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CURVATURE ASPECTS OF GRAPHS

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ABSTRACT. We prove the Lichnerowicz type lower bound estimates for finite connected graphs with positive Ricci curvature lower bound.

1. INTRODUCTION

The Ricci curvature on Riemannian manifolds plays a very important rule in geometric analysis. For a diffusion operator on measure metric space, the curvature dimension conditions are defined via the Γ operator and the iterated operator denoted by Γ_2, which was initiated in Bakry and Émery [1]. The curvature dimension condition on graphs, in the nondiffusion case, was introduced by Lin and Yau [7] and serves as a combination of a lower bound of Ricci curvature and an upper bound of the dimension; see Section 2 below. For bounded Laplacians on graphs, Bauer et al. [3] introduced the involved curvature dimension condition, the so-called $CDE(K,n)$ condition and $CDE'(K,n)$ condition, to imply the Li-Yau inequality on graphs.

In this paper, we discuss different aspects of *Ricci curvature* on finite weighted graphs, either in the sense of D. Bakry and M. Emery [1] and [7] or in the sense of Y. Ollivier [9]; see also [8]. We give estimates of nonzero eigenvalues of the associated Laplacian via the positive curvature values, together with some examples to show that these bounds can be sharp. Bauer and Horn also obtained a similar estimate under the $CDE(K,n)$ condition [2] by using the maximum principle argument.

The basic setting is as follows. Denote by G a finite nonoriented connected graph composed of a vertex set V with an edge set E, and $\rho(x,y)$ the distance function which equals the minimal number of edges in any path connecting x and y in V. Write $x \sim y$ when x is adjacent to y, in particular, a loop $x \sim x$ is possible. In this paper, we use the notation $\sum\limits_{y\sim x}$ to mean summing over all edges adjacent to x. We also use (x,y) to denote an edge in E connecting vertices x and y, and $\sum\limits_{(x,y)\in E}$ to mean summing over all edges in E.

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Let's equip G with a *weight* $\mu_\bullet$ which is a symmetric function on $V \times V$ such that $\mu_{xy} > 0$ if $x \sim y$ and $\mu_{xy} = 0$ otherwise. Then $(G, \mu_\bullet)$ becomes a weighted graph. $\mu_\bullet$ is called *standard* if $\mu_{xy} = 1$ for any $x \sim y$ and $\mu_{xx} = 0$. Denote by $d_x = \sum_{y \sim x} \mu_{xy}$ the *degree* at x, and $\mathrm{Vol}G = \sum_{x \in V} d_x$ the *volume* of G. Define the *transition matrix* (or *Markov operator*) M by

$$M(x,y) := \frac{\mu_{xy}}{d_x},$$

which satisfies that

$$\sum_{y \sim x} M(x,y) = 1, \qquad M(x,y)d_x = M(y,x)d_y.$$

Define V^R to be the space of real valued functions on V, and Δ the Laplace operator acting on V^R by

$$\Delta := M - \mathrm{Id},$$

which means for any $f \in V^R$ that

$$-\Delta f(x) = \frac{1}{d_x} \sum_{y \sim x} \mu_{xy}[f(x) - f(y)].$$

Suppose a function $f\colon V \to R$ satisfies

$$(-\Delta) f(x) = \lambda f(x);$$

then f is called an eigenfunction of the Laplace operator on G with eigenvalue λ. Note that 0 is a trivial eigenvalue of $-\Delta$ associated to the constant eigenfunction.

Let $\lambda > 0$ be a nontrivial eigenvalue of $-\Delta$. In Section 2, we define the Ricci curvature in the sense of Bakry and Emery, and give an estimate $\lambda \geqslant \frac{mK}{m-1}$ through the *curvature-dimension type inequality* $CD(m, K)$ for some $m > 1$ and $K > 0$. There is a similar bound for an eigenvalue in a compact Riemannian manifold with a positive Ricci curvature lower bound proved by Lichnerowicz. In Section 3, we introduce the Ricci curvature from Ollivier, and give another estimate $\lambda \in [\kappa, 2\kappa]$ via the curvature's lower bound κ. We also prove that any finite weighted connected graph can be equipped with a new distance function and transition matrix such that it has a positive Ricci curvature.

2. The eigenvalue bound in terms of a positive Ricci curvature in the sense of Bakry and Émery

According to Bakry and Émery [1], define a bilinear operator $\Gamma : V^R \times V^R \to V^R$ by

$$\Gamma(f,g)(x) := \frac{1}{2}\{\Delta(f(x)g(x)) - f(x)\Delta g(x) - g(x)\Delta f(x)\},$$

and then the Ricci curvature operator on graphs Γ_2 by iterating Γ as

$$\Gamma_2(f,g)(x) := \frac{1}{2}\{\Delta\Gamma(f,g)(x) - \Gamma(f, \Delta g)(x) - \Gamma(g, \Delta f)(x)\}.$$

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More explicitly, we have

$$\Gamma(f,f)(x) = \frac{1}{2}\frac{1}{d_x}\sum_{y\sim x}\mu_{xy}|f(x)-f(y)|^2.$$

From the proof of Theorem 1.2 in [7] we have the following formula for the Ricci curvature operator on graphs:

$$\begin{aligned}\Gamma_2(f,f)(x) &= \frac{1}{4}\frac{1}{d_x}\sum_{y\sim x}\frac{\mu_{xy}}{d_y}\sum_{z\sim y}\mu_{yz}[f(x)-2f(y)+f(z)]^2\\ &\quad -\frac{1}{2}\frac{1}{d_x}\sum_{y\sim x}\mu_{xy}[f(x)-f(y)]^2+\frac{1}{2}[\frac{1}{d_x}\sum_{y\sim x}\mu_{xy}(f(x)-f(y))]^2.\end{aligned}$$

We say that the Laplacian Δ satisfies the *curvature-dimension type inequality* $CD(m,K)$ for some $m>1$ if for any $f\in V^R$ and for any $x\in V$,

$$\Gamma_2(f,f)(x) \geqslant \frac{1}{m}(\Delta f)(x)^2 + K\Gamma(f,f)(x). \tag{2.1}$$

Here m is called the *dimension* of Δ, and K the lower bound of the Ricci curvature of Δ. In particular, if $\Gamma_2(x)\geqslant K\Gamma(x)$, we say that Δ satisfies $CD(\infty,K)$. Correspondingly, for the Laplace-Beltrami operator Δ on a complete m-dimensional Riemannion manifold, it fulfills $CD(m,K)$ iff the Ricci curvature of the Riemanian manifold is bounded below by a constant K.

We proved in [7] that the Ricci flat graphs defined by F. Chung and Yau in [4] and [5] have the nonnegative Ricci curvature in the sense of Bakry-Emery, and also that any locally finite connected graph satisfies either $CD(2,\frac{1}{d_*}-1)$ if d_* is finite or $CD(2,-1)$ if d_* is infinite, where

$$d_* := \sup_{x\in V}\sup_{y\sim x}\frac{d_x}{\mu_{xy}}.$$

Moreover, we have

Theorem 2.1. *Suppose that Δ satisfies a curvature-dimension type inequality $CD(m,K)$ with finite $m>1$ and $K>0$. Then any nonzero eigenvalue λ of $-\Delta$ has a lower bound $\frac{mK}{m-1}$. In particular, if $m=\infty$, any nonzero eigenvalue λ of $-\Delta$ has a lower bound K.*

Proof. Suppose f is an eigenfunction satisfying

$$-\Delta f(x) = \lambda f(x).$$

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We consider

$$\begin{aligned}
\sum_x d_x \Gamma_2(f,f)(x) &= \frac{1}{4}\sum_x d_x \Delta |\nabla f|^2(x) + \lambda \sum_x d_x \Gamma(f,f)(x) \\
&= \lambda \sum_x d_x \Gamma(f,f)(x) \\
&= \frac{\lambda}{2}\sum_x d_x |\nabla f|^2(x) \\
&= \frac{\lambda}{2}\sum_x \sum_{y\sim x} (f(x)-f(y))^2 \\
&= \lambda \sum_{x\sim y} (f(x)-f(y))^2 \\
&= \lambda \sum_x f(x)(-\Delta f(x)) d_x \\
&= \lambda^2 \sum_x f(x)^2 d_x .
\end{aligned}$$

In the first item, we use the following fact:

$$\begin{aligned}
\sum_x d_x \Delta f(x) &= \sum_x \sum_{y\sim x} \mu_{xy}[f(x)-f(y)] \\
&= \sum_x \sum_{y\sim x} \mu_{xy} f(x) - \sum_x \sum_{y\sim x} \mu_{xy} f(y) \\
&= 2[\sum_{(x,y)\in E} \mu_{xy} f(x) - \sum_{(x,y)\in E} \mu_{xy} f(y)] \\
&= 2[\sum_{(x,y)\in E} \mu_{xy} f(x) - \sum_{(y,x)\in E} \mu_{yx} f(x)] \\
&= 0.
\end{aligned}$$

Combining this with (2.1), we have

$$\begin{aligned}
\lambda^2 \sum_x f(x)^2 d_x &\geqslant \frac{1}{m}\sum_x d_x \lambda^2 f(x)^2 + K \sum_x d_x \Gamma(f,f)(x) \\
&= \frac{\lambda^2}{m}\sum_x f(x)^2 d_x + K\sum_{x\sim y}(f(x)-f(y))^2 \\
&= (\frac{\lambda^2}{m} + K\lambda)\sum_x f(x)^2 d_x .
\end{aligned}$$

Thus we have

$$\lambda \geqslant \frac{mK}{m-1}.$$

We can also see from the last inequality that the eigenvalue 0 does not work in the proof of the theorem. □

We give an alternative proof of Theorem 2.1 using a maximum principle argument.

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Proof. Suppose f is an eigenfunction satisfying

$$\Delta f(x) = -\lambda f(x)$$

for all $x \in V$. We define the function

$$Q(x) = \Gamma(f,f)(x) + \frac{\lambda}{m} f^2(x).$$

At the maximum point x^* of Q we have $\Delta Q(x^*) \le 0$. Thus we have

$$\begin{aligned} 0 &\ge \Delta Q(x^*) \\ &= 2\Gamma_2(f,f)(x^*) + 2\Gamma(f,\Delta f)(x^*) + \frac{\lambda}{m}(2f\Delta f(x^*) + 2\Gamma(f,f)(x^*)) \\ &\ge 2K\Gamma(f,f)(x^*) - 2\lambda\Gamma(f,f)(x^*) + 2\frac{\lambda}{m}\Gamma(f,f)(x^*). \end{aligned}$$

Rearranging yields

$$\lambda \ge \frac{m}{m-1} K.$$

□

We calculate the curvature-dimension type inequalities for some graphs such as a path, cube or square. One can find details in Appendix A.

Example 1. Let $G = \{a, b\}$ be a path. Then it has a nonzero eigenvalue $\lambda = 2$ and satisfies $CD(2, 1)$, which means $m = 2$, $K = 1$ and $\frac{mK}{m-1} = 2$. Here the estimate in Theorem 2.1 is sharp for m finite.

Example 2. Let $G = \{a, b, c\}$ be a path. Then it has two nonzero eigenvalues $\lambda = 1$ or 2 and satisfies $CD(4, \frac{1}{2})$, which means $m = 4$, $K = \frac{1}{2}$ and $\frac{mK}{m-1} = \frac{2}{3}$.

Example 3. Let G_1 and G_2 be two graphs as in Figure 1 and Figure 2 together with standard weights. Then G_1 has a nonzero eigenvalue $\lambda = \frac{2}{3}$ and satisfies $CD(\infty, \frac{2}{3})$, so the estimate in Theorem 2.1 is sharp for $m = \infty$. G_2 satisfies $CD(\infty, \frac{1}{6})$.

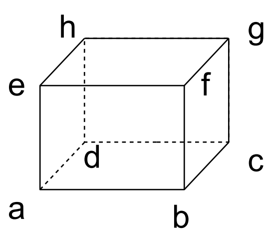

figure1

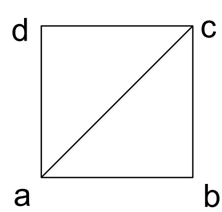

figure2

3. The eigenvalue bound in terms of positive Ricci curvature in the sense of Ricci-Wasserstein

The Ricci curvature or *Ricci-Wasserstein curvature* for Markov chains was introduced recently by Y. Ollivier [9]. In general, let (X, d) be a separable and complete metric space, $\mathrm{Lip}_1(d)$ the set of 1-Lipschitz functions, $\mathcal{P}(X)$ the set of all Borel probability measures, and $\mathcal{C}(\mu, \nu)$ the set of *couplings* of any μ and $\nu \in \mathcal{P}(X)$. Here, a coupling in $\mathcal{C}(\mu, \nu)$ is a probability measure on $X \times X$ associated with two

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marginals μ and ν respectively. Let $m = \{m_x\}_{x \in X}$ be a family in $\mathcal{P}(X)$. Technically, suppose m_x depends measurably on x and has a finite first moment, i.e. $\int d(o,y)dm_x(y) < \infty$ for some $o \in X$. Then m is called a *random walk* on (X,d).

Define the L^1 *transportation distance* (or *Wasserstein distance*) between m_x and m_y as

$$\mathcal{T}_1(m_x, m_y) := \inf_{\pi \in \mathcal{C}(m_x, m_y)} \int_{X \times X} d(\xi, \eta)\, d\pi(\xi, \eta).$$

$(\mathcal{P}(X), \mathcal{T}_1)$ becomes a complete metric space. Equivalently, via the *Kantorovich duality*,

$$\mathcal{T}_1(m_x, m_y) = \sup_{f \in \mathrm{Lip}_1(d)} \int f\, dm_x - \int f\, dm_y.$$

One can find more details in C. Villani [10].

According to [9], define the Ricci curvature of (X,d,m) as

$$\kappa(x,y) := 1 - \frac{\mathcal{T}_1(m_x, m_y)}{d(x,y)}.$$

When (X,d) is a finite weighted connected graph $(G, \rho, \mu_\bullet)$, we can define the transition family $m_x(y) := \mu_{xy}/d_x$. In [7], we proved that the Ricci curvature in the sense of Ollivier is bounded below; see also [8] for some modification of Ollivier's Ricci curvature. In this paper, we can estimate the eigenvalues associated to $-\Delta$ by the lower bound of $\kappa(x,y)$; see also Proposition 30 in [9].

Theorem 3.1. *Suppose that the Ricci curvature of a finite weighted connected graph $(G, \rho, \mu_\bullet)$ is at least κ. Then any nonzero eigenvalue λ of $-\Delta$ falls in $[\kappa, 2-\kappa]$.*

Proof. Let $f \in \mathrm{Lip}_1(\rho)$ be an eigenfunction satisfying $-\Delta f = \lambda f$. We have

$$f(x) - \int f\, dm_x = \frac{1}{d_x} \sum_{y \sim x} \mu_{xy}(f(x) - f(y)) = -\Delta f(x) = \lambda f(x),$$

which implies by the definition of Ricci curvature $\kappa(x,y)$ for any $x \sim y$ that

$$1 - \kappa \geqslant 1 - \kappa(x,y) \geqslant \left| \int f\, dm_x - \int f\, dm_y \right| \Big/ \rho(x,y) = |(1-\lambda)(f(x) - f(y))|.$$

Since there exist x and y such that $f(x) - f(y) = 1$, we obtain $\kappa \leqslant \lambda \leqslant 2 - \kappa$. $\square$

Now we give an instance to show that two interval end-points can be attained.

Example 4. Let $G = \{a, b, c\}$ be a complete graph equipped with the usual distance ρ and two transition matrices, respectively,

$$M_1 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Then, we calculate that (G, ρ, M_1) has a Ricci curvature at least $\frac{1}{2}$ and double eigenvalues $\frac{3}{2}$ and that (G, ρ, M_2) has a Ricci curvature at least $\frac{3}{4}$ and double eigenvalues $\frac{3}{4}$.

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We can apply Theorem 3.1 to general complete graphs.

Corollary 3.2. *Let G be a complete graph with n vertices satisfying that $n \geqslant 2$ and $\mu_{xy} = \frac{1}{n-1}$ for any $x \neq y$. Then the associated operator $-\Delta$ has a unique nonzero eigenvalue $\lambda = \frac{n}{n-1}$.*

Proof. Let $p \in [0,1)$. We define a family of "lazy" transition matrices by

$$M_p := \begin{pmatrix} p & \frac{1-p}{n-1} & \cdots & \frac{1-p}{n-1} & \frac{1-p}{n-1} \\ \frac{1-p}{n-1} & p & \cdots & \frac{1-p}{n-1} & \frac{1-p}{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{1-p}{n-1} & \frac{1-p}{n-1} & \cdots & p & \frac{1-p}{n-1} \\ \frac{1-p}{n-1} & \frac{1-p}{n-1} & \cdots & \frac{1-p}{n-1} & p \end{pmatrix},$$

which corresponds to the laplacian $\Delta_p = M_p - \mathrm{Id}$. Clearly, $\Delta_p = (1-p)\Delta$, in particular, $\Delta_0 = \Delta$. So $-\Delta_p$ has a nonzero eigenvalue $(1-p)\lambda$.

Define $m_{p,x}(y) = M_p(x,y)$. Then

$$\mathcal{T}_1(m_{p,x}, m_{p,y}) = \sup_{f \in \mathrm{Lip}_1(\rho)} \left| pf(x) + \frac{1-p}{n-1} f(y) - pf(y) - \frac{1-p}{n-1} f(x) \right| \leqslant \frac{|np-1|}{n-1},$$

which means (G, ρ, m_p) has a Ricci curvature at least $\kappa = 1 - \frac{|np-1|}{n-1}$. By Theorem 3.1, we have

$$1 - \frac{|np-1|}{n-1} \leqslant (1-p)\lambda \leqslant 1 + \frac{|np-1|}{n-1}.$$

Taking $p = n^{-1}$, we obtain $\lambda = \frac{n}{n-1}$. □

Remark 3.3. When $p = n^{-1}$, the Ricci curvature $\kappa(x,y)$ attains the maximum 1 everwhere.

In fact, every finite weighted connected graph G always has a positive Ricci curvature with some kind of distance function and random walk. Let μ be the normalized volume measure and $\mathcal{E}$ the associated quadratic form, that is,

$$\mu(x) := \frac{d_x}{\mathrm{Vol}G}, \quad \mathcal{E}(f,f) := \frac{1}{2\mathrm{Vol}G} \sum_{x \sim y} \mu_{xy} |f(x) - f(y)|^2 = -\int f(x) \cdot \Delta f(x) d\mu(x).$$

Write $\mathcal{E}[f] = \mathcal{E}(f,f)$. Define the *effective resistance*

$$R(x,y) := \sup_{\mathcal{E}[f] \neq 0} \frac{|f(x) - f(y)|^2}{\mathcal{E}[f]}.$$

Note that $\sqrt{R(x,y)}$ is a metric. Define the heat semigroup $P_t = e^{t\Delta}$ for any $t \geqslant 0$, and a new random walk $m^* = \{m_x^*\}_{x \in V}$ (depending on α) by

$$m_x^*(y) := \int_0^\infty \alpha e^{-\alpha t} P_t(x,y) dt.$$

Alternatively, recall the resolvent family $\{G_\alpha\}_{\alpha > 0}$ in [6]; we denote $\int f \, dm_x^* =: \alpha G_\alpha f(x)$.

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Theorem 3.4. *$(G, \sqrt{R}, m^*)$ yields a Ricci curvature at least $\kappa > 0$ provided that for some $\alpha > 0$ and $v \in V$ there holds $(2\alpha \int R(v,x)d\mu(x))^{1/2} \leqslant 1 - \kappa$.*

Proof. For any f satisfying $|f(x) - f(y)| \leqslant \sqrt{R(x,y)}$,

$$\frac{\left|\int f\, dm_x^* - \int f\, dm_y^*\right|}{\sqrt{R(x,y)}} = \frac{|\alpha G_\alpha f(x) - \alpha G_\alpha f(y)|}{\sqrt{R(x,y)}} \leqslant \sqrt{\mathcal{E}[\alpha G_\alpha f]}.$$

Without loss of generality, let $f(v) = 0$ for some v. Since $\mathcal{E}[\alpha G_\alpha f] = \alpha(f - \alpha G_\alpha f, \alpha G_\alpha f)$ according to [6], we estimate that

$$|f(x) - \alpha G_\alpha f(x)| \leqslant \int \sqrt{R(x,y)} dm_x^*(y), \quad |\alpha G_\alpha f(x)| \leqslant \int \sqrt{R(v,y)} dm_x^*(y).$$

Denote $g(x) = \int \sqrt{R(v,y)} dm_x^*(y)$; we have by using the Hölder inequality that

$$\mathcal{E}[\alpha G_\alpha f] \leqslant \alpha \int \left(\sqrt{R(v,x)} g(x) + g^2(x)\right) d\mu(x) \leqslant 2\alpha \int R(v,x) d\mu(x).$$

Recall the definition of Ricci curvature; it follows from above estimates. □

Corollary 3.5. *With the above conditions, any nonzero eigenvalue λ of $-\Delta$ has a lower bound $\frac{\kappa\alpha}{1-\kappa}$.*

Proof. Let $f \in \mathrm{Lip}_1(\sqrt{R})$ be an eigenfunction satisfying $-\Delta f = \lambda f$, thus $\alpha G_\alpha f = \frac{\alpha}{\alpha+\lambda} f$. By the same argument as Theorem 3.1, we have $1 - \kappa \geqslant \frac{\alpha}{\alpha+\lambda}$. □

Remark 3.6. It is not hard to obtain another lower bound $(\int R(v,x)d\mu(x))^{-1}$ better than $\frac{\kappa\alpha}{1-\kappa}$.

Appendix A. Calculations of examples in Section 2

Recall the formulas of Γ and Γ_2.

1. For Example 1, consider path P_1 with vertices a and b:

$$\begin{aligned}
\Gamma_2(f,f)(a) &= \frac{1}{4}|f(a) - 2f(b) + f(a)|^2 - \frac{1}{2}|f(a) - f(b)|^2 + \frac{1}{2}|f(a) - f(b)|^2 \\
&= |f(a) - f(b)|^2 \\
&= \frac{1}{2}|f(a) - f(b)|^2 + \frac{1}{2}|f(a) - f(b)|^2 \\
&= \frac{1}{2}(\Delta f(a))^2 + \Gamma(f,f)(a).
\end{aligned}$$

So $m = 2$, $K = 1$.

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Example 2 can be proved similarly.

2. Consider the cube in Figure 1:

$$
\begin{aligned}
&\Gamma_2(\phi,\phi)(a)\\
=\ & \frac{1}{4}\cdot\frac{1}{3}\sum_{y\sim a}\frac{1}{3}\sum_{z\sim y}|\phi(a)-2\phi(y)+\phi(z)|^2-\frac{1}{2}\cdot\frac{1}{3}\sum_{y\sim a}|\phi(a)-\phi(y)|^2\\
&+\frac{1}{2}\left(\frac{1}{3}\sum_{y\sim a}(\phi(a)-\phi(y))\right)^2\\
=\ & \frac{1}{36}\left(\sum_{z\sim b}|\phi(a)-2\phi(b)+\phi(z)|^2+\sum_{z\sim d}|\phi(a)-2\phi(d)+\phi(z)|^2\right.\\
&\left.\qquad+\sum_{z\sim e}|\phi(a)-2\phi(e)+\phi(z)|^2\right)\\
&-\frac{1}{6}\sum_{y\sim a}|\phi(a)-\phi(y)|^2+\frac{1}{18}\left(\sum_{y\sim a}(\phi(a)-\phi(y))\right)^2\\
=\ & \frac{1}{36}\left(|2\phi(a)-2\phi(b)|^2+|\phi(a)-2\phi(b)+\phi(c)|^2+|\phi(a)-2\phi(b)+\phi(f)|^2\right.\\
&+|2\phi(a)-2\phi(d)|^2+|\phi(a)-2\phi(d)+\phi(c)|^2+|\phi(a)-2\phi(d)+\phi(h)|^2\\
&\left.+|2\phi(a)-2\phi(e)|^2+|\phi(a)-2\phi(e)+\phi(f)|^2+|\phi(a)-2\phi(e)+\phi(h)|^2\right)\\
&-\frac{1}{6}\sum_{y\sim a}|\phi(a)-\phi(y)|^2+\frac{1}{18}|3\phi(a)-\phi(b)-\phi(d)-\phi(e)|^2\\
\geqslant\ & \frac{1}{36}\left(2|\phi(b)-\phi(d)|^2+2|\phi(b)-\phi(e)|^2+2|\phi(d)-\phi(e)|^2\right)\\
&+\left(\frac{4}{36}-\frac{1}{6}\right)\sum_{y\sim a}|\phi(a)-\phi(y)|^2+\frac{1}{18}|3\phi(a)-\phi(b)-\phi(d)-\phi(e)|^2\\
=\ & \frac{1}{9}\sum_{y\sim a}|\phi(a)-\phi(y)|^2\ =\ \frac{2}{3}\Gamma(\phi,\phi)(a).
\end{aligned}
$$

So $m=\infty$, $K=\frac{2}{3}$.

The square in Figure 2 can be proved similarly.

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