamond$-**gmod**. It takes also the natural graded indecomposable injective objects in $\bar{A}$-**gmod** to natural graded indecomposable tilting ones in $\bar{A}^\diamond$-**gmod**.

Let $\mathbf{C}$ be a highest weight category over a field k and $I \subset \mathrm{Irr}(\mathbf{C})$ be an ideal. Put $J = \mathrm{Irr}(\mathbf{C}) \setminus I$. Assume that $\mathbf{C}$ is standard Koszul.

The category $\mathbf{C}[I]$ has a Koszul grading by Lemma 2.2 and Remark 2.5. The natural bijection $\mathrm{Irr}(\mathbf{C}) \to \mathrm{Irr}(\mathbf{C}^!)$ is an anti-isomorphism of posets. Thus, the images $I^!$, $J^!$ of I, J are respectively a coideal and an ideal of $\mathrm{Irr}(\mathbf{C}^!)$.

Lemma 2.10. *For each ideal $I \subset \mathrm{Irr}(\mathbf{C})$ we have $\mathbf{C}[I]^! = \mathbf{C}^!(I^!)$.*

Proof. Set $L_I = \bigoplus_{L \in I} L$. By Lemma 2.1 and Remark 2.5, we have $\mathbf{C}^! = \mathrm{Ext}_{\mathbf{C}}(L)$-**mod** and $\mathbf{C}[I]^! = \mathrm{Ext}_{\mathbf{C}}(L_I)$-**mod**. Let $e \in \mathrm{End}_{\mathbf{C}}(L)$ be the projection from L to L_I. We have $\mathrm{Ext}_{\mathbf{C}}(L_I) = e\, \mathrm{Ext}_{\mathbf{C}}(L)\, e$.

The full subcategory $K_I \subset \mathbf{C}^!$ consisting of the modules killed by e is a Serre subcategory and there is an equivalence $\mathbf{C}^!/K_I \to \mathbf{C}[I]^!$, $M \mapsto eM$. In other words, the restriction with respect to the obvious inclusion $\mathrm{Ext}_{\mathbf{C}}(L_I) \subset \mathrm{Ext}_{\mathbf{C}}(L)$, yields a quotient functor $\mathbf{C}^! \to \mathbf{C}[I]^!$ whose kernel is K_I.

Since $\mathbf{C}$ is standard Koszul, its Koszul dual $\mathbf{C}^!$ is a highest weight category. We have $\mathbf{C}^!(I^!) = \mathbf{C}^!/\mathbf{C}^![J^!]$, because $J^! = \mathrm{Irr}(\mathbf{C}^!) \setminus I^!$. Since $\mathbf{C}^![J^!]$ is the Serre subcategory of $\mathbf{C}^!$ generated by $J^!$, it consists of the $\mathrm{Ext}_{\mathbf{C}}(L)$-modules killed by e. This implies the lemma. □

3. Affine Lie algebras and the parabolic category O

3.1. Lie algebras

Let $\mathfrak{g}$ be a simple Lie $\mathbb{C}$-algebra and let G be a connected simple group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Let $T \subset G$ be a maximal tori and let $\mathfrak{t} \subset \mathfrak{g}$ be its Lie algebra. Let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra containing $\mathfrak{t}$.

The elements of $\mathfrak{t}$, $\mathfrak{t}^*$ are called *coweights* and *weights* respectively. Given a root $\alpha \in \mathfrak{t}^*$ let $\check{\alpha} \in \mathfrak{t}$ denote the corresponding coroot. Let $\Pi \subset \mathfrak{t}^*$ be the set of roots of $\mathfrak{g}$, $\Pi^+ \subset \Pi$ the set of roots of $\mathfrak{b}$, and $\mathbb{Z}\Pi$ be the root lattice. Let ρ be half the sum of the positive roots. Let $\Phi = \{\alpha_i;\ i \in I\}$ be the set of simple roots in Π^+. Let W be the Weyl group. Let N be the dual Coxeter number of $\mathfrak{g}$.

3.2. Affine Lie algebras

Let $\mathbf{g}$ be the affine Lie algebra associated with $\mathfrak{g}$. Recall that $\mathbf{g} = \mathbb{C}\partial \oplus \widehat{L\mathfrak{g}}$, where $\widehat{L\mathfrak{g}}$ is a central extension of $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ and $\partial = t\partial_t$ is a derivation of $\widehat{L\mathfrak{g}}$ acting trivially on the canonical central element $\mathbf{1}$ of $\widehat{L\mathfrak{g}}$. Consider the Lie subalgebras $\mathbf{b} = \mathfrak{b} \oplus \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathbb{C}\partial \oplus \mathbb{C}\mathbf{1}$ and $\mathbf{t} = \mathbb{C}\partial \oplus \mathfrak{t} \oplus \mathbb{C}\mathbf{1}$.

Let $\widehat{\Pi}$, $\widehat{\Pi}^+$ be the sets of roots of $\mathbf{g}$, $\mathbf{b}$ respectively. We'll call an element of $\widehat{\Pi}$ an *affine root*. The set of simple roots in $\widehat{\Pi}^+$ is $\widehat{\Phi} = \{\alpha_i;\ i \in \{0\} \cup I\}$.

The elements of $\mathbf{t}$, $\mathbf{t}^*$ are called *affine coweights* and *affine weights* respectively. Let $(\bullet : \bullet) : \mathbf{t}^* \times \mathbf{t} \to \mathbb{C}$ be the canonical pairing. Let δ, Λ_0, $\hat{\rho}$ be the affine weights given by $(\delta : \partial) = (\Lambda_0 : \mathbf{1}) = 1$, $(\Lambda_0 : \mathbb{C}\partial \oplus \mathfrak{t}) = (\delta : \mathfrak{t} \oplus \mathbb{C}\mathbf{1}) = 0$ and $\hat{\rho} = \rho + N\Lambda_0$.

An element of $\mathbf{t}^*/\mathbb{C}\delta$ is called a *classical affine weight*. Let $cl : \mathbf{t}^* \to \mathbf{t}^*/\mathbb{C}\delta$ denote the obvious projection.

Let e be an integer $\neq 0$. We set $\mathbf{t}^*_e = \{\lambda \in \mathbf{t}^*;\ (\lambda : \mathbf{1}) = -e - N\}$. We'll use the identification $\mathbf{t}^* = \mathbb{C} \times \mathfrak{t}^* \times \mathbb{C}$ such that $\alpha_i \mapsto (0, \alpha_i, 0)$ if $i \neq 0$, $\Lambda_0 \mapsto (0, 0, 1)$ and $\delta \mapsto (1, 0, 0)$.

Let $\check{\alpha} \in \mathbf{t}$ be the affine coroot associated with the real affine root α. Let $\langle \bullet : \bullet \rangle$ be the non-degenerate symmetric bilinear form on $\mathbf{t}^*$ such that $(\lambda : \check{\alpha}_i) = 2\langle \lambda : \alpha_i \rangle / \langle \alpha_i : \alpha_i \rangle$ and $(\lambda : \mathbf{1}) = \langle \lambda : \delta \rangle$. Using $\langle \bullet : \bullet \rangle$ we identify $\check{\alpha}$ with an element of $\mathbf{t}^*$ for any real affine root α.

Let $\widehat{W} = W \ltimes \mathbb{Z}\Pi$ be the affine Weyl group and let $\mathcal{S} = \{s_i = s_{\alpha_i};\ \alpha_i \in \widehat{\Phi}\}$ be the set of simple affine reflections. The group $\widehat{W}$ acts on $\mathbf{t}^*$. For $w \in W$, $\tau \in \mathbb{Z}\Pi$ we have $w(\Lambda_0) = \Lambda_0$, $w(\delta) = \delta$, $\tau(\delta) = \delta$, $\tau(\lambda) = \lambda - \langle \tau : \lambda \rangle \delta$ and $\tau(\Lambda_0) = \tau + \Lambda_0 - \langle \tau : \tau \rangle \delta / 2$.

The $\bullet$-*action* on $\mathbf{t}^*$ is given by $w \bullet \lambda = w(\lambda + \hat{\rho}) - \hat{\rho}$. It factors to a $\widehat{W}$-action on $\mathbf{t}^*/\mathbb{C}\delta$. Two (classical) affine weights λ, μ are *linked* if they belong to the same orbit of the $\bullet$-action, and we write $\lambda \sim \mu$. Let W_λ be the stabilizer of an affine weight λ under the $\bullet$-action. We say that λ is *regular* if $W_\lambda = \{1\}$.

Set $\mathcal{C}^\pm = \{\lambda \in \mathbf{t}^*;\ \langle \lambda + \hat{\rho} : \alpha \rangle \geqslant 0,\ \alpha \in \widehat{\Pi}^\pm\}$. An element of $\mathcal{C}^-$ (resp. of $\mathcal{C}^+$) is called an *antidominant affine weight* (resp. a *dominant affine weight*). We write again $\mathcal{C}^\pm$ for $cl(\mathcal{C}^\pm)$. We have the following basic fact, see e.g., [25, lem. 2.10].

Lemma 3.1. *Let λ be an integral affine weight of level $-e - N$. We have*
(a) $\sharp(\widehat{W} \bullet \lambda \cap \mathcal{C}^-) = 1$ and $\sharp(\widehat{W} \bullet \lambda \cap \mathcal{C}^+) = 0$ if $e > 0$,
(b) $\sharp(\widehat{W} \bullet \lambda \cap \mathcal{C}^+) = 1$ and $\sharp(\widehat{W} \bullet \lambda \cap \mathcal{C}^-) = 0$ if $e < 0$.

We say that λ is *negative* in the first case, and *positive* in the second case.

For $\lambda \in \mathcal{C}^\pm$ the subgroup W_λ of $\widehat{W}$ is finite and is a standard parabolic subgroup. It is isomorphic to the Weyl group of the root system $\{\alpha \in \widehat{\Pi};\ \langle \lambda + \hat{\rho} : \alpha \rangle = 0\}$.

3.3. The parabolic category $\mathbf{O}$

Let $\mathcal{P}$ be the set of proper subsets of $\widehat{\Phi}$. An element of $\mathcal{P}$ is called a *parabolic type*. Fix a parabolic type ν. If ν is the empty set we say that ν *is regular*, and we write $\nu = \phi$.

Let $\mathbf{p}_\nu \subset \mathbf{g}$ be the unique parabolic subalgebra containing $\mathbf{b}$ whose set of roots is generated by $\widehat{\Phi} \cup (-\nu)$. Let Π_ν be the root system of a Levi subalgebra of $\mathbf{p}_\nu$. Set $\Pi^+_\nu = \Pi^+ \cap \Pi_\nu$. Let $W_\nu \subset \widehat{W}$ be the Weyl group of Π_ν. Let w_ν be the longest element in W_ν. Let $\tilde{\mathbf{O}}^\nu$ be the category of all $\mathbf{g}$-modules M such that $M = \bigoplus_{\lambda \in \mathbf{t}^*} M_\lambda$ with $M_\lambda = \{m \in M;\ xm = \lambda(x)m,\ x \in \mathbf{t}\}$ and $U(\mathbf{p}_\nu)m$ is finite dimensional for each $m \in M$. An affine weight λ is ν-*dominant* if $(\lambda : \check{\alpha}) \in \mathbb{N}$ for all $\alpha \in \Pi^+_\nu$. For any

ν-dominant affine weight λ, let $V^\nu(\lambda)$, $L(\lambda)$ be the parabolic Verma module with the highest weight λ and its simple top.

Recall that e is an integer $\neq 0$. For any weight $\lambda \in \mathfrak{t}^*$, let $\lambda_e = \lambda - (e+N)\Lambda_0$ and $z_\lambda = \langle \lambda : 2\rho + \lambda \rangle / 2e$. Let $\mathbf{O}^\nu \subset \tilde{\mathbf{O}}^\nu$ be the full subcategory of the modules such that the highest weight of any of its simple subquotients is of the form $\tilde{\lambda}_e = \lambda_e + z_\lambda \delta$, where λ is a ν-dominant integral weight. We'll abbreviate $V^\nu(\lambda_e) = V^\nu(\tilde{\lambda}_e)$ and $L(\lambda_e) = L(\tilde{\lambda}_e)$.

Now, fix $\mu \in \mathcal{P}$ and assume that $e > 0$. We use the following notation

- $\mathrm{o}_{\mu,-}$ is an antidominant integral classical affine weight of level $-e-N$ whose stabilizer for the $\bullet$-action of $\widehat{W}$ is equal to W_μ,
- $\mathrm{o}_{\mu,+} = -\mathrm{o}_{\mu,-} - 2\hat{\rho}$ is a dominant integral classical affine weight of level $e-N$ whose stabilizer for the $\bullet$-action of $\widehat{W}$ is equal to W_μ,
- $\mathbf{O}^\nu_{\mu,\pm} \subset \mathbf{O}^\nu$ is the full subcategory consisting of the modules such that the highest weight of any of its simple subquotients is linked to $\mathrm{o}_{\mu,\pm}$.

Let $I^{\min}_\mu, I^{\max}_\mu \subset \widehat{W}$ be the sets of minimal and maximal length representatives of the left cosets in $\widehat{W}/W_\mu$. Let $\leqslant$ be the Bruhat order. We'll consider the posets

- $I_{\mu,-} = (I^{\max}_\mu, \preccurlyeq)$ where $\preccurlyeq = \leqslant$, and $I^\nu_{\mu,-} = \{x \in I_{\mu,-};\ x \bullet \mathrm{o}_{\mu,-} \text{ is } \nu\text{-dominant}\}$,
- $I_{\mu,+} = (I^{\min}_\mu, \preccurlyeq)$ where $\preccurlyeq = \geqslant$, and $I^\nu_{\mu,+} = \{x \in I_{\mu,+};\ x \bullet \mathrm{o}_{\mu,+} \text{ is } \nu\text{-dominant}\}$.

They have the following properties.

Lemma 3.2. *For any μ, $\nu \in \mathcal{P}$ we have*
(a) $x \in I^\mu_{\phi,+} \Leftrightarrow x^{-1} \in I_{\mu,+}$,
(b) $x \in I^\mu_{\phi,-} \Leftrightarrow x^{-1} \in I_{\mu,-}$,
(c) $I^\mu_{\phi,\pm} \cap I_{\nu,\mp} = \{x w_\nu;\ x \in I^\mu_{\nu,\pm}\}$,
(d) $x \in I^\mu_{\nu,+} \Leftrightarrow w_\mu x w_\nu \in I^\mu_{\nu,-}$.

Proof. To prove (a) note that, since $\mathrm{o}_{\phi,+}$ is dominant regular, we have

$$\begin{aligned} I^\mu_{\phi,+} &= \{x \in \widehat{W};\ x \bullet \mathrm{o}_{\phi,+} \text{ is } \mu\text{-dominant}\} \\ &= \bigl\{x \in \widehat{W};\ \bigl\langle \mathrm{o}_{\phi,+} + \hat{\rho} : x^{-1}(\check{\alpha})\bigr\rangle \geqslant 0,\ \forall \alpha \in \Pi^+_\mu\bigr\} \\ &= \bigl\{x \in \widehat{W};\ x^{-1}\bigl(\Pi^+_\mu\bigr) \subset \widehat{\Pi}^+\bigr\} \\ &= \bigl\{x \in \widehat{W};\ x^{-1} \in I_{\mu,+}\bigr\}. \end{aligned}$$

The proof of (b) is similar and is left to the reader. Now we prove part (c). Choose positive integers d, f such that $\pm(f-d) > -N$. Then, the translation functor $T_{\phi,\nu} : {}^z\mathbf{O}_{\phi,\pm} \to {}^z\mathbf{O}_{\nu,\pm}$ in Proposition 4.36 is well-defined for any $z \in I^{\max}_\nu$. By Proposition 4.36(d), (e), we have

$$I^{\mu}_{\nu,\pm} = \{x;\ xw_\nu \in I^{\mu}_{\phi,\pm},\ x \in I_{\nu,\pm}\} = \{x;\ xw_\nu \in I^{\mu}_{\phi,\pm} \cap I_{\nu,\mp}\}.$$

Part (d) is standard. More precisely, by [8, prop. 2.7.5], if $x \in I^{\mu}_{\nu,+}$ then we have $W_\mu \cap x(W_\nu) = \emptyset$ and $x \in (I^{\min}_\mu)^{-1} \cap I^{\min}_\nu$. Then, we also have $w_\mu x \in (I^{\max}_\mu)^{-1} \cap I^{\min}_\nu$, from which we deduce that $w_\mu x w_\nu \in I^{\mu}_{\nu,-}$. The lemma is proved. □

For each $x \in I^{\nu}_{\mu,\mp}$, we write $x_\pm = w_\nu x w_\mu \in I^{\nu}_{\mu,\pm}$. From Lemma 3.2 and its proof, we get the following.

Corollary 3.3. *(a) We have $I^{\mu}_{\nu,+} = \{x;\ xw_\nu \in (I^{\min}_\mu)^{-1} \cap I^{\max}_\nu\}$ and $I^{\mu}_{\nu,-} = \{x;\ xw_\nu \in (I^{\max}_\mu)^{-1} \cap I^{\min}_\nu\}$.*

(b) The map $I^{\nu}_{\mu,\pm} \to I^{\mu}_{\nu,\pm}$, $x \mapsto x^{-1}$ is an isomorphism of posets.

(c) The map $I^{\nu}_{\mu,\mp} \to I^{\nu}_{\mu,\pm}$, $x \mapsto x_\pm$ is an anti-isomorphism of posets.

The category $\mathbf{O}^{\nu}_{\mu,\pm}$ is a direct summand in $\mathbf{O}^{\nu}$ by the linkage principle, see [39, thm. 6.1], and we have $\mathrm{Irr}(\mathbf{O}^{\nu}_{\mu,\pm}) = \{L(x \bullet \mathrm{o}_{\mu,\pm});\ x \in I^{\nu}_{\mu,\pm}\}$. Further, for each $x \in I^{\nu}_{\mu,+}$, the simple module $L(x \bullet \mathrm{o}_{\mu,+})$ has a projective cover $P^{\nu}(x \bullet \mathrm{o}_{\mu,+})$ in $\mathbf{O}^{\nu}_{\mu,+}$. For each $x \in I^{\nu}_{\mu,-}$, there is a tilting module $T^{\nu}(x \bullet \mathrm{o}_{\mu,-})$ in $\mathbf{O}^{\nu}_{\mu,-}$ with highest weight $x \bullet \mathrm{o}_{\mu,-}$, see e.g., [39]. Note that $\mathbf{O}^{\nu}_{\mu,+}$ does not have tilting objects and $\mathbf{O}^{\nu}_{\mu,-}$ does not have projective objects. Let $\mathbf{O}^{\nu,\Delta}_{\mu,\pm}$ be the full subcategory of $\mathbf{O}^{\nu,\Delta}_{\mu,\pm}$ consisting of objects with a finite filtration by the parabolic Verma modules. The following is a version of Ringel equivalence for the categories $\mathbf{O}^{\nu}_{\mu,\pm}$.

Lemma 3.4. *There is an equivalence of exact categories $D : \mathbf{O}^{\nu,\Delta}_{\mu,+} \xrightarrow{\sim} (\mathbf{O}^{\nu,\Delta}_{\mu,-})^{\mathrm{op}}$ which maps $V^{\nu}(x \bullet \mathrm{o}_{\mu,+})$ to $V^{\nu}(x_- \bullet \mathrm{o}_{\mu,-})$, and $P^{\nu}(x \bullet \mathrm{o}_{\mu,+})$ to $T^{\nu}(x_- \bullet \mathrm{o}_{\mu,-})$.*

Proof. Note that $x_- \bullet \mathrm{o}_{\mu,-} = w_\nu x \bullet \mathrm{o}_{\mu,-} = -w_\nu(x \bullet \mathrm{o}_{\mu,+} + \hat\rho) - \hat\rho$. So the lemma is given by [39, thm. 6.6, proof of cor. 7.6]. □

Remark 3.5. Recall that the BGG-duality on $\mathbf{O}^{\nu}_{\mu,-}$ is an equivalence $\mathrm{d} : \mathbf{O}^{\nu}_{\mu,-} \xrightarrow{\sim} \mathbf{O}^{\nu,\mathrm{op}}_{\mu,-}$ which maps a simple module to itself, maps a parabolic Verma module to a dual parabolic Verma module, and maps a tilting module to itself. Let $\mathbf{O}^{\nu,\nabla}_{\mu,-}$ be the full subcategory of $\mathbf{O}^{\nu}_{\mu,-}$ consisting of objects with a finite filtration by dual parabolic Verma modules. Then we have an equivalence

$$\mathrm{d} \circ D : \mathbf{O}^{\nu,\Delta}_{\mu,+} \xrightarrow{\sim} \mathbf{O}^{\nu,\nabla}_{\mu,-}$$

which maps $V^{\nu}(x \bullet \mathrm{o}_{\mu,+})$ to $\mathrm{d}(V^{\nu}(x_- \bullet \mathrm{o}_{\mu,-}))$, and $P^{\nu}(x \bullet \mathrm{o}_{\mu,+})$ to $T^{\nu}(x_- \bullet \mathrm{o}_{\mu,-})$.

3.4. The truncated category **O**

Fix parabolic types $\mu, \nu \in \mathcal{P}$ and an integer $e > 0$. Fix an element $w \in \widehat{W}$. We define the *truncated parabolic category* **O** at the negative/positive level in the following way.

First, set ${}^w I^\nu_{\mu,-} = \{x \in I^\nu_{\mu,-};\ x \preccurlyeq w\}$ and let ${}^w\mathbf{O}^\nu_{\mu,-}$ be the full subcategory of $\mathbf{O}^\nu_{\mu,-}[{}^w I^\nu_{\mu,-}]$ consisting of the finitely generated modules. We have $\mathrm{Irr}({}^w\mathbf{O}^\nu_{\mu,-}) = \{L(x \bullet \mathrm{o}_{\mu,-});\ x \in {}^w I^\nu_{\mu,-}\}$. For each $x \in {}^w I^\nu_{\mu,-}$, the modules $V^\nu(x \bullet \mathrm{o}_{\mu,-})$ and $T^\nu(x \bullet \mathrm{o}_{\mu,-})$ belong to ${}^w\mathbf{O}^\nu_{\mu,-}$. We may write $T^\nu(x \bullet \mathrm{o}_{\mu,-}) = {}^w T^\nu(x \bullet \mathrm{o}_{\mu,-})$. The module $L(x \bullet \mathrm{o}_{\mu,-})$ has a projective cover in ${}^w\mathbf{O}^\nu_{\mu,-}$. We denoted it by ${}^w P^\nu(x \bullet \mathrm{o}_{\mu,-})$. Let ${}^w P^\nu_{\mu,-}$ be the minimal projective generator, ${}^w T^\nu_{\mu,-}$ the characteristic tilting module, and let ${}^w L^\nu_{\mu,-}$ be the sum of simple modules.

Definition 3.6. Set ${}^w\bar{A}^\nu_{\mu,-} = \mathrm{Ext}_{{}^w\mathbf{O}^\nu_{\mu,-}}({}^w L^\nu_{\mu,-})^{\mathrm{op}}$ and ${}^w A^\nu_{\mu,-} = \mathrm{End}_{{}^w\mathbf{O}^\nu_{\mu,-}}({}^w P^\nu_{\mu,-})^{\mathrm{op}}$.

Lemma 3.7. *The category ${}^w\mathbf{O}^\nu_{\mu,-} \simeq {}^w A^\nu_{\mu,-}$-**mod** is a highest weight category with the set of standard objects $\Delta({}^w\mathbf{O}^\nu_{\mu,-}) = \{V^\nu(x \bullet \mathrm{o}_{\mu,-});\ x \in {}^w I^\nu_{\mu,-}\}$ and the highest weight order given by the partial order $\preccurlyeq$ on ${}^w I^\nu_{\mu,-}$.*

Proof. First, the category $\mathbf{O}^\nu_{\mu,-}$ is a highest weight category in the sense of [10, def. 3.1], i.e., it satisfies the axioms in Section 2.4 although it is not equivalent to the module category of a finite dimensional algebra. By [10, thm. 3.5] the subcategory ${}^w\mathbf{O}^\nu_{\mu,-}$ is also highest weight. Since ${}^w\mathbf{O}^\nu_{\mu,-} \simeq {}^w A^\nu_{\mu,-}$-**mod** and ${}^w A^\nu_{\mu,-}$ is finite dimensional, the category ${}^w\mathbf{O}^\nu_{\mu,-}$ is a highest weight category in the sense of Section 2.4. □

Next, set ${}^w I^\nu_{\mu,+} = \{x \in I^\nu_{\mu,+};\ x \succcurlyeq w\}$ and let ${}^w\mathbf{O}^\nu_{\mu,+}$ be the full subcategory of $\mathbf{O}^\nu_{\mu,+}({}^w I^\nu_{\mu,+})$ consisting of the finitely generated objects. We have $\mathrm{Irr}({}^w\mathbf{O}^\nu_{\mu,+}) = \{L(x \bullet \mathrm{o}_{\mu,+});\ x \in {}^w I^\nu_{\mu,+}\}$. Let ${}^w L^\nu_{\mu,+} = \bigoplus_{x \in {}^w I^\nu_{\mu,+}} L(x \bullet \mathrm{o}_{\mu,+})$ and let ${}^w P^\nu_{\mu,+} = \bigoplus_{x \in {}^w I^\nu_{\mu,+}} P^\nu(x \bullet \mathrm{o}_{\mu,+})$.

Definition 3.8. Set ${}^w\bar{A}^\nu_{\mu,+} = \mathrm{Ext}_{{}^w\mathbf{O}^\nu_{\mu,+}}({}^w L^\nu_{\mu,+})^{\mathrm{op}}$ and ${}^w A^\nu_{\mu,+} = \mathrm{End}_{{}^w\mathbf{O}^\nu_{\mu,-}}({}^w P^\nu_{\mu,+})^{\mathrm{op}}$.

Consider the quotient functor

$$F = \mathrm{Hom}_{\mathbf{O}^\nu_{\mu,+}}\left({}^w P^\nu_{\mu,+}, \bullet\right) : \mathbf{O}^\nu_{\mu,+} \to {}^w A^\nu_{\mu,+}\text{-}\mathbf{Mod}.$$

Its kernel is the full subcategory generated by the modules $L(x \bullet \mathrm{o}_{\mu,+})$ with $x \notin {}^v I^\nu_{\mu,+}$. So, we have an equivalence of categories $\mathbf{O}^\nu_{\mu,+}({}^v I^\nu_{\mu,+}) \simeq {}^w A^\nu_{\mu,+}$-**Mod**. It restricts to an equivalence

$${}^w\mathbf{O}^\nu_{\mu,+} \simeq {}^w A^\nu_{\mu,+}\text{-}\mathbf{mod}.$$

For each $x \in {}^w I^\nu_{\mu,+}$, we'll view $V^\nu(x \bullet \mathrm{o}_{\mu,+})$ as an object in ${}^w\mathbf{O}^\nu_{\mu,+}$ by identifying it with the ${}^w A^\nu_{\mu,+}$-module $F(V^\nu(x \bullet \mathrm{o}_{\mu,+}))$. We denote ${}^w P^\nu(x \bullet \mathrm{o}_{\mu,+}) = F(P^\nu(x \bullet \mathrm{o}_{\mu,+}))$.

For each $x \in {}^w I^\nu_{\mu,\pm}$, let 1_x be the (obvious) idempotents in the $\mathbb{C}$-algebras ${}^w\bar{A}^\nu_{\mu,\pm}$ and ${}^w A^\nu_{\mu,\pm}$, associated with the modules $L(x \bullet \mathrm{o}_{\mu,\pm})$ and ${}^w P^\nu(x \bullet \mathrm{o}_{\mu,\pm})$. We'll abbreviate $1^\diamond_x = (1_x)^\diamond$, ${}^w A^{\nu,\diamond}_{\mu,-} = ({}^w A^\nu_{\mu,-})^\diamond$ and ${}^v\mathbf{O}^{\nu,\diamond}_{\mu,-} = ({}^v\mathbf{O}^\nu_{\mu,-})^\diamond$.

Proposition 3.9. *Assume that $w \in I^\nu_{\mu,+}$ and $v = w_- \in I^\nu_{\mu,-}$.*

*(a) The category ${}^w\mathbf{O}^\nu_{\mu,+} \simeq {}^wA^\nu_{\mu,+}$-**mod** is a highest weight category with the set of standard objects $\Delta({}^w\mathbf{O}^\nu_{\mu,+}) = \{V^\nu(x \bullet \mathrm{o}_{\mu,+});\ x \in {}^wI^\nu_{\mu,+}\}$ and the highest weight order given by the partial order $\preccurlyeq$ on ${}^wI^\nu_{\mu,+}$.*

(b) There is an equivalence of highest weight categories ${}^v\mathbf{O}^{\nu,\diamond}_{\mu,-} \simeq {}^w\mathbf{O}^\nu_{\mu,+}$ which takes $V^\nu(x \bullet \mathrm{o}_{\mu,-})^\diamond$ to $V^\nu(x_+ \bullet \mathrm{o}_{\mu,+})$ for any $x \in {}^vI^\nu_{\mu,-}$.

(c) There is a $\mathbb{C}$-algebra isomorphism ${}^wA^{\nu,\diamond}_{\mu,-} \simeq {}^vA^\nu_{\mu,+}$ such that $1^\diamond_x \mapsto 1_{(x_+)}$ for any $x \in {}^vI^\nu_{\mu,-}$.

Proof. First, note that the anti-isomorphism of posets $I^\nu_{\mu,+} \to I^\nu_{\mu,-}$, $x \mapsto x_-$, in Corollary 3.3 restricts to an anti-isomorphism of posets

$$ {}^wI^\nu_{\mu,+} \xrightarrow{\sim} {}^vI^\nu_{\mu,-}. $$

Therefore by Remark 3.5 the functor $\mathrm{d} \circ D$ maps ${}^wP^\nu_{\mu,+}$ to ${}^vT^\nu_{\mu,-}$, and yields an algebra isomorphism $\mathrm{End}_{\mathbf{O}^\nu_{\mu,+}}({}^wP^\nu_{\mu,+})^{\mathrm{op}} \simeq \mathrm{End}_{\mathbf{O}^\nu_{\mu,-}}({}^vT^\nu_{\mu,-})^{\mathrm{op}}$. So, we have ${}^w\mathbf{O}^\nu_{\mu,+} = ({}^v\mathbf{O}^\nu_{\mu,-})^\diamond$. In particular ${}^w\mathbf{O}^\nu_{\mu,+}$ is a highest weight category because it is the Ringel dual of a highest weight category. The rest of the proposition follows from the generalities on Ringel duality in Section 2.5. □

We'll denote by ${}^wT^\nu(x \bullet \mathrm{o}_{\mu,+})$ the tilting object associated with $V^\nu(x \bullet \mathrm{o}_{\mu,+})$ in ${}^w\mathbf{O}^\nu_{\mu,+}$ and by ${}^wT^\nu_{\mu,+}$ the characteristic tilting module. If $\nu = \phi$, we also abbreviate ${}^w\mathbf{O}_{\mu,\pm} = {}^w\mathbf{O}^\phi_{\mu,\pm}$, suppressing ϕ everywhere in the notation.

Remark 3.10. The highest weight category ${}^w\mathbf{O}^\nu_{\mu,\pm}$ does not depend on the choice of $\mathrm{o}_{\mu,\pm}$ and e but only on μ, ν and on the sign of the level, see [13, thm. 11].

Remark 3.11. Write $D = \mathrm{d} \circ (\bullet)^\diamond$, where d is the BGG duality. Then, D restrict to an equivalence of categories ${}^w\mathbf{O}^{\nu,\Delta}_{\mu,+} \to ({}^v\mathbf{O}^{\nu,\Delta}_{\mu,-})^{\mathrm{op}}$.

3.5. Parabolic inclusion and truncation

Let $i = i_{\nu,\phi}$ be the canonical inclusion ${}^w\mathbf{O}^\nu_{\mu,\pm} \subset {}^w\mathbf{O}_{\mu,\pm}$. It is a fully faithful functor and it admits a left adjoint $\tau = \tau_{\phi,\nu}$ which takes an object of ${}^w\mathbf{O}_{\mu,\pm}$ to its largest quotient which lies in ${}^w\mathbf{O}^\nu_{\mu,\pm}$. We'll call i the *parabolic inclusion functor* and τ the *parabolic truncation functor*.

Since i is exact, the functor τ takes projectives to projectives. We have

(a) $i(L(x \bullet \mathrm{o}_{\mu,\pm})) = L(x \bullet \mathrm{o}_{\mu,\pm})$ for $x \in {}^wI^\nu_{\mu,\pm}$,
(b) $\tau({}^wP(x \bullet \mathrm{o}_{\mu,\pm})) = {}^wP^\nu(x \bullet \mathrm{o}_{\mu,\pm})$ for $x \in {}^wI^\nu_{\mu,\pm}$,
(c) $\tau({}^wP(x \bullet \mathrm{o}_{\mu,\pm})) = 0$ for $x \in {}^wI_{\mu,\pm} \setminus {}^wI^\nu_{\mu,\pm}$.

The same argument as in [5, thm. 3.5.3], using [17, thm. 5.5], implies that the functor i is injective on extensions in ${}^w\mathbf{O}^\nu_{\mu,-}$. See the proof of Proposition 4.47 for more details.

By Lemma 2.6, for each $w \in I^\nu_{\mu,-}$, the Ringel equivalence yields equivalences of triangulated categories $\mathbf{D}^b({}^v\mathbf{O}_{\mu,+}) \to \mathbf{D}^b({}^w\mathbf{O}_{\mu,-})$ and $\mathbf{D}^b({}^v\mathbf{O}^\nu_{\mu,+}) \to \mathbf{D}^b({}^w\mathbf{O}^\nu_{\mu,-})$, where $v = w_+ := w_\nu w w_\mu$, such that the diagram below commutes

$$\begin{array}{ccc} \mathbf{D}^b({}^v\mathbf{O}_{\mu,+}) & \longrightarrow & \mathbf{D}^b({}^w\mathbf{O}_{\mu,-}) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ \mathbf{D}^b({}^v\mathbf{O}^\nu_{\mu,+}) & \longrightarrow & \mathbf{D}^b({}^w\mathbf{O}^\nu_{\mu,-}). \end{array}$$

Thus, the parabolic inclusion functor i is also injective on extensions in ${}^v\mathbf{O}^\nu_{\mu,+}$.

3.6. The main result

Fix integers $e, f > 0$ and parabolic types $\mu, \nu \in \mathcal{P}$. We choose $\mathrm{o}_{\mu,\pm}$ of level $\pm e - N$ and $\mathrm{o}_{\nu,\pm}$ of level $\pm f - N$. Fix an element $w \in \widehat{W}$. Assume that $w \in I^\nu_{\mu,+}$, and set $v = w_-^{-1} \in I^\mu_{\nu,-}$.

The main result of this paper is the following.

Theorem 3.12. *We have $\mathbb{C}$-algebra isomorphisms ${}^wA^\nu_{\mu,+} = {}^v\bar{A}^\mu_{\nu,-}$ and ${}^w\bar{A}^\nu_{\mu,+} = {}^vA^\mu_{\nu,-}$ such that $1_x \mapsto 1_y$ with $y = x_-^{-1}$ for each $x \in {}^wI^\nu_{\mu,\mp}$. The graded $\mathbb{C}$-algebras ${}^w\bar{A}^\nu_{\mu,+}$ and ${}^v\bar{A}^\mu_{\nu,-}$ are Koszul and balanced. They are Koszul dual to each other, i.e., we have a graded $\mathbb{C}$-algebra isomorphism $({}^w\bar{A}^\nu_{\mu,+})^! = {}^v\bar{A}^\mu_{\nu,-}$ such that $1^!_x = 1_y$ for each $x \in {}^wI^\nu_{\mu,+}$. The categories ${}^w\mathbf{O}^\nu_{\mu,+}$, ${}^v\mathbf{O}^\mu_{\nu,-}$ are Koszul and are Koszul dual to each other.* $\square$

We'll abbreviate ${}^w\bar{A}^{\nu,!}_{\mu,\pm} = ({}^w\bar{A}^\nu_{\mu,\pm})^!$.

Remark 3.13. Since ${}^w\bar{A}^\nu_{\mu,+}$ and ${}^v\bar{A}^\mu_{\nu,-}$ are balanced, the Koszul duality commutes with the Ringel duality. In particular, for $w \in I^\nu_{\mu,-}$ and $v = w^{-1} \in I^\mu_{\nu,-}$, composing $(\bullet)^!$ and $(\bullet)^\diamond$ we get a graded $\mathbb{C}$-algebra isomorphism $({}^w\bar{A}^{\nu,\diamond}_{\mu,-})^! = {}^v\bar{A}^\mu_{\nu,-}$ such that $(1^\diamond_x)^! = 1_y$ with $y = x^{-1}$ for each $x \in {}^wI^\nu_{\mu,-}$.

4. Moment graphs, deformed category O and localization

First, we introduce some notation. Fix integers $e, f > 0$ and parabolic types $\mu, \nu \in \mathcal{P}$.

Let V be a finite dimensional $\mathbb{C}$-vector space, let S be the symmetric $\mathbb{C}$-algebra over V and let $\mathfrak{m} \subset S$ be the maximal ideal generated by V. We may regard S as a graded $\mathbb{C}$-algebra such that V has the degree 2.

Let R be a commutative, Noetherian, integral domain which is a (possibly graded) S-algebra with 1.

Let S_0 be the localization of S at the ideal $\mathfrak{m}$ and let k be the residue field of S_0. Note that $k = \mathbb{C}$ and that S_0 has no natural grading.

4.1. Moment graphs

Assume that $R = S$, viewed as a graded $\mathbb{C}$-algebra.

Definition 4.1. A *moment graph* over V is a tuple $\mathcal{G} = (I, H, \alpha)$ where (I, H) is a graph with a set of vertices I, a set of edges H, each edge joins two vertices, and α is a map $H \to \mathbb{P}(V)$, $h \mapsto k\alpha_h$.

We say that the moment graph $\mathcal{G}$ is a *GKM-graph* if we have $k\alpha_{h_1} \neq k\alpha_{h_2}$ for any edges $h_1 \neq h_2$ adjacent to the same vertex.

Definition 4.2. An order on $\mathcal{G}$ is a partial order $\preccurlyeq$ on I such that the two vertices joined by an edge are comparable.

Given an order $\preccurlyeq$ on $\mathcal{G}$ let h', h'' denote the *origin* and the *goal* of the edge h. In other words h' and h'' are two vertices joined by h and we have $h' \prec h''$.

Remark 4.3. We use the terminology in [21]. In [14], a moment graph is always assumed to be ordered. We'll also assume that $\mathcal{G}$ is *finite*, i.e., that the sets I and H are finite.

Definition 4.4. A *graded R-sheaf over $\mathcal{G}$* is a tuple $\mathcal{M} = (\mathcal{M}_x, \mathcal{M}_h, \rho_{x,h})$ with

- a graded R-module $\mathcal{M}_x$ for each $x \in I$,
- a graded R-module $\mathcal{M}_h$ for each $h \in H$ such that $\alpha_h \mathcal{M}_h = 0$,
- a graded R-module homomorphism $\rho_{x,h} : \mathcal{M}_x \to \mathcal{M}_h$ for h adjacent to x.

A *morphism* $\mathcal{M} \to \mathcal{N}$ is an $I \sqcup H$-tuple f of graded R-module homomorphisms $f_x : \mathcal{M}_x \to \mathcal{N}_x$ and $f_h : \mathcal{M}_h \to \mathcal{N}_h$ which are compatible with $(\rho_{x,h})$.

The R-modules $\mathcal{M}_x$ are called the *stalks* of $\mathcal{M}$. For $J \subset I$ we set

$$\mathcal{M}(J) = \big\{(m_x)_{x\in J};\ m_x \in \mathcal{M}_x,\ \rho_{h',h}(m_{h'}) = \rho_{h'',h}(m_{h''}),\ \forall h \text{ with } h', h'' \in J\big\}.$$

Note that $\mathcal{M}(\{x\}) = \mathcal{M}_x$. The *space of global sections* of the graded R-sheaf $\mathcal{M}$ is the graded R-module $\Gamma(\mathcal{M}) = \mathcal{M}(I)$.

We say that $\mathcal{M}$ is of *finite type* if all $\mathcal{M}_x$ and all $\mathcal{M}_h$ are finitely generated graded R-modules. If $\mathcal{M}$ is of finite type, then the graded R-module $\Gamma(\mathcal{M})$ is finitely generated, because R is Noetherian.

The *graded structural algebra* of $\mathcal{G}$ is the graded R-algebra $\bar{Z}_R = \{(a_x) \in R^{\oplus I};\ a_{h'} - a_{h''} \in \alpha_h R,\ \forall h\}$.

Definition 4.5. Let $\bar{\mathbf{F}}_R$ be the category consisting of the graded R-sheaves of finite type over $\mathcal{G}$ whose stalks are torsion free R-modules. Let $\bar{\mathbf{Z}}_R$ be the category consisting of the graded $\bar{Z}_R$-modules which are finitely generated and torsion free over R.

The categories $\bar{\mathbf{F}}_R$ and $\bar{\mathbf{Z}}_R$ are Krull–Schmidt graded R-categories (because they are hom-finite R-categories and each idempotent splits). Taking the global sections yields a functor $\Gamma : \bar{\mathbf{F}}_R \to \bar{\mathbf{Z}}_R$.

We'll call again graded R-sheaves the objects of $\bar{\mathbf{Z}}_R$, hoping it will not create any confusion.

The global sections functor Γ has a left adjoint $\mathcal{L}$ called the *localization functor* [14, thm. 3.6], [21, prop. 2.16, sec. 2.19].

We say that $\mathcal{M}$ is *generated by global sections* if the counit $\mathcal{L}\Gamma(\mathcal{M}) \to \mathcal{M}$ is an isomorphism, or, equivalently, if $\mathcal{M}$ belongs to the essential image of $\mathcal{L}$. This implies that the obvious map $\Gamma(\mathcal{M}) \to \mathcal{M}_x$ is surjective for each x. The functor Γ is fully faithful on the full subcategory of $\bar{\mathbf{F}}_R$ of the graded R-sheaves which are generated by global sections, see [14, sec. 3.7].

For each subsets $E \subset H$ and $J \subset I$, we set $\mathcal{M}_E = \bigoplus_{h\in E} \mathcal{M}_h$ and

$$\rho_{J,E} = \bigoplus_{x\in J} \bigoplus_{h\in E} \rho_{x,h} : \mathcal{M}(J) \to \mathcal{M}_E.$$

Let d_x be the set of edges with goal x, u_x be the set of edges with origin x and $e_x = d_x \bigsqcup u_x$. Given an order $\preccurlyeq$ on $\mathcal{G}$, we set $\mathcal{M}_{\partial x} = \mathrm{Im}(\rho_{\prec x, d_x})$, $\mathcal{M}^x = \mathrm{Ker}(\rho_{x,e_x})$ and $\mathcal{M}_{[x]} = \mathrm{Ker}(\rho_{x,d_x})$, where we abbreviate $\rho_{x,E} = \rho_{\{x\},E}$. Following [16, sec. 4.3], we call $\mathcal{M}^x$ the *costalk* of $\mathcal{M}$ at x. Note that $\mathcal{M}^x$, $\mathcal{M}_{[x]}$, $\mathcal{M}_x$ are graded $\bar{Z}_R$-modules such that $\mathcal{M}^x \subset \mathcal{M}_{[x]} \subset \mathcal{M}_x$.

Assume from now on that the moment graph $\mathcal{G}$ is a GKM graph. Let us quote the following definitions, see [21, lem. 3.2] and [14, lem. 4.8].

Definition 4.6. Assume that $\mathcal{M} \in \bar{\mathbf{F}}_R$ is generated by global sections. Then,

(*a*) $\mathcal{M}$ is *flabby* if $\mathrm{Im}(\rho_{x,d_x}) = \mathcal{M}_{\partial x}$ for each $x \in I$,

(*b*) $\mathcal{M}$ is Δ-*filtered* if it is flabby and if the graded $\bar{Z}_R$-module $\mathcal{M}_{[x]}$ is a free graded R-module for each $x \in I$.

Now, we can define the following categories.

Definition 4.7. Let $\bar{\mathbf{F}}^{\Delta}_{R,\preccurlyeq}$ be the full subcategory of $\bar{\mathbf{F}}_R$ consisting of the Δ-filtered objects. Let $\bar{\mathbf{Z}}^{\Delta}_{R,\preccurlyeq}$ be the essential image of $\bar{\mathbf{F}}^{\Delta}_{R,\preccurlyeq}$ by Γ.

We may abbreviate $\bar{\mathbf{F}}^{\Delta}_R = \bar{\mathbf{F}}^{\Delta}_{R,\preccurlyeq}$ and $\bar{\mathbf{Z}}^{\Delta}_R = \bar{\mathbf{Z}}^{\Delta}_{R,\preccurlyeq}$ when the order is clear from the context. The categories $\bar{\mathbf{F}}^{\Delta}_R$ and $\bar{\mathbf{Z}}^{\Delta}_R$ are Krull–Schmidt graded R-categories. Since Δ-filtered objects are generated by global sections, from the discussion above we deduce that $\mathcal{L}$, Γ are mutually inverse equivalences of graded R-categories between $\bar{\mathbf{F}}^{\Delta}_R$ and $\bar{\mathbf{Z}}^{\Delta}_R$.

For each $M \in \bar{\mathbf{Z}}_R^{\Delta}$ we write $M^x = \mathcal{L}(M)^x$, $M_{[x]} = \mathcal{L}(M)_{[x]}$, $M_x = \mathcal{L}(M)_x$, $M_{\prec x} = \mathcal{L}(M)_{\prec x}$, $M_{\partial x} = \mathcal{L}(M)_{\partial x}$ and $M_h = \mathcal{L}(M)_h$.

The categories $\bar{\mathbf{F}}_R^{\Delta}$ and $\bar{\mathbf{Z}}_R^{\Delta}$ are exact categories with the exact structures defined as follows. Following [14, lem. 4.5], we say that a sequence $0 \to M' \to M \to M'' \to 0$ in $\bar{\mathbf{Z}}_R^{\Delta}$ is exact if and only if the sequence of (free) R-modules $0 \to M'_{[x]} \to M_{[x]} \to M''_{[x]} \to 0$ is exact for each $x \in I$. This yields an exact structure on $\bar{\mathbf{Z}}_R^{\Delta}$. We transport it to an exact structure on $\bar{\mathbf{F}}_R^{\Delta}$ via the equivalence $(\mathcal{L}, \Gamma)$.

Remark 4.8. The forgetful functor $\bar{\mathbf{Z}}_R^{\Delta} \to \bar{\mathbf{Z}}_R$ is exact by [15, lem. 2.12]. Here $\bar{\mathbf{Z}}_R$ is equipped with the exact structure naturally associated with its structure of abelian category. To avoid confusion we'll call it the *stupid* exact structure.

Remark 4.9. Let $M \in \bar{\mathbf{Z}}_R^{\Delta}$. Then, we have $M \simeq \Gamma\mathcal{L}(M)$ and $M_x = \mathcal{L}(M)_x$. For each $J \subset I$, let M_J be the image of the obvious map $M = \Gamma\mathcal{L}(M) \to \bigoplus_{x\in J} M_x$, see also [15, def. 2.7]. It is a graded R-module, and we have $M_J = \mathcal{L}(M)(J)$ by [14, lem. 3.4]. Since $\bar{Z}_{R,x} = R$ for all x, the set $(\bar{Z}_R)_J$ is a graded R-subalgebra of $R^{\oplus J}$. So M_J is a graded $(\bar{Z}_R)_J$-module.

We'll use two different notions of projective objects in the exact category $\bar{\mathbf{Z}}_R^{\Delta}$, compare [21, sec. 3.8].

Definition 4.10. (a) A module $M \in \bar{\mathbf{Z}}_R^{\Delta}$ is *projective* if the functor $\mathrm{Hom}_{\bar{\mathbf{Z}}_R}(M, \bullet)$ on $\bar{\mathbf{Z}}_R^{\Delta}$ maps short exact sequences to short exact sequences.

(b) A module $M \in \bar{\mathbf{Z}}_R$ is *F-projective* if $\mathcal{L}(M)$ is flabby, M_x is a projective graded R-module for each x, $M_h = M_{h'}/\alpha_h M_{h'}$ and $\rho_{h',h}$ is the canonical map for each h.

Remark 4.11. If $M \in \bar{\mathbf{Z}}_R^{\Delta}$ is F-projective, then it is projective [14, prop. 5.1]. Note that [14, prop. 5.1] uses *reflexive* graded R-sheaves. However, Δ-filtered graded R-sheaves are automatically reflexive by [14, sec. 4].

We say that an object $M \in \bar{\mathbf{Z}}_R^{\Delta}$ is *tilting* if the contravariant functor $\mathrm{Hom}_{\bar{\mathbf{Z}}_R}(\bullet, M)$ on $\bar{\mathbf{Z}}_R^{\Delta}$ maps short exact sequences to short exact sequences.

For each graded R-module M, let M^* be its *dual graded R-module*, i.e., we set $M^* = \bigoplus_i (M^*)^i$ with $(M^*)^i = \mathrm{hom}_R(M, R\langle i\rangle)$. Here hom_R is the Hom space of graded R-modules. Since $\bar{Z}_R$ is commutative, the graded dual R-module M^* of a graded $\bar{Z}_R$-module M is a graded $\bar{Z}_R$-module.

As above, let the symbols Tilt and Proj denote the sets of indecomposable tilting and projective objects. By [14, sec. 4.6], the following holds.

Proposition 4.12. *The duality $D : \bar{\mathbf{Z}}_R \to \bar{\mathbf{Z}}_R$ such that $M \mapsto M^*$ restricts to an exact equivalence $\bar{\mathbf{Z}}_{R,\preccurlyeq}^{\Delta} \to (\bar{\mathbf{Z}}_{R,\succcurlyeq}^{\Delta})^{\mathrm{op}}$. In particular, it yields bijections* $\mathrm{Proj}(\bar{\mathbf{Z}}_{R,\preccurlyeq}^{\Delta}) \to \mathrm{Tilt}(\bar{\mathbf{Z}}_{R,\succcurlyeq}^{\Delta})$ *and* $\mathrm{Tilt}(\bar{\mathbf{Z}}_{R,\preccurlyeq}^{\Delta}) \to \mathrm{Proj}(\bar{\mathbf{Z}}_{R,\succcurlyeq}^{\Delta})$. □

We define the *support* of a graded R-sheaf M on $\mathcal{G}$ to be the set $\mathrm{supp}(M) = \{x \in I; M_x \neq 0\}$. We can now turn to the following important definition.

Proposition-Definition 4.13. *(See [14].) Let $(\mathcal{G}, \preccurlyeq)$ be an ordered GKM-graph. There is a unique object $\bar{B}_{R,\preccurlyeq}(x)$ in $\bar{\mathbf{Z}}_R$ which is indecomposable, F-projective, supported on the coideal $\{\succcurlyeq x\}$ and with $\bar{B}_{R,\preccurlyeq}(x)_x = R$. We call $\bar{B}_{R,\preccurlyeq}(x)$ a graded BM-sheaf.*

We may abbreviate $\bar{B}_R(x) = \bar{B}_{R,\preccurlyeq}(x)$ and $\bar{C}_R(x) = \bar{C}_{R,\preccurlyeq}(x) = D(\bar{B}_{R,\succcurlyeq}(x))$.

Remark 4.14. The existence and unicity of graded BM-sheaves is proved in [14, thm. 5.2] using the Braden–MacPherson algorithm [7, sec. 1.4]. See also [16, thm. 6.3]. The construction of $\bar{B}_R(x)$ is as follows.

- Set $\bar{B}_R(x)_y = 0$ for $y \not\succcurlyeq x$.
- Set $\bar{B}_R(x)_x = R$.
- Let $y \succ x$ and suppose we have already constructed $\bar{B}_R(x)_z$ and $\bar{B}_R(x)_h$ for any z, h such that $y \succ z, h', h'' \succcurlyeq x$. For $h \in d_y$ set
  - $\bar{B}_R(x)_h = \bar{B}_R(x)_{h'} / \alpha_h \bar{B}_R(x)_{h'}$ and $\rho_{h',h}$ is the canonical map,
  - $\bar{B}_R(x)_{\partial y} = \mathrm{Im}(\rho_{\prec y, d_y}) \subset \bar{B}_R(x)_{d_y}$,
  - $\bar{B}_R(x)_y$ is the projective cover of the graded R-module $\bar{B}_R(x)_{\partial y}$,
  - $\rho_{y,h}$ is the composition of the projective cover map $\bar{B}_R(x)_y \to \bar{B}_R(x)_{\partial y}$ with the obvious projection $\bar{B}_R(x)_{d_y} \to \bar{B}_R(x)_h$.

Proposition 4.15. *An F-projective object in $\bar{\mathbf{Z}}_R$ is a direct sum of objects of the form $\bar{B}_R(x)\langle j\rangle$ with $x \in I$ and $j \in \mathbb{Z}$.*

Proof. See [21, prop. 3.9]. □

Next, following [14, sec. 4.5], we introduce the following.

Proposition-Definition 4.16. *There is a unique object $\bar{V}_R(x) \in \bar{\mathbf{Z}}_R$ that is isomorphic to R as a graded R-module and on which the tuple $(z_y) \in \bar{Z}_R$ acts by multiplication with z_x. We call it a* graded Verma-sheaf.

Proposition 4.17. *(a) The graded Verma-sheaves are Δ-filtered and self-dual.*

(b) We have $\bar{V}_R(x)^y = \bar{V}_R(x)_y = \bar{V}_R(x)_{[y]} = R$ if $x = y$ and 0 else.

(c) For $M \in \bar{\mathbf{Z}}_R$ we have

$$\mathrm{Hom}_{Z_R}\big(\bar{V}_R(x)\langle i\rangle, M\big) = M^x\langle -i\rangle,$$
$$\mathrm{Hom}_{Z_R}\big(M, \bar{V}_R(y)\langle j\rangle\big) = (M_y)^*\langle j\rangle,$$
$$\mathrm{hom}_{\bar{Z}_R}\big(\bar{V}_R(x)\langle i\rangle, \bar{V}_R(y)\langle j\rangle\big) = \delta_{x,y}\, R^{i-j}.$$

Proof. Obvious, see e.g., [21, sec. 3.11]. □

Remark 4.18. A Δ-filtered graded R-sheaf M has a filtration $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ by graded $\bar{Z}_R$-submodules with [14, rk. 4.4, sec. 4.5]

$$\bigoplus_{r=1}^{n} M_r/M_{r-1} = \bigoplus_{y} M_{[y]}, \qquad M_r/M_{r-1} = \bar{V}_R(y_r)\langle i_r\rangle, \quad r \leqslant s \;\Rightarrow\; y_s \preccurlyeq y_r.$$

In particular, a Δ-filtered graded $\bar{Z}_R$-module is free and finitely generated as a graded R-module.

Remark 4.19. From Proposition 4.17(c) we deduce that $(M_y)^* = (M^*)^y$.

4.2. Ungraded version and base change

Assume that R is the localization of S with respect to some multiplicative set. Then, we define an R-sheaf over $\mathcal{G}$, its space of global sections, the structural algebra Z_R, the categories $\mathbf{F}_R$, $\mathbf{Z}_R$, etc., in the obvious way, by forgetting the grading in the definitions above. In particular, we define as above an exact category $\mathbf{Z}_R^\Delta$.

Now, assume that $R = S$. Forgetting the grading yields a faithful and faithfully exact functor $\bar{\mathbf{Z}}_R^\Delta \to \mathbf{Z}_R^\Delta$. See, e.g., [20], for details on faithfully exact functors. Let $V_R(x)$, $B_R(x)$, $C_R(x)$ be the images of $\bar{V}_R(x)$, $\bar{B}_R(x)$, $\bar{C}_R(x)$. We call $V_R(x)$ a Verma-sheaf and $B_R(x)$ a BM-sheaf.

Next, for each morphism of S-algebras $R \to R'$ we consider the *base change* functor $\bullet \otimes_R R' : R\text{-}\mathbf{mod} \to R'\text{-}\mathbf{mod}$ given by $M \mapsto R'M = M \otimes_R R'$.

Assume that R' is the localization of R with respect to some multiplicative subset. Then, the base change yields functors $\bullet \otimes_R R' : \mathbf{F}_R \to \mathbf{F}_{R'}$ and $\bullet \otimes_R R' : \mathbf{Z}_R \to \mathbf{Z}_{R'}$. These functors commute with Γ, $\mathcal{L}$ and the canonical map $R' \operatorname{Hom}_{\mathbf{Z}_R}(M, N) \to \operatorname{Hom}_{\mathbf{Z}_{R'}}(R'M, R'N)$ is an isomorphism, see [21, secs. 2.7, 2.18, 3.15]. Further, the base change commutes with the functor $M \mapsto M_{[x]}$, it preserves the flabby sheaves, the Verma sheaves, the F-projective ones and it yields an exact and faithful functor $\mathbf{Z}_R^\Delta \to \mathbf{Z}_{R'}^\Delta$, see [21, sec. 3.15].

Assume now that R is the localization of S with respect to some multiplicative set. Then, we define $V_R(x) = RV_S(x)$, $B_R(x) = RB_S(x)$ and $C_R(x) = RC_S(x)$. Note that $B_R(x)$, $C_R(x)$ are F-projective by [21, sec. 3.8], and that if $R = S_0$ then $B_R(x), C_R(x)$ are indecomposable by [21, sec. 3.16]. Further, any F-projective R-sheaf in $\mathbf{F}_R$ is a direct sum of objects of the form $B_R(x)$ with $x \in I$, see [21, prop. 3.13].

Forgetting the grading and taking the base change, we get the functor $\varepsilon : S\text{-}\mathbf{gmod} \to S_0\text{-}\mathbf{mod}$. It is faithful and faithfully exact. Indeed, if $\varepsilon(M) = 0$, then there is an element $f \in S \setminus \mathfrak{m}$ such that $fM = 0$. Since M is graded, this implies that $M = 0$. Hence ε is faithful, and by [20, thm. 1.1] it is also faithfully exact. Finally, note that a graded S-module M is free over S (as a graded module) if and only if $\varepsilon(M)$ is free over S_0.

Similarly, consider the functor $\varepsilon : \bar{\mathbf{Z}}_S \to \mathbf{Z}_{S_0}$ given by forgetting the grading and taking the base change. We have the following lemma.

Lemma 4.20. *(a) The functor ε commutes with the functor $M \mapsto M_{[x]}$.*
(b) The functor $\varepsilon : \bar{\mathbf{Z}}_S \to \mathbf{Z}_{S_0}$ is faithful and faithfully exact.
(c) For each $M \in \bar{\mathbf{Z}}_S$, we have $M \in \bar{\mathbf{Z}}_S^\Delta$ if and only if $\varepsilon(M) \in \mathbf{Z}_{S_0}^\Delta$. □

4.3. The moment graph of **O**

Fix a parabolic type $\mu \in \mathcal{P}$ and fix $w \in \widehat{W}$.

Unless specified otherwise, we'll set $V = \mathbf{t}$. In this section, we define a moment graph ${}^w\mathcal{G}_{\mu,\pm}$ over $\mathbf{t}$ which is naturally associated with the truncated categories ${}^w\mathbf{O}_{\mu,\pm}$. Then, we consider its category of (graded) R-sheaves over ${}^w\mathcal{G}_{\mu,\pm}$. In the graded case we'll assume that $R = S$, with the grading above. In the ungraded one, we'll assume that $R = S$ or S_0, see the previous section for details.

Definition 4.21. Let ${}^w\mathcal{G}_{\mu,\pm}$ be the moment graph over $\mathbf{t}$ whose set of vertices is ${}^wI_{\mu,\pm}$, with an edge between x, y labeled by $k\check{\alpha}$ if there is an affine reflection $s_\alpha \in \widehat{W}$ such that $s_\alpha y \in xW_\mu$. We equip ${}^w\mathcal{G}_{\mu,+}$ with the partial order $\preccurlyeq$ given by the opposite Bruhat order, and we equip ${}^w\mathcal{G}_{\mu,-}$ with the partial order $\preccurlyeq$ given by the Bruhat order.

Let ${}^w\bar{Z}_{S,\mu,\pm}$ be the graded structural algebra of ${}^w\mathcal{G}_{\mu,\pm}$. Let ${}^w\bar{\mathbf{Z}}_{S,\mu,\pm}$ be the category of graded ${}^w\bar{Z}_{S,\mu,\pm}$-modules which are finitely generated and torsion free over S. Let ${}^w\bar{\mathbf{Z}}^\Delta_{S,\mu,\pm}$ be the category of Δ-filtered graded S-sheaves on ${}^w\mathcal{G}_{\mu,\pm,\preccurlyeq}$. Let $\bar{V}_{S,\mu,\pm}(x) \in {}^w\bar{\mathbf{Z}}_{S,\mu,\pm}$ be the (graded) Verma-sheaf whose stalk at x is S. We'll use a similar notation for all other objects attached to ${}^w\mathcal{G}_{\mu,\pm}$.

We'll use the same notation as in Section 4.2. For example ${}^wB_{S,\mu,\pm}(x)$ is the nongraded analogue of ${}^w\bar{B}_{S,\mu,\pm}(x)$, and ${}^wB_{R,\mu,\pm}(x) = R\,{}^wB_{S,\mu,\pm}(x)$.

We also set ${}^w\bar{Z}_{k,\mu,\pm} = k\,{}^w\bar{Z}_{S,\mu,\pm}$, ${}^wZ_{k,\mu,\pm} = k\,{}^wZ_{S_0,\mu,\pm}$, ${}^w\bar{\mathbf{Z}}_{k,\mu,\pm} = {}^w\bar{Z}_{k,\mu,\pm}$-**gmod** and ${}^w\mathbf{Z}_{k,\mu,\pm} = {}^wZ_{k,\mu,\pm}$-**mod**. In the graded case, we put ${}^w\bar{B}_{k,\mu,\pm}(x) = k\,{}^w\bar{B}_{S,\mu,\pm}(x)$. In the non-graded case, we put ${}^wB_{k,\mu,\pm}(x) = k\,{}^wB_{S_0,\mu,\pm}(x)$. To unburden the notation we may omit the subscript k and write ${}^wZ_{\mu,\pm} = {}^wZ_{k,\mu,\pm}$, ${}^w\mathbf{Z}_{\mu,\pm} = {}^w\mathbf{Z}_{k,\mu,\pm}$, etc.

Proposition 4.22. *The (graded) BM-sheaves on ${}^w\mathcal{G}_{\mu,\pm}$ are Δ-filtered.*

Proof. The (graded) BM-sheaves on ${}^w\mathcal{G}_{\mu,-}$ are Δ-filtered by [14, thm. 5.2]. We claim that the (graded) BM-sheaves on ${}^w\mathcal{G}_{\mu,+}$ are also Δ-filtered.

By Section 4.2, a graded S-sheaf M is Δ-filtered if and only if the S_0-sheaf $\varepsilon(M)$ is Δ-filtered. Now, by Proposition 4.33(c) below, the BM-sheaves on ${}^w\mathcal{G}_{\mu,+}$ are Δ-filtered for $R = S_0$. This proves our claim. □

Let $l(x)$ be the length of an element $x \in \widehat{W}$. From Propositions 4.13, 4.22, we deduce the following.

Proposition 4.23. *For each $x \in {}^wI_{\mu,\pm}$, there is a unique Δ-filtered graded S-sheaf ${}^w\bar{B}_{S,\mu,\pm}(x) \in {}^w\bar{\mathbf{Z}}^{\Delta}_{S,\mu,\pm}$ which is indecomposable, F-projective, supported on the coideal $\{\succcurlyeq x\}$ and whose stalk at x is equal to $S\langle \pm l(x)\rangle$.* □

Assume that $w \in I_{\mu,+}$ and let $v = w_- \in I_{\mu,-}$. From the anti-isomorphism of posets ${}^wI_{\mu,+} \simeq {}^vI_{\mu,-}$ in Corollary 3.3(c), we deduce that ${}^w\mathcal{G}_{\mu,+}$ and ${}^v\mathcal{G}_{\mu,-}$ are the same moment graph with opposite orders. In particular, we have ${}^w\bar{Z}_{S,\mu,+} = {}^v\bar{Z}_{S,\mu,-}$ as graded algebras. Further, by Proposition 4.12, there is an exact equivalence

$$D : {}^w\bar{\mathbf{Z}}^{\Delta}_{S,\mu,+} \to \left({}^v\bar{\mathbf{Z}}^{\Delta}_{S,\mu,-}\right)^{\mathrm{op}}.$$

The same is true if + and − are switched everywhere. We define ${}^w\bar{C}_{S,\mu,\pm}(x) = D({}^v\bar{B}_{S,\mu,\mp}(y))$ with $y = x_\mp$, $v = w_\mp$ for each $x \in {}^wI_{\mu,\pm}$. It is a Δ-filtered graded S-sheaf on ${}^w\mathcal{G}_{\mu,\pm}$ which is indecomposable, tilting, supported on the ideal $\{\preccurlyeq x\}$ and whose costalk at x is equal to $S\langle \pm l(x)\rangle$. We abbreviate ${}^w\bar{B}_{S,\mu,\pm} = \bigoplus_x {}^w\bar{B}_{S,\mu,\pm}(x)$ and ${}^w\bar{C}_{S,\mu,\pm} = \bigoplus_x {}^w\bar{C}_{S,\mu,\pm}(x)$.

By Remark 4.18 we have the following lemma.

Lemma 4.24. *(a) The graded S-sheaf ${}^w\bar{B}_{S,\mu,\pm}(x)$ is filtered by graded Verma-sheaves. The top term in this filtration is $\bar{V}_{S,\mu,\pm}(x)\langle \pm l(x)\rangle$, which is a quotient of ${}^w\bar{B}_{S,\mu,\pm}(x)$. The other subquotients are of the form $\bar{V}_{S,\mu,\pm}(y)\langle j\rangle$ with $y \succ x$ and $j \in \mathbb{Z}$.*

(b) The graded S-sheaf ${}^w\bar{C}_{S,\mu,\pm}(x)$ is filtered by graded Verma-sheaves. The first term in this filtration is the sub-object $\bar{V}_{S,\mu,\pm}(x)\langle \pm l(x)\rangle$. The other subquotients are of the form $\bar{V}_{S,\mu,\pm}(y)\langle j\rangle$ with $y \prec x$ and $j \in \mathbb{Z}$.

Since ${}^w\bar{Z}_{S,\mu,+} = {}^v\bar{Z}_{S,\mu,-}$ we may also regard ${}^v\bar{C}_{S,\mu,-}(x_-) = D({}^w\bar{B}_{S,\mu,+}(x))$ as a graded ${}^w\bar{Z}_{S,\mu,+}$-module. The following proposition says that ${}^w\bar{B}_{S,\mu,+}(x)$ is self-dual.

Proposition 4.25. *For each $x \in {}^wI_{\mu,+}$, we have ${}^w\bar{B}_{S,\mu,+}(x) = {}^v\bar{C}_{S,\mu,-}(x_-)$ as graded ${}^w\bar{Z}_{S,\mu,+}$-modules.*

Proof. If $\mu = \phi$ is regular then the claim is [15, thm. 6.1].

Assume now that μ is not regular. We abbreviate $\bar{B}_\mu(x) = {}^w\bar{B}_{S,\mu,+}(x)$. We must prove that $D(\bar{B}_\mu(x)) = \bar{B}_\mu(x)$. The regular case implies that $D(\bar{B}_\phi(x)) = \bar{B}_\phi(x)$. We can assume that $w \in I_{\mu,-}$ and that $x \in {}^wI_\mu$.

From Proposition 4.41(e), (f) below, we deduce that

$$\bigoplus_{y \in W_\mu} D\big(\bar{B}_\mu(x)\big)\big\langle 2l(y) - l(w_\mu)\big\rangle \oplus D(M) = \bigoplus_{y \in W_\mu} \bar{B}_\mu(x)\big\langle l(w_\mu) - 2l(y)\big\rangle \oplus M,$$

where M is a direct sum of objects of the form $\bar{B}_\mu(z)\langle j\rangle$ with $z < x$. We have also $D(\bar{B}_\mu(e)) = \bar{B}_\mu(e)$. Thus, by induction, we may assume that $D(\bar{B}_\mu(z)) = \bar{B}_\mu(z)$ for all $z < x$, and the equality above implies that $D(\bar{B}_\mu(x)) = \bar{B}_\mu(x)$. □

Remark 4.26. The graded sheaf ${}^w\bar{B}_{S,\mu,-}(x)$ is not self-dual, i.e., ${}^w\bar{B}_{S,\mu,-}(x)$ is not isomorphic to ${}^v\bar{C}_{S,\mu,+}(x_+)$ in general.

Let $P^{\mu,q}_{y,x}(t) = \sum_i P^{\mu,q}_{y,x,i}\, t^i$ be Deodhar's parabolic Kazhdan–Lusztig polynomial "of type q" associated with the parabolic subgroup W_μ of $\widehat{W}$.

Let $Q^{\mu,-1}_{x,y}(t) = \sum_i Q^{\mu,-1}_{x,y,i}\, t^i$ be Deodhar's inverse parabolic Kazhdan–Lusztig polynomial "of type -1".

We use the notation in [26, rk. 2.1], which is not the usual one. We abbreviate $P_{y,x} = P^{\phi,q}_{y,x}$ and $Q_{y,x} = Q^{\phi,-1}_{y,x}$.

Proposition 4.27. *For each $x, y \in {}^wI_{\mu,+}$, we have graded S-module isomorphisms*

(a) ${}^w\bar{B}_{S,\mu,+}(x)_y = \bigoplus_{i\geqslant 0}(S\langle l(x) - 2i\rangle)^{\oplus P^{\mu,q}_{y,x,i}}$,

(b) ${}^w\bar{B}_{S,\mu,+}(x)_{[y]} = \bigoplus_{i\geqslant 0}(S\langle 2l(y) - l(x) + 2i\rangle)^{\oplus P^{\mu,q}_{y,x,i}}$.

Proof. Any finitely generated projective S-module is free, and that the S-module ${}^w\bar{B}_{S,\mu,+}(x)_y$ is projective because ${}^w\bar{B}_{S,\mu,+}(x)$ is F-projective. Hence, part (a) follows from Proposition 4.43 below and [26, thm. 1.4]. Part (b) follows from (a) and Proposition 4.25 as in [15, prop. 7.1] (where it is proved for μ regular). More precisely, write $\bar{V}(x) = \bar{V}_{S,\phi}(x)$, $\bar{B}(x) = {}^w\bar{B}_{S,\mu,+}(x)$ and $Z = {}^wZ_{S,\mu,+}$. Then, we have

$$\bar{B}(x)^y = \mathrm{Ker}(\rho_{y,e_y}) \subset \bar{B}(x)_{[y]} = \mathrm{Ker}(\rho_{y,d_y}) \subset \bar{B}(x)_y.$$

In particular, for $\alpha_{u_y} = \prod_{h\in u_y} \alpha_h$, we have $\alpha_{u_y}\,\bar{B}(x)_{[y]} \subset \bar{B}(x)^y$.

Next, the graded S-module $\bar{B}(x)_y$ is free and $\bar{B}(x)_h = \bar{B}(x)_y/\alpha_h\bar{B}(x)_y$ for $h \in u_y$ because $\bar{B}(x)$ is F-projective. Thus we have $\bar{B}(x)^y \subset \alpha_{u_y}\,\bar{B}(x)_y$, because α_{h_1} is prime to α_{h_2} in S if $h_1 \neq h_2$.

We claim that $\bar{B}(x)^y = \alpha_{u_y}\,\bar{B}(x)_{[y]}$. Let $b \in \bar{B}(x)^y$. Write $b = \alpha_{u_y}\, b'$ with $b' \in \bar{B}(x)_y$. If $\rho_{y,h}(b') \neq 0$ with $h \in d_y$ then $\rho_{y,h}(b) = \alpha_{u_y}\,\rho_{y,h}(b') \neq 0$, because the α_{u_y}-torsion submodule of $\bar{B}(x)_h$ is zero (we have $\bar{B}(x)_h = \bar{B}(x)_{h'}/\alpha_h\bar{B}(x)_{h'}$, the S-module $\bar{B}(x)_{h'}$ is free, and α_h is prime to α_{u_y}). This implies the claim.

In particular, we have $\bar{B}(x)_{[y]} = \bar{B}(x)^y\langle 2l(y)\rangle$ because $\sharp u_y = l(y)$. Hence, by Remark 4.19 and Proposition 4.25 we have a graded S-module isomorphism

$$\big(\bar{B}(x)_y\big)^*\big\langle 2l(y)\big\rangle = \bar{B}(x)^y\big\langle 2l(y)\big\rangle = \bar{B}(x)_{[y]}. \qquad \square$$

Definition 4.28. Consider the graded S-algebra ${}^w\bar{\mathscr{A}}_{S,\mu,\pm} = \mathrm{End}_{{}^wZ_{S,\mu,\pm}}({}^w\bar{B}_{S,\mu,\pm})^{\mathrm{op}}$ and the graded k-algebra ${}^w\bar{\mathscr{A}}_{\mu,\pm} = k\,{}^w\bar{\mathscr{A}}_{S,\mu,\pm}$. Set also ${}^w\mathscr{A}_{S_0,\mu,\pm} = \mathrm{End}_{{}^wZ_{S_0,\mu}}({}^wB_{S_0,\mu,\pm})^{\mathrm{op}}$ and let ${}^w\mathscr{A}_{\mu,\pm}$ be the k-algebra underlying ${}^w\bar{\mathscr{A}}_{\mu,\pm}$.

For each $x \in {}^wI_{\mu,\pm}$, let $1_x \in {}^w\bar{\mathscr{A}}_{\mu,\pm}$ be the idempotent associated with the direct summand ${}^w\bar{B}_{S,\mu,\pm}(x)$ of ${}^w\bar{B}_{S,\mu,\pm}$.

Proposition 4.29. *The graded k-algebra ${}^w\bar{\mathscr{A}}_{\mu,\pm}$ is basic. Its Hilbert polynomial is*

$$P({}^w\bar{\mathscr{A}}_{\mu,+},t)_{x,x'} = \sum_{y\leqslant x,x'} P^{\mu,q}_{y,x}(t^{-2})\,P^{\mu,q}_{y,x'}(t^{-2})\,t^{l(x)+l(x')-2l(y)},$$
$$P({}^w\bar{\mathscr{A}}_{\mu,-},t)_{x,x'} = \sum_{y\geqslant x,x'} Q^{\mu,-1}_{x,y}(t^{-2})\,Q^{\mu,-1}_{x',y}(t^{-2})\,t^{2l(y)-l(x)-l(x')}.$$

If $\mu=\phi$ then the matrix equation $P({}^w\bar{\mathscr{A}}_{\phi,-},t)\,P({}^v\bar{\mathscr{A}}_{\phi,+},-t)=1$ holds. Here $v=w^{-1}$ and the sets of indices of the matrices in the left and right factors are identified through the map $x\mapsto x^{-1}$.

Proof. First, note that if $M,B\in\bar{\mathbf{Z}}^\Delta_S$ and if B is projective, then the filtration (M_r) of M in Remark 4.18 yields a filtration of the graded S-module $\mathrm{Hom}_{Z_S}(B,M)$ whose associated graded is, see Proposition 4.17,

$$\bigoplus_{r=1}^n \mathrm{Hom}_{Z_S}(B,M_r/M_{r-1}) = \bigoplus_{r=1}^n (B_{y_r})^*\langle i_r\rangle.$$

Further, for each x', the graded BM-sheaf ${}^w\bar{B}_{S,\mu,+}(x')$ is projective by Remark 4.11 and Proposition 4.23.

Therefore, for each x,x', there is a finite filtration of the graded S-module $\mathrm{Hom}_{{}^wZ_{S,\mu,+}}({}^w\bar{B}_{S,\mu,+}(x'),{}^w\bar{B}_{S,\mu,+}(x))$ whose associated graded is, see Proposition 4.27,

$$\bigoplus_y\bigoplus_{i\geqslant 0}\big({}^w\bar{B}_{S,\mu,+}(x')_y\big)^*\big\langle 2l(y)-l(x)+2i\big\rangle^{\oplus P^{\mu,q}_{y,x,i}}$$
$$=\bigoplus_y\bigoplus_{i,i'\geqslant 0} S\big\langle 2l(y)-l(x)-l(x')+2i+2i'\big\rangle^{\oplus P^{\mu,q}_{y,x,i}\,P^{\mu,q}_{y,x',i'}}.$$

Thus we have a graded k-vector space isomorphism

$$1_x\,{}^w\bar{\mathscr{A}}_{\mu,+}1_{x'} = \bigoplus_{y\leqslant x,x'}\bigoplus_{i,i'\geqslant 0} k\big\langle 2l(y)-l(x)-l(x')+2i+2i'\big\rangle^{\oplus P^{\mu,q}_{y,x,i}\,P^{\mu,q}_{y,x',i'}},$$

where y,x,x' run over ${}^wI_{\mu,+}$.

We deduce that

$$P\big({}^w\bar{\mathscr{A}}_{\mu,+},t\big)_{x,x'} = \sum_{y\leqslant x,x'} P^{\mu,q}_{y,x}(t^{-2})\,P^{\mu,q}_{y,x'}(t^{-2})\,t^{l(x)+l(x')-2l(y)}.$$

In other words, the matrix equation $P({}^w\bar{\mathscr{A}}_{\mu,+},t) = P_{\mu,+}(t)^T\,P_{\mu,+}(t)$ holds with $P_{\mu,+}(t)_{y,x}=P^{\mu,q}_{y,x}(t^{-2})\,t^{l(x)-l(y)}$ and $x,y\in{}^wI_{\mu,+}$.

Note that $P^{\mu,q}_{y,x,i}=0$ if $l(y)-l(x)+2i>0$ and that if $l(y)-l(x)+2i=0$ then $P^{\mu,q}_{y,x,i}=0$ unless $y=x$. Thus we have $P({}^w\bar{\mathscr{A}}_{\mu,+},t)_{x,x'}\in\delta_{x,x'}+t\mathbb{N}[[t]]$. Hence the graded k-algebra ${}^w\bar{\mathscr{A}}_{\mu,+}$ is basic.

Set $Q_{\mu,-}(t)_{x,y} = Q^{\mu,-1}_{x,y}(t^{-2})\,t^{l(y)-l(x)}$ for each $x, y \in {}^wI_{\mu,+}$. The matrix equation $P({}^w\bar{\mathscr{A}}_{\mu,-}, t) = Q_{\mu,-}(t)\,Q_{\mu,-}(t)^T$ is proved in Proposition A.5 below. Hence ${}^w\bar{\mathscr{A}}_{\mu,-}$ is also basic.

Next, for each $x, x', y \in {}^wI_{\mu,+}$ and $a = -1, q$ we have, see e.g., [26, (2.39)]

$$\sum_{x\leqslant y\leqslant x'} (-1)^{l(y)-l(x)} Q^{\mu,a}_{x,y}(t)\,P^{\mu,a}_{y,x'}(t) = \delta_{x,x'}.$$

Further, if $\mu = \phi$ is regular then we have $P_{y,x}(t) = P^{\mu,q}_{y,x}(t)$ and $Q_{x,y}(t) = Q^{\mu,q}_{x,y}(t)$. So, we have the matrix equations $Q_{\phi,-}(t)\,P_{\phi,+}(-t) = 1$, $P_{\phi,+}(-t)\,Q_{\phi,-}(t) = 1$. We deduce that

$$P\big({}^w\bar{\mathscr{A}}_{\phi,-}, t\big)P\big({}^w\bar{\mathscr{A}}_{\phi,+}, -t\big) = Q_{\phi,-}(t)\,Q_{\phi,-}(t)^T\,P_{\phi,+}(-t)^T\,P_{\phi,+}(-t) = 1.$$

The matrix equation in the proposition follows easily, using the fact that $P_{y,x} = P_{y^{-1},x^{-1}}$ and $Q_{x,y} = Q_{x^{-1},y^{-1}}$. □

Remark 4.30. The grading on ${}^w\bar{\mathscr{A}}_{\mu,\pm}$ is positive, because $P^{\mu,q}_{y,x,i} = 0$ if $l(y) - l(x) + 2i > 0$ and $P^{\mu,q}_{y,x,i} = 0$ if $l(y) - l(x) + 2i = 0$ and $y \neq x$.

Remark 4.31. The results in this section have obvious analogues in finite type. In this case, we may omit the truncation and Proposition 4.27, Corollary A.4 yield

$$\bar{B}_{S,\mu,+}(x)_y = \bigoplus_{i\geqslant 0} S\big\langle l(x) - 2i\big\rangle^{\oplus P^{\mu,q}_{y,x,i}}$$

$$\bar{B}_{S,\mu,-}(x)_y\big\{l(w_0)\big\} = \bigoplus_{i\geqslant 0} S\big\langle l(w_0x) - 2i\big\rangle^{\oplus Q^{\mu,-1}_{x,y,i}},$$

where w_0 is the longest element in the Weyl group W. Indeed, we have

$$\omega^*\bar{B}_{S,\mu,+}(w_0xw_\mu) = \bar{B}_{S,\mu,-}(x)\big\langle l(w_0w_\mu)\big\rangle,$$

where $\omega : \mathcal{G}_{\mu,-} \to \mathcal{G}_{\mu,+}$ is the ordered moment graph isomorphism induced by the bijections $I_{\mu,+} \to I_{\mu,+}$, $x \mapsto w_0xw_\mu$ and $\mathbf{t} \to \mathbf{t}$, $h \mapsto w_0(h)$. Note that [26, props. 2.4, 2.6] and Kazhdan–Lusztig's inversion formula give $Q^{\mu,-1}_{x,y} = P^{\mu,q}_{w_0yw_\mu,w_0xw_\mu}$.

4.4. *Deformed category* $\mathbf{O}$

Assume that $R = S_0$ or k. If $R = k$ we may drop it from the notation.

We define $\mathbf{O}_{R,\mu,\pm}$ as the category consisting of the $(\mathbf{g}, R)$-bimodules M such that

- $M = \bigoplus_{\lambda\in\mathbf{t}^*} M_\lambda$ with $M_\lambda = \{m \in M;\ xm = (\lambda(x) + x)\cdot m,\ x \in \mathbf{t}\}$,
- $U(\mathbf{b})\,(R\cdot m)$ is finitely generated over R for each $m \in M$,
- the highest weight of any simple subquotient is linked to $\mathrm{o}_{\mu,\pm}$.

Here, for each $r \in R$ and each element m of an R-module M, the element $r \cdot m$ is the action of r on m. Further, in the symbol $\lambda(x) + x$ we view x as the element of R given by the image of x under the canonical map $S = S(\mathbf{t}) \to R$. The morphisms are the $(\mathbf{g}, R)$-bimodule homomorphisms.

Next, consider the following categories

- ${}^w\mathbf{O}_{R,\mu,-}$ is the Serre subcategory of $\mathbf{O}_{R,\mu,-}$ consisting of the finitely generated modules such that the highest weight of any simple subquotient is of the form $\lambda = x \bullet o_{\mu,-}$ with $x \preccurlyeq w$ and $x \in I_{\mu,-}$.
- ${}^w\mathbf{O}_{R,\mu,+}$ is the category of the finitely generated objects in the Serre quotient of $\mathbf{O}_{R,\mu,+}$ by the Serre subcategory consisting of the modules such that the highest weight of any simple subquotient is of the form $\lambda = x \bullet o_{\mu,+}$ with $x \not\succcurlyeq w$ and $x \in I_{\mu,+}$. We'll use the same notation for a module in $\mathbf{O}_{R,\mu,+}$ and a module in the quotient category ${}^w\mathbf{O}_{R,\mu,+}$, hoping it will not create any confusion.

For each $\lambda \in \mathbf{t}^*$ let R_λ be the $(\mathbf{t}, R)$-bimodule which is free of rank one over R and such that the element $x \in \mathbf{t}$ acts by multiplication by the image of the element $\lambda(x) + x$ by the canonical map $S \to R$. The *deformed Verma module* with highest weight λ is the $(\mathbf{g}, R)$-bimodule given by $V_R(\lambda) = U(\mathbf{g}) \otimes_{U(\mathbf{b})} R_\lambda$.

Proposition 4.32. *(a) The category ${}^w\mathbf{O}_{R,\mu,\pm}$ is a highest weight R-category. The standard objects are the deformed Verma modules $V_R(x \bullet o_{\mu,\pm})$ with $x \in {}^wI_{\mu,\pm}$.*

(b) Assume that $w \in I^\nu_{\mu,\pm}$. The Ringel equivalence gives an equivalence $(\bullet)^\diamond$: ${}^w\mathbf{O}^\Delta_{R,\mu,\pm} \to {}^v\mathbf{O}^\nabla_{R,\mu,\mp}$ where $v = w_\mp$.

Proof. First, we consider the category ${}^w\mathbf{O}_{R,\mu,-}$. The deformed Verma modules are split, i.e., their endomorphism ring is R. Further, the R-category ${}^w\mathbf{O}_{R,\mu,-}$ is Hom finite. Thus we must check that ${}^w\mathbf{O}_{R,\mu,-}$ has a projective generator and that the projective modules are Δ-filtered. Both statements follow from [12, thm. 2.7].

Next, we consider the category ${}^w\mathbf{O}_{R,\mu,+}$. Once again it is enough to check that ${}^w\mathbf{O}_{R,\mu,+}$ has a projective generator and that the projective modules are Δ-filtered. By [12, thm. 2.7], a simple module $L(x \bullet o_{\mu,+})$ in $\mathbf{O}_{R,\mu,+}$ has a projective cover $P_R(x \bullet o_{\mu,+})$. Note that the deformed category O in [12, thm. 2.7] is indeed a subcategory of $\mathbf{O}_{R,\mu,+}$ containing all finitely generated modules. Since $P_R(x \bullet o_{\mu,+})$ is finitely generated, the functor $\mathrm{Hom}_{\mathbf{O}_{R,\mu,+}}(P_R(x \bullet o_{\mu,+}), \bullet)$ commutes with direct limits. Thus, since any module in $\mathbf{O}_{R,\mu,+}$ is the direct limit of its finitely generated submodules, the module $P_R(x \bullet o_{\mu,+})$ is again projective in $\mathbf{O}_{R,\mu,+}$. Now, a standard argument using [32, thm. 3.1] shows that the functor

$$\mathrm{Hom}_{\mathbf{O}_{R,\mu,+}}\left(\bigoplus_x P_R(x \bullet o_{\mu,+}), \bullet\right), \quad x \in {}^wI_{\mu,+},$$

factors to an equivalence of abelian R-categories ${}^w\mathbf{O}_{R,\mu,+} \to A$-**mod**, where A is the finite projective R-algebra

$$A = \mathrm{End}_{\mathbf{O}_{R,\mu,+}}\Big(\bigoplus_x P_R(x \bullet \mathrm{o}_{\mu,+})\Big)^{\mathrm{op}}.$$

Therefore by Proposition 3.9(a) and [35, thm. 4.15] the category ${}^w\mathbf{O}_{R,\mu,+}$ is a highest weight category over R. This finishes the proof of (a).

By [13, sec. 2.6] there is an equivalence of exact categories $\mathbf{O}^{\Delta}_{R,\mu,+} \xrightarrow{\sim} (\mathbf{O}^{\Delta}_{R,\mu,-})^{\mathrm{op}}$ analogue to the one in Lemma 3.4. It maps $V_R(x\bullet\mathrm{o}_{\mu,+})$ to $V_R(x_-\bullet\mathrm{o}_{\mu,-})$, and $P_R(x\bullet\mathrm{o}_{\mu,+})$ to $T_R(x_-\bullet\mathrm{o}_{\mu,-})$. We deduce that $({}^v\mathbf{O}_{R,\mu,-})^\diamond = {}^w\mathbf{O}_{R,\mu,+}$ for $w \in I_{\mu,+}$ and $v = w_- \in I_{\mu,-}$. Now part (b) follows from generalities on Ringel equivalences, see Section 2.5. □

Let ${}^wP_R(x \bullet \mathrm{o}_{\mu,\pm})$ and ${}^wT_R(x \bullet \mathrm{o}_{\mu,\pm})$ be respectively the projective and the tilting objects in the highest weight categories ${}^w\mathbf{O}_{R,\mu,\pm}$ as given by Lemma 2.3. Set ${}^wP_{R,\mu,\pm} = \bigoplus_x {}^wP_R(x \bullet \mathrm{o}_{\mu,\pm})$ and ${}^wT_{R,\mu,\pm} = \bigoplus_x {}^wT_R(x \bullet \mathrm{o}_{\mu,\pm})$, where x runs over the set ${}^wI_{\mu,\pm}$. Composing the Ringel duality and the BGG duality, we get an equivalence

$$D = \mathrm{d} \circ (\bullet)^\diamond : {}^w\mathbf{O}^{\Delta}_{R,\mu,\pm} \to \big({}^v\mathbf{O}^{\Delta}_{R,\mu,\mp}\big)^{\mathrm{op}}. \tag{4.1}$$

Consider the base change functor ${}^w\mathbf{O}_{S_0,\mu,\pm} \to {}^w\mathbf{O}_{k,\mu,\pm}$ such that $M \mapsto kM$. By Proposition 2.4 we have $k{}^wP_{S_0}(x \bullet \mathrm{o}_{\mu,\pm}) = {}^wP(x \bullet \mathrm{o}_{\mu,\pm})$ and $k{}^wT_{S_0}(x \bullet \mathrm{o}_{\mu,\pm}) = {}^wT(x \bullet \mathrm{o}_{\mu,\pm})$ for each $x \in {}^wI_{\mu,\pm}$.

4.5. The functor $\mathbb{V}$

Assume that $R = S_0$ or k. There is a well defined functor

$$\mathbb{V}_R = \mathrm{Hom}_{{}^w\mathbf{O}_{R,\mu,-}}\big({}^wP(\mathrm{o}_{\mu,-}), \bullet\big) : {}^w\mathbf{O}^{\Delta}_{R,\mu,-} \to {}^w\mathbf{Z}_{R,\mu,-},$$

called the *structure functor*, see [13, sec. 3.4].

At the positive level, following [13, sec. before rem. 6], we define the structure functor $\mathbb{V}_R$ as the composition

$${}^w\mathbf{O}^{\Delta}_{R,\mu,+} \xrightarrow{D} ({}^v\mathbf{O}^{\Delta}_{R,\mu,-})^{\mathrm{op}} \xrightarrow{\mathbb{V}_R} ({}^v\mathbf{Z}_{R,\mu,-})^{\mathrm{op}} \xrightarrow{D} {}^w\mathbf{Z}_{R,\mu,+}.$$

Proposition 4.33. *Let $R = S_0$ or k. The following hold*

(a) $\mathbb{V}_R$ is exact on ${}^w\mathbf{O}_{R,\mu,-}$ and on ${}^w\mathbf{O}^{\Delta}_{R,\mu,+}$,
(b) $\mathbb{V}_R$ commutes with base change,
(c) the functor $\mathbb{V}_{S_0}$ on ${}^w\mathbf{O}^{\Delta}_{S_0,\mu,\pm}$ is such that
(i) $\mathbb{V}_{S_0} \circ D = D \circ \mathbb{V}_{S_0}$,
(ii) *$\mathbb{V}_{S_0}$ is an equivalence of exact categories ${}^w\mathbf{O}^{\Delta}_{S_0,\mu,\pm} \to {}^w\mathbf{Z}^{\Delta}_{S_0,\mu,\pm}$,*

(iii) *for each* $x \in {}^wI_{\mu,\pm}$ *we have*
- $\mathbb{V}_{S_0}(V_{S_0}(x \bullet o_{\mu,\pm})) = V_{S_0,\mu,\pm}(x)$,
- $\mathbb{V}_{S_0}({}^wP_{S_0}(x \bullet o_{\mu,\pm})) = {}^wB_{S_0,\mu,\pm}(x)$,
- $\mathbb{V}_{S_0}({}^wT_{S_0}(x \bullet o_{\mu,\pm})) = {}^wC_{S_0,\mu,\pm}(x)$.

Proof. Parts $(a), (b)$ and the first claim of (c) follow from the construction of $\mathbb{V}_R$.

The second claim of (c) follows from [14, thm. 7.1]. Note that we use here the exact structure on ${}^w\mathbf{Z}^{\Delta}_{S_0,\mu,\pm}$ defined before Remark 4.8, see [14, sec. 7.1].

The first claim of (c)(iii) is given by [13, prop. 2(2)]. The second one is proved in [14, rk. 7.6]. The third one follows from the second and (c)(i). □

Now, we compare the k-algebra ${}^wA_{\mu,\pm}$ in Definition 3.6 with the k-algebra ${}^w\mathscr{A}_{\mu,\pm}$ in Definition 4.28.

Corollary 4.34. *We have a* k*-algebra isomorphism* ${}^wA_{\mu,\pm} \xrightarrow{\sim} {}^w\mathscr{A}_{\mu,\pm}$ *such that* $1_x \mapsto 1_x$ *for each* $x \in {}^wI_{\mu,\pm}$.

Proof. By applying Proposition 2.4 to base change functor ${}^w\mathbf{O}_{S_0,\mu,\pm} \to {}^w\mathbf{O}_{k,\mu,\pm}$, we get ${}^wA_{\mu,\pm} = k\,\mathrm{End}_{{}^w\mathbf{O}_{S_0,\mu,\pm}}({}^wP_{S_0,\mu,\pm})^{\mathrm{op}}$. Thus, Proposition 4.33$(c)$ yields

$$ {}^wA_{\mu,\pm} = k\,\mathrm{End}_{{}^wZ_{S_0,\mu,\pm}}\left({}^wB_{S_0,\mu,\pm}\right)^{\mathrm{op}} = k\,\mathrm{End}_{{}^wZ_{S,\mu,\pm}}\left({}^wB_{S,\mu,\pm}\right)^{\mathrm{op}} = {}^w\mathscr{A}_{\mu,\pm}. \qquad \square $$

4.6. Translation functors in $\mathbf{O}$

Assume that $R = S_0$ or k.

Fix positive integers d, e. Assume that $o_{\mu,-}$, $o_{\phi,-}$ have the level $-e-N$, $-d-N$ respectively. Thus, the integral affine weights $o_{\mu,+}$, $o_{\phi,+}$ have the level $e-N$, $d-N$ respectively. In particular, the integral affine weight $o_{\mu,\pm} - o_{\phi,\pm}$ has the level $\pm(e-d)$. Assume that $e-d \neq \mp N$, hence we have $\pm(e-d) \neq -N$. The affine weight $o_{\mu,\pm} - o_{\phi,\pm}$ is positive if $\pm(e-d) > -N$ and it is negative else, see Section 3.2 for the terminology.

First, we consider the translation functors on the (deformed) Δ-filtered modules. We have the following.

Proposition 4.35. *Let* $R = S_0$ *or* k*. Let* $w \in \widehat{W}$ *and* $z \in I_{\mu,-}$*. Assume that* $wW_\mu = zW_\mu$ *and that* $\pm(e-d) > -N$*. We have* k*-linear functors* $T_{\phi,\mu} : {}^z\mathbf{O}^{\Delta}_{R,\phi,\pm} \to {}^w\mathbf{O}^{\Delta}_{R,\mu,\pm}$ *and* $T_{\mu,\phi} : {}^w\mathbf{O}^{\Delta}_{R,\mu,\pm} \to {}^z\mathbf{O}^{\Delta}_{R,\phi,\pm}$ *such that*

(a) $T_{\phi,\mu}$, $T_{\mu,\phi}$ *are exact,*
(b) $T_{\phi,\mu}$, $T_{\mu,\phi}$ *are bi-adjoint,*
(c) $T_{\phi,\mu}$, $T_{\mu,\phi}$ *commute with base change.*

Proof. The functors $T_{\phi,\mu}$, $T_{\mu,\phi}$ on $\mathbf{O}^{\Delta}_{R,\phi,\pm}$, $\mathbf{O}^{\Delta}_{R,\mu,\pm}$ are constructed in [12], see also [13, thm. 4(2)]. Since $z \in I_{\mu,-}$, the functors $T_{\phi,\mu}$, $T_{\mu,\phi}$ preserve the subcategories ${}^z\mathbf{O}^{\Delta}_{R,\phi,-}$,

${}^w\mathbf{O}^{\Delta}_{R,\mu,-}$ by [13, thm. 4(2)]. For the same reason the functors $T_{\phi,\mu}$, $T_{\mu,\phi}$ factor to the categories ${}^z\mathbf{O}^{\Delta}_{R,\phi,+}$, ${}^w\mathbf{O}^{\Delta}_{R,\mu,+}$. The claims (a), (b), (c) are proved in [13, thm. 4(1),(6)(4)]. □

Next, in the non-deformed setting, we consider the translation functors on the whole category O. We have the following.

Proposition 4.36. *Let $R = k$. Let $w \in \widehat{W}$ and $z \in I_{\mu,-}$. Assume that $wW_\mu = zW_\mu$ and that $\pm(e - d) > -N$. We have a k-linear functor $T_{\phi,\mu} : {}^z\mathbf{O}_{\phi,\pm} \to {}^w\mathbf{O}_{\mu,\pm}$ which coincides with the functors in Proposition 4.35 on ${}^z\mathbf{O}^{\Delta}_{\phi,\pm}$ and such that the following hold*

(a) $T_{\phi,\mu}$ has a left adjoint functor $T_{\mu,\phi}$, both functors are exact,
(b) $T_{\phi,\mu}$, $T_{\mu,\phi}$ take projectives to projectives,
(c) $T_{\phi,\mu}$, $T_{\mu,\phi}$ preserve the parabolic category O and commute with i, τ,
(d) $T_{\phi,\mu}(L(xw_\mu \bullet \mathrm{o}_{\phi,\pm})) = L(x \bullet \mathrm{o}_{\mu,\pm})$ for each $x \in {}^zI_{\mu,\pm}$,
(e) $T_{\phi,\mu}(L(xw_\mu \bullet \mathrm{o}_{\phi,\pm})) = 0$ for each $x \in {}^zI_{\phi,\pm} \setminus {}^zI_{\mu,\pm}$,
(f) $T_{\phi,\mu}({}^zL^{\nu}_{\phi,\pm}) = {}^wL^{\nu}_{\mu,\pm}$,
(g) $T_{\mu,\phi}({}^wP^{\nu}(x \bullet \mathrm{o}_{\mu,\pm})) = {}^zP^{\nu}(xw_\mu \bullet \mathrm{o}_{\phi,\pm})$ for each $x \in {}^wI^{\nu}_{\mu,\pm}$.

Proof. The definition of the translation functor $T_{\phi,\mu} : \mathbf{O}_{\phi,\pm} \to \mathbf{O}_{\mu,\pm}$ is well-known, see e.g., [25, sec. 3]. Its restriction to Δ-filtered objects coincides with the functor in Proposition 4.35 if $R = k$, see [12,25] for details. Formulas (d), (e) are proved in [25, prop. 3.8]. Since $z \in I_{\mu,-}$, they imply that $T_{\phi,\mu}$ factors to a functor ${}^z\mathbf{O}_{\phi,\pm} \to {}^w\mathbf{O}_{\mu,\pm}$ which satisfies again (d), (e).

The existence of the left adjoint $T_{\mu,\phi}$ follows from the following general fact.

Claim 4.37. *Let A, B be finite dimensional k-algebras and $T : A$-**mod** $\to B$-**mod** be an exact k-linear functor. Then T has a left and a right adjoint.*

The exactness of $T_{\phi,\mu}$ is obvious by construction, see e.g., [25, sec. 3]. It is easy to check that $T_{\phi,\mu}$ commutes with the BGG duality. Thus, its right adjoint is the conjugate of $T_{\mu,\phi}$ by the duality. So, to check that $T_{\mu,\phi}$ is exact it is enough to prove that $T_{\phi,\mu}$ takes projectives to projectives. This follows from Proposition 4.35.

Part (b) follows from the proof of (a).

Part (c) for $T_{\phi,\mu}$ is obvious by construction. For $T_{\mu,\phi}$, it follows by adjunction.

To prove (f), note that (d) implies that $T_{\phi,\mu}({}^zL_{\phi,\pm}) = {}^wL_{\mu,\pm}$. Thus (c) gives $T_{\phi,\mu}({}^zL^{\nu}_{\phi,\pm}) \subset {}^wL^{\nu}_{\mu,\pm}$. Further, for each $x \in {}^zI^{\nu}_{\mu,\pm}$ part (e) yields

$$L(xw_\mu \bullet \mathrm{o}_{\phi,\pm}) \subset T_{\mu,\phi}T_{\phi,\mu}L(xw_\mu \bullet \mathrm{o}_{\phi,\pm}) = T_{\mu,\phi}L(x \bullet \mathrm{o}_{\mu,\pm}). \tag{4.2}$$

Thus, by adjunction, we have a surjective map $T_{\phi,\mu}L(xw_\mu \bullet \mathrm{o}_{\phi,\pm}) \to L(x \bullet \mathrm{o}_{\mu,\pm})$. Now, since the right hand side of (4.2) is in ${}^w\mathbf{O}^{\nu}_{\phi,\pm}$ by (c), the left hand side is also in ${}^z\mathbf{O}^{\nu}_{\phi,\pm}$. Thus, we have a surjective map $T_{\phi,\mu}({}^zL^{\nu}_{\phi,\pm}) \to {}^wL^{\nu}_{\mu,\pm}$.

Part (g) is a consequence of (a), (d), (e). □

Remark 4.38. Let $w \in \widehat{W}$ and $w \in I^\nu_{\mu,+}$. Set $z = ww_\mu$. We have $z \in I^\nu_{\phi,+} \cap I_{\mu,-}$. Assume that $e - d > -N$. Then, the functor $T_{\mu,\phi} : {}^w\mathbf{O}^\nu_{\mu,+} \to {}^z\mathbf{O}^\nu_{\phi,+}$ is faithful. To prove this it is enough to check that $T_{\mu,\phi}(L) \neq 0$ for any simple module L. This follows from Proposition 4.36(a), (g).

4.7. *Translation functors in* $\mathbf{Z}$

Fix elements $w \in \widehat{W}$ and $z \in I_{\mu,-}$.

The algebra ${}^z\bar{Z}_{S,\phi,+} = {}^z\bar{Z}_{S,\phi,-}$ consists of the tuples (a_x) of elements of S labeled by elements $x \in \widehat{W}$ such that $x \leqslant z$ and $a_x - a_y \in \check{\alpha}S$ for all x, y such that $x = s_\alpha y$. Since z is maximal in the left coset zW_μ, switching the coordinates yields a left W_μ-action on such tuples such that $y \cdot (a_x) = (a_{xy})$ for each $y \in W_\mu$. This yields a left W_μ-action on the algebra ${}^z\bar{Z}_{S,\phi,\pm}$.

Assume that w belongs to the left coset zW_μ. Then, we have a map ${}^w\bar{Z}_{S,\mu,\pm} \to {}^z\bar{Z}_{S,\phi,\pm}$ such that $(a_x) \mapsto (b_{xy})$, where we define $b_{xy} = a_x$ for each $x \in {}^wI_{\mu,\pm}$ and $y \in W_\mu$. This map identifies the algebra ${}^w\bar{Z}_{S,\mu,\pm}$ with the set of W_μ-invariant elements in the algebra ${}^z\bar{Z}_{S,\phi,\pm}$.

Lemma 4.39. *For* $z \in I_{\mu,-}$ *and* $w \in I_{\mu,\pm}$ *such that* $w \in zW_\mu$ *there are graded* ${}^w\bar{Z}_{S,\mu,\pm}$*-module isomorphisms*

(a) ${}^z\bar{Z}_{S,\phi,\pm} \simeq \bigoplus_{y \in W_\mu} {}^w\bar{Z}_{S,\mu,\pm}\langle -2l(y)\rangle$,

(b) ${}^z\bar{Z}_{S,\phi,\pm}\langle 2l(w_\mu)\rangle \simeq \mathrm{Hom}_{{}^w\bar{Z}_{S,\mu,\pm}}({}^z\bar{Z}_{S,\phi,\pm}, {}^w\bar{Z}_{S,\mu,\pm})$.

Proof. It is enough to prove the lemma for ${}^w\bar{Z}_{S,\mu,+}$ because we have an isomorphism of graded algebra ${}^w\bar{Z}_{S,\mu,-} = {}^v\bar{Z}_{S,\mu,+}$ for $w \in I_{\mu,-}$ and $v = w_+ \in I_{\mu,+}$.

By Proposition 4.43(a) below, the graded S-algebra ${}^w\bar{Z}_{S,\mu,+}$ is isomorphic to the equivariant cohomology (graded) algebra $H_T(\bar{X}_{\mu,w})$ of the affine Schubert variety $\bar{X}_{\mu,w} = \bar{X}_{\mu,z}$. Therefore, the Bruhat decomposition yields

$$ {}^w\bar{Z}_{S,\mu,+} = H_T(\bar{X}_{\mu,z}) = \bigoplus_{x \in I_{\mu,-}, x \leqslant z} H_T(X_{\mu,x})\big\langle 2l(z) - 2l(x)\big\rangle, $$

$$ {}^z\bar{Z}_{S,\phi,+} = H_T(\bar{X}_{\phi,z}) = \bigoplus_{y \in W_\mu} \bigoplus_{x \in I_{\mu,-}, x \leqslant z} H_T(X_{\phi,xy})\big\langle 2l(z) - 2l(y) - 2l(x)\big\rangle. $$

Since the obvious projection $X_{\phi,xW_\mu} \to X_{\mu,x}$ is a P_μ/B-fibration, we have $H_T(X_{\mu,x}) = \bigoplus_{y \in W_\mu} H_T(X_{\phi,xy})$. This implies part ($a$).

Finally, part (b) follows from (a), because

$$ \mathrm{Hom}_{{}^w\bar{Z}_{S,\mu,+}}({}^z\bar{Z}_{S,\phi,+}, {}^w\bar{Z}_{S,\mu,+}) = \bigoplus_{y \in W_\mu} \mathrm{Hom}_{{}^w\bar{Z}_{S,\mu,+}}\big({}^w\bar{Z}_{S,\mu,+}\big\langle -2l(y)\big\rangle, {}^w\bar{Z}_{S,\mu,+}\big) $$

$$= \bigoplus_{y \in W_\mu} {}^w\bar{Z}_{S,\mu,+}\langle 2l(y)\rangle$$
$$= {}^z\bar{Z}_{S,\phi,+}\langle 2l(w_\mu)\rangle. \qquad \square$$

Remark 4.40. An algebraic proof of Lemma 4.39 is given in [15, lem. 5.1] in the particular case where $\sharp W_\mu = 2$.

Let $\bar{\theta}_{\phi,\mu} : {}^z\bar{\mathbf{Z}}_{S,\phi,\pm} \to {}^w\bar{\mathbf{Z}}_{S,\mu,\pm}$ and $\bar{\theta}_{\mu,\phi} : {}^w\bar{\mathbf{Z}}_{S,\mu,\pm} \to {}^z\bar{\mathbf{Z}}_{S,\phi,\pm}$ be the restriction and induction functors with respect to the inclusion ${}^w\bar{Z}_{S,\mu,\pm} \subset {}^z\bar{Z}_{S,\phi,\pm}$.

Next, let $R = S$ or S_0. Forgetting the gradings, we define in the same way as above the functors $\theta_{\phi,\mu} : {}^z\mathbf{Z}_{R,\phi,\pm} \to {}^w\mathbf{Z}_{R,\mu,\pm}$ and $\theta_{\mu,\phi} : {}^w\mathbf{Z}_{R,\mu,\pm} \to {}^z\mathbf{Z}_{R,\phi,\pm}$.

Proposition 4.41. *Assume that $w \in zW_\mu$ and $z \in I_{\mu,-}$. Then, we have*

(a) $\mathbb{V}_{S_0} \circ T_{\phi,\mu} = \theta_{\phi,\mu} \circ \mathbb{V}_{S_0}$ on ${}^z\mathbf{O}^\Delta_{S_0,\phi,\pm}$, and $\mathbb{V}_{S_0} \circ T_{\mu,\phi} = \theta_{\mu,\phi} \circ \mathbb{V}_{S_0}$ on ${}^w\mathbf{O}^\Delta_{S_0,\mu,\pm}$,

(b) $\bar{\theta}_{\phi,\mu}$ and $\bar{\theta}_{\mu,\phi}$ are exact functors ${}^z\bar{\mathbf{Z}}^\Delta_{S,\phi,\pm} \to {}^w\bar{\mathbf{Z}}^\Delta_{S,\mu,\pm}$ and ${}^w\bar{\mathbf{Z}}^\Delta_{S,\mu,\pm} \to {}^z\bar{\mathbf{Z}}^\Delta_{S,\phi,\pm}$,

(c) $(\bar{\theta}_{\mu,\phi}, \bar{\theta}_{\phi,\mu}, \bar{\theta}_{\mu,\phi}\langle 2l(w_\mu)\rangle)$ is a triple of adjoint functors,

(d) $\bar{\theta}_{\phi,\mu} \circ D = D \circ \bar{\theta}_{\phi,\mu}$ and $\bar{\theta}_{\mu,\phi} \circ D = D \circ \bar{\theta}_{\mu,\phi} \circ \langle 2l(w_\mu)\rangle$,

(e) for each $x \in {}^wI_{\mu,+}$, there is a summand M of ${}^w\bar{B}_{S,\mu,+}(t)\langle j\rangle$'s with $t < x$ such that

$$\bar{\theta}_{\phi,\mu}\big({}^z\bar{B}_{S,\phi,+}(xw_\mu)\big) = \bigoplus_{y \in W_\mu} {}^w\bar{B}_{S,\mu,+}(x)\big\langle l(w_\mu) - 2l(y)\big\rangle \bigoplus M,$$

(f) We have

(i) $\bar{\theta}_{\mu,\phi}({}^w\bar{B}_{S,\mu,+}(x)) = {}^z\bar{B}_{S,\phi,+}(xw_\mu)\langle -l(w_\mu)\rangle$ *for* $x \in {}^wI_{\mu,+}$,

(ii) $\bar{\theta}_{\mu,\phi}({}^w\bar{B}_{S,\mu,-}(x)) = {}^z\bar{B}_{S,\phi,-}(xw_\mu)$ *for* $x \in {}^wI_{\mu,-}$.

Proof. Part (a) is proved in [13, thm. 9].

For (b), note that the base change with respect to $S \to S_0$ commutes with $\theta_{\phi,\mu}$, $\theta_{\mu,\phi}$ and with the functor $M \mapsto M_{[x]}$, see Section 4.2. Now, by (a) and by Propositions 4.33(c), 4.35(a), the functors $\theta_{\phi,\mu}$, $\theta_{\mu,\phi}$ preserve ${}^z\mathbf{Z}^\Delta_{S_0,\phi,\pm}$, ${}^w\mathbf{Z}^\Delta_{S_0,\mu,\pm}$. Hence, by Lemma 4.20$(c)$, the functor $\bar{\theta}_{\phi,\mu}$, $\bar{\theta}_{\mu,\phi}$ preserve ${}^z\bar{\mathbf{Z}}^\Delta_{S,\phi,\pm}$, ${}^w\bar{\mathbf{Z}}^\Delta_{S,\mu,\pm}$.

Next, to prove the exactness of $\bar{\theta}_{\phi,\mu}$, $\bar{\theta}_{\mu,\phi}$ it is enough to check the exactness of $\theta_{\phi,\mu}$, $\theta_{\mu,\phi}$, because ε is faithfully exact by Lemma 4.20(b). This follows from (a) and Propositions 4.33(c), 4.35(a).

The proof of part (c) is similar to [15, prop. 5.2]. More precisely, by Lemma 4.39(b) there is an isomorphism of functors

$${}^z\bar{Z}_{S,\phi,\pm}\big\langle 2l(w_\mu)\big\rangle \otimes_{{}^w\bar{Z}_{S,\mu,\pm}} \bullet \simeq \mathrm{Hom}_{{}^w\bar{Z}_{S,\mu,\pm}}\big({}^z\bar{Z}_{S,\phi,\pm}, {}^w\bar{Z}_{S,\mu,\pm}\big) \otimes_{{}^w\bar{Z}_{S,\mu,\pm}} \bullet$$
$$\simeq \mathrm{Hom}_{{}^w\bar{Z}_{S,\mu,\pm}}\big({}^z\bar{Z}_{S,\phi,\pm}, \bullet\big).$$

Therefore $(\bar{\theta}_{\mu,\phi}, \bar{\theta}_{\phi,\mu}, \bar{\theta}_{\mu,\phi}\langle 2l(w_\mu)\rangle)$ is a triple of adjoint of functors.

Part (d) follows from (c). Indeed, since $\bar\theta_{\phi,\mu}(M) = M$ as a graded S-module, we have $\bar\theta_{\phi,\mu} \circ D = D \circ \bar\theta_{\phi,\mu}$. Then, part (c) implies $\bar\theta_{\mu,\phi} \circ D = D \circ \bar\theta_{\mu,\phi}\langle 2l(w_\mu)\rangle$ by unicity of adjoint functors.

Now, we prove (e). To unburden the notation let us temporarily abbreviate $\bar B_\mu(x) = {}^w\bar B_{S,\mu,+}(x)$, $\bar B_\phi(x) = {}^z\bar B_{S,\phi,+}(x)$, $\bar Z_\mu = {}^w\bar Z_{S,\mu,+}$, $\bar Z_\phi = {}^z\bar Z_{S,\phi,+}$ and $\bar{\mathbf{Z}}_\phi = {}^z\bar{\mathbf{Z}}_{S,\phi,+}$. First, note that $\bar\theta_{\phi,\mu}(\bar B_\phi(xw_\mu))$ is a direct sum of objects of the form $\bar B_\mu(z)\langle j\rangle$ by Corollary 4.44 below. Next, for each subset $J \subset {}^zI_{\phi,+}$ and each object $M \in \bar{\mathbf{Z}}_\phi^\Delta$, we have defined a graded S-module $M_J = \mathcal{L}(M)(J)$ in Remark 4.9. Setting $J = xW_\mu$, we claim that

$$\bar B_\phi(xw_\mu)_{xW_\mu} = \bar Z_{\phi,xW_\mu}\big\langle l(xw_\mu)\big\rangle. \tag{4.3}$$

To prove this, we use the geometric approach to sheaves over moment graphs recalled in Section 4.8 below. We also use the notation given there (for the dual root system). The $\bar Z_\phi$-module $\bar B_\phi(xw_\mu) \in \bar{\mathbf{Z}}_\phi^\Delta$ is a graded BM-sheaf on ${}^z\mathcal{G}_{\phi,+}$. Since $x \in I_{\mu,+}$, the set xW_μ is an ideal of the poset ${}^{xw_\mu}I_{\mu,+}$. Thus, by Lemma 4.46, we have $\bar B_\phi(xw_\mu)_{xW_\mu} = IH_T(X_{\phi,xW_\mu})$. Since X_{ϕ,xW_μ} is a smooth open subset of $\bar X_{\phi,xw_\mu}$ (see comments before Lemma 4.46), we deduce that

$$\bar B_\phi(xw_\mu)_{xW_\mu} = IH_T(X_{\phi,xW_\mu}) = H_T(X_{\phi,xW_\mu})\big\langle l(xw_\mu)\big\rangle = \bar Z_{\phi,xW_\mu}\big\langle l(xw_\mu)\big\rangle.$$

Now, by [15, prop. 5.3], for each $x \in {}^zI_{\mu,+}$ we have $\bar\theta_{\phi,\mu}(\bar B_\phi(xw_\mu))_x = \bar B_\phi(xw_\mu)_{xW_\mu}$. Thus, by (4.3) and Lemma 4.39(a) we have an isomorphism of graded S-modules

$$\begin{aligned}\bar\theta_{\phi,\mu}\big(\bar B_\phi(xw_\mu)\big)_x &= (\bar Z_{\phi,xW_\mu})_{xW_\mu}\big\langle l(xw_\mu)\big\rangle \\ &= \bigoplus_{y\in W_\mu} (\bar Z_{\mu,x})_x\big\langle l(xw_\mu) - 2l(y)\big\rangle \\ &= \bigoplus_{y\in W_\mu} S\big\langle l(xw_\mu) - 2l(y)\big\rangle.\end{aligned}$$

Finally, since for each $t \in I_{\mu,+}$ we have

$$t \nleq x \quad \Rightarrow \quad \bar\theta_{\phi,\mu}\big(\bar B_\phi(xw_\mu)\big)_t = \bar B_\phi(xw_\mu)_{tW_\mu} = 0,$$

the support condition in Proposition 4.23 implies that the decomposition of $\bar\theta_{\phi,\mu}(\bar B_\phi(xw_\mu))$ into direct sums of $\bar B_\mu(z)\langle j\rangle$'s has the form as predicted in part (e).

Finally, we prove (f). By Propositions 4.33(c), 4.36(g) and part (a) we have $\theta_{\mu,\phi}({}^wB_{S,\mu,\pm}(x)) = {}^zB_{S,\phi,\pm}(xw_\mu)$ for each $x \in {}^wI_{\mu,\pm}$. To identify the gradings, note that, by [15, prop. 5.3], we have

$$\bar\theta_{\mu,\phi}\big({}^w\bar B_{S,\mu,\pm}(x)\big)_{xW_\mu} = {}^z\bar Z_{\phi,xW_\mu} \otimes_{{}^w\bar Z_{\mu,x}} {}^w\bar B_{S,\mu,\pm}(x)_x = {}^z\bar Z_{\phi,xW_\mu}\big\langle \pm l(x)\big\rangle,$$

because ${}^w\bar Z_{\mu,x} = S$ and ${}^w\bar B_{S,\mu,\pm}(x)_x = S\langle \pm l(x)\rangle$. Since $({}^z\bar Z_{\phi,x})_y = S$ for all $y \in xW_\mu$, we deduce that $\bar\theta_{\mu,\phi}({}^w\bar B_{S,\mu,+}(x))_{xw_\mu} = S\langle l(x)\rangle$ and $\bar\theta_{\mu,\phi}({}^w\bar B_{S,\mu,-}(x))_x = S\langle -l(x)\rangle$. □

Remark 4.42. Now, we consider the case $R = k$. Recall the algebras ${}^wZ_\mu$, ${}^w\bar{Z}_\mu$ and the categories ${}^w\mathbf{Z}_\mu$, ${}^w\bar{\mathbf{Z}}_\mu$ from the beginning of Section 4.3, and the translation functors $T_{\mu,\phi}$, $T_{\phi,\mu}$ in Section 4.6.

We define $\bar{\theta}_{\phi,\mu} : {}^z\bar{\mathbf{Z}}_\phi \to {}^w\bar{\mathbf{Z}}_\mu$ and $\bar{\theta}_{\mu,\phi} : {}^w\bar{\mathbf{Z}}_\mu \to {}^z\bar{\mathbf{Z}}_\phi$ to be the restriction and induction functors with respect to the inclusion ${}^w\bar{Z}_\mu \subset {}^z\bar{Z}_\phi$. They are exact (for the stupid exact structure).

In the non-graded case, we define $\theta_{\mu,\phi}$, $\theta_{\phi,\mu}$ in a similar way. Then, we have $\mathbb{V}_k \circ T_{\phi,\mu} = \theta_{\phi,\mu} \circ \mathbb{V}_k$ and $\mathbb{V}_k \circ T_{\mu,\phi} = \theta_{\mu,\phi} \circ \mathbb{V}_k$ on Δ-filtered modules by Proposition 4.41(*a*), because $\theta_{\phi,\mu}$, $\theta_{\mu,\phi}$, $T_{\phi,\mu}$, $T_{\mu,\phi}$ and $\mathbb{V}$ commute with base change by Propositions 4.33, 4.35.

4.8. Moment graphs and the flag ind-scheme

Fix a parabolic type $\mu \in \mathcal{P}$. Let $P_\mu \subset G(k((t)))$ be the parabolic subgroup with Lie algebra $\mathbf{p}_\mu$. Write $B = P_\phi$ and let T be the torus with Lie algebra $\mathbf{t}$. Note that T is the product of $k^\times \times k^\times$ by the maximal torus of G.

Let X' be the partial (affine) flag ind-scheme $G(k((t)))/P_\mu$. The affine Bruhat cells are indexed by the cosets $\widehat{W}/W_\mu$, which we identify with $I_{\mu,+}(= I_\mu^{\min})$. For each $w \in I_{\mu,+}$ let $X_w \subset \bar{X}_w \subset X'$ be the corresponding finite dimensional affine Bruhat cell and Schubert variety. To avoid confusions we may write $X'_\mu = X'$, $X_{\mu,w} = X_w$ and $\bar{X}_{\mu,w} = \bar{X}_w$. For any subset $J \subset I_{\mu,+}$ we abbreviate $X_J = \bigcup_{w\in J} X_w$.

The group T acts on $\bar{X}_w$, with the first copy of $k^\times$ acting by rotating the loop and the last one acting trivially. The varieties $\bar{X}_w$ form an inductive system of complex projective varieties with closed embeddings and X' is represented by the ind-scheme $\mathrm{ind}_w \bar{X}_w$. Recall that $\bar{X}_w$ has dimension $l(w)$ and $\bar{X}_w = \bigcup_{x\in {}^wI_{\mu,+}} X_x$.

Let $\mathbf{D}^b(\bar{X}_w)$ be the bounded derived category of constructible sheaves of k-vector spaces on $\bar{X}_w$, which are locally constant along the B-orbits, and let $\mathbf{P}(\bar{X}_w)$ be the full lubcategory of perverse sheaves. Let $\mathbf{D}^b_T(\bar{X}_w)$ and $\mathbf{P}_T(\bar{X}_w)$ betheir T-equivariant analogue.

For each $x \in {}^wI_{\mu,+}$, let ${}^wIC(\bar{X}_x)$ be the intersection cohomology complex in $\mathbf{P}(\bar{X}_w)$ associated with $\bar{X}_x$ and let ${}^wIC_T(\bar{X}_x)$ be its T-equivariant analogue. Let $IH(\bar{X}_x)$ be the intersection cohomology of $\bar{X}_x$ and let $IH_T(\bar{X}_x)$ be its T-equivariant analogue. Finally, let $H(\bar{X}_x)$, $H_T(\bar{X}_x)$ denote the (T-equivariant) cohomology of $\bar{X}_x$. See Section A.1 for details.

In this section we set $V = \mathbf{t}^*$. Since S is the symmetric algebra over V, we have $S = H_T(\bullet)$. Let $S^\vee$ denote the symmetric algebra over $V^* = \mathbf{t}$.

Let ${}^w\mathcal{G}^\vee_{\mu,\pm}$ be the moment graph over V whose set of vertices is ${}^wI_{\mu,\pm}$, with an edge labeled by $k\,\alpha$ between x, y if there is an affine reflection s_α such that $s_\alpha y \in xW_\mu$.

We define ${}^w\bar{Z}^\vee_{S,\mu,\pm}$, ${}^w\bar{\mathbf{Z}}^\vee_{S,\mu,\pm}$, ${}^w\bar{B}_{S,\mu,\pm}(x)^\vee$, ${}^w\bar{C}_{S,\mu,\pm}(x)^\vee$, etc., in the obvious way. We'll also write ${}^w\bar{\mathbf{F}}^\vee_{S,\mu,\pm}$ for the category of graded S-sheaves of finite type over ${}^w\mathcal{G}^\vee_{\mu,\pm}$ whose stalks are torsion free as S-modules. We also write ${}^w\bar{B}^\vee_{\mu,\pm}(x) = k\,{}^w\bar{B}^\vee_{S,\mu,\pm}(x)$, ${}^w\bar{C}^\vee_{\mu,\pm}(x) = k\,{}^w\bar{C}^\vee_{S,\mu,\pm}(x)$, ${}^w\bar{Z}^\vee_{\mu,\pm} = k{}^w\bar{Z}^\vee_{S,\mu,\pm}$, ${}^w\bar{\mathbf{Z}}^\vee_{\mu,\pm} = {}^w\bar{Z}^\vee_{\mu,\pm}$-**mod**, etc. In the non-graded case we use a similar notation.

Proposition 4.43. *For $w \in I_{\mu,+}$ and $x \in {}^wI_{\mu,+}$ we have*

(a) $H_T(\bar{X}_w) = {}^w\bar{Z}^\vee_{S,\mu,+}$ *and* $H(\bar{X}_w) = {}^w\bar{Z}^\vee_{\mu,+}$ *as graded k-algebras,*
(b) $IH_T(\bar{X}_x) = {}^w\bar{B}^\vee_{S,\mu,+}(x)$ *as graded* ${}^w\bar{Z}^\vee_{S,\mu,+}$*-modules,*
(c) $IH(\bar{X}_x) = {}^w\bar{B}^\vee_{\mu,+}(x)$ *as graded* ${}^w\bar{Z}^\vee_{\mu,+}$*-modules.*

Proof. Part (a) follows from [19], parts (b), (c) from [7, thm. 1.5, 1.6, 1.8]. □

Corollary 4.44. *Assume that $w \in I_{\mu,+}$ and $z = w_- \in I_{\mu,-}$. For each $x \in {}^zI_{\phi,+}$, the graded ${}^w\bar{Z}^\vee_{S,\mu,+}$-module $\bar{\theta}_{\phi\mu}({}^z\bar{B}^\vee_{S,\phi,+}(x))$ is a direct sum of graded modules of the form ${}^w\bar{B}^\vee_{S,\mu,+}(y)\langle j\rangle$ with $y \in {}^wI_{\mu,+}$ and $j \in \mathbb{Z}$.*

Proof. The obvious projection $\pi : \bar{X}_{\phi,z} \to \bar{X}_{\mu,w}$ is proper. By Proposition 4.43, for each $\mathcal{E} \in \mathbf{P}_T(\bar{X}_{\phi,z})$, we may regard the cohomology spaces $H(\mathcal{E})$, $H(\pi_*\mathcal{E})$ as graded modules over ${}^z\bar{Z}^\vee_{S,\phi,+}$ and ${}^w\bar{Z}^\vee_{S,\mu,+}$ respectively. Since $\bar{\theta}_{\phi,\mu}$ is the restriction of graded modules with respect to the inclusion ${}^w\bar{Z}^\vee_{S,\mu,+} \subset {}^z\bar{Z}^\vee_{S,\phi,+}$, we have an obvious isomorphism of graded ${}^w\bar{Z}^\vee_{S,\mu,+}$-modules $H(\pi_*\mathcal{E}) \simeq \bar{\theta}_{\phi\mu}H(\mathcal{E})$. Setting $\mathcal{E} = {}^wIC_T(\bar{X}_{\phi,x})$, we obtain a graded module isomorphism $H(\pi_*({}^zIC_T(\bar{X}_{\phi,x}))) \simeq \bar{\theta}_{\phi\mu}({}^z\bar{B}^\vee_{S,\phi,+}(x))$. Now, by the decomposition theorem, the complex $\pi_*({}^zIC_T(\bar{X}_{\phi,x}))$ is a direct sum of complexes of the form ${}^wIC_T(\bar{X}_{\mu,y})[j]$, with $y \in {}^wI_{\mu,+}$ and $j \in \mathbb{Z}$. We deduce that $\bar{\theta}_{\phi\mu}({}^z\bar{B}^\vee_{S,\phi,+}(x))$ is a direct sum of graded ${}^w\bar{Z}^\vee_{S,\mu,+}$-modules of the form ${}^w\bar{B}^\vee_{S,\mu,+}(y)\langle j\rangle$. □

Remark 4.45. The counterpart of Proposition 4.43 for ${}^w\bar{Z}^\vee_{S,\mu,-}$ and ${}^w\bar{B}^\vee_{S,\mu,-}(x)$ is proved in Proposition A.3 below.

Recall that the poset ${}^xI_{\mu,+}$ is equipped with the opposite Bruhat order. An ideal $J \subset {}^xI_{\mu,+}$ is a coideal in the set $\{y \in I^{\min}_\mu;\ y \leqslant x\}$ equipped with the Bruhat order. Hence the variety X_J is a T-stable open subvariety of $\bar{X}_x$. We have the following proposition.

Lemma 4.46. *Let $w \in I_{\mu,+}$. For any $x \in {}^wI_{\mu,+}$ and any ideal $J \subset {}^xI_{\mu,+}$ there is an isomorphism of graded $({}^w\bar{Z}^\vee_{S,\mu,+})_J$-modules ${}^w\bar{B}^\vee_{S,\mu,+}(x)_J = IH_T(X_J)$.*

Proof. The T-variety X_J satisfies the assumption in [7, sec. 1.1]. Hence, by [7, sec. 1.1], one can associate a moment graph $\mathcal{G}^\vee_J$ to it. By construction, the graph $\mathcal{G}^\vee_J$ is the same as the subgraph of ${}^w\mathcal{G}^\vee_{\mu,+}$ consisting of the vertices which belong to J and the edges h with $h', h'' \in J$. Let $\bar{Z}^\vee_{S,J}$ be the graded structural algebra of $\mathcal{G}^\vee_J$. We have $\bar{Z}^\vee_{S,J} = ({}^w\bar{Z}^\vee_{S,\mu,+})_J$ as graded rings, see Remark 4.9.

For any $y \in J$, let $\bar{B}^\vee_{S,J}(y)$ be the set of sections of the graded BM-sheaf on $\mathcal{G}^\vee_J$ associated with y, as defined in Proposition-Definition 4.13. By [7, thm. 1.5,1.7], we have a graded ring isomorphism $\bar{Z}^\vee_{S,J} = H_T(X_J)$ and a graded $\bar{Z}^\vee_{S,J}$-module isomorphism $\bar{B}^\vee_{S,J}(y) = IH_T(\bar{X}_y \cap X_J)$. In particular, we have $\bar{B}^\vee_{S,J}(x) = IH_T(X_J)$.

Next, since $J \subset {}^xI_{\mu,+}$ is an ideal, it follows from the construction of BM-sheaves recalled in Remark 4.14 that we have canonical identifications $\bar{B}^\vee_{S,J}(x)_y = {}^w\bar{B}^\vee_{S,\mu,+}(x)_y$, $\bar{B}^\vee_{S,J}(x)_h = {}^w\bar{B}^\vee_{S,\mu,+}(x)_h$ for any y, $h \in \mathcal{G}^\vee_J$ which are compatible with the maps $\rho_{y,h}$. This implies that $\bar{B}^\vee_{S,J}(x) = {}^w\bar{B}^\vee_{S,\mu,+}(x)_J$ as graded $\bar{Z}^\vee_{S,J}$-modules. We deduce that ${}^w\bar{B}^\vee_{S,\mu,+}(x)_J = IH_T(X_J)$ as $({}^w\bar{Z}^\vee_{S,\mu,+})_J$-modules. □

Proposition 4.47. *Assume that $w \in I_{\mu,+}$. Let $v = w_-^{-1} \in I^\mu_{\phi,-}$. There is an equivalence of abelian categories $\Phi = \Phi^\mu : {}^v\mathbf{O}^\mu_{\phi,-} \to \mathbf{P}(\bar{X}_{\mu,w})$ such that $\Phi^\mu(L(y \bullet \mathrm{o}_{\phi,-})) = {}^wIC(\bar{X}_{\mu,x})$ with $x = y_+^{-1}$ for each $y \in {}^vI^\mu_{\phi,-}$.*

Proof. Set $z = v^{-1} \in I_{\mu,-}$. We have $w = zw_\mu$ and $x = y^{-1}w_\mu$. By Corollary 3.3(*b*), (*c*), the assignment $y \mapsto x$ yields a bijection ${}^vI^\mu_{\phi,-} \to {}^wI_{\mu,+}$.

By [17, thm. 5.5], see also [4, thm. 7.15, 7.16] and [24], we have an equivalence of abelian categories $\Phi^\phi : {}^v\mathbf{O}_{\phi,-} \to \mathbf{P}(\bar{X}_{\phi,z})$ such that $L(u \bullet \mathrm{o}_{\phi,-}) \mapsto {}^zIC(\bar{X}_{u^{-1}})$ for each $u \in {}^vI_{\phi,-}$. Composing Φ^ϕ and the parabolic inclusion functor $i_{\mu,\phi} : {}^v\mathbf{O}^\mu_{\phi,-} \to {}^v\mathbf{O}_{\phi,-}$ yields an embedding of abelian categories $\Phi^\phi \circ i_{\mu,\phi} : {}^v\mathbf{O}^\mu_{\phi,-} \to \mathbf{P}(\bar{X}_{\phi,z})$. It takes $L(y \bullet \mathrm{o}_{\phi,-})$ to ${}^zIC(\bar{X}_{y^{-1}})$ for each $y \in {}^vI^\mu_{\phi,-}$.

Finally, since $z \in I_{\mu,-}$, the obvious projection $\pi : \bar{X}_{\phi,z} \to \bar{X}_{\mu,w}$ is smooth. Hence, the functor $\pi^{!*} = \pi^*[\dim \pi]$ yields an embedding of abelian categories $\mathbf{P}(\bar{X}_{\mu,w}) \to \mathbf{P}(\bar{X}_{\phi,z})$ such that ${}^wIC(\bar{X}_{\mu,x}) \mapsto {}^zIC(\bar{X}_{\phi,xw_\mu})$ for each $x \in {}^wI_{\mu,+}$. The essential images of the functors $\Phi^\phi \circ i_{\mu,\phi}$ and $\pi^{!*}$ are Serre subcategories of $\mathbf{P}(\bar{X}_{\phi,v^{-1}})$ generated by the same set of simple objects. Thus, they are equivalent. Hence, the categories ${}^v\mathbf{O}^\mu_{\phi,-}$ and $\mathbf{P}(\bar{X}_{\mu,w})$ are also equivalent. □

By Proposition 4.43, composing Φ^μ and the cohomology we get a functor $\mathbb{H} = \mathbb{H}^\mu : {}^v\mathbf{O}^\mu_{\phi,-} \to {}^w\mathbf{Z}^\vee_{\mu,+}$ for each $w \in I_{\mu,+}$, $v \in I^\mu_{\phi,-}$ such that $v = w_-^{-1}$.

Proposition 4.48. *Let v, w be as above and $z = v^{-1}$. Let $y, t \in {}^vI^\mu_{\phi,-}$. Set $x = y_+^{-1}$ and $s = t_+^{-1}$. We have*

(*a*) *$\mathbb{H}^\mu(L(y \bullet \mathrm{o}_{\phi,-})) = {}^wB^\vee_{\mu,+}(x)$,*

(*b*) *we have graded k-vector space isomorphisms*

$$\begin{aligned}\mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}\big(L(y \bullet \mathrm{o}_{\phi,-}), L(t \bullet \mathrm{o}_{\phi,-})\big) &= k\,\mathrm{Hom}_{{}^wZ^\vee_{S,\mu,+}}\big({}^w\bar{B}^\vee_{S,\mu,+}(x), {}^w\bar{B}^\vee_{S,\mu,+}(s)\big) \\ &= \mathrm{Hom}_{{}^wZ^\vee_{\mu,+}}\big({}^w\bar{B}^\vee_{\mu,+}(x), {}^w\bar{B}^\vee_{\mu,+}(s)\big),\end{aligned}$$

(*c*) *there is a morphism of functors $\theta_{\mu,\phi} \circ \mathbb{H}^\mu \to \mathbb{H}^\phi \circ i_{\mu,\phi}$ which yields a ${}^zZ^\vee_{\phi,+}$-module isomorphism $\theta_{\mu,\phi}\mathbb{H}^\mu(L) \simeq \mathbb{H}^\phi i_{\mu,\phi}(L)$ for each $L \in \mathrm{Irr}({}^v\mathbf{O}^\mu_{\phi,-})$.*

Proof. Part (*a*) follows from Propositions 4.43, 4.47. To prove part (*b*), note that, by Proposition 4.47, we have a graded k-vector space isomorphism

$$\mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}\big(L(y \bullet \mathrm{o}_{\phi,-}), L(t \bullet \mathrm{o}_{\phi,-})\big) = \mathrm{Ext}_{\mathbf{D}^b(\bar{X}_w)}\big({}^wIC(\bar{X}_x), {}^wIC(\bar{X}_s)\big).$$

Further, by Propositions A.1, 4.43, we have graded k-vector space isomorphisms

$$\mathrm{Ext}_{\mathbf{D}^b_T(\bar{X}_w)}\big({}^wIC_T(\bar{X}_x), {}^wIC_T(\bar{X}_s)\big) = \mathrm{Hom}_{{}^wZ^\vee_{S,\mu,+}}\big({}^w\bar{B}^\vee_{S,\mu,+}(x), {}^w\bar{B}^\vee_{S,\mu,+}(s)\big),$$
$$\mathrm{Ext}_{\mathbf{D}^b(\bar{X}_w)}\big({}^wIC(\bar{X}_x), {}^wIC(\bar{X}_s)\big) = k\,\mathrm{Ext}_{\mathbf{D}^b_T(\bar{X}_w)}\big({}^wIC_T(\bar{X}_x), {}^wIC_T(\bar{X}_s)\big).$$

This proves the first isomorphism in (b). To prove the second one, note that we have ${}^w\bar{B}^\vee_{\mu,+}(x) = k{}^w\bar{B}^\vee_{S,\mu,+}(x)$. Thus, by Propositions 4.43, A.1, we have graded k-vector space isomorphisms

$$\begin{aligned} k\,\mathrm{Hom}_{{}^wZ^\vee_{S,\mu,+}}\big({}^w\bar{B}^\vee_{S,\mu,+}(x), {}^w\bar{B}^\vee_{S,\mu,+}(s)\big) &= k\,\mathrm{End}_{H_T(\bar{X}_w)}\big(IH_T(\bar{X}_x), IH_T(\bar{X}_s)\big),\\ &= \mathrm{End}_{H(\bar{X}_w)}\big(IH(\bar{X}_x), IH(\bar{X}_s)\big),\\ &= \mathrm{Hom}_{{}^wZ^\vee_{\mu,+}}\big({}^w\bar{B}^\vee_{\mu,+}(x), {}^w\bar{B}^\vee_{\mu,+}(s)\big). \end{aligned}$$

Now, we prove (c). By Proposition 4.43, taking the cohomology gives a functor $\mathbf{P}(\bar{X}_{\mu,w}) \to {}^wZ^\vee_{\mu,+}$-**mod** such that $\mathcal{E} \mapsto H(\mathcal{E})$. Since $z \in I_{\mu,-}$, the obvious projection $\pi : \bar{X}_{\phi,z} \to \bar{X}_{\mu,w}$ is smooth. By Proposition 4.43, for each $\mathcal{E} \in \mathbf{P}(\bar{X}_{\mu,w})$ we may regard the cohomology spaces $H(\mathcal{E})$, $H(\pi^*\mathcal{E})$ as modules over ${}^wZ^\vee_{\mu,+}$ and ${}^zZ^\vee_{\phi,+}$ respectively. The unit $1 \to \pi_*\pi^*$ yields a map $H(\mathcal{E}) \to H(\pi_*\pi^*\mathcal{E})$. By Proposition 4.43, it may be viewed as natural morphism of ${}^wZ^\vee_{\mu,+}$-modules $H(\mathcal{E}) \to \theta_{\phi,\mu}H(\pi^*\mathcal{E})$. Hence, by adjunction, we get a morphism of functors $\theta_{\mu,\phi} \circ H \to H \circ \pi^*$ from $\mathbf{P}(\bar{X}_{\mu,w})$ to ${}^zZ^\vee_{\phi,+}$-**mod**. Let $\Phi^\mu : {}^v\mathbf{O}^\mu_{\phi,-} \to \mathbf{P}(\bar{X}_{\mu,w})$ and $\Phi^\phi : {}^v\mathbf{O}_{\phi,-} \to \mathbf{P}(\bar{X}_{\phi,z})$ be as above. The proof of Proposition 4.47 implies that $\mathbb{H}^\mu = H \circ \Phi^\mu$, $\mathbb{H}^\phi = H \circ \Phi^\phi$ and $\Phi^\phi \circ i_{\mu,\phi} = \pi^{!*} \circ \Phi^\mu$. Hence, we have a morphism of functors $\theta_{\mu,\phi} \circ \mathbb{H}^\mu \to \mathbb{H}^\phi \circ i_{\mu,\phi}$.

We claim that, for each $L \in \mathrm{Irr}({}^v\mathbf{O}^\mu_{\phi,-})$, the corresponding ${}^zZ^\vee_{\phi,+}$-module homomorphism $f : \theta_{\mu,\phi}\mathbb{H}^\mu(L) \to \mathbb{H}^\phi i_{\mu,\phi}(L)$ is an isomorphism. Indeed, set $L = L(y \bullet o_{\phi,-})$ with $y \in {}^vI^\mu_{\phi,-}$. By part (a), we have $\mathbb{H}^\mu(L) = {}^zB^\vee_{\mu,+}(x)$, $\mathbb{H}^\phi i_{\mu,\phi}(L) = {}^zB^\vee_{\phi,+}(xw_\mu)$ with $x = y_+^{-1}$. Hence, by Proposition 4.41(f) and base change, we have $\theta_{\mu,\phi}\mathbb{H}^\mu(L) = \mathbb{H}^\phi i_{\mu,\phi}(L) = {}^zB^\vee_{\phi,+}(xw_\mu)$, see Remark 4.42. Hence, the map f can be regarded as an element in $\mathrm{End}_{{}^zZ^\vee_{\phi,+}}({}^zB^\vee_{\phi,+}(xw_\mu))$. Now, by part (b) and Propositions 2.4(b), 4.33(b), (c), we have

$$\begin{aligned} \mathrm{End}_{{}^zZ^\vee_{\phi,+}}\big({}^zB^\vee_{\phi,+}(xw_\mu)\big) &= k\,\mathrm{End}_{{}^zZ^\vee_{S,\phi,+}}\big({}^zB^\vee_{S,\phi,+}(xw_\mu)\big),\\ &= k\,\mathrm{End}_{{}^z\mathbf{O}^\vee_{S,\phi,+}}\big({}^zP^\vee_S(xw_\mu \bullet o_{\phi,+})\big),\\ &= \mathrm{End}_{{}^z\mathbf{O}^\vee_{\phi,+}}\big({}^zP^\vee(xw_\mu \bullet o_{\phi,+})\big). \end{aligned}$$

Here, the symbols $\mathbf{O}^\vee$ and $P^\vee$ denote the category O and the projective modules associated with the dual root system. Since the module ${}^zP^\vee(xw_\mu \bullet o_{\phi,+})$ is projective and indecomposable, we deduce that the k-algebra $\mathrm{End}_{{}^zZ^\vee_{\phi,+}}({}^zB^\vee_{\phi,+}(xw_\mu))$ is local. Thus, to prove the claim it is enough to observe that the map f is not nilpotent. □

We can now prove the following graded analogue of (part of) Corollary 4.34 which compares the graded k-algebra ${}^v\bar{A}^\mu_{\phi,-}$ in Definition 3.6 with the graded k-algebra ${}^w\bar{\mathscr{A}}_{\mu,+}$ in Definition 4.28. The comparison between ${}^u\bar{A}^\mu_{\phi,+}$ and ${}^z\bar{\mathscr{A}}_{\mu,-}$ will be done in Corollary 5.8 below.

Corollary 4.49. *Let $w \in I_{\mu,+}$ and $v = w_-^{-1} \in I^\mu_{\phi,-}$. We have a graded k-algebra isomorphism ${}^v\bar{A}^\mu_{\phi,-} \to {}^w\bar{\mathscr{A}}_{\mu,+}$ such that $1_y \mapsto 1_x$ with $x = y_+^{-1}$ for each $y \in {}^vI^\mu_{\phi,-}$.*

Proof. We have ${}^w\bar{\mathscr{A}}_{\mu,+} = k\,\mathrm{End}_{{}^wZ_{S^\vee,\mu,+}}({}^w\bar{B}_{S^\vee,\mu,+})^{\mathrm{op}}$, where ${}^w\bar{B}_{S^\vee,\mu,+} \in {}^w\bar{\mathbf{Z}}_{S^\vee,\mu,+}$ is the space of sections of a direct sum of graded BM-sheaves on ${}^w\mathcal{G}_{\mu,+}$. Next, since ${}^v\bar{A}^\mu_{\phi,-} = \mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}({}^vL^\mu_{\phi,-})^{\mathrm{op}}$ by Definition 3.6, we have a graded k-algebra isomorphism ${}^v\bar{A}^\mu_{\phi,-} = k\,\mathrm{End}_{{}^wZ^\vee_{S,\mu}}({}^w\bar{B}^\vee_{S,\mu,+})^{\mathrm{op}}$ by Proposition 4.48(b). Thus, we must check that $k\,\mathrm{End}_{{}^wZ^\vee_{S,\mu}}({}^w\bar{B}^\vee_{S,\mu,+}) \simeq k\,\mathrm{End}_{{}^wZ_{S^\vee,\mu}}({}^w\bar{B}_{S^\vee,\mu,+})$.

The choice of a $\widehat{W}$-invariant pairing on $\mathbf{t}$ yields a $\widehat{W}$-equivariant isomorphism $\mathbf{t}^* \simeq \mathbf{t}$. It induces a $\widehat{W}$-equivariant graded algebra isomorphism $S \simeq S^\vee$. The moment graphs ${}^w\mathcal{G}^\vee_{\mu,+}$ and ${}^w\mathcal{G}_{\mu,+}$ are indeed isomorphic. Using Remark 4.14, it is easy to check that this isomorphism identifies the graded BM-sheaves $\mathcal{L}({}^w\bar{B}^\vee_{S,\mu,+})$ and $\mathcal{L}({}^w\bar{B}_{S^\vee,\mu,+})$. This proves the corollary. □

Corollary 4.50. *For any $w \in I_{\mu,\pm}$, the following hold*

(a) we have graded k-algebra isomorphisms ${}^w\bar{\mathscr{A}}_{\mu,\pm} \to \mathrm{End}_{{}^wZ_{\mu,\pm}}({}^w\bar{B}_{\mu,\pm})^{\mathrm{op}}$,

(b) the functor $\mathbb{V}_k$ is fully-faithful on projective objects in ${}^w\mathbf{O}^\Delta_{\mu,\pm}$.

Proof. The obvious map ${}^w\bar{\mathscr{A}}_{\mu,+} \to \mathrm{End}_{{}^wZ_{\mu,+}}({}^w\bar{B}_{\mu,+})^{\mathrm{op}}$ is an isomorphism by Proposition 4.48(b) (applied to the dual root system). This proves (a) in the positive-level case. The negative one is proved in a similar way, see Corollary A.6.

Now, let us concentrate on part (b). By Proposition 4.33(b), (c), we have ${}^wB_{\mu,+} = \mathbb{V}_k({}^wP_{\mu,+})$ and ${}^w\mathscr{A}_{\mu,+} = k\,\mathrm{End}_{{}^w\mathbf{O}_{S_0,\mu,+}}({}^wP_{S_0,\mu,+})^{\mathrm{op}}$. The latter is isomorphic to $\mathrm{End}_{{}^w\mathbf{O}_{\mu,+}}({}^wP_{\mu,+})^{\mathrm{op}}$ by Proposition 2.4(b). We deduce that $\mathbb{V}_k$ yields an isomorphism $\mathrm{End}_{{}^wZ_{\mu,+}}(\mathbb{V}_k({}^wP_{\mu,+})) = \mathrm{End}_{{}^w\mathbf{O}_{\mu,+}}({}^wP_{\mu,+})$, which proves that $\mathbb{V}_k$ is fully-faithful on projectives in ${}^w\mathbf{O}^\Delta_{\mu,+}$.

The negative-level case is similar. Indeed, since ${}^wP_{\mu,-} = k\,{}^wP_{S_0,\mu,-}$ by Proposition 2.4(e), it follows from Proposition 4.33(b), (c) that ${}^wB_{\mu,-} = \mathbb{V}_k({}^wP_{\mu,-})$ and ${}^w\mathscr{A}_{\mu,-} = k\,\mathrm{End}_{{}^w\mathbf{O}_{S_0,\mu,-}}({}^wP_{S_0,\mu,-})^{\mathrm{op}}$. Thus $\mathrm{End}_{{}^wZ_{\mu,-}}(\mathbb{V}_k({}^wP_{\mu,-})) = \mathrm{End}_{{}^w\mathbf{O}_{\mu,-}}({}^wP_{\mu,-})$ by Proposition 2.4(b). We deduce that the functor $\mathbb{V}_k$ is fully faithful on projectives in ${}^w\mathbf{O}^\Delta_{\mu,-}$. □

Remark 4.51. Assume that $w \in I_{\mu,\pm}$. We have ${}^w\bar{C}_{S,\mu,\pm}(x) = D({}^v\bar{B}_{S,\mu,\mp}(y))$ with $y = x_\mp$, $v = w_\mp$ for each $x \in {}^wI_{\mu,\pm}$. Therefore, there are graded k-algebra isomorphisms ${}^v\bar{\mathscr{A}}_{\mu,\mp} = k\,\mathrm{End}_{{}^wZ_{S,\mu,\pm}}({}^w\bar{C}_{S,\mu,\pm})$. By Corollary 4.50, we also have graded k-algebra isomorphisms ${}^v\bar{\mathscr{A}}_{\mu,\mp} = \mathrm{End}_{{}^wZ_{\mu,\pm}}({}^w\bar{C}_{\mu,\pm})$.

5. Proof of the main theorem

5.1. The regular case

Fix integers $d, e > 0$ and fix $\mu \in \mathcal{P}$. Assume that the weights $\mathrm{o}_{\mu,-}$, $\mathrm{o}_{\phi,-}$ have the level $-e-N$ and $-d-N$ respectively.

Let $w \in I_{\mu,+}$ and put $u = w^{-1}$, $v = w_-^{-1}$ and $z = w_-$. Note that $u \in I^\mu_{\phi,+}$, $v \in I^\mu_{\phi,-}$ and $z \in I_{\mu,-}$.

The first step is to compare the algebras ${}^wA_{\mu,+} = \mathrm{End}_{{}^w\mathbf{O}_{\mu,+}}({}^wP_{\mu,+})^{\mathrm{op}}$ and ${}^v\bar{A}^\mu_{\phi,-} = \mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}({}^vL^\mu_{\phi,-})^{\mathrm{op}}$, and, then, the algebras ${}^w\bar{A}_{\mu,+} = \mathrm{Ext}_{{}^w\mathbf{O}_{\mu,+}}({}^wL_{\mu,+})^{\mathrm{op}}$ and ${}^vA^\mu_{\phi,-} = \mathrm{End}_{{}^v\mathbf{O}^\mu_{\phi,-}}({}^vP^\mu_{\phi,-})^{\mathrm{op}}$. More precisely, we prove the following.

Proposition 5.1. *If $x \in {}^wI_{\mu,+}$ and $y = x_-^{-1}$, then $y \in {}^vI^\mu_{\phi,-}$. We have a k-algebra isomorphism ${}^wA_{\mu,+} = {}^v\bar{A}^\mu_{\phi,-}$ such that $1_x \mapsto 1_y$ for each $x \in {}^wI_{\mu,+}$. We have a k-algebra isomorphism ${}^w\bar{A}_{\mu,+} = {}^vA^\mu_{\phi,-}$ such that $1_x \mapsto 1_y$ for each $x \in {}^wI_{\mu,+}$. The graded k-algebras ${}^w\bar{A}_{\mu,+}$ and ${}^v\bar{A}^\mu_{\phi,-}$ are Koszul and are Koszul dual to each other. Further, we have $1_x = 1^!_y$ for each $x \in {}^wI_{\mu,+}$.*

Proof. By Corollaries 4.34, 4.49, composing $\mathbb{H}$, $\mathbb{V}_k$ yields k-algebra isomorphisms

$$ {}^wA_{\mu,+} = {}^w\mathscr{A}_{\mu,+} = {}^v\bar{A}^\mu_{\phi,-}, \tag{5.1}$$

which identify the idempotents $1_y \in {}^v\bar{A}^\mu_{\phi,-}$ and $1_x \in {}^wA_{\mu,+}, {}^w\mathscr{A}_{\mu,+}$ for each $x \in {}^wI_{\mu,+}$ and $y = x_-^{-1}$.

Now, we claim that ${}^vA^\mu_{\phi,-}$ has a Koszul grading. By Lemma 2.2 and Section 3.5, it is enough to check that ${}^vA_{\phi,-}$ has a Koszul grading. This follows from the matrix equation in Proposition 4.29 and from [5, thm. 2.11.1], because ${}^vA_{\phi,-} = {}^v\mathscr{A}_{\phi,-}$ as k-algebras by Corollary 4.34.

Equip ${}^vA^\mu_{\phi,-}$ with the Koszul grading above. Then, Lemma 2.1 implies that

$$ {}^vA^{\mu,!}_{\phi,-} = \mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}\big({}^vL^\mu_{\phi,-}\big)^{\mathrm{op}} = {}^v\bar{A}^\mu_{\phi,-}. \tag{5.2}$$

Thus, the graded k-algebra ${}^v\bar{A}^\mu_{\phi,-}$ is also Koszul. Since ${}^wA_{\mu,+} = {}^v\bar{A}^\mu_{\phi,-}$ as a k-algebra, this implies that the k-algebra ${}^wA_{\mu,+}$ has a Koszul grading.

Applying Lemma 2.1 once again, we get

$$ {}^wA^!_{\mu,+} = \mathrm{Ext}_{{}^w\mathbf{O}_{\mu,+}}\big({}^wL_{\mu,+}\big)^{\mathrm{op}} = {}^w\bar{A}_{\mu,+}. \tag{5.3}$$

In particular, the graded k-algebra ${}^w\bar{A}_{\mu,+}$ is also Koszul.

Finally, using (5.2), (5.1) and (5.3) we get k-algebra isomorphisms

$$ {}^vA^\mu_{\phi,-} = {}^v\bar{A}^{\mu,!}_{\phi,-} = {}^wA^!_{\mu,+} = {}^w\bar{A}_{\mu,+}. $$

They identify the idempotent $1_y \in {}^vA^\mu_{\phi,-}$ with the idempotent $1_x \in {}^w\bar{A}_{\mu,+}$. □

Remark 5.2. The Koszul grading on ${}^{v}A^{\mu}_{\phi,-}$ can also be obtained using mixed perverse sheaves on the ind-scheme X' as in [5, thm. 4.5.4], [1]. Note that there is no analogue of [5, lem. 3.9.2] in our situation, because ${}^{v}R_{\phi,-}$ is not Koszul self-dual.

The second step consists of comparing the k-algebras ${}^{z}\bar{A}_{\mu,-} = \mathrm{Ext}_{{}^{z}\mathbf{O}_{\mu,-}}({}^{z}L_{\mu,-})^{\mathrm{op}}$, ${}^{u}A^{\mu}_{\phi,+} = \mathrm{End}_{{}^{u}\mathbf{O}^{\mu}_{\phi,+}}({}^{u}P^{\mu}_{\phi,+})^{\mathrm{op}}$ and the k-algebras ${}^{z}A_{\mu,-} = \mathrm{End}_{{}^{z}\mathbf{O}_{\mu,+}}({}^{w}P_{\mu,-})^{\mathrm{op}}$, ${}^{u}\bar{A}^{\mu}_{\phi,+} = \mathrm{Ext}_{{}^{u}\mathbf{O}^{\mu}_{\phi,+}}({}^{u}L^{\mu}_{\phi,+})^{\mathrm{op}}$.

We cannot argue as in the previous proposition, because we have no analogue of the localization functor Φ in Proposition 4.47 for positive levels. Hence, we have no analogue of Proposition 4.48 and Corollary 4.49.

To remedy this, we'll use the technic of standard Koszul duality. To do so, we need the following crucial result.

Lemma 5.3. *The quasi-hereditary k-algebra ${}^{w}A_{\mu,+}$ has a balanced grading.*

Now we can prove the second main result of this section.

Proposition 5.4. *If $x \in {}^{z}I_{\mu,-}$ and $y = x_{+}^{-1}$ then $y \in {}^{u}I^{\mu}_{\phi,+}$. We have a k-algebra isomorphism ${}^{z}\bar{A}_{\mu,-} = {}^{u}A^{\mu}_{\phi,+}$ such that $1_x \mapsto 1_y$ for each $x \in {}^{z}I_{\mu,-}$, and a k-algebra isomorphism ${}^{z}A_{\mu,-} = {}^{u}\bar{A}^{\mu}_{\phi,+}$ such that $1_x \mapsto 1_y$ for each $x \in {}^{z}I_{\mu,-}$. The graded k-algebras ${}^{u}\bar{A}^{\mu}_{\phi,+}$ and ${}^{z}\bar{A}_{\mu,-}$ are Koszul and are Koszul dual to each other. Further, we have $1_x = 1^{!}_{y}$ for each $x \in {}^{z}I_{\mu,-}$.*

Proof. By Proposition 3.9, the Ringel dual of ${}^{w}A_{\mu,+}$ is ${}^{w}A^{\diamond}_{\mu,+} = {}^{z}A_{\mu,-}$. Thus, since the k-algebra ${}^{w}A_{\mu,+}$ has a balanced grading by Lemma 5.3, we deduce that the k-algebra ${}^{z}A_{\mu,-}$ has a Koszul grading.

We equip ${}^{z}A_{\mu,-}$ with this grading. Then, Lemma 2.1 implies that

$$ {}^{z}A^{!}_{\mu,-} = \mathrm{Ext}_{{}^{z}\mathbf{O}_{\mu,-}}\big({}^{z}L_{\mu,-}\big)^{\mathrm{op}} = {}^{z}\bar{A}_{\mu,-}. \tag{5.4} $$

Hence, the graded k-algebra ${}^{z}\bar{A}_{\mu,-}$ is Koszul and balanced.

Therefore, [29, thm. 1], Proposition 3.9 and (5.4) yield a k-algebra isomorphism

$$ {}^{z}\bar{A}_{\mu,-} = {}^{z}A^{!}_{\mu,-} = \big(\big({}^{z}A^{\diamond}_{\mu,-}\big)^{!}\big)^{\diamond} = \big({}^{w}A^{!}_{\mu,+}\big)^{\diamond}. $$

Hence, using (5.3) and Propositions 5.1, 3.9, we get a k-algebra isomorphism

$$ {}^{z}\bar{A}_{\mu,-} = {}^{w}\bar{A}^{\diamond}_{\mu,+} = {}^{v}A^{\mu,\diamond}_{\phi,-} = {}^{u}A^{\mu}_{\phi,+} $$

such that $1_x \mapsto 1_y$. So, the k-algebra ${}^{u}A^{\mu}_{\phi,+}$ has a Koszul grading which is balanced.

We equip ${}^{u}A^{\mu}_{\phi,+}$ with this grading. Lemma 2.1 implies that

$$ {}^{u}A^{\mu,!}_{\phi,+} = \mathrm{Ext}_{{}^{u}\mathbf{O}^{\mu}_{\phi,+}}\big({}^{u}L^{\mu}_{\phi,+}\big)^{\mathrm{op}} = {}^{u}\bar{A}^{\mu}_{\phi,+}. \tag{5.5} $$

Hence, the graded k-algebra ${}^{u}\bar{A}^{\mu}_{\phi,+}$ is Koszul and balanced.

Therefore, [29, thm. 1], Proposition 3.9 yield

$$ {}^u\bar{A}^{\mu}_{\phi,+} = {}^uA^{\mu,!}_{\phi,+} = \left(\left({}^uA^{\mu,\diamond}_{\phi,+}\right)^!\right)^\diamond = \left({}^vA^{\mu,!}_{\phi,-}\right)^\diamond. $$

So, using (5.2) and Propositions 5.1, 3.9, we get a k-algebra isomorphism

$$ {}^u\bar{A}^{\mu}_{\phi,+} = \left({}^v\bar{A}^{\mu}_{\phi,-}\right)^\diamond = {}^wA^{\diamond}_{\mu,+} = {}^zA_{\mu,-}. \qquad \square $$

Now, let us prove Lemma 5.3.

Proof of Lemma 5.3. We equip ${}^wA_{\mu,+}$ with the grading given by ${}^v\bar{A}^{\mu}_{\phi,-}$, see Proposition 5.1. Since ${}^v\bar{A}^{\mu}_{\phi,-}$ is Koszul, we must prove that ${}^v\bar{A}^{\mu,!}_{\phi,-}$ is quasi-hereditary and that the grading on ${}^v\bar{A}^{\mu,\diamond}_{\phi,-}$ is positive, see Section 2.6.

We have ${}^v\bar{A}^{\mu,!}_{\phi,-} = {}^w\bar{A}_{\mu,+}$ by Proposition 5.1. Hence, it is quasi-hereditary. Next, the grading of ${}^z\bar{\mathscr{A}}_{\mu,-}$ is positive by Remark 4.30. Thus, to prove the lemma, it is enough to check that we have a graded k-algebra isomorphism ${}^v\bar{A}^{\mu,\diamond}_{\phi,-} = {}^z\bar{\mathscr{A}}_{\mu,-}$.

First, we check that ${}^wA^{\diamond}_{\mu,+} = {}^z\mathscr{A}_{\mu,-}$ as (ungraded) k-algebras.

To do this, note that Proposition 3.9(c) yields a k-algebra isomorphism ${}^wA^{\diamond}_{\mu,+} = {}^zA_{\mu,-}$. Further, we have ${}^zA_{\mu,-} = \operatorname{End}_{{}^z\mathbf{O}_{\mu,-}}({}^zP_{\mu,-})^{\mathrm{op}}$ and, by Proposition 3.9(b), the Ringel equivalence takes ${}^z\mathbf{O}^{\Delta}_{\mu,-}$ to ${}^z\mathbf{O}^{\Delta}_{\mu,+}$. Thus, Propositions 2.4(b), 4.33(c) yield

$$ \begin{aligned} {}^wA^{\diamond}_{\mu,+} &= \operatorname{End}_{{}^z\mathbf{O}_{\mu,+}}\left({}^zT_{\mu,+}\right)^{\mathrm{op}} \\ &= k\operatorname{End}_{{}^z\mathbf{O}_{S_0,\mu,+}}\left({}^zT_{S_0,\mu,+}\right)^{\mathrm{op}} \\ &= k\operatorname{End}_{{}^zZ_{S_0,\mu,+}}\left({}^zC_{S_0,\mu,+}\right)^{\mathrm{op}} \\ &= {}^z\mathscr{A}_{\mu,-}, \end{aligned} $$

where the last equality is Definition 4.28.

Now, we must identify the gradings of ${}^v\bar{A}^{\mu,\diamond}_{\phi,-}$ and ${}^z\bar{\mathscr{A}}_{\mu,-}$ under the isomorphism ${}^wA^{\diamond}_{\mu,+} = {}^z\mathscr{A}_{\mu,-}$ above.

First, we consider the grading on ${}^v\bar{A}^{\mu,\diamond}_{\phi,-}$. Let ${}^w\bar{T}_{\mu,+}$ be the graded ${}^v\bar{A}^{\mu}_{\phi,-}$-module equal to ${}^wT_{\mu,+}$ as an ${}^wA_{\mu,+}$-module, with the natural grading (this is well-defined, because ${}^v\bar{A}^{\mu}_{\phi,-}$ is Koszul). Then, by Section 2.6, we have ${}^v\bar{A}^{\mu,\diamond}_{\phi,-} = \operatorname{End}_{{}^wA_{\mu,+}}({}^w\bar{T}_{\mu,+})^{\mathrm{op}}$.

Next, we consider the grading on ${}^z\bar{\mathscr{A}}_{\mu,-}$. By Corollary A.6, we have ${}^z\bar{\mathscr{A}}_{\mu,-} = k\operatorname{End}_{{}^wZ_{S,\mu,+}}({}^w\bar{C}_{S,\mu,+}) = \operatorname{End}_{{}^wZ_{\mu,+}}({}^w\bar{C}_{\mu,+})$.

Let us identify ${}^wA_{\mu,+} = {}^w\mathscr{A}_{\mu,+}$ via (5.1). We must prove the following.

Claim 5.5. *There is a graded k-algebra isomorphism*

$$ \operatorname{End}_{{}^w\mathscr{A}_{\mu,+}}\left({}^w\bar{T}_{\mu,+}\right) = \operatorname{End}_{{}^wZ_{\mu,+}}\left({}^w\bar{C}_{\mu,+}\right). $$

To do that, we'll need some new material.

Consider the graded categories given by ${}^w\bar{\mathbf{O}}_{\mu,+} = {}^w\bar{\mathscr{A}}_{\mu,+}$-**gmod** and ${}^w\bar{\mathbf{O}}_{S,\mu,+} = {}^w\bar{\mathscr{A}}_{S,\mu,+}$-**gmod**. Since we have ${}^w\bar{T}_{\mu,+} \in {}^v\bar{A}^{\mu}_{\phi,-}$-**gmod** and ${}^w\bar{\mathscr{A}}_{\mu,+} = {}^v\bar{A}^{\mu}_{\phi,-}$ by Corollary 4.49, we may view ${}^w\bar{T}_{\mu,+}$ as an object of ${}^w\bar{\mathbf{O}}_{\mu,+}$.

We have an isomorphism of graded S-algebras ${}^w\bar{\mathscr{A}}_{S,\mu,+} = \mathrm{End}_{{}^w Z_{S,\mu,+}}({}^w\bar{B}_{S,\mu,+})^{\mathrm{op}}$. Consider the pair of adjoint functors $(\bar{\mathbb{V}}_S, \bar{\psi}_S)$ between ${}^w\bar{\mathbf{O}}_{S,\mu,+}$ and ${}^w\bar{Z}_{S,\mu,+}$-**gmod** given by

$$\bar{\mathbb{V}}_S = {}^w\bar{B}_{S,\mu,+} \otimes_{{}^w\bar{\mathscr{A}}_{S,\mu,+}} \bullet : {}^w\bar{\mathbf{O}}_{S,\mu,+} \to {}^w\bar{Z}_{S,\mu,+}\text{-}\mathbf{gmod},$$

$$\bar{\psi}_S = \mathrm{Hom}_{{}^w Z_{S,\mu,+}}({}^w\bar{B}_{S,\mu,+}, \bullet) : {}^w\bar{Z}_{S,\mu,+}\text{-}\mathbf{gmod} \to {}^w\bar{\mathbf{O}}_{S,\mu,+}.$$

We consider the object ${}^w\bar{T}_{S,\mu,+}$ of ${}^w\bar{\mathbf{O}}_{S,\mu,+}$ given by ${}^w\bar{T}_{S,\mu,+} = \bar{\psi}_S({}^w\bar{C}_{S,\mu,+})$.

Recall the functor $\mathbb{V}_{S_0} : {}^w\mathbf{O}^{\Delta}_{S_0,\mu,+} \to {}^w\mathbf{Z}^{\Delta}_{S_0,\mu,+}$ studied in Proposition 4.33. It yields an S_0-algebra isomorphism $\mathrm{End}_{{}^w\mathbf{O}_{S_0,\mu,+}}({}^wP_{S_0,\mu,+})^{\mathrm{op}} \to {}^w\mathscr{A}_{S_0,\mu,+}$. Thus, since ${}^wP_{S_0,\mu,+}$ is a pro-generator of ${}^w\mathbf{O}_{S_0,\mu,+}$, the functor $\phi_{S_0} = \mathrm{Hom}_{{}^w\mathbf{O}_{S_0,\mu,+}}({}^wP_{S_0,\mu,+}, \bullet)$ identifies ${}^w\mathbf{O}_{S_0,\mu,+}$ with the category ${}^w\mathscr{A}_{S_0,\mu,+}$-**mod**.

Consider the functor $\psi_{S_0} : {}^wZ_{S_0,\mu,+}$-**mod** $\to {}^w\mathscr{A}_{S_0,\mu,+}$-**mod** given by $\psi_{S_0} = \mathrm{Hom}_{{}^wZ_{S_0,\mu,+}}({}^wB_{S_0,\mu,+}, \bullet)$. We have the commutative triangle

$$\begin{array}{ccc} {}^w\mathbf{O}^{\Delta}_{S_0,\mu,+} & \xrightarrow[\sim]{\mathbb{V}_{S_0}} & {}^w\mathbf{Z}^{\Delta}_{S_0,\mu,+} \\ {\scriptstyle \phi_{S_0}}\big\downarrow & \swarrow{\scriptstyle \psi_{S_0}} & \\ {}^w\mathscr{A}_{S_0,\mu,+}\text{-}\mathbf{mod}, & & \end{array} \tag{5.6}$$

such that $\mathbb{V}_{S_0}({}^wT_{S_0,\mu,+}) = {}^wC_{S_0,\mu,+}$. Now, we define the module ${}^wT'_{S_0,\mu,+} = \psi_{S_0}({}^wC_{S_0,\mu,+})$ in ${}^w\mathscr{A}_{S_0,\mu,+}$-**mod**. It is identified with ${}^wT_{S_0,\mu,+}$ by ϕ_{S_0}, because (5.6) is commutative. By Proposition 4.33(c), the functor ψ_{S_0} yields a k-algebra isomorphism $\mathrm{End}_{{}^wZ_{S_0,\mu,+}}({}^wC_{S_0,\mu,+}) \to \mathrm{End}_{{}^w\mathscr{A}_{S_0,\mu,+}}({}^wT'_{S_0,\mu,+})$. Note that ϕ_{S_0} and ψ_{S_0} are both exact.

Since taking Hom's commutes with localization, we have $\varepsilon({}^w\bar{\mathscr{A}}_{S,\mu,+}) = {}^w\mathscr{A}_{S_0,\mu,+}$. Thus, forgetting the grading and taking base change yield a functor $\varepsilon : {}^w\bar{\mathbf{O}}_{S,\mu,+} \to {}^w\mathbf{O}_{S_0,\mu,+}$ which is faithful and faithfully exact, see Section 4.2. Let ${}^w\bar{\mathbf{O}}^{\Delta}_{S,\mu,+} \subset {}^w\bar{\mathbf{O}}_{S,\mu,+}$ be the full subcategory of the modules taken to ${}^w\mathbf{O}^{\Delta}_{S_0,\mu,+}$ by ε.

We identify ${}^w\mathbf{O}_{S_0,\mu,+}$ with ${}^w\mathscr{A}_{S_0,\mu,+}$-**mod** by ϕ_{S_0}. We have the following.

Claim 5.6. *We have the following commutative diagram*

$$\begin{array}{ccccc} {}^w\bar{\mathbf{O}}^{\Delta}_{S,\mu,+} & \xrightarrow{\bar{\mathbb{V}}_S} & {}^w\bar{\mathbf{Z}}^{\Delta}_{S,\mu,+} & \xrightarrow{\bar{\psi}_S} & {}^w\bar{\mathbf{O}}^{\Delta}_{S,\mu,+} \\ {\scriptstyle \varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon} & & {\scriptstyle \varepsilon}\big\downarrow \\ {}^w\mathbf{O}^{\Delta}_{S_0,\mu,+} & \xrightarrow{\mathbb{V}_{S_0}} & {}^w\mathbf{Z}^{\Delta}_{S_0,\mu,+} & \xrightarrow{\psi_{S_0}} & {}^w\mathbf{O}^{\Delta}_{S_0,\mu,+}. \end{array}$$

Since ${}^w\mathscr{A}_{S_0,\mu,+}$ is Noetherian, any finitely generated module has a finite presentation. By Proposition 4.33(c), the functor $\mathbb{V}_{S_0}$ is exact and we have $\mathbb{V}_{S_0}({}^wP_{S_0,\mu,+}) = {}^wB_{S_0,\mu,+}$. We identify ${}^w\mathbf{O}_{S_0,\mu,+}$ with ${}^w\mathscr{A}_{S_0,\mu,+}$-**mod** as above. By the five lemma, the canonical morphism ${}^wB_{S_0,\mu,+} \otimes_{{}^w\mathscr{A}_{S_0,\mu,+}} \bullet \to \mathbb{V}_{S_0}$ is an isomorphism. We deduce that the left square in Claim 5.6 is commutative.

Next, by definition we have $\varepsilon \circ \bar{\psi}_S = \psi_{S_0} \circ \varepsilon$. From the diagram 5.6, we deduce that ψ_{S_0} maps into ${}^w\mathbf{O}^{\Delta}_{S_0,\mu,+}$ and is a quasi-inverse to $\mathbb{V}_{S_0}$. Claim 5.6 follows.

From Claim 5.6 we deduce that $\varepsilon({}^w\bar{T}_{S,\mu,+}) = {}^wT_{S_0,\mu,+}$ and ${}^w\bar{T}_{S,\mu,+} \in {}^w\bar{\mathbf{O}}^{\Delta}_{S,\mu,+}$.

Next, since $\varepsilon : S$-**gmod** $\to S_0$-**mod** is faithfully exact by Section 4.2 and ψ_{S_0} gives an isomorphism $\mathrm{End}_{{}^wZ_{S_0,\mu,+}}({}^wC_{S_0,\mu,+}) \to \mathrm{End}_{{}^w\mathscr{A}_{S_0,\mu,+}}({}^wT_{S_0,\mu,+})$, we deduce from Claim 5.6 that $\bar{\psi}_S$ gives a graded k-algebra isomorphism $\mathrm{End}_{{}^wZ_{S,\mu,+}}({}^w\bar{C}_{S,\mu,+}) \to \mathrm{End}_{{}^w\mathscr{A}_{S,\mu,+}}({}^w\bar{T}_{S,\mu,+})$.

Finally, since the functor ψ_{S_0} is exact and ε is faithfully exact by Section 4.2, we deduce from Claim 5.6 that $\bar{\psi}_S$ is exact on ${}^w\bar{\mathbf{Z}}^{\Delta}_{S,\mu,+}$.

Now, by Definition 4.28, we have ${}^w\bar{\mathscr{A}}_{\mu,+} = k\,{}^w\bar{\mathscr{A}}_{S,\mu,+}$. Thus, the specialization at k gives the module $k\,{}^w\bar{T}_{S,\mu,+}$ in ${}^w\bar{\mathbf{O}}_{\mu,+} = {}^w\bar{\mathscr{A}}_{\mu,+}$-**gmod**.

Claim 5.7. *We have* $k\,{}^w\bar{T}_{S,\mu,+} = {}^w\bar{T}_{\mu,+}$.

The specialization gives a graded ring homomorphism $k\,\mathrm{End}_{{}^w\mathscr{A}_{S,\mu,+}}({}^w\bar{T}_{S,\mu,+}) \to \mathrm{End}_{{}^w\mathscr{A}_{\mu,+}}(k\,{}^w\bar{T}_{S,\mu,+})$. Since $\varepsilon({}^w\bar{T}_{S,\mu,+}) = {}^wT_{S_0,\mu,+}$, forgetting the grading it is taken to the obvious map $k\,\mathrm{End}_{{}^w\mathbf{O}_{S_0,\mu,+}}({}^wT_{S_0,\mu,+}) \to \mathrm{End}_{{}^w\mathbf{O}_{\mu,+}}(k\,{}^wT_{S_0,\mu,+})$. The latter is an isomorphism by Proposition 2.4(b). Since $k\,{}^w\bar{T}_{S,\mu,+} = {}^w\bar{T}_{\mu,+}$, we deduce that $k\,\mathrm{End}_{{}^w\mathscr{A}_{S,\mu,+}}({}^w\bar{T}_{S,\mu,+}) = \mathrm{End}_{{}^w\mathscr{A}_{\mu,+}}({}^w\bar{T}_{\mu,+})$.

Since we have $k\,\mathrm{End}_{{}^wZ_{S,\mu,+}}({}^w\bar{C}_{S,\mu,+}) = \mathrm{End}_{{}^wZ_{\mu}}({}^w\bar{C}_{\mu,+})$ and $\mathrm{End}_{{}^wZ_{S,\mu,+}}({}^w\bar{C}_{S,\mu,+}) = \mathrm{End}_{{}^w\mathscr{A}_{S,\mu,+}}({}^w\bar{T}_{S,\mu,+})$, this implies Claim 5.5.

Now, we prove Claim 5.7. The grading of ${}^w\bar{T}_{\mu,+}$ is characterized in the following way. First, we have a decomposition ${}^w\bar{T}_{\mu,+} = \bigoplus_x {}^w\bar{T}(x \bullet o_{\mu,+})$ where ${}^w\bar{T}(x \bullet o_{\mu,+})$ is the natural graded lift of ${}^wT(x \bullet o_{\mu,+})$. Since ${}^wT(x \bullet o_{\mu,+})$ is indecomposable in ${}^w\mathbf{O}_{\mu,+}$, it admits at most one graded lift in ${}^w\bar{\mathbf{O}}_{\mu,+}$, up to grading shift. Let $\bar{V}(x \bullet o_{\mu,+})$ be the graded ${}^v\bar{A}^{\mu}_{\phi,-}$-module equal to $V(x \bullet o_{\mu,+})$ as an ${}^wA_{\mu,+}$-module, with the natural grading (this is well-defined, because ${}^v\bar{A}^{\mu}_{\phi,-}$ is Koszul). The natural grading is characterized by the fact that there is an inclusion $\bar{V}(x \bullet o_{\mu,+}) \subset {}^w\bar{T}(x \bullet o_{\mu,+})$ which is homogeneous of degree 0.

The graded object ${}^w\bar{T}_{S,\mu,+}$ satisfies a similar property. Indeed, by Lemma 4.24, the graded S-sheaf ${}^w\bar{C}_{S,\mu,+}(x)$ is filtered by shifted Verma-sheaves, and the lower term of this filtration yields an inclusion $\bar{V}_{S,\mu}(x)\langle l(x)\rangle \subset {}^w\bar{C}_{S,\mu,+}(x)$. Further, consider the object ${}^w\bar{T}_S(x \bullet o_{\mu,+})$ in ${}^w\bar{\mathbf{O}}_{S,\mu,+}$ given by ${}^w\bar{T}_S(x \bullet o_{\mu,+}) = \bar{\psi}_S({}^w\bar{C}_{S,\mu,+}(x))$. Then, we have the decomposition ${}^w\bar{T}_{S,\mu,+} = \bigoplus_x {}^w\bar{T}_S(x \bullet o_{\mu,+})$.

For each $x \in {}^wI_{\mu,+}$, we consider also the object $\bar{V}_S(x \bullet o_{\mu,+})$ in ${}^w\bar{\mathbf{O}}_{S,\mu,+}$ given by $\bar{V}_S(x \bullet o_{\mu,+}) = \bar{\psi}_S(\bar{V}_{S,\mu}(x))\langle l(x)\rangle$. Since $\bar{\psi}_S$ is exact, we deduce that ${}^w\bar{T}_S(x \bullet o_{\mu,+})$

is filtered by $\bar{V}_S(y \bullet \mathrm{o}_{\mu,+})$'s, and the lower term of this filtration yields an inclusion $\bar{V}_S(x \bullet \mathrm{o}_{\mu,+}) \subset {}^w\bar{T}_S(x \bullet \mathrm{o}_{\mu,+})$ which is homogeneous of degree 0.

Now, by Proposition 4.33(*c*), we have $\mathbb{V}_{S_0}(V_{S_0}(x \bullet \mathrm{o}_{\mu,+})) = V_{S_0,\mu}(x)$. From the proof of Claim 5.6, we deduce that $\psi_{S_0}(V_{S_0,\mu}(x)) = V_{S_0}(x \bullet \mathrm{o}_{\mu,+})$. Thus

$$\varepsilon\big(\bar{V}_S(x \bullet \mathrm{o}_{\mu,+})\big) = \varepsilon\bar{\psi}_S\big(\bar{V}_{S,\mu}(x)\big) = \psi_{S_0}\varepsilon\big(\bar{V}_{S,\mu}(x)\big) = V_{S_0}(x \bullet \mathrm{o}_{\mu,+}).$$

Since $V_{S_0}(x\bullet\mathrm{o}_{\mu,+})$ is free over S_0, we deduce that $\bar{V}_S(x\bullet\mathrm{o}_{\mu,+})$ is free over S by Section 4.2. Therefore, the inclusion $\bar{V}_S(x \bullet \mathrm{o}_{\mu,+}) \subset {}^w\bar{T}_S(x \bullet \mathrm{o}_{\mu,+})$ gives an inclusion $k\bar{V}_S(x \bullet \mathrm{o}_{\mu,+}) \subset k{}^w\bar{T}_S(x \bullet \mathrm{o}_{\mu,+})$ which is homogeneous of degree 0.

Hence, to prove Claim 5.7 we are reduced to check that we have $\bar{V}(x \bullet \mathrm{o}_{\mu,+}) = k\bar{V}_S(x \bullet \mathrm{o}_{\mu,+})$ in ${}^w\bar{\mathbf{O}}_{\mu,+}$. Note that $\bar{V}(x \bullet \mathrm{o}_{\mu,+})$ and $k\bar{V}_S(x \bullet \mathrm{o}_{\mu,+})$ are both graded lifts of the Verma module $V(x \bullet \mathrm{o}_{\mu,+})$, which is indecomposable. Thus they coincide up to a grading shift.

To identify this shift, recall that by Lemma 4.24 we have a surjection ${}^w\bar{B}_{S,\mu,+}(x) \to \bar{V}_S(x)\langle l(x)\rangle$. We define ${}^w\bar{P}_S(x \bullet \mathrm{o}_{\mu,+}) = \bar{\psi}_S({}^w\bar{B}_{S,\mu,+}(x))$. Since $\bar{\psi}_S$ is exact, we have a surjection ${}^w\bar{P}_S(x \bullet \mathrm{o}_{\mu,+}) \to \bar{V}_S(x \bullet \mathrm{o}_{\mu,+})$.

Now, since ${}^v\bar{A}^{\mu}_{\phi,-}$ is Koszul, we can consider the natural graded lift ${}^w\bar{P}(x \bullet \mathrm{o}_{\mu,+})$ of ${}^wP(x \bullet \mathrm{o}_{\mu,+})$ in ${}^w\bar{\mathbf{O}}_{\mu,+}$. By definition of the natural grading, we have a surjection ${}^w\bar{P}(x \bullet \mathrm{o}_{\mu,+}) \to \bar{V}(x \bullet \mathrm{o}_{\mu,+})$.

So, we must prove that ${}^w\bar{P}(x \bullet \mathrm{o}_{\mu,+}) = k{}^w\bar{P}_S(x \bullet \mathrm{o}_{\mu,+})$ in ${}^w\bar{\mathbf{O}}_{\mu,+}$. This is obvious, because ${}^w\bar{P}(x \bullet \mathrm{o}_{\mu,+}) = {}^w\bar{\mathscr{A}}_{\mu,+}1_x$ by definition of the natural grading of ${}^wP(x \bullet \mathrm{o}_{\mu,+})$, and because Definition 4.28 yields the chain of isomorphisms

$${}^w\bar{P}_S(x \bullet \mathrm{o}_{\mu,+}) = \bar{\psi}_S\big({}^w\bar{B}_{S,\mu,+}(x)\big) = \mathrm{End}_{{}^wZ_{S,\mu,+}}\big({}^w\bar{B}_{S,\mu,+}\big)^{\mathrm{op}}1_x = {}^w\bar{\mathscr{A}}_{\mu,+}1_x. \qquad \square$$

We can now prove the following graded analogue of Corollary 4.34. See also Corollary 4.49.

Corollary 5.8. *Assume that $u \in I^{\mu}_{\phi,+}$. Let $z = u_-^{-1} \in I_{\mu,-}$. We have an isomorphism of graded k-algebras ${}^u\bar{A}^{\mu}_{\phi,+} \to {}^z\bar{\mathscr{A}}_{\mu,-}$ such that $1_y \mapsto 1_x$ with $x = y_-^{-1}$ for each $y \in {}^vI^{\mu}_{\phi,+}$.*

Proof. By [29, thm. 1] and Lemma 5.3, the graded k-algebra ${}^v\bar{A}^{\mu,\diamond}_{\phi,-} = {}^z\bar{\mathscr{A}}_{\mu,-}$ is Koszul. By Proposition 5.4, the graded k-algebra ${}^u\bar{A}^{\mu}_{\phi,+}$ is Koszul and is isomorphic to ${}^wA_{\mu,-}$ as a k-algebra. By Proposition 5.4, we have ${}^v\bar{A}^{\mu}_{\phi,-} = {}^wA_{\mu,+}$ as a k-algebra. Finally, by Proposition 3.9, the Ringel dual of ${}^wA_{\mu,+}$ is ${}^wA^{\diamond}_{\mu,+} = {}^zA_{\mu,-}$. Thus we have a k-algebra isomorphism ${}^zA_{\mu,-} = {}^z\mathscr{A}_{\mu,-}$, which lifts to a graded k-algebra isomorphism ${}^u\bar{A}^{\mu}_{\phi,+} = {}^z\bar{\mathscr{A}}_{\mu,-}$ by unicity of the Koszul grading [5, cor. 2.5.2]. $\square$

5.2. The general case

We can now complete the proof of Theorem 3.12. We first prove a series of preliminary lemmas.

Fix parabolic types $\mu, \nu \in \mathcal{P}$ and integers $d, e, f > 0$. Choose integral weights $o_{\mu,-}$, $o_{\nu,-}$, $o_{\phi,-}$ of level $-e-N$, $-f-N$ and $-d-N$ respectively.

Fix an element $w \in \widehat{W}$. Let $\tau_{\phi,\nu} : {}^w\mathbf{O}_{\mu,+} \to {}^w\mathbf{O}^\nu_{\mu,+}$ be the parabolic truncation functor, see Section 3.5. Applying the functor $\tau_{\phi,\nu}$ to the module ${}^wP_{\mu,+}$, we get a k-algebra homomorphism $\tau_{\phi,\nu} : {}^wA_{\mu,+} \to {}^wA^\nu_{\mu,+}$.

Lemma 5.9. *Assume that $w \in I^\nu_{\mu,+}$. Then, the k-algebra homomorphism $\tau_{\phi,\nu} : {}^wA_{\mu,+} \to {}^wA^\nu_{\mu,+}$ is surjective. Its kernel is the two-sided ideal generated by the idempotents 1_x with $x \in {}^wI_{\mu,+} \setminus {}^wI^\nu_{\mu,+}$. Further, we have $\tau_{\phi,\nu}(1_x) = 1_x$ for each $x \in {}^wI^\nu_{\mu,+}$.*

Proof. By Section 3.5, the functor $\tau_{\phi,\nu}$ takes the minimal projective generator of ${}^w\mathbf{O}_{\mu,+}$ to the minimal projective generator of ${}^w\mathbf{O}^\nu_{\mu,+}$. Let $i_{\nu,\phi}$ be the right adjoint of $\tau_{\phi,\nu}$. For any M the unit $M \to i_{\nu,\phi}\tau_{\phi,\nu}(M)$ is surjective. Hence, for any projective module P we have a surjective map

$$\mathrm{Hom}_{{}^w\mathbf{O}_{\mu,+}}(P, M) \to \mathrm{Hom}_{{}^w\mathbf{O}_{\mu,+}}\big(P, i_{\nu,\phi}\tau_{\phi,\nu}(M)\big) = \mathrm{Hom}_{{}^w\mathbf{O}^\nu_{\mu,+}}\big(\tau_{\phi,\nu}(P), \tau_{\phi,\nu}(M)\big).$$

Thus, the k-algebra homomorphism $\tau_{\phi,\nu}$ is surjective.

Let $I \subset {}^wA_{\mu,+}$ be the 2-sided ideal generated by the idempotents 1_x such that $x \in {}^wI_{\mu,+}$ and $\tau_{\phi,\nu}(1_x) = 0$. By Section 3.5(c), the latter are precisely the idempotents 1_x with $x \in {}^wI_{\mu,+} \setminus {}^wI^\nu_{\mu,+}$.

We have a k-algebra isomorphism ${}^wA_{\mu,+}/I \simeq {}^wA^\nu_{\mu,+}$, because ${}^w\mathbf{O}^\nu_{\mu,+}$ is the Serre subcategory of ${}^w\mathbf{O}_{\mu,+}$ generated by the simple modules killed by I. Under this isomorphism, the map $\tau_{\phi,\nu}$ is the canonical projection ${}^wA_{\mu,+} \to {}^wA_{\mu,+}/I$.

The last claim in the lemma follows from Section 3.5(b). □

Now, let $v \in I_{\nu,-}$ and $w = v_+^{-1}$. We have $w \in I^\nu_{\phi,+}$.

We equip the k-algebras ${}^vA_{\nu,-}$, ${}^vA_{\phi,-}$ with the gradings ${}^w\bar{A}^\nu_{\phi,+}$, ${}^w\bar{A}_{\phi,+}$, see Proposition 5.4. Consider the categories ${}^v\bar{\mathbf{O}}_{\nu,-} = {}^w\bar{A}^\nu_{\phi,+}$-**gmod** and ${}^v\bar{\mathbf{O}}_{\phi,-} = {}^w\bar{A}_{\phi,+}$-**gmod**, which are graded analogues of the categories ${}^v\mathbf{O}_{\nu,-}$ and ${}^v\mathbf{O}_{\phi,-}$.

To unburden the notation, we'll abbreviate $L_\nu = {}^vL_{\nu,-}$ and $L_\phi = {}^vL_{\phi,-}$. Let $\bar{L}_\nu$, $\bar{L}_\phi$ be the natural graded lifts in ${}^v\bar{\mathbf{O}}_{\nu,-}$, ${}^v\bar{\mathbf{O}}_{\phi,-}$ of the semi-simple modules L_ν, L_ϕ respectively.

For $v \in I_{\nu,-}$ and $d+N > f$, we'll use the following graded analogue of the translation functor $T_{\phi,\nu} : {}^v\mathbf{O}_{\phi,-} \to {}^v\mathbf{O}_{\nu,-}$ in Proposition 4.36.

Lemma 5.10. *Assume that $v \in I_{\nu,-}$ and $d+N > f$. Then, there is an exact functor of graded categories $\bar{T}_{\phi,\nu} : {}^v\bar{\mathbf{O}}_{\phi,-} \to {}^v\bar{\mathbf{O}}_{\nu,-}$ such that $\bar{T}_{\phi,\nu}(\bar{L}_\phi) = \bar{L}_\nu$ and $\bar{T}_{\phi,\nu}$ coincides with $T_{\phi,\nu}$ when forgetting the grading.*

Proof. First, note that the translation functor $T_{\phi,\nu} : {}^v\mathbf{O}_{\phi,-} \to {}^v\mathbf{O}_{\nu,-}$ is well-defined, because $d+N > f$ and $v \in I_{\nu,-}$. See Section 4.6 for more details.

Now, by Corollary 4.34, we have ${}^v\mathbf{O}_{\nu,-} = {}^v\mathscr{A}_{\nu,-}$-**mod** and ${}^v\mathbf{O}_{\phi,-} = {}^v\mathscr{A}_{\phi,-}$-**mod**. By Corollary 4.50, we can view ${}^vB_{\nu,-}$, ${}^vB_{\phi,-}$ as right modules over ${}^v\mathscr{A}_{\nu,-}$, ${}^v\mathscr{A}_{\phi,-}$ respectively.

Consider the functors $\mathbb{V}'_k : {}^v\mathbf{O}_{\nu,-} \to {}^v\mathbf{Z}_{\nu,-}$ and $\mathbb{V}'_k : {}^v\mathbf{O}_{\phi,-} \to {}^v\mathbf{Z}_{\phi,-}$ given by $\mathbb{V}'_k = {}^vB_{\nu,-} \otimes_{{}^v\mathscr{A}_{\nu,-}} \bullet$ and $\mathbb{V}'_k = {}^vB_{\phi,-} \otimes_{{}^v\mathscr{A}_{\phi,-}} \bullet$.

By Proposition 4.33, the functors $\mathbb{V}_k$ on ${}^v\mathbf{O}_{\nu,-}$, ${}^v\mathbf{O}_{\phi,-}$ are exact and we have $\mathbb{V}_k({}^v\mathscr{A}_{\nu,-}) = {}^vB_{\nu,-}$, $\mathbb{V}_k({}^v\mathscr{A}_{\phi,-}) = {}^vB_{\phi,-}$. Further, since ${}^v\mathscr{A}_{\nu,+}$, ${}^v\mathscr{A}_{\phi,+}$ are Noetherian, any finitely generated module has a finite presentation. Thus, by the five lemma, the obvious morphism of functors $\mathbb{V}'_k \to \mathbb{V}_k$ is invertible.

Next, by Corollary 5.8, we have ${}^w\bar{A}^{\nu}_{\phi,+} = {}^v\bar{\mathscr{A}}_{\nu,-}$ and ${}^w\bar{A}_{\phi,+} = {}^v\bar{\mathscr{A}}_{\phi,-}$ as graded k-algebras. Thus, we have ${}^v\bar{\mathbf{O}}_{\nu,-} = {}^v\bar{\mathscr{A}}_{\nu,-}$-**mod** and ${}^v\bar{\mathbf{O}}_{\phi,-} = {}^v\bar{\mathscr{A}}_{\phi,-}$-**mod**. Consider the functors on ${}^v\bar{\mathbf{O}}_{\nu,-}$, ${}^v\bar{\mathbf{O}}_{\phi,-}$ given by $\bar{\mathbb{V}}_k = {}^v\bar{B}_{\nu,-} \otimes_{{}^v\bar{\mathscr{A}}_{\nu,-}} \bullet$, $\bar{\mathbb{V}}_k = {}^v\bar{B}_{\phi,-} \otimes_{{}^v\bar{\mathscr{A}}_{\phi,-}} \bullet$ respectively. We have the commutative squares

$$\begin{array}{ccc} {}^v\bar{\mathbf{O}}_{\nu,-} & \xrightarrow[\bar{\mathbb{V}}_k]{} & {}^v\bar{\mathbf{Z}}_{\nu,-} \\ \downarrow & & \downarrow \\ {}^v\mathbf{O}_{\nu,-} & \xrightarrow{\mathbb{V}_k} & {}^v\mathbf{Z}_{\nu,-} \end{array} \qquad \begin{array}{ccc} {}^v\bar{\mathbf{O}}_{\phi,-} & \xrightarrow[\bar{\mathbb{V}}_k]{} & {}^v\bar{\mathbf{Z}}_{\phi,-} \\ \downarrow & & \downarrow \\ {}^v\mathbf{O}_{\phi,-} & \xrightarrow{\mathbb{V}_k} & {}^v\mathbf{Z}_{\phi,-}. \end{array}$$

Now, let ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-} \subset {}^v\bar{\mathbf{O}}_{\phi,-}$ be the full subcategory of the projective objects. We define ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\nu,-}$, ${}^v\mathbf{O}^{\mathrm{proj}}_{\nu,-}$ and ${}^v\mathbf{O}^{\mathrm{proj}}_{\phi,-}$ in a similar way.

The functor $\mathbb{V}_k$ is fully faithful on ${}^v\mathbf{O}^{\mathrm{proj}}_{\nu,-}$ and ${}^v\mathbf{O}^{\mathrm{proj}}_{\phi,-}$ by Corollary 4.50. Hence, the functor $\bar{\mathbb{V}}_k$ is fully faithful on ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\nu,-}$ and ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-}$. Therefore, we may identify ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\nu,-}$, ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-}$ with some full subcategories ${}^v\bar{\mathbf{Z}}^{\mathrm{proj}}_{\nu,-}$, ${}^v\bar{\mathbf{Z}}^{\mathrm{proj}}_{\phi,-}$ of ${}^v\bar{\mathbf{Z}}_{\nu,-}$, ${}^v\bar{\mathbf{Z}}_{\phi,-}$ via $\bar{\mathbb{V}}_k$

The functor $\bar{\theta}_{\phi,\nu}$ in Remark 4.42 gives an exact functor ${}^v\bar{\mathbf{Z}}^{\mathrm{proj}}_{\phi,-} \to {}^v\bar{\mathbf{Z}}^{\mathrm{proj}}_{\nu,-}$. We define the functor $\bar{T}_{\phi,\nu} : {}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-} \to {}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\nu,-}$ so that it coincides with $\bar{\theta}_{\phi,\nu}$ under $\bar{\mathbb{V}}_k$. It gives a functor of triangulated categories $\mathbf{K}^b({}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-}) \to \mathbf{K}^b({}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\nu,-})$, where $\mathbf{K}^b$ denotes the bounded homotopy category.

Since ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\nu,-}$ and ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-}$ have finite global dimensions, there are canonical equivalences $\mathbf{K}^b({}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-}) \simeq \mathbf{D}^b({}^v\bar{\mathbf{O}}_{\phi,-})$ and $\mathbf{K}^b({}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\nu,-}) \simeq \mathbf{D}^b({}^v\bar{\mathbf{O}}_{\nu,-})$. Thus, we can view $\bar{T}_{\phi,\nu}$ as a functor of triangulated categories $\mathbf{D}^b({}^v\bar{\mathbf{O}}_{\phi,-}) \to \mathbf{D}^b({}^v\bar{\mathbf{O}}_{\nu,-})$.

By Proposition 4.36(b), the functor $T_{\phi,\nu}$ gives an exact functor ${}^v\mathbf{O}^{\mathrm{proj}}_{\phi,-} \to {}^v\mathbf{O}^{\mathrm{proj}}_{\nu,-}$. By Remark 4.42, the functor $\bar{T}_{\phi,\nu}$ coincides with $T_{\phi,\nu}$ when forgetting the grading. The functor $T_{\phi,\nu}$ yields a functor of triangulated categories $\mathbf{D}^b({}^v\mathbf{O}_{\phi,-}) \to \mathbf{K}^b({}^v\mathbf{O}_{\nu,-})$ which is t-exact for the standard t-structures and which coincides with $T_{\phi,\nu}$ when forgetting the grading. Hence, the functor $\bar{T}_{\phi,\nu}$ is also t-exact. Thus, it gives an exact functor $\bar{T}_{\phi,\nu} : {}^v\bar{\mathbf{O}}_{\phi,-} \to {}^v\bar{\mathbf{O}}_{\nu,-}$ which coincides with $T_{\phi,\nu}$ when forgetting the grading.

Now, we concentrate on the equality $\bar{T}_{\phi,\nu}(\bar{L}_\phi) = \bar{L}_\nu$.

Since $\bar{T}_{\phi,\nu}$ coincides with $T_{\phi,\nu}$ when forgetting the grading, by Proposition 4.36(d), for each $x \in {}^wI_{\nu,-}$ there is an integer j such that $\bar{T}_{\phi,\nu}(\bar{L}(xw_\nu \bullet o_{\phi,-})) = \bar{L}(x \bullet o_{\nu,-})\langle j\rangle$. We must check that $j = 0$.

Let $\bar{T}_{\nu,\phi}$ be the left adjoint to $\bar{T}_{\phi,\nu}$, see Remark 5.11 below. By Corollary 5.8 we have ${}^v\bar{\mathbf{O}}_{\nu,-} = {}^v\bar{\mathscr{A}}_{\nu,-}$-**mod** and ${}^v\bar{\mathbf{O}}_{\phi,-} = {}^v\bar{\mathscr{A}}_{\phi,-}$-**mod**. The graded k-algebras ${}^w\bar{A}^{\nu}_{\phi,+}$, ${}^w\bar{A}_{\phi,+}$ ($= {}^v\bar{\mathscr{A}}_{\nu,-}$, ${}^v\bar{\mathscr{A}}_{\phi,-}$) are Koszul. The natural graded indecomposable projective modules are of the form ${}^v\bar{\mathscr{A}}_{\nu,-}1_x$, ${}^v\bar{\mathscr{A}}_{\phi,-}1_x$ with $x \in {}^vI_{\nu,-}$, ${}^vI_{\phi,-}$ respectively. We must check that $\bar{T}_{\nu,\phi}({}^v\bar{\mathscr{A}}_{\nu,-}1_x) = {}^v\bar{\mathscr{A}}_{\phi,-}1_{xw_\nu}$ for each $x \in {}^wI_{\nu,-}$.

By definition of $\bar{T}_{\phi,\nu}$ we have an isomorphism $\bar{\theta}_{\phi,\nu}\circ\bar{\mathbb{V}}_k \simeq \bar{\mathbb{V}}_k\circ\bar{T}_{\phi,\nu}$ of functors on ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-}$. Further, by definition of $\bar{\mathbb{V}}_k$, we have $\bar{\mathbb{V}}_k({}^v\bar{\mathscr{A}}_{\nu,-}1_x) = {}^v\bar{B}_{\nu,-}(x)$ and $\bar{\mathbb{V}}_k({}^v\bar{\mathscr{A}}_{\phi,-}1_{xw_\nu}) = {}^v\bar{B}_{\phi,-}(xw_\nu)$. Therefore, it is enough to check that $\bar{\theta}_{\nu,\phi}({}^v\bar{B}_{\nu,-}(x)) = {}^v\bar{B}_{\phi,-}(xw_\nu)$.

By Remark 4.42, this follows from Proposition 4.41(f) by base change. □

Remark 5.11. General facts imply that the functor $\bar{T}_{\phi,\nu}$ has a left adjoint $\bar{T}_{\nu,\phi} : {}^v\bar{\mathbf{O}}_{\nu,-} \to {}^v\bar{\mathbf{O}}_{\phi,-}$, see the proof of Proposition 4.36. By definition of $T_{\nu,\phi}$ and the unicity of the left adjoint, the functor $\bar{T}_{\nu,\phi}$ coincides with $T_{\nu,\phi}$ when forgetting the grading.

Now, we prove the following lemma which is dual to Lemma 5.9.

Lemma 5.12. *Assume that $v \in I_{\nu,-}$ and $d + N > f$. Then, the functor $T_{\phi,\nu} : {}^v\mathbf{O}_{\phi,-} \to {}^v\mathbf{O}_{\nu,-}$ induces a surjective graded k-algebra homomorphism $T_{\phi,\nu} : {}^v\bar{A}^\mu_{\phi,-} \to {}^v\bar{A}^\mu_{\nu,-}$. Its kernel contains the two-sided ideal generated by $\{1_x;\ x \in {}^vI^\mu_{\phi,-},\ x \notin I_{\nu,+}\}$. Further, for $x \in {}^vI^\mu_{\phi,-} \cap I_{\nu,+}$ we have $xw_\nu \in {}^vI^\mu_{\nu,-}$ and $T_{\phi,\nu}(1_x) = 1_{xw_\nu}$.*

Proof. First, note that, since $v \in I_{\nu,-}$ and $d+N > f$, the functor $T_{\phi,\nu} : {}^v\mathbf{O}_{\phi,-} \to {}^v\mathbf{O}_{\nu,-}$ is well-defined and it takes ${}^v\mathbf{O}^\mu_{\phi,-}$ into ${}^v\mathbf{O}^\mu_{\nu,-}$ by Proposition 4.36(c).

We'll abbreviate $L_\phi = {}^vL^\mu_{\phi,-}$ and $L_\nu = {}^vL^\mu_{\nu,-}$. By Proposition 4.36($f$), we have $T_{\phi,\nu}(L_\phi) = L_\nu$. Thus, since $T_{\phi,\nu}$ is exact, it induces a graded k-algebra homomorphism $T_{\phi,\nu} : {}^v\bar{A}^\mu_{\phi,-} = \mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}(L_\phi)^{\mathrm{op}} \to {}^v\bar{A}^\mu_{\nu,-} = \mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\nu,-}}(L_\nu)^{\mathrm{op}}$.

Composing $T_{\phi,\nu}$ with its left adjoint $T_{\nu,\phi}$, we get the functor $\Theta = T_{\nu,\phi}\circ T_{\phi,\nu}$. To prove that $T_{\phi,\nu}$ is surjective, we must prove that the counit $\Theta \to \mathbf{1}$ yields a surjective map $\mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}(L_\phi) \to \mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}(\Theta(L_\phi), L_\phi)$.

The parabolic inclusion ${}^v\mathbf{O}^\mu_{\phi,-} \subset {}^v\mathbf{O}_{\phi,-}$ is injective on extensions by Section 3.5. So we must prove that the counit yields a surjective map $\mathrm{Ext}_{{}^v\mathbf{O}_{\phi,-}}(L_\phi) \to \mathrm{Ext}_{{}^v\mathbf{O}_{\phi,-}}(\Theta(L_\phi), L_\phi)$.

Let us consider the graded analogue of this statement. Set $\bar{\Theta} = \bar{T}_{\nu,\phi}\circ\bar{T}_{\phi,\nu}$, where $\bar{T}_{\nu,\phi}$, $\bar{T}_{\phi,\nu}$ are as in Lemma 5.10 and Remark 5.11. We have

$$\mathrm{Ext}_{{}^v\mathbf{O}_{\phi,-}}(L_\phi) = \bigoplus_j \mathrm{Ext}_{{}^v\bar{\mathbf{O}}_{\phi,-}}\big(\bar{L}_\phi, \bar{L}_\phi\langle j\rangle\big),$$

$$\mathrm{Ext}_{{}^v\mathbf{O}_{\phi,-}}\big(\Theta(L_\phi), L_\phi\big) = \bigoplus_j \mathrm{Ext}_{{}^v\bar{\mathbf{O}}_{\phi,-}}\big(\bar{\Theta}(\bar{L}_\phi), \bar{L}_\phi\langle j\rangle\big).$$

Thus we must prove that for each i, j the counit $\eta : \bar{\Theta} \to \mathbf{1}$ yields a surjective map

$$\mathrm{Ext}^i_{{}^v\bar{\mathbf{O}}_{\phi,-}}\big(\bar{L}_\phi, \bar{L}_\phi\langle j\rangle\big) \to \mathrm{Ext}^i_{{}^v\bar{\mathbf{O}}_{\phi,-}}\big(\bar{\Theta}(\bar{L}_\phi), \bar{L}_\phi\langle j\rangle\big). \tag{5.7}$$

By Lemma 5.10, we have $\bar{T}_{\phi,\nu}(\bar{L}_\phi) = \bar{L}_\nu$. Further, since ${}^v\bar{\mathbf{O}}_{\nu,-} = {}^w\bar{A}^\nu_{\phi,+}$**-gmod** and ${}^v\bar{\mathbf{O}}_{\phi,-} = {}^w\bar{A}_{\phi,+}$**-gmod**, the gradings on ${}^v\bar{\mathbf{O}}_{\phi,-}$ and ${}^v\bar{\mathbf{O}}_{\nu,-}$ are Koszul by Proposition 5.4. Hence, since the right hand side of (5.7) is $\mathrm{Ext}^i_{{}^v\bar{\mathbf{O}}_{\nu,-}}(\bar{L}_\nu, \bar{L}_\nu\langle j\rangle)$, it is zero unless $i = j$.

Now, we define the integer $\ell = \min\{d;\ \bar{\Theta}(\bar{L}_\phi)^d \neq 0\}$. Recall that the grading on ${}^w\bar{A}_{\phi,+}$ is positive. Further, since ${}^v\bar{\mathbf{O}}_{\phi,-} = {}^w\bar{A}_{\phi,+}$-**gmod**, we can view $\bar{\Theta}(\bar{L}_\phi)$ as a graded ${}^w\bar{A}_{\phi,+}$-module. Thus $\bar{\Theta}(\bar{L}_\phi)^\ell$ is a quotient of $\bar{\Theta}(\bar{L}_\phi)$ which is killed by the radical of ${}^w\bar{A}_{\phi,+}$. We deduce that $\bar{\Theta}(\bar{L}_\phi)^\ell \subset \operatorname{top}(\bar{\Theta}(\bar{L}_\phi))$.

Next, we claim that for any simple graded ${}^w\bar{A}_{\phi,+}$-module $\bar{L}$ such that $\bar{T}_{\phi,\nu}(\bar{L}) \neq 0$, the map $\eta(\bar{L}) : \bar{\Theta}(\bar{L}) \to \bar{L}$ yields an isomorphism $\operatorname{top}(\bar{\Theta}(\bar{L})) \to \bar{L}$. Indeed, $\eta(\bar{L})$ is surjective because it is non-zero, and for any simple quotient $\bar{\Theta}(\bar{L}) \to \bar{L}'$ we have

$$0 \neq \operatorname{Hom}_{{}^v\bar{\mathbf{O}}_{\phi,-}}\big(\bar{\Theta}(\bar{L}), \bar{L}'\big) = \operatorname{Hom}_{{}^v\bar{\mathbf{O}}_{\nu}}\big(\bar{T}_{\phi,\nu}(\bar{L}), \bar{T}_{\phi,\nu}(\bar{L}')\big).$$

By Proposition 4.36(d), (e) this implies that $\bar{T}_{\phi,\nu}(\bar{L}) = \bar{T}_{\phi,\nu}(\bar{L}')$, and that it is non-zero. Therefore, we have $\bar{L} = \bar{L}'$, proving the claim.

Recall that $\bar{L}_\phi$ is a semi-simple module, see Section 3.4. Applying the claim to the simple summands $\bar{L} \subset \bar{L}_\phi$ such that $\bar{T}_{\phi,\nu}(\bar{L}) \neq 0$, we get that $\operatorname{top}(\bar{\Theta}(\bar{L}_\phi)) = \operatorname{Im}\eta(\bar{L}_\phi)$. In particular $\operatorname{top}(\bar{\Theta}(\bar{L}_\phi))$ is pure of degree zero. This implies that we have $\ell = 0$. Therefore, the kernel of $\eta(\bar{L}_\phi)$ lives in degrees > 0. Hence, by Koszulity of the grading of ${}^v\bar{\mathbf{O}}_{\phi,-}$, we get

$$\operatorname{Ext}^i_{{}^v\bar{\mathbf{O}}_{\phi,-}}\big(\operatorname{Ker}\eta(\bar{L}_\phi), \bar{L}_\phi\langle i\rangle\big) = 0.$$

Hence the surjectivity of (5.7) for $i = j$ follows from the long exact sequence of Ext's groups associated with the exact sequence

$$0 \to \operatorname{Ker}\big(\eta(\bar{L}_\phi)\big) \to \bar{\Theta}(\bar{L}_\phi) \to \bar{L}_\phi.$$

This proves that the map $T_{\phi,\nu}$ is surjective, proving the first part of the lemma.

Next, we have $I^\mu_{\nu,-} = \{xw_\nu;\ x \in I^\mu_{\phi,-} \cap I_{\nu,+}\}$ by Corollary 3.3(a). Further, we have $T_{\phi,\nu}(1_{xw_\nu}) = 0$ for $x \notin I_{\nu,-}$ by Proposition 4.36(e), and $xw_\nu \in I_{\nu,+}$ if and only if $x \in I_{\nu,-}$. This proves the second claim of the lemma. Finally, the last claim of the lemma follows from Proposition 4.36(d). $\square$

Next, we prove the following.

Lemma 5.13. *Assume that $w \in I^\nu_{\mu,+}$. Let $v = w_-^{-1}$. We have $v \in I^\mu_{\nu,-}$. Assume also that $d + N > f$ and $e + N > d$. There is a k-algebra isomorphism $p_{\mu,\nu} : {}^wA^\nu{}_{\mu,+} \to {}^v\bar{A}^\mu_{\nu,-}$ such that $p_{\mu,\nu}(1_x) = 1_y$ for each $x \in {}^wI^\nu_{\mu,+}$, where $y = x_-^{-1}$, and such that the following square is commutative*

$$\begin{array}{ccc} {}^{w_\nu w}A_{\mu,+} & \overset{(5.1)}{=\!=\!=} & {}^v\bar{A}^\mu_{\phi,-} \\ \Big\downarrow \tau_{\phi,\nu} & & T_{\phi,\nu} \Big\downarrow \\ {}^wA^\nu_{\mu,+} & \overset{p_{\mu,\nu}}{=\!=\!=} & {}^v\bar{A}^\mu_{\nu,-}. \end{array} \tag{5.8}$$

Proof. We have $v = w_\mu w^{-1} w_\nu$. Note that $w_\nu w \in I^\nu_{\phi,-} \cap I_{\mu,+}$, because by Lemma 3.2 we have $w^{-1} \in I^\mu_{\nu,+}$, hence $w^{-1}w_\nu \in I^\mu_{\phi,+} \cap I_{\nu,-}$ and $w_\nu w = (w^{-1}w_\nu)^{-1} \in I_{\mu,+} \cap I^\nu_{\phi,-}$.

Let $\pi_{\mu,\nu} : {}^{w_\nu w}A_{\mu,+} \to {}^v\bar{A}^\mu_{\nu,-}$ be the composition of the k-algebra homomorphism $T_{\phi,\nu} : {}^v\bar{A}^\mu_{\phi,-} \to {}^v\bar{A}^\mu_{\nu,-}$ in Lemma 5.12 and of the k-algebra isomorphism ${}^{w_\nu w}A_{\mu,+} = {}^v\bar{A}^\mu_{\phi,-}$ in (5.1).

Note that ${}^{w_\nu w}A^\nu_{\mu,+} = {}^wA^\nu_{\mu,+}$. We must construct a k-algebra isomorphism $p_{\mu,\nu} : {}^wA^\nu_{\mu,+} \to {}^v\bar{A}^\mu_{\nu,-}$ such that $\pi_{\mu,\nu} = p_{\mu,\nu} \circ \tau_{\phi,\nu}$.

Let $x \in {}^{w_\nu w}I_{\mu,+}$. Thus $w_\mu x^{-1} \in {}^vI^\mu_{\phi,-}$. By Lemma 5.12, we have

$$\pi_{\mu,\nu}(1_x) \neq 0 \quad \Leftrightarrow \quad T_{\phi,\nu}(1_{w_\mu x^{-1}}) \neq 0 \quad \Leftrightarrow \quad w_\mu x^{-1} \in I^\mu_{\phi,-} \cap I_{\nu,+}.$$

By Lemma 5.9, we have

$$\tau_{\phi,\nu}(1_x) \neq 0 \quad \Leftrightarrow \quad x \in I^\nu_{\mu,+}.$$

Again by Lemma 3.2, we have

$$w_\mu x^{-1} \in I^\mu_{\phi,-} \cap I_{\nu,+} \quad \Leftrightarrow \quad x w_\mu \in I^\nu_{\phi,+} \cap I_{\mu,-} \quad \Leftrightarrow \quad x \in I^\nu_{\mu,+}.$$

Hence, we have $\tau_{\phi,\nu}(1_x) = 0$ if and only if $\pi_{\mu,\nu}(1_x) = 0$. Thus, we have $\mathrm{Ker}(\tau_{\phi,\nu}) \subset \mathrm{Ker}(\pi_{\mu,\nu})$, because the left hand side is generated by the 1_x's killed by $\tau_{\phi,\nu}$ and the right hand side contains the 1_x's killed by $T_{\phi,\nu}$.

This proves the existence of a k-algebra homomorphism $p_{\mu,\nu}$ such that $\pi_{\mu,\nu} = p_{\mu,\nu} \circ \tau_{\phi,\nu}$ and $p_{\mu,\nu}(1_x) = 1_{w_\mu x^{-1}}$ for each $x \in {}^wI^\nu_{\mu,+}$. The map $p_{\mu,\nu}$ is surjective, because the map $T_{\phi,\nu}$ is surjective by Lemma 5.12.

Now, we prove that $p_{\mu,\nu}$ is injective. By Section 3.5, the parabolic inclusion functor $i_{\mu,\phi} : {}^v\mathbf{O}^\mu_{\nu,-} \to {}^v\mathbf{O}_{\nu,-}$ yields a graded k-algebra homomorphism $i_{\mu,\phi} : {}^v\bar{A}^\mu_{\nu,-} \to {}^v\bar{A}_{\nu,-}$. Set $z = v^{-1} = w_\nu w w_\mu$. The following is proved below.

Claim 5.14. *We have the commutative diagram*

$$\begin{array}{ccccc}
{}^{w_\nu w}A_{\mu,+} & \overset{(5.1)}{=\!=\!=} & {}^v\bar{A}^\mu_{\phi,-} & \xrightarrow{T_{\phi,\nu}} & {}^v\bar{A}^\mu_{\nu,-} \\
\downarrow{\scriptstyle T_{\mu,\phi}} & & \downarrow{\scriptstyle i_{\mu,\phi}} & & \downarrow{\scriptstyle i_{\mu,\phi}} \\
{}^zA_{\phi,+} & \overset{(5.1)}{=\!=\!=} & {}^v\bar{A}_{\phi,-} & \xrightarrow{T_{\phi,\nu}} & {}^v\bar{A}_{\nu,-}.
\end{array} \tag{5.9}$$

Note that $z \in I_{\mu,-}$, hence the map $T_{\mu,\phi}$ is well-defined.

Now, by Proposition 4.36(g), the translation functor $T_{\mu,\phi}$ yields a map ${}^wA^\nu_{\mu,+} \to {}^{ww_\mu}A^\nu_{\phi,+}$. Consider the diagram

$$\begin{array}{ccccc} {}^{w_\nu w}A_{\mu,+} & \xrightarrow[\tau_{\phi,\nu}]{} & {}^{w}A^{\nu}_{\mu,+} & \xrightarrow[p_{\mu,\nu}]{} & {}^{v}\bar{A}^{\mu}_{\nu,-} \\ \Big\downarrow{\scriptstyle T_{\mu,\phi}} & & \Big\downarrow{\scriptstyle T_{\mu,\phi}} & & {\scriptstyle i_{\mu,\phi}}\Big\downarrow \\ {}^{z}A_{\phi,+} & \xrightarrow{\tau_{\phi,\nu}} & {}^{ww_\mu}A^{\nu}_{\phi,+} & \xrightarrow{p_{\phi,\nu}} & {}^{v}\bar{A}_{\nu,-}. \end{array} \tag{5.10}$$

By Claim 5.14, the outer rectangle in (5.9) is commutative. Thus, the outer rectangle in (5.10) is commutative. The left square in (5.10) is commutative, by Proposition 4.36(c). Thus, since $\tau_{\phi,\nu}$ is surjective, the right square in (5.10) is also commutative.

The middle vertical map in (5.10) is injective by Remark 4.38. Therefore, to prove that $p_{\mu,\nu}$ is injective it is enough to check that $p_{\phi,\nu}$ is injective.

Now, it is easy to see that the map $p_{\phi,\nu}$ is indeed invertible, because it is surjective by the discussion above (applied to the choice $\nu = \phi$) and $\dim({}^{ww_\mu}A_{\phi,+}{}^{\nu}) = \dim({}^{v}\bar{A}_{\nu,-})$ by Proposition 5.4.

Finally, to finish the proof of Lemma 5.13 we must check that $p_{\mu,\nu}(1_x) = 1_y$ for each $x \in {}^{w}I^{\nu}_{\mu,+}$, where $y = x_-^{-1}$.

To do that, it suffices to observe that $x_- = w_\nu x w_\mu$, and that, by Proposition 5.1 and Lemmas 5.9, 5.12, the square of maps in (5.8) gives the following diagram

$$\begin{array}{ccc} 1_x & \xmapsto{(5.1)} & 1_{w_\mu x^{-1}} \\ \Big\downarrow{\scriptstyle \tau_{\phi,\nu}} & & {\scriptstyle T_{\phi,\nu}}\Big\downarrow \\ 1_x & \xmapsto{p_{\mu,\nu}} & 1_{w_\mu x^{-1} w_\nu}. \end{array}$$

Now, we prove Claim 5.14. The right square in (5.9) is commutative, by Proposition 4.36(c).

Let us concentrate on the left square. We must prove that the isomorphisms ${}^{w_\nu w}A_{\mu,+} = {}^{v}\bar{A}^{\mu}_{\phi,-}$ and ${}^{w}A_{\phi,+} = {}^{v}\bar{A}_{\phi,-}$ in Proposition 5.1 yield a commutative square

$$\begin{array}{ccc} {}^{w_\nu w}A_{\mu,+} & \overset{}{\underset{(5.1)}{=\!=\!=}} & {}^{v}\bar{A}^{\mu}_{\phi,-} \\ \Big\downarrow{\scriptstyle T_{\mu,\phi}} & & {\scriptstyle i}\Big\downarrow \\ {}^{z}A_{\phi,+} & \overset{(5.1)}{=\!=\!=} & {}^{v}\bar{A}_{\phi,-}. \end{array} \tag{5.11}$$

By Proposition 4.36(g), the module $T_{\mu,\phi}({}^{w_\nu w}P_{\mu,+})$ is a direct summand of ${}^{z}P_{\phi,+}$. By Proposition 4.41(f) and Remark 4.42, the sheaf $\theta_{\mu,\phi}({}^{w}B_{\mu,+})$ is a direct summand of ${}^{w}B_{\phi,+}$. Thus, we have the following diagram

$$\begin{array}{ccc} {}^{w_\nu w}A_{\mu,+} & \overset{\mathbb{V}_k}{=\!=\!=} & \mathrm{End}_{{}^{w_\nu w}Z_{\mu,+}}({}^{w_\nu w}B_{\mu,+})^{\mathrm{op}} \\ \downarrow{\scriptstyle T_{\mu,\phi}} & & \downarrow{\scriptstyle \theta_{\mu,\phi}} \\ {}^{z}A_{\phi,+} & \overset{\mathbb{V}_k}{=\!=\!=} & \mathrm{End}_{{}^{z}Z_{\phi,+}}({}^{z}B_{\phi,+})^{\mathrm{op}}. \end{array}$$

Note that the horizontal maps are invertible by Corollary 4.50. This diagram is commutative by Remark 4.42, see also Proposition 4.41(a). Next, by Proposition 4.48(c) and Corollary 4.49, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{End}_{{}^{w_\nu w}Z_{\mu,+}}({}^{w_\nu w}B_{\mu,+})^{\mathrm{op}} & \overset{\mathbb{H}}{=\!=\!=} & {}^{v}\bar{A}^{\mu}_{\phi,-} \\ \downarrow{\scriptstyle \theta_{\mu,\phi}} & & \downarrow{\scriptstyle i} \\ \mathrm{End}_{{}^{z}Z_{\phi,+}}({}^{z}B_{\phi,+})^{\mathrm{op}} & \overset{\mathbb{H}}{=\!=\!=} & {}^{v}\bar{A}_{\phi,-}. \end{array}$$

Finally, the horizontal maps in (5.11) are equal to the composition of $\mathbb{H}$ and $\mathbb{V}_k$. □

Finally, we prove our main theorem.

Proof of Theorem 3.12. Since the highest weight categories ${}^{w}\mathbf{O}^{\nu}_{\mu,+}$, ${}^{v}\mathbf{O}^{\mu}_{\nu,-}$ do not depend on e, f by Remark 3.10, we can assume that there is a positive integer d such that $d+N>f$ and $e+N>d$. Thus the hypothesis of Lemma 5.13 is satisfied.

By Propositions 5.1, 5.4 the k-algebra ${}^{w}A_{\mu,\pm}$ has a Koszul grading. Thus, by Lemma 2.2 and Section 3.5, the k-algebra ${}^{w}A^{\nu}_{\mu,\pm}$ has also a Koszul grading.

Let us equip ${}^{w}A^{\nu}_{\mu,\pm}$ with this grading. By Lemma 2.1, we have ${}^{w}A^{\nu,!}_{\mu,\pm}={}^{w}\bar{A}^{\nu}_{\mu,\pm}$ as graded k-algebras. Therefore, the graded k-algebra ${}^{w}\bar{A}^{\nu}_{\mu,\pm}$ is Koszul and its Koszul dual is isomorphic to ${}^{w}A^{\nu}_{\mu,\pm}$ as a k-algebra.

We have ${}^{w}\bar{A}^{\nu,!}_{\mu,+}={}^{w}A^{\nu}_{\mu,+}$ as a k-algebra. By Lemma 5.13, we have also a k-algebra isomorphism ${}^{w}A^{\nu}_{\mu,+}={}^{v}\bar{A}^{\mu}_{\nu,-}$. Thus, we have ${}^{w}\bar{A}^{\nu,!}_{\mu,+}={}^{v}\bar{A}^{\mu}_{\nu,-}$ as k-algebras. By unicity of the Koszul grading, we deduce that ${}^{w}\bar{A}^{\nu,!}_{\mu,+}={}^{v}\bar{A}^{\mu}_{\nu,-}$ as graded k-algebras.

The involutivity of the Koszul duality implies that we have also ${}^{w}\bar{A}^{\nu,!}_{\mu,-}={}^{v}\bar{A}^{\mu}_{\nu,+}$ as graded k-algebras, and ${}^{w}A^{\nu}_{\mu,-}={}^{v}\bar{A}^{\mu}_{\nu,+}$ as k-algebras.

Finally, we must check that under the isomorphism ${}^{w}\bar{A}^{\nu,!}_{\mu,+}={}^{v}\bar{A}^{\mu}_{\nu,-}$ we have $1^{!}_{x}=1_{y}$ with $y=x_{-}^{-1}$ for each $x\in{}^{w}I^{\nu}_{\mu,+}$.

If $\nu=\phi$ this is Proposition 5.1. If $\mu=\phi$ this is Proposition 5.4.

The isomorphism ${}^{w}\bar{A}^{\nu,!}_{\mu,+}={}^{w}A^{\nu}_{\mu,+}$ above, which is given by Lemma 2.1, identifies the idempotents $1^{!}_{x}$ and 1_{x} for each $x\in{}^{w}I^{\nu}_{\mu,+}$. Thus, by Lemma 5.13, the isomorphism ${}^{w}\bar{A}^{\nu,!}_{\mu,+}={}^{v}\bar{A}^{\mu}_{\nu,-}$ identifies the idempotents $1^{!}_{x}$ and 1_{y}, where $y=x_{-}^{-1}$. □

6. Type A and applications to CRDAHA

Fix integers $e,\ell,N>0$. Let $\mathfrak{g}=\mathfrak{gl}(N)$.

6.1. Koszul duality in the type A case

Let $\mathfrak{b}, \mathfrak{t} \subset \mathfrak{g}$ be the Borel subalgebra of upper triangular matrices and the maximal torus of diagonal matrices. Let (ϵ_i) be the canonical basis of $\mathbb{C}^N$. We identify $\mathfrak{t}^* = \mathbb{C}^N$, $\mathfrak{t} = \mathbb{C}^N$ and $W = \mathfrak{S}_N$ in the obvious way. Put $\rho = (0, -1, \ldots, 1-N)$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for each $i \in [1, N)$.

We define the affine Lie algebra $\mathbf{g}$ of $\mathfrak{g}$ as in Section 3.2. For any subset $X \subset \mathbb{C}^\ell$ and any $d \in \mathbb{C}$ let $X(d) = \{(x_1, \ldots, x_\ell) \in X;\ \sum_i x_i = d\}$. Set $\mathscr{C}^\ell_d = \mathbb{N}^\ell(d)$ and fix an element $\nu \in \mathscr{C}^\ell_N$. Let ν denote also the parabolic type $\{\alpha_i;\ i \neq \nu_1, \nu_1 + \nu_2, \ldots\}$. Let $\mathbf{p}_\nu \subset \mathbf{g}$ be the corresponding parabolic subalgebra. The Levi subalgebra of $\mathbf{p}_\nu$ is the Lie subalgebra $\mathfrak{g}_\nu = \mathfrak{gl}(\nu_1) \oplus \ldots \oplus \mathfrak{gl}(\nu_\ell)$ of $\mathfrak{g}$ consisting of block diagonal elements.

Let $P = \mathbb{Z}^N$ be the set of integral weights of $\mathfrak{g}$ and let $P^\nu \subset P$ be the subset of ν-dominant integral weights. Fix an element $\mu \in \mathscr{C}^e_N$. Consider the N-tuple $1_\mu = (1^{\mu_1} 2^{\mu_2} \cdots e^{\mu_e})$. Since $1_\mu \in P + \rho$, the affine weight $o_{\mu,-} = (1_\mu - \rho)_e$ is an antidominant integral classical affine weight of $\mathbf{g}$ of level $-e-N$, see Section 3.3.

Recall that $W_\mu \subset W$ is the parabolic subgroup generated by the simple affine reflections s_i with $\alpha_i \notin \mu$. Let $\widetilde{W} = \mathfrak{S}_N \ltimes \mathbb{Z}^N$ be the extended affine symmetric group. We define the *e-action of $\widetilde{W}$ on* $\mathbb{Z}^N$ to be such that $\mathfrak{S}_N$ acts by permutation of the entries of an N-tuple, while $\tau \in \mathbb{Z}^N$ acts by translation by the N-tuple $-e\tau$. Let $w \cdot_e x$ be the result of the e-action of w on the element $x \in \mathbb{Z}^N$. The stabilizer of the N-tuple 1_μ is equal to W_μ. It is a standard parabolic subgroup.

Consider the analogue for $\mathfrak{g}$ of the abelian category $\mathbf{O}^\nu_{\mu,-}$ introduced in Section 3.3. See below for more details. It is canonically equivalent to an inductive limit of sums of blocks of truncated categories O of the affine Lie algebra associated with $\mathfrak{sl}(N)$, because $\mathfrak{sl}(N)$ differs from $\mathfrak{g}$ by a central element. Hence, all the results above can be applied to the category $\mathbf{O}^\nu_{\mu,-}$. We'll write $\mathbf{O}^\nu_{\mu,-e} = \mathbf{O}^\nu_{\mu,-}$ and $\mathbf{O}^\nu_{-e} = \bigoplus_{\mu \in \mathscr{C}^e_N} \mathbf{O}^\nu_{\mu,-e}$. The classes $[V^\nu(\lambda_e)]$ with $\lambda \in P^\nu$ form a $\mathbb{C}$-basis of the complexified Grothendieck group $[\mathbf{O}^\nu_{-e}]$. Composing the Koszul equivalence in Theorem 3.12 and the tilting equivalence, we get a $\mathbb{C}$-linear isomorphism $K : [\mathbf{O}^\nu_{\mu,-e}] \to [\mathbf{O}^\mu_{\nu,-\ell}]$.

6.2. The level-rank duality

In this section, we'll relate the map K above to the level-rank duality. Consider the Lie algebra $\mathfrak{sl}(e)$. We identify the set of weights of $\mathfrak{sl}(e)$ with $\mathbb{C}^e/\mathbb{C}1^e$. Thus, elements of $\mathbb{C}^e$ can be viewed as weights of $\mathfrak{sl}(e)$, or equivalently, as level 0 classical affine weights of the affine Kac–Moody algebra $\widehat{\mathfrak{sl}}(e)$. Let $\{\varepsilon_i;\ i \in [1, e]\}$ be the canonical basis of $\mathbb{C}^e$. For each $\lambda \in P$ and each $k \in [1, N]$, we decompose the k-th entry of $\lambda + \rho$ as the sum $\lambda_k + \rho_k = i_k + e\,r_k$ with $i_k \in [1, e]$ and $r_k \in \mathbb{Z}$. Then, we can view the sum $\widehat{\mathrm{wt}}_e(\lambda) = \sum_{k=1}^N \varepsilon_{i_k} + (\sum_{k=1}^N r_k)\,\delta$ as a level 0 affine weight of $\widehat{\mathfrak{sl}}(e)$, and the sum $\mathrm{wt}_e(\lambda) = \sum_{k=1}^N \varepsilon_{i_k}$ as a weight of $\mathfrak{sl}(e)$.

The vector spaces $V_e = \mathbb{C}^e \otimes \mathbb{C}[z^{-1}, z]$ carries an obvious level zero action of $\widehat{\mathfrak{sl}}(e)$. It induces an action of $\widehat{\mathfrak{sl}}(e)$ on the space $\bigwedge^\nu(V_e) = \bigotimes_{p=1}^\ell \bigwedge^{\nu_p}(V_e)$ of tensor product of wedge powers. Write $v_{i+er} = \varepsilon_i \otimes z^r \in V_e$ for each $i \in [1, e]$, $r \in \mathbb{Z}$. Note that a tuple $a \in \mathbb{Z}^N$ of the form $a = (a_{1,1}, a_{1,2}, \dots, a_{\ell,\nu_\ell})$ belongs to $P^\nu + \rho$ if and only if we have $a_{p,1} > a_{p,2} > \cdots > a_{p,\nu_p}$ for each $p \in [1, \ell]$. Set $\wedge^\nu(a) = \bigotimes_{p=1}^{\ell}(v_{a_{p,1}} \wedge v_{a_{p,2}} \wedge \ldots \wedge v_{a_{p,\nu_p}})$. Then the set $\{\wedge^\nu(a);\ a \in P^\nu + \rho\}$ is a basis of $\bigwedge^\nu(V_e)$. Further, for each $\lambda \in P^\nu$ the element $\wedge^\nu(\lambda + \rho)$ has the weight $\widehat{\mathrm{wt}}_e(\lambda)$. Let $\bigwedge^\nu(V_e)_\mu$ be the weight subspace of $\bigwedge^\nu(V_e)$ of weight $\bar{\mu} = \sum_{i=1}^e \mu_i \varepsilon_i$ with respect to the action of $\mathfrak{sl}(e)$. The set $\{\wedge^\nu(a);\ a \in (P^\nu + \rho) \cap (\widetilde{W} \cdot_e 1_\mu)\}$ is a basis of $\bigwedge^\nu(V_e)_\mu$. In a similar way, we equip the vector space $V_\ell = \mathbb{C}^\ell \otimes \mathbb{C}[z^{-1}, z]$ with the basis $\{v_{p+\ell r};\ p \in [1, \ell], r \in \mathbb{Z}\}$. Then, we define the $\widehat{\mathfrak{sl}}(\ell)$-module $\bigwedge^\mu(V_\ell)$ as above.

Next, consider the $\widehat{\mathfrak{sl}}(e) \times \widehat{\mathfrak{sl}}(\ell)$-module $\bigwedge^N(V_{e,\ell})$ with $V_{e,\ell} = \mathbb{C}^e \otimes \mathbb{C}^\ell \otimes \mathbb{C}[z^{-1}, z]$. Let $\bar{\nu} = \sum_{p=1}^\ell \nu_p \varepsilon_p$, viewed as a weight of $\mathfrak{sl}(\ell)$. The weight subspace of $\bigwedge^N(V_{e,\ell})$ of weight $\bar{\mu}$ for the $\mathfrak{sl}(e)$-action is isomorphic to the $\widehat{\mathfrak{sl}}(\ell)$-module $\bigwedge^\mu(V_\ell)$, and the weight subspace of weight $\bar{\nu}$ for the $\mathfrak{sl}(\ell)$-action is isomorphic to the $\widehat{\mathfrak{sl}}(e)$-module $\bigwedge^\nu(V_e)$.

Since $\bigwedge^\nu(V_e)_\mu$ and $\bigwedge^\mu(V_\ell)_\nu$ are both canonically identified with the weight $(\bar{\mu}, \bar{\nu})$ subspace $\bigwedge^N(V_{e,\ell})_{\mu,\nu}$ of $\bigwedge^N(V_{e,\ell})$ for the $\mathfrak{sl}(e) \times \mathfrak{sl}(\ell)$-action, we have a canonical linear isomorphism $LR : \bigwedge^\nu(V_e)_\mu \to \bigwedge^\mu(V_\ell)_\nu$. More precisely, we have an isomorphism $f_{\mu,\nu} : \bigwedge^\nu(V_e)_\mu \to \bigwedge^N(V_{e,\ell})_{\mu,\nu}$ which takes the element $\wedge^\nu(a)$, with $a = (a_{p,k})$ in $P^\nu + \rho$, to the monomial $\bigwedge_{(p,k)}(v_{i_{p,k}} \otimes v_p \otimes z^{r_{p,k}})$. Here $i_{p,k} \in [1, e]$, $r_{p,k} \in \mathbb{Z}$ are such that $a_{p,k} = i_{p,k} + e\, r_{p,k}$, and the pair (p, k) runs from $(1, 1)$ to (ℓ, ν_ℓ) in lexicographic order. Next, there is an obvious isomorphism of $\mathfrak{sl}(e) \times \mathfrak{sl}(\ell)$-modules $\tau : \bigwedge^N(V_{e,\ell}) \to \bigwedge^N(V_{\ell,e})$ which exchanges the vector spaces $\mathbb{C}^e$ and $\mathbb{C}^\ell$. Then, we define $LR = (f_{\nu,\mu})^{-1} \circ \tau \circ f_{\mu,\nu}$. We'll call LR the *level-rank duality.*

Our next result compares the isomorphisms LR and K. To do so, we consider the $\mathbb{C}$-linear isomorphism $\theta : [\mathbf{O}^\nu_{-e}] \to \bigwedge^\nu(V_e)$ such that $\theta([V^\nu(\lambda_e)]) = \wedge^\nu(\lambda + \rho)$ for each $\lambda \in P^\nu$. It takes $[\mathbf{O}^\nu_{\mu,-e}]$ onto the weight subspace $\bigwedge^\nu(V_e)_\mu$. We can now prove the following.

Proposition 6.1. *We have a commutative square*

$$\begin{array}{ccc} [\mathbf{O}^\nu_{\mu,-e}] & \xrightarrow{K} & [\mathbf{O}^\mu_{\nu,-\ell}] \\ \theta\downarrow & & \downarrow\theta \\ \bigwedge^\nu(V_e)_\mu & \xrightarrow{LR} & \bigwedge^\mu(V_\ell)_\nu. \end{array}$$

Proof. Let triv_μ, sgn_μ be the idempotents of the $\mathbb{C}$-algebra of the parabolic subgroup $W_\mu \subset \widetilde{W}$ associated with the trivial representation and the signature.

For each tuple $a \in \mathbb{Z}^N$ we write $v(a) = \bigotimes_{k=1}^N v_{a_k}$. Let $(V_e)_\mu^{\otimes N}$ be the subspace of $(V_e)^{\otimes N}$ of weight $\bar{\mu}$ relatively to the $\mathfrak{sl}(e)$-action. The element $v(a)$ of $(V_e)^{\otimes N}$ belongs to

$(V_e)^{\otimes N}_\mu$ if and only if $a \in \widetilde{W} \cdot_e 1_\mu$. Therefore, the assignment $x \cdot \mathrm{triv}_\mu \mapsto v(x \cdot_e 1_\mu)$ with $x \in \widetilde{W}$ yields a $\mathbb{C}$-linear isomorphism $\mathbb{C}\widetilde{W} \cdot \mathrm{triv}_\mu \to (V_e)^{\otimes N}_\mu$.

Since $x \bullet \mathrm{o}_{\mu,-} = (x \cdot_e 1_\mu - \rho)_e$, we have $x \in I^\nu_{\mu,-}$ if and only if the N-tuple $x \cdot_e 1_\mu$ lies in $P^\nu + \rho$. We deduce that the isomorphism above factors to a $\mathbb{C}$-linear isomorphism $B_{\nu,\mu} : \mathrm{sgn}_\nu \cdot \mathbb{C}\widetilde{W} \cdot \mathrm{triv}_\mu \to \bigwedge^\nu (V_e)_\mu$ such that $\mathrm{sgn}_\nu \cdot x \cdot \mathrm{triv}_\mu \mapsto \wedge^\nu (x \cdot_e 1_\mu)$. Under this isomorphism, the basis $\{\wedge^\nu(a);\ a \in (P^\nu + \rho) \cap (\widetilde{W} \cdot_e 1_\mu)\}$ of $\bigwedge^\nu (V_e)_\mu$ is identified with $\{\mathrm{sgn}_\nu \cdot x \cdot \mathrm{triv}_\mu;\ x \in I^\nu_{\mu,-}\}$.

By Corollary 3.3, we have a bijection $I^\nu_{\mu,-} \to I^\mu_{\nu,-}$ such that $x \mapsto x^{-1}$. We claim that, under the isomorphisms $B_{\nu,\mu}$ and $B_{\mu,\nu}$, the map LR is identified with the $\mathbb{C}$-linear map $\mathrm{sgn}_\nu \cdot \mathbb{C}\widetilde{W} \cdot \mathrm{triv}_\mu \to \mathrm{sgn}_\mu \cdot \mathbb{C}\widetilde{W} \cdot \mathrm{triv}_\nu$ such that $\mathrm{sgn}_\nu \cdot x \cdot \mathrm{triv}_\mu \mapsto \mathrm{sgn}_\mu \cdot x^{-1} \cdot \mathrm{triv}_\nu$. Since, by Remark 3.13, the isomorphism K takes the element $[V^\nu(x \bullet \mathrm{o}_{\mu,-})]$ to $[V^\mu(x^{-1} \bullet \mathrm{o}_{\nu,-})]$, this finishes the proof of the proposition.

The claim is a direct consequence of the definition of the map LR. □

6.3. The CRDAHA

Let $\Gamma \subset \mathbb{C}^\times$ be the group of the ℓ-th roots of 1. Fix $\nu \in \mathbb{Z}^\ell(N)$ and fix an integer $d > 0$.

Let $\Gamma_d = \mathfrak{S}_d \ltimes \Gamma^d$. It is a complex reflection group. Let $\mathscr{P}_d$ be the set of *partitions of* d. Write $|\lambda| = d$ and let $l(\lambda)$ be the *length* of λ. Let $\mathscr{P}^\ell_d$ be the set of *ℓ-partitions of* d, i.e., the set of ℓ-tuples $\lambda = (\lambda_p)$ of partitions with $\sum_p |\lambda_p| = d$. There is a bijection between the set of irreducible representations of Γ_d and $\mathscr{P}^\ell_d$, see e.g. [35, sec. 6].

We set $h = 1/e$ and $h_p = \nu_{p+1}/e - p/\ell$ for each $p \in \mathbb{Z}/\ell\mathbb{Z}$. Let $H^\nu(d)$ be the CRDAHA of Γ_d with parameters h and (h_p). It is the quotient of the smash product of $\mathbb{C}\Gamma_d$ and the tensor algebra of $(\mathbb{C}^2)^{\oplus d}$ by the relations

$$[y_i, x_i] = 1 - k \sum_{j \neq i} \sum_{\gamma \in \Gamma} s^\gamma_{ij} - \sum_{\gamma \in \Gamma \setminus \{1\}} c_\gamma \gamma_i,$$

$$[y_i, x_j] = k \sum_{\gamma \in \Gamma} \gamma s^\gamma_{ij} \quad \text{if } i \neq j,$$

$$[x_i, x_j] = [y_i, y_j] = 0.$$

The parameters are such that $k = -h$ and $-c_\gamma = \sum_{p=0}^{\ell-1} \gamma^{-p}(h_p - h_{p-1})$ for $\gamma \neq 1$.

Let $\mathcal{O}^\nu_{-e}\{d\}$ be the category O of $H^\nu(d)$. It is a highest weight category with set of standard modules $\Delta(\mathcal{O}^\nu_{-e}\{d\}) = \{\Delta(\lambda);\ \lambda \in \mathscr{P}^\ell_d\}$, see [38, secs. 3.3, 3.6], [40, sec. 3.6]. To avoid confusions we may write $\Delta^\nu_e(\lambda) = \Delta(\lambda)$. Let $S^\nu_e(\lambda)$ be the top of $\Delta^\nu_e(\lambda)$.

We have the block decomposition $\mathcal{O}^\nu_{-e}\{d\} = \bigoplus_\mu \mathcal{O}^\nu_\mu\{d\}$, where $\mu = (\mu_1, \ldots, \mu_e)$ is identified with $\bar{\mu} = \sum_{i=1}^e \mu_i \varepsilon_i$ and it runs over the set of all integral weights of $\mathfrak{sl}(e)$. By [28], these blocks are determined by the following combinatorial rule.

For each $\lambda \in \mathscr{P}^\ell_d$, an *$(i,\nu)$-node* in λ is a triple (x, y, p) with $x, y > 0$ and $y \leqslant (\lambda_p)_x$ such that $y - x + \nu_p = i$ modulo e. Let $n^\nu_i(\lambda)$ be the number of (i,ν)-nodes in λ. Then, we have $\Delta^\nu_e(\lambda) \in \mathcal{O}^\nu_\mu\{d\}$ if and only if $\lambda \in \Lambda^\nu_\mu\{d\}$, where

$$\Lambda^\nu_\mu\{d\} = \left\{\lambda \in \mathscr{P}^\ell_d;\ \sum_{p=1}^{\ell} \omega_{\nu_p} - \sum_{i=1}^{e-1} \bigl(n^\nu_i(\lambda) - n^\nu_0(\lambda)\bigr)\,\alpha_i = \bar{\mu}\right\}. \tag{6.1}$$

The expression above should be regarded as an equality of integral weights of $\mathfrak{sl}(e)$. More precisely, the symbols $\omega_1, \omega_2, \dots, \omega_{e-1}$ and $\alpha_1, \alpha_2, \dots, \alpha_{e-1}$ are respectively the fundamental weights and the simple roots of $\mathfrak{sl}(e)$, and the subscript ν_p in (6.1) should be viewed as the residue class of ν_p in $[1, e)$. See [38, lem. 5.16] for details.

By [36, rem. 7.2], the condition (6.1) is equivalent to

$$\Lambda^\nu_\mu\{d\} = \bigl\{\lambda \in \mathscr{P}^\ell_d;\ \mathrm{wt}_e(\lambda + \rho_\nu - \rho) = \bar{\mu}\bigr\}.$$

Note that an integral weight of $\mathfrak{sl}(e)$ can be represented by an element of $\mathbb{Z}^e(k)$ if and only if it lies in $\omega_k + e\mathbb{Z}\Pi$. Thus, since $\nu \in \mathbb{Z}^\ell(N)$, if $\Lambda^\nu_\mu\{d\} \neq \emptyset$ then μ can be represented by an e-tuple in $\mathbb{Z}^e(N)$. Indeed, it is not difficult to check that if $\Lambda^\nu_\mu\{d\} \neq \emptyset$ then $\mu \in \mathscr{C}^e_N$.

For each $\mu \in \mathscr{C}^e_N$ and $a \in \mathbb{N}$, we write $\mathcal{O}^\nu_\mu[a] = \mathcal{O}^\nu_\mu\{d\}$ and $\Lambda^\nu_\mu[a] = \Lambda^\nu_\mu\{d\}$, where $d = a\,e + \langle \sum_{p=1}^{\ell} \omega_{\nu_p} - \mu : \rho\rangle$. For each $\lambda \in \mathscr{P}^\ell$, we have $\lambda \in \Lambda^\nu_\mu[a]$ if and only if $n^\nu_0(\lambda) = a$ and $\mathrm{wt}_e(\lambda + \rho_\nu - \rho) = \bar{\mu}$.

We are interested by the following conjecture [9, conj. 6].

Conjecture 6.2. *The blocks $\mathcal{O}^\nu_\mu[a]$ and $\mathcal{O}^\mu_\nu[a]$ have a (standard) Koszul grading. The Koszul dual of $\mathcal{O}^\nu_\mu[a]$ is equivalent to the Ringel dual of $\mathcal{O}^\mu_\nu[a]$.*

6.4. The Schur category

Fix a composition $\nu \in \mathscr{C}^\ell_N$. Let $\mathscr{P}^\nu_d = \{\lambda \in \mathscr{P}^\ell_d;\ l(\lambda_p) \leqslant \nu_p\}$. There is an inclusion $\mathscr{P}^\nu_d \subset P^\nu \subset \mathbb{Z}^N$ such that

$$\lambda \mapsto \bigl(\lambda_1 0^{\nu_1 - l(\lambda_1)}, \lambda_2 0^{\nu_2 - l(\lambda_2)}, \dots, \lambda_\ell 0^{\nu_\ell - l(\lambda_\ell)}\bigr),$$

where the partition λ_p is viewed as an element in $\mathbb{Z}^{l(\lambda_p)}$.

Consider the ν-dominant weight $\rho_\nu = (\nu_1, \nu_1 - 1, \dots, 1, \nu_2, \nu_2 - 1, \dots, \nu_\ell, \dots 1) \in P^\nu$. For each $\lambda \in \mathscr{P}^\nu$, we abbreviate $V^\nu_e(\lambda) = V^\nu((\lambda + \rho_\nu - \rho)_e)$ and $L^\nu_e(\lambda) = L^\nu((\lambda + \rho_\nu - \rho)_e)$.

Following [40], let $\mathbf{A}^\nu_{-e}\{d\} \subset \mathbf{O}^\nu_{-e}$ be the Serre subcategory consisting of the finite length $\mathbf{g}$-modules of level $-e - N$ whose constituents belong to the set $\{L^\nu_e(\lambda);\ \lambda \in \mathscr{P}^\nu_d\}$. By [40], it is a highest weight category with the set of standard modules $\Delta(\mathbf{A}^\nu_{-e}\{d\}) = \{V^\nu_e(\lambda);\ \lambda \in \mathscr{P}^\nu_d\}$.

We have the following conjecture [40, conj. 8.8].

Conjecture 6.3. *There is a quotient functor $\mathcal{O}^\nu_{-e}\{d\} \to \mathbf{A}^\nu_{-e}\{d\}$ taking $S^\nu_e(\lambda)$ to $L^\nu_e(\lambda)$ if $\lambda \in \mathscr{P}^\nu_d$ and to 0 else. If $\nu_p \geqslant d$ for each p, then we have $\mathscr{P}^\nu_d = \mathscr{P}_d$ and the functor above is an equivalence of highest weight categories.*

A proof of Conjecture 6.3 is given in [36].

6.5. Koszul duality of the Schur category

Fix an integer $d \geqslant 0$ and compositions $\mu \in \mathscr{C}^e_N$, $\nu \in \mathscr{C}^\ell_N$. Let $\mathbf{A}^\nu_\mu\{d\}$, $\mathbf{A}^\nu_\mu[a]$ be the Serre subcategories of $\mathbf{A}^\nu_{-e} = \bigoplus_{d\in\mathbb{N}} \mathbf{A}^\nu_{-e}\{d\}$ generated by the modules $L^\nu_e(\lambda)$ such that $\lambda \in \Lambda^\nu_\mu\{d\}$, $\Lambda^\nu_\mu[a]$ respectively. Write $\mathbf{A}^\nu_{\mu,-e} = \bigoplus_{d\in\mathbb{N}} \mathbf{A}^\nu_{\mu,-e}\{d\}$.

In view of Conjecture 6.3, the following claim can be regarded as an analogue of Conjecture 6.2.

Theorem 6.4. *The category $\mathbf{A}^\nu_\mu[a]$ has a (standard) Koszul grading. The Koszul dual of $\mathbf{A}^\nu_\mu[a]$ is equivalent to the Ringel dual of $\mathbf{A}^\mu_\nu[a]$.*

Proof. Recall the $\mathbb{C}$-linear map $K : [\mathbf{O}^\nu_{\mu,-e}] \to [\mathbf{O}^\mu_{\nu,-\ell}]$. First, we claim that $K([\mathbf{A}^\nu_{\mu,-e}]) = [\mathbf{A}^\mu_{\nu,-\ell}]$. By Proposition 6.1 and the definition of the Schur category, we must compute the element $LR(\wedge^\nu(a))$, for each $a = \lambda + \rho_\nu$ such that $\lambda \in \mathscr{P}^\nu$ and $\wedge^\nu(a) \in \bigwedge^\nu(V_e)_\mu$.

First, we consider the $\mathbb{C}$-vector space $LR(\bigwedge^\nu(V_e)_\mu)$. The map $f_{\mu,\nu}$ takes the monomial $\bigwedge^\nu(a)$, with $a = (a_{p,k}) \in P^\nu + \rho$, to the monomial $\bigwedge_{(p,k)}(v_{i_{p,k}} \otimes v_p \otimes z^{r_{p,k}})$ such that $a_{p,k} = i_{p,k} + e\, r_{p,k}$. The weight subspace $\bigwedge^\nu(V_e)_\mu$ is spanned by the monomials $\bigwedge^\nu(a)$ as above such that $\mu_i = \sharp\{(p,k);\ i_{p,k} = i\}$ for each $i \in [1,e]$. Therefore, we have $f_{\mu,\nu}(\bigwedge^\nu(V_e)_\mu) = \bigwedge^N(V_{e,\ell})_{\mu,\nu}$, which is the subspace of $\bigwedge^N(V_{e,\ell})$ spanned by all the monomials $\bigwedge^N(x) = \bigwedge_{(i',p',r')}(v_{i'} \otimes v_{p'} \otimes z^{r'})$ where (i',p',r') runs over the entries of an N-tuple $x \in ([1,e] \times [1,\ell] \times \mathbb{Z})^N$ such that $\mu_i = \sharp\{(i',p',r') \in x;\ i' = i\}$ and $\nu_p = \sharp\{(i',p',r') \in x;\ p' = p\}$ for each $i \in [1,e]$, $p \in [1,\ell]$.

Next, we consider the subspace $LR \circ \theta([\mathbf{A}^\nu_{\mu,-e}])$. The subspace $\theta([\mathbf{A}^\nu_{\mu,-e}])$ of $\bigwedge^\nu(V_e)_\mu$ is spanned by the monomials $\wedge^\nu(a)$ such that there is an ℓ-partition $\lambda \in \Lambda^\nu_\mu$ with $a_{p,k} = \lambda_{p,k} + \nu_p + 1 - k$ for each p,k. Here $\lambda_{p,k}$ is the k-th part of the p-th partition λ_p of λ. Thus, the map $f_{\mu,\nu}$ takes $\theta([\mathbf{A}^\nu_{\mu,-e}])$ to the subspace of $\bigwedge^N(V_{e,\ell})_{\mu,\nu}$ spanned by all the monomials $\bigwedge^N(x)$ as above such that $x \in ([1,e] \times [1,\ell] \times \mathbb{N})^N$. This implies in particular that $LR \circ \theta([\mathbf{A}^\nu_{\mu,-e}]) = \theta([\mathbf{A}^\mu_{\nu,-\ell}])$, proving the claim.

Now, to finish the proof of the theorem, it is enough to observe that $\theta([\mathbf{A}^\nu_{-e}[a]])$ is the sum, over all affine weights $\hat{\mu} \in P + a\delta$, of the intersection of $\theta([\mathbf{A}^\nu_{-e}])$ with the subspace of $\bigwedge^\nu(V_e)$ of weight $\hat{\mu}$ for the $\widehat{\mathfrak{sl}}(e)$-action. □

The following is obvious.

Corollary 6.5. *Conjecture 6.3 implies Conjecture 6.2.* □

6.6. Koszulity of the q-Schur algebra

Now, we consider the case $\ell = 1$.

By the Kazhdan–Lusztig equivalence [27, thm. 38.1], the category $\mathbf{A}^{(N)}_{-e}\{d\}$ is equivalent to the module category of the q-Schur algebra. This equivalence is an equivalence of highest-weight categories. This follows from the fact that an equivalence of abelian categories $\mathbf{C}_1 \to \mathbf{C}_2$ such that $\mathbf{C}_1$, $\mathbf{C}_2$ are of highest weight and that the induced bijection $\mathrm{Irr}(\mathbf{C}_1) \to \mathrm{Irr}(\mathbf{C}_2)$ is increasing is an equivalence of highest weight categories, i.e., it induces a bijection $\Delta(\mathbf{C}_1) \to \Delta(\mathbf{C}_2)$. See also [40, thm. A.5.1] for a more detailed proof.

Therefore, Theorem 6.4 implies the following.

Corollary 6.6. *The q-Schur algebra is Morita equivalent to a Koszul algebra which is balanced.* □

See also [9] for a different approach to this result. Note that our proof gives an explicit description of the Koszul dual of the q-Schur algebra in term of affine Lie algebras.

Appendix A. Finite codimensional affine Schubert varieties

By a scheme we always mean a scheme over the field k, and by a variety we'll mean a reduced scheme of finite type which is quasi-projective. Let T be a torus. A T*-scheme* is a scheme with an algebraic T-action.

Fix a contractible topological T-space ET with a topologically free T-action. For any T-scheme X, we set $X_T = X \times_T ET$. There are obvious projections $p : X \times ET \to X$ and $q : X \times ET \to X_T$.

We'll identify S with the T-equivariant cohomology k-space $H_T(\bullet)$ of a point.

A.1. Equivariant perverse sheaves on finite dimensional varieties

Fix a T-variety X. Let $\mathbf{D}^b_T(X)$ be the T-equivariant bounded derived category. It is the full subcategory of the bounded derived category $\mathbf{D}^b(X_T)$ of sheaves of k-vector spaces on X_T (for the analytic topology on X), consisting of the complexes of sheaves $\mathcal{F}$ with an isomorphism $q^*\mathcal{F} \simeq p^*\mathcal{F}_X$ for some $\mathcal{F}_X \in \mathbf{D}^b(X)$.

The cohomology of an object $\mathcal{F} \in \mathbf{D}^b_T(X)$ is the graded S-module $H(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}^b_T(X)}(k_X, \mathcal{F}[i])$. For each $\mathcal{E}, \mathcal{F} \in \mathbf{D}^b_T(X)$ we have an object $\mathcal{RH}om(\mathcal{E}, \mathcal{F})$ in $\mathbf{D}^b(X_T)$ such that $\mathrm{Ext}_{\mathbf{D}^b_T(X)}(\mathcal{E}, \mathcal{F}) = H(\mathcal{RH}om(\mathcal{E}, \mathcal{F}))$.

If $Y \subset X$ is a T-equivariant embedding, let $\bar{Y}$ be its closure in X. Let $IC_T(\bar{Y})$ be the minimal extension of $k_Y[\dim Y]$ (the shifted equivariant constant sheaf). It is a perverse sheaf on X supported on $\bar{Y}$.

Let $IH_T(\bar{Y})$ be the k-space of equivariant intersection cohomology of $\bar{Y}$ and let $H_T(\bar{Y})$ be its equivariant cohomology. We have $IH_T(\bar{Y}) = H(IC_T(\bar{Y}))$ and $H_T(\bar{Y}) = H(k_{\bar{Y}})$. Forgetting the T-action we define $IC(\bar{Y})$, $IH(\bar{Y})$ and $H(\bar{Y})$ in the same way.

We'll say that X is *good* if the following hold

- X has a Whitney stratification $X = \bigsqcup_x X_x$ by T-stable subvarieties,
- $X_x = \mathbb{A}^{l(x)}$ with a linear T-action, for some integer $l(x) \in \mathbb{N}$,
- there are integers $n_{x,y,i} \in \mathbb{N}$ such that

$$j_y^* IC_T(\bar{X}_x) = \bigoplus_p k_{X_y}\big[l(y)\big]\big[l(x) - l(y) - 2p\big]^{\oplus n_{x,y,p}} \tag{A.1}$$

where j_x is the inclusion $X_x \subset X$. Equivalently, we have

$$j_y^! IC_T(\bar{X}_x) = \bigoplus_q k_{X_y}\big[l(y)\big]\big[l(y) - l(x) + 2q\big]^{\oplus n_{x,y,q}}. \tag{A.2}$$

We call the third property the *parity vanishing.*

Proposition A.1. *If X is a good T-variety, then the following hold*

(a) $\dim \operatorname{Ext}^i_{\mathbf{D}^b(X)}(IC(\bar{X}_x), IC(\bar{X}_y)) = \sum_{z,p,q} n_{x,z,p} n_{y,z,q}$ *where z, p, q runs over the set of triples such that $2l(z) - l(y) - l(x) + 2p + 2q = i$,*

(b) $\operatorname{Ext}_{\mathbf{D}^b(X)}(IC(\bar{X}_x), IC(\bar{X}_y)) = k \operatorname{Ext}_{\mathbf{D}^b_T(X)}(IC_T(\bar{X}_x), IC_T(\bar{X}_y))$*,*

(c) $\operatorname{Ext}_{\mathbf{D}^b_T(X)}(IC_T(\bar{X}_x), IC_T(\bar{X}_y)) = \operatorname{Hom}_{H_T(X)}(IH_T(\bar{X}_x), IH_T(\bar{X}_y))$*,*

(d) $\operatorname{Ext}_{\mathbf{D}^b(X)}(IC(\bar{X}_x), IC(\bar{X}_y)) = \operatorname{Hom}_{H(X)}(IH(\bar{X}_x), IH(\bar{X}_y))$*,*

(e) $IH(\bar{X}_x)$ vanishes in degrees $\not\equiv l(x)$ modulo 2 and $IH(\bar{X}_x) = kIH_T(\bar{X}_x)$.

Proof. Part (a) is [5, thm. 3.4.1]. We sketch briefly the proof for the comfort of the reader. Set $X_p = \bigsqcup_{p=l(z)} X_z$. Let j_p be the inclusion of X_p into X. For each $\mathcal{F} \in \mathbf{D}^b(X)$, there is a spectral sequence $E_1^{p,q} = H^{p+q}(j_p^! \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{F})$. Therefore, if $\mathcal{F} = \mathcal{RH}om(IC(\bar{X}_x), IC(\bar{X}_y))$, we get a spectral sequence

$$E_1^{p,q} = \bigoplus_z H^{p+q} \mathcal{RH}om\big(j_z^* IC(\bar{X}_x), j_z^! IC(\bar{X}_y)\big) \quad \Rightarrow \quad \operatorname{Ext}^{p+q}_{\mathbf{D}^b(X)}\big(IC(\bar{X}_x), IC(\bar{X}_y)\big),$$

where z runs over the set of elements with $l(z) = p$. By (A.1), (A.2) the spectral sequence degenerates at E_1, and we get

$$\dim \operatorname{Ext}^i_{\mathbf{D}^b(X)}\big(IC(\bar{X}_x), IC(\bar{X}_y)\big) = \sum_{z,p,q} n_{x,z,p} n_{y,z,q},$$

where z, p, q are as in (a).

Now, we prove (b). For each $\mathcal{F} \in \mathbf{D}^b_T(X)$, there is a spectral sequence $E_2^{p,q} = S^p \otimes H^q(\mathcal{F}_X) \Rightarrow H^{p+q}(\mathcal{F})$, see [19, sec. 5.5]. Thus, if $\mathcal{F} = \mathcal{RH}om(IC_T(\bar{X}_x), IC_T(\bar{X}_y))$, we get a spectral sequence

$$E_2^{p,q} = S^p \otimes \operatorname{Ext}^q_{\mathbf{D}^b(X)}\big(IC(\bar{X}_x), IC(\bar{X}_y)\big) \quad \Rightarrow \quad \operatorname{Ext}^{p+q}_{\mathbf{D}^b_T(X)}\big(IC_T(\bar{X}_x), IC_T(\bar{X}_y)\big).$$

Now, we have $\operatorname{Ext}^q_{\mathbf{D}^b(X)}(IC(\bar{X}_x), IC(\bar{X}_y)) = 0$ for $q \not\equiv l(x) + l(y)$ modulo 2 by (a). Therefore, since S vanishes in odd degrees, the spectral sequence degenerates at E_2. Thus, we have

$$S \otimes \operatorname{Ext}_{\mathbf{D}^b(X)}\big(IC(\bar{X}_x), IC(\bar{X}_y)\big) = \operatorname{Ext}_{\mathbf{D}^b_T(X)}\big(IC_T(\bar{X}_x), IC_T(\bar{X}_y)\big).$$

The proof of (d) is as in [18, sec. 1], [6, sec. 3.3]. Since our setting is slightly different we sketch briefly the main arguments. Fix a partial order on the set of strata such that $\bar{X}_x = X_{\leqslant x} = \bigsqcup_{y \leqslant x} X_y$. Consider the obvious inclusions $j_{\leqslant x} : X_{\leqslant x} \to X$, $i = j_{<x} : X_{<x} \to X_{\leqslant x}$ and $j = j_x : X_x \to X_{\leqslant x}$. For each $y \leqslant x$, we set $\mathcal{F}_1 = j^*_{\leqslant x} IC(\bar{X}_x)$ and $\mathcal{F}_2 = j^*_{\leqslant x} IC(\bar{X}_y)$. For $s = 1, 2$ and $p \in \mathbb{N}$, there are integers d_s and $d_{s,y,p}$ such that

$$j_y^* \mathcal{F}_s = \bigoplus_p k_{X_y}[d_s - 2p]^{\oplus d_{s,y,p}}, \qquad j_y^! \mathcal{F}_s = \bigoplus_q k_{X_y}\big[2l(y) - d_s + 2q\big]^{\oplus d_{s,y,q}}. \quad \text{(A.3)}$$

Consider the diagram of graded k-vector spaces given by

$$\begin{array}{ccc}
\operatorname{Ext}_{\mathbf{D}^b(X_{<x})}(i^*\mathcal{F}_1, i^!\mathcal{F}_2) & \longrightarrow & \operatorname{Hom}_{H(X_{<x})}(H(i^*\mathcal{F}_1), H(i^!\mathcal{F}_2)) \\
\downarrow & & \downarrow a \\
\operatorname{Ext}_{\mathbf{D}^b(X_{\leqslant x})}(\mathcal{F}_1, \mathcal{F}_2) & \longrightarrow & \operatorname{Hom}_{H(X_{\leqslant x})}(H(\mathcal{F}_1), H(\mathcal{F}_2)) \\
\downarrow & & \downarrow b \\
\operatorname{Ext}_{\mathbf{D}^b(X_x)}(j^*\mathcal{F}_1, j^*\mathcal{F}_2) & \longrightarrow & \operatorname{Hom}_{H(X_x)}(H(j^*\mathcal{F}_1), H(j^*\mathcal{F}_2)).
\end{array}$$

In the right side sequence, the Hom's are the k-spaces of graded module homomorphisms over the graded k-algebras $H(X_{<x})$, $H(X_{\leqslant x})$, $H(X_x)$ respectively, which are computed in the category of non-graded modules. The horizontal maps are given by taking hypercohomology.

We'll prove that the middle map is invertible by induction on x. This proves part (d). The short exact sequence associated with the left side is exact by (A.3), see e.g., [16, lem. 5.3] and the references there. The lower map is obviously invertible, because $j^*\mathcal{F}_s$ are constant sheaves on X_x for each $s = 1, 2$. The complexes $i^*\mathcal{F}_s$ and $i^!\mathcal{F}_s$ on $X_{<x}$ satisfy again (A.3) for any stratum $X_y \subset X_{<x}$. Thus, by induction, we may assume that the upper map is invertible. Then, to prove that the middle map is invertible it is enough to check that a is injective and that $\operatorname{Im}(a) = \operatorname{Ker}(b)$.

By (A.3), the distinguished triangles $i_*i^! \to 1 \to j_*j^* \xrightarrow{+1}$ and $j_!j^* \to 1 \to i_*i^* \xrightarrow{+1}$ yield exact sequences of $H(X_{\leqslant x})$-modules

$$0 \to H\big(i^!\mathcal{F}_s\big) \xrightarrow{\alpha_s} H(\mathcal{F}_s) \xrightarrow{\beta_s} H\big(j^*\mathcal{F}_s\big) \to 0,$$
$$0 \to H_c\big(j^*\mathcal{F}_s\big) \xrightarrow{\gamma_s} H(\mathcal{F}_s) \xrightarrow{\delta_s} H\big(i^*\mathcal{F}_s\big) \to 0,$$

where $H_c(j^*\mathcal{F}_s) = H(j_!j^*\mathcal{F}_s)$ is the cohomology with compact supports of $j^*\mathcal{F}_s$. Thus, the map a is injective, because $a(\phi) = \alpha_2 \circ \phi \circ \delta_1$ for each ϕ, α_2 is injective and δ_1 is surjective.

Next, fix an element $\phi \in \operatorname{Ker}(b)$. Since $b(\phi) \circ \beta_1 = \beta_2 \circ \phi$ by construction and β_1 is surjective, we have $\beta_2 \circ \phi = 0$. Hence, there is an $H(X_{\leqslant x})$-linear map $\phi' : H(\mathcal{F}_1) \to H(i^!\mathcal{F}_2)$ such that $\phi = \alpha_2 \circ \phi'$. We must check that $\phi' \circ \gamma_1 = 0$.

To do that, set $d = \dim X_x$. Since $X_x \simeq k^d$, the fundamental class ω of X_x belongs to $H_c^{2d}(X_x)$. The cup product with ω restricts to 0 on $X_{<x}$, because there is an exact sequence $0 \to H_c(X_x) \to H(X_{\leqslant x}) \to H(X_{<x}) \to 0$. Thus, it yields a map $g_s : H(\mathcal{F}_s) \to H_c(j^*\mathcal{F}_s)[2d]$ which vanishes on $\mathrm{Im}(\alpha_s)$. We deduce that $g_2 \circ \phi = 0$. Since $H_c(X_x) \subset H_c(X_{\leqslant x})$ and ϕ is $H_c(X_{\leqslant x})$-linear, we have also $\phi \circ g_1 = 0$. Since g_1 is surjective, we deduce that $\phi \circ \gamma_1 = 0$. Hence, we have $\phi' \circ \gamma_1 = 0$, because α_2 is injective.

Part (c) is proved as part (d), using equivariant cohomology.

Finally, we prove (e). By parity vanishing, the spectral sequence

$$E_1^{p,q} = H^{p+q}\big(j_p^! IC(\bar{X}_x)\big) \quad \Rightarrow \quad IH^{p+q}(\bar{X}_x)$$

degenerates at E_1. This yields the first claim, as in part (a). The second one follows from the first one, because the spectral sequence $E_2^{p,q} = S^p \otimes IH^q(\bar{X}_x) \Rightarrow IH^{p+q}(\bar{X}_x)$ degenerates at E_2. □

A.2. Equivariant perverse sheaves on infinite dimensional varieties

This section is a reminder on equivariant perverse sheaves on infinite dimensional varieties, to be used in the next section.

Let X be an *essentially smooth* T-scheme, in the sense of [23, sec. 1.6]. In [23, sec. 2] the derived category of constructible complexes on X and perverse sheaves on X (for the analytic topology) are defined. Note that the convention for perverse sheaves here differs from the convention for perverse sheaves on varieties (as in Section A.1): indeed $k_Y[-\operatorname{codim} Y]$ is perverse if Y is an essentially smooth T-scheme, while $k_Y[\dim Y]$ is perverse for a smooth variety Y. We'll use the same terminology as in [23, sec. 2] and we formulate our results in the T-equivariant setting. The equivariant version of the constructions in [23] is left to the reader.

Let $\mathbf{D}_T(X)$ be the T-equivariant derived category on X. If X, Y are essentially smooth and $Y \to X$ is a T-equivariant embedding of finite presentation, let $IC_T(\overline{Y})$ be the minimal extension of the equivariant constant shifted sheaf $k_Y[-\operatorname{codim} Y]$ on Y. It is a perverse sheaf on X supported on the Zariski closure $\bar{Y}$ in X. If $\mathcal{F} \in \mathbf{D}_T(X)$, we set $H(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_T(X)}(k_X, \mathcal{F}[i])$, a graded S-module. We abbreviate $IH_T(\bar{Y}) = H(IC_T(\bar{Y}))$ and $H_T(Y) = H(k_Y)$. We call $H_T(Y)$ the *equivariant cohomology of* Y. It is an $H_T(X)$-module.

Recall that for a morphism $f : Z \to X$ of essentially smooth T-schemes there is a functor $f^* : \mathbf{D}_T(X) \to \mathbf{D}_T(Z)$, see [23, sec. 3.7]. If f is the inclusion of a T-stable subscheme we write $\mathcal{F}_Z = i^*\mathcal{F}$ and $IH_T(\overline{Y})_Z = H(IC_T(\overline{Y})_Z)$.

Now, assume that X is the limit of a projective system of smooth schemes

$$X = \varprojlim_n X_n \tag{A.4}$$

as in [23, sec. 1.3]. Let $p_n : X \to X_n$ be the projection and set $Y_n = p_n(Y)$. Assume that Y_n is locally closed in X_n. Since $k_Y = p_n^* k_{Y_n}$ we have an obvious map

$$\begin{aligned} H_T(Y_n) &= \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{D}_T(X_n)}\big(k_{X_n}, k_{Y_n}[i]\big) \\ &\to \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{D}_T(X)}\big(p_n^* k_{X_n}, p_n^* k_{Y_n}[i]\big) \\ &\to \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{D}_T(X)}\big(k_X, k_Y[i]\big) \\ &\to H_T(Y). \end{aligned}$$

It yields an isomorphism $H_T(Y) = \varinjlim_n H_T(Y_n)$.

A.3. Moment graphs and the Kashiwara flag manifold

In this section we consider the localization on Kashiwara's flag manifold. The main motivation is Corollary A.6 below that we used in the rest of text, in particular in Corollary 4.50 to prove that the functor $\mathbb{V}_k$ is fully-faithful on projectives in ${}^w\mathbf{O}^{\Delta}_{\mu,-}$.

Let P_μ be the parabolic subgroup corresponding to the Lie algebra $\mathbf{p}_\mu$. Let $X = X_\mu = \mathcal{G}/P_\mu$ be the Kashiwara partial flag manifold associated with $\mathbf{g}$ and $\mathbf{p}_\mu$, see [22]. Here $\mathcal{G}$ is the schematic analogue of $G(k((t)))$ defined in [22], which has a locally free right action of the group-k-scheme P_μ and a locally free left action of the group-k-scheme B^-, the Borel subgroup opposite to B. Recall that X is an essentially smooth, not quasi-compact, T-scheme, which is covered by T-stable quasi-compact open subsets isomorphic to $\mathbb{A}^\infty = \operatorname{Spec} k[x_k;\ k \in \mathbb{N}]$.

Let $e_X = P_\mu/P_\mu$ be the origin of X. For each $x \in I_{\mu,+}$, we set $X^x = B^- x e_X = B^- x P_\mu/P_\mu$. Note that X^x is a locally closed T-stable subscheme of X of codimension $l(x)$ which is isomorphic to $\mathbb{A}^\infty$. Consider the T-stable subschemes

$$\overline{X^x} = X^{\geqslant x} = \bigsqcup_{y\geqslant x} X^y, \qquad X^{\leqslant x} = \bigsqcup_{y\leqslant x} X^y, \qquad X^{<x} = \bigsqcup_{y<x} X^y.$$

We call $\overline{X^x}$ a *finite-codimensional affine Schubert variety.* We call $X^{\leqslant x}$ an *admissible open set.*

If Ω is an admissible open set, there are canonical isomorphisms

$$IC_T\big(\overline{X^x}\big)_\Omega = IC_T\big(\overline{X^x} \cap \Omega\big), \qquad IH_T\big(\overline{X^x}\big)_\Omega = IH_T\big(\overline{X^x} \cap \Omega\big).$$

We can view Ω as the limit of a projective system of smooth schemes (Ω_n) as in [23, lem. 4.4.3]. So, the projection $p_n : \Omega \to \Omega_n$ is a good quotient by a congruence subgroup B_n^- of B^-. Let n be large enough. Then, we have $IC_T(\overline{X^x} \cap \Omega) = p_n^* IC_T(\overline{X^x_{\Omega,n}})$ with $\overline{X^x_{\Omega,n}} = p_n(\overline{X^x} \cap \Omega)$ by [23, sec. 2.6]. Thus we have a map

$$IH_T\big(\overline{X^x_{\Omega,n}}\big) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{D}_T(\Omega_n)}\big(k_{\Omega_n}, IC_T\big(\overline{X^x_{\Omega,n}}\big)[i]\big)$$

$$\to \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_T(\Omega)}\big(k_\Omega, p_n^* IC_T\big(\overline{X^x_{\Omega,n}}\big)[i]\big)$$
$$\to IH_T\big(\overline{X^x}\cap\Omega\big)$$

which yields an isomorphism

$$IH_T\big(\overline{X^x}\cap\Omega\big) = \varinjlim_n IH_T\big(\overline{X^x_{\Omega,n}}\big).$$

For $\Omega = X^{\leqslant w}$ and $x \leqslant w$ we abbreviate $X^{[x,w]} = \overline{X^x}\cap X^{\leqslant w}$ and $X_n^{[x,w]} = p_n(X^{[x,w]})$. Since p_n is a good quotient by B_n^- and since X^x is B_n^--stable, we have an algebraic stratification $X_n^{\leqslant w} = \bigsqcup_{x\leqslant w} X_n^x$, where X_n^x is an affine space whose Zariski closure is $X_n^{[x,w]}$.

Lemma A.2. *(a) The T-variety $X_n^{\leqslant w}$ is smooth and good.*

(b) It is covered by T-stable open affine subsets with an attractive fixed point. The fixed points subset is naturally identified with ${}^wI_{\mu,+}$.

(c) There is a finite number of one-dimensional orbits. The closure of each of them is smooth. Two fixed points are joined by a one-dimensional orbit if and only if the corresponding points in ${}^wI_{\mu,+}$ are joined by an edge in ${}^w\mathcal{G}_\mu$.

Proof. The T-variety $X_n^{\leqslant w}$ is smooth by [23], because $X^{\leqslant w}$ is smooth and p_n is a B_n^--torsor for n large enough. We claim that it is also quasi-projective.

Let X_0 be the stack of G-bundles on $\mathbb{P}^1$. We may assume that P_μ is maximal parabolic. Then, by the Drinfeld–Simpson theorem, a k-point of X is the same as a k-point of X_0 with a trivialization of its pullback to $\mathrm{Spec}(k[[t]])$. Here t is regarded as a local coordinate at $\infty \in \mathbb{P}^1$ and we identify B^- with the Iwahori subgroup in $G(k[[t]])$. We may choose B_n^- to be the kernel of the restriction $G(k[[t]]) \to G(k[t]/t^n)$. Then, a k-point of $X_n = X/B_n^-$ is the same as a k-point of X_0 with a trivialization of its pullback to $\mathrm{Spec}(k[t]/t^n)$. We'll prove that there is an increasing system of open subsets $\mathcal{U}_m \subset X_0$ such that for each m and for $n \gg 0$ the fiber product $X_n \times_{X_0} \mathcal{U}_m$ is representable by a quasi-projective variety. This implies our claim.

Choosing a faithful representation $G \subset SL_r$ we can assume that $G = SL_r$. So a k-point of X_0 is the same as a rank r vector bundle on $\mathbb{P}^1$ of degree 0. For an integer $m > 0$ let $\mathcal{U}_m(k)$ be the set of V in $X_0(k)$ with $H^1(\mathbb{P}^1, V\otimes\mathcal{O}(m)) = 0$ which are generated by global sections. It is the set of k-points of an open substack $\mathcal{U}_m$ of X_0. Note that $\mathcal{U}_m \subset \mathcal{U}_{m+1}$ and $X_0 = \bigcup_m \mathcal{U}_m$. Now, the set $\mathcal{Y}_m(k)$ of pairs (V, b) where $V \in \mathcal{U}_m(k)$ and b is a basis of $H^0(\mathbb{P}^1, V\otimes\mathcal{O}(m))$ is the set of k-points of a quasi-projective variety $\mathcal{Y}_m$ by the Grothendieck theory of Quot-schemes. Further, there is a canonical $GL_{r(m+1)}$-action on $\mathcal{Y}_m$ such that the morphism $\mathcal{Y}_m \to \mathcal{U}_m$, $(V, b) \mapsto V$ is a $GL_{r(m+1)}$-bundle. Now, for $n \gg 0$ the fiber product $X_n \times_{X_0} \mathcal{U}_m$ is representable by a quasi-projective variety, see e.g., [41, thm. 5.0.14].

Next, note that $X_n^{\leqslant w}$ is recovered by the open subsets $V_n^x = p_n(V^x)$ with $x \leqslant w$. Each of them contains a unique fixed point under the T-action and finitely many one-dimensional orbits.

Finally, the parity vanishing holds: since $IC(X^{[x,w]}) = p_n^* IC(X_n^{[x,w]})$ we have

$$IC\big(X_n^{[x,w]}\big)_{X_n^y} = \bigoplus_i k_{X_n^y}\big[-l(y)\big]\big[l(y) - l(x) - 2i\big]^{\oplus Q_{x,y,i}^{\mu,-1}}$$

by [26, thm. 1.3]. The change in the degrees with respect to Section A.1 is due to the change of convention for perverse sheaves mentioned above. □

Now we set $V = \mathbf{t}^*$ and we consider the moment graph ${}^w\mathcal{G}^\vee$.

Proposition A.3. *We have*

(a) $H_T(X^{\leqslant w}) = {}^w\bar{Z}^\vee_{S,\mu,-}$ *and* $H(X^{\leqslant w}) = {}^w\bar{Z}^\vee_{\mu,-}$ *as graded k-algebras,*

(b) $IH_T(X^{[x,w]}) = {}^w\bar{B}^\vee_{S,\mu,-}(x)$ *as a graded* ${}^w\bar{Z}_{S,\mu,-}$*-module,*

(c) $IH(X^{[x,w]}) = {}^w\bar{B}^\vee_{\mu,-}(x)$ *as a graded* ${}^w\bar{Z}^\vee_{\mu,-}$*-module.*

Proof. Assuming n to be large enough we may assume that $H_T(X^{\leqslant w}) = H_T(X_n^{\leqslant w})$, $IH_T(X^{[x,w]}) = IH_T(X_n^{[x,w]})$, etc. By Lemma A.2 the S-module $H_T(X_n^{\leqslant w})$ is free. Thus we can apply the localization theorem [19, thm. 6.3], which proves (a).

Now, we concentrate on (b). The graded k-module $IH(X_n^{[x,w]})$ vanishes in odd degree by Proposition A.1 and Lemma A.2. Thus, applying [7] to $X_n^{[x,w]}$, we get a graded ${}^w\bar{Z}_{S,\mu,-}$-module isomorphism $IH_T^*(X_n^{[x,w]}) = {}^w\bar{B}^\vee_{S,\mu,-}(x)$.

Part (c) follows from (b), Proposition A.1 and Lemma A.2. □

Corollary A.4. *We have a graded S-module isomorphism*

$${}^w\bar{B}^\vee_{S,\mu,-}(x)_y = \bigoplus_{i\geqslant 0}\big(S\big\langle -l(x) - 2i\big\rangle\big)^{\oplus Q_{x,y,i}^{\mu,-1}}.$$

Proof. Apply Proposition A.3 and [26, thm. 1.3(i)]. □

Proposition A.5. *For each $x, y \leqslant w$ we have*

(a) $\sum_{i\in\mathbb{Z}} t^i \dim k \operatorname{Ext}^i_{\mathbf{D}_T(X^{\leqslant w})}(IC_T(X^{[x,w]}), IC_T(X^{[y,w]})) = \sum_z Q_\mu(t)_{x,z} Q_\mu(t)_{y,z}$,

(b) $\operatorname{Ext}_{\mathbf{D}_T(X^{\leqslant w})}(IC_T(X^{[x,w]}), IC_T(X^{[y,w]})) = \operatorname{Hom}_{H_T(X^{\leqslant w})}(IH_T(X^{[x,w]}), IH_T(X^{[y,w]}))$,

(c) $\operatorname{Ext}_{\mathbf{D}(X^{\leqslant w})}(IC(X^{[x,w]}), IC(X^{[y,w]})) = \operatorname{Hom}_{H(X^{\leqslant w})}(IH(X^{[x,w]}), IH(X^{[y,w]}))$,

(d) $\operatorname{Ext}_{\mathbf{D}(X^{\leqslant w})}(IC(X^{[x,w]}), IC(X^{[y,w]})) = k \operatorname{Ext}_{\mathbf{D}_T(X^{\leqslant w})}(IC_T(X^{[x,w]}), IC_T(X^{[y,w]}))$.

Proof. Apply Proposition A.1 and Lemma A.2. □

Finally, we obtain the following.

Corollary A.6. *We have a graded k-algebra isomorphism* ${}^w\bar{\mathscr{A}}^\vee_{\mu,-} = \mathrm{End}_{{}^wZ^\vee_\mu}({}^w\bar{B}^\vee_{\mu,-})^{\mathrm{op}}$.

Proof. Apply Propositions A.3, A.5. □

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