J. Math. Anal. Appl. 415 (2014) 25–32

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Harnack inequalities for graphs with non-negative Ricci curvature

Fan Chung [a], Yong Lin [b,*,1], S.-T. Yau [c]

[a] University of California, San Diego, United States
[b] Renmin University of China, China
[c] Harvard University, United States

ARTICLE INFO

Article history:
Received 17 March 2013
Available online 21 January 2014
Submitted by S.A. Fulling

Keywords:
The Laplace operator for graphs
The Harnack inequalities
Eigenvalues
Diameter

ABSTRACT

We establish a Harnack inequality for finite connected graphs with non-negative Ricci curvature. As a consequence, we derive an eigenvalue lower bound, extending previous results for Ricci flat graphs.

1. Introduction

Let G be an undirected finite connected weighted graph with vertex set V and edge set E. The edge weight of an edge $\{x, y\}$ is denoted by w_{xy} and the degree d_x is the sum of all w_{xy} over all y adjacent to x. The Laplace operator Δ of a graph G is defined by

$$\Delta f(x) = \frac{1}{d_x} \sum_{y \sim x} w_{xy} \big(f(y) - f(x) \big)$$

for any function $f \in V^R = \{f | f : V \to R\}$ and any vertex $x \in V$.

Suppose a function $f \in V^R$ satisfies that, for every vertex $x \in V$,

$$(-\Delta) f(x) = \frac{1}{d_x} \sum_{y \sim x} w_{xy} \big(f(x) - f(y) \big) = \lambda f(x).$$

Then f is called a harmonic eigenfunction of the Laplace operator Δ on G with eigenvalue λ. For a finite graph, it is straightforward to verify that λ is an eigenvalue for the (normalized) Laplacian $\mathcal{L}$ as a matrix defined by

* Corresponding author.
[1] Supported by the National Natural Science Foundation of China (Grant No. 11271011).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2014.01.044

$$\mathcal{L} = -\Delta = I - D^{-1/2}AD^{-1/2}$$

where D is the diagonal degree matrix and A is the weighted adjacency matrix with $A(x,y) = w_{xy}$. Because of the positivity of $\mathcal{L}$, a connected graph has all eigenvalues positive except for one eigenvalue *zero*.

From this definition, at each vertex x, the eigenfunction locally stretches the incident edges in a balanced fashion. Globally, it is desirable to have some tools to capture the notion that adjacent vertices are close to each other.

A crucial part of spectral graph theory concerns understanding the behavior of eigenfunctions. Harnack inequalities are one of the main methods for dealing with eigenfunctions. The Harnack inequalities for certain special families of graphs, called Ricci flat graphs (see [3]), are formulated as follows:

$$\frac{1}{d_x}\sum_{y\sim x}\big(f(x)-f(y)\big)^2 \leqslant 8\lambda \max_z f^2(z) \tag{1.1}$$

for any eigenfunction f with eigenvalue $\lambda > 0$.

In general, the above inequality does not hold for all graphs. An easy counterexample is the graph formed by joining two complete graphs of the same size by a single edge [2].

In this paper, we will establish a Harnack inequality for general graphs. We will show the following:

$$\frac{1}{d_x}\sum_{y\sim x}\big(f(x)-f(y)\big)^2 \leqslant \left(\left(8-\frac{2}{m}\right)\lambda - 4\kappa\right)\max_z f^2(z) \tag{1.2}$$

for any graph with Ricci curvature bounded below by κ. The definition of the Ricci curvature for graphs will be given in the next section.

For a graph G, the diameter of G is the least number D such that any two vertices in G are joined by a path with at most D edges. By using the above Harnack inequality, we will derive the following eigenvalue/diameter inequality for graphs with non-negative Ricci curvature.

$$\lambda \geqslant \frac{1+4\kappa d D^2}{8d\cdot D^2}$$

where d is the maximum degree and D denotes the diameter of G.

2. The Ricci curvature for graphs

In [3] and [4], Chung and Yau defined Ricci flat graphs and proved that inequality (1.1) and (1.4) hold for a large family of Ricci flat graphs. There are several ways to define Ricci curvature for a general graph. In this paper, we will use the definition of Ricci curvature for graphs in the sense of Bakry and Emery [1], as introduced in [6]. We note that a different notion of Ricci curvature was introduced by Ollivier [7].

To define the Ricci curvature of a graph, we begin with a bilinear operator $\Gamma : V^R \times V^R \to V^R$, defined by

$$\Gamma(f,g)(x) = \frac{1}{2}\big\{\Delta\big(f(x)g(x)\big) - f(x)\Delta g(x) - g(x)\Delta f(x)\big\}.$$

According to Bakry and Emery [1], the Ricci curvature operator Γ_2 is defined by:

$$\Gamma_2(f,g)(x) = \frac{1}{2}\big\{\Delta\Gamma(f,g)(x) - \Gamma(f,\Delta g)(x) - \Gamma(g,\Delta f)(x)\big\}.$$

For simplicity, we will omit the variable x in the following equations. Note that all the equations hold locally for every $x \in V$.

Definition 2.1. The operator Δ satisfies the curvature-dimension type inequality $CD(m, \kappa)$ for $m \in (1, +\infty)$ and $\kappa \in \mathbb{R}'$ if

$$\Gamma_2(f, f) \geqslant \frac{1}{m}(\Delta f)^2 + \kappa \Gamma(f, f).$$

We call m the dimension of the operator Δ and κ a lower bound of the Ricci curvature of the operator Δ. If $\Gamma_2 \geqslant \kappa \Gamma$, we say that Δ satisfies $CD(\infty, \kappa)$.

It is easy to see that for $m < m'$, the operator Δ satisfies the curvature-dimension type inequality $CD(m', K)$ if it satisfies the curvature-dimension type inequality $CD(m, K)$.

Here we list a number of helpful facts concerning Γ, Γ_2 and the Ricci curvature that will be useful later.

From the definition of Γ, we can express Γ in the following alternative formulation. The derivation is straightforward and we omit the proof here.

Lemma 2.2.

$$\Gamma(f, g)(x) = \frac{1}{2d_x} \sum_{y \sim x} w_{xy}\big(f(x) - f(y)\big)\big(g(x) - g(y)\big), \tag{2.1}$$

$$\Gamma(f, f)(x) = \frac{1}{2d_x} \sum_{y \sim x} w_{xy}\big[f(x) - f(y)\big]^2 = \frac{1}{2}|\nabla f|^2(x). \tag{2.2}$$

For Laplace–Beltrami operator Δ on a complete m dimensional Riemannion manifold, the operator Δ satisfies $CD(m, K)$ if the Ricci curvature of the Riemannian manifold is bounded below by κ. For graphs, a similar bound can be established as follows.

Lemma 2.3. *In a connected graph G, let λ denote a non-trivial eigenvalue. Then the Ricci curvature κ of G satisfies*

$$\kappa \leqslant \lambda.$$

Proof. Let f denote a harmonic eigenvector associated with eigenvalue λ. Consider the vertex x which achieves the maximum of $|\nabla f|^2$. Then we have $\Delta|\nabla f|^2 \leqslant 0$ and therefore

$$\begin{aligned} \Gamma_2(f, f) &\leqslant -\Gamma(f, \Delta f) \\ &= \lambda \Gamma(f, f). \end{aligned}$$

From the definition of κ, we have

$$\lambda \Gamma(f, f) \geqslant \frac{1}{m}(\Delta f)^2 + \kappa \Gamma(f, f).$$

Thus we have $\kappa \leqslant \lambda$. $\square$

We remark that Lemma 2.3 can be slightly improved to $\lambda \geqslant \kappa(1 + 1/(m - 1))$ as seen in [5].

It was proved in [6] that the Ricci flat graphs as defined in [3] and [4] are graphs satisfy $CD(\infty, 0)$. In [6], it was shown that any locally finite connected G satisfy the $CD(\frac{1}{2}, \frac{1}{d} - 1)$, if the maximum degree d is finite, or $CD(2, -1)$ if d is infinite. Thus, the Ricci curvature of a graph G has a lower bound -1.

3. Harnack inequality and eigenvalue estimate

First, we will establish several basic facts for graphs with non-negative Ricci curvature.

Lemma 3.1. *Suppose G is a finite connected graph satisfying $CD(m,\kappa)$. Then for $x \in V$ and $f \in V^R$, we have*

$$\left(\frac{4}{m}-2\right)\left(\Delta f(x)\right)^2+(2+2\kappa)|\nabla f|^2(x) \leqslant \frac{1}{d_x}\sum_{y\sim x}\frac{w_{xy}}{d_y}\sum_{z\sim y}w_{yz}\left[f(x)-2f(y)+f(z)\right]^2.$$

Proof. We consider $\Delta(\Gamma(f,f))$. By straightforward manipulation and (2.2), we have

$$\begin{aligned}\Delta\bigl(\Gamma(f,f)\bigr)(x) &= \frac{1}{2d_x}\sum_{y\sim x}\frac{w_{xy}}{d_y}\sum_{z\sim y}w_{yz}\left(-\left[f(x)-f(y)\right]^2+\left[f(y)-f(z)\right]^2\right)\\ &= \frac{1}{2d_x}\sum_{y\sim x}\frac{w_{xy}}{d_y}\sum_{z\sim y}w_{yz}\left[f(x)-2f(y)+f(z)\right]^2\\ &\quad -\frac{1}{d_x}\sum_{y\sim x}\frac{w_{xy}}{d_y}\sum_{z\sim y}w_{yz}\left[f(x)-2f(y)+f(z)\right]\left[f(x)-f(y)\right],\end{aligned}$$

and by (2.1) we have

$$\Gamma(f,\Delta f)(x)=\frac{1}{2}\cdot\frac{1}{d_x}\sum_{y\sim x}w_{xy}\left[f(y)-f(x)\right]\cdot\left[\Delta f(y)-\Delta f(x)\right].$$

By the definition of $\Gamma_2(f,f)$, we have

$$\begin{aligned}\Gamma_2(f,f)(x) &= \frac{1}{4}\frac{1}{d_x}\sum_{y\sim x}\frac{w_{xy}}{d_y}\sum_{z\sim y}w_{yz}\left[f(x)-2f(y)+f(z)\right]^2\\ &\quad -\frac{1}{2}\frac{1}{d_x}\sum_{y\sim x}w_{xy}\left[f(x)-f(y)\right]^2+\frac{1}{2}\left(\frac{1}{d_x}\sum_{y\sim x}w_{xy}\bigl(f(x)-f(y)\bigr)\right)^2\\ &= \frac{1}{4}\frac{1}{d_x}\sum_{y\sim x}\frac{w_{xy}}{d_y}\sum_{z\sim y}w_{yz}\left[f(x)-2f(y)+f(z)\right]^2-\frac{1}{2}|\nabla f|^2(x)+\frac{1}{2}(\Delta f)^2. \end{aligned} \tag{3.1}$$

Since G satisfies $CD(m,\kappa)$, we have

$$\Gamma_2(f,f)\geqslant\frac{1}{m}(\Delta f)^2+\kappa\Gamma(f,f).$$

From above inequality, we obtain

$$\left(\frac{1}{m}-\frac{1}{2}\right)(\Delta f)^2+\frac{1+\kappa}{2}|\nabla f|^2\leqslant\frac{1}{4}\frac{1}{d_x}\sum_{y\sim x}\frac{w_{xy}}{d_y}\sum_{z\sim y}w_{yz}\left[f(x)-2f(y)+f(z)\right]^2$$

as desired. □

By using Lemma 3.1, we can prove the following Harnack type inequality. The idea of proof comes from [3].

Theorem 3.2. *Suppose that a finite connected graph G satisfies $CD(m,\kappa)$ and $f \in V^R$ is a harmonic eigenfunction of Laplacian Δ with eigenvalue λ. Then the following inequality holds for all $x \in V$ and $\alpha \geqslant 2 - 2\kappa/\lambda$*

$$|\nabla f|^2(x) + \alpha\lambda f^2(x) \leqslant \frac{(\alpha^2 - \frac{4}{m})\lambda + 2\kappa\alpha}{(\alpha-2)\lambda + 2\kappa}\lambda \max_{z\in V} f^2(z).$$

Proof. Using Lemmas 2.2 and 3.1, we have

$$\begin{aligned}
(-\Delta)|\nabla f|^2(x) &= -\frac{1}{d_x}\sum_{y\sim x}\frac{w_{xy}}{d_y}\sum_{z\sim y} w_{yz}\big[f(x) - 2f(y) + f(z)\big]^2 \\
&\quad + \frac{2}{d_x}\sum_{y\sim x}\frac{w_{xy}}{d_y}\sum_{z\sim y} w_{yz}\big[f(x) - 2f(y) + f(z)\big]\cdot\big[f(x) - f(y)\big] \\
&\leqslant -(2+2\kappa)\cdot|\nabla f|^2(x) + \left(2 - \frac{4}{m}\right)\cdot\big(\Delta f(x)\big)^2 + 2\cdot|\nabla f|^2(x) \\
&\quad + \frac{2}{d_x}\sum_{y\sim x} w_{xy}\big(f(x) - f(y)\big)\cdot\frac{1}{d_y}\sum_{z\sim y} w_{yz}\big[f(z) - f(y)\big] \\
&= -2\kappa\cdot|\nabla f|^2(x) + \left(2 - \frac{4}{m}\right)\big[-\lambda f(x)\big]^2 + \frac{2}{d_x}\sum_{y\sim x} w_{xy}\big(f(x) - f(y)\big)\cdot\big(-\lambda f(y)\big) \\
&= -2\kappa\cdot|\nabla f|^2(x) + \left(2 - \frac{4}{m}\right)\lambda^2 f^2(x) \\
&\quad + \frac{2}{d_x}\sum_{y\sim x} w_{xy}\big(f(x) - f(y)\big)\big(-\lambda f(y) + \lambda f(x) - \lambda f(x)\big) \\
&= (2\lambda - 2\kappa)\cdot|\nabla f|^2(x) - \frac{4}{m}\lambda^2 f^2(x).
\end{aligned}$$

Now we consider

$$\begin{aligned}
(-\Delta)f^2(x) &= \frac{1}{d_x}\sum_{y\sim x} w_{xy}\big[f^2(x) - f^2(y)\big] \\
&= \frac{2}{d_x}\sum_{y\sim x} w_{xy} f(x)\big[f(x) - f(y)\big] - \frac{1}{d_x}\sum_{y\sim x} w_{xy}\big[f(x) - f(y)\big]^2 \\
&= 2\lambda f^2(x) - |\nabla f|^2(x).
\end{aligned}$$

Combining the above inequalities, we have, for any positive α, the following:

$$\begin{aligned}
(-\Delta)\big(|\nabla f|^2(x) + \alpha\lambda f^2(x)\big) &\leqslant (2\lambda - 2\kappa)|\nabla f|^2(x) - \frac{4}{m}\lambda^2 f^2(x) + 2\alpha\lambda^2 f^2(x) - \alpha\lambda|\nabla f|^2(x) \\
&= (2\lambda - \alpha\lambda - 2\kappa)|\nabla f|^2(x) + \left(2\alpha - \frac{4}{m}\right)\lambda^2 f^2(x).
\end{aligned}$$

We choose a vertex v, which maximizes the expression

$$|\nabla f|^2(x) + \alpha\lambda f^2(x)$$

over all $x \in V$. Then we have

$$\begin{aligned} 0 &\leqslant (-\Delta)\big(|\nabla f|^2(v) + \alpha\lambda f^2(v)\big) \\ &\leqslant (2\lambda - \alpha\lambda - 2\kappa) \cdot |\nabla f|^2(v) + \left(2\alpha - \frac{4}{m}\right)\lambda^2 f^2(v). \end{aligned}$$

This implies

$$|\nabla f|^2(v) \leqslant \frac{2\alpha - \frac{4}{m}}{(\alpha - 2)\lambda + 2\kappa} \cdot \lambda^2 \cdot f^2(v)$$

for $\alpha > 2 - \frac{2\kappa}{\lambda}$.

Therefore for every $x \in V$, we have

$$\begin{aligned} \big|\nabla f^2\big|(x) + \alpha\lambda f^2(x) &\leqslant |\nabla f|^2(v) + \alpha\lambda f^2(v) \\ &\leqslant \frac{2\alpha - \frac{4}{m}}{(\alpha - 2)\lambda + 2\kappa} \cdot \lambda^2 f^2(v) + \alpha\lambda f^2(v) \\ &\leqslant \frac{(\alpha^2 - \frac{4}{m})\lambda + 2\kappa\alpha}{(\alpha - 2)\lambda + 2\kappa} \cdot \lambda \cdot \max_{z \in V} f^2(z) \end{aligned}$$

as desired. □

From Lemma 2.3, we can choose $\alpha = 4 - \frac{2\kappa}{\lambda} \geqslant 0$. By substituting into the statement of Theorem 3.2, we have

Theorem 3.3. *Suppose a finite connected graph G satisfies the $CD(m, \kappa)$ and $f \in V^R$ is a harmonic eigenfunction of Laplacian Δ with nontrivial eigenvalue λ. Then the following inequality holds for all $x \in V$*

$$|\nabla f|^2(x) \leqslant \left(\left(8 - \frac{2}{m}\right)\lambda - 4\kappa\right) \cdot \max_{z \in V} f^2(z).$$

If G is a non-negative Ricci curvature graph, i.e. $\kappa = 0$. Then we have the following result:

Corollary 3.4. *Suppose a finite connected graph G satisfies $CD(m, 0)$ and $f \in V^R$ is a harmonic eigenfunction of Laplacian Δ with nontrivial eigenvalue λ. Then the following Harnack inequality holds for all $x \in V$*

$$|\nabla f|^2(x) \leqslant \left(8 - \frac{2}{m}\right) \cdot \lambda \cdot \max_{z \in V} f^2(z).$$

We can use the Harnack inequality in Theorem 3.3 to derive the following eigenvalue estimate.

Theorem 3.5. *Suppose a finite connected graph G satisfies $CD(m, \kappa)$ and λ is a non-zero eigenvalue of Laplace operator Δ on G. Then*

$$\lambda \geqslant \frac{1 + 4\kappa d D^2}{d \cdot (8 - \frac{2}{m}) \cdot D^2}$$

where d is the maximum degree and D denotes the diameter of G.

Proof. Let f be the eigenfunction of Laplacian Δ with eigenvalue $\lambda \neq 0$. That is, for all $x \in V$,

$$(-\Delta) f(x) = \lambda f(x).$$

Then

$$\begin{aligned}\sum_{x\in V} d_x f(x) &= \frac{1}{\lambda}\sum_{x\in V} d_x(-\Delta) f(x)\\ &= \frac{1}{\lambda}\sum_{x\in V}\sum_{y\sim x} w_{xy}\big[f(x)-f(y)\big]\\ &= 0.\end{aligned}$$

We can assume that

$$\sup_{z\in V} f(z) = 1 > \inf_{z\in V} f(z) = \beta,$$

where $\beta < 0$.

Choose $x_0, x_t \in V$ such that $f(x_0) = \sup_{z\in V} f(z) = 1$, $f(x_t) = \inf_{z\in V} f(z) = \beta < 0$ and let $x_0, x_1, \dots, x_t$ be the shortest path connecting x_0 and x_t, where $x_i \sim x_{i+1}$. Then $n \leqslant D$ where D is the diameter of G. From Corollary 3.4 we have

$$\big[f(x_{i-1}) - f(x_i)\big]^2 + \big[f(x_i) - f(x_{i+1})\big]^2 \leqslant d\cdot|\nabla f|^2(x_i) \leqslant d\cdot\left(\left(8-\frac{2}{m}\right)\cdot\lambda - 4\kappa\right).$$

Therefore

$$\sum_{i=0}^{t-1}\big[f(x_i) - f(x_{i+i})\big]^2 \leqslant dD\cdot\left(\left(8-\frac{2}{m}\right)\lambda - 4\kappa\right).$$

On the other hand, by using the Cauchy–Schwarz inequality we have

$$\begin{aligned}\sum_{i=0}^{t-1}\big[f(x_i) - f(x_{i+1})\big]^2 &\geqslant \frac{1}{D}\big(f(x_t) - f(x_1)\big)^2\\ &\geqslant \frac{1}{D}.\end{aligned}$$

Together we have

$$\lambda \geqslant \frac{1+4\kappa dD^2}{d\cdot(8-\frac{2}{m})\cdot D^2}.$$

This completes the proof of Theorem 3.5. □

Remark 3.6. We note that Theorem 3.5 gives an eigenvalue lower bound for graphs with Ricci curvature κ satisfying

$$\kappa > -\frac{1}{4dD^2}.$$

As an immediately consequence, we have the following:

Corollary 3.7. *Suppose a finite connected graph G satisfies $CD(m,0)$ and λ is a non-zero eigenvalue of Laplace operator Δ on G. Then*

$$\lambda \geqslant \frac{1}{d \cdot (8 - \frac{2}{m}) \cdot D^2}$$

where d is the maximum degree and D denotes the diameter of G.

Chung and Yau (see [3]) proved that $\lambda \geqslant \frac{1}{d \cdot 8 \cdot D^2}$ for Ricci flat graphs. Since Ricci flat graphs satisfies $CD(\infty, 0)$, our results extend and strengthen the results in [3].

References

[1] D. Bakry, M. Emery, Diffusions hypercontractives, in: Sem. de Prob., XIX, in: Lecture Notes in Math., vol. 1123, Springer-Verlag, 1985, pp. 177–206.
[2] Fan R.K. Chung, Spectral Graph Theory, CBMS Reg. Conf. Ser. Math., vol. 92, American Mathematical Society, 1997.
[3] Fan R.K. Chung, S.T. Yau, A Harnack inequality for homogeneous graphs and subgraphs, Comm. Anal. Geom. 2 (1994) 628–639.
[4] Fan R.K. Chung, S.T. Yau, Logarithmic Harnack inequalities, Math. Res. Lett. 3 (1996) 793–812.
[5] Fan Chung, Yong Lin, Yuan Liu, Curvature aspects of graphs, preprint.
[6] Y. Lin, S.T. Yau, Ricci curvature and eigenvalue estimation on locally finite graphs, Math. Res. Lett. 17 (2) (2010) 345–358.
[7] Y. Ollivier, Ricci curvature of Markov chains on metric spaces, J. Funct. Anal. 256 (3) (2009) 810–864.