KOTTWITZ-RAPOPORT CONJECTURE ON UNIONS OF AFFINE DELIGNE-LUSZTIG VARIETIES

LA CONJECTURE DE KOTTWITZ-RAPOPORT SUR LES UNIONS DE VARIÉTÉS DE DELIGNE-LUSZTIG AFFINES

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ABSTRACT. In this paper, we prove a conjecture of Kottwitz and Rapoport on a union of (generalized) affine Deligne-Lusztig varieties $X(\mu, b)_J$ for a p-adic group G and its parahoric subgroup P_J. We show that $X(\mu, b)_J \neq \emptyset$ if and only if the group-theoretic version of Mazur's inequality is satisfied. In the process, we obtain a generalization of Grothendieck's conjecture on the closure relation of σ-conjugacy classes of a twisted loop group.

Dans cet article nous prouvons une conjecture de Kottwitz et Rapoport sur l'union de variétés de Deligne-Lusztig affines (généralisées) $X(\mu, b)_J$ pour G un groupe p-adique et P_J son sous-groupe parahorique. Nous montront que $X(\mu, b)_J$ est non vide si et seulement si la version de l'inégalité de Mazur pour les groupes est satisfaite. Au cours de la preuve, nous obtenons une généralisation de la conjecture de Grothendieck sur les inclusions des adhérences de classes de σ-conjugaison d'un groupe de lacets tordu.

INTRODUCTION

0.1. The motivation of this paper comes from the reduction of Shimura varieties with a parahoric level structure. On the special fiber, there are two important stratifications:

- Newton stratification, indexed by specific σ-conjugacy classes in the associated p-adic group G.

- Kottwitz-Rapoport stratification, indexed by specific double cosets in $W_J \backslash \tilde{W} / W_J$, where $\tilde{W}$ is the Iwahori-Weyl group of G and W_J is the Weyl group of the parahoric subgroup P_J.

A fundamental question is to determine which Kottwitz-Rapoport strata and which Newton strata are nonempty, in other words, to determine the double cosets of $W_J \backslash \tilde{W} / W_J$ and the subset of σ-conjugacy classes that appear in the reduction of Shimura varieties.

It consists of two parts: local theory and global theory. In this paper, we focus on local theory.

0.2. In [29] and [23], Pappas and Zhu give a group-theoretic definition of "local models" of Shimura varieties and show that the subset of $W_J \backslash \tilde{W} / W_J$ for the local model is the admissible set $\mathrm{Adm}_J(\mu)$ (defined in §1.5).

The next question is to describe the σ-conjugacy classes arises in the reduction of Shimura varieties. Based on some foundational relations between Newton strata, Kottwitz-Rapoport strata and affine Deligne-Lusztig varieties, we study the set $X(\mu, b)_J$, a union of generalized affine Deligne-Lusztig varieties indexed by $\mathrm{Adm}_J(\mu)$. It is defined as follows. Let L be the completion of the maximal unramified extension of a p-adic field and $b \in G(L)$, set

$$X(\mu, b)_J = \{gP_J \in G(L)/P_J; g^{-1}b\sigma(g) \in \cup_{w \in \mathrm{Adm}_J(\mu)} P_J w P_J\}.$$

Kottwitz and Rapoport introduced a set $B(G, \mu)$ of acceptable σ-conjugacy classes, defined by the group-theoretic version of Mazur's theorem. The main purpose of this paper is to prove the following result, conjectured by Kottwitz and Rapoport in [19] and [24].

Theorem A. *Suppose that G splits over a tamely ramified extension of F. Then $X(\mu, b)_J \neq \emptyset$ if and only if $[b] \in B(G, \mu)$.*

0.3. The direction

$$X(\mu, b)_J \neq \emptyset \Rightarrow [b] \in B(G, \mu)$$

is the group-theoretic version of Mazur's inequality between the Hodge polygon of an F-crystal and the Newton polygon of its underlying F-isocrystal. The case where G is an unramified group and P_J is a hyperspecial maximal subgroup, is proved by Rapoport and Richartz in [25, Theorem 4.2]. Another proof is given by Kottwitz in [18]. The case where G is an unramified group and P_J is an Iwahori subgroup, is proved in [24, Notes added June 2003, (7)].

The other direction

$$X(\mu, b)_J \neq \emptyset \Leftarrow [b] \in B(G, \mu)$$

is the "converse to Mazur's inequality" and was proved by Wintenberger in [28] in case G is quasi-split.

0.4. Another related question is to determine the non-emptiness pattern for a single affine Deligne-Lusztig variety.

If G is quasi-split and P_J is a special maximal parahoric subgroup, then the non-emptiness pattern of a single affine Deligne-Lusztig variety is still governed by Mazur's inequality. It is conjectured and proved for $G = GL_n$ or GSp_{2n} by Kottwitz and Rapoport in [19]. It is then proved by Lucarelli [15] for classical split groups and then by Gashi [1] for unramified cases. The general case is proved in [12, Theorem 7.1]. Notice that if P_J is a special maximal parahoric subgroup and μ is minuscule with respect to $\tilde{W}$, $X(\mu, b)_J$ is in fact a single affine Deligne-Lusztig variety.

If P_J is an Iwahori subgroup and b is basic, a conjecture on the non-emptiness pattern (for split groups) is given by Görtz, Haines, Kottwitz, and Reuman in [3] in terms of P-alcoves in [3] and the generalization of this conjecture to any tamely ramified groups is proved in [4]. The non-emptiness pattern for basic b and other parahoric subgroups can then be deduced from Iwahori case easily.

However, such information is not useful for the study of $X(\mu, b)_J$. The reason is that for b basic, it is very easy to determine whether $X(\mu, b)_J$ is empty (by checking the image under Kottwitz map) and for other b, and non-special parahoric subgroup J, very little is known about the non-emptiness pattern for a single affine Deligne-Lusztig variety.

0.5. Now we discuss the strategy of the proof of Theorem A. The key ingredients are

- the partial order on $B(G)$;
- some nice properties on the admissible set $\mathrm{Adm}_J(\mu)$;
- the fact that the maximal element in $B(G, \mu)$ is represented by an element in the admissible set.

We discuss the first ingredient in this subsection and the second and third ingredients in the next subsection.

The starting point is the natural map

$$\Psi : B(\tilde{W}, \sigma) \to B(G)$$

from the set of σ-conjugacy classes of $\tilde{W}$ to the set of σ-conjugacy classes of $G(L)$. This map is surjective, but not injective in general. However, there exists a natural section of Ψ given by the straight σ-conjugacy classes of $\tilde{W}$ (see §2.2).

On the set of straight σ-conjugacy classes of $\tilde{W}$, there is a natural partial order $\preceq_\sigma$ (defined in §3.2). On $B(G)$, there are two partial orders, given by the closure relation between the σ-conjugacy classes and given by the dominance order of the corresponding Newton polygons. A generalization of Grothendieck conjecture says that the two partial orders on $B(G)$ coincide. We prove in Theorem 3.1 that

Theorem B. *For any twisted loop group, the partial order $\preceq_\sigma$ on the set of straight σ-conjugacy classes coincides with both partial orders on $B(G)$ via the map $\Psi : B(\tilde{W}, \sigma) \to B(G)$. In particular, the two partial orders on $B(G)$ coincide.*

The proof is based on the reductive method in [12] à la Deligne and Lusztig, some remarkable combinatorial properties on $\tilde{W}$ established in [13] and the Grothendieck conjecture for split groups proved by Viehmann in [27].

0.6. By definition,

$$X(\mu, b)_J \neq \emptyset \Leftrightarrow [b] \cap \cup_{w \in \mathrm{Adm}_J(\mu)} P_J w P_J \neq \emptyset.$$

Using a similar argument as in the proof of Theorem B, the latter condition is equivalent to $[b] \in \Psi(\mathrm{Adm}_J(\mu))$.

Notice that Mazur's inequality is defined using the dominance order on the Newton polygons. For quasi-split groups, it is easy to see that μ is the unique maximal element in $B(G, \mu)$ with respect to the dominance order. Thus the converse to Mazur's inequality follows from the coincides between the partial order $\preceq_\sigma$ on the set of straight σ-conjugacy classes and the dominance order on the Newton polygons. For non quasi-split groups, the maximal element in $B(G, \mu)$ is harder to understand and we use [14] on the properties of this element.

The proof of Mazur's inequality is based on two properties of the admissible sets:

- The additivity of the admissible sets (Theorem 5.1), proved by Zhu's global Schubert varieties [29].
- The compatibility of admissible sets (Theorem 6.1), proved by the "partial conjugation method" in [10].

[**Note added 09/29/2015**: The "tamely ramified" assumption is used only in the proof of Theorem 5.1, in which the global Schubert varieties of X. Zhu [29] are used. Haines and I [5] recently gave a proof of Theorem 5.1 for any group G over L. Hence the "tamely ramified" assumption is not necessary for the results of this paper.]

1. Preliminaries

1.1. Let $\mathbb{F}_q$ be the finite field with q elements. Let $\Bbbk$ be an algebraic closure of $\mathbb{F}_q$. Let F be a finite field extension of $\mathbb{Q}_p$ with residue class field $\mathbb{F}_q$ and uniformizer ε or $F = \mathbb{F}_q((\epsilon))$ be the field of Laurent series over $\mathbb{F}_q$. Let L be the completion of the maximal unramified extension of F.

Let G be a connected reductive group over F. Let σ be the Frobenius automorphism of L/F. We also denote the induced automorphism on $G(L)$ by σ.

Let S be a maximal L-split torus that is defined over F and let T be its centralizer. By Steinberg's theorem, G is quasi-split over L. Thus T is a maximal torus. Let N be its normalizer. The *finite Weyl group* associated to S is

$$W_0 = N(L)/T(L).$$

The Iwahori-Weyl group associated to S is

$$\tilde{W} = N(L)/T(L)_1,$$

where $T(L)_1$ denotes the unique Iwahori subgroup of $T(L)$. The Frobenius morphism σ induces an action on $\tilde{W}$, which we still denote by σ.

For any $w \in \tilde{W}$, we choose a representative in $N(L)$ and also write it as w.

1.2. Let $\mathcal{A}$ be the apartment of G_L corresponding to S. Since σ induces a permutation of finite order on the set of alcoves in $\mathcal{A}$, there exists a σ-invariant alcove $\mathbf{a}$ in $\mathcal{A}$. Let I be the corresponding Iwahori subgroup. Let $\tilde{\mathbb{S}}$ be the set of simple reflections of $\tilde{W}$. The set $\tilde{\mathbb{S}}$ is equipped with an action of σ. For any $J \subset \tilde{\mathbb{S}}$, let $W_J \subset \tilde{W}$ be the subgroup generated by the simple reflections in J and by ${}^J\tilde{W}$ (resp. $\tilde{W}^J$) the set of minimal length elements for the cosets $W_J \backslash \tilde{W}$ (resp. $\tilde{W}/W_J$). We simply write ${}^J\tilde{W}^{J'}$ for ${}^J\tilde{W} \cap \tilde{W}^{J'}$.

We follow [6]. Let $\Gamma_F = \mathrm{Gal}(\bar{L}/F)$ be the absolute Galois group of F and $\Gamma = \mathrm{Gal}(\bar{L}/L)$ the inertia group. The Iwahori-Weyl group $\tilde{W}$ contains the affine Weyl group W_a as a normal subgroup and we have a short exact sequence

$$0 \to W_a \to \tilde{W} \to \pi_1(G)_\Gamma \to 0,$$

where $\pi_1(G)$ denotes algebraic fundamental group of G and $\pi_1(G)_\Gamma$ its coinvariants under the action of σ. The choice of the alcove $\mathbf{a}$ splits this extension, and

$$\tilde{W} = W_a \rtimes \Omega,$$

where Ω is the normalizer of $\mathbf{a}$, and is isomorphic to $\pi_1(G)_\Gamma$. The length function and Bruhat order on W_a extend in a natural way to $\tilde{W}$.

We have another exact sequence

$$0 \to X_*(T)_\Gamma \to \tilde{W} \to W_0 \to 0.$$

We choose a special vertex of $\mathbf{a}$ and represent $\tilde{W}$ as a semidirect product

$$\tilde{W} = X_*(T)_\Gamma \rtimes W_0 = \{t^\lambda w; \lambda \in X_*(T)_\Gamma, w \in W_0\}.$$

1.3. For $b, b' \in G(L)$, we say that b and b' are σ-conjugate if there exists $g \in G(L)$ such that $b' = g^{-1}b\sigma(g)$. Let $B(G)$ be the set of σ-conjugacy classes. The classification of the σ-conjugacy classes is obtained by Kottwitz in [16] and [17]. The description is as follows.

Let $\kappa_G : B(G) \to \pi_1(G)_{\Gamma_F}$ be the Kottwitz map [17, §7]. This gives one invariant. Another invariant is obtained by the Newton map. An element $b \in G(L)$ determines a homomorphism $\mathbb{D} \to G_L$, where $\mathbb{D}$ is the pro-algebraic torus whose character group is $\mathbb{Q}$. This homomorphism determines an element ν_b in the closed dominant chamber $X_*(T)_\mathbb{Q}^+$. The element ν_b is called the *Newton point* of b and the map $b \mapsto \nu_b$ is called the *Newton map*. Note that for any b, $\sigma(\nu_b) = \nu_b$. By [17, §4.13], the map

$$f : B(G) \to X_*(T)_\mathbb{Q}^+ \times \pi_1(G)_{\Gamma_F}, \qquad b \mapsto (\nu_b, \kappa_G(b))$$

is injective.

1.4. Write σ as $\sigma = \tau \circ \sigma_0$, where σ_0 is a diagram automorphism of $G(L)$ such that σ_0 fixes $\tilde{\mathbb{S}} - \mathbb{S}$ and the induced action of τ on the adjoint group G_{ad} is inner.

For $\nu, \nu' \in X_*(T)_\mathbb{Q}^+$, we write $\nu \leqslant \nu'$ if $\nu' - \nu$ is a non-negative $\mathbb{Q}$-linear combination of positive relative coroots. This is called the dominance order on $X_*(T)_\mathbb{Q}^+$.

Let $\mu \in X_*(T)^+$, we set

$$\mu^\diamond = \frac{1}{N}\sum_{i=0}^{N-1}\sigma_0^i(\mu) \in X_*(T)_\mathbb{Q}^+,$$

where N is the order of σ_0. A σ-conjugacy class $[b]$ is called *(neutral) acceptable* for μ if $\nu_b \leqslant \mu^\diamond$ and $\kappa_G(b) = \mu^\sharp$, where $\mu^\sharp$ is the image of μ in $\pi_1(G)_{\Gamma_F}$. Let $B(G, \mu)$ be the set of (neutral) acceptable elements for μ.

1.5. The μ-*admissible set* is defined as

$$\mathrm{Adm}(\mu) = \{w \in \tilde{W}; w \leqslant t^{x(\underline{\mu})} \text{ for some } x \in W_0\},$$

where $\underline{\mu}$ is the image of μ in $X_*(T)_\Gamma$.

More generally, let $J \subset \tilde{\mathbb{S}}$ such that $\sigma(J) = J$ and W_J is finite. The μ-admissible set associated to J is

$$\mathrm{Adm}^J(\mu) = W_J \mathrm{Adm}(\mu) W_J \subset \tilde{W}.$$

It is the inverse image under the natural map $\tilde{W} \to W_J \backslash \tilde{W} / W_J$ of $\mathrm{Adm}_J(\mu)$ in [19, (3.6)].

1.6. Let $J \subset \tilde{\mathbb{S}}$ such that $\sigma(J) = J$ and W_J is finite. Let $P_J \supset I$ be the standard parahoric subgroup corresponding to J. For any $w \in W_J \backslash \tilde{W} / W_J$ and $b \in G(L)$, the generalized affine Deligne-Lusztig variety

$$X_{J,w}(b) = \{g \in G(L)/P_J; g^{-1} b\sigma(g) \in P_J w P_J\}.$$

In this paper, we are mainly interested in the following finite union of affine Deligne-Lusztig varieties:

$$X(\mu, b)_J = \{g \in G(L)/P_J; g^{-1} b\sigma(g) \in Y_{J,\mu}\}$$
$$= \cup_{w \in \mathrm{Adm}^J(\mu)} X_{J,w}(b),$$

where $Y_{J,\mu} = \cup_{w \in \mathrm{Adm}(\mu)} P_J w P_J = \cup_{w \in \mathrm{Adm}^J(\mu)} I w I$.

Let $J' \subset \tilde{\mathbb{S}}$ such that $\sigma(J') = J'$ and $W_{J'}$ is finite and $J \subset J'$. Then $Y_{J',\mu} = P_{J'} Y_{J,\mu} P_{J'}$ and hence the projection map $G(L)/P_J \to G(L)/P_{J'}$ induces

$$\pi_{J,J'} : X(\mu, b)_J \to X(\mu, b)_{J'}.$$

The main result of this paper is

Theorem 1.1. *Suppose that G splits over a tamely ramified extension of F. Let $b \in G(L)$, $\mu \in X_*(T)^+$ and $J \subset J'$ be σ-stable subsets of $\tilde{\mathbb{S}}$ with $W_{J'}$ finite. Then*
(1) $X(\mu, b)_J \neq \emptyset$ if and only if $[b] \in B(G, \mu)$.
(2) The map $\pi_{J,J'}$ is surjective.

2. THE MAP $\Psi : B(\tilde{W}, \sigma) \to B(G)$

2.1. We first recall the definition of straight elements of $\tilde{W}$.

Let $w \in \tilde{W}$. Then there exists a positive integer n such that $(w\sigma)^n = t^\lambda \in \tilde{W} \rtimes \langle \sigma \rangle$ for some $\lambda \in X_*(T)_\Gamma$. Let $\nu_{w,\sigma} = \lambda/n$ and $\bar{\nu}_{w,\sigma}$ be the unique dominant element in the W_0-orbit of $\nu_{w,\sigma}$. It is known that $\bar{\nu}_{w,\sigma}$ is independent of the choice of n and is Γ-invariant.

We say that an element w is σ-*straight* $\ell(w) = \langle \bar{\nu}_{w,\sigma}, 2\rho \rangle$, where ρ is the half sum of all positive roots in the root system of the affine Weyl

group W_a. This is equivalent to $\ell((w\sigma)^n) = n\ell(w)$, where we regard $w\sigma$ as an element in $\tilde{W} \rtimes \langle\sigma\rangle$. A σ-conjugacy class of $\tilde{W}$ is called *straight* if it contains a σ-straight element.

2.2. Let $B(\tilde{W}, \sigma)$ be the set of σ-conjugacy classes of $\tilde{W}$ and $B(\tilde{W}, \sigma)_{str}$ be the set of straight σ-conjugacy classes of $\tilde{W}$. Following [12], there exists a commutative diagram

(a)

$$
\begin{array}{ccc}
B(\tilde{W}, \sigma) & \xrightarrow{\quad\Psi\quad} & B(G) \\
& \searrow^{f} \qquad f\swarrow & \\
& X_*(T)^+_{\mathbb{Q}} \times \pi_1(G)_{\Gamma_F} &
\end{array}
\quad,
$$

where $\Psi : B(\tilde{W}, \sigma) \to B(G)$ is induced from the natural inclusion $N(L) \to G(L)$.

By [12, §3], the restriction of Ψ to $B(\tilde{W}, \sigma)_{str}$ is a bijection. For any straight σ-conjugacy class $\mathcal{O}$ of $\tilde{W}$, we denote by $[\mathcal{O}]$ the corresponding σ-conjugacy class in $G(L)$. We also set $\nu_{\mathcal{O}} = \bar{\nu}_{w,\sigma}$ for any $w \in \mathcal{O}$.

2.3. By definition, for $w \in \tilde{W}$, $X_{\emptyset,w}(b) \neq \emptyset$ if and only if $[b] \in IwI \neq \emptyset$. If $\Psi(w) = [b]$, then automatically $[b] \cap IwI \neq \emptyset$, i.e. $X_{\emptyset,w}(b) \neq \emptyset$. The converse, is far from being true. In [12, Theorem 6.1], we give a criterion about the non-emptiness pattern of affine Deligne-Lusztig varieties in affine flag varieties in terms of class polynomials of affine Hecke algebras. The computation of class polynomials, however, is very hard in general.

The main result of this section is the following simple criterion of the non-emptiness criterion for "closed" affine Deligne-Lusztig varieties in affine flag varieties.

Theorem 2.1. *Let $b \in G(L)$ and $w \in \tilde{W}$. Then $\cup_{w' \leqslant w} X_{\emptyset,w'}(b) \neq \emptyset$ if and only if $[b] \in \cup_{w' \leqslant w} \Psi(w')$.*

To prove this theorem, we combine the method for the finite case [10, Proposition 5.8] and [11, Proposition 2.5], with the reduction method [12, Section 3]. The proof will be given in §2.7.

2.4. For $w, w' \in \tilde{W}$ and $s \in \tilde{\mathbb{S}}$, we write $w \xrightarrow{s}_\sigma w'$ if $w' = sw\sigma(s)$ and $\ell(w') \leqslant \ell(w)$. We write $w \to_\sigma w'$ if there is a sequence $w = w_0, w_1, \cdots, w_n = w'$ of elements in $\tilde{W}$ such that for any k, $w_{k-1} \xrightarrow{s}_\sigma w_k$ for some $s \in \tilde{\mathbb{S}}$. We write $w \approx_\sigma w'$ if $w \to_\sigma w'$ and $w' \to_\sigma w$ and write $w \tilde{\approx}_\sigma w'$ if $w \approx_\sigma \tau w' \sigma(\tau)^{-1}$ for some $\tau \in \Omega$. It is easy to see that $w \approx_\sigma w'$ if $w \to_\sigma w'$ and $\ell(w) = \ell(w')$.

For any σ-conjugacy class $\mathcal{O}$ in $\tilde{W}$, we denote by $\mathcal{O}_{\min}$ the set of minimal length elements in $\mathcal{O}$. Now we recall some properties on the minimal length elements, obtained in [13, §2].

Theorem 2.2. *Let $\mathcal{O}$ be a σ-conjugacy class of $\tilde{W}$ and $w \in \mathcal{O}$. Then there exists $w' \in \mathcal{O}_{\min}$ such that*
  (1) $w \to_\sigma w'$;
  (2) There exists $J \subset \tilde{\mathbb{S}}$ with W_J finite, an σ-straight element $x \in \tilde{W}$ with $x \in {}^J\tilde{W}^{\sigma(J)}$ and $x\sigma(J) = J$, and $u \in W_J$, such that $w' = ux$.

Theorem 2.3. *Let $\mathcal{O}$ be a straight σ-conjugacy class of $\tilde{W}$ and $w, w' \in \mathcal{O}_{\min}$. Then $w\tilde{\approx}_\sigma w'$.*

2.5. For $g, g' \in G(L)$, we write $g \cdot_\sigma g' = gg'\sigma(g)^{-1}$. The subset $G(L) \cdot_\sigma IwI$ is studied in [12, §3]. Now we recollect some results that will be used here.
  (1) If $w\tilde{\approx}_\sigma w'$, then $G(L) \cdot_\sigma IwI = G(L) \cdot_\sigma Iw'I$.
  (2) If $w \in W$ and $s \in \tilde{\mathbb{S}}$ such that $sw\sigma(s) < w$, then $G(L) \cdot_\sigma IwI = G(L) \cdot_\sigma IswI \cup G(L) \cdot_\sigma Isw\sigma(s)I$.
  (3) If $w \in W$ is a minimal length element in its σ-conjugacy class, then $G(L) \cdot_\sigma IwI$ is a single σ-conjugacy class in $G(L)$.
  (4) Let $J \subset \tilde{\mathbb{S}}$ with W_J finite, and $x \in \tilde{W}$ with $x \in {}^J\tilde{W}^{\sigma(J)}$ and $x\sigma(J) = J$. Then for any $u \in W_J$, $G(L) \cdot_\sigma IuxI = G(L) \cdot_\sigma IxI$.

2.6. Let $w \in \tilde{W}$ and $\mathcal{O}$ be a straight σ-conjugacy class in $\tilde{W}$. We write $\mathcal{O} \preceq_\sigma w$ if there exists a minimal length element $w' \in \mathcal{O}$ such that $w' \leqslant w$ in the usual Bruhat order.
  Now we discuss some properties on $\preceq_\sigma$.

Proposition 2.4. *Let $w, w' \in \tilde{W}$ with $w \to_\sigma w'$. Let $\mathcal{O}$ be a straight σ-conjugacy class of $\tilde{W}$. If $\mathcal{O} \preceq_\sigma w'$, then $\mathcal{O} \preceq_\sigma w$.*

Remark 2.5. The proof is similar to the finite case [10, Lemma 4.4]. We include the proof here for completeness.

Proof. It suffices to prove the case where $w' = sw\sigma(s)$ for some $s \in \tilde{\mathbb{S}}$. Let $x \in \mathcal{O}_{\min}$ with $x \leqslant w'$.
  If $w > w'$, then $x < w$ and hence $\mathcal{O} \preceq_\sigma w$. Now we assume that $\ell(w) = \ell(w')$. Without loss of generalization, we may assume that $sw < w$ and $w\sigma(s) > w$.
  If $sx < x$, then $\ell(sx\sigma(s)) \leqslant \ell(x)$. Since $x \in \mathcal{O}_{\min}$, $sx\sigma(s) \in \mathcal{O}_{\min}$. By [20, Corollary 2.5], $sx \leqslant sw$ and $sx\sigma(s) \leqslant sw\sigma(s)$. Hence $\mathcal{O} \preceq_\sigma w$.
  If $sx > x$, then [20, Corollary 2.5], $x \leqslant sw$ and hence $x < sw\sigma(s)$. We also have that $\mathcal{O} \preceq_\sigma w$. $\square$

Corollary 2.6. *Let $\mathcal{O}$ be a straight σ-conjugacy class of $\tilde{W}$ and $w \in \mathcal{O}$. Then v is of minimal length in $\mathcal{O}$ if and only if v is a minimal element in $\mathcal{O}$ with respect to the Bruhat order.*

Corollary 2.7. *Let $\mathcal{O}$ be a straight σ-conjugacy class of $\tilde{W}$ and $w \in \tilde{W}$. Then $\mathcal{O} \preceq_\sigma w$ if and only if there exists $x \in \mathcal{O}$ with $x \leqslant w$.*

2.7. Now we prove Theorem 2.1.

By definition, $\cup_{\mathcal{O} \preceq_\sigma w}[\mathcal{O}] \subset \cup_{w' \leqslant w} \Psi(w') \subset \cup_{w' \leqslant w} G(L) \cdot Iw'I$.

Now we show that $\cup_{w' \leqslant w} G(L) \cdot Iw'I \subset \cup_{\mathcal{O} \preceq_\sigma w}[\mathcal{O}]$. By induction, it suffices to show that

$$G(L) \cdot IwI \subset \cup_{\mathcal{O} \preceq_\sigma w}[\mathcal{O}].$$

We argue by induction on $\ell(w)$.

If w is of minimal length in its σ-conjugacy class, then by Theorem 2.2 (2), then there exists $J \subset \tilde{\mathbb{S}}$ with W_J finite, $x \in \tilde{W}$ be an σ-straight element with $x \in {}^J\tilde{W}^{\sigma(J)}$ and $x\sigma(J) = J$, and $u \in W_J$ such that $w \approx_\sigma ux$. Let $\mathcal{O}_x$ be the σ-conjugacy class of x. Then $\mathcal{O}_x \preceq_\sigma ux$. By Proposition 2.4, $\mathcal{O}_x \preceq_\sigma w$. By §2.5 (1), (3) & (4),

$$G(L) \cdot_\sigma IwI = G(L) \cdot_\sigma IuxI = G(L) \cdot_\sigma IxI = [\mathcal{O}_x] \subset \cup_{\mathcal{O} \preceq_\sigma w}[\mathcal{O}].$$

If w is not of minimal length in its σ-conjugacy class, then by Theorem 2.2 (1), there exists $w' \in \tilde{W}$ with $w \approx_\sigma w'$ and $s \in \tilde{\mathbb{S}}$ with $sw'\sigma(s) < w'$. By §2.5 (1) & (2),

$$G(L) \cdot_\sigma IwI = G(L) \cdot_\sigma Iw'I = G(L) \cdot_\sigma Isw'I \cup G(L) \cdot_\sigma Isw'\sigma(s)I.$$

By induction hypothesis on sw' and $sw'\sigma(s)$,

$$G(L) \cdot_\sigma IwI \subset \cup_{\mathcal{O} \preceq_\sigma sw' \text{ or } \mathcal{O} \preceq_\sigma sw'\sigma(s)}[\mathcal{O}] \subset \cup_{\mathcal{O} \preceq_\sigma w'}[\mathcal{O}].$$

By Proposition 2.4, $\mathcal{O} \preceq_\sigma w'$ if and only if $\mathcal{O} \preceq_\sigma w$. Hence $G(L) \cdot_\sigma IwI \subset \cup_{\mathcal{O} \preceq_\sigma w}[\mathcal{O}]$. The statement is proved.

Corollary 2.8. *Let $w \in \tilde{W}$. Then $\cup_{w' \leqslant w} \Psi(w') = \cup_{\mathcal{O} \preceq_\sigma w}[\mathcal{O}]$.*

The following special case of Theorem 2.1 is useful in this paper.

Corollary 2.9. *Let $b \in G(L)$, $\mu \in X_*(T)^+$ and $J \subset \tilde{\mathbb{S}}$ such that $\sigma(J) = J$ and W_J is finite. Then $X(\mu, b)_J \neq \emptyset$ if and only if $[b] \in \Psi(\mathrm{Adm}^J(\mu))$.*

Proof. By definition, $X(\mu, b)_J \neq \emptyset$ if and only if

$$[b] \subset \cup_{w \in \mathrm{Adm}^J(\mu)} G(L) \cdot_\sigma IwI.$$

Notice that $\mathrm{Adm}^J(\mu)$ is of the form $\cup_i \{w \in \tilde{W}; w \leqslant x_i\}$ for finitely many x_i's. The statement follows from Theorem 2.1. $\qquad\square$

3. Three partial orders

3.1. In this section, we assume that $F = \mathbb{F}_q((\epsilon))$. Recall the commutative diagram in §2.2 (a):

$$B(\tilde{W}, \sigma)_{str} \xrightarrow{\quad \Psi \quad} B(G)$$

$$f \searrow \qquad \swarrow f$$

$$X_*(T)_{\mathbb{Q}}^+ \times \pi_1(G)_{\Gamma_F} \qquad ,$$

We will introduce partial orders on these sets and show that these partial orders are compatible.

3.2. Let $\mathcal{O}, \mathcal{O}' \in B(\tilde{W}, \sigma)_{str}$. We write $\mathcal{O}' \preceq_\sigma \mathcal{O}$ if for some $w \in \mathcal{O}_{\min}$, $\mathcal{O}' \preceq_\sigma w$. By Theorem 2.3 and Proposition 2.4, if $\mathcal{O}' \preceq_\sigma \mathcal{O}$, then $\mathcal{O}' \preceq_\sigma x$ for any $x \in \mathcal{O}_{\min}$. Hence $\preceq_\sigma$ is a partial order on $B(\tilde{W}, \sigma)_{str}$.

For $(v_1, \kappa_1), (v_2, \kappa_2) \in X_*(T)_{\mathbb{Q}}^+ \times \pi_1(G)_{\Gamma_F}$, we write $(v_1, k_1) \leqslant (v_2, k_2)$ if $v_1 \leqslant v_2$ (the dominance order) and $k_1 = k_2$.

Following Grothendieck, we introduce admissible subscheme of $G(L)$ and show that each σ-conjugacy class of $G(L)$ is a locally closed admissible subscheme of $G(L)$ (see Appendix). The closure relation between the σ-conjugacy classes of $G(L)$ gives a partial order on $B(G)$.

The main result of this section is

Theorem 3.1. *Let $\mathcal{O}, \mathcal{O}' \in B(\tilde{W}, \sigma)_{str}$. The following conditions are equivalent:*

(1) $\mathcal{O} \preceq_\sigma \mathcal{O}'$.
(2) $[\mathcal{O}] \subset \overline{[\mathcal{O}']}$.
(3) $f(\mathcal{O}) \leqslant f(\mathcal{O}')$, i.e. $\kappa(\mathcal{O}) = \kappa(\mathcal{O}')$ and $\nu_{\mathcal{O}} \leqslant \nu_{\mathcal{O}'}$.

Proof. We first prove $(1) \Leftrightarrow (2)$.

Let w' be a σ-straight element of $\mathcal{O}'$. Then

$$\overline{[\mathcal{O}']} = \overline{G(L) \cdot_\sigma Iw'I} = \cup_{w \leqslant w'} G(L) \cdot_\sigma IwI$$

$$= \cup_{\mathcal{O}_1 \preceq_\sigma w'} [\mathcal{O}_1] = \cup_{\mathcal{O}_1 \preceq \mathcal{O}'} [\mathcal{O}_1].$$

Here the first equality follows from §2.5 (3), the second equality follows from Theorem A.3, the third equality follows from Theorem 2.1 and Corollary 2.8 and the last equality follows from §3.2.

Next we prove $(1) \Rightarrow (3)$.

If $\mathcal{O} \preceq_\sigma \mathcal{O}'$, then there exists $w \in \mathcal{O}_{\min}$ and $w' \in \mathcal{O}'_{\min}$ such that $w \leqslant w'$. In particular, $wW_a = w'W_a$. Hence $\kappa(\mathcal{O}) = \kappa(\mathcal{O}')$. Moreover, w and w' are σ-straight elements. So for any n, $\ell((w\sigma)^n) =$

$n\ell(w)$ and $\ell((w'\sigma)^n) = n\ell(w')$. Thus $(w\sigma)^n \leqslant (w'\sigma)^n$. In particular, $t^{m\nu_{w,\sigma}} \leqslant t^{m\nu_{w',\sigma}}$ for sufficiently divisible integer m. In particular, $m\bar{\nu}_{w,\sigma} \leqslant m\bar{\nu}_{w',\sigma}$. So $\bar{\nu}_{w,\sigma} \leqslant \bar{\nu}_{w',\sigma}$.

Now we prove $(3) \Rightarrow (1)$.

Suppose that $\kappa(\mathcal{O}) = \kappa(\mathcal{O}')$ and $\nu_{\mathcal{O}} \leqslant \nu_{\mathcal{O}'}$. Let $\tilde{W}_{ad}$ be the Iwahori-Weyl group of the adjoint group G_{ad}. The natural projection $\pi : \tilde{W} \to \tilde{W}_{ad}$ send $\mathcal{O}$ to $\mathcal{O}_{ad}$ and $\mathcal{O}'$ to $\mathcal{O}'_{ad}$. As π preserves length, $\mathcal{O}_{ad}$ and $\mathcal{O}'_{ad}$ are straight σ-conjugacy classes of $\tilde{W}_{ad}$. Moreover, $\kappa(\mathcal{O}_{ad}) = \kappa(\mathcal{O}'_{ad})$ and $\nu_{\mathcal{O}_{ad}} \leqslant \nu_{\mathcal{O}'_{ad}}$.

We may write σ as $\sigma = \mathrm{Ad}(\tau) \circ \sigma_0$, where τ is a length-zero element in $\tilde{W}_{ad}$ and σ_0 is a diagram automorphism of $\tilde{W}_{ad}$ such that σ_0 fixes $\tilde{\mathbb{S}} - \mathbb{S}$. Then $\mathcal{O}_{ad}\tau$ and $\mathcal{O}'_{ad}\tau$ are straight σ_0-conjugacy classes of $\tilde{W}_{ad}$. Moreover, $\nu_{\mathcal{O}_{ad}\tau} \leqslant \nu_{\mathcal{O}'_{ad}\tau}$.

We associate a quasi-split unramified group H to the pair $(\tilde{W}_{ad}, \sigma_0)$. We regard $[\mathcal{O}_{ad}\tau]$ and $[\mathcal{O}'_{ad}\tau]$ as σ_0-conjugacy classes of $H(L)$. By [27, Theorem 2] and [8, Theorem 1.1][1], $[\mathcal{O}_{ad}\tau] \subset \overline{[\mathcal{O}'_{ad}\tau]}$. By the equivalence $(1) \Leftrightarrow (2)$ for G_{ad}, $\mathcal{O}_{ad}\tau \preceq_{\sigma_0} \mathcal{O}'_{ad}\tau$. This is equivalence to $\mathcal{O}_{ad} \preceq_\sigma \mathcal{O}'_{ad}$.

By definition, there exists $w_{ad} \in (\mathcal{O}_{ad})_{\min}$ and $w'_{ad} \in (\mathcal{O}'_{ad})_{\min}$ such that $w_{ad} \leqslant w'_{ad}$. Let $w \in \mathcal{O}$ and $w' \in \mathcal{O}'$ such that $\pi(w) = w_{ad}$, $\pi(w') = w'_{ad}$ and $wW_a = w'W_a$. Then $w \leqslant w'$. Hence $\mathcal{O} \preceq_\sigma \mathcal{O}'$. $\square$

4. Converse to Mazur's inequality

Proposition 4.1. *Let $\mu \in X_*(T)^+$ and $\mathcal{O}$ be a straight σ-conjugacy class of $\tilde{W}$. If $\kappa(\mathcal{O}) = \mu^\sharp$ and $\nu_{\mathcal{O}} \leqslant \mu^\diamond$, then $\mathcal{O} \cap \mathrm{Adm}(\mu) \neq \emptyset$.*

Proof. By [14], the set $\{\nu_{\mathcal{O}}; \kappa(\mathcal{O}) = \mu^\sharp, \nu_{\mathcal{O}} \leqslant \mu^\diamond\}$ contains a unique maximal element ν and there exists $x \in \mathrm{Adm}(\mu)$ with $\bar{\nu}_x = \nu$.

Let $\mathcal{O}$ be a straight σ-conjugacy class $\mathcal{O}$ of $\tilde{W}$ with $\kappa(\mathcal{O}) = \mu^\sharp$ and $\nu_{\mathcal{O}} \leqslant \mu^\diamond$. Then $\nu_{\mathcal{O}} \leqslant \nu$. By Theorem 3.1 and Corollary 2.6, $\nu_{\mathcal{O}} \leqslant \mu^\diamond$, $\mathcal{O} \preceq_\sigma x$. In other words, there exists $w \in \mathcal{O}_{\min}$ such that $w \leqslant x$. Since $\mathrm{Adm}(\mu)$ is closed under the Bruhat order, $w \in \mathrm{Adm}(\mu)$. $\square$

Now we prove the converse to Mazur's inequality.

Theorem 4.2. *Let $b \in G(L)$, $\mu \in X_*(T)^+$ and $J \subset \tilde{\mathbb{S}}$ such that $\sigma(J) = J$ and W_J is finite. If $b \in B(G, \mu)$, then $X(\mu, b)_J \neq \emptyset$.*

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[1] The statement in [8] is for PEL type Shimura varieties. The argument still holds for any unramified loop groups over function fields. It is based on Viehmann's strategy in [27, Proof of Theorem 20] (see also [8, Proposition 5.13], using the dimension formula of affine Deligne-Lusztig varieties [7] and the purity Theorem [27, Corollary 18] and [8, Proposition 5.4].

Proof. Let $b \in B(G, \mu)$. Then $[b]$ is represented by a straight σ-conjugacy class $\mathcal{O}$ of $\tilde{W}$. By Proposition 4.1, $\mathcal{O} \cap \mathrm{Adm}(\mu) \neq \emptyset$. Note that $\mathrm{Adm}(\mu) \subset \mathrm{Adm}^J(\mu)$. Hence $\mathcal{O} \cap \mathrm{Adm}^J(\mu) \neq \emptyset$. By Corollary 2.9, $X(\mu, b)_J \neq \emptyset$. $\square$

5. Mazur's inequality: Iwahori case

To prove Mazur's inequality in the Iwahori case, we need the following additivity property of admissible sets due to Xinwen Zhu [30].

Theorem 5.1. *Suppose that G splits over a tamely ramified extension of F. Let $\mu, \mu' \in X_*(T)^+$. Then*

$$\mathrm{Adm}(\mu) \, \mathrm{Adm}(\mu') = \mathrm{Adm}(\mu + \mu').$$

Proof. We first show that $\mathrm{Adm}(\mu + \mu') \subset \mathrm{Adm}(\mu) \, \mathrm{Adm}(\mu')$.

Let $z \in \mathrm{Adm}(\mu + \mu')$. By definition, $z \leqslant t^{x(\underline{\mu}+\underline{\mu}')}$ for some $x \in W_0$. Notice that $t^{x(\underline{\mu}+\underline{\mu}')} = t^{x(\underline{\mu})}t^{x(\underline{\mu}')}$ and $\ell(t^{x(\underline{\mu}+\underline{\mu}')}) = \ell(t^{x(\underline{\mu})})\ell(t^{x(\underline{\mu}')})$. In other words, there exists a reduced expression of $t^{x(\underline{\mu}+\underline{\mu}')}$ consisting of two parts, the first part is a reduced expression of $t^{x(\underline{\mu})}$ and the second part is a reduced expression of $t^{x(\underline{\mu}')}$. Hence there exists $z_1 \leqslant t^{x(\underline{\mu})} \in \mathrm{Adm}(\mu)$ and $z_2 \leqslant t^{x(\underline{\mu}')} \in \mathrm{Adm}(\mu')$ such that $z = z_1 z_2$.

The proof of the other direction $\mathrm{Adm}(\mu) \, \mathrm{Adm}(\mu') \subset \mathrm{Adm}(\mu + \mu')$ is based on the theory of global Schubert varieties of Zhu [29]. We first recall the definition.

Let $L = \bar{\mathbb{F}}_q((\epsilon))$ and G be a connected reductive group over L, split over a tamely ramified extension, and with Iwahori-Weyl group $\tilde{W}$. Let $\mathcal{G}$ be the Iwahori group scheme over $\mathcal{O}_L$. The element $\mu \in X_*(T)$ defines a section s_μ of the global affine Grassmannian $Gr_{\mathcal{G}}$ as in [29, Proposition 3.4]. The global Schubert variety $\overline{Gr_{\mathcal{G},\mu}}$ is the scheme-theoretic closure of the $\mathcal{L}^+\mathcal{G} \cdot s_\mu$ in $Gr_{\mathcal{G}}$, where $\mathcal{L}^+\mathcal{G}$ is the positive loop group. It is a scheme over $\mathcal{O}_F$. One of the main result of [29] is that the special fiber of $\overline{Gr_{\mathcal{G},\mu}}$ is isomorphic to $\sqcup_{w \in \mathrm{Adm}(\mu)} IwI/I$.

Now we take the convolution product of $\overline{Gr_{\mathcal{G},\mu}}$ with $\overline{Gr_{\mathcal{G},\mu}}$ as in [29, §6]. By definition, the special fiber of the convolution product is isomorphic to $\cup_{w \in \mathrm{Adm}(\mu), w' \in \mathrm{Adm}(\mu')} IwIw'I/I \supset \cup_{z \in \mathrm{Adm}(\mu)\,\mathrm{Adm}(\mu')} IzI/I$. On the other hand, it is proved in [29, §6] that the special fiber is isomorphic to $\sqcup_{z \in \mathrm{Adm}(\mu+\mu')} IzI$. Hence $\mathrm{Adm}(\mu) \, \mathrm{Adm}(\mu') \subset \mathrm{Adm}(\mu + \mu')$. $\square$

Now we prove Mazur's inequality in the Iwahori case.

Theorem 5.2. *Let $b \in G(L)$ and $\mu \in X_*(T)^+$. If $X(\mu, b)_\emptyset \neq \emptyset$, then $b \in B(G, \mu)$.*

Proof. Recall that $\sigma = \tau \circ \sigma_0$, where σ_0 is a diagram automorphism of $G(L)$ such that σ_0 fixes $\tilde{\mathbb{S}} - \mathbb{S}$ and the induced action of τ on the adjoint group G_{ad} is inner. For any $\mu \in X_*(T)^+$, $\sigma_0(\mathrm{Adm}(\mu)) = \mathrm{Adm}(\sigma_0(\mu))$. Note that $\tau(\mu) = x(\mu)$ for some $x \in W_0$. Thus $\tau(\mathrm{Adm}(\mu)) = \mathrm{Adm}(\mu)$. Therefore

$$\sigma(\mathrm{Adm}(\mu)) = \mathrm{Adm}(\sigma_0(\mu)).$$

By Theorem 2.1, $X(\mu, b)_{\emptyset} \neq \emptyset$ implies that $w \in [b]$ for some $w \in \mathrm{Adm}(\mu)$. Let n_0 be the order of σ in $\mathrm{Aut}(\tilde{W})$ and $n = n_0 \sharp(W_0)$. We regard $w\sigma$ as an element in $\tilde{W} \rtimes \langle \sigma \rangle$. Then $(w\sigma)^{n_0} \in \tilde{W}$ and $(w\sigma)^n = t^\lambda$ for some $\lambda \in X_*(T)$. By definition, λ lies in the W_0-orbit of $n\nu_{\mathbb{O}}$. On the other hand,

$$\begin{aligned}
(w\sigma)^n &= w\sigma(w)\sigma^2(w) \cdots \sigma^{n-1}(w) \\
&\in \mathrm{Adm}(\mu)\,\mathrm{Adm}(\sigma_0(\mu))\,\mathrm{Adm}(\sigma_0^2(\mu)) \cdots \mathrm{Adm}(\sigma_0^{n-1}(\mu)) \\
&= \mathrm{Adm}(\mu + \sigma_0(\mu) + \cdots + \sigma_0^{n-1}(\mu)) \\
&= \mathrm{Adm}(n\mu^{\diamond})
\end{aligned}$$

Hence $t^\lambda \in \mathrm{Adm}(n\mu^{\diamond})$ and $\bar{\lambda} \leqslant n\mu^{\diamond}$. Thus $\nu_{\mathbb{O}} \leqslant \mu^{\diamond}$. $\square$

6. Mazur's inequality: General case

6.1. To pass from Iwahori case to the general case, we need part (2) of Theorem 1.1. There are two key ingredients in the proof.

(a) A suitable stratification of $Y_{J,\mu}$ with respect to the σ-conjugation action of P_J.

(b) A compatibility property of admissible sets.

6.2. We discuss §6.1(a) first. The stratification is established in [11, §2] and [2, §3], generalizing Lusztig's G-stable piece decomposition for the finite case.

Let $J = \sigma(J) \subset \tilde{\mathbb{S}}$ with W_J finite. For any $w \in {}^J\tilde{W}$, we consider the subset $P_J \cdot_\sigma IwI$ of $G(L)$. Then

(1) $G(L) = \sqcup_{w \in {}^J\tilde{W}} P_J \cdot_\sigma IwI$.

(2) $Y_{J,\mu} = \sqcup_{w \in {}^J\tilde{W} \cap \mathrm{Adm}^J(\mu)} P_J \cdot_\sigma IwI$.

(3) If $F = \mathbb{F}_q((\epsilon))$, then for $w \in {}^J\tilde{W}$,

$$\overline{P_J \cdot_\sigma IwI} = \sqcup_{w'} P_J \cdot_\sigma Iw'I,$$

where w' runs over elements in ${}^J\tilde{W}$ such that there exists $x \in W_J$ with $xw\sigma(x)^{-1} \leqslant w'$.

Then we discuss the following compatibility result on the sets ${}^J\tilde{W} \cap \mathrm{Adm}^J(\mu)$.

Theorem 6.1. *Let $\mu \in X_*(T)^+$ and $J \subset \tilde{\mathbb{S}}$ with W_J finite. Then $^J\tilde{W} \cap \mathrm{Adm}^J(\mu) = {}^J\tilde{W} \cap \mathrm{Adm}(\mu)$.*

Proof. Let Φ be the relative root system and Φ_a be the affine root system, which is a set of affine functions on $V = X_*(S) \otimes \mathbb{R}$ of the form $\beta + r$ for $\beta \in \Phi$ and $r \in \mathbb{R}$.

Let $w \in {}^J\tilde{W} \cap \mathrm{Adm}^J(\mu)$. Then $w \leqslant \max(W_J t^\gamma W_J)$ for some $\gamma \in W_0 \cdot \underline{\mu}$.

We first show that

(a) $w \leqslant \max(t^\lambda W_J)$ for some $\lambda \in W_0 \cdot \underline{\mu}$ with $t^\lambda \in {}^J\tilde{W}$.

For $y \in {}^J\tilde{W}$, we set $I(J, y) = \max\{K \subset J; y(K) = K\}$. By [10, Corollary 2.6], t^γ is conjugate by an element in W_J to an element $z = xw_1$, where $w_1 \in {}^J\tilde{W}$ and $x \in W_{I(J,w_1)}$. Since z is conjugate to t^γ, it is of the form t^λ for some $\lambda \in W_0 \cdot \underline{\mu}$.

Let Φ_1 be the root system associated to $I(J, w_1)$. By definition, for any $\alpha \in \Phi_1$, $t^\lambda(\alpha) \in \Phi_1$. Therefore $t^\lambda(\alpha) - \alpha = \langle \lambda, \alpha \rangle$ is in the root lattice of Φ_1. However, any nonzero $r \in \Phi_a$ is not spanned by K for any $K \subset \tilde{\mathbb{S}}$ with W_K finite. Hence $\langle \lambda, \alpha \rangle = 0$ and $t^\lambda(\alpha) = \alpha$ for all $\alpha \in \Phi_1$. In particular, $t^\lambda \in {}^{I(J,w_1)}\tilde{W}$. Since $w_1 \in {}^{I(J,w_1)}\tilde{W}$ and $t^\lambda \in W_{I(J,w_1)} w_1$, we must have $x = 1$.

(a) is proved.

We may write $\max(t^\lambda W_J)$ as ab, where $a \in W_J$ and $b \in {}^J\tilde{W}$. Since $t^\lambda \in {}^J\tilde{W}$, $b = t^\lambda y$ for some $y \in W_J$ with $\ell(t^\lambda y) = \ell(t^\lambda) + \ell(y)$. If $y \neq 1$, then $s_i y < y$ for some $i \in J$. Let α_i be the simple root associated to s_i. Since $t^\lambda \in {}^J\tilde{W}$, $t^{-\lambda} \in \tilde{W}^J$. Hence $t^{-\lambda}(\alpha_i) = \alpha_i - \langle \lambda, \alpha \rangle$ is a positive affine root. Hence $\langle \lambda, \alpha_i \rangle \leqslant 0$.

If $\langle \lambda, \alpha_i \rangle < 0$, then $t^\lambda(\alpha_i)$ is a negative affine root and $t^\lambda s_i < t^\lambda$, which contradicts the fact that $\ell(t^\lambda y) = \ell(t^\lambda) + \ell(y)$. If $\langle \lambda, \alpha_i \rangle = 0$, then $t^\lambda y = s_i t^\lambda(s_i y)$, which contradicts the fact that $t^\lambda y \in {}^J\tilde{W}$. Therefore $y = 1$ and $w \leqslant t^\lambda$. $\qquad\square$

6.3. We prove Theorem 1.1 (2).

Let $J \subset \tilde{\mathbb{S}}$ such that $\sigma(J) = J$ and W_J is finite. Recall that $Y_{J,\mu} = \cup_{w \in \mathrm{Adm}(\mu)} P_J w P_J$. By §6.2 and Theorem 6.1,

$$Y_{J,\mu} = \cup_{x \in \mathrm{Adm}^J(\mu) \cap {}^J\tilde{W}} P_J \cdot_\sigma IxI$$
$$= \cup_{x \in \mathrm{Adm}(\mu) \cap \tilde{W}} P_J \cdot_\sigma IxI$$
$$\subset P_J \cdot_\sigma Y_{\emptyset,\mu}.$$

Therefore,

(a) $$Y_{J,\mu} = P_J \cdot_\sigma Y_{\emptyset,\mu}.$$

For any $J \subset J' \subset \tilde{\mathbb{S}}$ with $\sigma(J) = J$, $\sigma(J') = J'$ and $W_{J'}$ finite, we have

(b) $\qquad Y_{J',\mu} = P_{J'} \cdot_\sigma Y_{\emptyset,\mu} = P_{J'} \cdot_\sigma (P_J \cdot_\sigma Y_{\emptyset,\mu}) = P_{J'} \cdot_\sigma Y_{J,\mu}.$

Now

$$\pi_{J,J'}(X(\mu,b)_J) = \{gP_{J'} \in G(L)/P_{J'}; g^{-1}b\sigma(g) \in P_{J'} \cdot_\sigma Y_{J,\mu}\} = X(\mu,b)_{J'}.$$

In other words, $\pi_{J,J'}$ is surjective.

6.4. Now we prove Mazur's inequality for J.

If $X(\mu,b)_J \neq \emptyset$, then by Theorem 1.1 (2), $X(\mu,b)_\emptyset \neq \emptyset$. By Theorem 5.2, $b \in B(G,\mu)$.

APPENDIX A. ADMISSIBILITY

A.1. In the appendix, we assume that $F = \mathbb{F}_q((\epsilon))$. We first recall the Moy-Prasad filtration [21].

Let v be a generic point in the base alcove $\mathbf{a}$. For any $r \geqslant 0$, let I_r be the subgroup of I generated by a suitable subgroup of $T(L)_1$ and U_ϕ, where ϕ runs over all the affine roots with $\phi(v) \geqslant r$. By definition, if $x \in \tilde{W}$ with $\ell(x) < r$, then $hI_rh^{-1} \subset I_{r-\ell(x)} \subset I$ for any $h \in IxI$.

A.2. A subset V of $G(L)$ is called *admissible* if for any $w \in \tilde{W}$, there exists $r \geqslant 0$ such that $\cup_{w' \leqslant w}(V \cap Iw'I)$ is stable under the right action of I_r. This is equivalent to say that for any $w \in \tilde{W}$, there exists $r' \geqslant 0$ such that $V \cap IwI$ is stable under the right action of $I_{r'}$.

An admissible subset V of $G(L)$ is a locally closed subscheme if for any $w \in \tilde{W}$ and $r \geqslant 0$ such that $\cup_{w' \leqslant w}(V \cap Iw'I)$ is stable under the right action of I_r, $\cup_{w' \leqslant w}(V \cap Iw'I)/I_r$ is a locally closed subscheme of $\overline{IwI/I_r} = \cup_{w' \leqslant w}Iw'I/I_r \subset G(L)/I_r$.

We define the closure of a locally closed subscheme V in G as follows.

Let $w \in \tilde{W}$. Let $r \geqslant 0$ such that $\cup_{w' \leqslant w}(V \cap Iw'I)$ is stable under the right action of I_r. Let V_w be the inverse image under the projection $G(L) \to G(L)/I_r$ of the closure of $\cup_{w' \leqslant w}(V \cap Iw'I)/I_r$ in $G(L)/I_r$. Then it is easy to see that V_w is independent of the choice of r. Moreover, if $w' \leqslant w$, then $V_{w'} \subset V_w$. Set

$$\overline{V} = \varinjlim_w V_w.$$

Theorem A.1. *Let $[b]$ be a σ-conjugacy class of $G(L)$. Then $[b]$ is admissible.*

Remark A.2. For split groups, this is first proved by Hartl and Viehmann in [9].

Proof. Let w be a σ-straight element in $[b]$. By §2.5 (3), $[b] = G(L) \cdot_\sigma IwI$. Let $y \in \tilde{W}$ such that $[b] \cap IyI \neq \emptyset$. By [26, Theorem 1.4], there exists $n \in \mathbb{N}$ such that for any $g \in [b] \cap IyI$, $h^{-1}g\sigma(h) \in IwI$ for some $h \in IzI$ with $\ell(z) < n$. By §A.1, $h^{-1}gI_n\sigma(h) \subset h^{-1}g\sigma(h)I_{n-\ell(z)} \subset IwI$. Hence $gI_n \subset G(L) \cdot_\sigma IwI = [b]$. The theorem is proved. $\square$

Another admissibility result we need is the following:

Theorem A.3. *Let $w \in \tilde{W}$. Then $G(L) \cdot_\sigma IwI$ is admissible and*

$$\overline{G(L) \cdot_\sigma IwI} = \cup_{w' \leqslant w} G(L) \cdot_\sigma Iw'I.$$

Proof. Set $V = G(L) \cdot_\sigma IwI$ and $V' = \cup_{w' \leqslant w} G(L) \cdot_\sigma Iw'I$. By Theorem 2.1, both V and V' are finite unions of σ-conjugacy classes and $V' = \sqcup_{\mathcal{O} \preceq_\sigma w} [\mathcal{O}]$. By Theorem A.1, V and V' are admissible.

Let $x \in \tilde{W}$. By [26, Theorem 1.4], there exists $n \in \mathbb{N}$ such that

$$V \cap \overline{IxI} = (\cup_{z \in \tilde{W}, \ell(z) < n} IzI) \cdot_\sigma IwI \cap \overline{IxI};$$
$$V' \cap \overline{IxI} = (\cup_{z \in \tilde{W}, \ell(z) < n} IzI) \cdot_\sigma \overline{IwI} \cap \overline{IxI}.$$

Define the action of I_n on $(\cup_{z \in \tilde{W}, \ell(z) < n} IzI) \times G(L)/I_n$ by $h \cdot (g, g') = (gh^{-1}, hg')$. We denote by $(\cup_{z \in \tilde{W}, \ell(z) < n} IzI) \times^{I_n} G(L)/I_n$ its quotient. Consider the map

$$(\cup_{z \in \tilde{W}, \ell(z) < n} IzI) \times G(L)/I_n \to G(L)/I, \qquad (g, g') \mapsto gg'\sigma(g)^{-1}.$$

By §A.1, it is well-defined. It induces a map

$$\pi : (\cup_{z \in \tilde{W}, \ell(z) < n} IzI) \times^{I_n} \overline{IwI}/I_n \to G(L)/I.$$

This is a proper map. Hence the image is closed in $G(L)/I$ and is the closure of the image of $(\cup_{z \in \tilde{W}, \ell(z) < n} IzI) \times^{I_n} IwI/I_n$.

Therefore $V' \cap \overline{IxI}$ is closed and is the closure of $V \cap \overline{IxI}$. In other words, $V_x = V' \cap \overline{IxI}$. Hence

$$\overline{V} = \varinjlim_x V_x = V'.$$

$\square$

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