arXiv:1401.1565v1 [math.GT] 8 Jan 2014

Four-ball genus bounds and a refinement of the Ozsváth-Szabó tau-invariant

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Abstract

Based on work of Rasmussen [Ras03], we construct a concordance invariant associated to the knot Floer complex, and exhibit examples in which this invariant gives arbitrarily better bounds on the 4-ball genus than the Ozsváth-Szabó τ invariant.

1 Introduction

The 4-*ball genus* of a knot $K \subset S^3$ is

$$g_4(K) = \min\{g(\Sigma) \mid \Sigma \text{ smoothly embedded in } B^4 \text{ with } \partial\Sigma = K\},$$

where $g(\Sigma)$ denotes the genus of the surface Σ. The 4-ball genus gives a lower bound on the unknotting number of a knot (that is, the minimal number of crossing changes needed to obtain the unknot). We say knots K_1 and K_2 are *concordant* if $g_4(K_1 \# -K_2) = 0$, where $-K_2$ denotes the reverse of the mirror image of K_2.

In [OS03c], Ozsváth-Szabó defined a concordance invariant, τ, that gives a lower bound for the 4-ball genus of a knot. This invariant is sharp on torus knots, giving a new proof of the Milnor conjecture, originally proved by Kronheimer-Mrowka using gauge theory [KM93]

The knot Floer homology package [OS04a, Ras03] associates to a knot K a $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex over the ring $\mathbb{F}[U, U^{-1}]$, where $\mathbb{F}$ denotes the field of two elements and U is a formal variable. We denote this complex $CFK^\infty(K)$. The invariant τ depends only on a single $\mathbb{Z}$-filtration, and forgets the module structure. By studying the module structure together with the full $\mathbb{Z} \oplus \mathbb{Z}$-filtration, we obtain a concordance invariant, ν^+, which gives a better bound on the 4-ball genus than τ, in the sense that

$$\tau(K) \leq \nu^+(K) \leq g_4(K). \tag{1.1}$$

Moreover, the gap between τ and ν^+ can be made arbitrarily large.

Theorem 1. *For any positive integer p, there exists a knot K with $\tau(K) \geq 0$ and*

$$\tau(K) + p \leq \nu^+(K) = g_4(K).$$

Remark 1.1. The invariant ν^+ is closely related to the sequence of local h invariants of Rasmussen [Ras03, Section 7], which Rasmussen uses to give bounds on the 4-ball genus; indeed, ν^+ corresponds to the first place in the sequence where a zero appears.

In Proposition 3.7, we also show that the gap between ν^+ and the knot signature can be made arbitrarily large.

In the case of alternating knots (or, more generally, quasi-alternating knots), the invariant ν^+ is completely determined by the signature of the knot.

Theorem 2. *Let $K \subset S^3$ be a quasi-alternating knot. Then,*

$$\nu^+(K) = \begin{cases} 0 & \text{if } \sigma(K) \geq 0, \\ -\frac{\sigma(K)}{2} & \text{if } \sigma(K) < 0. \end{cases}$$

We also have the following result when K is strongly quasipositive. See [Hed10] for background on strongly quasipositive knots.

Proposition 3. *If K is strongly quasipositive, then*

$$\nu^+(K) = \tau(K) = g_4(K) = g(K).$$

Proof. [Hed10, Theorem 1.2] states that $\tau(K) = g_4(K) = g(K)$ if and only if K is strongly quasipositive. Since $\tau(K) \leq \nu^+(K) \leq g_4(K)$, the result follows. ☐

Organization. In Section 2, we define the invariant ν^+ and prove various properties. In Section 3, we construct an infinite family of knots in order to prove Theorem 1. Throughout, we work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Acknowledgements. The first author was partially supported by NSF grant DMS-1307879. The second author would like to thank Hiroshi Goda for helpful email communications.

2 The invariant ν^+

Heegaard Floer homology, introduced by Ozsváth and Szabó [OS04b], is an invariant for closed oriented Spin^c 3–manifolds $(Y, \mathfrak{s})$, taking the form of a collection of related homology groups: $\widehat{HF}(Y,\mathfrak{s})$, $HF^{\pm}(Y,\mathfrak{s})$, and $HF^{\infty}(Y,\mathfrak{s})$. There is a U–action on the Heegaard Floer homology groups $HF^{\pm}$ and HF^{∞}. When $\mathfrak{s}$ is torsion, there is an absolute Maslov $\mathbb{Q}$–grading on the Heegaard Floer homology groups. The U–action decreases the grading by 2.

For a rational homology 3–sphere Y with a Spin^c structure $\mathfrak{s}$, $HF^+(Y,\mathfrak{s})$ can be decomposed as the direct sum of two groups: the first group is the image of $HF^{\infty}(Y,\mathfrak{s}) \cong \mathbb{F}[U,U^{-1}]$ in $HF^+(Y,\mathfrak{s})$, which is isomorphic to $\mathcal{T}^+ = \mathbb{F}[U,U^{-1}]/U\mathbb{F}[U]$, and its minimal absolute $\mathbb{Q}$–grading is an invariant of $(Y,\mathfrak{s})$, denoted by $d(Y,\mathfrak{s})$, the *correction term* [OS03a]; the second group is the quotient modulo the above image and is denoted by $HF_{\mathrm{red}}(Y,\mathfrak{s})$. Altogether, we have

$$HF^+(Y,\mathfrak{s}) = \mathcal{T}^+ \oplus HF_{\mathrm{red}}(Y,\mathfrak{s}).$$

We briefly recall the large N surgery formula of [OS04a, Theorem 4.4]. We use the notation of [NW10]. Let $CFK^{\infty}(K)$ denote the knot Floer complex of K, which takes the form of a $\mathbb{Z}\oplus\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complex over $\mathbb{F}[U,U^{-1}]$. The U-action lowers each filtration by one. We will be particularly interested in the quotient complexes

$$A_k^+ = C\{\max\{i, j-k\} \geq 0\} \quad \text{and} \quad B^+ = C\{i \geq 0\}$$

where i and j refer to the two filtrations. The complex B^+ is isomorphic to $CF^+(S^3)$. There is a map

$$v_k^+ : A_k^+ \to B^+$$

defined by projection. One can also define a map

$$h_k^+ : A_k^+ \to B^+$$

defined by projection to $C\{j \geq k\}$, followed by shifting to $C\{j \geq 0\}$ via the U-action, and concluding with a chain homotopy equivalence between $C\{j \geq 0\}$ and $C\{i \geq 0\}$. These maps correspond to the maps induced on HF^+ by the two handle cobordism from $S^3_N(K)$ to S^3 [OS04a, Theorem 4.4].

Similarly, one can consider the subquotient complexes

$$\widehat{A}_k = C\{\max\{i, j-k\} = 0\} \quad \text{and } \widehat{B} = C\{i = 0\} \cong \widehat{CF}(S^3)$$

and the maps

$$\widehat{v}_k : \widehat{A}_k \to \widehat{B} \quad \text{and} \quad \widehat{h}_k : \widehat{A}_k \to \widehat{B}.$$

The invariant τ is defined in [OS03c] to be

$$\tau(K) = \min\{k \in \mathbb{Z} \mid \iota_k \text{ induces a nontrivial map on homology}\},$$

where $\iota_k : C\{i = 0, j \leq k\} \to \widehat{CF}(S^3)$ denotes inclusion. A slightly stronger concordance invariant, ν, is defined in [OS11, Definition 9.1] to be

$$\nu(K) = \min\{k \in \mathbb{Z} \,|\, \widehat{v}_k : \widehat{A}_k \to \widehat{CF}(S^3) \text{ induces a nontrivial map in homology}\}.$$

The invariant $\nu(K)$ gives a lower bound for $g_4(K)$ and is equal to either $\tau(K)$ or $\tau(K)+1$; in particular, in many cases ν gives a better 4-ball genus than τ.

We can further refine these bounds by considering maps on CF^+ rather than $\widehat{CF}$.

Definition 2.1. Define $\nu^+(K)$ by

$$\nu^+(K) = \min\{k \in \mathbb{Z} \,|\, v_k^+ : A_k^+ \to CF^+(S^3),\ \ v_k^+(1) = 1\}.$$

Here, 1 denotes the lowest graded generator of the subgroup $\mathcal{T}^+$ in the homology of the complex.

According to [NW10], the definition of $\nu^+(K)$ is equivalent to the smallest k such that $V_k = 0$, where V_k is the U-exponent of v_k^+ at sufficiently high gradings. We can define H_k similarly in terms of h_k^+. By [NW12, Equation (13)] and [HLZ12, Lemma 2.5], the V_k's and H_k's satisfy

$$H_k = V_{-k} \tag{2.1}$$
$$H_k = V_k + k \tag{2.2}$$
$$V_k - 1 \leq V_{k+1} \leq V_k \tag{2.3}$$

and are related to the correction terms in the surgery formula [NW10, Proposition 1.6]:

Proposition 2.2. *Suppose $p, q > 0$, and fix $0 \leq i \leq p-1$. Then*

$$d(S^3_{p/q}(K), i) = d(L(p,q), i) - 2\max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}\}. \tag{2.4}$$

We have the following properties for ν^+.

Proposition 2.3. *The invariant ν^+ satisfies:*

1. *ν^+ is a smooth concordance invariant.*
2. *$\nu^+(K) \geq 0$, and the equality holds if and only if $V_0 = 0$.*
3. *$\nu^+(K) \geq \nu(K) \geq \tau(K)$.*

Proof. To see 1, note that V's are determined by the d-invariants of the surgered manifolds $S^3_n(K)$ [NW10, Proposition 2.11], and the d-invariants are concordance invariants. To see 2, note that $V_{-1} > H_{-1} = V_1 \geq 0$ by Equations (2.1) and (2.2). To see 3, chase the commutative diagram

$$\begin{array}{ccc} \widehat{A}_k & \xrightarrow{j_A} & A_k^+ \\ {\scriptstyle \widehat{v}_k}\big\downarrow & & {\scriptstyle v_k^+}\big\downarrow \\ \widehat{B} & \xrightarrow{j_B} & B^+. \end{array}$$

□

The ν^+ invariant can be computed explicitly for quasi-alternating knots, a generalization of alternating knots introduced in [MO08]. In fact, Theorem 2 states that ν^+ is completely determined by the signature of the knot, just as the τ invariant:

$$\nu^+(K) = \begin{cases} 0 & \text{if } \sigma(K) \geq 0, \\ -\frac{\sigma(K)}{2} & \text{if } \sigma(K) < 0. \end{cases}$$

Proof of Theorem 2. Let K be quasi-alternating. By [OS03b, Corollary 1.5] and [MO08, Theorem 2], $d(S^3_1(K)) = 0$ when $\sigma(K) \geq 0$. This proves that $\nu^+(K) = 0$ when $\sigma(K) \geq 0$. On the other hand, the proof of Theorem 1.4 of [OS03b], together with [MO08, Theorem 2], implies that for any $s > 0$,

$$H_{\leq s+\frac{\sigma}{2}-2}(A^+_s) \cong HF^+_{\leq s+\frac{\sigma}{2}-2}(S^3).$$

In particular, if we let $s = -\sigma/2$ when $\sigma(K) < 0$, then

$$H_{\leq -2}(A^+_s) \cong HF^+_{\leq -2}(S^3) \cong 0.$$

Here, the gradings of the homology of both sides are inherited from the grading on $CFK^\infty(K)$. Thus, the generator of $\mathcal{T}^+ \subset H_*(A^+_s)$ has grading $-2V_s$. In light of the vanishing of the homology group $H_{\leq -2}(A^+_s)$, we must have $V_s = 0$. So

$$\nu^+(K) \leq s = -\sigma(K)/2$$

from the definition. We also know that

$$\nu^+(K) \geq \tau(K) = -\sigma(K)/2$$

for a quasi-alternating knot K. Hence, $\nu^+(K) = -\sigma(K)/2$. □

Next, we show that ν^+ also give a lower bound for the four-ball genus of a knot.

Proposition 2.4. $\nu^+(K) \leq g_4(K)$

Proof. This follows from [Ras03, Corollary 7.4]. The function $h_k(K)$ in [Ras03] is the same as V_k in [NW10]. □

Remark 2.5. [Ras03, Corollary 7.4] states that $g_4(K) \geq V_k + k$ for all $k \leq g_4(K)$, so one might wonder if other V_k's can give stronger 4-ball genus bounds. However, since $V_k - 1 \leq V_{k+1} \leq V_k$, it follows that ν^+ is the best 4-ball genus bound obtainable from the sequence of V_k's.

3 Four-ball genus bound

In this section, we exhibit some examples of knots whose ν^+ invariant is arbitrarily better than the corresponding τ invariant. Hence, the ν^+ invariant indeed gives us significantly improved four-ball genus bound for some particular knots. We will show that for any integer $n \geq 2$, there exists a knot K with $\tau(K) \geq 0$ and

$$\tau(K) + n = \nu^+(K) = g_4(K).$$

Let $K_{p,q}$ denote the (p, q)-cable of K, where p denotes the longitudinal winding. Without loss of generality, we will assume throughout that $p > 0$. Let $T_{p,q}$ denote the (p, q)-torus knot (that is, the (p, q)-cable of the unknot), and $T_{p,q;m,n}$ the (m, n)-cable of $T_{p,q}$. We begin with a single example of a knot for which ν^+ gives a better 4-ball genus bound than τ.

Proposition 3.1. *Let K be the knot $T_{2,9}\# - T_{2,3;2,5}$. We have*

$$\tau(K) = 0, \qquad \nu(K) = 1, \quad \text{and} \quad \nu^+(K) = 2.$$

Proof. The torus knot $T_{2,9}$ is an L-space knot, as is $T_{2,3;2,5}$ [Hed09, Theorem 1.10], so their knot Floer complexes are completely determined by their Alexander polynomials [OS05, Theorem 1.2] (cf. [Hom11, Remark 6.6]). We have that

$$\Delta_{T_{2,9}}(T) = t^8 - t^7 + t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$$

and

$$\begin{aligned}\Delta_{T_{2,3;2,5}}(t) &= \Delta_{T_{2,3}}(t^2) \cdot \Delta_{T_{2,5}}(t) \\ &= t^8 - t^7 + t^4 - t + 1.\end{aligned}$$

Furthermore, we have that $CFK^\infty(-K) \cong CFK^\infty(K)^*$ [OS04a, Section 3.5], where $CFK^\infty(K)^*$ denotes the dual of $CFK^\infty(K)$. Thus, $CFK^\infty(-T_{2,3;2,5})$ is generated over $\mathbb{F}[U, U^{-1}]$ by

$$[y_0, 0, -4], \quad [y_1, -1, -4], \quad [y_1, -1, -1], \quad [y_3, -3, -1], \quad [y_4, -4, 0],$$

where we write $[y, i, j]$ to denote that the generator y has filtration level (i, j). The differential is given by

$$\begin{aligned}\partial y_0 &= y_1 \\ \partial y_2 &= y_1 + y_3 \\ \partial y_4 &= y_3.\end{aligned}$$

The complex $CFK^\infty(T_{2,9})$ is generated by

$$[x_0, 0, 4], \quad [x_1, 1, 4], \quad [x_2, 1, 3], \quad [x_3, 2, 3], \quad [x_4, 2, 2],$$
$$[x_5, 3, 2], \quad [x_6, 3, 1], \quad [x_7, 4, 1], \quad [x_8, 4, 0].$$

The differential is given by

$$\begin{aligned}\partial x_1 &= x_0 + x_2 \\ \partial x_3 &= x_2 + x_4 \\ \partial x_5 &= x_4 + x_6 \\ \partial x_7 &= x_6 + x_8.\end{aligned}$$

The complexes $CFK^\infty(-T_{2,3;2,5})$ and $CFK^\infty(T_{2,9})$ are depicted in Figures 1 and 2, respectively. (More precisely, CFK^∞ consists of the complexes pictured tensored with $\mathbb{F}[U, U^{-1}]$, where U lowers i and j each by 1.) In particular, we see that $\tau(-T_{2,3;2,5}) = -4$ since y_0 generates the vertical homology, and that $\tau(T_{2,9}) = 4$ since x_0 generates the vertical homology. Since τ is additive under connected sum, it follows that

$$\tau(-T_{2,3;2,5} \# T_{2,9}) = 0,$$

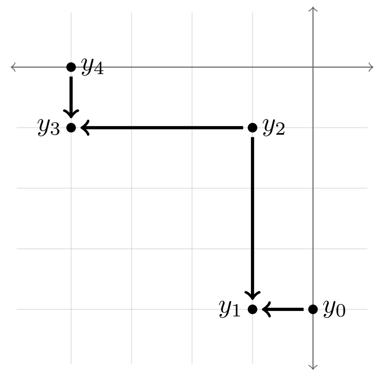

Figure 1: $CFK^\infty(-T_{2,3;2,5})$

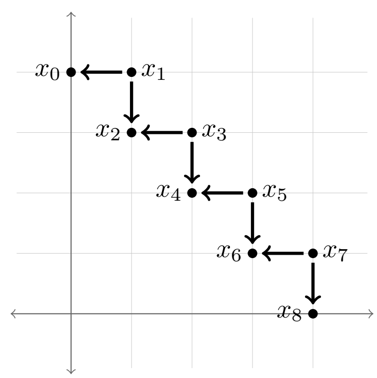

Figure 2: $CFK^\infty(T_{2,9})$

as desired.

The knot Floer complex satisfies a Künneth formula [OS04a, Theorem 7.1]:

$$CFK^\infty(K_1 \# K_2) \cong CFK^\infty(K_1) \otimes_{\mathbb{F}[U,U^{-1}]} CFK^\infty(K_2).$$

In particular, we may compute $CFK^\infty(T_{2,9} \# -T_{2,3;2,5})$ as the tensor product of $CFK^\infty(T_{2,9})$ and $CFK^\infty(-T_{2,3;2,5})$, where

$$[x, i, j] \otimes [y, k, \ell] = [xy, i + k, j + \ell].$$

The generators, filtration levels, and differentials in the tensor product are listed below.

$$\partial[x_0y_0, 0, 0] = x_0y_1$$
$$\partial[x_1y_0, 1, 0] = x_1y_1 + x_0y_0 + x_2y_0$$

$$\partial[x_2y_0,1,-1]=x_2y_1$$
$$\partial[x_3y_0,2,-1]=x_3y_1+x_2y_0+x_4y_0$$
$$\partial[x_4y_0,2,-2]=x_4y_1$$
$$\partial[x_5y_0,3,-2]=x_5y_1+x_4y_0+x_6y_0$$
$$\partial[x_6y_0,3,-3]=x_6y_1$$
$$\partial[x_7y_0,4,-3]=x_7y_1+x_6y_0+x_8y_0$$
$$\partial[x_8y_0,4,-4]=x_8y_1$$
$$\partial[x_0y_1,-1,0]=0$$
$$\partial[x_1y_1,0,0]=x_0y_1+x_2y_1$$
$$\partial[x_2y_1,0,-1]=0$$
$$\partial[x_3y_1,1,-1]=x_2y_1+x_4y_1$$
$$\partial[x_4y_1,1,-2]=0$$
$$\partial[x_5y_1,2,-2]=x_4y_1+x_6y_1$$
$$\partial[x_6y_1,2,-3]=0$$
$$\partial[x_7y_1,3,-3]=x_6y_1+x_8y_1$$
$$\partial[x_8y_1,3,-4]=0$$
$$\partial[x_0y_2,-1,3]=x_0y_1+x_0y_3$$
$$\partial[x_1y_2,0,3]=x_1y_1+x_1y_3+x_0y_2+x_2y_2$$
$$\partial[x_2y_2,0,2]=x_2y_1+x_2y_3$$
$$\partial[x_3y_2,1,2]=x_3y_1+x_3y_3+x_2y_2+x_4y_2$$
$$\partial[x_4y_2,1,1]=x_4y_1+x_4y_3$$
$$\partial[x_5y_2,2,1]=x_5y_1+x_5y_3+x_4y_2+x_6y_2$$
$$\partial[x_6y_2,2,0]=x_6y_1+x_6y_3$$
$$\partial[x_7y_2,3,0]=x_7y_1+x_7y_3+x_6y_2+x_8y_2$$
$$\partial[x_8y_2,3,-1]=x_8y_1+x_8y_3$$
$$\partial[x_0y_3,-4,3]=0$$
$$\partial[x_1y_3,-3,3]=x_0y_3+x_2y_3$$
$$\partial[x_2y_3,-3,2]=0$$
$$\partial[x_3y_3,-2,2]=x_2y_3+x_4y_3$$
$$\partial[x_4y_3,-2,1]=0$$
$$\partial[x_5y_3,-1,1]=x_4y_3+x_6y_3$$
$$\partial[x_6y_3,-1,0]=0$$
$$\partial[x_7y_3,0,0]=x_6y_3+x_8y_3$$
$$\partial[x_8y_3,0,-1]=0$$
$$\partial[x_0y_4,-4,4]=x_0y_3$$
$$\partial[x_1y_4,-3,4]=x_1y_3+x_0y_4+x_2y_4$$
$$\partial[x_2y_4,-3,3]=x_2y_3$$

$$
\begin{aligned}
\partial[x_3y_4, -2, 3] &= x_3y_3 + x_2y_4 + x_4y_4\\
\partial[x_4y_4, -2, 2] &= x_4y_3\\
\partial[x_5y_4, -1, 2] &= x_5y_3 + x_4y_4 + x_6y_4\\
\partial[x_6y_4, -1, 1] &= x_6y_3\\
\partial[x_7y_4, 0, 1] &= x_7y_3 + x_6y_4 + x_8y_4\\
\partial[x_8y_4, 0, 0] &= x_8y_3
\end{aligned}
$$

We perform the following change of basis on $CFK^\infty(T_{2,9}\# - T_{2,3;2,5})$. In the linear combinations below, we have ordered the terms so that the first basis element has the greatest filtration and thus determines the filtration level of the linear combination.

$$
\begin{aligned}
z_0 &= x_0y_0\\
z_1 &= x_0y_1\\
z_2 &= x_0y_2 + x_1y_3 + x_3y_3 + x_4y_4\\
z_3 &= x_1y_2\\
z_4 &= x_2y_2 + x_3y_3 + x_1y_1 + x_4y_4\\
z_5 &= x_3y_2 + x_5y_4 + x_1y_0\\
z_6 &= x_4y_2 + x_5y_3 + x_3y_1 + x_6y_4 + x_2y_0\\
z_7 &= x_5y_2 + x_7y_4 + x_3y_0\\
z_8 &= x_6y_2 + x_7y_3 + x_5y_1 + x_4y_0\\
z_9 &= x_7y_2\\
z_{10} &= x_8y_2 + x_7y_1 + x_4y_0 + x_5y_1\\
z_{11} &= x_8y_3\\
z_{12} &= x_8y_4\\
w_0^i &= x_{2i+1}y_4 && i = 0,1,2,3\\
w_1^i &= x_{2i}y_4 && i = 0,1,2,3\\
w_2^i &= x_{2i}y_3 && i = 0,1,2,3\\
w_3^i &= x_{2i+1}y_3 + x_{2i+2}y_4 && i = 0,1,2,3\\
w_0^{i+4} &= x_{2i+1}y_0 && i = 0,1,2,3\\
w_1^{i+4} &= x_{2i+1}y_1 + x_{2i}y_0 && i = 0,1,2,3\\
w_2^{i+4} &= x_{2i+2}y_1 && i = 0,1,2,3\\
w_3^{i+4} &= x_{2i+2}y_0 && i = 0,1,2,3.
\end{aligned}
$$

See Figure 3.

Notice that the basis elements $\{z_i\}_{i=0}^{12}$ generate a direct summand C of $CFK^\infty(T_{2,9}\# - T_{2,3;2,5})$. See Figure 4. Since the total homology of this summand is non-zero, this summand determines both ν and ν^+. We write $\widehat{A}_s$ and A_s^+ to refer to the associated subquotient complexes of C.

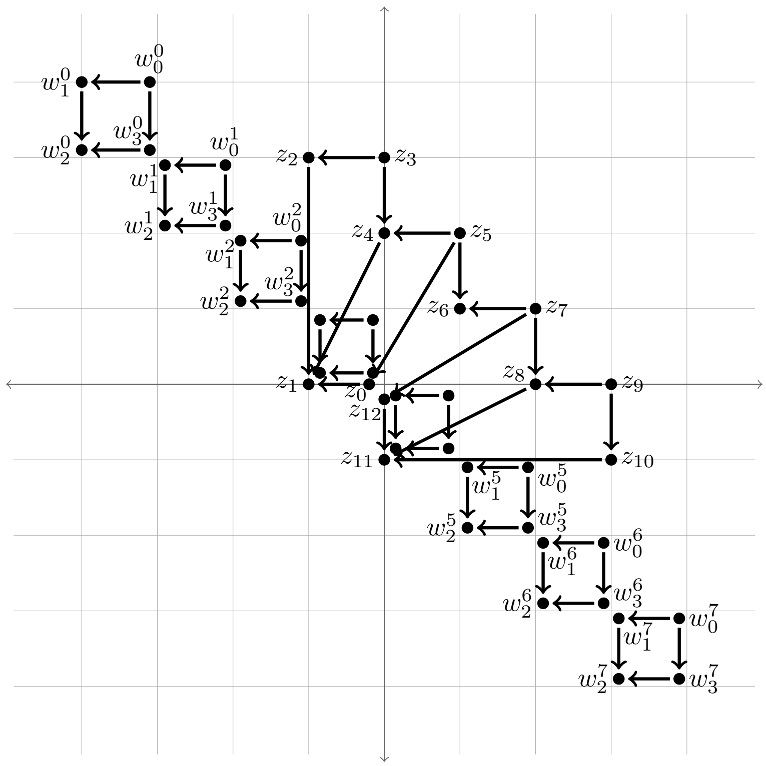

Figure 3: $CFK^\infty(T_{2,9}\# - T_{2,3;2,5})$ after a change of basis

The vertical homology of C is generated by z_0. The generator z_0 in $C\{i = 0\}$ is not the image of any cycle in $\widehat{A}_0$. On the other hand, z_0 is non-zero in $H_*(\widehat{A}_1)$. Hence $\nu(T_{2,9}\# - T_{2,3;2,5}) = 1$.

The cycle z_6 generates $H_*(C)$. Moreover, the cycle Uz_6 is non-zero in $H_*(A_1^+)$; see Figure 5. The cycle Uz_6 is a boundary in A_2^+ as in Figure 6, while the cycle z_6 is non-zero in $H_*(A_2^+)$. It follows that $\nu^+(T_{2,9}\# - T_{2,3;2,5}) = 2$, as desired. ☐

Corollary 3.2. *Let* $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. *Then*

$$\tau(K) = 0, \qquad \nu(K) = 1, \qquad \text{and} \qquad \nu^+(K) = 2.$$

Proof. By [HKL, Theorem B.1],

$$CFK^\infty(T_{2,5}\#2T_{2,3}) \cong CFK^\infty(T_{2,9}) \oplus A,$$

where A is acyclic (i.e., its total homology vanishes). Since acyclic summands do not affect τ, ν, and ν^+, the result follows. ☐

Lemma 3.3. *Let* $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. *Then* $g_4(K) = 2$.

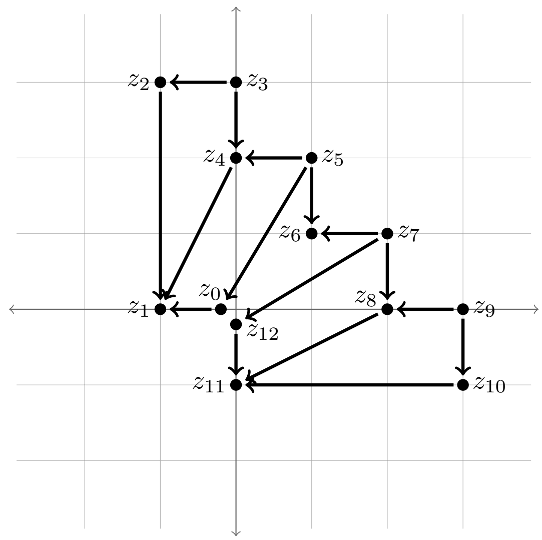

Figure 4: The relevant summand of $CFK^\infty(T_{2,9}\# -T_{2,3;2,5})$

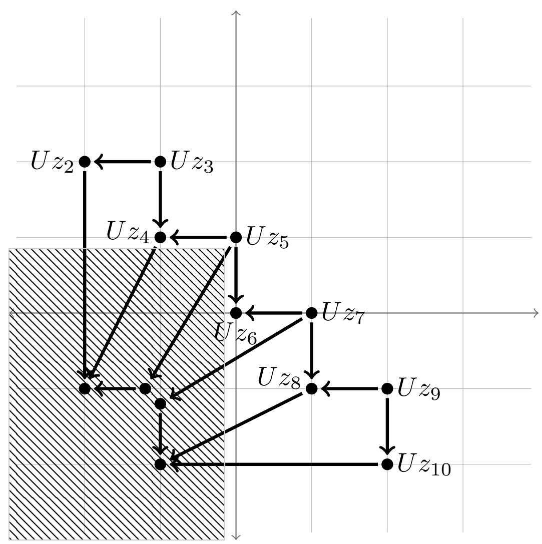

Figure 5: The generators $\{Uz_i\}$ in A_1^+

Proof. When $p, q > 0$, the genus of $T_{p,q}$ is equal to $\frac{(p-1)(q-1)}{2}$. We can construct a genus 4 Seifert surface F for $-T_{2,3;2,5} = (-T_{2,3})_{-2,5}$ by taking two parallel copies of the genus one Seifert surface for $-T_{2,3}$ and connecting them with 5 half-twisted bands. The knot $-T_{2,3}\#T_{-2,5}$ sits on F. To see this, consider one copy of the Seifert surface for $-T_{2,3}$ together with the half-twisted bands and a

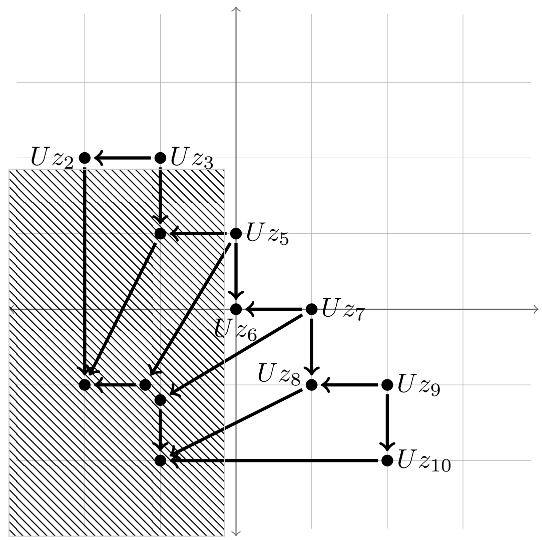

Figure 6: The generators $\{Uz_i\}$ in A_2^+

small neighborhood of a segment connecting the ends of the bands.

Take the boundary sum of F with the genus two Seifert surface for $T_{2,5}$ and with two copies of the genus one Seifert surface for $T_{2,3}$ to obtain a surface F'. The surface F' is a genus 8 Seifert surface for K. The genus 6 slice knot $J = -T_{2,3}\#T_{-2,5}\#T_{2,3}\#T_{2,5}$ sits on this surface. Performing surgery along J on F' in B^4 yields a genus two slice surface for K. Since $\nu^+(K) = 2$ and $\nu^+(K) \leq g_4(K)$, it follows that $g_4(K) = 2$. □

In order to prove the main theorem, we will consider certain cables of the knot $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. We first compute τ of these cables.

Lemma 3.4. *Let K be the knot $T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. Then*

$$\tau(K_{p,3p-1}) = \frac{3p(p-1)}{2}.$$

Proof. Recall from [Hom12, Definition 3.4] that the invariant $\varepsilon(K)$ is defined to be -1 if $\tau(K) < \nu(K)$. The equality then follows from [Hom12, Theorem 1], which states that if $\varepsilon(K) = -1$, then

$$\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q+1)}{2}.$$

□

Proposition 3.5. *Let K be the knot $T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. Then*

$$\nu^+(K_{p,3p-1}) = g_4(K_{p,3p-1}) = \frac{p(3p-1)}{2} + 1.$$

Proof. Let $p, q > 0$. For a cable knot $K_{p,q}$, there is a reducible surgery

$$S^3_{pq}(K_{p,q}) \cong S^3_{q/p}(K) \# L(p,q).$$

We apply the surgery formula (2.4) for the above knot surgery when K is the unknot. Note that $\max\{V_i, H_{i-pq}\} = V_i$ when $0 \leq i \leq \frac{pq}{2}$ since $V_i = H_{-i}$ and $H_{i-1} \leq H_i$. Thus, we have

$$d(L(pq,1),i) - 2V_i(T_{p,q}) = d(L(q,p),p_1(i)) + d(L(p,q),p_2(i)) \tag{3.1}$$

for all $0 \leq i \leq \frac{pq}{2}$.

Here, we identify the Spin^c structure of a rational homology sphere by an integer i as in [NW10], and $p_1(i)$ and $p_2(i)$ are the projection of the Spin^c structure to the two factors of the reducible manifold. In particular, we can identify $p_1(i)$ with some integers between 0 and $q-1$ and $p_2(i)$ with some integers between 0 and $p-1$.

We can also apply (2.4) for an arbitrary knot K. We have

$$\begin{aligned} d(L(pq,1),i) - 2V_i(K_{p,q}) &= d(L(q,p),p_1(i)) - 2\max\{V_{\lfloor \frac{p_1(i)}{p} \rfloor}(K), H_{\lfloor \frac{p_1(i)-q}{p} \rfloor}(K)\} \\ &\quad + d(L(p,q),p_2(i)). \end{aligned}$$

for all $i \leq \frac{pq}{2}$.

Compared with Equation (3.1) and using the fact $V_i(T_{p,q}) \geq 0$, we deduce that for all $i \leq \frac{pq}{2}$,

$$\begin{aligned} V_i(K_{p,q}) &= V_i(T_{p,q}) + \max\{V_{\lfloor \frac{p_1(i)}{p} \rfloor}(K), H_{\lfloor \frac{p_1(i)-q}{p} \rfloor}(K)\} \\ &\geq \max\{V_{\lfloor \frac{p_1(i)}{p} \rfloor}(K), H_{\lfloor \frac{p_1(i)-q}{p} \rfloor}(K)\} \end{aligned}$$

From now on, let us specialize to the case when K is the knot $T_{2,5} \# 2T_{2,3} \# -T_{2,3;2,5}$ and $q = 3p-1$. We claim that

$$\max\{V_{\lfloor \frac{p_1(i)}{p} \rfloor}, H_{\lfloor \frac{p_1(i)-q}{p} \rfloor}\} > 0.$$

To see this, note that $V_0(K), V_1(K) > 0$ as $\nu^+(K) = 2$. When $0 \leq p_1(i) < 2p$, $V_{\lfloor \frac{p_1(i)}{p} \rfloor}(K) > 0$. Otherwise, $2p \leq p_1(i) < q = 3p-1$, and then $H_{\lfloor \frac{p_1(i)-q}{p} \rfloor}(K) > 0$ since $H_{-k} = V_k$ and $V_0(K), V_1(K) > 0$.

Hence, $V_i(K_{p,q}) > 0$ for all $i \leq \frac{pq}{2}$. This implies that

$$\nu^+(K_{p,3p-1}) \geq \frac{p(3p-1)}{2} + 1.$$

On the other hand,

$$g_4(K_{p,q}) \leq p g_4(K) + \frac{(p-1)(q-1)}{2},$$

since one can construct a slice surface for $K_{p,q}$ from p parallel copies of a slice surface for K together with $(p-1)q$ half-twisted bands. By Lemma 3.3, $g_4(K) = 2$, so when $q = 3p-1$, the right-hand side of the above inequality is $\frac{p(3p-1)}{2}+1$. Hence

$$\frac{p(3p-1)}{2}+1 \leq \nu^+(K_{p,3p-1}) \leq g_4(K_{p,3p-1}) \leq \frac{p(3p-1)}{2}+1,$$

so $\nu^+(K_{p,3p-1}) = g_4(K_{p,3p-1}) = \frac{p(3p-1)}{2}+1$. □

Note that $\nu^+(K_{p,3p-1}) - \tau(K_{p,3p-1}) = p+1$ for $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. This proves Theorem 1.

A similar argument shows that ν^+ gives a sharp four-ball genus bound for certain other cable knots as well.

Proposition 3.6. *Let K be a knot with $\nu^+(K) = g_4(K) = n$, then*

$$\nu^+(K_{p,(2n-1)p-1}) = g_4(K_{p,(2n-1)p-1}) = \frac{p((2n-1)p-1)}{2}+1.$$

Proof. Let $q = (2n-1)p-1$. We proved

$$V_i(K_{p,q}) \geq \max\{V_{\lfloor \frac{p_1(i)}{p} \rfloor}(K), H_{\lfloor \frac{p_1(i)-q}{p} \rfloor}(K)\}.$$

We claim that

$$\max\{V_{\lfloor \frac{p_1(i)}{p} \rfloor}, H_{\lfloor \frac{p_1(i)-q}{p} \rfloor}\} > 0.$$

To see this, note that $V_i(K) > 0$ for all $i < n$. When $0 \leq p_1(i) < np$, $V_{\lfloor \frac{p_1(i)}{p} \rfloor}(K) > 0$. Otherwise, $np \leq p_1(i) < q = (2n-1)p-1$, and then $H_{\lfloor \frac{p_1(i)-q}{p} \rfloor}(K) > 0$. Hence, $V_i(K_{p,q}) > 0$ for all $i \leq \frac{pq}{2}$. This implies that

$$\begin{aligned} \nu^+(K_{p,q}) &\geq \frac{pq}{2}+1 \\ &= \frac{p((2n-1)p-1)}{2}+1. \end{aligned}$$

On the other hand,

$$\begin{aligned} g_4(K_{p,q}) &\leq pg_4(K) + \frac{(p-1)(q-1)}{2} \\ &= pn + \frac{(p-1)((2n-1)p-2)}{2} \\ &= \frac{p((2n-1)p-1)}{2}+1. \end{aligned}$$

So $\nu^+(K_{p,(2n-1)p-1}) = g_4(K_{p,(2n-1)p-1}) = \frac{p((2n-1)p-1)}{2}+1$. □

We conclude by showing that the knot signature cannot detect the four-ball genus of the knots used in Theorem 1. Recall that

$$\frac{1}{2}|\sigma(K)| \leq g_4(K).$$

Proposition 3.7. *Let $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. Then for $p > 0$,*

$$\frac{1}{2}|\sigma(K_{p,3p-1})| + 2p - 2 \leq g_4(K_{p,3p-1}).$$

Proof. We have that $\sigma(T_{2,q}) = 1 - q$. By [Shi71, Theorem 9],

$$\sigma(K_{p,q}) = \begin{cases} \sigma(T_{p,q}) & \text{if } p \text{ is even} \\ \sigma(K) + \sigma(T_{p,q}) & \text{if } p \text{ is odd.} \end{cases}$$

Thus, $\sigma(T_{2,3;2,5}) = -4$ and since signature is additive under connected sum,

$$\begin{aligned} \sigma(T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}) &= -4 + 2(-2) - (-4) \\ &= -4. \end{aligned}$$

We showed in Lemma 3.3 that $g_4(K) = 2$, so for K, the signature is indeed strong enough to detect the four-ball genus. However, we will now show that it is not strong enough to detect the four-ball genus of $K_{p,3p-1}$. We have that

$$\begin{aligned} |\sigma(K_{p,3p-1})| &\leq |\sigma(K)| + |\sigma(T_{p,3p-1})| \\ &\leq 4 + (p-1)(3p-2) \\ &= 3p^2 - 5p + 6, \end{aligned}$$

where the second inequality follows from the fact that when $p, q > 0$,

$$|\sigma(T_{p,q})| \leq 2g_4(T_{p,q}) = (p-1)(q-1).$$

On the other hand,

$$2g_4(K_{p,3p-1}) = 3p^2 - p + 2,$$

so

$$|\sigma(K_{p,3p-1})| + 4p - 4 \leq 2g_4(K_{p,3p-1}).$$

□

Recall from Proposition 3.5 that $g_4(K_{p,3p-1}) = \nu^+(K_{p,3p-1})$. A consequence of Proposition 3.7 is that the gap between $\frac{1}{2}\sigma$ and ν^+ can be made arbitrarily large.

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