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Hypergeometric SLE: Conformal Markov Characterization and Applications

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Abstract: This article pertains to the classification of pairs of simple random curves with conformal Markov property and symmetry. We give the complete classification of such curves: conformal Markov property and symmetry single out a two-parameter family of random curves—Hypergeometric SLE—denoted by $\mathrm{hSLE}_\kappa(\nu)$ for $\kappa \in (0, 4]$ and $\nu < \kappa - 6$. The proof relies crucially on Dubédat's commutation relation (Commun Pure Appl Math 60(12):1792–1847, 2007) and a uniqueness result proved in Miller and Sheffield (Ann Probab 44(3):1647–1722, 2016). The classification indicates that hypergeometric SLE is the only possible scaling limit of the interfaces in critical lattice models (conjectured or proved to be conformally invariant) in topological rectangles with alternating boundary conditions. We also prove various properties of $\mathrm{hSLE}_\kappa(\nu)$ with $\kappa \in (0, 8)$: continuity, reversibility, target-independence, and conditional law characterization. As by-products, we give two applications of these properties. The first one is about the critical Ising interfaces. We prove the convergence of the Ising interface in rectangles with alternating boundary conditions. This result was first proved by Izyurov (Commun Math Phys 337(1):225–252, 2015), and our proof is different. Our method is based on the properties of hSLE and is easy to generalize to more complicated boundary conditions and to other models. The second application is the existence of the so-called pure partition functions of multiple SLEs. Such existence was proved for $\kappa \in (0, 8)\backslash\mathbb{Q}$ in Kytölä and Peltola (Commun Math Phys 346(1):237–292, 2016), and it was later proved for $\kappa \in (0, 4]$ in Peltola and Wu (Commun. Math. Phys. 366(2):469–536, 2019). We give a new proof of the existence for $\kappa \in (0, 6]$ using the properties of hSLE.

1. Introduction

Conformal invariance and critical phenomena in two-dimensional lattice models play a central role in mathematical physics in the last few decades. We take Ising model as

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an example (see details in Sect. 5). Suppose Ω is a simply connected domain and x, y are distinct boundary points. When one considers the critical Ising model in $\Omega \cap \mathbb{Z}^2$ with Dobrushin boundary conditions: $\oplus$ along the boundary arc (xy) and $\ominus$ along the boundary arc (yx), an interface from x to y appears naturally which separates $\oplus$-spin from $\ominus$-spin. The scaling limit of the interface is believed to satisfy conformal invariance and domain Markov property. We call the combination of the two as *conformal Markov property*. Thus, to understand the scaling limit of interfaces in critical lattice models, one needs to understand random curves with conformal Markov property.

In [Sch00], O. Schramm introduced SLE which is a random growth process in simply connected domain starting from one boundary point to another boundary point. This is a one-parameter family of random curves, denoted by SLE_κ with $\kappa \geq 0$. This family is the only one with conformal Markov property, and is conjectured to be the scaling limits of interfaces in critical models. Since its introduction, this conjecture has been rigorously proved for several models: percolation [Smi01,CN07], loop-erased random walk and uniform spanning tree [LSW04], level lines of the discrete Gaussian free field [SS09,SS13], and the critical Ising and FK-Ising models [CS12,CDCH+14].

SLE process corresponds to the scaling limit of interface in critical model with Dobrushin boundary conditions. It is natural to consider critical model with more complicated boundary conditions. In this article, we focus on the alternating boundary conditions in topological rectangles (*quads* for short). We take Ising model as an example again. Suppose Ω is a simply connected domain and x^R, y^R, y^L, x^L are four distinct boundary points in counterclockwise order. Consider critical Ising model in $\Omega \cap \mathbb{Z}^2$ with alternating boundary conditions: $\oplus$ along the boundary arcs $(x^R y^R)$ and $(y^L x^L)$, and $\ominus$ along the arcs $(x^L x^R)$ and $(y^R y^L)$. With such boundary conditions, a pair of interfaces appears naturally. This pair of interfaces connects between the four points x^R, y^R, y^L, x^L and the two interfaces cannot cross, see Fig. 1. The scaling limit of the pair of interfaces, if exists, should satisfy conformal Markov property (see Definition 1.2). This article concerns probability measures on pairs of simple curves with conformal Markov property, and they should describe scaling limits of pairs of interfaces in critical lattice model with alternating boundary conditions in quads.

In the case of Dobrushin boundary conditions, there are two boundary points, and conformal Markov property determines the one-parameter family of random curves

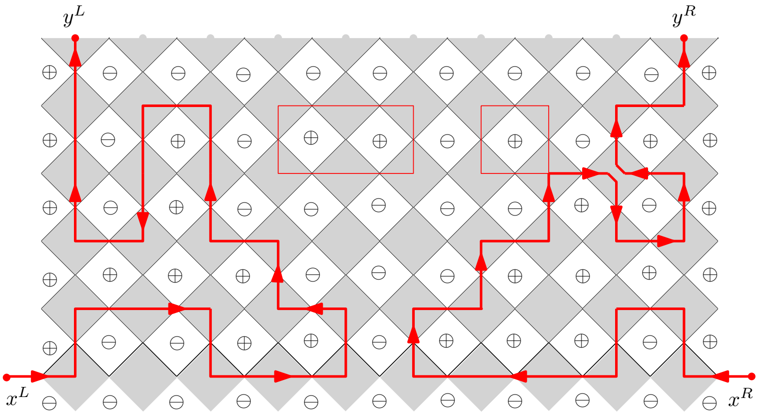

Fig. 1. The Ising interface with alternating boundary conditions

SLE_κ. However, in the case of alternating boundary conditions in quads, there are four boundary points, and conformal Markov property is not sufficient to naturally single out random processes. We go back to the critical Ising model. As described before, there is a pair of interfaces when the boundary conditions are alternating. The scaling limit of such pair should satisfy conformal Markov property; at the same time, it is clear that the pair of curves also satisfy a particular symmetry (see Definition 1.3). To understand the scaling limit of such pair, it is then natural to require the symmetry as well as conformal Markov property.

It turns out that the combination of conformal Markov property and symmetry determines a two-parameter family of pairs of curves. These curves are *hypergeometric SLE*s.

1.1. Hypergeometric SLE. Hypergeometric SLE is a two-parameter family of random curves in quad. The two parameters are $\kappa \in (0, 8)$ and $\nu \in \mathbb{R}$, and we denote it by $\mathrm{hSLE}_\kappa(\nu)$. We denote it by hSLE_κ when $\nu = 0$. For a quad $(\Omega; x_1, x_2, x_3, x_4)$ where the four boundary points x_1, x_2, x_3, x_4 are in counterclockwise order, $\mathrm{hSLE}_\kappa(\nu)$ is a random process from x_1 to x_4 with two marked points (x_2, x_3). We will give definition of this process in Sect. 3, and the main theorem of Sect. 3 is continuity and reversibility of hypergeometric SLEs.

Theorem 1.1. *Fix $\kappa \in (0, 8)$, $\nu > (-4) \vee (\kappa/2 - 6)$, and $x_1 < x_2 < x_3 < x_4$. Let η be the $\mathrm{hSLE}_\kappa(\nu)$ in $\mathbb{H}$ from x_1 to x_4 with marked points (x_2, x_3). The process η is almost surely generated by a continuous curve for all times. Moreover, the process η enjoys reversibility for $\nu \geq \kappa/2 - 4$: the time reversal of η is the $\mathrm{hSLE}_\kappa(\nu)$ in $\mathbb{H}$ from x_4 to x_1 with marked points (x_3, x_2).*

Here we briefly summarize the relation between hSLE and SLE_κ (or $\mathrm{SLE}_\kappa(\rho)$) process. Fix $x_1 = 0 < x_2 < x_3 < x_4 = \infty$. Suppose η is $\mathrm{hSLE}_\kappa(\nu)$ in $\mathbb{H}$ from 0 to ∞ with marked points (x_2, x_3).

- When $\nu = -2$, the law of η equals SLE_κ.
- When $\kappa = 4$, the law of η equals $\mathrm{SLE}_4(\nu + 2, -\nu - 2)$ with force points (x_2, x_3).
- When $x_3 \to x_4$, the law of η converges weakly to the law of $\mathrm{SLE}_\kappa(\nu + 2)$ with force point x_2. See Lemma 3.7. In particular, the reversibility in Theorem 1.1 implies that the time reversal of $\mathrm{SLE}_\kappa(\nu + 2)$ is $\mathrm{hSLE}_\kappa(\nu)$.
- When $\kappa \in (4, 8)$ and $\nu = \kappa - 6$, the law of η equals the law of SLE_κ conditioned to avoid the interval (x_2, x_3). See Proposition 3.9.

From these relations, we see that $\mathrm{hSLE}_\kappa(\nu)$ is a generalization of $\mathrm{SLE}_\kappa(\rho)$ process. In general, the driving function of hSLE has a drift term which involves a hypergeometric function. When $\nu > (-4) \vee (\kappa/2 - 6)$, the process is almost surely generated by a continuous curve from x_1 to x_4. The process is also defined when $\nu \leq (-4) \vee (\kappa/2 - 6)$. In this case, it is defined up to the swallowing time x_2. When $\kappa = 4$, the hypergeometric term becomes zero, and the process coincides with $\mathrm{SLE}_4(\nu + 2, -\nu - 2)$ process. See more discussion in Sect. 3.4.

1.2. Conformal Markov characterization. We denote by $\mathcal{Q}$ the collection of all quads, and for each quad $q = (\Omega; x^R, y^R, y^L, x^L)$, we denote by $X_0(\Omega; x^R, y^R, y^L, x^L)$ the collection of pairs of disjoint simple curves $(\eta^L; \eta^R)$ such that η^R connects x^R and y^R and η^L connects x^L and y^L. The following definition concerns conformal Markov property for pairs of simple curves. See Fig. 2 for an illustration.

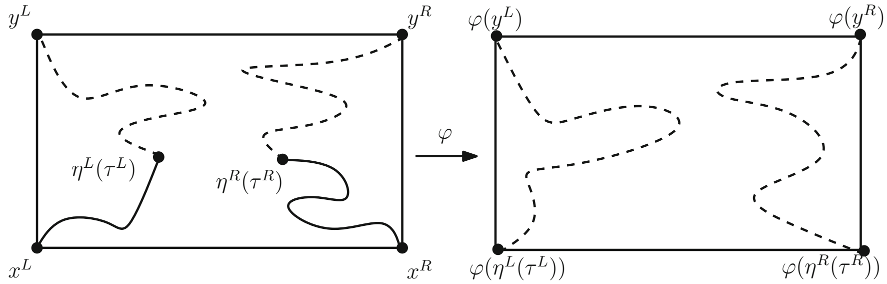

Fig. 2. Suppose the pair $(\eta^L; \eta^R)$ satisfies CMP. For any η^L-stopping time τ^L and any η^R-stopping time τ^R, let φ be a conformal map from $\Omega\backslash(\eta^L[0, \tau^L] \cap \eta^R[0, \tau^R])$ onto a quad $\tilde{q} = (\tilde{\Omega}; \tilde{x}^R, \tilde{y}^R, \tilde{y}^L, \tilde{x}^L)$ such that $\varphi(\eta^R(\tau^R)) = \tilde{x}^R, \varphi(y^R) = \tilde{y}^R, \varphi(y^L) = \tilde{y}^L, \varphi(\eta^L(\tau^L)) = \tilde{x}^L$. Then the conditional law of $(\varphi(\eta^L); \varphi(\eta^R))$ given $\eta^L[0, \tau^L] \cup \eta^R[0, \tau^R]$ is the same as $\mathbb{P}_{\tilde{q}}$

Definition 1.2. Suppose $(\mathbb{P}_q, q \in \mathcal{Q})$ is a family of probability measures on pairs of disjoint simple curves $(\eta^L; \eta^R) \in X_0(\Omega; x^R, y^R, y^L, x^L)$. We say that $(\mathbb{P}_q, q \in \mathcal{Q})$ satisfies conformal Markov property (CMP) if it satisfies the following two properties.

- Conformal invariance. Suppose $q = (\Omega; x^R, y^R, y^L, x^L), \tilde{q} = (\tilde{\Omega}; \tilde{x}^R, \tilde{y}^R, \tilde{y}^L, \tilde{x}^L) \in \mathcal{Q}$, and $\psi : \Omega \to \tilde{\Omega}$ is a conformal map with $\psi(x^R) = \tilde{x}^R, \psi(y^R) = \tilde{y}^R, \psi(y^L) = \tilde{y}^L, \psi(x^L) = \tilde{x}^L$. Then for $(\eta^L; \eta^R) \sim \mathbb{P}_q$, we have $(\psi(\eta^L); \psi(\eta^R)) \sim \mathbb{P}_{\tilde{q}}$.
- Domain Markov property. Suppose $(\eta^L; \eta^R) \sim \mathbb{P}_q$. Then for every η^L-stopping time τ^L and η^R-stopping time τ^R, the conditional law of $(\eta^L|_{t\geq\tau^L}; \eta^R|_{t\geq\tau^R})$ given $\eta^L[0, \tau^L]$ and $\eta^R[0, \tau^R]$ is the same as $\mathbb{P}_{q_{\tau^L,\tau^R}}$ where
$$q_{\tau^L,\tau^R} = (\Omega\backslash(\eta^L[0, \tau^L] \cup \eta^R[0, \tau^R]); \eta^R(\tau^R), y^R, y^L, \eta^L(\tau^L)).$$

In Definition 1.2, we need to specify what happens when $\eta^R[0, \tau^R]$ disconnects y^R from y^L (resp. $\eta^L[0, \tau^L]$ disconnects y^L from y^R). In this case, we think the CMP in Definition 1.2 becomes the CMP for $\eta^L|_{t\geq\tau^L}$ (resp. $\eta^R|_{t\geq\tau^R}$) with three marked points, as in Definition 2.7.

The following definition concerns symmetries. For pairs of simple curves in $X_0(\Omega; x^R, y^R, y^L, x^L)$, there are two symmetries: left-right symmetry and top-bottom symmetry. To distinguish them, we call the former as symmetry, and the latter as reversibility.

Definition 1.3. Suppose $(\mathbb{P}_q, q \in \mathcal{Q})$ is a family of probability measures on pairs of disjoint simple curves $(\eta^L; \eta^R) \in X_0(\Omega; x^R, y^R, y^L, x^L)$.

- We say that $(\mathbb{P}_q, q \in \mathcal{Q})$ satisfies symmetry if for all $q \in \mathcal{Q}$ the following is true. Suppose $(\eta^L; \eta^R) \sim \mathbb{P}_q$, and $\psi : \Omega \to \Omega$ is the anti-conformal map which swaps x^L, y^L and x^R, y^R. Then $(\psi(\eta^R); \psi(\eta^L)) \sim \mathbb{P}_q$.
- We say that $(\mathbb{P}_q, q \in \mathcal{Q})$ satisfies reversibility if for all $q \in \mathcal{Q}$ the following is true. Suppose $(\eta^L; \eta^R) \sim \mathbb{P}_q$, and $\psi : \Omega \to \Omega$ is the anti-conformal map which swaps x^L, x^R and y^L, y^R. Then $(\psi(\eta^R); \psi(\eta^L)) \sim \mathbb{P}_q$.

It turns out that the combination of CMP and the symmetry determines a two-parameter family of pairs of curves—hSLE$_\kappa(\nu)$. In Theorem 1.4, we consider pairs of random curves with CMP and the symmetry, and we also require "Condition C1". This is a technical requirement concerning certain regularity of the curves and its definition is in Sect. 2.3.

Theorem 1.4. *Suppose* $(\mathbb{P}_q, q \in \mathcal{Q})$ *satisfies CMP in Definition 1.2, the symmetry in Definition 1.3 and Condition C1. Then there exist* $\kappa \in (0,4]$ *and* $\nu < \kappa - 6$ *such that, for* $q = (\Omega; x^R, y^R, y^L, x^L) \in \mathcal{Q}$ *and* $(\eta^L; \eta^R) \sim \mathbb{P}_q$*, the marginal law of* η^R *(up to the first hitting time of* $[y^R y^L]$*) equals* $\mathrm{hSLE}_\kappa(\nu)$ *in* Ω *from* x^R *to* x^L *with marked points* (y^R, y^L) *conditioned to hit* $[y^R y^L]$ *(up to the first hitting time of* $[y^R y^L]$*).*

The combination of CMP, the symmetry and the reversibility singles out a one-parameter family of pairs of curves.

Corollary 1.5. *Suppose* $(\mathbb{P}_q, q \in \mathcal{Q})$ *satisfies CMP in Definition 1.2, the symmetry and the reversibility in Definition 1.3, and Condition C1. Then there exists* $\kappa \in (0,4]$ *such that, for any* $q = (\Omega; x^R, y^R, y^L, x^L) \in \mathcal{Q}$ *and* $(\eta^L; \eta^R) \sim \mathbb{P}_q$*, the marginal law of* η^R *equals* hSLE_κ *in* Ω *from* x^R *to* y^R *with marked points* (x^L, y^L)*.*

1.3. Convergence of critical planar Ising interfaces. Let us go back to the critical Ising model. We take it as an example to explain the interest in pairs of random curves and the motivation for the definition of conformal Markov property and symmetries. We find that the combination of conformal Markov property and symmetries singles out hypergeometric SLEs. In this section, we point out that hypergeometric SLE DOES correspond to the scaling limit of critical Ising model with alternating boundary conditions.

Proposition 1.6. *Let* $(\Omega_\delta; x^R_\delta, y^R_\delta, y^L_\delta, x^L_\delta)$ *be a sequence of quads on* $\delta\mathbb{Z}^2$ *converging to a quad* $q = (\Omega; x^R, y^R, y^L, x^L)$ *in the Carathéodory sense as* $\delta \to 0$ *(see Sect. 2.1). Consider the critical Ising model in* Ω_δ *with the following boundary conditions:*

$$\ominus \text{ along } (x^L_\delta x^R_\delta), \quad \oplus \text{ along } (x^R_\delta y^R_\delta) \cup (y^L_\delta x^L_\delta), \quad \xi \in \{\ominus, \text{free}\} \text{ along } (y^R_\delta y^L_\delta).$$

Denote by $\mathcal{C}^\ominus_v(q)$ *the event that the quad is vertically crossed by* $\ominus$ *and by* $\mathcal{C}^\oplus_h(q)$ *the event that the quad is horizontally crossed by* $\oplus$*. See Fig. 5 and Fig. 6.*

- *Suppose* $\xi = \ominus$*. On the event* $\mathcal{C}^\ominus_v(q)$*, let* η_δ *be the interface connecting* x^R_δ *and* y^R_δ*. Then the law of* η_δ *converges weakly to* hSLE_3 *in* Ω *from* x^R *to* y^R *with marked points* (x^L, y^L) *as* $\delta \to 0$*.*
- *Suppose* $\xi = \text{free}$*. On the event* $\mathcal{C}^\ominus_v(q)$*, let* η_δ *be the interface connecting* x^R_δ *and* y^R_δ*. Then the law of* η_δ *(up to the first hitting time of* $[y^R_\delta y^L_\delta]$*) converges weakly to* $\mathrm{hSLE}_3(-7/2)$ *in* Ω *from* x^R *to* x^L *conditioned to hit* $[y^R y^L]$ *(up to the first hitting time of* $[y^R y^L]$*) as* $\delta \to 0$*.*
- *Suppose* $\xi = \text{free}$*. On the event* $\mathcal{C}^\oplus_h(q)$*, let* η_δ *be the interface connecting* x^R_δ *and* x^L_δ*. Then the law of* η_δ *converges weakly to* $\mathrm{hSLE}_3(-3/2)$ *in* Ω *from* x^R *to* x^L *with marked points* (y^R, y^L) *as* $\delta \to 0$*.*

The conclusions in Proposition 1.6 are not new. They were proved by K. Izyurov [Izy15], and we will give a new proof in Sect. 5. There are three features on the method developed there.

- No need to construct new observable. Constructing holomorphic observable is the usual way to prove the convergence of interfaces in the critical lattice models (as in [Izy15]); however, our method does not require new observable. The only input we need is the convergence of the interface with Dobrushin boundary conditions.

- The result is "global". There are many works on multiple SLEs trying to study the scaling limit of interfaces in critical lattice model with alternating boundary conditions, see [Dub07,BBK05,KP16,Izy15], and their works study the local growth of these interfaces. Whereas, we prove the convergence of the entire interface.
- Easy to generalize. Our method can be generalized to more complicated boundary conditions, and the method also works for other critical lattice models including FK-Ising model and percolation, see [BPW18].

1.4. Pure partition functions of multiple SLEs. The motivation to study hypergeometric SLE is to understand the scaling limits of interfaces in critical lattice models in quad with alternating boundary conditions. It is natural to consider the interfaces in general polygon. We call $(\Omega; x_1, \ldots, x_{2N})$ a polygon if $\Omega \subsetneq \mathbb{C}$ is simply connected and $x_1, \ldots, x_{2N}$ are $2N$ boundary points in counterclockwise order. We take Ising model as an example again. Suppose $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$ are discrete domains on $\delta\mathbb{Z}^2$ that approximate some polygon $(\Omega; x_1, \ldots, x_{2N})$. Consider the critical Ising model in Ω^δ with alternating boundary conditions:

$$\oplus \text{ on } (x_{2j-1}^\delta, x_{2j}^\delta), \quad \text{for } j \in \{1, \ldots, N\}; \quad \ominus \text{ on } (x_{2j}^\delta, x_{2j+1}^\delta), \quad \text{for } j \in \{0, 1, \ldots, N\},$$

with the convention that $x_0 = x_{2N}$ and $x_{2N+1} = x_1$. Then N interfaces $(\eta_1^\delta, \ldots, \eta_N^\delta)$ arise in the model and they connect the $2N$ boundary points $x_1^\delta, \ldots, x_{2N}^\delta$, forming a planar connectivity. We describe the connectivities by planar pair partitions $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$ where $\{a_1, b_1, \ldots, a_N, b_N\} = \{1, 2, \ldots, 2N\}$. We call such α link patterns and we denote the set of them by LP_N. We denote $\mathrm{LP} = \sqcup_{N\geq 0}\mathrm{LP}_N$. Given a link pattern $\alpha \in \mathrm{LP}_N$ and $\{a, b\} \in \alpha$, we denote by $\alpha/\{a, b\}$ the link pattern in LP_{N-1} obtained by removing $\{a, b\}$ from α and then relabelling the remaining indices so that they are the first $2(N-1)$ integers.

It turns out that the scaling limits of $(\eta_1^\delta, \ldots, \eta_N^\delta)$ are the Loewner chains associated to the so-called pure partition functions: Fix $\kappa \in (0, 8)$, multiple SLE *pure partition functions* are a collection of positive smooth functions

$$\mathcal{Z}_\alpha : \mathfrak{X}_{2N} \to \mathbb{R}_+, \quad \alpha \in \mathrm{LP}_N$$

defined on the space $\mathfrak{X}_{2N} := \{(x_1, \ldots, x_{2N}) : x_1 < \cdots < x_{2N}\}$ with following three properties:

- PDE system (PDE): for all $i \in \{1, \ldots, 2N\}$,

$$\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{2h}{(x_j - x_i)^2}\right)\right]\mathcal{Z}(x_1, \ldots, x_{2N}) = 0. \tag{1.1}$$

- Conformal covariance (COV): for all Möbius maps φ of $\mathbb{H}$ such that $\varphi(x_1) < \cdots < \varphi(x_{2N})$,

$$\mathcal{Z}(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N}\varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1), \ldots, \varphi(x_{2N})), \quad \text{where } h = \frac{6-\kappa}{2\kappa}. \tag{1.2}$$

– Asymptotics (ASY): for all $\alpha \in \mathrm{LP}_N$ and for all $j \in \{1, \ldots, 2N-1\}$ and $\xi \in (x_{j-1}, x_{j+2})$,

$$\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{(x_{j+1} - x_j)^{-2h}} = \begin{cases} 0 & \text{if } \{j, j+1\} \notin \alpha \\ \mathcal{Z}_{\hat{\alpha}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}) & \text{if } \{j, j+1\} \in \alpha \end{cases} \tag{1.3}$$

where $\hat{\alpha} = \alpha / \{j, j+1\} \in \mathrm{LP}_{N-1}$.

The appearance of such three properties is natural. Assuming the existence of scaling limits of interfaces in polygon, the Loewner chain of interfaces should satisfies the so-called "commutation relation" which gives rise to the PDE system. The conformal covariance comes from the conformal invariance of the scaling limit. The asymptotics correspond to "comptability" for the system of functions for different N. See [Pel19] for the background from statistical mechanics and from conformal field theory. Although the scaling limits of interfaces in polygon lead to the introduction of pure partition functions, it is far from clear why such functions exist, and we will discuss the existence of such functions in the following theorem.

Theorem 1.7. *Let $\kappa \in (0, 6]$. There exists a unique collection $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ of smooth functions $\mathcal{Z}_\alpha : \mathfrak{X}_{2N} \to \mathbb{R}_+$, for $\alpha \in \mathrm{LP}_N$, satisfying the normalization $\mathcal{Z}_\emptyset = 1$ and* PDE (1.1)*,* COV (1.2)*,* ASY (1.3) *and, for all $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N$, the power law bound*

$$0 < \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \le \prod_{j=1}^{N} |x_{b_j} - x_{a_j}|^{-2h}. \tag{1.4}$$

The uniqueness is a deep result and it follows from results in [FK15, Lemma 1] for all $\kappa \in (0, 8)$. The existence part was proved for $\kappa \in (0, 8) \setminus \mathbb{Q}$ in [KP16] using Coulomb gas techniques. The difficulty with the Coulomb gas techniques is that the authors could not show the positivity of the constructed functions, neither the upper bound in (1.4). The existence was later proved for $\kappa \in (0, 4]$ in [PW19] using the construction of global multiple SLEs. Since the construction used Brownian loop soup, it only gives the existence for $\kappa \le 4$. In this paper, we will give a new proof of the existence for $\kappa \in (0, 6]$ using properties of hypergeometric SLE. We will construct the pure partition by cascade relation and then show that they satisfies all the requirements. The main obstacle in this construction is checking the PDE, and this is obtained using properties of hypergeometric SLEs.

Outline and relation to previous work. We will give preliminaries on SLEs in Sect. 2. We will introduce hypergeometric SLE in Sect. 3. Hypergeometric SLEs were previously introduced by D. Zhan [Zha10] and W. Qian [Qia18] with different motivations and definitions: D. Zhan introduced it to describe the time-reversal of $\mathrm{SLE}_\kappa(\rho)$ and W. Qian introduced it to describe the boundary of the so-called trichordal restriction samples. Our motivation is to describe the scaling limits of interfaces in critical lattice models in quad. Our definition is different from the one in [Qia18]. The definition in [Zha10] is a particular case of ours. We will prove Theorem 1.1 in Sect. 3 and many other interesting properties of hSLE. We prove Theorem 1.4 in Sect. 4. We introduce Ising model in Sect. 5 and prove Proposition 1.6. We complete the proof of Theorem 1.7 in Sect. 6.

2. Preliminaries

2.1. Space of curves. A planar curve is a continuous mapping from $[0, 1]$ to $\mathbb{C}$ modulo reparameterization. Let X be the set of planar curves. The metric d on X is defined by

$$d(\eta_1, \eta_2) = \inf_{\varphi_1,\varphi_2} \sup_{t\in[0,1]} |\eta_1(\varphi_1(t)) - \eta_2(\varphi_2(t))|,$$

where the inf is over increasing homeomorphisms $\varphi_1, \varphi_2 : [0, 1] \to [0, 1]$. The metric space (X, d) is complete and separable. A simple curve is a continuous injective mapping from $[0, 1]$ to $\mathbb{C}$ modulo reparameterization. Let X_{simple} be the subspace of simple curves in X and denote by X_0 its closure. The curves in X_0 may have multiple points but they do not have self-crossings.

We call $(\Omega; x_1, \ldots, x_n)$ a *(topological) polygon* if Ω is a non-empty simply connected proper subset of $\mathbb{C}$ and $x_1, \ldots, x_n$ are boundary points appearing in counterclockwise order and lying on locally connected boundary segments. If the points $x_1, \ldots, x_n$ of the polygon $(\Omega; x_1, \ldots, x_n)$ lie on sufficiently regular boundary segments (e.g. $C^{1+\epsilon}$ for some $\epsilon > 0$), we call $(\Omega; x_1, \ldots, x_n)$ a *nice polygon.* Let $(\Omega; x_1, \ldots, x_n)$ be a bounded polygon. We say that a sequence of polygons $(\Omega^\delta; x_1^\delta, \ldots, x_n^\delta)$ converges to $(\Omega; x_1, \ldots, x_n)$ as $\delta \to 0$ in the *Carathéodory sense* if there exist conformal maps f^δ from the unit disc $\mathbb{U}$ to Ω^δ and conformal map f from $\mathbb{U}$ to Ω such that $f^\delta \to f$ uniformly on any compact subset of $\mathbb{U}$, and $\lim_{\delta\to 0}(f^\delta)^{-1}(x_j^\delta) = f^{-1}(x_j)$ for all $j \in \{1, \ldots, n\}$.

We call a polygon $(\Omega; x_1, \ldots, x_n)$ a *Dobrushin domain* if $n = 2$. Given a Dobrushin domain $(\Omega; x, y)$, denote by (xy) the arc of $\partial\Omega$ from x to y counterclockwise, and by $[xy]$ the closed arc. We call $(\Omega; x_1, \ldots, x_n)$ a *triangle* if $n = 3$, and we denote by $\mathcal{T}$ the collection of all triangles $(\Omega; x_1, x_2, x_3)$ with $x_1 \neq x_3$. We call $(\Omega; x_1, \ldots, x_n)$ a *quad* if $n = 4$, and we denote by $\mathcal{Q}$ the collection of all quads $(\Omega; x_1, x_2, x_3, x_4)$ with $x_1 \neq x_4$. Given a quad $(\Omega; a, b, c, d)$, we denote by $d_\Omega((ab), (cd))$ the extremal distance between (ab) and (cd) in Ω.

Given a Dobrushin domain $(\Omega; x, y)$, let $X_{\text{simple}}(\Omega; x, y)$ be the space of simple curves η such that $\eta(0) = x$, $\eta(1) = y$, and $\eta(0, 1) \subset \Omega$. Denote by $X_0(\Omega; x, y)$ the closure of $X_{\text{simple}}(\Omega; x, y)$.

Given a quad $(\Omega; x^L, x^R, y^R, y^L)$, let $X_{\text{simple}}(\Omega; x^L, x^R, y^R, y^L)$ be the collection of pairs of simple curves $(\eta^L; \eta^R)$ such that $\eta^L \in X_{\text{simple}}(\Omega; x^L, y^L)$ and $\eta^R \in X_{\text{simple}}(\Omega; x^R, y^R)$ and that $\eta^L \cap \eta^R = \emptyset$. The definition of $X_0(\Omega; x^L, x^R, y^R, y^L)$ is a little bit complicated. Given $\epsilon > 0$, let $X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L)$ be the set of pairs of curves $(\eta^L; \eta^R)$ such that

- $\eta^L \in X_0(\Omega; x^L, y^L)$ and $\eta^R \in X_0(\Omega; x^R, y^R)$;
- $d_{\Omega^L}(\eta^L, (x^R y^R)) \geq \epsilon$ where Ω^L is the connected component of $\Omega\setminus\eta^L$ with $(x^R y^R)$ on the boundary, and η^R is contained in the closure of Ω^L ;
- $d_{\Omega^R}(\eta^R, (y^L x^L)) \geq \epsilon$ where Ω^R is the connected component of $\Omega\setminus\eta^R$ with $(y^L x^L)$ on the boundary, and η^L is contained in the closure of Ω^R.

Define the metric on $X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L)$ by

$$\mathcal{D}((\eta_1^L, \eta_1^R), (\eta_2^L, \eta_2^R)) = \max\{d(\eta_1^L, \eta_2^L), d(\eta_1^R, \eta_2^R)\}.$$

One can check $\mathcal{D}$ is a metric and the space $X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L)$ with $\mathcal{D}$ is complete and separable. Finally, set

$$X_0(\Omega; x^L, x^R, y^R, y^L) = \bigcup_{\epsilon>0} X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L).$$

Note that $X_0(\Omega; x^L, x^R, y^R, y^L)$ is no longer complete.

Suppose E is a metric space and $\mathcal{B}_E$ is the Borel σ-field. Let $\mathcal{P}$ be the space of probability measures on $(E, \mathcal{B}_E)$. The Prohorov metric $d_{\mathcal{P}}$ on $\mathcal{P}$ is defined by

$$d_{\mathcal{P}}(\mathbb{P}_1, \mathbb{P}_2) = \inf\left\{\epsilon > 0 : \mathbb{P}_1[A] \le \mathbb{P}_2[A^\epsilon] + \epsilon, \mathbb{P}_2[A] \le \mathbb{P}_1[A^\epsilon] + \epsilon, \forall A \in \mathcal{B}_E\right\},$$

where A^ϵ is the ϵ-neighborhood of the set A. When E is complete and separable, the space $\mathcal{P}$ is complete and separable ([Bil99, Theorem 6.8]); moreover, a sequence $\mathbb{P}_n$ in $\mathcal{P}$ converges weakly to $\mathbb{P}$ if and only if $d_{\mathcal{P}}(\mathbb{P}_n, \mathbb{P}) \to 0$.

Let Σ be a family of probability measures on $(E, \mathcal{B}_E)$. We call Σ *relatively compact* if every sequence of elements in Σ contains a weakly convergent subsequence. We call Σ *tight* if, for every $\epsilon > 0$, there exists a compact set K_ϵ such that $\mathbb{P}[K_\epsilon] \ge 1 - \epsilon$ for all $\mathbb{P} \in \Sigma$. By Prohorov's Theorem ([Bil99, Theorem 5.2]), when E is complete and separable, relative compactness is equivalent to tightness.

2.2. Loewner chain and SLE. We call a compact subset K of $\overline{\mathbb{H}}$ an $\mathbb{H}$-*hull* if $\mathbb{H}\setminus K$ is simply connected. Riemann's Mapping Theorem asserts that there exists a unique conformal map g_K from $\mathbb{H}\setminus K$ onto $\mathbb{H}$ such that $\lim_{z\to\infty} |g_K(z) - z| = 0$. We call such g_K the conformal map from $\mathbb{H}\setminus K$ onto $\mathbb{H}$ normalized at ∞ and we call $a(K) := \lim_{z\to\infty} z(g_t(z) - z)$ the *half-plane capacity* of K seen from ∞.

Loewner chain is a collection of $\mathbb{H}$-hulls $(K_t, t \ge 0)$ associated with the family of conformal maps $(g_t, t \ge 0)$ obtained by solving the Loewner equation: for each $z \in \mathbb{H}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

where $(W_t, t \ge 0)$ is a one-dimensional continuous function which we call the driving function. Let T_z be the *swallowing time* of z defined as $\sup\{t \ge 0 : \min_{s\in[0,t]} |g_s(z) - W_s| > 0\}$. Let $K_t := \overline{\{z \in \mathbb{H} : T_z \le t\}}$. Then g_t is the unique conformal map from $H_t := \mathbb{H}\setminus K_t$ onto $\mathbb{H}$ normalized at ∞. Since the half-plane capacity of K_t is $2t$ for all $t \ge 0$, we say that the process $(K_t, t \ge 0)$ is parameterized by the half-plane capacity. We say that $(K_t, t \ge 0)$ can be generated by the continuous curve $(\eta(t), t \ge 0)$ if for any t, the unbounded connected component of $\mathbb{H}\setminus\eta[0, t]$ coincides with $H_t = \mathbb{H}\setminus K_t$.

Indeed, a continuous simple curve under mild constraints does solve the Loewner equation with continuous driving function. Suppose $T \in (0, \infty]$ and $\eta : [0, T) \to \overline{\mathbb{H}}$ is a continuous simple curve with $\eta(0) = 0$. Assume η satisfies the following: for every $t \in (0, T)$,

- $\eta(t, T)$ is contained in the closure of the unbounded connected component of $\mathbb{H}\setminus\eta[0, t]$ and
- $\eta^{-1}(\eta[0, t] \cup \mathbb{R})$ has empty interior in (t, T).

For each $t > 0$, let g_t be the conformal map which maps the unbounded connected component of $\mathbb{H}\setminus\eta[0, t]$ onto $\mathbb{H}$ normalized at ∞. After reparameterization, $(g_t, t \ge 0)$ solves the above Loewner equation with continuous driving function [Law05, Section 4.1].

Here we discuss the evolution of a point $y \in \mathbb{R}$ under g_t. We assume $y \ge 0$. There are two possibilities: if y is not swallowed by K_t, then we define $Y_t = g_t(y)$; if y is swallowed by K_t, then we define Y_t to be the image of the rightmost of point of $K_t \cap \mathbb{R}$ under g_t. Suppose that $(K_t, t \ge 0)$ is generated by a continuous curve $(\eta(t), t \ge 0)$

and that the Lebesgue measure of $\eta[0,\infty]\cap\mathbb{R}$ is zero. Then the process Y_t is uniquely characterized by the following equation:

$$Y_t = y + \int_0^t \frac{2ds}{Y_s - W_s}, \quad Y_t \geq W_t, \quad \forall t \geq 0.$$

In this paper, we may write $g_t(y)$ for the process Y_t.

Schramm Loewner Evolution SLE_κ is the random Loewner chain $(K_t, t\geq 0)$ driven by $W_t = \sqrt{\kappa}B_t$ where $(B_t, t\geq 0)$ is a standard one-dimensional Brownian motion. In [RS05], the authors prove that $(K_t, t\geq 0)$ is almost surely generated by a continuous transient curve, i.e. there almost surely exists a continuous curve η such that for each $t\geq 0$, H_t is the unbounded connected component of $\mathbb{H}\backslash\eta[0,t]$ and that $\lim_{t\to\infty}|\eta(t)| = \infty$. There are phase transitions at $\kappa = 4$ and $\kappa = 8$: SLE_κ are simple curves when $\kappa\in(0,4]$; they have self-touchings when $\kappa\in(4,8)$; and they are space-filling when $\kappa\geq 8$.

For any Dobrushin domain $(\Omega; x, y)$, SLE_κ in $(\Omega; x, y)$ is defined via conformal image: Let φ be any conformal map from Ω onto $\mathbb{H}$ that sends x to 0 and y to ∞. Then SLE_κ in $(\Omega; x, y)$ is $\varphi^{-1}(\eta)$ where η is an SLE_κ in $\mathbb{H}$ from 0 to ∞. For $\kappa\in(0,8)$, the curves SLE_κ enjoy *reversibility*: let η be an SLE_κ in Ω from x to y, then the time-reversal of η has the same law as SLE_κ in Ω from y to x. The reversibility for $\kappa\in(0,4]$ was proved in [Zha08], and it was proved for $\kappa\in(4,8)$ in [MS16c].

2.3. Convergence of curves. In this section, we first recall the main result of [KS17] and then show a similar result for pairs of curves. Suppose $(Q; a, b, c, d)$ is a quad. We say that a curve η *crosses* Q if there exists a subinterval $[s,t]$ such that $\eta(s,t)\subset Q$ and $\eta[s,t]$ intersects both (ab) and (cd). Given a Dobrushin domain $(\Omega; x, y)$, for any curve η in $X_0(\Omega; x, y)$ and any time τ, define Ω_τ to be the connected component of $\Omega\backslash\eta[0,\tau]$ with y on the boundary. Consider a quad $(Q; a, b, c, d)$ in Ω_τ such that (bc) and (da) are contained in $\partial\Omega_\tau$. We say that Q is *avoidable* if it does not disconnect $\eta(\tau)$ from y in Ω_τ.

Definition 2.1. A family Σ of probability measures on curves in $X_{\mathrm{simple}}(\Omega; x, y)$ is said to satisfy **Condition C2** if, for any $\epsilon\in(0,1)$, there exists a constant $c(\epsilon)>0$ such that for any $\mathbb{P}\in\Sigma$, any stopping time τ, and any avoidable quad $(Q; a, b, c, d)$ in Ω_τ such that $d_Q((ab),(cd))\geq c(\epsilon)$, we have

$$\mathbb{P}[\eta[\tau,1] \text{ crosses } Q \mid \eta[0,\tau]] \leq 1-\epsilon.$$

If the above property holds for $\tau = 0$, we say that the family satisfies **Condition C1**.

It is clear that the combination of Condition C1 and CMP implies Condition C2.

Theorem 2.2 [KS17, Corollary 1.7, Proposition 2.6]. *Fix a Dobrushin domain $(\Omega; x, y)$. Suppose that $\{\eta_n\}_{n\in\mathbb{N}}$ is a sequence of random curves in $X_{\mathrm{simple}}(\Omega; x, y)$ satisfying Condition C2. Denote by $(W_n(t), t\geq 0)$ the driving process of η_n. Then*

- *the family of laws of $\{W_n\}_{n\in\mathbb{N}}$ is tight in the metrisable space of continuous functions on $[0,\infty)$ with the topology of uniform convergence on compact subsets of $[0,\infty)$;*
- *the family of laws of $\{\eta_n\}_{n\in\mathbb{N}}$ is tight in the space of curves X;*
- *the family of laws of $\{\eta_n\}_{n\in\mathbb{N}}$, when each curve is parameterized by the half-plane capacity, is tight in the metrisable space of continuous functions on $[0,\infty)$ with the topology of uniform convergence on compact subsets of $[0,\infty)$.*

Moreover, if the sequence converges in any of the topologies above it also converges in the two other topologies and the limits agree in the sense that the limiting random curve is driven by the limiting driving function.

Next, we will explain a similar result for pairs of curves. Fix a quad $(\Omega; x^L, x^R, y^R, y^L)$.

Definition 2.3. A family Σ of probability measures on pairs of curves in $X_{\text{simple}}(\Omega; x^L, x^R, y^R, y^L)$ is said to satisfy **Condition C2** if, for any $\epsilon \in (0, 1)$, there exists a constant $c(\epsilon) > 0$ such that for any $\mathbb{P} \in \Sigma$, the following holds. Given any η^L-stopping time τ^L and any η^R-stopping time τ^R, and any avoidable quad $(Q^R; a^R, b^R, c^R, d^R)$ for η^R in $\Omega\backslash(\eta^L[0, \tau^L] \cup \eta^R[0, \tau^R])$ such that $d_{Q^R}((a^R b^R), (c^R d^R)) \geq c(\epsilon)$, and any avoidable quad $(Q^L; a^L, b^L, c^L, d^L)$ for η^L in $\Omega\backslash(\eta^L[0, \tau^L] \cup \eta^R[0, \tau^R])$ such that $d_{Q^L}((a^L b^L), (c^L d^L)) \geq c(\epsilon)$, we have

$$\mathbb{P}\left[\eta^R[\tau^R, 1] \text{ crosses } Q^R \mid \eta^L[0, \tau^L], \eta^R[0, \tau^R]\right] \leq 1 - \epsilon,$$
$$\mathbb{P}\left[\eta^L[\tau^L, 1] \text{ crosses } Q^L \mid \eta^L[0, \tau^L], \eta^R[0, \tau^R]\right] \leq 1 - \epsilon.$$

If the above property holds for $\tau^L = \tau^R = 0$, we say that the family satisfies **Condition C1**.

Theorem 2.4 *Suppose that $\{(\eta_n^L; \eta_n^R)\}_{n\in\mathbb{N}}$ is a sequence of pairs of random curves in $X_{\text{simple}}(\Omega; x^L, x^R, y^R, y^L)$ and denote their laws by $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$. Let Ω_n^L be the connected component of $\Omega\backslash\eta_n^L$ with $(x^R y^R)$ on the boundary and Ω_n^R be the connected component of $\Omega\backslash\eta_n^R$ with $(y^L x^L)$ on the boundary. Define, for each n,*

$$\mathcal{D}_n^L = d_{\Omega_n^L}(\eta_n^L, (x^R y^R)), \quad \mathcal{D}_n^R = d_{\Omega_n^R}(\eta_n^R, (y^L x^L)).$$

Assume that the family of laws of $\{(\eta_n^L; \eta_n^R)\}_{n\in\mathbb{N}}$ satisfies Condition C2 and that the family of laws of $\{(\mathcal{D}_n^L; \mathcal{D}_n^R)\}_{n\in\mathbb{N}}$ is tight in the following sense: for any $u > 0$, there exists $\epsilon > 0$ such that

$$\mathbb{P}_n\left[\mathcal{D}_n^L \geq \epsilon, \mathcal{D}_n^R \geq \epsilon\right] \geq 1 - u, \quad \forall n.$$

Then the sequence $\{(\eta_n^L; \eta_n^R)\}_{n\in\mathbb{N}}$ is relatively compact in $X_0(\Omega; x^L, x^R, y^R, y^L)$.

Proof. By Theorem 2.2, there is subsequence $n_k \to \infty$ such that $\eta_{n_k}^L$ (resp. $\eta_{n_k}^R$) converges weakly in all three topologies in Theorem 2.2. By Skorohod Representation Theorem, we could couple all $(\eta_{n_k}^L; \eta_{n_k}^R)$ in a common space so that $\eta_{n_k}^L \to \eta^L$ and $\eta_{n_k}^R \to \eta^R$ almost surely. For $\epsilon > 0$, define

$$K_\epsilon = \Big\{(\eta^L; \eta^R) \in X_{\text{simple}}(\Omega; x^L, x^R, y^R, y^L) : d_{\Omega^L}(\eta^L, (x^R y^R)) \geq \epsilon,$$
$$d_{\Omega^R}(\eta^R, (y^L x^L)) \geq \epsilon\Big\}.$$

From the assumption, we know that, for any $u > 0$, there exists $\epsilon > 0$ such that $\inf_n \mathbb{P}_n[K_\epsilon] \geq 1 - u$. Therefore, with probability at least $1 - u$, the sequence $(\eta_{n_k}^L; \eta_{n_k}^R)$ converges to $(\eta^L; \eta^R)$ in $X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L) \subset X_0(\Omega; x^L, x^R, y^R, y^L)$. This is true for any $u > 0$, thus we have $(\eta_{n_k}^L; \eta_{n_k}^R)$ converges to $(\eta^L; \eta^R)$ in $X_0(\Omega; x^L, x^R, y^R, y^L)$ almost surely. $\square$

2.4. Conformal Markov characterization of $\mathrm{SLE}_\kappa(\rho)$. $\mathrm{SLE}_\kappa(\rho)$ *processes* are variants of SLE_κ where one keeps track of one extra point on the boundary. $\mathrm{SLE}_\kappa(\rho)$ process with force point $w \in \mathbb{R}$ is the Loewner evolution driven by W_t which is the solution to the system of integrated SDEs:

$$W_t = \sqrt{\kappa} B_t + \int_0^t \frac{\rho ds}{W_s - V_s}, \quad V_t = w + \int_0^t \frac{2ds}{V_s - W_s},$$

where B_t is one-dimensional Brownian motion. For $\rho \in \mathbb{R}$, the process is well-defined up to the first time that w is swallowed. When $\rho > -2$, the process is well-defined for all time and it is generated by a continuous transient curve. Assume $w \geq 0$. When $\rho \geq \kappa/2 - 2$, the curve never hits the interval $[w, \infty)$; when $\rho < \kappa/2 - 2$, the curve hits the interval $[w, \infty)$ at finite time; and when $\rho \leq \kappa/2 - 4$, the curve accumulates at the point w almost surely. We define $\mathrm{SLE}_\kappa(\rho)$ in any triangle via conformal image.

Lemma 2.5. *Fix* $\kappa \in (0, 8)$ *and* $\rho > (-2) \vee (\kappa/2 - 4)$. *Then* $\mathrm{SLE}_\kappa(\rho)$ *satisfies Condition C1.*

Proof. Suppose η is an $\mathrm{SLE}_\kappa(\rho)$ in $\mathbb{H}$ from 0 to ∞ with force point $w \in \mathbb{R}$. Then there exists a function $p(\delta) \to 0$ as $\delta \to 0$ such that

$$\mathbb{P}[\eta \text{ hits } B(1, \delta)] \leq p(\delta), \tag{2.1}$$

and that p depends only on κ, ρ and is uniform over w, see for instance [Wu18, Lemma A.5].

Suppose $(Q; a, b, c, d)$ is an avoidable quad for η. It is explained in [KS17, Eq.(12) in the proof of Theorem 1.10] that $\{\eta \text{ crosses } Q\}$ implies $\{\eta \text{ hits } B(u, r)\}$ for some $u \in \mathbb{R}$, $r > 0$ such that

$$\frac{r}{|u|} = \left(\frac{\exp(\pi d_Q((ab), (cd)))}{16} - 1 \right)^{-1}.$$

Combining with (2.1), it implies that η satisfies Condition C1. □

Lemma 2.6 [SW05, Theorem 3]. *Fix* $\kappa > 0$ *and* $\rho \in \mathbb{R}$ *and a triangle* $(\Omega; x, w, y)$. *Let* η *be an* $\mathrm{SLE}_\kappa(\rho)$ *in* Ω *from* x *to* y *with force point* w. *Then* η *has the same law as* $\mathrm{SLE}_\kappa(\kappa - 6 - \rho)$ *in* Ω *from* x *to* w *with force point* y, *up to the first time that the curve disconnects* w *from* y.

Next, we explain the conformal Markov characterization of $\mathrm{SLE}_\kappa(\rho)$ derived in [MS16b]. Recall that $\mathcal{T}$ is the collection of all triangles $(\Omega; x_1, x_2, x_3)$ with $x_1 \neq x_3$.

Definition 2.7. Suppose $(\mathbb{P}_c, c \in \mathcal{T})$ is a family of probability measures on continuous curves from x to y in Ω. We say that $(\mathbb{P}_c, c \in \mathcal{T})$ satisfies conformal Markov property (CMP) if it satisfies the following two properties.

- Conformal invariance. Suppose that $c = (\Omega; x, w, y)$, $\tilde{c} = (\tilde{\Omega}; \tilde{x}, \tilde{w}, \tilde{y}) \in \mathcal{T}$, and $\psi : \Omega \to \tilde{\Omega}$ is the conformal map with $\psi(x) = \tilde{x}$, $\psi(w) = \tilde{w}$, $\psi(y) = \tilde{y}$. Then for $\eta \sim \mathbb{P}_c$, we have $\psi(\eta) \sim \mathbb{P}_{\tilde{c}}$.
- Domain Markov property. Suppose $\eta \sim \mathbb{P}_c$, then for every η-stopping time τ, the conditional law of $(\eta|_{t \geq \tau})$ given $\eta[0, \tau]$ is the same as $\mathbb{P}_{c_\tau}$ where $c_\tau = (\Omega_\tau; \eta(\tau), w_\tau, y)$. Here Ω_τ is the connected component of $\Omega \setminus \eta[0, \tau]$ with y on the boundary, and $w_\tau = w$ if w is not swallowed by $\eta[0, \tau]$ and w_τ is the last point of $\eta[0, \tau] \cap (xy)$ if w is swallowed by $\eta[0, \tau]$.

Theorem 2.8 [MS16b, Theorem 1.4]. *Suppose* $(\mathbb{P}_c, c \in \mathcal{T})$ *satisfies CMP in Definition 2.7 and Condition C1, then there exist* $\kappa \in (0, 8)$ *and* $\rho > (-2) \vee (\kappa/2 - 4)$ *such that, for each* $c = (\Omega; x, w, y) \in \mathcal{T}$, $\mathbb{P}_c$ *is the law of* $\mathrm{SLE}_\kappa(\rho)$ *in* Ω *from* x *to* y *with force point* w.

In [MS16b, Theorem 1.4], the authors do not require Condition C1; instead, they require the assumption that, when $\partial\Omega$ is smooth, the Lebesgue measure of $\eta \cap \partial\Omega$ is zero almost surely. Note that Condition C1 implies this latter assumption, and we find Condition C1 is more natural, since it is the continuum counterpart of Russo-Symour-Welsh bound for critical lattice models, see Proposition 5.1.

2.5. SLE with multiple force points. $\mathrm{SLE}_\kappa(\rho)$ *processes* are variants of SLE_κ where one keeps track of multiple points on the boundary. Suppose $\underline{y} = (0 \le y_1 < y_2 < \cdots < y_n)$ and $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ with $\rho_i \in \mathbb{R}$. An $\mathrm{SLE}_\kappa(\underline{\rho})$ process with force points $\underline{y}$ is the Loewner evolution driven by W_t which is the solution to the following system of integrated SDEs:

$$W_t = \sqrt{\kappa} B_t + \sum_{i=1}^n \int_0^t \frac{\rho_i ds}{W_s - V_s^i}, \quad V_t^i = y_i + \int_0^t \frac{2ds}{V_s^i - W_s}, \quad \text{for } 1 \le i \le n,$$

where B_t is an one-dimensional Brownian motion. Note that the process V_t^i is the time evolution of the point y_i, and we may write $g_t(y_i)$ for V_t^i. We define the *continuation threshold* of the $\mathrm{SLE}_\kappa(\underline{\rho})$ to be the infimum of the time t for which

$$\sum_{i: V_t^i = W_t} \rho_i \le -2.$$

By [MS16a, Theorem 1.3], the $\mathrm{SLE}_\kappa(\underline{\rho})$ process is well-defined up to the continuation threshold, and it is almost surely generated by a continuous curve up to and including the continuation threshold. The Radon-Nikodym derivative between $\mathrm{SLE}_\kappa(\underline{\rho})$ and SLE_κ is given by the following lemma.

Lemma 2.9 [SW05]. *The process* $\mathrm{SLE}_\kappa(\underline{\rho})$ *with force points* $\underline{y}$ *is the same as* SLE_κ *process weighted by the following local martingale, up to the first time that* y_1 *is swallowed:*

$$M_t = \prod_{1 \le i \le n} \left(g_t'(y_i)^{\rho_i(\rho_i + 4 - \kappa)/(4\kappa)} (g_t(y_i) - W_t)^{\rho_i/\kappa} \right) \times \prod_{1 \le i < j \le n} (g_t(y_j) - g_t(y_i))^{\rho_i \rho_j/(2\kappa)}.$$

3. Hypergeometric SLE: Basic Properties

3.1. Definition of hSLE. We first define hSLE in the upper-half plane $\mathbb{H}$. Fix $\kappa \in (0, 8)$ and $\nu \in \mathbb{R}$, and four boundary points $x_1 < x_2 < x_3 < x_4$. We are interested in Euler's hypergeometric differential equation

$$z(1-z)F''(z) + \left(\frac{2\nu + 8}{\kappa} - \frac{2\nu + 2\kappa}{\kappa} z \right) F'(z) - \frac{2(\nu + 2)(\kappa - 4)}{\kappa^2} F(z) = 0. \quad (3.1)$$

When $\nu > (-4) \vee (\kappa/2 - 6)$, define F to be the hypergeometric function (see Appendix A):

$$F(z) := {}_2F_1\left(\frac{2\nu+4}{\kappa}, 1-\frac{4}{\kappa}, \frac{2\nu+8}{\kappa}; z\right). \tag{3.2}$$

When $\nu \leq (-4) \vee (\kappa/2 - 6)$, define F to be the following:

$$F(z) := (1-z)^{8/\kappa-1} G(1-z), \quad \text{where} \quad G(z) = {}_2F_1\left(\frac{2\nu+12-\kappa}{\kappa}, \frac{4}{\kappa}, \frac{8}{\kappa}; z\right). \tag{3.3}$$

Note that the functions F defined in both (3.2) and (3.3) are solutions to (3.1).

Lemma 3.1. *Fix $\kappa \in (0, 8)$.*

– *When $\nu > (-4) \vee (\kappa/2 - 6)$, the function F defined in (3.2) is uniformly bounded for $z \in [0, 1]$:*

$$0 < 1 \wedge F(1) \leq F(z) \leq 1 \vee F(1) < \infty, \quad \forall z \in [0, 1].$$

– *When $\nu \leq (-4) \vee (\kappa/2 - 6)$, the function G defined in (3.3) is uniformly bounded for $z \in [0, 1]$:*

$$0 < 1 \wedge G(1) \leq G(z) \leq 1 \vee G(1) < \infty, \quad \forall z \in [0, 1].$$

Proof. Denote by

$$A = \frac{2\nu+4}{\kappa}, \quad B = 1 - \frac{4}{\kappa}, \quad C = \frac{2\nu+8}{\kappa}.$$

When $\nu > (-4) \vee (\kappa/2 - 6)$, we have

$$C > 0, \quad C > A, \quad C > B, \quad C > A + B.$$

Then $F(1) \in (0, \infty)$ by (A.1). If $AB > 0$, F is increasing by Lemma A.1. If $AB = 0$, we have $F \equiv 1$. If $AB < 0$, F is decreasing by Lemma A.2. In summary, F is monotone, and it is bounded by $F(0) = 1$ and $F(1)$.

Note that

$$G(z) = {}_2F_1(C - B, C - A, 1 + C - A - B; z).$$

When $\nu \leq (-4) \vee (\kappa/2 - 6)$, we have

$$1 + C - A - B > 0, \quad 1 + C - A - B > C - B, \quad 1 + C - A - B > C - A,$$
$$1 + C - A - B > 2C - A - B.$$

Then $G(1) \in (0, \infty)$ by (A.1). Similarly, G is monotone, and it is bounded by $G(0) = 1$ and $G(1)$. □

Set

$$h = \frac{6-\kappa}{2\kappa}, \quad a = \frac{\nu+2}{\kappa}, \quad b = \frac{(\nu+2)(\nu+6-\kappa)}{4\kappa}. \tag{3.4}$$

For $x_1 < x_2 < x_3 < x_4$, define partition function

$$\begin{aligned} &\mathcal{Z}_{\kappa,\nu}(x_1, x_2, x_3, x_4) = (x_4 - x_1)^{-2h}(x_3 - x_2)^{-2b} z^a F(z), \\ &\text{where} \quad z = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)}. \end{aligned} \tag{3.5}$$

Suppose $q = (\Omega; x_1, x_2, x_3, x_4)$ is a nice quad, then we may extend the above definition to q via conformal image:

$$\begin{aligned} &\mathcal{Z}_{\kappa,\nu}(\Omega; x_1, x_2, x_3, x_4) \\ &= |\varphi'(x_1)|^h |\varphi'(x_2)|^b |\varphi'(x_3)|^b |\varphi'(x_4)|^h \mathcal{Z}_{\kappa,\nu}(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4)), \end{aligned} \tag{3.6}$$

where φ is any conformal map from Ω onto $\mathbb{H}$ such that $\varphi(x_1) < \varphi(x_2) < \varphi(x_3) < \varphi(x_4)$.

The process $\mathrm{hSLE}_\kappa(\nu)$ in $\mathbb{H}$ from x_1 to x_4 with marked points (x_2, x_3) is the Loewner chain driven by W_t which is the solution to the following SDEs:

$$\begin{aligned} dW_t &= \sqrt{\kappa} dB_t + \kappa(\partial_1 \log \mathcal{Z}_{\kappa,\nu})(W_t, g_t(x_2), g_t(x_3), g_t(x_4))dt, \\ \partial_t g_t(x_i) &= \frac{2}{g_t(x_i) - W_t}, \quad \text{for } i = 2, 3, 4. \end{aligned} \tag{3.7}$$

In particular, this implies that the law of η is the same as SLE_κ in $\mathbb{H}$ from x_1 to ∞ weighted by the following local martingale:

$$M_t = g_t'(x_2)^b g_t'(x_3)^b g_t'(x_4)^h \mathcal{Z}_{\kappa,\nu}(W_t, g_t(x_2), g_t(x_3), g_t(x_4)). \tag{3.8}$$

In the above definition, hSLE is well-defined up to the swallowing time of x_2. We will define the whole process in the following way.

- When $\kappa \in (0, 4]$ and $\nu > -4$, the process is well-defined for all times from (3.7); moreover, it is generated by a continuous curve. See Proposition 3.2.
- When $\kappa \in (4, 8)$ and $\nu > \kappa/2 - 6$, the process is well-defined up to and including the swallowing time of x_3 which is finite; moreover, it is generated by a continuous curve up to and including the same time. See Proposition 3.2. After the swallowing time of x_3, we continue the process as a standard SLE_κ towards x_4.
- When $\nu \le (-4) \vee (\kappa/2 - 6)$, the process is well-defined up to the swallowing time x_2 and we stop the process there. The process is generated by a continuous curve up to and including the same time, see Proposition 3.3.

As $\mathcal{Z}_{\kappa,\nu}$ in (3.5) is scaling covariant, hSLE in $\mathbb{H}$ is scaling invariant. hSLE in general quad is defined via conformal image. For any quad $q = (\Omega; x_1, x_2, x_3, x_4)$, $\mathrm{hSLE}_\kappa(\nu)$ in Ω from x_1 to x_4 with marked points (x_2, x_3) is $\varphi^{-1}(\eta)$ where φ is any conformal map from Ω onto $\mathbb{H}$ such that $\varphi(x_1) < \varphi(x_2) < \varphi(x_3) < \varphi(x_4)$ and η is an $\mathrm{hSLE}_\kappa(\nu)$ in $\mathbb{H}$ from $\varphi(x_1)$ to $\varphi(x_4)$ with marked points $(\varphi(x_2), \varphi(x_3))$.

Recall that we write hSLE_κ for $\mathrm{hSLE}_\kappa(0)$ with $\nu = 0$. When $\nu = 0$, the partition function defined in (3.5) is the same as the partition function for two SLEs defined in [KL07, Section 3.3] and in [Dub06, Section 4.1].

We end this section with a discussion on the phase transition of the two parameters κ and ν in the definition of hSLE. From (3.8), the partition function $\mathcal{Z}_{\kappa,\nu}$ gives the Radon-Nikodym derivative between $\text{hSLE}_\kappa(\nu)$ and standard SLE_κ. As a Radon-Nikodym derivative, it is important to understand whether it is positive and bounded. Thus, it is important to consider the positivity and bound of the hypergeometric function F in the definition of $\mathcal{Z}_{\kappa,\nu}$ in (3.5). As in the proof of Lemma 3.1, in order for F to be positive and bounded on $[0, 1]$, we need $C > 0, C > A, C > B, C > A + B$. This explains the phase transition for ν at $(-4) \vee (\kappa/2 - 6)$.

3.2. Continuity of hSLE. To derive the continuity of the process, it is more convenient to work with hSLE in $\mathbb{H}$ from 0 to ∞ with two marked points $0 < x < y$. The process $\text{hSLE}_\kappa(\nu)$ in $\mathbb{H}$ from 0 to ∞ with marked points (x, y) is the random Loewner chain driven by W which is the solution to the following system of SDEs:

$$\begin{aligned} dW_t &= \sqrt{\kappa} dB_t + \frac{(\nu+2)dt}{W_t - V_t^x} + \frac{-(\nu+2)dt}{W_t - V_t^y} - \kappa \frac{F'(Z_t)}{F(Z_t)} \left(\frac{1 - Z_t}{V_t^y - W_t} \right) dt, \\ dV_t^x &= \frac{2dt}{V_t^x - W_t}, \quad dV_t^y = \frac{2dt}{V_t^y - W_t}, \quad \text{where} \quad Z_t = \frac{V_t^x - W_t}{V_t^y - W_t}, \end{aligned} \tag{3.9}$$

where B_t is one-dimensional Brownian motion, and the initial values are $W_0 = 0$, $V_0^x = x$ and $V_0^y = y$. Denote by T_x the swallowing time of x and by T_y the swallowing time of y.

Proposition 3.2. *Fix $\kappa \in (0, 8)$, $\nu > (-4) \vee (\kappa/2 - 6)$ and $0 < x < y$. Consider* $\text{hSLE}_\kappa(\nu)$ *in $\mathbb{H}$ from 0 to ∞ with marked points (x, y) defined from (3.9).*

- *When $\kappa \in (0, 4]$, it is well-defined for all times. Moreover, it is generated by a continuous transient curve almost surely.*
- *When $\kappa \in (4, 8)$, it is well-defined up to T_y. Moreover, it is generated by a continuous curve up to and including T_y almost surely.*
- *When $\nu \geq \kappa/2 - 4$, it never hits the interval $[x, y]$ almost surely.*

Before proving Proposition 3.2, let us compare $\text{hSLE}_\kappa(\nu)$ with $\text{SLE}_\kappa(\nu+2, \kappa-6-\nu)$ process. By Lemma 2.9 and (3.8), the law of $\text{hSLE}_\kappa(\nu)$ with marked points (x, y) is the same as the law of $\text{SLE}_\kappa(\nu+2, \kappa-6-\nu)$ with force points (x, y) weighted by R_t/R_0 where

$$R_t = (g_t(y) - W_t)^{4/\kappa - 1} F(Z_t), \quad \text{and } Z_t = \frac{g_t(x) - W_t}{g_t(y) - W_t}.$$

Note that $0 \leq Z_t \leq 1$ for all t and $F(z)$ is bounded for $z \in [0, 1]$. For $n \geq 1$, define

$$S_n = \inf\{t : g_t(y) - W_t \leq 1/n \text{ or } g_t(y) - W_t \geq n\}.$$

Then R_{S_n} is bounded. Therefore, the law of $\text{hSLE}_\kappa(\nu)$ is absolutely continuous with respect to the law of $\text{SLE}_\kappa(\nu+2, \kappa-6-\nu)$ up to S_n. Since $\text{SLE}_\kappa(\nu+2, \kappa-6-\nu)$ is generated by a continuous curve up to T_y, $\text{hSLE}_\kappa(\nu)$ is generated by a continuous curve up to S_n. Let $n \to \infty$, $\text{hSLE}_\kappa(\nu)$ is generated by a continuous curve up to $T_y = \lim_n S_n$. However, the absolute continuity is not preserved as $n \to \infty$, since R_t may be no longer bounded away from 0 or ∞ as $t \to T_y$. Thus the difficulty in proving Proposition 3.2 is to analyze the behavior of $\text{hSLE}_\kappa(\nu)$ as $t \to T_y$.

Proof of Proposition 3.2. When $\kappa \in (0,8)$, $\nu > (-4) \vee (\kappa/2-6)$, the function $F(z)$ defined in (3.2) is uniformly bounded for $z \in [0,1]$ between $F(0)=1$ and $F(1) \in (0,\infty)$. Since $\mathrm{hSLE}_\kappa(\nu)$ is scaling invariant, we may assume $y=1$ and $x \in (0,1)$, and denote T_y by T. We will analyze the behavior of $\mathrm{hSLE}_\kappa(\nu)$ as $t \to T$. To this end, we perform a standard change of coordinates and parameterize the process according the capacity seen from the point 1, see [SW05, Theorem 3].

Set $f(z) = z/(1-z)$. Clearly, f is the Möbius transform of $\mathbb{H}$ sending the points $(0,1,\infty)$ to $(0,\infty,-1)$. Consider the image of $(K_t, 0 \le t \le T)$ under f, denoted by $(\tilde{K}_s, 0 \le s \le \tilde{S})$, where we parameterize this curve by its capacity $s(t)$ seen from ∞. Let $(\tilde{g}_s)$ be the corresponding family of conformal maps and $(\tilde{W}_s)$ be the driving function. Let f_t be the Möbius transform of $\mathbb{H}$ such that $\tilde{g}_s \circ f = f_t \circ g_t$ where $s = s(t)$. By expanding $\tilde{g}_s = f_t \circ g_t \circ f^{-1}$ around ∞ and comparing the coefficients in both sides, we have

$$f_t(z) = -1 - \frac{g_t''(1)}{2g_t'(1)} + \frac{g_t'(1)}{g_t(1) - z}.$$

Thus, with $s = s(t)$,

$$\tilde{W}_s = f_t(W_t) = -1 - \frac{g_t''(1)}{2g_t'(1)} + \frac{g_t'(1)}{g_t(1) - W_t},$$
$$d\tilde{W}_s = \frac{(\kappa-6)g_t'(1)dt}{(g_t(1)-W_t)^3} + \frac{g_t'(1)dW_t}{(g_t(1)-W_t)^2}.$$

Define

$$\tilde{V}_s^x = \tilde{g}_s(\tilde{x}) = f_t(V_t^x), \quad \tilde{V}_s^\infty = \tilde{g}_s(-1) = f_t(\infty), \quad \tilde{Z}_s = \frac{\tilde{V}_s^x - \tilde{W}_s}{\tilde{V}_s^x - \tilde{V}_s^\infty} = Z_t.$$

Plugging in the time change

$$\dot{s}(t) = f_t'(W_t)^2 = \frac{g_t'(1)^2}{(g_t(1)-W_t)^4},$$

we obtain

$$d\tilde{W}_s = \sqrt{\kappa}d\tilde{B}_s + \frac{(\nu+2)ds}{\tilde{W}_s - \tilde{V}_s^x} + \frac{(\kappa-6)ds}{\tilde{W}_s - \tilde{V}_s^\infty} - \kappa \frac{F'(\tilde{Z}_s)}{F(\tilde{Z}_s)} \frac{ds}{\tilde{V}_s^x - \tilde{V}_s^\infty},$$

where $\tilde{B}_s$ is one-dimensional Brownian motion. By Girsanov's Theorem, the law of $\tilde{K}$ is the law of $\mathrm{SLE}_\kappa(\kappa-6;\nu+2)$ with force points $(-1; \tilde{x} := x/(1-x))$ weighted by R_s/R_0 where

$$R_s = F(\tilde{Z}_s)\,(\tilde{g}_s(\tilde{x}) - \tilde{g}_s(-1))^{a(4-\kappa)/2}, \quad \text{and } \tilde{Z}_s = \frac{\tilde{g}_s(\tilde{x}) - \tilde{W}_s}{\tilde{g}_s(\tilde{x}) - \tilde{g}_s(-1)}.$$

Note that $0 \le \tilde{Z}_s \le 1$ and $F(z)$ is bounded for $z \in [0,1]$; and that the process $\tilde{g}_s(\tilde{x}) - \tilde{g}_s(-1)$ is increasing, thus $\tilde{g}_s(\tilde{x}) - \tilde{g}_s(-1) \ge 1/(1-x)$. Let $\tilde{S}$ be the swallowing time of -1. Define, for $n \ge 1$,

$$\tilde{S}_n = \inf\{t : \tilde{K}_t \text{ exits } B(0,n)\}.$$

Then R_s is bounded up to $\tilde{S} \wedge \tilde{S}_n$. The process $\mathrm{SLE}_\kappa(\kappa-6;\nu+2)$ with force points $(-1; \tilde{x} = x/(1-x))$ is generated by a continuous curve up to and including the continuation threshold. Moreover, it has the following properties.

(a) When $\kappa \in (0, 4]$, since $\kappa - 6 \le \kappa/2 - 4$ and $\nu + 2 > -2$, the curve accumulates at the point -1 in finite time almost surely.
(b) When $\kappa \in (4, 8)$, since $\kappa - 6 \in (-2, \kappa/2-2)$ and $\nu+2 > -2$, the curve accumulates at a point on the interval $(-\infty, -1)$ in finite time almost surely.
(c) When $\nu \ge \kappa/2 - 4$, since $\nu + 2 \ge \kappa/2 - 2$, the curve does not hit the interval $[\tilde{x}, \infty)$ almost surely.

From items (a) and (b), $\lim_{n\to\infty} \tilde{S} \wedge \tilde{S}_n = \tilde{S} < \infty$. Thus $\tilde{K}$ is generated by a continuous curve up to and including $\tilde{S}$. This implies that our original $\mathrm{hSLE}_\kappa(\nu)$ process $(K_t, t \ge 0)$ is generated by a continuous curve up to and including T. When $\kappa \le 4$, $\mathrm{hSLE}_\kappa(\nu)$ process goes to ∞ without touching the interval $[1, \infty)$, thus $T = \infty$. When $\kappa \in (4, 8)$, $\mathrm{hSLE}_\kappa(\nu)$ process accumulates at a point on the interval $(1, \infty)$ in finite time, thus $T < \infty$. From item (c), $\mathrm{hSLE}_\kappa(\nu)$ process does not hit the interval $[x, 1]$ when $\nu \ge \kappa/2 - 4$. □

Proposition 3.3. *Fix $\kappa \in (0, 8)$ and $\nu \le (-4) \vee (\kappa/2 - 6)$ and $0 < x < y$. $\mathrm{hSLE}_\kappa(\nu)$ in $\mathbb{H}$ from 0 to ∞ with marked points (x, y) is well-defined up to T_x. Moreover, it is generated by a continuous curve up to and including T_x, and it accumulates at a point on $[x, y)$ as $t \to T_x$.*

Proof. Suppose $\tilde{\eta}$ is an $\mathrm{SLE}_\kappa(\nu + 2, \nu + 2)$ in $\mathbb{H}$ from 0 to ∞ with force points (x, y). The law of η is the same as the law of $\tilde{\eta}$ weighted by R_t/R_0 where

$$R_t = (g_t(y) - W_t)^{a(\kappa/2-6-\nu)}(1 - Z_t)^{8/\kappa-1-a(\nu+4-\kappa/2)}G(1 - Z_t), \quad \text{and}$$
$$Z_t = \frac{g_t(x) - W_t}{g_t(y) - W_t}.$$

Here G is defined in (3.3). For $n \ge 1$, define S_n to be the minimum of

$$\inf\{t : \tilde{\eta}(t) \text{ exits } B(0, n)\}, \quad \text{and} \quad \inf\{t : g_t(y) - g_t(x) \le 1/n\}.$$

Then $R_{T_x \wedge S_n}$ is bounded. Thus η is continuous up to $T_x \wedge S_n$.

First, we assume $\kappa \in (4, 8)$ and $\nu \le \kappa/2 - 6$. Since $\nu + 2 \le \kappa/2 - 4$ and $2\nu + 4 \le \kappa/2 - 4$, the process $\tilde{\eta}$ accumulates at the point x as $t \to T_x$ (see [Dub09, Lemma 15]). Combining the fact that it is generated by a continuous curve up to and including T_x, we have $R_{T_x \wedge S_n} \to R_{T_x} \in (0, \infty)$ as $n \to \infty$. Therefore, η is generated by a continuous curve up to and including T_x and it accumulates at the point x as $t \to T_x$.

Next, we assume $\kappa \in (0, 4]$ and $\nu \le -4$. Since $\nu+2 < \kappa/2-2$ and $2\nu+4 \le \kappa/2-4$, the process $\tilde{\eta}$ accumulates at a point on $[x, y]$ as $t \to T_x$ (see [Dub09, Lemma 15]). In fact, we can further derive that $\tilde{\eta}$ accumulates at a point on $[x, y)$ as $t \to T_x$. Let φ be the Möbius transform of $\mathbb{H}$ sending the triple $(0, x, y)$ to $(0, x, \infty)$. Then the law of $\varphi(\tilde{\eta})$ is $\mathrm{SLE}_\kappa(\kappa - 10 - 2\nu; \nu + 2)$ from 0 to ∞ with force points $(\varphi(\infty); x)$. Since $\kappa - 10 - 2\nu \ge \kappa/2 - 2$ and $\nu + 2 < \kappa/2 - 2$, the curve $\varphi(\tilde{\eta})$ almost surely hits $[x, \infty)$ before reaching ∞. This implies that $\tilde{\eta}$ accumulates at a point on $[x, y)$ as $t \to T_x$. Combing the fact that it is generated by a continuous curve up to and including T_x, we have $R_{T_x \wedge S_n} \to R_{T_x} \in (0, \infty)$ as $n \to \infty$. Therefore, η is generated by a continuous curve up to and including T_x and it accumulates at a point on $[x, y)$. This completes the proof. □

3.3. Reversibility of hSLE. In this section, we still work with hSLE in $\mathbb{H}$ from 0 to ∞. In this case, the local martingale in (3.8) has a more explicit expression.

Lemma 3.4. *Fix $\kappa \in (0, 8)$, $\nu \in \mathbb{R}$ and $0 < x < y$. Suppose η is an SLE_κ in $\mathbb{H}$ from 0 to ∞ and $(g_t, t \geq 0)$ is the corresponding family of conformal maps. Let T_x be the swallowing time of x. Define, for $t < T_x$,*

$$J_t = \frac{g_t'(x)g_t'(y)}{(g_t(y) - g_t(x))^2}, \quad Z_t = \frac{g_t(x) - W_t}{g_t(y) - W_t}.$$

Then the following process is a local martingale:

$$M_t := Z_t^a J_t^b F(Z_t) \mathbb{1}_{\{t<T_x\}},$$

where a, b are defined through (3.4) and F is defined through (3.2) or (3.3).

Proposition 3.5. *Fix $\kappa \in (0, 8)$, $\nu \geq \kappa/2 - 4$ and $0 < x < y$. The local martingale defined in Lemma 3.4 is a uniformly integrable martingale for η; and the law of η weighted by M_∞ is the same as $\mathrm{hSLE}_\kappa(\nu)$ with marked points (x, y). Furthermore,*

$$M_\infty = (H_D(x, y))^b \, \mathbb{1}_{\{\eta\cap[x,y]=\emptyset\}},$$

where D is the connected component of $\mathbb{H}\backslash\eta$ with (xy) on the boundary, and $H_D(x, y)$ is the boundary Poisson kernel.[1]

Proof. We first argue that M_t is a uniformly integrable martingale. Note that J_t is decreasing in t, thus $J_t \leq J_0$. Therefore M_t is bounded as long as J_t and Z_t are bounded from below. Define, for $n \geq 1$,

$$S_n = \inf\{t : J_t \leq 1/n \text{ or } Z_t \leq 1/n\}.$$

Denote by $\mathbb{P}$ the law of η. Define $\mathbb{P}_n^*$ by $d\mathbb{P}_n^*/d\mathbb{P} = M_{S_n}/M_0$. Then $\mathbb{P}_n^*$ is the same as $\mathrm{hSLE}_\kappa(\nu)$ up to S_n. Since $\{\mathbb{P}_n^*\}_n$ are compatible in n, there exists a probability $\mathbb{P}^*$ such that, under $\mathbb{P}^*$, and for each n, the process is the same as $\mathrm{hSLE}_\kappa(\nu)$ up to S_n. By Proposition 3.2, $\mathrm{hSLE}_\kappa(\nu)$ is generated by a continuous transient curve and the curve never hits the interval $[x, y]$ when $\nu \geq \kappa/2 - 4$. Hence $\mathbb{P}^*$ is the same as the law of $\mathrm{hSLE}_\kappa(\nu)$. This implies that M_t is a uniformly integrable martingale.

It remains to derive the explicit expression of M_∞. As $t \to \infty$, we find

$$Z_t \to 1, \quad J_t \to J_\infty := \frac{g'(x)g'(y)}{(g(y) - g(x))^2},$$

where g is any conformal map from D onto $\mathbb{H}$. In fact, the quantity J_∞ is the boundary Poisson kernel $H_D(x, y)$. Thus we have almost surely $M_\infty = \lim_{t\to\infty} M_t = H_D(x, y)^b$ as desired. □

Proof of Theorem 1.1. We have shown that $\mathrm{hSLE}_\kappa(\nu)$ is generated by a continuous transient curve in Proposition 3.2. Thus, to show Theorem 1.1, it remains to show the reversibility when $\nu \geq \kappa/2 - 4$. By Proposition 3.5, the Radon-Nikodym derivative of the law of $\mathrm{hSLE}_\kappa(\nu)$ with marked points (x, y) with respect to the law of SLE_κ is given by M_∞/M_0 where M_∞ is the boundary Poison kernel to the power b. Combining the reversibility of standard SLE_κ and the conformal invariance of the boundary Poisson kernel, we have the reversibility of $\mathrm{hSLE}_\kappa(\nu)$. □

[1] Fix a nice Dobrushin domain $(\Omega; x, y)$. The boundary Poisson kernel $H_\Omega(x, y)$ is a conformally covariant function which, in $\mathbb{H}$ with $x, y \in \mathbb{R}$ is given by $H_{\mathbb{H}}(x, y) = |y - x|^{-2}$, and in Ω it is defined via conformal image: we may set $H_\Omega(x, y) = |\varphi'(x)\varphi'(y)| H_{\mathbb{H}}(\varphi(x), \varphi(y))$ for any conformal map $\varphi : \Omega \to \mathbb{H}$.

From the above analysis, we obtain the reversibility of $\mathrm{hSLE}_\kappa(\nu)$ for $\nu \geq \kappa/2-4$. In fact, we believe the reversibility holds for all $\nu > (-4)\vee(\kappa/2-6)$.

Conjecture 3.6. Fix $\kappa \in (0,8)$ and $\nu > (-4)\vee(\kappa/2-6)$ and a quad $(\Omega; x_1, x_2, x_3, x_4)$. Let η be an $\mathrm{hSLE}_\kappa(\nu)$ in Ω from x_1 to x_4 with marked points (x_2, x_3). The time-reversal of η has the same law as $\mathrm{hSLE}_\kappa(\nu)$ in Ω from x_4 to x_1 with marked points (x_3, x_2).

3.4. Relation to $\mathrm{SLE}_\kappa(\rho)$. In the following lemma, we explain the relation between $\mathrm{hSLE}_\kappa(\nu)$ and $\mathrm{SLE}_\kappa(\rho)$.

Lemma 3.7. *Fix $\kappa \in (0,8)$, $\nu \in \mathbb{R}$ and $x_1 < x_2 < x_3 < x_4$. When $x_3 \to x_4$, the process* $\mathrm{hSLE}_\kappa(\nu)$ *in $\mathbb{H}$ from x_1 to x_4 with marked points (x_2, x_3) converges weakly to* $\mathrm{SLE}_\kappa(\nu+2)$ *in $\mathbb{H}$ from x_1 to x_4 with force point x_2.*

Proof. We may assume $x_1 = 0$ and $x_4 = \infty$, and the two marked points are $0 < x < y$. Let η be $\mathrm{hSLE}_\kappa(\nu)$ in $\mathbb{H}$ from 0 to ∞ with marked points (x, y). Let $\tilde{\eta}$ be $\mathrm{SLE}_\kappa(\nu+2)$ in $\mathbb{H}$ from 0 to ∞ with force point x. The law of η is the same as the law of $\tilde{\eta}$ weighted by the following Radon-Nikodym derivative

$$\frac{R_t}{R_0} = g_t'(y)^b \left(\frac{g_t(y)-W_t}{y}\right)^{-a} \frac{F(Z_t)}{F(Z_0)}, \quad \text{where} \quad Z_t = \frac{g_t(x)-W_t}{g_t(y)-W_t},$$

and F is the function in (3.2) or (3.3).

Let T be the continuation threshold of $\tilde{\eta}$. For $n \geq 1$, let S_n be the first time that $\tilde{\eta}$ exits the ball $B(0,n)$. Fix n, and let $y \to \infty$, we see that $Z_0, Z_{T\wedge S_n} \to 0$ and $R_{T\wedge S_n}/R_0 \to 1$. Furthermore, $R_{T\wedge S_n}/R_0$ is uniformly bounded when y is large enough. Thus, for any fixed $n \geq 1$, the law of η up to $T \wedge S_n$ converges weakly to the law of $\tilde{\eta}$ up to the same time. This gives the conclusion. □

The special case: $\kappa = 4$

When $\kappa = 4$, the hypergeometric SLE process degenerates. When $\nu > -4$, we have $F \equiv 1$ in (3.2). From (3.9), it is clear that $\mathrm{hSLE}_4(\nu)$ is the same as $\mathrm{SLE}_4(\nu+2, -\nu-2)$. When $\nu \leq -4$, although $\mathrm{hSLE}_4(\nu)$ is distinct from $\mathrm{SLE}_4(\nu+2, -\nu-2)$ in this case, they are still closely related. To explain the relation, we first do a calculation with $\mathrm{SLE}_4(\nu+2, -\nu-2)$.

Suppose $\tilde{\eta}$ is an $\mathrm{SLE}_4(\nu+2, -\nu-2)$ in $\mathbb{H}$ from x_1 to x_4 with force points (x_2, x_3). In this case, the process $\tilde{\eta}$ can be viewed as the level line of Gaussisan Free Field with the following boundary data ($\lambda = \pi/2$): (see [SS13,WW17])

$$\begin{aligned} &-\lambda \text{ on } (-\infty, x_1), \quad \lambda \text{ on } (x_1, x_2), \quad \lambda(\nu+3) \text{ on } (x_2, x_3), \\ &\lambda \text{ on } (x_3, x_4), \quad -\lambda \text{ on } (x_4, \infty). \end{aligned}$$

In particular, the process $\tilde{\eta}$ is generated by a continuous curve up to and including the continuation threshold, denoted by T. When $\nu+3 \leq -1$, the curve $\tilde{\eta}$ may terminate at either x_2 or x_4. Furthermore, we can calculate the probabilities of these two events.

Lemma 3.8. *Fix $\nu \leq -4$ and set $\alpha = -(\nu+2)/2 \geq 1$. Suppose $\tilde{\eta}$ is an* $\mathrm{SLE}_4(\nu+2, -\nu-2)$ *in $\mathbb{H}$ from x_1 to x_4 with force points (x_2, x_3). Let T be its continuation threshold. We have*

$$\mathbb{P}[\tilde{\eta}(T) = x_2] = 1 - z^\alpha, \quad \textit{and} \quad \mathbb{P}[\tilde{\eta}(T) = x_4] = z^\alpha, \quad \textit{where} \quad z = \frac{(x_2-x_1)(x_4-x_3)}{(x_3-x_1)(x_4-x_2)}.$$

Proof. Recall that the driving function of $\tilde{\eta}$ satisfies the following:

$$dW_t = 2dB_t + \frac{-(\nu+2)dt}{g_t(x_2) - W_t} + \frac{(\nu+2)dt}{g_t(x_3) - W_t} + \frac{2dt}{g_t(x_4) - W_t}.$$

Define

$$M_t = z_t^{\alpha}, \quad \text{where} \quad z_t = \frac{(g_t(x_2) - W_t)(g_t(x_4) - g_t(x_3))}{(g_t(x_3) - W_t)(g_t(x_4) - g_t(x_2))}. \tag{3.10}$$

By Itô's Formula, one can check that M_t is a local martingale for $\tilde{\eta}$. We see that, as $t \to T$,

$$M_t \to 0, \quad \text{if } \tilde{\eta}(t) \to x_2; \quad \text{and} \quad M_t \to 1, \quad \text{if } \tilde{\eta}(t) \to x_4.$$

Note that $0 \le M_t \le 1$. Thus Optional Stopping Theorem implies that

$$\mathbb{P}[\tilde{\eta}(T) = x_4] = \mathbb{E}[M_T] = M_0 = z^{\alpha}.$$

This gives the conclusion. □

When $\nu \le -4$, from (3.3), we have $F(z) = 1 - z^{\alpha}$. In this case, the law of $\text{hSLE}_4(\nu)$ in $\mathbb{H}$ from x_1 to x_4 with marked points (x_2, x_3) is the same as $\tilde{\eta}$ weighted by $1 - M_t$ where M_t is the martingale defined in (3.10). Equivalently, the law of $\text{hSLE}_4(\nu)$ is the same as $\tilde{\eta}$ conditioned on the event $\{\tilde{\eta}(T) = x_2\}$.

3.5. Relation between different hSLE*'s.* Recall that hSLE in general quad $(\Omega; x_1, x_2, x_3, x_4)$ is defined via conformal image as in the end of Sect. 3.1. Denote by $\mathbb{P}_{\kappa,\nu}(\Omega; x_1, x_2, x_3, x_4)$ the law of $\text{hSLE}_\kappa(\nu)$ in Ω from x_1 to x_4 with marked points (x_2, x_3). Proposition 3.9 derives the relation between hSLEs with different ν's. Proposition 3.10 derives the relation between hSLEs in different domains. Proposition 3.11 derives the relation between hSLEs with different target points.

Proposition 3.9. *Fix $\kappa \in (0, 8)$, $\nu \in \mathbb{R}$ and a quad $(\Omega; x_1, x_2, x_3, x_4)$. When $\nu \ge \kappa/2 - 4$, we have $\eta \cap [x_2x_3] = \emptyset$ almost surely. When $(-4) \vee (\kappa/2 - 6) < \nu < \kappa/2 - 4$, the event $\{\eta \cap [x_2x_3] = \emptyset\}$ has positive chance which is given by*

$$\frac{\mathcal{Z}_{\kappa,\kappa-8-\nu}(\Omega; x_1, x_2, x_3, x_4)\Gamma((2\nu+8)/\kappa)\Gamma((\kappa-4-2\nu)/\kappa)}{\mathcal{Z}_{\kappa,\nu}(\Omega; x_1, x_2, x_3, x_4)\Gamma((2\nu+12-\kappa)/\kappa)\Gamma((2\kappa-8-2\nu)/\kappa)}. \tag{3.11}$$

Moreover, for $(-4) \vee (\kappa/2 - 6) < \nu < \kappa/2 - 4$, we have

$$\mathbb{P}_{\kappa,\nu}(\Omega; x_1, x_2, x_3, x_4)[\cdot \mid \eta \cap [x_2x_3] = \emptyset] = \mathbb{P}_{\kappa,\kappa-8-\nu}(\Omega; x_1, x_2, x_3, x_4)[\cdot].$$

In particular, when $\kappa \in (4, 8)$, the law of SLE_κ *from x_1 to x_4 conditioned to avoid (x_2x_3) is the same as* $\text{hSLE}_\kappa(\kappa - 6)$ *from x_1 to x_4 with marked points (x_2, x_3).*

Proof. We may assume $\Omega = \mathbb{H}$ and $x_1 = 0 < x_2 = x < x_3 = y < x_4 = \infty$. Let η be an $\text{hSLE}_\kappa(\nu)$ from 0 to ∞ with marked points (x, y). Denote by T_x the swallowing time of x and by T_y the swallowing time of y. The fact that $\eta \cap [x, y] = \emptyset$ when $\nu \ge \kappa/2 - 4$ is proved in Proposition 3.2. In the following, we assume $(-4) \vee (\kappa/2 - 6) < \nu < \kappa/2 - 4$.

Set $\hat{\nu} = \kappa - 8 - \nu$ and $\hat{a} = (\hat{\nu} + 2)/\kappa$, and let $\hat{\eta}$ be an hSLE$_\kappa(\hat{\nu})$ from 0 to ∞ with marked points (x, y). The following process is a local martingale for η:

$$M_t = z_t^{\hat{a}-a} \frac{\hat{F}(z_t)}{F(z_t)} \mathbb{1}_{\{t<T_x\}}, \quad \text{where} \quad Z_t = \frac{g_t(x) - W_t}{g_t(y) - W_t},$$

and F is defined through (3.2) and

$$\hat{F}(z) = {}_2F_1\left(\frac{2(\kappa - 6 - \nu)}{\kappa}, 1 - \frac{4}{\kappa}, \frac{2(\kappa - 4 - \nu)}{\kappa}; z\right).$$

Moreover, the law of η weighted by M is the same as $\hat{\eta}$ up to T_x. Since $\hat{\nu} \geq \kappa/2 - 4$, $\hat{\eta}$ does not hit the closed interval $[x, y]$ and thus $T_x = T_y$. Using a similar argument as in the proof of Proposition 3.5, M is a uniformly integrable martingale for η. As $t \to T_y$, we have $Z_t \to 1$. Thus the law of $\hat{\eta}$ is the same as η weighted by $\mathbb{1}_{\{\eta\cap[x,y]=\emptyset\}}$. In particular, we have

$$\mathbb{P}[\eta \cap [x, y] = \emptyset] = z^{\hat{a}-a} \frac{\hat{F}(z)/\hat{F}(1)}{F(z)/F(1)}, \quad \text{where} \quad z = \frac{x}{y}.$$

This gives (3.11). □

Next, we derive the boundary perturbation property of the hSLE$_\kappa(\nu)$, which is a generalization of the boundary perturbation property of SLE$_\kappa$ derived in [LSW03, Section 5]. Suppose $\hat{\Omega} \subset \Omega$ such that $\hat{\Omega}$ is simply connected and agrees with Ω in a neighborhood of the arc (x_1x_4). In the following proposition, we will derive the relation between $\mathbb{P}_{\kappa,\nu}(\hat{\Omega}; x_1, x_2, x_3, x_4)$ and $\mathbb{P}_{\kappa,\nu}(\Omega; x_1, x_2, x_3, x_4)$. To this end, we need to introduce Brownian loop measure.

The *Brownian loop measure* is a conformally invariant measure on unrooted Brownian loops in the plane. In the present article, we will not need the precise definition of this measure, so we content ourselves with referring to the literature for the definition: see, e.g., [Law09] or [LW04, Sections 3 and 4]. Given a non-empty simply connected domain $\Omega \subsetneq \mathbb{C}$ and two disjoint subsets $V_1, V_2 \subset \Omega$, we denote by $\mu(\Omega; V_1, V_2)$ the Brownian loop measure of loops in Ω that intersect both V_1 and V_2. This quantity is conformally invariant: $\mu(\varphi(\Omega); \varphi(V_1), \varphi(V_2)) = \mu(\Omega; V_1, V_2)$ for any conformal transformation $\varphi : \Omega \to \varphi(\Omega)$. In general, the Brownian loop measure is an infinite measure. By [Law09, Corollary 4.6], we have $0 \leq \mu(\Omega; V_1, V_2) < \infty$ when both of V_1, V_2 are closed, one of them is compact, and $\mathrm{dist}(V_1, V_2) > 0$.

Proposition 3.10. *Fix $\kappa \in (0, 4]$, $\nu > -4$ and a quad $(\Omega; x_1, x_2, x_3, x_4)$. Assume that $\hat{\Omega} \subset \Omega$ is simply connected and it agrees with Ω in a neighbourhood of the arc (x_1x_4). Then* hSLE$_\kappa(\nu)$ *in $\hat{\Omega}$ is absolutely continuous with respect to* hSLE$_\kappa(\nu)$ *in Ω, and the Radon-Nikodym derivative is given by*

$$\frac{d\mathbb{P}_{\kappa,\nu}(\hat{\Omega}; x_1, x_2, x_3, x_4)}{d\mathbb{P}_{\kappa,\nu}(\Omega; x_1, x_2, x_3, x_4)} = \frac{\mathcal{Z}_{\kappa,\nu}(\Omega; x_1, x_2, x_3, x_4)}{\mathcal{Z}_{\kappa,\nu}(\hat{\Omega}; x_1, x_2, x_3, x_4)} \mathbb{1}_{\{\eta\subset\hat{\Omega}\}} \exp(c\mu(\Omega; \eta, \Omega\backslash\hat{\Omega})),$$

where $c = (3\kappa - 8)(6 - \kappa)/(2\kappa)$.

When $\nu = 0$, the same conclusion appeared in [KL07, Section 3].

Proof. We may assume $\Omega = \mathbb{H}$ and $x_1 = 0 < x_2 = x < x_3 = y < x_4 = \infty$. Let ϕ be the conformal map from $\hat{\Omega}$ onto $\mathbb{H}$ such that $\phi(0) = 0$ and $\lim_{z\to\infty} \phi(z)/z = 1$. Suppose η (resp. $\hat{\eta}$) is $\mathrm{hSLE}_\kappa(\nu)$ in $\mathbb{H}$ (resp. in $\hat{\Omega}$) from 0 to ∞ with marked points (x, y). Let $(g_t, t \geq 0)$ be the corresponding family of conformal maps, and V_t^x, V_t^y are the evolutions of x, y respectively. Let T be the first time when η exits $\hat{\Omega}$. We will study the law of $\tilde{\eta}(t) = \phi(\eta(t))$ for $t < T$. Define $\tilde{g}_t$ to be the conformal map from $\mathbb{H}\backslash\tilde{\eta}[0, t]$ onto $\mathbb{H}$ normalized at ∞ and let φ_t be the conformal map from $\mathbb{H}\backslash g_t(K)$ onto $\mathbb{H}$ such that $\varphi_t \circ g_t = \tilde{g}_t \circ \phi$. One can check that the following process is a local martingale for η:

$$\begin{aligned} M_t :=& \mathbb{1}_{\{t<T\}} \varphi_t'(W_t)^h \varphi_t'(V_t^x)^b \varphi_t'(V_t^y)^b \\ & \left(\frac{\varphi_t(V_t^y) - \varphi_t(V_t^x)}{V_t^y - V_t^x}\right)^{-2b} \exp(c\mu(\mathbb{H}; \eta[0, t], \mathbb{H}\backslash\hat{\Omega})) \\ & \times \left(\frac{\varphi_t(V_t^x) - \varphi_t(W_t)}{\varphi_t(V_t^y) - \varphi_t(W_t)} \frac{V_t^y - W_t}{V_t^x - W_t}\right)^a \times \frac{F\left(\frac{\varphi_t(V_t^x)-\varphi_t(W_t)}{\varphi_t(V_t^y)-\varphi_t(W_t)}\right)}{F\left(\frac{V_t^x-W_t}{V_t^y-W_t}\right)}, \end{aligned}$$

where a, b, h are defined through (3.4) and F is defined through (3.2). Moreover, the law of η weighted by M is the same as $\hat{\eta}$ up to T. Since $\kappa \leq 4$, the process $\hat{\eta}$ never exits $\hat{\Omega}$ and goes to ∞. Using a similar argument as in the proof of Proposition 3.5, M is a uniformly integrable martingale for η and the law of η weighted by M_T/M_0 is the same as $\hat{\eta}$ where

$$M_T := \lim_{t\to T} M_t = \mathbb{1}_{\{\eta\subset\hat{\Omega}\}} \exp(c\mu(\mathbb{H}; \eta, \mathbb{H}\backslash\hat{\Omega})).$$

This completes the proof. □

Proposition 3.11. *Fix* $\kappa \in (0, 8)$ *and a quad* $(\Omega; x_1, x_2, x_3, x_4)$. *Let* η *be an* $\mathrm{hSLE}_\kappa(\kappa - 8)$ *in* Ω *from* x_1 *to* x_4 *with marked points* (x_2, x_3). *Let* $\tilde{\eta}$ *be an* hSLE_κ *in* Ω *from* x_1 *to* x_2 *with marked points* (x_4, x_3). *Then* $\tilde{\eta}$ *(up to the first hitting time of* $[x_2x_3]$*) has the same law as* η *conditioned to hit* $[x_2x_3]$ *(up to the first hitting time of* $[x_2x_3]$*).*

Proof. We may assume $\Omega = \mathbb{H}$ and $x_1 < x_2 < x_3 < x_4$. For η, let T be its swallowing time of x_2. Denote $X_{j1} = g_t(x_j) - W_t$ for $2 \leq j \leq 4$ and $X_{ij} = g_t(x_j) - g_t(x_i)$ for $2 \leq i < j \leq 4$. When $\nu = \kappa - 8$, we have

$$a = \frac{\nu + 2}{\kappa} = -2h, \quad b = \frac{(\nu + 2)(\nu + 6 - \kappa)}{4\kappa} = h.$$

First, we assume $\kappa \in (4, 8)$. In this case, we have $\kappa - 8 > \kappa/2 - 6$. Define

$$F(z) := {}_2F_1\left(2 - \frac{12}{\kappa}, 1 - \frac{4}{\kappa}, 2 - \frac{8}{\kappa}; z\right), \quad \tilde{F}(z) := {}_2F_1\left(\frac{4}{\kappa}, 1 - \frac{4}{\kappa}, \frac{8}{\kappa}; z\right).$$

In this case, both F and $\tilde{F}$ are bounded for $z \in [0, 1]$. The law of η is the same as SLE_κ in $\mathbb{H}$ from x_1 to ∞ weighted by the following local martingale:

$$M_t = g_t'(x_2)^h g_t'(x_3)^h g_t'(x_4)^h X_{41}^{-2h} X_{32}^{-2h} z_t^{-2h} F(z_t), \quad \text{where} \quad z_t = \frac{X_{21}X_{43}}{X_{31}X_{42}}.$$

The law of $\tilde{\eta}$ is the same as SLE_κ in $\mathbb{H}$ from x_1 to ∞ weighted by the following local martingale:

$$\tilde{M}_t = g_t'(x_2)^h g_t'(x_3)^h g_t'(x_4)^h X_{21}^{-2h} X_{43}^{-2h} s_t^{2/\kappa} \tilde{F}(s_t), \quad \text{where} \quad s_t = \frac{X_{32}X_{41}}{X_{31}X_{42}}.$$

Comparing these two local martingales, we see that the law of $\tilde{\eta}$ is the same as η weighted by the following local martingale up to T:

$$N_t = (1-z_t)^{8/\kappa-1}\frac{\tilde{F}(1-z_t)}{F(z_t)}.$$

We have the following observation.

- On the event $\{\eta \cap [x_2, x_3] \neq \emptyset\}$, the curve η accumulates at a point on (x_2, x_3) as $t \to T$. Thus $z_t \to 0$ as $t \to T$.
- On the event $\{\eta \cap [x_2, x_3] = \emptyset\}$, the curve η accumulates at a point on (x_3, x_4) as $t \to T$. Thus $z_t \to 1$ as $t \to T$.

Combining these two facts, we see that the law of $\tilde{\eta}$ (up to the first hitting time of $[x_2, x_3]$) is the same as η conditioned on $\{\eta \cap [x_2, x_3] \neq \emptyset\}$ (up to the first hitting time of $[x_2, x_3]$).

Next, we assume $\kappa \in (0, 4]$. In this case, we have $\kappa - 8 \leq -4$. Define

$$G(z) := {}_2F_1\left(1-\frac{4}{\kappa}, \frac{4}{\kappa}, \frac{8}{\kappa}; z\right).$$

The law of η is the same as SLE_κ in $\mathbb{H}$ from x_1 to ∞ weighted by the following local martingale up to T:

$$M_t = g_t'(x_2)^h g_t'(x_3)^h g_t'(x_4)^h X_{41}^{-2h} X_{32}^{-2h} z_t^{-2h}(1-z_t)^{8/\kappa-1} G(1-z_t).$$

Therefore, the law of $\tilde{\eta}$ is the same as η weighted by the following local martingale up to T:

$$N_t = \frac{\tilde{F}(1-z_t)}{G(1-z_t)}.$$

With the analysis in the end the proof of Proposition 3.3, since $\nu + 2 = \kappa - 6 \leq \kappa/2 - 4$, the curve η accumulates at the point x_2 almost surely. Thus $z_t \to 0$ as $t \to T$ almost surely. Therefore, the law of $\tilde{\eta}$ is the same as η. □

We end this section with a discussion on the definition of hSLE. In the definition of hSLE in Sect. 3.1, it is important that the process in (3.8) is a local martingale. This is equivalent to that the function F in (3.5) needs to satisfy (3.1). Whereas, there is a two-dimensional solution space for (3.1). The readers may wonder why we choose the particular solution as in (3.2) or (3.3). Indeed, there is freedom in choosing F as long as it is in the solution space. But we choose the one as in (3.2) or (3.3) with the following consideration.

When $\nu > (-4) \vee (\kappa/2-6)$, we choose F as in (3.2). First of all, when $\nu = 0$, this is consistent with the hypergeometric SLE discussed in [KL07, Section 3.3] and in [Dub06, Section 4.1]. Second, it is consistent with the definition of $\mathrm{SLE}_\kappa(\rho)$ in the following sense: it is believed that the time-reversal of $\mathrm{SLE}_\kappa(\rho)$ when $\rho > (-2) \vee (\kappa/2-4)$ is $\mathrm{hSLE}_\kappa(\rho-2)$, as in Conjecture 3.6.

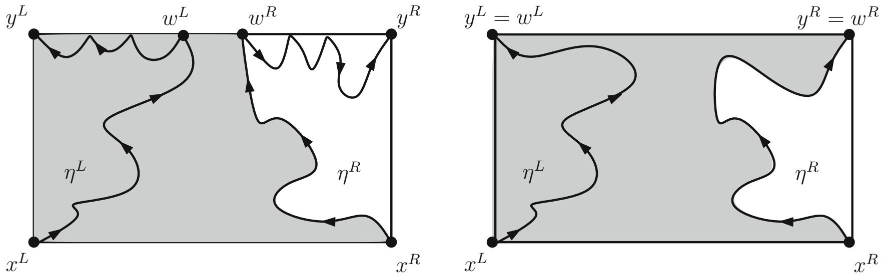

Fig. 3. Fix a quad $q = (\Omega; x^R, y^R, y^L, x^L)$ and consider disjoint continuous simple curves $(\eta^L; \eta^R)$ in the space $X_0(\Omega; x^R, y^R, y^L, x^L)$. Let T^R be the first time that η^R hits the closed arc $[y^R y^L]$, and denote by w^R the point $\eta^R(T^R)$, and by Ω^R the connected component of $\Omega \backslash \eta^R[0, T^R]$ with $(y^L x^L)$ on the boundary. In the figure, the gray part indicates Ω^R. Note that T^R, w^R, Ω^R are deterministic functions of η^R. Let T^L be the first time that η^L hits the closed arc $[y^R y^L]$, and denote by w^L the point $\eta^L(T^L)$, and by Ω^L the connected component of $\Omega \backslash \eta^L[0, T^L]$ with $(x^R y^R)$ on the boundary

When $\nu \leq (-4) \vee (\kappa/2 - 6)$, we choose F as in (3.3) with the following reason: the corresponding $\mathrm{hSLE}_\kappa(\nu)$ process accumulates at a point on the interval $[x_2, x_3)$ almost surely, as proved in Proposition 3.3. This makes the answer in Theorem 1.4 explicit: if $\kappa \in (0, 4]$ and $\nu \leq -4$, the marginal law of η^R up to the first hitting time of $[y^R y^L]$ equals $\mathrm{hSLE}_\kappa(\nu)$ up to the same time (without conditioning).

4. Hypergeometric SLE: Conformal Markov Characterization

The focus of this section is to give characterization of pairs of simple random curves in quad, and then to prove Theorem 1.4. Fix a quad $q = (\Omega; x^R, y^R, y^L, x^L)$, consider disjoint continuous simple curves $(\eta^L; \eta^R) \in X_0(\Omega; x^R, y^R, y^L, x^L)$. We will show in Propositions 4.1 and 4.2 that the joint law on such pairs are uniquely characterized by the conditional laws. These results play an important role in proving Theorem 1.4. To state the main results, we first introduce some notations in Fig. 3.

Proposition 4.1. *Assume the same notations as in Fig. 3. Fix $\kappa \in (0, 4]$ and $\rho^L > -2$, $\rho^R > -2$ and a quad $q = (\Omega; x^R, y^R, y^L, x^L)$.*

- *(Existence and Uniqueness) There exists a unique probability measure on disjoint continuous simple curves $(\eta^L; \eta^R) \in X_0(\Omega; x^R, y^R, y^L, x^L)$ such that the conditional law of η^R given η^L is $\mathrm{SLE}_\kappa(\rho^R)$ in Ω^L from x^R to y^R with force point x^R_+; and that the conditional law of η^L given η^R is $\mathrm{SLE}_\kappa(\rho^L)$ in Ω^R from x^L to y^L with force point x^L_-.*
- *(Identification) Under this probability measure, when $\rho^L = 0$, the marginal law of η^L is $\mathrm{hSLE}_\kappa(\rho^R)$ in Ω from x^L to y^L with marked points (x^R, y^R).*

Proposition 4.2. *Assume the same notations as in Fig. 3. Fix $\kappa \in (0, 4]$ and $\rho > -2$ and a quad $q = (\Omega; x^R, y^R, y^L, x^L)$.*

- *(Existence and Uniqueness) There exists a unique probability measure on disjoint continuous simple curves $(\eta^L; \eta^R) \in X_0(\Omega; x^R, y^R, y^L, x^L)$ such that the conditional law of η^R given η^L is $\mathrm{SLE}_\kappa(\rho)$ in Ω^L from x^R to y^R with force point w^L; and that the conditional law of η^L given η^R is $\mathrm{SLE}_\kappa(\rho)$ in Ω^R from x^L to y^L with force point w^R. We denote this probability measure by $\mathbb{Q}_q(\kappa, \rho)$.*

– *(Identification) Under this probability measure, the marginal of η^R stopped at the first hitting time of $[y^R y^L]$ is the same as* $\mathrm{hSLE}_\kappa(\kappa-8-\rho)$ *in Ω from x^R to x^L with marked points (y^R, y^L) conditioned to hit $[y^R y^L]$, stopped at the first hitting time of $[y^R y^L]$.*

We will prove Proposition 4.1 in Sect. 4.1 and prove Proposition 4.2 in Sect. 4.2. We will prove Theorem 1.4 in Sect. 4.3.

4.1. Proof of Proposition 4.1. The uniqueness in Proposition 4.1 was proved in [MS16b, Theorem 4.1] and [MSW16, Appendix A], we only need to show the existence and the identification. To construct the pair $(\eta^L; \eta^R)$ in Proposition 4.1, we need to introduce boundary perturbation property of $\mathrm{SLE}_\kappa(\rho)$ process. This is a particular case of Proposition 3.10.

Lemma 4.3 [WW13, Section 3]. *Fix $\kappa \in (0, 4]$, $\rho > -2$ and a Dobrushin domain $(\Omega; x, y)$. Assume that $\hat{\Omega} \subset \Omega$ is simply connected and it agrees with Ω in a neighborhood of the arc (xy). Then $\mathrm{SLE}_\kappa(\rho)$ in $\hat{\Omega}$ from x to y with force point x_+ is absolutely continuous with respect to $\mathrm{SLE}_\kappa(\rho)$ in Ω from x to y with force point x_+, and the Radon-Nikodym derivative is given by*

$$\mathbb{1}_{\{\eta\subset\hat{\Omega}\}}|\varphi'(x)\varphi'(y)|^{-b}\exp(c\mu(\Omega; \eta, \Omega\backslash\hat{\Omega})),$$

where

$$b = \frac{(\rho+2)(\rho+6-\kappa)}{4\kappa}, \quad c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa},$$

and μ is Brownian loop measure, and φ is any conformal map from $\hat{\Omega}$ onto Ω fixing x and y.

Proof of Proposition 4.1, Existence and Identification. First, we will construct a probability measure on $(\eta^L; \eta^R) \in X_0(\Omega; x^L, x^R, y^R, y^L)$. By conformal invariance, it suffices to give the construction for the quad $(\mathbb{H}; 0, x, y, \infty)$ with $0 < x < y$. Denote by $\mathbb{P}_L$ the law of $\mathrm{SLE}_\kappa(\rho^L)$ in $\mathbb{H}$ from 0 to ∞ with force point 0_- and denote by $\mathbb{P}_R$ the law of $\mathrm{SLE}_\kappa(\rho^R)$ in $\mathbb{H}$ from x to y with force point x_+. Define measure $\mathcal{M}$ on $X_0(\mathbb{H}; 0, x, y, \infty)$ by

$$\mathcal{M}[d\eta^L, d\eta^R] = \mathbb{1}_{\{\eta^L\cap\eta^R=\emptyset\}}\exp\left(c\mu(\mathbb{H}; \eta^L, \eta^R)\right)\mathbb{P}_L\left[d\eta^L\right]\otimes\mathbb{P}_R\left[d\eta^R\right].$$

We argue that the total mass of $\mathcal{M}$, denoted by $|\mathcal{M}|$, is finite. Given $\eta^L \in X_0(\mathbb{H}; 0, \infty)$, denote by D the connected component of $\mathbb{H}\backslash\eta^L$ with (xy) on the boundary and let g be any conformal map from D onto $\mathbb{H}$. Then

$$\begin{aligned}|\mathcal{M}| &= \mathbb{E}_L\otimes\mathbb{E}_R\left[\mathbb{1}_{\{\eta^L\cap\eta^R=\emptyset\}}\exp\left(c\mu(\mathbb{H}; \eta^L, \eta^R)\right)\right]\\ &= \mathbb{E}_L\left[\left(\frac{g'(x)g'(y)}{(g(x)-g(y))^2}\right)^b\right] && \text{(by Lemma 4.3)}\\ &\le (y-x)^{-2b} && \text{(where } b = (\rho^R+2)(\rho^R+6-\kappa)/(4\kappa).)\end{aligned}$$

This implies that $|\mathcal{M}|$ is positive and finite. We define the probability measure $\mathcal{M}^\sharp$ to be $\mathcal{M}/|\mathcal{M}|$.

Second, we show that, under $\mathcal{M}^\sharp$, the conditional law of η^R given η^L is $\mathrm{SLE}_\kappa(\rho^R)$. By symmetry in the definition of $\mathcal{M}$, the conditional law of η^L given η^R is $\mathrm{SLE}_\kappa(\rho^L)$. Given η^L, denote by D the connected component of $\mathbb{H}\backslash\eta^L$ with (xy) on the boundary and let g be any conformal map from D onto $\mathbb{H}$. Denote by $\mathbb{P}_R$ the law of $\mathrm{SLE}_\kappa(\rho^R)$ in $\mathbb{H}$ from x to y and by $\tilde{\mathbb{P}}_R$ the law of $\mathrm{SLE}_\kappa(\rho^R)$ in D from x to y. By Lemma 4.3, for any bounded continuous function $\mathcal{F}$ on continuous curves, we have

$$\begin{aligned}\mathcal{M}^\sharp\left[\mathcal{F}(\eta^R)\,|\,\eta^L\right] &= |\mathcal{M}|^{-1}\mathbb{E}_R\left[\mathbb{1}_{\{\eta^L\cap\eta^R=\emptyset\}}\exp\left(c\mu(\mathbb{H};\eta^L,\eta^R)\right)\mathcal{F}(\eta^R)\right]\\ &= |\mathcal{M}|^{-1}\left(\frac{g'(x)g'(y)}{(g(x)-g(y))^2}\right)^b\tilde{\mathbb{E}}_R\left[\mathcal{F}(\eta^R)\right].\end{aligned}$$

This implies that the conditional law of η^R given η^L is $\mathrm{SLE}_\kappa(\rho^R)$ in D.

Finally, we show that, under $\mathcal{M}^\sharp$ and fixing $\rho^L = 0$, the marginal law of η^L is $\mathrm{hSLE}_\kappa(\rho^R)$. In fact, the above equation implies that the law of η^L is the law of SLE_κ in $\mathbb{H}$ from 0 to ∞ weighted by

$$\left(\frac{g'(x)g'(y)}{(g(x)-g(y))^2}\right)^b.$$

By Proposition 3.5, the law of η^L coincides with $\mathrm{hSLE}_\kappa(\rho^R)$ as desired. □

4.2. Proof of Proposition 4.2. We will prove the existence (plus identification) and uniqueness in Proposition 4.2 separately. For the existence, our proof relies on Dubédat's commutation relation, which is explained in Appendix B, and the continuity of hSLE proved in Sect. 3. For the uniqueness, although our setup is different from the one in [MS16b, Theorem 4.1], their proof also works in our setting with minor modification, as detailed below.

Proof of Proposition 4.2, Existence. Note that, given $\eta^L[0,T^L]$, the conditional law of the remaining part of η^L is $\mathrm{SLE}_\kappa(\rho)$ from w^L to y^L with force point w^L_+; given $\eta^R[0,T^R]$, the conditional law of the remaining part of η^R is $\mathrm{SLE}_\kappa(\rho)$ from w^R to y^R with force point w^R_-. Thus, to show the existence of the pair $(\eta^L;\eta^R)$ in Proposition 4.2, it is sufficient to show the existence of the pair $(\eta^L|_{[0,T^L]};\eta^R|_{[0,T^R]})$.

Set $\nu := \kappa-8-\rho$. Let η^R be $\mathrm{hSLE}_\kappa(\nu)$ in Ω from x^R to x^L with marked points (y^R,y^L) conditioned to hit (y^Ry^L) (since $\nu<\kappa/2-4$, this event has positive chance). For $\epsilon>0$, let T^R_ϵ be the first time that η^R hits the ϵ-neighborhood of (y^Ry^L). Given $\eta^R[0,T^R_\epsilon]$, let η^L be $\mathrm{hSLE}_\kappa(\nu)$ in $\Omega\backslash\eta^R[0,T^R_\epsilon]$ from x^L to $\eta^R(T^R_\epsilon)$ with marked points (y^L,y^R) conditioned to hit (y^Ry^L). Let T^L_ϵ be the first time that η^L hits the ϵ-neighborhood of (y^Ry^L). Here we obtain a pair of continuous simple curves $(\eta^L|_{[0,T^L_\epsilon]};\eta^R|_{[0,T^R_\epsilon]})$. We could also sample the pair by first sampling η^R and then sampling η^L conditionally on η^R. Corollary B.3 and Lemma B.1 guarantee that the law on the pair $(\eta^L|_{[0,T^L_\epsilon]};\eta^R|_{[0,T^R_\epsilon]})$ does not depend on the sampling order. Here it is important that $\kappa\le 4$ and the curves do not hit each other almost surely.

Let $\epsilon\to 0$. The continuity of $\mathrm{hSLE}_\kappa(\nu)$ in Propositions 3.2 and 3.3 implies that the law on the pair $(\eta^L|_{[0,T^L]};\eta^R|_{[0,T^R]})$ does not depend on the sampling order. Consider the pair $(\eta^L|_{[0,T^L]};\eta^R|_{[0,T^R]})$, by Lemma 3.7, the conditional law of η^L given $\eta^R[0,T^R]$ is $\mathrm{SLE}_\kappa(\nu+2)$ in Ω^R from x^L to w^R with force point y^L up to the first hitting time of

$(y^R y^L)$. By Lemma 2.6, this is the same as $\mathrm{SLE}_\kappa(\rho)$ in Ω^R from x^L to y^L with force point w^R up to the first hitting time of $(y^R y^L)$. Similarly, the conditional law of η^R given $\eta^L[0, T^L]$ is $\mathrm{SLE}_\kappa(\rho)$ in Ω^L from x^R to y^R with force point w^L up to the first hitting time of $(y^R y^L)$. This implies the existence part (as well as the identification part) of Proposition 4.2. $\square$

Proof of Proposition 4.2, Uniqueness. The uniqueness part could be proved similarly as the proof of [MS16b, Theorem 4.1]. We will briefly summarize the proof and point out the different places. We construct a Markov chain on configurations in $X_0(\Omega; x^R, y^R, y^L, x^L)$: one transitions from one configuration $(\eta^L; \eta^R)$ by picking $i \in \{L, R\}$ uniformly and then resampling η^i according to the conditional law given the other one. The uniqueness of $\mathbb{Q}_q(\kappa, \rho)$ will follow from the uniqueness of the stationary measure of this Markov chain. The ϵ-Markov chain is defined similarly except in each step we resample the paths conditioned on them staying in $X_0^\epsilon(\Omega; x^R, y^R, y^L, x^L)$. Denote by P_ϵ the transition kernel for the ϵ-Markov chain. It suffices to show that there is a unique stationary distribution for the ϵ-Markov chain. Sending $\epsilon \to 0$ implies that the original chain has a unique stationary distribution.

It is proved in [MS16b] that the transition kernel for ϵ-Markov chain is continuous. In this part, the requirements are that the conditional law—$\mathrm{SLE}_\kappa(\rho)$— can be sampled as flow lines of GFF, and that the two curves do not hit each other almost surely. In our case, the conditional law is $\mathrm{SLE}_\kappa(\rho)$ with force point $w^L \in (y^R y^L)$ or $w^R \in (y^R y^L)$, thus the two curves do not hit for all $\rho > -2$ as long as $\kappa \le 4$. So our setting satisfies the two requirements. Let μ be any stationary distribution of the Markov chain, and let μ_ϵ be μ conditioned on $X_0^\epsilon(\Omega; x^R, y^R, y^L, x^L)$. Then μ_ϵ is stationary for the ϵ-Markov chain. Let $\mathcal{S}_\epsilon$ be the set of all such stationary probability measures. Then $\mathcal{S}_\epsilon$ is convex and compact by the continuity of the transition kernel of the ϵ-Markov chain. By Choquet's Theorem, the measure μ_ϵ can be uniquely expressed as a superposition of extremal elements of $\mathcal{S}_\epsilon$. To show that $\mathcal{S}_\epsilon$ consists of a single element, it suffices to show that there is only one extremal in $\mathcal{S}_\epsilon$. Suppose that $\nu, \tilde{\nu}$ are two extremal elements in $\mathcal{S}_\epsilon$. By Lebesgue decomposition theorem, one can uniquely write $\nu = \nu_0 + \nu_1$ such that ν_0 is absolutely continuous and ν_1 is singular with respect to $\tilde{\nu}$. If ν_0 and ν_1 are both nonzero, since $\nu = \nu_0 P_\epsilon + \nu_1 P_\epsilon$, by the uniqueness of the Lebesgue decomposition, we see that ν_0 and ν_1 are both stationary and thus can be normalized as stationary distributions for the ϵ-Markov chain. This contradicts that ν is an extremal measure. This implies that either ν is absolutely continuous with respect to $\tilde{\nu}$ or singular.

Next, it is proved in [MS16b] that it is impossible for ν to be absolutely continuous with respect to $\tilde{\nu}$. The same proof for this part also works here. The last part is showing that ν cannot be singular with respect to $\tilde{\nu}$. Suppose $(\eta_0^L; \eta_0^R) \sim \nu$ and $(\tilde{\eta}^L; \tilde{\eta}^R) \sim \tilde{\nu}$ are two initial states for the ϵ-Markov chain. Then they argued that it is possible to couple $(\eta_2^L; \eta_2^R)$ and $(\tilde{\eta}_2^L; \tilde{\eta}_2^R)$ such that the event $(\eta_2^L; \eta_2^R) = (\tilde{\eta}_2^L; \tilde{\eta}_2^R)$ has positive chance. This implies that ν and $\tilde{\nu}$ cannot be singular. The key ingredient in this part is [MS16b, Lemma 4.2] which needs to be replaced by Lemma 4.4 in our setting. $\square$

Lemma 4.4. *Fix $\kappa \in (0, 8)$ and $\rho > (-2) \vee (\kappa/2 - 4)$. Suppose $(\Omega; x^R, y^R, y^L, x^L)$ is a quad, $w^R \in (y^L y^R)$ is a boundary point, and $\tilde{\Omega} \subset \Omega$ is such that $\tilde{\Omega}$ agrees with Ω in a neighborhood of $(w^R x^L)$. Let η be an $\mathrm{SLE}_\kappa(\rho)$ in Ω from x^L to y^L with force point y^R and let $\tilde{\eta}$ be an $\mathrm{SLE}_\kappa(\rho)$ in $\tilde{\Omega}$ from x^L to y^L with force point w^R. Then there exists a coupling between η and $\tilde{\eta}$ such that the event $\{\eta = \tilde{\eta}\}$ has positive chance.*

Proof. Although our setting is different from that of the proof of [MS16b, Lemma 4.2], the same proof works here. We can view η (resp. $\tilde{\eta}$) as the flow line of a GFF h in Ω

(resp. the flow line of a GFF $\tilde{h}$ in $\tilde{\Omega}$). The key point is that the boundary value of h and $\tilde{h}$ agree in a neighborhood U of $(w^R x^L)$. Therefore $h|_U$ and $\tilde{h}|_U$ are mutually absolutely continuous. Since the flow lines are deterministic functions of the GFF, this implies that the laws of η and $\tilde{\eta}$ stopped upon first exiting U are mutually absolutely continuous. Since there is a positive chance for η to stay in U, the absolute continuity implies the conclusion. □

4.3. Proof of Theorem 1.4 and Corollary 1.5.

Proof of Theorem 1.4. Suppose $(\eta^L; \eta^R) \sim \mathbb{P}_q$ with $q = (\Omega; x^R, y^R, y^L, x^L)$, and assume the same notations as in Fig. 3. Since $(\eta^L; \eta^R)$ satisfies CMP, the conditional law of η^L given η^R satisfies CMP in Definition 2.7. By Theorem 2.8, we know that the conditional law of η^L given η^R is $\mathrm{SLE}_\kappa(\rho)$ with force point w^R for some κ and ρ. Since we require the curves to be simple, we have $\kappa \in (0, 4]$ and $\rho > -2$. Similarly, the conditional law of η^R given η^L is $\mathrm{SLE}_{\tilde{\kappa}}(\tilde{\rho})$ for some $\tilde{\kappa} \in (0, 4]$ and $\tilde{\rho} > -2$. By the symmetry in Definition 1.3, we have $\tilde{\kappa} = \kappa$ and $\tilde{\rho} = \rho$. This implies the only possible candidate for $\mathbb{P}_q$ is the probability measure $\mathbb{Q}_q(\kappa, \rho)$ in Proposition 4.2. To finish the proof, we still need to argue that $\mathbb{Q}_q(\kappa, \rho)$ does satisfy all the requirements in Theorem 1.4.

First, we show that the pair $(\eta^L; \eta^R) \sim \mathbb{Q}_q(\kappa, \rho)$ satisfies CMP. For every η^L-stopping time τ^L and every η^R-stopping time τ^R, consider the conditional law of $(\eta^L|_{t\geq\tau^L}; \eta^R|_{t\geq\tau^R})$ given $\eta^L[0, \tau^L]$ and $\eta^R[0, \tau^R]$, it is clear that the conditional law of $\eta^R|_{t\geq\tau^R}$ given $\eta^L|_{t\geq\tau^L}$ is $\mathrm{SLE}_\kappa(\rho)$ and the conditional law of $\eta^L|_{t\geq\tau^L}$ given $\eta^R|_{t\geq\tau^R}$ is $\mathrm{SLE}_\kappa(\rho)$. Therefore, the pair $(\eta^L; \eta^R)$ satisfies CMP.

Next, we show that the pair $(\eta^L; \eta^R) \sim \mathbb{Q}_q(\kappa, \rho)$ satisfies Condition C1. We only need to show that η^L satisfies Condition C1. Suppose $(Q; a, b, c, d)$ is an avoidable quad for η^L. By the comparison principle of extremal distance (see [Ahl10, Section 4-3]), we have

$$d_{Q\setminus\eta^R}((ab), (cd)) \geq d_Q((ab), (cd)).$$

Note that the conditional law of η^L given η^R is $\mathrm{SLE}_\kappa(\rho)$ and $\mathrm{SLE}_\kappa(\rho)$ satisfies Condition C1 (by Lemma 2.5), combining with the above inequality, η^L satisfies Condition C1. □

Next, we will show Corollary 1.5. To this end, we first discuss the reversibility of $\mathrm{SLE}_\kappa(\rho)$ processes. Suppose $x \leq w \leq y$, and let η be an $\mathrm{SLE}_\kappa(\rho)$ in $\mathbb{H}$ from x to y with force point w. The process η does not have reversibility when $x < w < y$, see Lemma 4.5; but it enjoys reversibility when $w = x_+$, see [MS16b, Theorem 1.1] and [MS16c, Theorem 1.2]. The reversibility for $w = x_+$ is a deep result and it is a particular case of Conjecture 3.6 when $x_1 = x_2$ and $x_3 = x_4$.

Lemma 4.5. *Fix $\kappa \in (0, 8)$, $\rho > -2$ and $x < w, \tilde{w} < y$. Suppose η is an $\mathrm{SLE}_\kappa(\rho)$ in $\mathbb{H}$ from x to y with force point w. Then the time-reversal of η is an $\mathrm{SLE}_{\tilde{\kappa}}(\tilde{\rho})$ from y to x with force point $\tilde{w}$ if and only if $\tilde{\kappa} = \kappa$ and $\rho = \tilde{\rho} = 0$.*

Proof. Let $\hat{\eta}$ be the time-reversal of η. If $\hat{\eta}$ has the law of $\mathrm{SLE}_{\tilde{\kappa}}(\tilde{\rho})$, since the dimension of $\mathrm{SLE}_\kappa(\rho)$ process is $1 + \kappa/8$ [Bef08], we have $\tilde{\kappa} = \kappa$. It remains to show $\tilde{\rho} = \rho = 0$. Let $\tilde{\eta}$ be an $\mathrm{SLE}_\kappa(\tilde{\rho})$ in $\mathbb{H}$ from y to x with force point $\tilde{w}$.

When $\kappa \in (4, 8)$ and $\rho \geq \kappa/2-2$, we have $\eta \cap (w, y) = \emptyset$ and $\tilde{\eta} \cap (w, y) \neq \emptyset$ almost surely. Thus $\hat{\eta}$ cannot have the same law as $\tilde{\eta}$. When $\kappa \in (4, 8)$ and $\rho \in (\kappa/2-4, \kappa/2-2)$, we have almost surely (see [MW17, Theorem 1.6])

$$\dim(\eta \cap (w \vee \tilde{w}, y)) = 1 - (\rho + 2)(\rho + 4 - \kappa/2)/\kappa,$$
$$\dim(\tilde{\eta} \cap (w \vee \tilde{w}, y)) = 1 - (8 - \kappa)/\kappa.$$

If $\hat{\eta}$ has the same law as $\tilde{\eta}$, then these two dimensions have to coincide, and hence $\rho = 0$ and therefore $\tilde{\rho} = 0$. When $\kappa \in (4, 8)$ and $\rho \in (-2, \kappa/2 - 4]$, the curve η fills the interval $(w \vee \tilde{w}, y)$, whereas $\tilde{\eta} \cap (w \vee \tilde{w}, y)$ has no interior point. Thus the $\hat{\eta}$ cannot have the same law as $\tilde{\eta}$.

When $\kappa \in (0, 4]$ and $\rho < \kappa/2 - 2$, we have $\eta \cap (w \vee \tilde{w}, y) \neq \emptyset$ and $\tilde{\eta} \cap (w \vee \tilde{w}, y) = \emptyset$ almost surely. Thus $\hat{\eta}$ cannot have the same law as $\tilde{\eta}$. When $\kappa \in (0, 4]$ and $\rho \geq \kappa/2 - 2$, it is proved in Theorem 1.1 that $\hat{\eta}$ is $\mathrm{hSLE}_\kappa(\rho - 2)$ in $\mathbb{H}$ from y to x with marked points (y_-, w) which equals $\mathrm{SLE}_\kappa(\tilde{\rho})$ process from y to x with force point $\tilde{w}$ if and only if $\rho = \tilde{\rho} = 0$. □

Now, we are ready to show Corollary 1.5.

Proof of Corollary 1.5. Suppose $(\eta^L; \eta^R) \sim \mathbb{P}_q$ for $q = (\Omega; x^R, y^R, y^L, x^L)$. By the proof of Theorem 1.4, there exists $\kappa \in (0, 4]$ and $\rho > -2$ such that the conditional law of η^L given η^R is $\mathrm{SLE}_\kappa(\rho)$ in Ω^R from x^L to y^L with force point w^R. Denote by $\hat{\eta}^L$ the time-reversal of η^L and by $\hat{\eta}^R$ the time-reversal of η^R. By the reversibility in Definition 1.3, we can apply Theorem 1.4 on the pair $(\hat{\eta}^L, \hat{\eta}^R)$, then there exists $\tilde{\kappa} \in (0, 4]$ and $\tilde{\rho} > -2$ such that the conditional law of $\hat{\eta}^L$ given $\hat{\eta}^R$ is $\mathrm{SLE}_{\tilde{\kappa}}(\tilde{\rho})$ from y^L to x^L with force point x^R. From Lemma 4.5, we see that $\tilde{\kappa} = \kappa, \tilde{\rho} = \rho = 0$. Therefore, the conditional law of η^L given η^R is SLE_κ. Similarly, the conditional law of η^R given η^L is SLE_κ. By Proposition 4.1, there exists a unique such probability measure and the marginal of η^R is hSLE_κ. □

We end this section with several remarks on Propositions 4.1 and 4.2.

- When $\rho^L \neq 0$ or $\rho^R \neq 0$, the pair $(\eta^L; \eta^R)$ in Proposition 4.1 does not satisfy CMP in Definition 1.2; whereas, it satisfies the reversibility in Definition 1.3, and it satisfies the symmetry in Definition 1.3 when $\rho^L = \rho^R$.
- We compare Proposition 4.2 with $\rho = 0$ and Proposition 4.1 with $\rho^L = \rho^R = 0$. In this case, the two propositions describe the same law on the pair $(\eta^L; \eta^R)$. From Proposition 4.2, we see that the marginal law of η^R is $\mathrm{hSLE}_\kappa(\kappa - 8)$ from x^R to x^L; whereas, from Proposition 4.1, the marginal law of η^R is hSLE_κ from x^R to y^R. This implies that hSLE_κ from x^R to y^R has the same law as $\mathrm{hSLE}_\kappa(\kappa - 8)$ from x^R to x^L. This is consistent with the target-independence proved in Proposition 3.11.
- I expect the conclusions in Propositions 4.1 and 4.2 also hold for $\kappa \in (4, 8)$. When $\kappa \in (4, 8)$, the uniqueness follows from [MSW16, Appendix A]; the existence when $\rho^L = \rho^R = 0$ in Proposition 4.1 and the existence when $\rho = 0$ in Proposition 4.2 are given by Proposition 6.10; whereas, the existence in general case is not clear to me. For Proposition 4.1, the construction in Sect. 4.1 relies essentially on the fact that the two curves do not intersect. For Proposition 4.2, the construction in Sect. 4.2 is based on Commutation Relation, and it does not allow the two curves to hit each other. These give the restriction on $\kappa \leq 4$.
- Theorem 1.4 holds for $\kappa \in (4, 8)$ as long as Proposition 4.2 holds. Corollary 1.5 holds for $\kappa \in (4, 8)$. In the above proof of Corollary 1.5, we only need to replace Proposition 4.1 by Proposition 6.10 when $\kappa \in (4, 8)$.
- It is clear that the uniqueness in Propositions 4.1 and 4.2 fails for $\kappa \geq 8$.

5. Convergence of Ising Interfaces to Hypergeometric SLE

5.1 Ising model.

Notation and terminology. We focus on the square lattice $\mathbb{Z}^2$. Two vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are neighbors if $|x_1 - y_1| + |x_2 - y_2| = 1$, and we write $x \sim y$. The *dual square lattice* $(\mathbb{Z}^2)^*$ is the dual graph of $\mathbb{Z}^2$. The vertex set is $(1/2, 1/2) + \mathbb{Z}^2$ and the edges are given by nearest neighbors. The vertices and edges of $(\mathbb{Z}^2)^*$ are called dual-vertices and dual-edges. For each edge e of $\mathbb{Z}^2$, it is associated to a dual edge, denoted by e^*. The dual edge e^* crosses e in the middle. For a finite subgraph G, we define G^* to be the subgraph of $(\mathbb{Z}^2)^*$ with edge-set $E(G^*) = \{e^* : e \in E(G)\}$ and vertex set given by the end-points of these dual-edges. The *medial lattice* $(\mathbb{Z}^2)^\diamond$ is the graph with the centers of edges of $\mathbb{Z}^2$ as vertex set, and edges connecting nearest vertices. This lattice is a rotated and rescaled version of $\mathbb{Z}^2$. The vertices and edges of $(\mathbb{Z}^2)^\diamond$ are called medial-vertices and medial-edges. We identify the faces of $(\mathbb{Z}^2)^\diamond$ with the vertices of $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$. A face of $(\mathbb{Z}^2)^\diamond$ is said to be black if it corresponds to a vertex of $\mathbb{Z}^2$ and white if it corresponds to a vertex of $(\mathbb{Z}^2)^*$.

Let Ω be a finite subset of $\mathbb{Z}^2$. The Ising model with free boundary conditions is a random assignment $\sigma \in \{\ominus, \oplus\}^\Omega$ of spins $\sigma_x \in \{\ominus, \oplus\}$, where σ_x denotes the spin at the vertex x. The Hamiltonian of the Ising model is defined by

$$H_\Omega^{\mathrm{free}}(\sigma) = -\sum_{x\sim y} \sigma_x \sigma_y.$$

The Ising measure is the Boltzmann measure with Hamiltonian H_Ω^{free} and inverse-temperature $\beta > 0$:

$$\mu_{\beta,\Omega}^{\mathrm{free}}[\sigma] = \frac{\exp(-\beta H_\Omega^{\mathrm{free}}(\sigma))}{Z_{\beta,\Omega}^{\mathrm{free}}}, \quad \text{where } Z_{\beta,\Omega}^{\mathrm{free}} = \sum_\sigma \exp(-\beta H_\Omega^{\mathrm{free}}(\sigma)).$$

For a graph Ω and $\tau \in \{\ominus, \oplus\}^{\mathbb{Z}^2}$, one may also define the Ising model with boundary conditions τ by the Hamiltonian

$$H_\Omega^\tau(\sigma) = -\sum_{x\sim y, \{x,y\}\cap\Omega\neq\emptyset} \sigma_x\sigma_y, \quad \text{if } \sigma_x = \tau_x, \forall x \notin \Omega.$$

Suppose that $(\Omega; a, b)$ is a Dobrushin domain. The *Dobrushin boundary conditions* is the following: $\oplus$ along (ab), and $\ominus$ along (ba).

The set $\{\ominus, \oplus\}^\Omega$ is equipped with a partial order: $\sigma \le \sigma'$ if $\sigma_x \le \sigma'_x$ for all $x \in \Omega$. A random variable X is increasing if $\sigma \le \sigma'$ implies $X(\sigma) \le X(\sigma')$. An event $\mathcal{A}$ is increasing if $\mathbb{1}_\mathcal{A}$ is increasing. The Ising model satisfies FKG inequality: Let Ω be a finite subset and τ be boundary conditions, and $\beta > 0$. For any two increasing events $\mathcal{A}$ and $\mathcal{B}$, we have $\mu_{\beta,\Omega}^\tau[\mathcal{A}\cap\mathcal{B}] \ge \mu_{\beta,\Omega}^\tau[\mathcal{A}]\mu_{\beta,\Omega}^\tau[\mathcal{B}]$. As a consequence of FKG inequality, we have the comparison between boundary conditions: For boundary conditions $\tau_1 \le \tau_2$ and an increasing event $\mathcal{A}$, we have

$$\mu_{\beta,\Omega}^{\tau_1}[\mathcal{A}] \le \mu_{\beta,\Omega}^{\tau_2}[\mathcal{A}]. \tag{5.1}$$

The critical Ising model ($\beta = \beta_c$) is conformally invariant in the scaling limit, see [DC13] for general background. We only collect several properties of the critical Ising model that will be useful later: strong RSW and the convergence of the interface.

Given a quad $(Q; a, b, c, d)$ on the square lattice, we denote by $d_Q((ab), (cd))$ the discrete extermal distance between (ab) and (cd) in Q, see [Che16, Section 6]. The

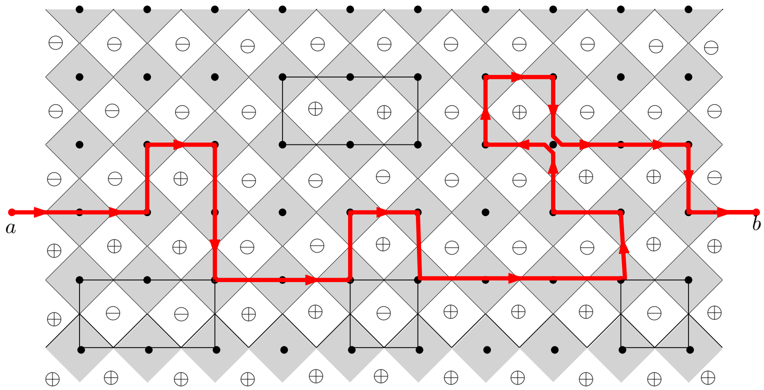

Fig. 4. The Ising interface with Dobrushin boundary conditions

discrete extremal distance is uniformly comparable to and converges to its continuous counterpart—the classical extremal distance. The quad $(Q; a, b, c, d)$ is crossed by $\oplus$ in an Ising configuration σ if there exists a path of $\oplus$ going from (ab) to (cd) in Q. We denote this event by $(ab) \overset{\oplus}{\longleftrightarrow} (cd)$.

Proposition 5.1 [CDCH16, Corollary 1.7]. *For each $L > 0$ there exists $c(L) > 0$ such that the following holds: for any quad $(Q; a, b, c, d)$ with $d_Q((ab), (cd)) \geq L$,*

$$\mu^{mixed}_{\beta_c, Q}\left[(ab) \overset{\oplus}{\longleftrightarrow} (cd)\right] \leq 1 - c(L),$$

where the boundary conditions are free *on $(ab) \cup (cd)$ and $\ominus$ on $(bc) \cup (da)$.*

For $\delta > 0$, we consider the rescaled square lattice $\delta\mathbb{Z}^2$. The definitions of dual lattice, medial lattice and Dobrushin domains extend to this context, and they will be denoted by $(\Omega_\delta; a_\delta, b_\delta)$, $(\Omega^*_\delta; a^*_\delta, b^*_\delta)$, $(\Omega^\diamond_\delta; a^\diamond_\delta, b^\diamond_\delta)$ respectively. Consider the critical Ising model on $(\Omega^*_\delta; a^*_\delta, b^*_\delta)$. The boundary $\partial\Omega^*_\delta$ is divided into two parts $(a^*_\delta b^*_\delta)$ and $(b^*_\delta a^*_\delta)$. We fix the Dobrushin boundary conditions: $\ominus$ on $(b^*_\delta a^*_\delta)$ and $\oplus$ on $(a^*_\delta b^*_\delta)$. Define the *interface* as follows. It starts from $a^\diamond_\delta$, lies on the primal lattice and turns at every vertex of Ω_δ is such a way that it has always dual vertices with spin $\ominus$ on its left and $\oplus$ on its right. If there is an indetermination when arriving at a vertex (this may happen on the square lattice), turn left. See Fig. 4. We have the convergence of the interface:

Theorem 5.2 [CDCH+14]. *Let $(\Omega^\diamond_\delta; a^\diamond_\delta, b^\diamond_\delta)$ be a sequence of Dobrushin domains converging to a Dobrushin domain $(\Omega; a, b)$ in the Carathéodory sense as $\delta \to 0$. The interface of the critical Ising model in $(\Omega^*_\delta; a^*_\delta, b^*_\delta)$ with Dobrushin boundary conditions converges weakly to* SLE_3 *as $\delta \to 0$.*

Theorem 5.3. *Let $(\Omega^\diamond_\delta; a^\diamond_\delta, w^\diamond_\delta, b^\diamond_\delta)$ be a sequence of triangles converging to a triangle $(\Omega; a, w, b)$ in the Carathéodory sense as $\delta \to 0$. The interface of the critical Ising model in $(\Omega^*_\delta; a^*_\delta, w^*_\delta, b^*_\delta)$ with the boundary conditions $\ominus$ along $(b^*_\delta a^*_\delta)$, $\oplus$ along $(a^*_\delta w^*_\delta)$ and* free *along $(w^*_\delta b^*_\delta)$ converges weakly to* $\mathrm{SLE}_3(-3/2)$ *as $\delta \to 0$.*

Proof. [HK13, Theorem 1] proves that the initial segment of the interface, i.e. the interface stopped at the first hitting time of the free segment $(w^*_\delta b^*_\delta)$, converges weakly to $\mathrm{SLE}_3(-3/2)$. Based on this result and crossing estimates in [CDCH16], the convergence of the whole process is obtained in [BDCH16, Theorem 4]. $\square$

5.1. Proof of Proposition 1.6. Let $(\Omega_\delta; x_\delta^R, y_\delta^R, y_\delta^L, x_\delta^L)$ be a sequence of quads on $\delta\mathbb{Z}^2$ converging to a quad $q = (\Omega; x^R, y^R, y^L, x^L)$ in the Carathéodory sense as $\delta \to 0$. Consider the critical Ising model in Ω_δ^* with alternating boundary conditions:

$$\ominus \text{ along } (x_\delta^L x_\delta^R) \cup (y_\delta^R y_\delta^L), \qquad \xi^R \in \{\oplus, \text{free}\} \text{ along } (x_\delta^R y_\delta^R),$$
$$\xi^L \in \{\oplus, \text{free}\} \text{ along } (y_\delta^L x_\delta^L). \tag{5.2}$$

The quad is vertically crossed by $\ominus$ if there exists a path of $\ominus$ going from $(x_\delta^L x_\delta^R)$ to $(y_\delta^R y_\delta^L)$. The quad is horizontally crossed by $\oplus$ in an Ising configuration if there exists a path of $\oplus$ going from $(y_\delta^L x_\delta^L)$ to $(x_\delta^R y_\delta^R)$. We denote these events by

$$\mathcal{C}_v^\ominus(q) = \left\{(x_\delta^L x_\delta^R) \overset{\ominus}{\longleftrightarrow} (y_\delta^R y_\delta^L)\right\}, \quad \mathcal{C}_h^\oplus(q) = \left\{(y_\delta^L x_\delta^L) \overset{\oplus}{\longleftrightarrow} (x_\delta^R y_\delta^R\right\}.$$

Suppose there is a vertical crossing of $\ominus$. Let η_δ^L be the interface starting from x_δ^L lying on the primal lattice. It turns at every vertex in the way that it has spin $\oplus$ on its left and $\ominus$ on its right, and that it turns left when there is ambiguity. Let η_δ^R be the interface starting from x_δ^R lying on the primal lattice. It turns at every vertex in the way that it has spin $\ominus$ to its left and $\oplus$ to its right, and turns right when there is ambiguity. Then η_δ^L will end at y_δ^L and η_δ^R will end at y_δ^R. See Fig. 1. Let Ω_δ^L be the connected component of $\Omega_\delta \backslash \eta_\delta^L$ with $(x_\delta^R y_\delta^R)$ on the boundary and denote by $\mathcal{D}_\delta^L$ the discrete extremal distance between η_δ^L and $(x_\delta^R y_\delta^R)$ in Ω_δ^L. Define Ω_δ^R and $\mathcal{D}_\delta^R$ similarly.

Lemma 5.4. *The family of random variables $\{(\mathcal{D}_\delta^L; \mathcal{D}_\delta^R)\}_{\delta>0}$ is tight in the following sense: for any $u > 0$, there exists $\epsilon > 0$ such that*

$$\mathbb{P}\left[\mathcal{D}_\delta^L \geq \epsilon, \mathcal{D}_\delta^R \geq \epsilon \,|\, \mathcal{C}_v^\ominus(q)\right] \geq 1 - u, \quad \forall \delta > 0.$$

Proof. Since $(\Omega_\delta; x_\delta^L, x_\delta^R, y_\delta^R, y_\delta^L)$ approximates $(\Omega; x^L, x^R, y^R, y^L)$, by Proposition 5.1 and (5.1), the probability $\mathbb{P}[\mathcal{C}_v^\ominus(q)]$ can be bounded from below by some quantity that depends only on the extremal distance in Ω between $(x^L x^R)$ and $(y^R y^L)$ and that is uniform over δ. Thus, it is sufficient to show that $\mathbb{P}\left[\{\mathcal{D}_\delta^L \leq \epsilon\} \cap \mathcal{C}_v^\ominus(q)\right]$ is small for $\epsilon > 0$ small. Given η_δ^L and on the event $\{\mathcal{D}_\delta^L \leq \epsilon\}$, combining Proposition 5.1 and (5.1), the probability to have a vertical crossing of $\ominus$ in Ω_δ^L is bounded by $c(\epsilon)$ which only depends on ϵ and goes to zero as $\epsilon \to 0$. Thus $\mathbb{P}\left[\{\mathcal{D}_\delta^L \leq \epsilon\} \cap \mathcal{C}_v^\ominus(q)\right] \leq c(\epsilon)$. This implies the conclusion. □

Lemma 5.5. *Conditionally on the event $\mathcal{C}_v^\ominus(q)$, there exists a pair of interfaces $(\eta_\delta^L; \eta_\delta^R)$ where η_δ^L (resp. η_δ^R) is the interface connecting x_δ^L to y_δ^L (resp. connecting x_δ^R to y_δ^R). The law of the pair $(\eta_\delta^L; \eta_\delta^R)$ converges weakly to the pair of* SLE *curves in Proposition 4.1 as $\delta \to 0$ where $\kappa = 3$ and $\xi^R, \xi^L, \rho^R, \rho^L$ are related in the following way: for $q \in \{L, R\}$,*

$$\rho^q = 0, \quad \textit{if } \xi^q = \oplus; \quad \rho^q = -3/2, \quad \textit{if } \xi^q = \text{free}.$$

Proof. We only prove the conclusion for $\xi^R = \xi^L = \oplus$, and the other cases can be proved similarly (by replacing Theorem 5.2 with 5.3 when necessary). Combining the crossing estimates in [CDCH16] (see also [CDCH+14, Remark 4]) and Lemma 5.4, the sequence $\{(\eta_\delta^L; \eta_\delta^R)\}_{\delta>0}$ satisfies the requirements in Theorem 2.4, so the sequence is

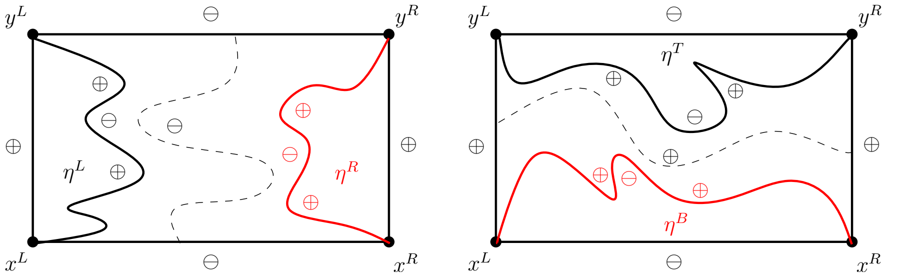

Fig. 5. Consider the critical Ising model in Ω_δ with the following boundary conditions: $\ominus$ along $(x_\delta^L x_\delta^R) \cup (y_\delta^R y_\delta^L)$ and $\oplus$ along $(x_\delta^R y_\delta^R) \cup (y_\delta^L x_\delta^L)$. In the left panel, there is a vertical crossing of $\ominus$. Then there exists a pair of interfaces $(\eta_\delta^L; \eta_\delta^R)$: η_δ^R connects x_δ^R to y_δ^R and η_δ^L connects x_δ^L to y_δ^L. In the right panel, there is a horizontal crossing of $\oplus$. Then there exists a pair of interfaces $(\eta_\delta^B; \eta_\delta^T)$: η_δ^B connects x_δ^R to x_δ^L and η_δ^T connects y_δ^R to y_δ^L

relatively compact. Suppose $(\eta^L; \eta^R) \in X_0(\Omega; x^L, x^R, y^R, y^L)$ is any sub-sequential limit and, for some $\delta_k \to 0$,

$$(\eta_{\delta_k}^L; \eta_{\delta_k}^R) \longrightarrow (\eta^L; \eta^R) \quad \text{in } X_0(\Omega; x^L, x^R, y^R, y^L).$$

For convenience, we couple them in the same space so that there is almost sure convergence. Since $\eta_{\delta_k}^L \to \eta^L$, by Theorem 2.2, we have the convergence in all three topologies. In particular, this implies the convergence of $\Omega_{\delta_k}^L$ in the Carathéodory sense. Note that the conditional law of $\eta_{\delta_k}^R$ in $\Omega_{\delta_k}^L$ given $\eta_{\delta_k}^L$ is the interface of the critical planar Ising model with Dobrushin boundary conditions. Combining with Theorem 5.2, we derive that the conditional law of η^R in Ω^L given η^L is SLE_3. By symmetry, the conditional law of η^L in Ω^R given η^R is SLE_3. By Proposition 4.1, there exists a unique such measure. Thus it has to be the unique sub-sequential limit. This proves the convergence of the whole sequence. $\square$

Corollary 5.6. *Suppose $\xi^L = \xi^R = \oplus$ in (5.2).*

- *On the event $\mathcal{C}_v^{\ominus}(q)$, let η_δ be the interface connecting x_δ^R and y_δ^R. Then the law of η_δ converges weakly to* hSLE_3 *in Ω from x^R to y^R with marked points (x^L, y^L) as $\delta \to 0$.*
- *On the event $\mathcal{C}_h^{\oplus}(q)$, let η_δ be the interface connecting x_δ^R and x_δ^L. Then the law of η_δ converges weakly to* hSLE_3 *in Ω from x^R to x^L with marked points (y^R, y^L) as $\delta \to 0$.*

Proof. On the event $\mathcal{C}_v^{\ominus}(q)$, there is a pair of Ising interfaces $(\eta_\delta^L; \eta_\delta^R)$, as indicated in Fig. 5. By Lemma 5.5, the sequence $(\eta_\delta^L; \eta_\delta^R)$ converges weakly to the pair of SLEs in Proposition 4.1 with $\kappa = 3$ and $\rho^L = \rho^R = 0$. In particular, the law of η_δ^R conditioned on $\mathcal{C}_v^{\ominus}(q)$ converges weakly to hSLE_3 in Ω from x^R to y^R. The other case can be proved similarly. $\square$

Corollary 5.7. *Consider the critical Ising model in Ω_δ with the following boundary conditions:*

$$\ominus \textit{ along } (x_\delta^L x_\delta^R), \quad \oplus \textit{ along } (x_\delta^R y_\delta^R) \cup (y_\delta^L x_\delta^L), \quad \text{free } \textit{along } (y_\delta^R y_\delta^L).$$

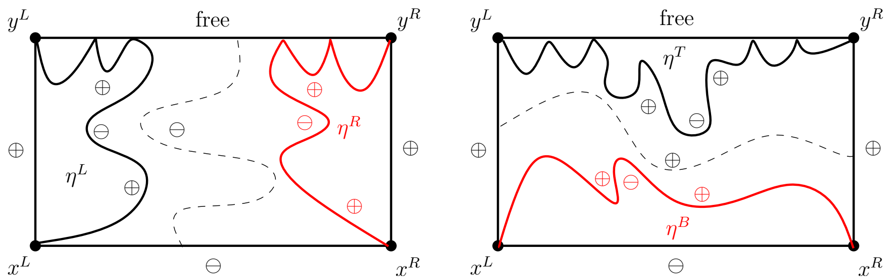

Fig. 6. Consider the critical Ising model in Ω_δ with the following boundary conditions: $\ominus$ along $(x^L_\delta x^R_\delta)$, $\oplus$ along $(x^R_\delta y^R_\delta) \cup (y^L_\delta x^L_\delta)$, and free along $(y^R_\delta y^L_\delta)$. In the left panel, there is a vertical crossing of $\ominus$. Then there exists a pair of interfaces $(\eta^L_\delta; \eta^R_\delta)$: η^R_δ connects x^R_δ to y^R_δ and η^L_δ connects x^L_δ to y^L_δ. In the right panel, there is a horizontal crossing of $\oplus$. Then there exists a pair of interfaces $(\eta^B_\delta; \eta^T_\delta)$: η^B_δ connects x^R_δ to x^L_δ and η^T_δ connects y^R_δ to y^L_δ

- *On the event $\mathcal{C}^{\ominus}_v(q)$, let η_δ be the interface connecting x^R_δ and y^R_δ. Then the law of η_δ (up to the first hitting time of $[y^R_\delta y^L_\delta]$) converges weakly to* $\mathrm{hSLE}_3(-7/2)$ *from x^R to x^L conditioned to hit $[y^R y^L]$ (up to the first hitting time of $[y^R y^L]$) as $\delta \to 0$.*
- *On the event $\mathcal{C}^{\oplus}_h(q)$, let η_δ be the interface connecting x^R_δ and x^L_δ. Then the law of η_δ converges weakly to* $\mathrm{hSLE}_3(-3/2)$ *from x^R to x^L as $\delta \to 0$.*

Proof. On the event $\mathcal{C}^{\ominus}_v(q)$, there is a pair of Ising interfaces $(\eta^L_\delta; \eta^R_\delta)$, as indicated in Fig. 6. By a similar argument as in Lemma 5.5, the sequence $(\eta^L_\delta; \eta^R_\delta)$ converges weakly to the pair of SLEs in Proposition 4.2 with $\kappa = 3$ and $\rho = -3/2$. In particular, the law of η^R_δ conditioned on $\mathcal{C}^{\ominus}_v(q)$ converges weakly to $\mathrm{hSLE}_3(-7/2)$ in Ω from x^R to x^L conditioned to hit $[y^R y^L]$ (here is $\mathrm{hSLE}_3(-7/2)$ from x^R to x^L, this is not a typo).

On the event $\mathcal{C}^{\oplus}_h(q)$, there is a pair of Ising interfaces $(\eta^B_\delta; \eta^T_\delta)$ as indicated in Fig. 6. By Lemma 5.5, the sequence $(\eta^B_\delta; \eta^T_\delta)$ converges weakly to the pair of SLEs in Proposition 4.1 (rotated by 90 degrees counterclockwise) with $\kappa = 3$ and $\rho = -3/2$. In particular, the law of η^B_δ conditioned on $\mathcal{C}^{\oplus}_h(q)$ converges weakly to $\mathrm{hSLE}_3(-3/2)$ in Ω from x^R to x^L. $\square$

Proof of Proposition 1.6. Proposition 1.6 is a collection of Corollaries 5.6 and 5.7. $\square$

6. Pure Partition Functions of Multiple SLEs

In this section, we will prove Theorem 1.7. Recall that the multiple SLE pure partition functions is the collection $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ of positive smooth functions $\mathcal{Z}_\alpha : \mathfrak{X}_{2N} \to \mathbb{R}_+$ for $\alpha \in \mathrm{LP}_N$, satisfying $\mathcal{Z}_\emptyset = 1$, PDE (1.1), COV (1.2), ASY (1.3), and the power law bound (1.4). To state the main result of this section, we need to introduce some notations and properties first.

Fix the constants in this section:

$$\kappa \in (0, 6], \quad h = \frac{6-\kappa}{2\kappa}.$$

The pure partition functions introduced in Sect. 1.4 are only defined for the upper half-plane, we may extend the definition to general polygon via conformal image. Suppose $(\Omega; x_1, \ldots, x_{2N})$ is a nice polygon. Define, for $\alpha \in \mathrm{LP}_N$,

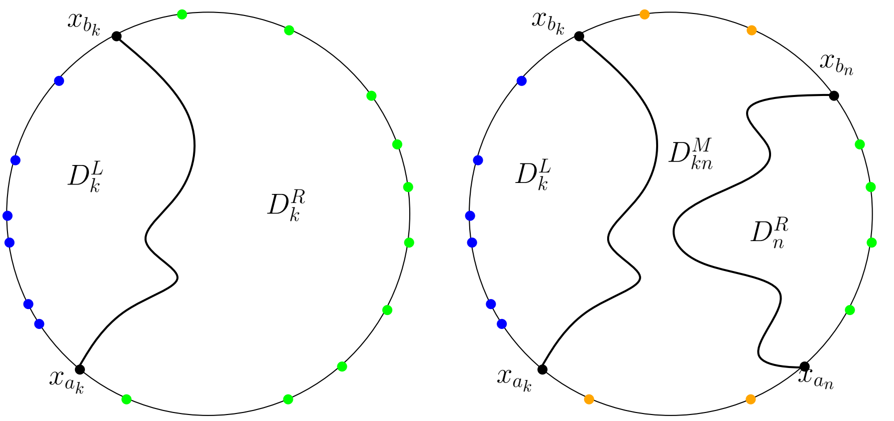

Fig. 7. In the left panel, the blue marked points correspond to α_k^L and the green marked points correspond to α_k^R. In the right panel, the orange marked points correspond to α_{kn}^M and the green marked points correspond to α_n^R

$$\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N}) = \prod_{j=1}^{2N} |\varphi'(x_j)|^h \times \mathcal{Z}_\alpha(\varphi(x_1), \ldots, \varphi(x_{2N})), \tag{6.1}$$

where φ is any conformal map from Ω onto $\mathbb{H}$ with $\varphi(x_1) < \cdots < \varphi(x_{2N})$.

Next, we introduce the cascade relation of the pure partition functions. Suppose $(\Omega; x_1, \ldots, x_{2N})$ is a nice polygon. Suppose $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N$, and assume $a_j < b_j$ for all $1 \le j \le N$. For $1 \le k \le N$, let η_k be an SLE_κ in Ω from x_{a_k} to x_{b_k}. The link $\{a_k, b_k\}$ divides the link pattern α into two sub-link patterns, connecting $\{a_k+1, \ldots, b_k-1\}$ and $\{b_k+1, \ldots, a_k-1\}$ respectively. After relabelling the indices, we denote these two link patterns by α_k^R and α_k^L, see Fig. 7.

We first explain the cascade relation when $\kappa \in (0, 4]$ as the notations in this case are simpler. Consider the set $\Omega \setminus \eta_k$, denote by D_k^R the connected component having $(x_{a_k+1} x_{b_k-1})$ on the boundary, and denote by D_k^L the connected component having $(x_{b_k+1} x_{a_k-1})$ on the boundary, see Fig. 7. We expect the following *cascade relation* of the pure partition functions:

$$\begin{aligned}&\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})\\&\quad = H_\Omega(x_{a_k}, x_{b_k})^h \mathbb{E}\left[\mathcal{Z}_{\alpha_k^R}(D_k^R; x_{a_k+1}, \ldots, x_{b_k-1}) \times \mathcal{Z}_{\alpha_k^L}(D_k^L; x_{b_k+1}, \ldots, x_{a_k-1})\right].\end{aligned}$$

We then explain the cascade relation when $\kappa \in (4, 6]$. The idea is similar as above, but the situation is more complicated as η_k may hit boundary segments in this case. Consider the set $\Omega \setminus \eta_k$. If η_k does not hit the boundary segments $(x_{a_k+1} x_{b_k-1})$ nor $(x_{b_k+1} x_{a_k-1})$, then we define D_k^R and D_k^L as above. Whereas, it is also possible that η_k does hit these boundary segments. We say that η_k *is allowed by* α if, for all $j \ne k$, the points x_{a_j} and x_{b_j} lie on the boundary of the same connected component of $\Omega \setminus \eta_k$. In other words, η_k is allowed by α if it does not disconnect any pair of points $\{x_{a_j}, x_{b_j}\}$ for $j \ne k$. We denote this event by $\mathcal{E}_\alpha^k$. On the event $\mathcal{E}_\alpha^k$, the points $x_{a_k+1}, \ldots, x_{b_k-1}$ are divided into smaller groups. We denote the connected components of $\Omega \setminus \eta_k$ having these points on the boundary by $D_k^{R,1}, \ldots, D_k^{R,r}$ in counterclockwise order. We denote by D_k^R the union of $D_k^{R,1}, \dots, D_k^{R,r}$. The sub-link pattern α_k^R are divided into smaller sub-link patterns, after relabelling the indices, we denote these link patterns by $\alpha_k^{R,1}, \dots, \alpha_k^{R,r}$. Now define

$$\mathcal{Z}_{\alpha_k^R}(D_k^R; x_{a_k+1}, \dots, x_{b_k-1}) = \mathcal{Z}_{\alpha_k^{R,1}}(D_k^{R,1}; \dots) \times \cdots \times \mathcal{Z}_{\alpha_k^{R,r}}(D_k^{R,r}; \dots),$$

where the points on the boundary of $D_k^{R,i}$ are clear and we omit them from the notation. We define $D_k^{L,1}, \dots, D_k^{L,l}, D_k^L, \alpha_k^{L,1}, \dots, \alpha_k^{L,l}$ in similar way, and set

$$\mathcal{Z}_{\alpha_k^L}(D_k^L; x_{b_k+1}, \dots, x_{a_k-1}) = \mathcal{Z}_{\alpha_k^{L,1}}(D_k^{L,1}; \dots) \times \cdots \times \mathcal{Z}_{\alpha_k^{L,l}}(D_k^{L,l}; \dots).$$

Finally, the cascade relation is the following:

$$\begin{aligned}&\mathcal{Z}_\alpha(\Omega; x_1, \dots, x_{2N}) \\ &\quad = H_\Omega(x_{a_k}, x_{b_k})^h \mathbb{E}\left[\mathcal{Z}_{\alpha_k^R}(D_k^R; x_{a_k+1}, \dots, x_{b_k-1}) \times \mathcal{Z}_{\alpha_k^L}(D_k^L; x_{b_k+1}, \dots, x_{a_k-1}) \mathbb{1}_{\mathcal{E}_\alpha^k}\right].\end{aligned} \tag{6.2}$$

The main result of this section is the following.

Proposition 6.1. *Let* $\kappa \in (0, 6]$. *For each* $N \geq 1$, *there exists a collection* $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_n, n \leq N\}$ *of smooth functions* $\mathcal{Z}_\alpha : \mathfrak{X}_{2n} \to \mathbb{R}_+$, *for* $\alpha \in \mathrm{LP}_n$, *satisfying the normalization* $\mathcal{Z}_\emptyset = 1$, PDE (1.1), COV (1.2), ASY (1.3), *the power law bound* (1.4), *and the cascade relation* (6.2).

The uniqueness in Proposition 6.1 follows from [FK15, Lemma 1]. In fact, the uniqueness proved in [FK15, Lemma 1] is much stronger. Using our notations, [FK15, Lemma 1] reads as follows: Let $\kappa \in (0, 8)$. Suppose $F : \mathfrak{X}_{2N} \to \mathbb{C}$ is a smooth function satisfying PDE (1.1), COV (1.2), and the following two properties.

- There exists constants $C > 0$ and $p > 0$ such that, for all $(x_1, \dots, x_{2N}) \in \mathfrak{X}_{2N}$, we have

$$|F(x_1, \dots, x_{2N})| \leq C \prod_{1 \leq i < j \leq 2N} (x_j - x_i)^{\mu_{ij}(p)}, \quad \text{where}$$

$$\mu_{ij}(p) = \begin{cases} p, & \text{if } |x_j - x_i| > 1, \\ -p, & \text{if } |x_j - x_i| \leq 1. \end{cases}$$

- The asymptotics:

$$\lim_{x_j, x_{j+1} \to \xi} \frac{F(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-2h}} = 0, \quad \text{for all } j \in \{2, 3, \dots, 2N-1\} \text{ and } \xi \in (x_{j-1}, x_{j+2})$$

(with the convention that $x_0 = -\infty$ and $x_{2N+1} = \infty$).

Then $F \equiv 0$. From this result, the uniqueness part in Proposition 6.1 is immediate. We focus on the existence part in this section. Note that the existence part in Proposition 6.1 is different from the one in Theorem 1.7: In Proposition 6.1, we also require the cascade relation in the statement. In fact, the cascade relation plays an essential role in our proof.

Before we proceed, we collect some basic properties here. Recall that, given a nice Dobrushin domain $(\Omega; x, y)$, the notation $H_\Omega(x, y)$ denotes the boundary Poisson kernel. With the general definition of $\mathcal{Z}_\alpha$ in (6.1) for general nice polygon $(\Omega; x_1, \dots, x_{2N})$, we can rewrite ASY (1.3) as follows: for all $\alpha \in \mathrm{LP}_N$ and for all $j \in \{1, \ldots, 2N\}$ and $\xi \in (x_{j-1} x_{j+2})$,

$$\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{H_\Omega(x_j, x_{j+1})^h}$$
$$= \begin{cases} 0 & \text{if } \{j, j+1\} \notin \alpha \\ \mathcal{Z}_{\hat{\alpha}}(\Omega; x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}) & \text{if } \{j, j+1\} \in \alpha \end{cases} \tag{6.3}$$

where $\hat{\alpha} = \alpha / \{j, j+1\} \in \mathrm{LP}_{N-1}$.

Define, for $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N$,

$$\mathcal{B}_\alpha(\Omega; x_1, \ldots, x_{2N}) = \prod_{j=1}^{N} H_\Omega(x_{a_j}, x_{b_j})^{1/2}.$$

Then the power law bound (1.4) can be written as follows:

$$0 < \mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N}) \le \mathcal{B}_\alpha(\Omega; x_1, \ldots, x_{2N})^{2h}. \tag{6.4}$$

The boundary Poisson kernel has monotonicity: suppose $(\Omega; x, y)$ is a nice Dobrushin domain and suppose $U \subset \Omega$ is simply connected and agrees with Ω in neighborhoods of x and y. Then $H_U(x, y) \le H_\Omega(x, y)$. As a consequence, we have the monotonicity of $\mathcal{B}_\alpha$: suppose $(\Omega; x_1, \ldots, x_{2N})$ is a nice polygon and suppose $U \subset \Omega$ is simply connected and agrees with Ω in neighborhoods of $\{x_1, \ldots, x_{2N}\}$. Then, for any $\alpha \in \mathrm{LP}_N$,

$$\mathcal{B}_\alpha(U; x_1, \ldots, x_{2N}) \le \mathcal{B}_\alpha(\Omega; x_1, \ldots, x_{2N}). \tag{6.5}$$

6.1. Proof of Proposition 6.1. We will prove the existence in Proposition 6.1 by induction on N. It is immediate to check the existence for $N = 1$ and $N = 2$. When $N = 1$, for $x < y$ and $= \{\{1, 2\}\}$,

$$\mathcal{Z}(x, y) = (y - x)^{-2h}.$$

When $N = 2$, we obtain for $= \{\{1, 4\}, \{2, 3\}\}$ and $= \{\{1, 2\}, \{3, 4\}\}$, and for $x_1 < x_2 < x_3 < x_4$,

$$\begin{aligned} \mathcal{Z}(x_1, x_2, x_3, x_4) &= (x_4 - x_1)^{-2h} (x_3 - x_2)^{-2h} z^{2/\kappa} \frac{F(z)}{F(1)}, \\ \mathcal{Z}(x_1, x_2, x_3, x_4) &= (x_2 - x_1)^{-2h} (x_4 - x_3)^{-2h} (1 - z)^{2/\kappa} \frac{F(1 - z)}{F(1)}, \end{aligned} \tag{6.6}$$

where z is the cross-ratio and F is the hypergeometric function in (3.2) with $\nu = 0$:

$$z = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_4 - x_2)(x_3 - x_1)}, \quad F(z) := {}_2F_1\left(\frac{4}{\kappa}, 1 - \frac{4}{\kappa}, \frac{8}{\kappa}; z\right).$$

Suppose the collection of pure partition functions exists up to N, and consider LP_{N+1}. Assume $\alpha = \{\{a_1, b_1\}, \ldots, \{a_{N+1}, b_{N+1}\}\}$ and $a_j < b_j$ for all $1 \le j \le N + 1$. Suppose $(\Omega; x_1, \ldots, x_{2N+2})$ is a nice polygon. For $1 \le k \le N + 1$, let η_k be an SLE_κ in Ω

from x_{a_k} to x_{b_k}. We denote by $\mathcal{E}_\alpha^k$ the event that η_k is allowed by α and we define $D_k^{R,1}, \ldots, D_k^{R,r}, D_k^R, D_k^{L,1}, \ldots, D_k^{L,l}, D_k^L, \alpha_k^{R,1}, \ldots, \alpha_k^{R,r}, \alpha_k^{L,1}, \ldots, \alpha_k^{L,l}$ in the same way as before. As the collection of pure partition functions exists up to N, the following two functions are well-defined:

$$\mathcal{Z}_{\alpha_k^R}(D_k^R; x_{a_k+1}, x_{a_k+2}, \ldots, x_{b_k-1}), \quad \mathcal{Z}_{\alpha_k^L}(D_k^L; x_{b_k+1}, x_{b_k+2}, \ldots, x_{a_k-1}).$$

Then, we define

$$\begin{aligned} &\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2}) \\ &\quad = H_\Omega(x_{a_k}, x_{b_k})^h \mathbb{E}\left[\mathcal{Z}_{\alpha_k^R}(D_k^R; x_{a_k+1}, \ldots, x_{b_k-1}) \times \mathcal{Z}_{\alpha_k^L}(D_k^L; x_{b_k+1}, \ldots, x_{a_k-1}) \mathbb{1}_{\mathcal{E}_\alpha^k}\right]. \end{aligned} \tag{6.7}$$

Eq. (6.7) is analog of (6.2) for $2N+2$ marked points. The expectation in the right hand side is finite, see Lemma 6.7. Thus the function $\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2})$ in (6.7) is well-defined. When $\Omega = \mathbb{H}$, we denote $\mathcal{Z}_\alpha^{(k)}(\mathbb{H}; x_1, \ldots, x_{2N+2})$ by $\mathcal{Z}_\alpha^{(k)}(x_1, \ldots, x_{2N+2})$. From above definition, $\mathcal{Z}_\alpha^{(k)}$ depends on the choice of $k \in \{1, \ldots, N+1\}$, but we will show that it does not:

Lemma 6.2. *Suppose Proposition 6.1 holds up to N. The function $\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2})$ defined in (6.7) does not depend on the choice of k.*

This lemma is the one that we need to use properties of hypergeometric SLE. We leave its proof to Sect. 6.2. Next, we show that the functions $\mathcal{Z}_\alpha^{(k)}$ satisfy all the requirements in Proposition 6.1 one by one in Lemmas 6.3 to 6.7.

Lemma 6.3. *Suppose Proposition 6.1 holds up to N. The function $\mathcal{Z}_\alpha^{(k)}(x_1, \ldots, x_{2N+2})$ defined in (6.7) is smooth and satisfies two* PDE*s in (1.1) with $i = a_k$ and $i = b_k$.*

Proof. We only prove the conclusion for $i = a_k$ and the case when $i = b_k$ can be proved similarly as SLE is reversible.

Recall that, in the definition of $\mathcal{Z}_\alpha^{(k)}$, the curve η_k is an SLE$_\kappa$ in $\mathbb{H}$ from $x_{a_k}(= x_i)$ to x_{b_k}. We parametrize η_k by the half-plane capacity and denote by $(g_t, t \geq 0)$ the corresponding conformal maps in the Loewner chain. Let us calculate the conditional expectation $\mathbb{E}\left[\mathcal{Z}_{\alpha_k^R}(D_k^R; \ldots) \times \mathcal{Z}_{\alpha_k^L}(D_k^L; \ldots) \mathbb{1}_{\mathcal{E}_\alpha^k} \mid \eta_k[0,t]\right]$ for small $t > 0$. By the conformal covariance of $\mathcal{Z}_{\alpha_k^L}$ and $\mathcal{Z}_{\alpha_k^R}$ in the hypothesis, we have

$$\begin{aligned} &\mathbb{E}\left[\mathcal{Z}_{\alpha_k^R}(D_k^R; x_{i+1}, \ldots, x_{b_k-1}) \times \mathcal{Z}_{\alpha_k^L}(D_k^L; x_{b_k+1}, \ldots, x_{i-1}) \mathbb{1}_{\mathcal{E}_\alpha^k} \mid \eta_k[0,t]\right] \\ &\quad = \prod_{j \neq i, b_k} g_t'(x_j)^h \times \mathbb{E}\Big[\mathcal{Z}_{\alpha_k^R}\left(g_t(D_k^R); g_t(x_{i+1}), \ldots, g_t(x_{b_k-1})\right) \\ &\qquad\qquad \times \mathcal{Z}_{\alpha_k^L}\left(g_t(D_k^L); g_t(x_{b_k+1}), \ldots, g_t(x_{i-1})\right) \mathbb{1}_{\mathcal{E}_\alpha^k} \mid \eta_k[0,t]\Big] \\ &\quad = \prod_{j \neq i, b_k} g_t'(x_j)^h \times (g_t(x_{b_k}) - W_t)^{2h} \\ &\qquad \times \mathcal{Z}_\alpha^{(k)}(g_t(x_1), \ldots, g_t(x_{i-1}), W_t, g_t(x_{i+1}), \ldots, g_t(x_{2N+2})). \end{aligned}$$

Therefore, the following process is a martingale for η_k:

$$\prod_{j\neq i,b_k} g_t'(x_j)^h \times (g_t(x_{b_k}) - W_t)^{2h} \times \mathcal{Z}_\alpha^{(k)}(g_t(x_1), \ldots, g_t(x_{i-1}), W_t, g_t(x_{i+1}), \ldots, g_t(x_{2N+2})).$$

By Itô's formula, the function $\mathcal{Z}_\alpha^{(k)}$ satisfies the PDE (1.1) with $i = a_k$ in the distribution sense, see details in [PW19, Proof of Lemma 4.4]. By [Dub15, Lemma 5] (see also [PW19, Proposition 2.5]), the operator

$$\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{2h}{(x_j - x_i)^2}\right)$$

in PDE (1.1) is hypoelliptic. Therefore, the function $\mathcal{Z}_\alpha^{(k)}$ is a smooth solution to the PDE (1.1) with $i = a_k$. □

Lemma 6.4. *Suppose Proposition 6.1 holds up to N. The function $\mathcal{Z}_\alpha^{(k)}(x_1, \ldots, x_{2N+2})$ defined in (6.7) is smooth and satisfies $(2N+2)$ PDEs in (1.1).*

Proof. In Lemma 6.3, we show that $\mathcal{Z}_\alpha^{(k)}$ satisfies PDE (1.1) with $i = a_k$ and $i = b_k$. By Lemma 6.2, we have $\mathcal{Z}_\alpha^{(k)} = \mathcal{Z}_\alpha^{(n)}$ for any $n \neq k$. Combining with Lemma 6.3, $\mathcal{Z}_\alpha^{(k)} = \mathcal{Z}_\alpha^{(n)}$ also satisfies PDE (1.1) with $i = a_n$ and $i = b_n$. This completes the proof. □

Lemma 6.5. *Suppose Proposition 6.1 holds up to N. The function $\mathcal{Z}_\alpha^{(k)}(x_1, \ldots, x_{2N+2})$ defined in (6.7) satisfies* COV (1.2).

Proof. This is true because: (a) SLE_κ is conformally invariant; (b) the boundary Poisson kernel is conformally covariant; (c) the pure partition functions $\mathcal{Z}_{\alpha_k^R}$ and $\mathcal{Z}_{\alpha_k^L}$ are conformally covariant by the hypothesis. □

Lemma 6.6. *Suppose Proposition 6.1 holds up to N. The function $\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2})$ defined in (6.7) satisfies* ASY (6.3).

Proof. In order to prove ASY (6.3), we need to check the following cases: Case (a). $\{a_k, b_k\} = \{j, j+1\}$; Case (b). $a_k = j$ and $b_k \neq j+1$. The cases $\#\{a_k, b_k\} \cap \{j, j+1\} = 1$ can be proved similarly; Case (c). $\{a_k, b_k\} \cap \{j, j+1\} = \emptyset$.

Case (a). Suppose $\{a_k, b_k\} = \{j, j+1\}$. Note that η_k is the SLE_κ in $\mathbb{H}$ from x_j to x_{j+1}. In this case, $\alpha_k^R = \emptyset$ and $\alpha_k^L = \hat{\alpha} := \alpha/\{j, j+1\}$. Then we have

$$\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2}) = H_\Omega(x_j, x_{j+1})^h \mathbb{E}\left[\mathcal{Z}_{\hat{\alpha}}(D_k^L; x_{j+2}, \ldots, x_{j-1})\mathbb{1}_{\mathcal{E}_\alpha^k}\right].$$

By the power law bound in the hypothesis, (6.5) and $h \geq 0$, we have

$$\mathcal{Z}_{\hat{\alpha}}(D_k^L; x_{j+2}, \ldots, x_{j-1}) \leq \mathcal{B}_{\hat{\alpha}}(\Omega; x_{j+2}, \ldots, x_{j-1})^{2h}.$$

Bounded convergence theorem gives

$$\begin{aligned}
&\lim_{x_j, x_{j+1}\to\xi} \frac{\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2})}{H_\Omega(x_j, x_{j+1})^h} \\
&= \lim_{x_j, x_{j+1}\to\xi} \mathbb{E}\left[\mathcal{Z}_{\hat{\alpha}}(D_k^L; x_{j+2}, \ldots, x_{j-1})\mathbb{1}_{\mathcal{E}_\alpha^k}\right]
\end{aligned}$$

$$= \mathcal{Z}_{\hat{\alpha}}(\Omega; x_{j+2}, \ldots, x_{j-1}).$$

This completes the proof of Case (a).

Case (b). $a_k = j$ and $b_k \neq j+1$. In this case, we have

$$\begin{aligned}&\frac{\mathcal{Z}^{(k)}_{\alpha}(\Omega; x_1, \ldots, x_{2N+2})}{H_{\Omega}(x_j, x_{j+1})^h}\\ &= H_{\Omega}(x_{a_k}, x_{b_k})^h \mathbb{E}\left[\frac{\mathcal{Z}_{\alpha^R_k}(D^R_k; x_{a_k+1}, \ldots, x_{b_k-1})}{H_{\Omega}(x_j, x_{j+1})^h} \times \mathcal{Z}_{\alpha^L_k}(D^L_k; x_{b_k+1}, \ldots, x_{a_k-1}) \mathbb{1}_{\mathcal{E}^k_{\alpha}}\right].\end{aligned}$$

When $\kappa < 6$ (thus $h > 0$), by the power law bound in the hypothesis and (6.5), we have

$$\begin{aligned}&\mathcal{Z}_{\alpha^L_k}(D^L_k; x_{b_k+1}, \ldots, x_{a_k-1}) \le \mathcal{B}_{\alpha^L_k}(\Omega; x_{b_k+1}, \ldots, x_{a_k-1})^{2h};\\ &\frac{\mathcal{Z}_{\alpha^R_k}(D^R_k; x_{a_k+1}, \ldots, x_{b_k-1})}{H_{\Omega}(x_j, x_{j+1})^h} \le \frac{\mathcal{B}_{\alpha^R_k}(\Omega; x_{a_k+1}, \ldots, x_{b_k-1})^{2h}}{H_{\Omega}(x_j, x_{j+1})^h} \to 0, \quad \text{as } x_j, x_{j+1} \to \xi.\end{aligned}$$

Therefore, when $\kappa < 6$, we have

$$\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}^{(k)}_{\alpha}(\Omega; x_1, \ldots, x_{2N+2})}{H_{\Omega}(x_j, x_{j+1})^h} = 0.$$

When $\kappa = 6$ (thus $h = 0$), by the power law bound in the hypothesis, we have

$$\mathcal{Z}_{\alpha^L_k}(D^L_k; x_{b_k+1}, \ldots, x_{a_k-1}) \le 1, \quad \mathcal{Z}_{\alpha^R_k}(D^R_k; x_{a_k+1}, \ldots, x_{b_k-1}) \le 1.$$

Therefore, when $\kappa = 6$, we have

$$\lim_{x_j, x_{j+1} \to \xi} \mathcal{Z}^{(k)}_{\alpha}(\Omega; x_1, \ldots, x_{2N+2}) \le \lim_{x_j, x_{j+1} \to \xi} \mathbb{P}[\mathcal{E}^k_{\alpha}] = 0.$$

This completes the proof of Case (b).

Case (c). $\{a_k, b_k\} \cap \{j, j+1\} = \emptyset$. We may assume $a_k < j < j+1 < b_k$. In this case, we have

$$\begin{aligned}&\frac{\mathcal{Z}^{(k)}_{\alpha}(\Omega; x_1, \ldots, x_{2N+2})}{H_{\Omega}(x_j, x_{j+1})^h}\\ &= H_{\Omega}(x_{a_k}, x_{b_k})^h \mathbb{E}\left[\frac{\mathcal{Z}_{\alpha^R_k}(D^R_k; x_{a_k+1}, \ldots, x_{b_k-1})}{H_{\Omega}(x_j, x_{j+1})^h} \times \mathcal{Z}_{\alpha^L_k}(D^L_k; x_{b_k+1}, \ldots, x_{a_k-1}) \mathbb{1}_{\mathcal{E}^k_{\alpha}}\right].\end{aligned}$$

By the power law bound in the hypothesis, (6.5) and $h \ge 0$, we have

$$\begin{aligned}&\mathcal{Z}_{\alpha^L_k}(D^L_k; x_{b_k+1}, \ldots, x_{a_k-1}) \le \mathcal{B}_{\alpha^L_k}(\Omega; x_{b_k+1}, \ldots, x_{a_k-1})^{2h};\\ &\frac{\mathcal{Z}_{\alpha^R_k}(D^R_k; x_{a_k+1}, \ldots, x_{b_k-1})}{H_{\Omega}(x_j, x_{j+1})^h} \le \frac{\mathcal{B}_{\alpha^R_k}(\Omega; x_{a_k+1}, \ldots, x_{b_k-1})^{2h}}{H_{\Omega}(x_j, x_{j+1})^h}.\end{aligned}$$

If $\{j, j+1\} \in \alpha$, then we have $\{j, j+1\} \in \alpha_k^R$, denote by $\hat{\alpha} = \alpha/\{j, j+1\}$ and $\hat{\alpha}_k^R = \alpha_k^R/\{j, j+1\}$. We have

$$\frac{\mathcal{Z}_{\alpha_k^R}(D_k^R; x_{a_k+1}, \ldots, x_{b_k-1})}{H_\Omega(x_j, x_{j+1})^h} \le \frac{\mathcal{B}_{\alpha_k^R}(\Omega; x_{a_k+1}, \ldots, x_{b_k-1})^{2h}}{H_\Omega(x_j, x_{j+1})^h}$$
$$= \mathcal{B}_{\hat{\alpha}_k^R}(\Omega; x_{a_k+1}, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{b_k-1})^{2h}.$$

By the asymptotic in the hypothesis, we have almost surely on $\mathcal{E}_\alpha^k$,

$$\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}_{\alpha_k^R}(D_k^R; x_{a_k+1}, \ldots, x_{b_k-1})}{H_\Omega(x_j, x_{j+1})^h} = \mathcal{Z}_{\hat{\alpha}_k^R}(D_k^R; x_{a_k+1}, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{b_k-1}).$$

Bounded convergence theorem and the cascade relation in the hypothesis give

$$\begin{aligned}
&\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2})}{H_\Omega(x_j, x_{j+1})^h} \\
&= H_\Omega(x_{a_k}, x_{b_k})^h \\
&\times \mathbb{E}\left[\mathcal{Z}_{\hat{\alpha}_k^R}(D_k^R; x_{a_k+1}, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{b_k-1}) \times \mathcal{Z}_{\alpha_k^L}(D_k^L; x_{b_k+1}, \ldots, x_{a_k-1}) \mathbb{1}_{\mathcal{E}_\alpha^k}\right] \\
&= \mathcal{Z}_{\hat{\alpha}}(\Omega; x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}).
\end{aligned}$$

If $\{j, j+1\} \notin \alpha$ and $\kappa < 6$ (thus $h > 0$), we have

$$\frac{\mathcal{B}_{\alpha_k^R}(\Omega; x_{a_k+1}, \ldots, x_{b_k-1})^{2h}}{H_\Omega(x_j, x_{j+1})^h} \to 0, \quad \text{as } x_j, x_{j+1} \to \xi.$$

Thus

$$\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2})}{H_\Omega(x_j, x_{j+1})^h} = 0.$$

If $\{j, j+1\} \notin \alpha$ and $\kappa = 6$ (thus $h = 0$), by the power law bound and the asymptotic in the hypothesis, we have

$$\mathcal{Z}_{\alpha_k^L}(D_k^L; x_{b_k+1}, \ldots, x_{a_k-1}) \le 1;$$
$$\mathcal{Z}_{\alpha_k^R}(D_k^R; x_{a_k+1}, \ldots, x_{b_k-1}) \le 1; \qquad \lim_{x_j, x_{j+1} \to \xi} \mathcal{Z}_{\alpha_k^R}(D_k^R; x_{a_k+1}, \ldots, x_{b_k-1}) = 0.$$

Bounded convergence theorem gives

$$\lim_{x_j, x_{j+1} \to \xi} \mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2}) = 0.$$

This completes the proof of Case (c) and completes the proof of this lemma. □

Lemma 6.7. *Suppose Proposition 6.1 holds up to N. The function $\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2})$ defined in (6.7) satisfies the power law bound (6.4).*

Proof. By the power law bound in the hypothesis, (6.5) and $h \geq 0$, we have

$$\begin{aligned} &\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2}) \\ &\leq H_\Omega(x_{a_k}, x_{b_k})^h \times \mathcal{B}_{\alpha_k^R}(\Omega; x_{a_k+1}, \ldots, x_{b_k-1})^{2h} \times \mathcal{B}_{\alpha_k^L}(\Omega; x_{b_k+1}, \ldots, x_{a_k-1})^{2h} \\ &= \mathcal{B}_\alpha(\Omega; x_1, \ldots, x_{2N+2})^{2h}. \end{aligned}$$

□

Now, we are ready to prove the conclusion.

Proof of Proposition 6.1—Existence. It is clear that the conclusion holds for $N = 1$. Suppose the conclusion holds up to N. For $1 \leq k \leq N+1$, define $\mathcal{Z}_\alpha^{(k)}$ as in (6.7). By Lemma 6.2, it does not depend on the choice of k. Thus we denote it by $\mathcal{Z}_\alpha$. Consider the functions $\{\mathcal{Z}_\alpha, \alpha \in \mathrm{LP}_{N+1}\}$. By Lemma 6.4, they satisfy $(2N+2)$ PDEs in (1.1). By Lemma 6.5, they satisfy COV (1.2). By Lemma 6.6, they satisfy ASY (1.3). By Lemma 6.7, they satisfy the power law bound (1.4). Combining Lemma 6.2 and (6.7), we obtain the cascade relation (6.2). These complete the proof. □

It is clear that Proposition 6.1 implies Theorem 1.7. Moreover, as a consequence of Proposition 6.1, we also obtain the cascade relation of the pure partition functions.

Corollary 6.8. *The collection of pure partition functions in Theorem 1.7 also satisfies the cascade relation (6.2).*

In fact, the proof of Lemma 6.6 implies the following refined asymptotic. We do not need this refined asymptotic in this paper, but it is very useful when one tries to derive probabilities for certain crossing events in related models, see [PW19, Section 5] and [PW18]. So we record this result here.

Corollary 6.9. *The collection of pure partition functions in Theorem 1.7 also satisfies the following refined asymptotic: for all $\alpha \in \mathrm{LP}_N$ and for all $j \in \{1, \ldots, 2N-1\}$ and $x_1 < x_2 < \cdots < x_{j-1} < \xi < x_{j+2} < \cdots < x_{2N}$,*

$$\begin{aligned} &\lim_{\substack{\tilde{x}_j, \tilde{x}_{j+1} \to \xi, \\ \tilde{x}_i \to x_i \text{ for } i \neq j, j+1}} \frac{\mathcal{Z}_\alpha(\tilde{x}_1, \ldots, \tilde{x}_{2N})}{(\tilde{x}_{j+1} - \tilde{x}_j)^{-2h}} \\ &= \begin{cases} 0 & \textit{if } \{j, j+1\} \notin \alpha \\ \mathcal{Z}_{\hat{\alpha}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}) & \textit{if } \{j, j+1\} \in \alpha \end{cases}, \end{aligned}$$

where $\hat{\alpha} = \alpha / \{j, j+1\}$.

Finally, let us discuss the range of κ in Theorem 1.7. The proofs of Lemmas 6.2, 6.4, and 6.5 hold for all $\kappa \in (0, 8)$; whereas, the proofs of Lemmas 6.6 and 6.7 only hold for $\kappa \in (0, 6]$ because we use $h \geq 0$ in various places.

6.2. Proof of Lemma 6.2. To show Lemma 6.2, we need the following property of hypergeometric SLE and Proposition 3.5.

Proposition 6.10. *Fix $\kappa \in (0, 8)$ and a quad $q = (\Omega; x^R, y^R, y^L, x^L)$.*

– *(Existence and Uniqueness) There exists a unique probability measure on pairs of continuous curves* $(\eta^L; \eta^R) \in X_0(\Omega; x^R, y^R, y^L, x^L)$ *such that the conditional law of* η^R *given* η^L *is* SLE_κ *in* Ω^L *from* x^R *to* y^R *and the conditional law of* η^L *given* η^R *is* SLE_κ *in* Ω^R *from* x^L *to* y^L.
– *(Identification) Under this probability measure, the marginal law of* η^L *is* hSLE_κ *in* Ω *from* x^L *to* y^L *with marked points* (x^R, y^R).

Proof. When $\kappa \le 4$, this proposition is a special case of Proposition 4.1 when $\rho^L = \rho^R = 0$. When $\kappa \in (4, 8)$, the existence and the uniqueness were proved in [MS16c] and [MSW16, Appendix A]. In [BPW18], the authors provided another perspective for the existence and the uniqueness with $\kappa \in (4, 6]$. We define *global* 2-SLE_κ to be this unique probability measure. It remains to derive the marginal law of η^L in global 2-SLE_κ. Such question is included in some form in previous papers: [BBK05, Section 8], [Dub06, Section 4], and [MW18, Section 4]. Let us briefly summarize how they derived the marginal law.

Suppose $(\eta^L; \eta^R) \in X_0(\Omega; x^R, y^R, y^L, x^L)$ is the global 2-SLE_κ. Suppose $U_1, \ldots, U_4$ are neighborhoods of the points x^L, y^L, y^R, x^R respectively such that $\Omega \setminus U_j$ are simply connected and $U_j \cap U_k = \emptyset$ for $j \neq k$. Let γ_1 be the part of η^L that starts from x^L and ends at exiting U_1; let γ_2 be the part of η^R that starts from x^R and ends at exiting U_2; let γ_3 be the part of the time-reversal of η^R that starts from y^R and ends at exiting U_3; let γ_4 be the part of the time-reversal of η^L that starts from y^L and ends at exiting U_4. By the conformal invariance of the global 2-SLE_κ and the reversibility of SLE_κ, we could argue that $(\gamma_1, \ldots, \gamma_4)$ is a local 2-SLE_κ, as described in [KP16, Theorem A.4]. By the commutation relation in [Dub07] and a complete classification summarized in [KP16, Theorem A.4], we know that γ_1 has the law of hSLE_κ. In other words, the law of η^L restricted to U_1 has the law of hSLE_κ. This is true for any localization neighborhoods $(U_1, \ldots, U_4)$. This implies that η^L is an hSLE_κ up to any stopping time τ as long as $\eta^L[0, \tau]$ has positive distance from the points $\{x^R, y^R, y^L\}$. By Proposition 3.2, hSLE_κ in Ω from x^L to y^L with marked points (x^R, y^R) is generated by continuous transient curuve and it does have positive distance from the points $\{x^R, y^R\}$ almost surely, thus the whole process η^L has the law of hSLE_κ as desired. □

Proof of Lemma 6.2. Pick $n \neq k$, we will show that $\mathcal{Z}_\alpha^{(k)} = \mathcal{Z}_\alpha^{(n)}$. Assume $a_k < a_n < b_n < b_k$. Recall that η_k is an SLE_κ in Ω from x_{a_k} to x_{b_k}, and that

$$\begin{aligned}
&\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2}) \\
&\quad = H_\Omega(x_{a_k}, x_{b_k})^h \mathbb{E}\left[\mathcal{Z}_{\alpha_k^R}(D_k^R; x_{a_k+1}, \ldots, x_{b_k-1}) \times \mathcal{Z}_{\alpha_k^L}(D_k^L; x_{b_k+1}, \ldots, x_{a_k-1}) \mathbb{1}_{\mathcal{E}_\alpha^k}\right].
\end{aligned}$$

Let η_n be an SLE_κ in D_k^R from x_{a_n} to x_{b_n}. Define $\mathcal{E}_\alpha^{kn}$ to be the event that η_n is allowed by α_k^R in D_k^R. On $\mathcal{E}_\alpha^{kn}$, let D_n^R be the union of the connected components of $D_k^R \setminus \eta_n$ having $x_{a_n+1}, \ldots, x_{b_n-1}$ on the boundary, and D_{kn}^M be the union of the connected components of $D_k^R \setminus \eta_n$ having $x_{a_k+1}, \ldots, x_{a_n-1}, x_{b_n+1}, \ldots, x_{b_k-1}$ on the boundary, see Fig. 7.

The links $\{a_k, b_k\}$ and $\{a_n, b_n\}$ divide the link pattern α into three sub-link patterns, connecting $\{b_k + 1, \ldots, a_k - 1\}$, $\{a_k + 1, \ldots, a_n - 1, b_n + 1, \ldots, b_k - 1\}$, and $\{a_n + 1, \ldots, b_n - 1\}$ respectively. After relabelling the remaining indices, we denote these link patterns by $\alpha_k^L, \alpha_{kn}^M, \alpha_n^R$. The marked points of the domains D_k^L, D_{kn}^M, D_n^R are clear, so we omit them from the notation.

By the cascade relation in the hypothesis, we have

$$\mathcal{Z}_{\alpha_k^R}(D_k^R; x_{a_k+1}, \ldots, x_{b_k-1}) = H_{D_k^R}(x_{a_n}, x_{b_n})^h \mathbb{E}\left[\mathcal{Z}_{\alpha_n^R}(D_n^R; \ldots) \times \mathcal{Z}_{\alpha_{kn}^M}(D_{kn}^M; \ldots)\mathbb{1}_{\mathcal{E}_\alpha^{kn}}\right].$$

Plugging into the definition of $\mathcal{Z}_\alpha^{(k)}$, we have

$$\begin{aligned}
&\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2}) \\
&= H_\Omega(x_{a_k}, x_{b_k})^h \\
&\times \mathbb{E}\left[H_{D_k^R}(x_{a_n}, x_{b_n})^h \times \mathcal{Z}_{\alpha_n^R}(D_n^R; \ldots) \times \mathcal{Z}_{\alpha_{kn}^M}(D_{kn}^M; \ldots) \times \mathcal{Z}_{\alpha_k^L}(D_k^L; \ldots)\mathbb{1}_{\mathcal{E}_\alpha^k \cap \mathcal{E}_\alpha^{kn}}\right].
\end{aligned}$$

Here $\mathbb{E}$ corresponds to the following probability measure: sample η_k as SLE_κ in Ω from x_{a_k} to x_{b_k}; given η_k and on $\mathcal{E}_\alpha^k$, sample η_n as SLE_κ in D_k^R from x_{a_n} to x_{b_n}. Note that $\mathcal{E}_\alpha^k \cap \mathcal{E}_\alpha^{kn}$ can be written as $\mathcal{E}_\alpha^k \cap \mathcal{E}_\alpha^n \cap \mathcal{F}_\alpha^{kn}$ where the event $\mathcal{F}_\alpha^{kn}$ is that η_k stays to the left of η_n.

From Proposition 3.5, the law of η_k weighted by $H_{D_k^R}(x_{a_n}, x_{b_n})^h$ becomes hSLE_κ in Ω from x_{a_k} to x_{b_k} with marked points (x_{a_n}, x_{b_n}). Moreover, the Radon-Nikodym derivative between hSLE_κ and SLE_κ is the following

$$\frac{H_\Omega(x_{a_k}, x_{b_k})^h H_{D_k^R}(x_{a_n}, x_{b_n})^h}{\mathcal{Z}_{\frown}(\Omega; x_{a_k}, x_{a_n}, x_{b_n}, x_{b_k})}$$

where $\mathcal{Z}_{\frown}$ is defined in (6.6).

Denote by $q = (\Omega; x_{a_n}, x_{b_n}, x_{b_k}, x_{a_k})$ and denote by $\mathbb{Q}_q$ the following probability measure: sample η_k as hSLE_κ in Ω from x_{a_k} to x_{b_k} with marked points (x_{a_n}, x_{b_n}); given η_k, sample η_n as SLE_κ in D_k^R from x_{a_n} to x_{b_n}. Then we have

$$\begin{aligned}
&\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2}) \\
&= \mathcal{Z}_{\frown}(\Omega; x_{a_k}, x_{a_n}, x_{b_n}, x_{b_k}) \\
&\times \mathbb{Q}_q\left[\mathcal{Z}_{\alpha_n^R}(D_n^R; \ldots) \times \mathcal{Z}_{\alpha_{kn}^M}(D_{kn}^M; \ldots) \times \mathcal{Z}_{\alpha_k^L}(D_k^L; \ldots)\mathbb{1}_{\mathcal{E}_\alpha^k \cap \mathcal{E}_\alpha^n \cap \mathcal{F}_\alpha^{kn}}\right].
\end{aligned} \tag{6.8}$$

By Proposition 6.10, $\mathbb{Q}_q$ is the same as the unique probability measure there. In particular, it is symmetric in η_k and η_n. Therefore, the function $\mathcal{Z}_\alpha^{(n)}(\Omega; x_1, \ldots, x_{2N+2})$ can be expanded in the same way as the right hand side of (6.8). As a consequence,

$$\mathcal{Z}_\alpha^{(k)}(\Omega; x_1, \ldots, x_{2N+2}) = \mathcal{Z}_\alpha^{(n)}(\Omega; x_1, \ldots, x_{2N+2}),$$

as desired. □

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A. Appendix: Hypergeometric Functions

For $A, B, C \in \mathbb{R}$, the hypergeometric function is defined for $|z| < 1$ by the power series:

$$F(z) = {}_2F_1(A, B, C; z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{(C)_n} \frac{z^n}{n!},$$

where $(x)_n$ is the Pochhammer symbol $(x)_n := x(x+1)\cdots(x+n-1)$ for $n \ge 1$ and $(x)_n = 1$ for $n = 0$. The power series is well-defined when $C \notin \{0, -1, -2, -3, \ldots\}$, and it is absolutely convergent on $z \in [0, 1]$ when $C > A + B$. When $C > A + B$ and $C \notin \{0, -1, -2, -3, \ldots\}$, we have

$$F(1) = \frac{\Gamma(C)\Gamma(C - A - B)}{\Gamma(C - A)\Gamma(C - B)}, \tag{A.1}$$

where Γ is Gamma Function. The hypergeometric function is a solution of Euler's hypergeometric differential equation

$$z(1-z)F''(z) + (C - (A + B + 1)z)F'(z) - ABF(z) = 0. \tag{A.2}$$

Lemma A.1. *When $C > 0$ and $AB > 0$, the function $F(z)$ is increasing for $z \in [0, 1)$.*

Proof. We have $F(0) = 1$ and $F'(0) = AB/C > 0$. If the conclusion is false, then there exists $z_0 \in (0, 1)$ such that F is increasing for $z \in (0, z_0)$ and is decreasing for $z \in (z_0, z_0+\epsilon)$ for some $\epsilon > 0$. This implies that z_0 is a local maximum and thus $F(z_0) \ge 1$, $F'(z_0) = 0$ and $F''(z_0) \le 0$. However, by (A.2), we have $z_0(1-z_0)F''(z_0) = ABF(z_0)$, contradiction. □

Lemma A.2. *When $C > 0$, $C > A$, $C > B$, $C > A+B$ and $AB < 0$, the function $F(z)$ is decreasing for $z \in [0, 1]$.*

Proof. We assume $B < 0 < A$. There exists $n \in \{1, 2, \ldots\}$ such that $1 > B + n \ge 0$. By [AS92, Eq. (15.2.2)], we have, for $j \ge 1$,

$$F^{(j)}(z) = \frac{(A)_j (B)_j}{(C)_j} {}_2F_1(A + j, B + j, C + j; z).$$

To get the monotinicity of F, we will consider the sign and monotonicity of $F^{(j)}$ with $1 \le j \le n$. Note that,

$$(A)_j > 0, \quad \mathrm{sign}((B)_j) = (-1)^j, \quad (C)_j > 0, \quad \text{for } 1 \le j \le n.$$

Since $C > A + B + n - 1$, by (A.1), we have

$${}_2F_1(A + j, B + j, C + j; 1) \in (0, \infty), \quad \text{for } 0 \le j \le n - 1.$$

Since $A + n > 0$, $B + n \ge 0$, $C + n > 0$, the function ${}_2F_1(A + n, B + n, C + n; \cdot)$ is increasing, thus

$${}_2F_1(A + n, B + n, C + n; z) \ge 1, \quad \text{for } z \in [0, 1).$$

If n is even, we have $F^{(n)}(z) \geq 0$. Thus $F^{(n-1)}(\cdot)$ is increasing. In particular,

$$F^{(n-1)}(z) \leq F^{(n-1)}(1) = \frac{(A)_{n-1}(B)_{n-1}}{(C)_{n-1}} {}_2F_1(A+n-1, B+n-1, C+n-1; 1) \leq 0.$$

Thus $F^{(n-2)}(\cdot)$ is decreasing and $F^{(n-2)}(z) \geq F^{(n-2)}(1) \geq 0$. In this way, we could argue that $F^{(n-j)}(\cdot)$ is decreasing for even j and it is increasing for odd j. In particular, F is decreasing.

If n is odd, we have $F^{(n)}(z) \leq 0$. Thus $F^{(n-1)}(\cdot)$ is decreasing, and

$$F^{(n-1)}(z) \geq F^{(n-1)}(1) = \frac{(A)_{n-1}(B)_{n-1}}{(C)_{n-1}} {}_2F_1(A+n-1, B+n-1, C+n-1; 1) \geq 0.$$

Thus $F^{(n-2)}(\cdot)$ is increasing and $F^{(n-2)}(z) \geq F^{(n-2)}(0) \geq 0$. In this way, we could argue that $F^{(n-j)}(\cdot)$ is increasing for even j and it is decreasing for odd j. In particular, F is decreasing. □

B. Appendix: Commutation Relation

In [Dub07] and [KP16, Appendix A], the authors studied local multiple SLEs and classified them according to the so-called partition functions. Following the same idea, we will define a local SLE that describes two initial segments with two extra marked points.

Fix a quad $q = (\Omega; x_1, x_2, x_3, x_4)$. We will study a local SLE in Ω that describes two initial segments γ_1 and γ_4 starting from x_1 and x_4 respectively, with two extra marked points x_2 and x_3, up to exiting some neighborhoods U_1 and U_4. The localization neighborhoods U_1 and U_4 are assumed to be closed subsets of $\overline{\Omega}$ such that $\Omega \setminus U_j$ are simply connected for $j = 1, 4$ and that $U_1 \cap U_4 = \emptyset$ and that $\mathrm{dist}(\{x_2, x_3\}, U_1 \cup U_4) > 0$. A local SLE_κ in Ω, started from (x_1, x_4) and localized in (U_1, U_4) with two marked points (x_2, x_3), is a probability measure on two curves (γ_1, γ_4) such that, for $j \in \{1, 4\}$, the curve $\gamma_j : [0, 1] \to U_j$ starts at $\gamma_j(0) = x_j$ and ends at $\gamma_j(1) \in \partial U_j$. A local SLE_κ is the indexed collection

$$P = \left(P_{(q; U_1, U_4)}\right)_{q; U_1, U_4}.$$

This collection of probability measures is required to satisfy the following three properties.

- Conformal invariance. Suppose that $q = (\Omega; x_1, x_2, x_3, x_4)$, $\tilde{q} = (\tilde{\Omega}; \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) \in \mathcal{Q}$, and $\psi : \Omega \to \tilde{\Omega}$ is a conformal map with $\psi(x_j) = \tilde{x}_j$ for $j \in \{1, 2, 3, 4\}$. Then for $(\gamma_1, \gamma_4) \sim P_{(q; U_1, U_4)}$, we have $(\psi(\gamma_1), \psi(\gamma_4)) \sim P_{(\tilde{q}; \psi(U_1), \psi(U_4))}$.
- Domain Markov property. Suppose that τ_1 is a stopping time for γ_1 and τ_4 is a stopping time for γ_4. The conditional law of $(\gamma_1|_{t \geq \tau_1}, \gamma_4|_{t \geq \tau_4})$, given the initial segments $\gamma_1[0, \tau_1]$ and $\gamma_4[0, \tau_4]$, is the same as $P_{(\tilde{q}; \tilde{U}_1, \tilde{U}_4)}$ where $\tilde{q} = (\tilde{\Omega}; \gamma_1(\tau_1), x_2, x_3, \gamma_4(\tau_4))$ and $\tilde{\Omega}$ is the connected component of $\Omega \setminus (\gamma_1[0, \tau_1] \cup \gamma_4[0, \tau_4])$ with $(x_2 x_3)$ on the boundary, and $\tilde{U}_j = U_j \cap \tilde{\Omega}$ for $j \in \{1, 4\}$.
- Absolute continuity of the marginals. Define

$$\mathfrak{X}_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 < x_2 < x_3 < x_4\}.$$

  There exist smooth functions $F_j : \mathfrak{X}_4 \to \mathbb{R}$, for $j \in \{1, 4\}$, such that for the domain $\Omega = \mathbb{H}$, boundary points $x_1 < x_2 < x_3 < x_4$, and localization neighborhoods

U_1 and U_4, the marginal law of γ_j under $P_{(\mathbb{H};x_1,x_2,x_3,x_4;U_1,U_4)}$ is the Loewner chain driven by the solution to the following SDEs:

$$\begin{aligned}
&\text{for } \gamma_1: \quad dW_t = \sqrt{\kappa}dB_t + F_1(W_t, V_t^2, V_t^3, V_t^4)dt,\\
&\quad dV_t^i = \frac{2dt}{V_t^i - W_t}, \quad \text{for } i = 2,3,4;\\
&\text{for } \gamma_4: \quad d\tilde{W}_t = \sqrt{\kappa}d\tilde{B}_t + F_4(\tilde{V}_t^1, \tilde{V}_t^2, \tilde{V}_t^3, \tilde{W}_t)dt,\\
&\quad d\tilde{V}_t^i = \frac{2dt}{\tilde{V}_t^i - W_t}, \quad \text{for } i = 1,2,3;
\end{aligned} \tag{B.1}$$

where $W_0 = x_1$, $V_0^2 = x_2$, $V_0^3 = x_3$ and $V_0^4 = x_4$ and $\tilde{W}_0 = x_4$, $\tilde{V}_0^1 = x_1$, $\tilde{V}_0^2 = x_2$, $\tilde{V}_0^3 = x_3$.

Lemma B.1. *Suppose both (U_1, U_4) and (V_1, V_4) are localization neighborhoods for quad $q = (\Omega; x_1, x_2, x_3, x_4)$ and that $V_j \subset U_j$ for $j \in \{1, 4\}$. Suppose $(\gamma_1, \gamma_4) \sim P_{(q;U_1,U_4)}$ and let τ_j be γ_j's first time to exit V_j for $j \in \{1, 4\}$. Then $(\gamma_1|_{[0,\tau_1]}, \gamma_4|_{[0,\tau_4]}) \sim P_{(q;V_1,V_4)}$.*

Proof. It is clear that the restriction measures also satisfy all the required three properties. □

It turns out that the existence of local SLE with two extra points is related to positive functions $\mathcal{Z} : \mathfrak{X}_4 \to \mathbb{R}_+$ which satisfy a certain PDE system and conformal covariance: $h = (6-\kappa)/(2\kappa)$ and b is a constant parameter,

– PDE system (PDE):

$$\begin{aligned}
&\frac{\kappa}{2}\partial_{x_1}^2\mathcal{Z} + \sum_{2\le i\le 4}\frac{2\partial_{x_i}\mathcal{Z}}{x_i - x_1} + \left(\frac{-2h}{(x_4-x_1)^2} + \frac{-2b}{(x_2-x_1)^2} + \frac{-2b}{(x_3-x_1)^2}\right)\mathcal{Z} = 0,\\
&\frac{\kappa}{2}\partial_{x_4}^2\mathcal{Z} + \sum_{1\le i\le 3}\frac{2\partial_{x_i}\mathcal{Z}}{x_i - x_4} + \left(\frac{-2h}{(x_1-x_4)^2} + \frac{-2b}{(x_2-x_4)^2} + \frac{-2b}{(x_3-x_4)^2}\right)\mathcal{Z} = 0,
\end{aligned} \tag{B.2}$$

– Conformal covariance (COV): for all Möbius maps φ of $\mathbb{H}$ such that $\varphi(x_1) < \varphi(x_2) < \varphi(x_3) < \varphi(x_4)$,

$$\begin{aligned}
\mathcal{Z}(x_1, x_2, x_3, x_4) = &|\varphi'(x_1)|^h|\varphi'(x_2)|^b|\varphi'(x_3)|^b|\varphi'(x_4)|^h\\
&\times\mathcal{Z}(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4)).
\end{aligned} \tag{B.3}$$

Proposition B.2. *We have the following correspondence between local* SLE *with two extra marked points and positive solutions to* PDE (B.2) *and* COV (B.3).

(a) Suppose $\mathcal{Z} : \mathfrak{X}_4 \to \mathbb{R}$ is a positive solution to PDE (B.2) *and* COV (B.3)*. Then there exists a local* SLE$_\kappa$ *with two extra marked points such that the drift terms in* (B.1) *are given by $F_1 = \kappa\partial_{x_1}\log\mathcal{Z}$ and $F_4 = \kappa\partial_{x_4}\log\mathcal{Z}$.*
(b) Suppose there exists a local SLE$_\kappa$ *with two extra marked points. Then there exists a positive solution $\mathcal{Z} : \mathfrak{X}_4 \to \mathbb{R}$ to* PDE (B.2) *and* COV (B.3) *such that the drift terms in* (B.1) *are given by $F_1 = \kappa\partial_{x_1}\log\mathcal{Z}$ and $F_4 = \kappa\partial_{x_4}\log\mathcal{Z}$.*

Proof of Proposition B.2—Part (a). There are two ways to sample γ_1 and γ_4: Method 1—sample γ_1 first, and Method 2—sample γ_4 first.

Method 1. Since $\mathcal{Z}$ satisfies PDE (B.2), the following process is a local martingale with respect to the law of SLE_κ in $\mathbb{H}$ from x_1 to ∞:

$$M_t^{(1)} = g_t'(x_2)^b g_t'(x_3)^b g_t'(x_4)^h \mathcal{Z}(W_t, g_t(x_2), g_t(x_3), g_t(x_4)).$$

We sample γ_1 according to the law of SLE_κ in $\mathbb{H}$ from x_1 to ∞ weighted by the local martingale $M_t^{(1)}$, up to the first time σ_1 that the process exits U_1. Let $G = g_{\sigma_1}$ and denote by

$$\tilde{x}_1 = G(\gamma_1(\sigma_1)), \quad \tilde{x}_2 = G(x_2), \quad \tilde{x}_3 = G(x_3), \quad \tilde{x}_4 = G(x_4).$$

Since $\mathcal{Z}$ satisfies PDE (B.2), the following process is a local martingale with respect to the law of SLE_κ in $\mathbb{H}$ from x_4 to ∞:

$$\tilde{M}_s^{(4)} = \tilde{g}_s'(\tilde{x}_1)^h \tilde{g}_s'(\tilde{x}_2)^b \tilde{g}_s'(\tilde{x}_3)^b \mathcal{Z}(\tilde{g}_s(\tilde{x}_1), \tilde{g}_s(\tilde{x}_2), \tilde{g}_s(\tilde{x}_3), \tilde{W}_s).$$

We sample $\tilde{\gamma}_4$ according to the the law of SLE_κ in $\mathbb{H}$ from $\tilde{x}_4$ to ∞ weighted by the local martingale $\tilde{M}_s^{(4)}$, up to the first time $\tilde{\sigma}_4$ that the process exits $G(U_4)$. Finally, set $\gamma_4 = G^{-1}(\tilde{\gamma}_4)$.

Method 2. This is defined in the same way as in Method 1 except we switch the roles of γ_1 and γ_4.

According to the local commutation relation in [Dub07, Theorem 7.1], these two methods give the same law on pairs (γ_1, γ_4). The probability measure defined by the sampling procedure clearly satisfies the domain Markov property and the absolute continuity of the marginals. By COV (B.3), we could define the law on (γ_1, γ_4) in any simple connected domain via conformal image. This implies the conformal invariance. $\square$

Proof of Proposition B.2—Part (b). Since a local SLE with extra two points is conformally invariant, we could assume $x_2 = \infty$, $x_3 = 0$, $x_4 = x$, $x_1 = y$ for $0 < x < y$. By [Dub07, Theorem 7.1], the existence of local SLE_κ in neighborhoods of x and y with two extra marked points 0 and ∞ implies that there exists a positive function $\psi : \mathfrak{X}_2 \to \mathbb{R}$ that solves the following PDE system:

$$\begin{aligned}
&\frac{\kappa}{2}\partial_x^2\psi + \frac{2}{x}\partial_x\psi + \left(\frac{2}{x} + \frac{2}{y-x}\right)\partial_y\psi + \left(\frac{-2h}{(y-x)^2} + \frac{-\mu}{x^2}\right)\psi = 0,\\
&\frac{\kappa}{2}\partial_y^2\psi + \frac{2}{y}\partial_y\psi + \left(\frac{2}{y} + \frac{2}{x-y}\right)\partial_x\psi + \left(\frac{-2h}{(x-y)^2} + \frac{-\mu}{y^2}\right)\psi = 0,
\end{aligned} \tag{B.4}$$

where μ is a constant parameter, and ψ is homogeneous of some fixed degree. Moreover, the marginal laws of γ_1, γ_4 are the Loewner chains driven by the solutions to the following SDEs:

$$\begin{aligned}
\text{for } \gamma_1 : dW_t &= \sqrt{\kappa}dB_t + \kappa(\partial_y \log\psi)(V_t^4 - V_t^3, W_t - V_t^3)dt,\\
dV_t^i &= \frac{2dt}{V_t^i - W_t}, \quad i = 3, 4;\\
\text{for } \gamma_4 : d\tilde{W}_t &= \sqrt{\kappa}d\tilde{B}_t + \kappa(\partial_x \log\psi)(\tilde{W}_t - \tilde{V}_t^3, \tilde{V}_t^1 - \tilde{V}_t^3)dt,\\
d\tilde{V}_t^i &= \frac{2dt}{\tilde{V}_t^i - \tilde{W}_t}, \quad i = 1, 3.
\end{aligned}$$

Suppose ψ is homogeneous of degree $-\aleph$. Then there exists a positive function $f : (0, 1) \to \mathbb{R}$ such that $\psi(x, y) = (y - x)^{-\aleph} f(x/y)$. Then the two PDEs in (B.4) become

$$\frac{\kappa}{2}z^2 f''(z) + \frac{z}{1-z}\,(2 + (\kappa\aleph - 4)z)\, f'(z) + \left(-\mu + \frac{z^2}{(1-z)^2}\left(\frac{\kappa}{2}\aleph(\aleph+1) - 2\aleph - 2h\right)\right) f = 0,$$
$$\frac{\kappa}{2}z^2 f''(z) + \frac{z}{1-z}\,(\kappa\aleph + \kappa - 4 + (2-\kappa)z)\, f'(z) + \left(-\mu + \frac{1}{(1-z)^2}\left(\frac{\kappa}{2}\aleph(\aleph+1) - 2\aleph - 2h\right)\right) f = 0.$$

In order to have non-zero solution, we must have $\aleph = 2h$ and f satisfies the following ODE:

$$\frac{\kappa}{2}z^2 f''(z) + \frac{z}{1-z}(2 + (2-\kappa)z) f'(z) - \mu f(z) = 0. \tag{B.5}$$

Define, for $x_1 < x_2 < x_3 < x_4$,

$$\mathcal{Z}(x_1, x_2, x_3, x_4) := (x_4 - x_1)^{-2h}(x_3 - x_2)^{-\mu} f(z), \quad \text{where } z = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)}.$$

One can check that $\mathcal{Z}$ satisfies PDE (B.2) and COV (B.3) with $b = \mu/2$. □

Corollary B.3. *For any $\kappa \in (0, 8)$ and $\nu \in \mathbb{R}$, there exists a local* SLE_κ *with two extra marked points such that the drift term in* (B.1) *are given by $F_1 = \kappa\partial_{x_1} \log \mathcal{Z}_{\kappa,\nu}$ and $F_4 = \kappa\partial_{x_4} \log \mathcal{Z}_{\kappa,\nu}$ where $\mathcal{Z}_{\kappa,\nu}$ is defined in* (3.5)*. In particular, the marginal law of γ_1 is* $\mathrm{hSLE}_\kappa(\nu)$ *in $\mathbb{H}$ from x_1 to x_4 with marked points (x_2, x_3) stopped at the first exiting time of U_1, and the marginal law of γ_4 is* $\mathrm{hSLE}_\kappa(\nu)$ *in $\mathbb{H}$ from x_4 to x_1 with marked points (x_3, x_2) stopped at the first exiting time of U_4.*

Proof. The function $\mathcal{Z}_{\kappa,\nu}$ defined in (3.5) satisfies PDE (B.2) and COV (B.3) for

$$b = (\nu + 2)(\nu + 6 - \kappa)/(4\kappa).$$

Combining with Proposition B.2—Part (a), we obtain the conclusion. □

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