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A PRODUCT OF TENSOR PRODUCT L-FUNCTIONS OF QUASI-SPLIT CLASSICAL GROUPS OF HERMITIAN TYPE

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Abstract. A family of global zeta integrals representing a product of tensor product (partial) L-functions:

$$L^S(s,\pi\times\tau_1)L^S(s,\pi\times\tau_2)\cdots L^S(s,\pi\times\tau_r)$$

is established in this paper, where π is an irreducible cuspidal automorphic representation of a quasi-split classical group of Hermitian type and $\tau_1,\ldots,\tau_r$ are irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{a_1},\ldots,\mathrm{GL}_{a_r}$, respectively. When $r=1$ and the classical group is an orthogonal group, this family was studied by Ginzburg et al. (Mem Am Math Soc 128: viii+218, 1997). When π is generic and $\tau_1,\ldots,\tau_r$ are not isomorphic to each other, such a product of tensor product (partial) L-functions is considered by Ginzburg et al. (The descent map from automorphic representations of $\mathrm{GL}(n)$ to classical groups, World Scientific, Singapore, 2011) in with different kind of global zeta integrals. In this paper, we prove that the global integrals are eulerian and finish the explicit calculation of unramified local zeta integrals in a certain case (see Section 4 for detail), which is enough to represent the product of unramified tensor product local L-functions. The remaining local and global theory for this family of global integrals will be considered in our future work.

1 Introduction

We study a finite product of tensor product (partial) automorphic L-functions for quasi-split unitary or orthogonal groups and general linear groups via global zeta integral method.

Let G_n be a quasi-split group, which is either $\mathrm{U}_{n,n}$, $\mathrm{U}_{n+1,n}$, SO_{2n+1}, or SO_{2n}, defined over a number field F. Let E be a quadratic extension of F when we discuss unitary groups and E be equal to F when we discuss orthogonal groups. Let $\mathbb{A}_E$ be the ring of adeles of E and $\mathbb{A}$ be the ring of adeles of F. Take τ to be an irreducible

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generic automorphic representation of $\mathrm{Res}_{E/F}(\mathrm{GL}_a)(\mathbb{A}) = \mathrm{GL}_a(\mathbb{A}_E)$ of isobaric type, i.e.

$$\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r \tag{1.1}$$

where $a = \sum_{i=1}^r a_i$ is a partition of a and τ_i is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{a_i}(\mathbb{A}_E)$. Let π be an irreducible cuspidal automorphic representation of $G_n(\mathbb{A})$. We consider a family of global zeta integrals (see Section 3 for definition), which represents a finite product of the tensor product (partial) automorphic L-functions:

$$L^S(s, \pi \times \tau) = L^S(s, \pi \times \tau_1)L^S(s, \pi \times \tau_2) \cdots L^S(s, \pi \times \tau_r). \tag{1.2}$$

It is often interesting and important in number theory and arithmetic to consider simultaneous behavior at a particularly given point $s = s_0$ of the L-functions $L^S(s, \pi \times \tau_i)$ with $i = 1, 2, \ldots, r$. For instance, one may consider the nonvanishing at $s = \frac{1}{2}$, the center of the symmetry of the L-functions $L^S(s, \pi \times \tau_1)$, $L^S(s, \pi \times \tau_2)$, $\cdots$, $L^S(s, \pi \times \tau_r)$, or particularly, take $\tau_1 = \tau_2 = \cdots = \tau_r$ and consider the r-th power $L^S(s, \pi \times \tau_1)^r$ at a given value $s = s_0$ for all positive integers r. As remarked at the end of this paper, the arguments and the methods still work if one replaces the single variable s by multi-variable $(s_1, \ldots, s_r)$. However, we focus on the case of single variable s in this paper.

We use a family of the Bessel periods (discussed in Section 2) to define the family of global zeta integrals, following closely the formulation of Ginzburg et al. in [GPR97], where the case when $r = 1$ and G_n is an orthogonal group was considered. When π is generic, i.e. has a nonzero Whittaker-Fourier coefficient, and $\tau_1, \ldots, \tau_r$ are not isomorphic to each other, this family of tensor product L-functions was studied by Ginzburg et al. in their recent book [GRS11]. However, the global zeta integrals studied in [GRS11] are formulated in a different way and can not cover the general situation considered in this paper. It is worthwhile to remark that the global zeta integrals here are the most general version of this kind, the origin of which goes back to the pioneering work of Piatetski-Shapiro and Rallis and of Gelbart and Piatetski-Shapiro [GPR87]. Other special cases of this kind were studied earlier by various authors, and we refer to the relevant discussions in [GPR97] and [GRS11].

In addition to the potential application towards the simultaneous nonvanishing of the central values of the tensor product L-functions, the basic relation between the product of the tensor product (partial) L-functions and the family of global zeta integrals is also an important ingredient in the proof of the nonvanishing of the certain explicit constructions of endoscopy correspondences as indicated for some special cases in the work of Ginzburg in [Gin08], and as generally formulated in the work of the first named author [Jia11] and [Jia13]. We will come back to this topic in our future work [JZ13].

In general, the meromorphic continuation to the whole complex plane of the product of the tensor product (partial) L-functions is known from the work of R. Langlands on the explicit calculation of the constant terms of Eisenstein series [Lan71].

However, when π is not generic, i.e. has no nonzero Whittaker-Fourier coefficients, the Langlands conjecture on the standard functional equation and the finite number of poles for $\Re(s) \geq \frac{1}{2}$ is still not known [Sha10].

According to the work of Arthur [Art13] and also of C.-P. Mok [Mok13], any irreducible cuspidal automorphic representation π of $G_n(\mathbb{A})$ has a global Arthur parameter ψ, which determines an irreducible automorphic representation π_ψ of the corresponding general linear group $\mathrm{GL}_N(\mathbb{A}_E)$, where the integer N depends on the type of G_n. The mapping from π to π_ψ is called the Arthur–Langlands transfer, which is the weak Langlands functorial transfer from G_n to GL_N. This means that the global transfer from π to π_ψ is compatible with the corresponding local Langlands functorial transfers at all unramified local places of π. Hence we have an identity for partial L-functions

$$L^S(s, \pi \times \tau) = L^S(s, \pi_\psi \times \tau).$$

The partial L-function on the right hand side is the Rankin-Selberg convolution L-function for general linear groups studied by Jacquet et al. [JPS83]. When π has a generic global Arthur parameter, the Arthur–Langlands transfer from G_n to general linear groups is compatible with the corresponding local Langlands functorial transfer at all local places. Hence one may define the complete tensor product L-function by

$$L(s, \pi \times \tau) := L(s, \pi_\psi \times \tau),$$

just as in [Art13] and [Mok13].

However, when the global Arthur parameter ψ is not generic, there exists irreducible cuspidal automorphic representation π with Arthur parameter ψ, whose local component π_ν at some ramified local place ν may not be transferred to the corresponding ramified local component $(\pi_\psi)_\nu$ under the local Langlands functorial transfer at ν. Hence it is impossible to define the local tensor product L-factors (and also γ-factors and ϵ-factors) of the pair (π_ν, τ_ν) in terms of those of the pair $((\pi_\psi)_\nu, \tau_\nu)$ at such ramified local places ν. Therefore, it is still an *open problem* to define the local ramified L-factors (and also γ-factors and ϵ-factors) for an irreducible cuspidal automorphic representation π of $G_n(\mathbb{A})$ when π has a non-generic global Arthur parameter. At this point, it seems that the integral representation of Rankin-Selberg type for automorphic L-functions is the only available method to attack this open problem.

For quasi-split classical groups of skew-Hermitian type, some preliminary work has been done in [GJRS11], using Fourier-Jacobi periods. Further work is in progress, including the work of X. Shen in his PhD thesis in University of Minnesota, 2013, which has produced two preprints [She12a, She12b]. A parallel theory for this case will also be considered in future.

In Section 2, we introduce a family of Eisenstein series, whose Bessel periods are needed to formulate the family of global zeta integrals as mentioned above. A basic analytic property of such global zeta integrals is stated in Proposition 2.1. We note that the construction of the global zeta integrals has two integers j and ℓ involved, depending on the cuspidal data of the Eisenstein series and the structure of the Bessel periods.

Section 3 finishes the first step in the global theory for the family of global zeta integrals, which proves that they are eulerian, i.e. they are expressed as an eulerian product of the corresponding local zeta integrals at all local places of F (Theorem 3.8). The argument is standard, although it is technically quite involved. Based on an explicit calculation of generalized Bruhat decomposition in [GRS11, Section 4.2], we calculate in Section 3.1 the Bessel-Fourier coefficients of the Eisenstein series used in the global zeta integrals. Then we use [GRS11, Section 4.4] to carry out a long calculation in Section 3.2, which proves Theorem 3.8. We note that Sections. 2 and 3 are for both unitary groups and orthogonal groups. Following the general understanding of the Rankin-Selberg method, after expressing the local zeta integrals in terms of the expected local L-functions, which is the key part of the local theory for global zeta integrals, the global analytic properties of the global zeta integrals will be transferred to the expected complete (or partial) L-functions. Although for any pair (j, ℓ) of integers, the global zeta integrals are eulerian (Theorem 3.8), it seems that only in the case where $j = \ell + 1$ the local zeta integral is better understood and is enough to reach the local L-factors of the tensor product type as we expected. The remaining cases will be considered in future.

The local theory starts in Section 4. In Section 4.1, we reformulate the local zeta integrals from the eulerian product in Theorem 3.8 in terms of the uniqueness of local Bessel functionals and related them to the corresponding twisted Jacquet modules. We show that the local zeta integrals converges absolutely when the real part of the complex parameter s is sufficiently large (Lemma 4.1). The twisted Jacquet modules are explicitly calculated following closely [GRS11, Chapter 5]. This is necessary for the development of the local theory at all finite local places. In Sections. 4.2, 4.3, 4.4, and 4.5, we only consider the unramified case. In Section 4.2, we write down explicitly the unramified local L-factors of tensor product type for unitary groups in terms of the corresponding Satake parameters of the unramified representations. Section 4.3 shows that the unramified local zeta integrals are rational functions in q_E^{-s} following the Bernstein rationality theorem. Starting with Section 4.4, we assume that $j = \ell+1$ and are concentrated on the proof of Theorem 3.9, i.e. the explicit calculation of the unramified local zeta integrals in terms of the expected local L-factors. The main arguments used here can be viewed as natural extension of those used in [GPR97] for orthogonal group case to unitary group case. Hence we only discuss the unitary group case, since the orthogonal groups case was treated in [GPR97]. Sections 4.4 and 4.5 are quite technical and are devoted to the understanding of the denominator and numerator of the rational function from Section 4.3. The result is stated in Theorem 4.12, which is the main local result of the paper.

The main global result in this paper is Theorem 4.13, which is stated at the end of Section 4. In order to carry out the complete understanding of the family of global zeta integrals, one has to develop the complete theory for the local zeta integrals at all local places, which is in fact our main concern and is considered in our work in progress.

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2 Certain Eisenstein series and Bessel periods

We introduce a family of Eisenstein series which will be used in the definition of a family of global zeta integrals, representing the family of the product of the tensor product L-functions as discussed in the introduction. The global zeta integrals are basically a family of Bessel periods of those Eisenstein series. We recall first the general notion of the Bessel periods of automorphic forms from [GPR97], [GJR09], [BS09] and [GRS11].

Let F be a number field. Define $E = F$ or $E = F(\sqrt{\rho})$, a quadratic extension of F, depending on that the classical group we considered is orthogonal or unitary, accordingly. It follows that the Galois group of E/F is either trivial or generated by a non-trivial automorphism $x \mapsto \bar{x}$. The ring of adeles of F is denoted by $\mathbb{A}$, while the ring of adeles of E is denoted by $\mathbb{A}_E$.

Let V be an E-vector space of dimension m with a non-degenerate quadratic form $q_{_V}$ if $E = F$ or a non-degenerate Hermitian form (also denoted by $q_{_V}$) if $E = F(\sqrt{\rho})$. Let $\mathrm{U}(q_{_V})$ be the connected component of isometry group of $(V, q_{_V})$ defined over F. It follows that $\mathrm{U}(q_{_V})$ is a special orthogonal group or a unitary group. Let $\tilde{m} = \mathrm{Witt}(V)$ be the Witt index of V. Let V^+ be a maximal totally isotropic subspace of V and V^- be its dual, so that V has the following polar decomposition

$$V = V^+ \oplus V_0 \oplus V^-,$$

where $V_0 = (V^+ \oplus V^-)^\perp$ denotes the anisotropic kernel of V. We choose a basis $\{e_1, e_2, \ldots, e_{\tilde{m}}\}$ of V^+ and a basis $\{e_{-1}, e_{-2}, \ldots, e_{-\tilde{m}}\}$ of V^- such that $q_{_V}(e_i, e_{-j}) = \delta_{i,j}$ for all $1 \le i, j \le \tilde{m}$.

We assume in this paper that the algebraic F-group $\mathrm{U}(q_{_V})$ is F-quasi-split. Then the anisotropic kernel V_0 is at most two dimensional. More precisely, when $E = F$, if $\dim_E V = m$ is even, then $\dim_E V_0$ is either 0 or 2, and if $\dim_E V = m$ is odd, then $\dim_E V_0$ is 1; and when $E = F(\sqrt{\rho})$, $\dim_E V_0$ is 0 or 1 according to that $\dim_E V = m$ is even or odd.

When $\dim V_0 = 2$, we choose an orthogonal basis $\{e_0^{(1)}, e_0^{(2)}\}$ of V_0 with the property that

$$q_{_{V_0}}(e_0^{(1)}, e_0^{(1)}) = 1, \qquad q_{_{V_0}}(e_0^{(2)}, e_0^{(2)}) = -c,$$

where $c \in F^\times$ is not a square and $q_{_{V_0}} = q_{_V}|_{V_0}$. When $\dim V_0 = 1$, we choose an anisotropic basis vector e_0 for V_0. We put the basis in the following order

$$e_1, e_2, \ldots, e_{\tilde{m}}, e_0^{(1)}, e_0^{(2)}, e_{-\tilde{m}}, \ldots, e_{-2}, e_{-1} \tag{2.1}$$

if $\dim_E V_0 = 2$;

$$e_1, e_2, \ldots, e_{\tilde{m}}, e_0, e_{-\tilde{m}}, \ldots, e_{-2}, e_{-1} \tag{2.2}$$

if $\dim_E V_0 = 1$; and

$$e_1, e_2, \ldots, e_{\tilde{m}}, e_{-\tilde{m}}, \ldots, e_{-2}, e_{-1} \tag{2.3}$$

if $\dim_E V_0 = 0$.

With the choice of the ordering of the basis vectors, the F-rational points $\mathrm{U}(q_{_V})(F)$ of the algebraic group $\mathrm{U}(q_{_V})$ are realized as an algebraic subgroup of $\mathrm{GL}_m(E)$. Define $n = [\frac{m}{2}]$ and put $G_n = \mathrm{U}(q_{_V})$. This agrees with the definition of G_n which was given in the introduction. From now on, for any F-algebraic subgroup H of G_n, the F-rational points $H(F)$ of H are regarded as a subgroup of $\mathrm{GL}_m(E)$. Similarly, the $\mathbb{A}$-rational points $H(\mathbb{A})$ of H are regarded as a subgroup of $\mathrm{GL}_m(\mathbb{A}_E)$.

The corresponding standard flag of V (with respect to the given ordering of the basis vectors) defines an F-Borel subgroup B. We may write $B = TN$ with T a maximal F-torus, whose elements are diagonal matrices, and with N the unipotent radical of B, whose elements are upper-triangular matrices. Let T_0 be the maximal F-split torus of G_n contained in T. We define the root system $\Phi(T_0, G_n)$ with the set of positive roots $\Phi^+(T_0, G_n)$ corresponding to the Borel subgroup given above.

Take ℓ an integer between 1 and $\tilde{m}$. Let $V_\ell^\pm$ be the totally isotropic subspace generated by $\{e_{\pm 1}, e_{\pm 2}, \ldots, e_{\pm \ell}\}$ and $P_\ell = M_\ell U_\ell$ be a standard maximal parabolic subgroup of G_n, which stabilizes V_ℓ^+. The Levi subgroup M_ℓ is isomorphic to $\mathrm{GL}(V_\ell^+) \times G_{n-\ell}$. Here $\mathrm{GL}(V_\ell^+) = \mathrm{Res}_{E/F}(\mathrm{GL}_\ell)$ and $G_{n-\ell} = \mathrm{U}(q_{_{W_\ell}})$ with $q_{_{W_\ell}} = q_{_V}|_{W_\ell}$ and $W_\ell = (V_\ell^+ \oplus V_\ell^-)^\perp$.

Let $\underline{\ell} := [\ell_1 \ell_2 \ldots \ell_p]$ be a partition of ℓ. Then $P_{\underline{\ell}} = M_{\underline{\ell}} U_{\underline{\ell}}$ is a standard parabolic subgroup of G_n, whose Levi subgroup

$$M_{\underline{\ell}} \cong \mathrm{Res}_{E/F}\mathrm{GL}_{\ell_1} \times \mathrm{Res}_{E/F}\mathrm{GL}_{\ell_2} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{\ell_p} \times G_{n-\ell}.$$

2.1 Bessel periods. Define N_ℓ to be the unipotent subgroup of G_n consisting of elements of following type,

$$N_\ell = \left\{ n = \begin{pmatrix} z & y & x \\ & I_{m-2\ell} & y' \\ & & z^* \end{pmatrix} \in G_n \mid z \in Z_\ell \right\}, \tag{2.4}$$

where Z_ℓ is the standard maximal (upper-triangular) unipotent subgroup of $\mathrm{Res}_{E/F}\mathrm{GL}_\ell$. It is clear that $N_\ell = U_{[1^\ell]}$ where $[1^\ell]$ is the partition of ℓ with 1 repeated ℓ times.

Fix a nontrivial character ψ_0 of $F\backslash\mathbb{A}_F$ and define a character ψ of $E\backslash\mathbb{A}_E$ by

$$\psi(x) := \begin{cases} \psi_0(x) & \text{if } E = F; \\ \psi_0(\frac{1}{2}\mathrm{tr}_{E/F}(\frac{x}{\sqrt{\rho}})) & \text{if } E = F(\sqrt{\rho}). \end{cases} \tag{2.5}$$

Then take w_0 to be an anisotropic vector in W_ℓ and define a character ψ_{ℓ,w_0} of N_ℓ by

$$\psi_{\ell,w_0}(n) := \psi\left(\sum_{i=1}^{\ell-1} z_{i,i+1} + q_{W_\ell}(y_\ell, w_0)\right), \tag{2.6}$$

where y_ℓ is the last row of y in $n \in N_\ell$ as defined in (2.4), which is regarded as a vector in W_ℓ.

If $\ell = \tilde{m}$, ψ_{ℓ,w_0} is a generic character on the maximal unipotent group $N = N_{\tilde{m}}$. We will not consider this case here and hence we always assume that $\ell < \tilde{m}$ from now on.

For $\kappa \in F^\times$, we choose

$$w_0 = y_\kappa = e_{\tilde{m}} + (-1)^{m+1}\frac{\kappa}{2}e_{-\tilde{m}}, \tag{2.7}$$

which implies that $q(y_\kappa, y_\kappa) = (-1)^{m+1}\kappa$ and that the corresponding character is

$$\psi_{\ell,\kappa}(n) = \psi_{\ell,w_0}(n) = \psi\left(\sum_{i=1}^{\ell-1} z_{i,i+1} + y_{\ell,\tilde{m}-\ell} + (-1)^{m+1}\frac{\kappa}{2}y_{\ell,m-\tilde{m}-\ell+1}\right). \tag{2.8}$$

The Levi subgroup $M_{[1^\ell]} = (\mathrm{Res}_{E/F}\mathrm{GL}_1)^{\times\ell} \times G_{n-\ell}$ normalizes the unipotent subgroup N_ℓ, and acts on the set of the characters of $N_\ell(\mathbb{A})$. Each orbit for this action contains a character of the form $\psi_{\ell,\kappa}$, with $\kappa \in F^\times$. The $M_{[1^\ell]}(F)$-orbits are classified by the Witt Theorem and give all F-generic characters of $N_\ell(\mathbb{A})$. The stabilizer of ψ_{ℓ,w_0} in the Levi subgroup $M_{[1^\ell]}$ is the subgroup

$$L_{\ell,w_0} = \left\{\begin{pmatrix} I_\ell & & \\ & \gamma & \\ & & I_\ell \end{pmatrix} \in G_n \mid \gamma J_{m-2\ell}w_0 = J_{m-2\ell}w_0\right\} \cong H_{n-\ell}, \tag{2.9}$$

where $H_{n-\ell}$ is defined to be $\mathrm{U}(q_{W_\ell \cap w_0^\perp})$ with $q_{W_\ell \cap w_0^\perp} = q_V|_{W_\ell \cap w_0^\perp}$, and J_k is the $k \times k$ matrix defined inductively by $J_k = \left(\begin{smallmatrix} & 1 \\ J_{k-1} & \end{smallmatrix}\right)$ and $J_1 = 1$. Define

$$R_{\ell,w_0} := H_{n-\ell}N_\ell = \mathrm{U}(q_{W_\ell \cap w_0^\perp})N_\ell. \tag{2.10}$$

Note that $\dim_E V$ and $\dim_E W_\ell \cap w_0^\perp$ have the different parity. If $\ell = 0$, the unipotent subgroup N_0 is trivial and we have that

$$R_{0,w_0} = \mathrm{U}(q_{V \cap w_0^\perp}).$$

When taking $w_0 = y_\kappa$, we will use the notation $\psi_{\ell,y_\kappa} = \psi_{\ell,\kappa}$, $L_{\ell,y_\kappa} = L_{\ell,\kappa}$ and $R_{\ell,y_\kappa} = R_{\ell,\kappa}$, respectively.

Let ϕ be an automorphic form on $G_n(\mathbb{A})$. Define the **Bessel-Fourier coefficient** (or Gelfand-Graev coefficient) of ϕ by

$$\mathcal{B}^{\psi_{\ell,w_0}}(\phi)(h) := \int_{N_\ell(F)\backslash N_\ell(\mathbb{A})} \phi(nh)\psi_{\ell,w_0}^{-1}(n)\,\mathrm{d}n. \tag{2.11}$$

This defines an automorphic function on the stabilizer $L_{\ell,w_0}(\mathbb{A}) = H_{n-\ell}(\mathbb{A})$. Take a cuspidal automorphic form φ on $H_{n-\ell}(\mathbb{A})$ and define the $(\psi_{\ell,w_0}, \varphi)$-**Bessel period** or simply **Bessel period** of ϕ by

$$\mathcal{P}^{\psi_{\ell,w_0}}(\phi,\varphi) := \int_{H_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})} \mathcal{B}^{\psi_{\ell,w_0}}(\phi)(h)\varphi(h)\,\mathrm{d}h. \tag{2.12}$$

We will apply this Bessel period to a family of Eisenstein series next.

2.2 Eisenstein series. We follow the notation of [MW95] to define a family of Eisenstein series.

For some j with $1 \le j \le \tilde{m}$, let $P_j = M_jU_j$ be a standard maximal parabolic F-subgroup of G_n with the Levi subgroup

$$M_j = \mathrm{Res}_{E/F}(\mathrm{GL}_j) \times G_{n-j}.$$

When $j = \tilde{m}$, the group $G_{n-\tilde{m}}$ is trivial and can be omitted, if $\dim_E V_0 = 0$, or $\dim_E V_0 = 1$ and $E = F$. Following [MW95, Page 5], the space X_{M_j} of all continuous homomorphisms from $M_j(\mathbb{A})$ to $\mathbb{C}^\times$, which is trivial on the subgroup $M_j(\mathbb{A})^1$ (defined in Chapter 1 [MW95]), can be identified with $\mathbb{C}$ by the mapping $\lambda_s \leftrightarrow s$, which is normalized as in [Sha10].

Let τ be an irreducible unitary generic automorphic representation of $\mathrm{GL}_j(\mathbb{A}_E)$ of the following isobaric type:

$$\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r, \tag{2.13}$$

where $\underline{j} = [j_1j_2\cdots j_r]$ is a partition of j and τ_i is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{j_i}(\mathbb{A}_E)$. Let σ be an irreducible automorphic representation of $G_{n-j}(\mathbb{A})$ (we do not assume that σ is cuspidal). Note that σ is irrelevant if $j = \tilde{m}$ and the group $G_{n-\tilde{m}}$ disappears. Following the definition of automorphic forms in [MW95, I.2.17], take an automorphic form

$$\phi = \phi_{\tau\otimes\sigma} \in \mathcal{A}(U_j(\mathbb{A})M_j(F)\backslash G_n(\mathbb{A}))_{\tau\otimes\sigma}. \tag{2.14}$$

For $\lambda_s \in X_{M_j}$, the Eisenstein series associated to $\phi(g)$ is defined by

$$E(\phi,s)(g) = E(\phi_{\tau\otimes\sigma},\lambda_s)(g) = \sum_{\delta\in P_j(F)\backslash G_n(F)} \lambda_s\phi(\delta g). \tag{2.15}$$

It is absolutely convergent for $\Re(s)$ large and uniformly converges for g over any compact subset of $G_n(\mathbb{A})$, has meromorphic continuation to $s \in \mathbb{C}$ and satisfies the standard functional equation.

Recall that $H_{n-\ell}$ is defined to be $\mathrm{U}(q_{W_\ell \cap w_0^\perp})$ and that $\dim_E V$ and $\dim_E W_\ell \cap w_0^\perp$ have the different parity. Let π be an irreducible *cuspidal* automorphic representation of $H_{n-\ell}(\mathbb{A})$ and take a cuspidal automorphic form

$$\varphi \in \mathcal{A}_0(H_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A}))_\pi. \tag{2.16}$$

The global zeta integral $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ is defined to be the following Bessel period

$$\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0}) := \mathcal{P}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma},s),\varphi_\pi). \tag{2.17}$$

Because φ_π is cuspidal, following a standard argument as in [CP04] and [BS09] for instance, one can easily prove the following.

PROPOSITION 2.1. *The global zeta integral $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ converges absolutely and uniformly in vertical strips in $\mathbb{C}$, away from the possible poles of the Eisenstein series $E(\phi_{\tau\otimes\sigma},s)$, and hence is meromorphic in $s\in\mathbb{C}$ with possible poles at the locations where the Eisenstein series has poles.*

We remark that after the Eisenstein series $E(\phi_{\tau\otimes\sigma},s)$ is properly normalized, the functional equation for $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ relating s to $-s$ follows from that for the Eisenstein series $E(\phi_{\tau\otimes\sigma},s)$. Of course, it is an interesting problem to understand the poles of $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ in terms of the structure of the global Arthur parameter ψ of π [Art13] and [Mok13] and/or in terms of the explicit construction of the endoscopy transfer in [Jia13]. This is in fact a long term project outlined in [Jia13].

3 The eulerian property of the global integrals

We prove here that the global zeta integral $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ will be expressed as an eulerian product of local zeta integrals. When $j=n=[\frac{m}{2}]$, such global zeta integrals with generic π have been studied in [GRS11, Chapter 10]. Hence we assume from now on that $j<n$ and also $\ell<\tilde{m}\leq n$.

We first calculate the Bessel-Fourier coefficients of the Eisenstein series and state the result in Proposition 3.3. Then, by using cuspidality, we prove that after the Eisenstein series is fully unfolded, the global zeta integral ends up with one possible non-zero term (Proposition 3.6). Then by considering certain Fourier developments to the integrands, we show that the global zeta integral is eulerian (Theorem 3.8).

Recall from (2.17) that $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ is the $(\psi_{\ell,w_0},\varphi_\pi)$-**Bessel period** of the Eisenstein series $E(\phi_{\tau\otimes\sigma},\lambda_s)(g)$, which is given by

$$\int_{H_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})} \mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma},s))(h)\varphi_\pi(h)\,\mathrm{d}h, \tag{3.1}$$

where the Bessel-Fourier coefficient $\mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma},s))(h)$ is given, as in (2.11), by

$$\mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma},s))(h) := \int_{N_\ell(F)\backslash N_\ell(\mathbb{A})} E(\phi_{\tau\otimes\sigma},s)(nh)\psi^{-1}_{\ell,w_0}(n)\,\mathrm{d}n. \tag{3.2}$$

We first calculate the Bessel-Fourier coefficient $\mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma},s))$.

3.1 Calculation of Bessel-Fourier coefficients. In order to calculate the Bessel-Fourier coefficient $\mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma},s))$, i.e. the integral in (3.2), we assume that $\Re(s)$ is large, and unfold the Eisenstein series. This leads to consider the double cosets decomposition $P_j\backslash G_n/P_\ell$, whose set of representatives $\epsilon_{\alpha,\beta}$ is explicitly given in [GRS11, Section 4.2]. In our situation, we put it into three cases for discussion.

Case (1): G_n is not the F-split even special orthogonal group. In this case, the set of representatives $\epsilon_{\alpha,\beta}$ of the double coset decomposition $P_j\backslash G_n/P_\ell$ is in bijection with the set of pairs of nonnegative integers

$$\{(\alpha,\beta)\mid 0\le\alpha\le\beta\le j \text{ and } j\le\ell+\beta-\alpha\le\tilde{m}\}. \tag{3.3}$$

Recall that $\tilde{m}$ is the Witt index of $(V,q_{_V})$ defining G_n.

In the next two cases, the set of pairs (α,β) is also given in (3.3).

Case (2-1): G_n is the F-split even special orthogonal group and $\ell+\beta-\alpha<\tilde{m}=n$. In this case, the set of representatives $\epsilon_{\alpha,\beta}$ of the double coset decomposition $P_j\backslash G_n/P_\ell$ is in bijection with the set of pairs of nonnegative integers

$$\{(\alpha,\beta)\mid 0\le\alpha\le\beta\le j \text{ and } j\le\ell+\beta-\alpha\le n-1\}. \tag{3.4}$$

Case (2-2): G_n is the F-split even special orthogonal group and $\ell+\beta-\alpha=n$. In this case, there are two double cosets corresponding to each pair (α,β), and hence we may choose representatives $\epsilon_{\alpha,\beta}$ and $\tilde{\epsilon}_{\alpha,\beta}=w_q\epsilon_{\alpha,\beta}w_q$ of the two double cosets corresponding to such pairs (α,β).

In all cases, we denote by $P_\ell^{\epsilon_{\alpha,\beta}}:=\epsilon_{\alpha,\beta}^{-1}P_j\epsilon_{\alpha,\beta}\cap P_\ell$ the stabilizer in P_ℓ, whose elements have the following form as matrices in $\mathrm{GL}_m(E)$:

$$g_\ell^{(\alpha,\beta)}=\begin{pmatrix} a & x_1 & x_2 & y_1 & y_2 & y_3 & z_1 & z_2 & z_3\\ 0 & b & x_3 & 0 & y_4 & y_5 & 0 & z_4 & z_2'\\ 0 & 0 & c & 0 & 0 & y_6 & 0 & 0 & z_1'\\ &&& d & u & v & y_6' & y_5' & y_3'\\ &&& 0 & e & u' & 0 & y_4' & y_2'\\ &&& 0 & 0 & d^* & 0 & 0 & y_1'\\ &&&&&& c^* & x_3' & x_2'\\ &&&&&& 0 & b^* & x_1'\\ &&&&&& 0 & 0 & a^* \end{pmatrix} \tag{3.5}$$

where the block sizes are determined by $a,a^*\in\mathrm{GL}_\alpha$, $b,b^*\in\mathrm{GL}_{\ell-\alpha-j+\beta}$, $c,c^*\in\mathrm{GL}_{j-\beta}$, $d,d^*\in\mathrm{GL}_{\beta-\alpha}$, and $e\in\mathrm{GL}_{m-2(\ell+\beta-\alpha)}$. In the case $i=0$, GL_i disappears.

The stabilizer in P_j consists of elements of the following form, which are the indicated matrices conjugated by w_q^t where $t=j-\beta$:

$$g_j^{(\alpha,\beta)} = \epsilon_{\alpha,\beta} g \epsilon_{\alpha,\beta}^{-1} = \begin{pmatrix} a & y_1 & z_1 & x_1 & y_2 & z_2 & x_2 & y_3 & z_3 \\ 0 & d & y_6' & 0 & u & y_5' & 0 & v & y_3' \\ 0 & 0 & c^* & 0 & 0 & x_3' & 0 & 0 & x_2' \\ & & & b & y_4 & z_4 & x_3 & y_5 & z_2' \\ & & & 0 & e & y_4' & 0 & u' & y_2' \\ & & & 0 & 0 & b^* & 0 & 0 & x_1' \\ & & & & & & c & y_6 & z_1' \\ & & & & & & 0 & d^* & y_1' \\ & & & & & & 0 & 0 & a^* \end{pmatrix}^{w_q^t} \tag{3.6}$$

with the block sizes as before and w_q^t being the t-th power of the element w_q. Also, when (V, q_V) is Hermitian, $w_q = I_m$; when $E = F$ and (V, q_V) is of odd dimension, $w_q = -I_m$; when $E = F$ and the anisotropic kernel (V_0, q_{V_0}) is of dimension two, take $w_q = \mathrm{diag}(I_{\tilde{m}}, w_q^0, I_{\tilde{m}})$, where $w_q^0 = \mathrm{diag}\{1, -1\}$; and finally, when $E = F$ and the anisotropic kernel (V_0, q_{V_0}) is a zero space, take $w_q = \mathrm{diag}(I_{\tilde{m}-1}, w_q^0, I_{\tilde{m}-1})$, where $w_q^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Note that $\ell, j < n = [\frac{m}{2}]$, where $m = \dim_E V$ and $\tilde{m}$ is the Witt index of V.

In **Case (2-2)**, i.e. G_n is the F-split even special orthogonal group and $\ell+\beta-\alpha = \tilde{m}$, we have two double cosets corresponding to each pair (α, β). For the double coset $P_j \epsilon_{\alpha,\beta} P_\ell$, we get exactly the same form for the stabilizer as above. For the other double coset $P_j \tilde{\epsilon}_{\alpha,\beta} P_\ell$, the stabilizer in P_ℓ consists of all elements of the form $(g_\ell^{(\alpha,\beta)})^{w_q}$.

To continue the calculation, we consider further double cosets decomposition $P_\ell^{\epsilon_{\alpha,\beta}} \backslash P_\ell / R_{\ell,w_0}$. Recall that $H_{n-\ell} = \mathrm{U}(q_{W_\ell \cap w_0^\perp})$, $\dim_E V$ and $\dim_E W_\ell \cap w_0^\perp$ have different parity, and $R_{\ell,w_0} = H_{n-\ell} N_\ell$ with $H_{n-\ell} \cong L_{\ell,w_0}$. By [GRS11, Section 5.1], we may choose a set of representatives of form:

$$\eta_{\epsilon,\gamma} := \begin{pmatrix} \epsilon & & \\ & \gamma & \\ & & \epsilon^* \end{pmatrix} \tag{3.7}$$

where ϵ is a representative in the quotient of Weyl groups

$$W_{\mathrm{GL}_\alpha \times \mathrm{GL}_{\ell-\alpha-t} \times \mathrm{GL}_t} \backslash W_{\mathrm{GL}_\ell}$$

and γ is a representative $P_w' \backslash G_{n-\ell} / H_{n-\ell}$, where

$$P_w' := G_{n-\ell} \cap \epsilon_{\alpha,\beta}^{-1} P_j \epsilon_{\alpha,\beta}.$$

We are going to show that P_w' is the maximal parabolic subgroup of $G_{n-\ell}$ as follows.

In **Case (1)** or **Case (2-1)**, i.e. when G_n is not the F-split even special orthogonal group or when G_n is the F-split even special orthogonal group with $\ell+\beta-\alpha < n$, then P'_w is the parabolic subgroup of $G_{n-\ell}$, which preserves the standard $\beta-\alpha$ dimensional totally isotropic subspace $V^+_{\ell,\beta-\alpha}$ of W_ℓ, where

$$V^{\pm}_{\ell,f} = \mathrm{Span}_E \left\{ e_{\pm(\ell+1)}, \ldots, e_{\pm(\ell+f)} \right\}, \tag{3.8}$$

for $1 \leq f \leq m-\ell$.

In **Case (2-2)**, i.e. when G_n is the F-split even special orthogonal group with $\ell+\beta-\alpha = n$ (with $j, \ell < n$), then, when $w = \epsilon_{\alpha,\beta}$, P'_w is the parabolic subgroup of $G_{n-\ell}$, which preserves $V^+_{\ell,m-\ell}$; and when $w = \tilde{\epsilon}_{\alpha,\beta}$, P'_w is the parabolic subgroup of $G_{n-\ell}$, which preserves $w_q V^+_{\ell,m-\ell}$.

Denote the stabilizer in $H_{n-\ell}$ of the double coset $P'_w\gamma H_{n-\ell}$ with $\eta_{\epsilon,\gamma}$ as defined in (3.7) by

$$H^{\eta_{\epsilon,\gamma}}_{n-\ell} = H^{\gamma}_{n-\ell} = H_{n-\ell} \cap \gamma^{-1} P'_w \gamma = L_{\ell,w_0} \cap \gamma^{-1} P'_w \gamma. \tag{3.9}$$

With the above preparation, we are ready to calculate the Bessel-Fourier coefficient $\mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma},\lambda))(h)$ by assuming that $\Re(s)$ is large so that we are able to unfold the Eisenstein series.

$$\begin{aligned}
&\mathcal{B}^{\psi_{\ell,w_0}}(E(\phi,s))(h) \\
&= \int_{N_\ell(F)\backslash N_\ell(\mathbb{A})} E(\phi,s)(nh)\psi^{-1}_{\ell,w_0}(n)\,\mathrm{d}n \\
&= \sum_{\epsilon_{\alpha,\beta}\in\mathcal{E}_{j,\ell}} \int_{N_\ell(F)\backslash N_\ell(\mathbb{A})} \sum_{\delta\in P^{\epsilon_{\alpha,\beta}}_\ell(F)\backslash P_\ell(F)} \lambda\phi(\epsilon_{\alpha,\beta}\delta nh)\psi^{-1}_{\ell,w_0}(n)\,\mathrm{d}n,
\end{aligned}$$

where $\mathcal{E}_{j,\ell}$ is the set of representatives of $P_j(F)\backslash G_n(F)/P_\ell(F)$. Set $\mathcal{N}_{\alpha,\beta,\ell,w_0}$ to be the set of representatives of $P^{\epsilon_{\alpha,\beta}}_\ell(F)\backslash P_\ell(F)/R_{\ell,w_0}(F)$ and deduce that the above is equal to

$$\sum_{\epsilon_{\alpha,\beta}} \sum_{\eta\in\mathcal{N}_{\alpha,\beta,\ell,w_0}} \int_{N_\ell(F)\backslash N_\ell(\mathbb{A})} \sum_{\delta\in R^{\eta}_{\ell,w_0}(F)\backslash R_{\ell,w_0}(F)} \lambda\phi(\epsilon_{\alpha,\beta}\eta\delta nh)\psi^{-1}_{\ell,w_0}(n)\,\mathrm{d}n,$$

where $R^{\eta}_{\ell,w_0} = R_{\ell,w_0} \cap \eta^{-1} P^{\epsilon_{\alpha,\beta}}_\ell \eta$. Since $R_{\ell,w_0} = H_{n-\ell}N_\ell$, by re-arranging the summation in δ and the integration of $\mathrm{d}n$, we obtain that the above is equal to

$$\sum_{\epsilon_{\alpha,\beta}} \sum_{\eta} \sum_{\delta\in H^{\eta}_{n-\ell}(F)\backslash H_{n-\ell}(F)} \int_{N^{\eta}_\ell(F)\backslash N_\ell(\mathbb{A})} \lambda\phi(\epsilon_{\alpha,\beta}\eta\delta nh)\psi^{-1}_{\ell,w_0}(n)\,\mathrm{d}n,$$

where $N^{\eta}_\ell = N_\ell \cap \eta^{-1} P^{\epsilon_{\alpha,\beta}}_\ell \eta$. By factoring the integration of $\mathrm{d}n$, we obtain that the Bessel-Fourier coefficient $\mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma},s))(h)$, when $\Re(s)$ is large, is equal to

$$\sum_{\epsilon_{\alpha,\beta};\eta;\delta} \int_{N^{\eta}_\ell(\mathbb{A})\backslash N_\ell(\mathbb{A})} \int_{N^{\eta}_\ell(F)\backslash N^{\eta}_\ell(\mathbb{A})} \lambda\phi(\epsilon_{\alpha,\beta}\eta\delta unh)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u\,\mathrm{d}n. \tag{3.10}$$

In order to determine the vanishing of the summands in (3.10), we need the following two lemmas, which are the global version of Propositions 5.1 and 5.2 in [GRS11, Chapter 5].

LEMMA 3.1. *If $\alpha > 0$, then the inner integral in the summands of (3.10) has the following property:*

$$\int_{N_\ell^\eta(F)\backslash N_\ell^\eta(\mathbb{A})} \lambda\phi(\epsilon_{\alpha,\beta}\eta unh)\psi_{\ell,w_0}^{-1}(un)\,\mathrm{d}u = 0$$

for all choices of data.

Proof. Fix α, β and fix an $\epsilon \in W_{\mathrm{GL}_\alpha\times\mathrm{GL}_{\ell-\alpha-t}\times\mathrm{GL}_t}\backslash W_{\mathrm{GL}_\ell}$. If there exists a simple root subgroup U of Z_ℓ such that $\epsilon U\epsilon^{-1}$ lies inside $U_{\alpha,\ell-\alpha-t,t}$, then the subgroup $\epsilon_{\alpha,\beta}\eta_{\epsilon,\gamma}U(\epsilon_{\alpha,\beta}\eta_{\epsilon,\gamma})^{-1}$ lies inside U_j for every γ. Since the automorphic function $\lambda\phi$ is invariant on $U_j(\mathbb{A})$ and ψ_{ℓ,w_0} is not trivial on $U(\mathbb{A})$,

$$\int_{U(F)\backslash U(\mathbb{A})} \lambda\phi(\epsilon_{\alpha,\beta}\eta zunh)\psi_{\ell,w_0}^{-1}(z)\,\mathrm{d}z = \lambda\phi(\epsilon_{\alpha,\beta}\eta unh)\cdot\int_{E\backslash\mathbb{A}_E}\psi^{-1}(x)\,\mathrm{d}x$$

is identically zero.

If for each simple root subgroup U of GL_ℓ, $\epsilon U\epsilon^{-1}$ does not lie inside $U_{\alpha,\ell-\alpha-t,t}$, then according to [GRS11, Lemma 5.1], ϵ is uniquely determined modulo $W_{M_{\alpha,\ell-\alpha-t,t}}$, and we can take

$$\epsilon = \begin{pmatrix} & & I_\alpha \\ & I_{\ell-\alpha-t} & \\ I_t & & \end{pmatrix}. \tag{3.11}$$

Since $\alpha \neq 0$ (and $\ell < \tilde{m}$), we choose a nontrivial subgroup S of N_ℓ consisting of elements of form

$$\begin{pmatrix} I_{\ell-\alpha} & & & & \\ & I_\alpha & y & * & \\ & & I_{m-2\ell} & y' & \\ & & & I_\alpha & \\ & & & & I_{\ell-\alpha} \end{pmatrix}$$

where $y = (0_{r\times(\beta-\alpha)}\ y_2\ y_3)(w_q^{t'}\gamma)^{-1}$, and y_2 and y_3 are of size $\alpha\times(m-2(\ell+\beta-\alpha))$ and $\alpha\times(\beta-\alpha)$, respectively; and when G_n is split, even orthogonal, $\ell+\beta-\alpha = n$ and the representative w in the double coset of $P_j\backslash G_n/P_\ell$ is $\epsilon_{\alpha,\beta}^{w_q}$, we have that $t' = 1$, otherwise, we always have that $t' = 0$. Since w_0 is anisotropic, w_0 is not orthogonal to $V_0 \oplus V_{\ell,\beta-\alpha}^-$ and ψ_{ℓ,w_0} is not trivial on $S(\mathbb{A})$. By (3.6), we have $(\epsilon_{\alpha,\beta}\eta_{\epsilon,\gamma})S(\epsilon_{\alpha,\beta}\eta_{\epsilon,\gamma})^{-1}$ lies inside U_j and then

$$\int_{S(F)\backslash S(\mathbb{A})} \lambda\phi(\epsilon_{\alpha,\beta}\eta xunh)\psi^{-1}_{\ell,w_0}(x)\,\mathrm{d}x = \lambda\phi(\epsilon_{\alpha,\beta}\eta unh)\cdot \int_{S(F)\backslash S(\mathbb{A})} \psi^{-1}_{\ell,w_0}(x)\,\mathrm{d}x$$

is identically zero. This proves the lemma. □

By Lemma 3.1 and (3.10), when $\Re(s)$ is large, the Bessel-Fourier coefficient $\mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma},\lambda))(h)$ is equal to

$$\sum_{\epsilon_{0,\beta};\eta;\delta}\int_{N^\eta_\ell(\mathbb{A})\backslash N_\ell(\mathbb{A})}\int_{N^\eta_\ell(F)\backslash N^\eta_\ell(\mathbb{A})} \lambda\phi(\epsilon_{0,\beta}\eta\delta unh)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u\,\mathrm{d}n. \tag{3.12}$$

In particular, we may choose the ϵ in (3.11), which is part of the representation $\eta_{\epsilon,\gamma}$ in (3.7), to be of the form: $\epsilon = \begin{pmatrix} & I_{\ell-t}\\ I_t & \end{pmatrix}$. Note that ϵ is one of the representatives of $W_{\mathrm{GL}_{\ell-t}\times\mathrm{GL}_t}\backslash W_{\mathrm{GL}_\ell}$. The following lemma will help us to eliminate more terms in (3.12).

Lemma 3.2. *If $\beta > \max\{j-\ell,0\}$ and γw_0 is not orthogonal to $V^-_{\ell,\beta}$ for $\gamma \in P'_w\backslash G_{n-\ell}/H_{n-\ell}$, then the inner integral in the summands of (3.12) has the property:*

$$\int_{N^\eta_\ell(F)\backslash N^\eta_\ell(\mathbb{A})} \lambda\phi(\epsilon_{0,\beta}\eta_{\epsilon,\gamma}unh)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u = 0$$

for all choices of data.

Proof. Consider the subgroup S_w (depending on $w = \epsilon_{0,\beta}\eta_{\epsilon,\gamma}$) of N_ℓ consisting of elements of form

$$\begin{pmatrix} I_t &&&& \\ & I_{\ell-t} & y & * & \\ && I_{m-2\ell} & y' & \\ &&& I_{\ell-t} & \\ &&&& I_\beta \end{pmatrix},$$

where $y = (0_{(\ell-t)\times(m-2\ell-\beta)}\ y_5)(w^{t'}_q\gamma)^{-1}$ with t' as defined before, and y_5 is of size $(\ell-t)\times\beta$. By $\ell-t = \ell-j+\beta > 0$ and $\beta > 0$, y_5 is not trivial. Since γw_0 is not orthogonal to $V^-_{\ell,\beta}$, ψ_{ℓ,w_0} is not trivial on $S_w(\mathbb{A}_F)$. By (3.6), ϕ is invariant on $(\epsilon_{0,\beta}\eta_{\epsilon,\gamma})S_w(\mathbb{A})(\epsilon_{0,\beta}\eta_{\epsilon,\gamma})^{-1}$. It follows that the integral

$$\int_{S_w(F)\backslash S_w(\mathbb{A})} \lambda\phi(\epsilon_{0,\beta}\eta_{\epsilon,\gamma}xunh)\psi^{-1}_{\ell,w_0}(x)\,\mathrm{d}x = \lambda\phi(\epsilon_{0,\beta}\eta_{\epsilon,\gamma}unh)\cdot\int_{S_w(F)\backslash S_w(\mathbb{A})} \psi^{-1}_{\ell,w_0}(x)\,\mathrm{d}x$$

is identically zero. Since the previous integral factors through this one, this finishes the proof. □

To summarize the above calculation, we recall that $\mathcal{E}_{j,\ell}$ is the set of representatives of all double cosets in $P_j(F)\backslash G_n(F)/P_\ell(F)$ and $\mathcal{N}_{\beta,\ell,w_0}$ is the set of representatives of $P_\ell^{\epsilon_\beta}(F)\backslash P_\ell(F)/R_{\ell,w_0}(F)$ as defined before.

Proposition 3.3. *For $\Re(s)$ large, the Bessel-Fourier coefficient of the Eisenstein series as in (3.2), $\mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma},\lambda))(h)$, is equal to*

$$\sum_{\epsilon_\beta}\sum_{\eta}\sum_{\delta}\int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})}\int_{N_\ell^\eta(F)\backslash N_\ell^\eta(\mathbb{A})}\lambda\phi(\epsilon_\beta\eta\delta unh)\psi_{\ell,w_0}^{-1}(un)\,\mathrm{d}u\,\mathrm{d}n,$$

where

- *$\epsilon_\beta=\epsilon_{0,\beta}\in\mathcal{E}_{j,\ell}^0$, which is the subset of $\mathcal{E}_{j,\ell}$ consisting of elements with $\alpha=0$;*
- *$\eta=\mathrm{diag}(\epsilon,\gamma,\epsilon^*)$ belongs to $\mathcal{N}_{\beta,\ell,w_0}^0$, which is the subset of $\mathcal{N}_{\beta,\ell,w_0}$ consisting of elements with $\alpha=0$, $\epsilon=\begin{pmatrix} & I_{\ell-t}\\ I_t & \end{pmatrix}$, and $t=j-\beta$, and has the property that if $\beta>\max\{j-\ell,0\}$, then γw_0 is orthogonal to $V_{\ell,\beta}^-$ for $\gamma\in P_w'(F)\backslash G_{n-\ell}(F)/H_{n-\ell}(F)$;*
- *δ belongs to $H_{n-\ell}^\eta(F)\backslash H_{n-\ell}(F)$.*

We will apply the formula in Proposition 3.3 to the calculation of the global zeta integral $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ and use the cuspidality of φ_π to prove that the global zeta integral $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ is eulerian.

3.2 Global zeta integrals. By applying Proposition 3.3 to the global zeta integral in (3.1), we get

$$\begin{aligned}&\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})\\&=\int_{H_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})}\mathcal{B}^{\psi_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma},s))(h)\varphi_\pi(h)\,\mathrm{d}h\\&=\sum_{\epsilon_\beta;\eta;\delta}\int_{[H_{n-\ell}]}\varphi_\pi(h)\int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})}\int_{[N_\ell^\eta]}\lambda\phi(\epsilon_\beta\eta\delta unh)\psi_{\ell,w_0}^{-1}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h\end{aligned}\tag{3.13}$$

where $[H_{n-\ell}]:=H_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})$ and $[N_\ell^\eta]:=N_\ell^\eta(F)\backslash N_\ell^\eta(\mathbb{A})$; and the summations $\sum_{\epsilon_\beta;\eta;\delta}$ and other conditions for the representatives are given in Proposition 3.3.

We combine the summation on δ and the integration $\mathrm{d}h$ and obtain that $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ is equal to

$$\sum_{\epsilon_\beta;\eta}\int_{H_{n-\ell}^\eta(F)\backslash H_{n-\ell}(\mathbb{A})}\varphi_\pi(h)\int_n\int_{[N_\ell^\eta]}\lambda\phi(\epsilon_\beta\eta unh)\psi_{\ell,w_0}^{-1}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h,\tag{3.14}$$

where the integration $\int_n$ is over $N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})$. The following lemma is to make use of the cuspidality of φ_π.

LEMMA 3.4. *Let $\alpha = 0$ and γ be a representative in $P'_w\backslash G_{n-\ell}/H_{n-\ell}$. For a representative $\eta = \eta_{\epsilon,\gamma}$, if the stabilizer $H^{\eta}_{n-\ell}$ is a proper maximal parabolic subgroup of $H_{n-\ell}$, then the corresponding summand in (3.14) has the property:*

$$\int_{H^{\eta}_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})} \varphi_\pi(h) \int_n \int_{[N^{\eta}_{\ell}]} \lambda\phi(\epsilon_\beta \eta_{\epsilon,\gamma} unh)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h = 0$$

for all choices of data.

Proof. Let $H^{\eta}_{n-\ell} = M'U'$, where U' is the unipotent radical of the parabolic subgroup $H^{\eta}_{n-\ell}$ of $H_{n-\ell}$. Since ϕ is $P_j(F)$-invariant, ϕ is left-invariant with respect to the image under the adjoint action by $\epsilon_{0,\beta}\eta_{\epsilon,\gamma}$ of the unipotent radical $U'(\mathbb{A})$ of $H^{\eta}_{n-\ell}(\mathbb{A})$. Then we deduce that

$$\begin{aligned}
&\int_{H^{\eta}_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})} \varphi_\pi(h) \int_n \int_{[N^{\eta}_{\ell}]} \lambda\phi(\epsilon_\beta \eta_{\epsilon,\gamma} unh)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h \\
&= \int_h \int_{[U']} \varphi_\pi(u'h)\,\mathrm{d}u' \int_n \int_{[N^{\eta}_{\ell}]} \lambda\phi(\epsilon_\beta \eta_{\epsilon,\gamma} unh)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h
\end{aligned}$$

where $\int_h$ is over $M'(F)U'(\mathbb{A})\backslash H_{n-\ell}(\mathbb{A})$. By the cuspidality of π, we have that

$$\int_{U'(F)\backslash U'(\mathbb{A})} \varphi_\pi(u'h)\,\mathrm{d}u' = 0,$$

and hence the whole integral is zero. This proves the lemma. $\square$

To proceed with our calculation from (3.14) with Lemma 3.4, we discuss more explicitly each double coset.

By Proposition 3.3, the representatives ϵ_β have the restrictions that either $\beta = \max\{0, j-\ell\}$ or $\beta > \max\{0, j-\ell\}$ with γw_0 being orthogonal to $V^{-}_{\ell,\beta}$ for $\gamma \in P'_w(F)\backslash G_{n-\ell}(F)/H_{n-\ell}(F)$. In order to understand the double cosets decomposition $\gamma \in P'_w(F)\backslash G_{n-\ell}(F)/H_{n-\ell}(F)$, we recall the following descriptions.

LEMMA 3.5. (Proposition 4.4, [GRS11]) *Let X be a non-trivial totally isotropic subspace of W_ℓ and P be the maximal parabolic subgroup of $G_{n-\ell}$ preserving X. Then*

(1) *If $\dim_E X < \mathrm{Witt}(W_\ell)$, then the set $P\backslash G_{n-\ell}/H_{n-\ell}$ consists of two elements.*
(2) *Assume that $\mathrm{Witt}(w_0^{\perp}) = \dim_E X = \mathrm{Witt}(W_\ell)$.*
   (a) *If $G_{n-\ell}$ is unitary, then $P\backslash G_{n-\ell}/H_{n-\ell}$ consists of two elements.*
   (b) *If $G_{n-\ell}$ is orthogonal and $\dim W_\ell \geq 2\dim X + 2$, then $P\backslash G_{n-\ell}/H_{n-\ell}$ consists of two elements.*

(c) *If* $G_{n-\ell}$ *is orthogonal and* $\dim W_\ell = 2\dim X+1$, *then* $P\backslash G_{n-\ell}/H_{n-\ell}$ *consists of three elements.*

(3) *If* $\dim_E X = \mathrm{Witt}(W_\ell)$ *and* $\mathrm{Witt}(w_0^\perp) = \dim_E X-1$, *then* $P\backslash G_{n-\ell}/H_{n-\ell}$ *consists of one element.*

(4) *If* $\dim_E W_\ell = 2\dim_E X$, *then* $\mathrm{Witt}(w_0^\perp) = \dim X - 1$, *and, in particular,* $P\backslash G_{n-\ell}/H_{n-\ell}$ *consists of one element.*

We consider the case when $G_{n-\ell}$ is not the F-split even orthogonal group or the case when $G_{n-\ell}$ is the F-split even orthogonal group with $\ell+\beta<n$. In these cases, we must have that $\dim X = \beta$.

If $\ell+\beta<\tilde{m}$, then $P'_w\backslash G_{n-\ell}/H_{n-\ell}$ consists of two elements. It remains to consider that $\ell+\beta=\tilde{m}$. If $\ell+\beta<n$, we must have that $\ell+\beta=\tilde{m}<n$ and hence $G_{n-\ell}$ can not be the F-split even special orthogonal group.

In this case $\ell+\beta=\tilde{m}<n$, if $G_{n-\ell}$ is an F-quasisplit even unitary group, then $\mathrm{Witt}(W_\ell\cap y_\kappa^\perp) = \mathrm{Witt}(W_\ell)-1$ and $P'_w\backslash G_{n-\ell}/H_{n-\ell}$ has only one element; if $G_{n-\ell}$ is an odd special orthogonal group, then

$$^{\#}P'_w\backslash G_{n-\ell}/H_{n-\ell} = \begin{cases} 3, & \text{if } \mathrm{Witt}(w_0^\perp\cap W_\ell) = \tilde{m}-\ell, \\ 1, & \text{if } \mathrm{Witt}(w_0^\perp\cap W_\ell) = \tilde{m}-\ell-1; \end{cases}$$

and if $G_{n-\ell}$ is an F-quasisplit even special orthogonal group (with $\dim V_0=2$) or an F-quasisplit odd unitary group, then

$$^{\#}P'_w\backslash G_{n-\ell}/H_{n-\ell} = \begin{cases} 2, & \text{if } \mathrm{Witt}(w_0^\perp\cap W_\ell) = \tilde{m}-\ell, \\ 1, & \text{if } \mathrm{Witt}(w_0^\perp\cap W_\ell) = \tilde{m}-\ell-1. \end{cases}$$

It remains to consider the case when $G_{n-\ell}$ is an F-split even special orthogonal group with $\ell+\beta=n$. In this case, $P'_w\backslash G_{n-\ell}/H_{n-\ell}$ consists of two elements.

Now we continue the calculation from Equation (3.14) and write

$$\mathcal{Z}_{\beta,\eta} = \int_{H^\eta_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})} \varphi(h) \int_n \int_{[N^\eta_\ell]} \lambda\phi(\epsilon_{0,\beta}\eta unh)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h$$

for each summand in (3.14). Then, we apply Lemmas 3.4 and 3.5 to find the nonvanishing summand in the summation in (3.14).

For $\max\{0,j-\ell\}\le\beta<\tilde{m}-\ell$, $P'_w\backslash G_{n-\ell}/H_{n-\ell}$ consists of two elements γ_1 and γ_2 such that $\gamma_1 w_0$ is orthogonal to $V^-_{\ell,\beta}$ and $\gamma_2 w_0$ is not orthogonal to $V^-_{\ell,\beta}$. If γw_0 is orthogonal to $V^-_{\ell,\beta}$, the stabilizer $H^\gamma_{n-\ell} = H^\eta_{n-\ell}$ is a maximal parabolic subgroup of $H_{n-\ell}$, which preserves the isotropic subspace $w_q^t V^+_{\ell,\beta}\cap w_0^\perp$.

In this case, by Lemmas 3.2 and 3.4, there may be left with nonzero summands in the summation (3.14), which are with the representative ϵ_β for $\beta=\max\{0,j-\ell\}$ and with the representative $\eta=\eta_{\epsilon,\gamma}$ having the property that γw_0 is not orthogonal to $V^-_{\ell,\beta}$.

For $\beta = \tilde{m} - \ell$, there are six different cases. Also, we have that $\beta = \tilde{m} - \ell > \max\{0, j - \ell\}$.

If G_n is the F-split even special orthogonal group, then there are two (P_j, P_ℓ)-double cosets corresponding to the pair $(0, \beta)$ and the chosen representatives are $\epsilon_{0,\beta}$ and $\tilde{\epsilon}_{0,\beta}$. For these two cases, their stabilizer preserves two maximal isotropic subspace of W_ℓ with different orientations, and $P'_w \backslash G_{n-\ell}/H_{n-\ell}$ consists of one element in both cases with its stabilizer $H^\gamma_{n-\ell} = H^\eta_{n-\ell}$ being a maximal parabolic subgroup. Hence by Lemma 3.4, the corresponding summands are all zero.

If G_n is not the F-split even special orthogonal group and

$$\mathrm{Witt}(W_\ell \cap y_\kappa^\perp) = \mathrm{Witt}(W_\ell) - 1,$$

there is only one double coset whose stabilizer is a maximal parabolic subgroup of $H_{n-\ell}$. Hence by Lemma 3.4, the corresponding summand is zero.

If $\mathrm{Witt}(W_\ell \cap y_\kappa^\perp) = \mathrm{Witt}(W_\ell)$ and G_n is the odd unitary group or F-quasi-split even special orthogonal group, the stabilizers are similar to the case $\beta < \tilde{m} - \ell$ as discussed above. Hence by Lemmas 3.2 and 3.4, the corresponding summands are all zero.

If G_n is the odd special orthogonal group and

$$\mathrm{Witt}(W_\ell \cap y_\kappa^\perp) = \mathrm{Witt}(W_\ell) - 1,$$

then $P'_w \backslash G_{n-\ell}/H_{n-\ell}$ consists of three elements and the representatives are chosen in [GRS11, (4.33)]. Two stabilizers are maximal parabolic subgroups of $H_{n-\ell}$, and the third representative γ satisfies the property that γw_0 is not orthogonal to $V^-_{\ell,\beta}$. Hence by Lemmas 3.2 and 3.4, the corresponding summands are all zero.

By the discussions above, we deduce that the corresponding summands are all zero, because of Lemmas 3.2 and 3.4.

In conclusion, we are left with the case where $\beta = \max\{0, j - \ell\}$ and γ with the property that the corresponding stabilizer is not a proper maximal parabolic subgroup of $H^\gamma_{n-\ell}$, i.e. γw_0 is not orthogonal to $V^-_{\ell,\beta}$.

In this case, the representative $\eta = \eta_{\epsilon,\gamma}$ is uniquely determined by $\beta = \max\{0, j - \ell\}$. In fact, if $j \le \ell$, then $\beta = 0$. It follows that $\eta_1 = \eta_{\epsilon,\gamma}$ with $\gamma = I_{m-2\ell}$ and

$$\epsilon = \begin{pmatrix} & I_{\ell-j} \\ I_j & \end{pmatrix}; \tag{3.15}$$

and if $j > \ell$, then $\beta = j - \ell$. It implies that $\eta_2 = \eta_{\epsilon,\gamma}$ with $\epsilon = I_\ell$ and

$$\gamma = \begin{pmatrix} & I_{j-\ell} & & & \\ I_{\tilde{m}-j} & & & & \\ & & I_{V_0} & & \\ & & & & I_{\tilde{m}-j} \\ & & & I_{j-\ell} & \end{pmatrix}. \tag{3.16}$$

Therefore, we are left with only one summand in the summation in (3.14) with the above representative, accordingly.

Next we are going to write the only integral more explicitly (Proposition 3.6) and get ready to prove that it is eulerian in the next subsection.

If $j \le \ell$, then $\beta = 0$. In this case we have that $P'_w = G_{n-\ell}$ and $H^{\gamma}_{n-\ell} = H_{n-\ell}$ with ϵ and γ given above. Then the global zeta integral in (3.14) has the following expression:

$$\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) = \mathcal{Z}_{0,\eta_1} = \int_{[H_{n-\ell}]} \varphi_\pi(h) \int_{N^\eta_\ell(\mathbb{A})\backslash N_\ell(\mathbb{A})} \int_{[N^\eta_\ell]} \lambda\phi(\epsilon_{0,0}\eta_1 unh)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h. \tag{3.17}$$

where $[H_{n-\ell}] := H_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})$ and $[N^\eta_\ell] := N^\eta_\ell(F)\backslash N^\eta_\ell(\mathbb{A})$. Recall that $R_{\ell,w_0} = H_{n-\ell}N_\ell$ and $R^\eta_{\ell,w_0} = H^\eta_{n-\ell}N^\eta_\ell$. The stabilizers are, respectively, given by

$$R^\eta_{\ell,w_0} = \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ & b & y_4 & z_4 & 0 \\ & & e & y'_4 & 0 \\ & & & b^* & 0 \\ & & & & c^* \end{pmatrix} \tag{3.18}$$

where c, c^* is of size $j \times j$, b, b^* of size $(\ell - j) \times (\ell - j)$, and e of size $(m-2\ell)\times(m-2\ell)$; and

$$(\epsilon_{0,0}\eta_{\epsilon,\gamma})R^\eta_{\ell,w_0}(\epsilon_{0,0}\eta_{\epsilon,\gamma})^{-1} = \begin{pmatrix} c^* & 0 & 0 & 0 & 0 \\ & b & y_4 & z_4 & 0 \\ & & e & y'_4 & 0 \\ & & & b^* & 0 \\ & & & & c \end{pmatrix} \tag{3.19}$$

with $c \in Z_j$ and $b \in Z_{\ell-j}$. (Here Z_f is the maximal upper-triangular unipotent subgroup of GL_f.)

If $j > \ell$, then $\beta = j - \ell$. In this case, $\epsilon = I_\ell$ and γ is given in (3.16). The double coset decomposition $P'_w\backslash G_{n-\ell}/H_{n-\ell}$ produces two representatives which, as given in [GRS11, Section 4.4], are $\gamma = I_{m-2\ell}$ and the γ as given in (3.16).

For the representative $\gamma = I_{m-2\ell}$, the corresponding stabilizer $H^\gamma_{n-\ell}$ is a proper maximal parabolic subgroup. Then, the corresponding integral in (3.14) is zero by Lemma 3.4.

Now for the γ as given in (3.16), we have that the global zeta integral is expressed as

$$\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) = \mathcal{Z}_{j-\ell,\eta_2} = \int_{H^\eta_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})} \varphi_\pi(h) \int_{N^\eta_\ell(\mathbb{A})\backslash N_\ell(\mathbb{A})} \int_{[N^\eta_\ell]} \lambda\phi(\epsilon_\beta\eta_2 unh)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h, \tag{3.20}$$

where $[N_\ell^\eta] = N_\ell^\eta(F)\backslash N_\ell^\eta(\mathbb{A})$. The stabilizers are given, respectively,

$$\eta_{\epsilon,\gamma} R^\eta_{\ell,w_0} \eta^{-1}_{\epsilon,\gamma} = \begin{pmatrix} c & 0 & 0 & y_6 & 0 \\ & d & u & v & y_6' \\ & & e & u' & 0 \\ & & & d^* & 0 \\ & & & & c^* \end{pmatrix} \tag{3.21}$$

where c, c^* is of size $\ell \times \ell$ and $c \in Z_\ell$, d, d^* of size $(j-\ell)\times(j-\ell)$, and e of size $(m-2j)\times(m-2j)$; and

$$(\epsilon_{0,\beta}\eta_{\epsilon,\gamma}) R^\eta_{\ell,w_0} (\epsilon_{0,\beta}\eta_{\epsilon,\gamma})^{-1} = \begin{pmatrix} d & y_6' & u & 0 & v \\ & c^* & 0 & 0 & 0 \\ & & e & 0 & u' \\ & & & c & y_6 \\ & & & & d^* \end{pmatrix}. \tag{3.22}$$

We conclude this subsection with the following proposition which summarizes the calculations discussed up to this point.

Proposition 3.6. *Take notation as above. If $j \le \ell$, then $\beta = 0$ and the global zeta integral has the following expression:*

$$\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) = \int_{[H_{n-\ell}]} \varphi_\pi(h) \int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \int_{[N_\ell^\eta]} \lambda\phi(\epsilon_{0,0}\eta u n h)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h,$$

where $[H_{n-\ell}] := H_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})$ and $[N_\ell^\eta] := N_\ell^\eta(F)\backslash N_\ell^\eta(\mathbb{A})$; and with $\eta = \eta_1$ given explicitly above. If $j > \ell$, then $\beta = j - \ell$ and the global zeta integral has the following expression:

$$\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) = \int_{H^\eta_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})} \varphi_\pi(h) \int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \int_{[N_\ell^\eta]} \lambda\phi(\epsilon_{0,\beta}\eta u n h)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h,$$

where $[N_\ell^\eta] = N_\ell^\eta(F)\backslash N_\ell^\eta(\mathbb{A})$; and with $\eta = \eta_2$ given explicitly above.

We are going to show that the global zeta integrals are eulerian based on Proposition 3.6. This is done for the two cases, separately.

3.3 Eulerian products: $0 < \ell < j$ case. We must have that $\beta = j - \ell$. By Proposition 3.6, the global integral $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ is equal to the following integral

$$\int_h \varphi_\pi(h) \int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \int_{[N_\ell^\eta]} \lambda\phi(\epsilon_{0,\beta}\eta unh)\psi^{-1}_{\ell,w_0}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h, \tag{3.23}$$

where $h \in H^\eta_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})$; $[N_\ell^\eta] = N_\ell^\eta(F)\backslash N_\ell^\eta(\mathbb{A})$; and $\eta = \eta_{\epsilon,\gamma}$ is as given explicitly above.

In order to show that the integral in (3.23) is an eulerian product of local zeta integrals, we first show that the integral in (3.23) can be expressed as an adelic integration of certain Bessel periods, which is stated in Proposition 3.7, and then we show the resulting integral in Proposition 3.7 factorizes as an eulerian product by means of the uniqueness of Bessel functionals, which is Theorem 3.8.

First, we want to understand the Fourier coefficient of $\lambda\phi$:

$$\int_{[N_\ell^\eta]} \lambda\phi(\epsilon_{0,\beta}\eta uh)\psi^{-1}_{\ell,w_0}(u)\,\mathrm{d}u. \tag{3.24}$$

We identify $g \in \mathrm{Res}_{E/F}(\mathrm{GL}_j)$ with its embedding $\hat{g}$ of (g, I_{m-2j}, g^*) into the Levi subgroup $\mathrm{Res}_{E/F}(\mathrm{GL}_j) \times G_{n-j}$ of G_n. Then, $(\epsilon_{0,\beta}\eta)N_\ell^\eta(\epsilon_{0,\beta}\eta)^{-1}$ is the group Z'_ℓ, consisting of elements z' of the form

$$z' = \begin{pmatrix} I_\beta & y \\ & z \end{pmatrix} \in \mathrm{Res}_{E/F}(\mathrm{GL}_j) \tag{3.25}$$

with $z \in Z_\ell$. By conjugating the element $\epsilon_{0,\beta}\eta$ across the variable u and changing the variable by

$$(\epsilon_{0,\beta}\eta)u(\epsilon_{0,\beta}\eta)^{-1} \mapsto \hat{z'},$$

the Fourier coefficient in (3.24) reduces to

$$\int_{[Z'_\ell]} \lambda\phi(\hat{z'}\epsilon_{0,\beta}\eta h)\psi^{-1}_{\ell,w_0}((\epsilon_{0,\beta}\eta)^{-1}\hat{z'}(\epsilon_{0,\beta}\eta))\,\mathrm{d}z'. \tag{3.26}$$

It follows from the choice of the representatives $\epsilon_{0,\beta}$ and η that the character has following expression:

$$\psi^{-1}_{\ell,w_0}((\epsilon_{0,\beta}\eta)^{-1}\hat{z'}(\epsilon_{0,\beta}\eta)) = \psi(z_{1,2} + \cdots + z_{\ell-1,\ell} + (-1)^{m+1}\frac{\kappa}{2}y_{\beta,1}), \tag{3.27}$$

where $z = (z_{e,f})_{\ell\times\ell}$. If we write elements z' of Z'_ℓ as $z' = (z'_{e,f})_{j\times j}$, then this character can be written as

$$\psi_{Z'_\ell,\kappa}(z') := \psi((-1)^{m+1}\frac{\kappa}{2}z_{\beta,\beta+1} + z_{\beta+1,\beta+2} + \cdots + z_{j-1,j}). \tag{3.28}$$

In this way, the Fourier coefficient in (3.26) can be written as

$$\phi_\lambda^{\psi_{Z'_\ell,\kappa}}(h) := \int_{[Z'_\ell]} \lambda\phi(\hat{z'}h)\psi_{Z'_\ell,\kappa}(z')\,\mathrm{d}z'. \tag{3.29}$$

Hence the global zeta integral $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$, which is expressed as in (3.23), is equal to the following integral

$$\int_{H^\eta_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})} \varphi_\pi(h) \int_{N^\eta_\ell(\mathbb{A})\backslash N_\ell(\mathbb{A})} \phi_\lambda^{\psi_{Z'_\ell,\kappa}}(\epsilon_{0,\beta}\eta n h)\psi^{-1}_{\ell,w_0}(n)\,\mathrm{d}n\,\mathrm{d}h, \tag{3.30}$$

with $\eta=\eta_{\epsilon,\gamma}$ given explicitly above.

Next, we want to understand the structure of the subgroup $H^\eta_{n-\ell}$. By (3.9), $H^\eta_{n-\ell}=H_{n-\ell}\cap\gamma^{-1}P'_w\gamma$ with $\eta=\eta_{\epsilon,\gamma}$, and $P'_w=G_{n-\ell}\cap P_\ell^{\epsilon_{0,\beta}}$ is the parabolic subgroup of $G_{n-\ell}$, preserving the totally isotropic subspace $V^+_{\ell,\beta}$ as in (3.8). Denote by

$$P'^{\eta}_w = P'_w\cap\eta H_{n-\ell}\eta^{-1} = P'_w\cap\gamma H_{n-\ell}\gamma^{-1}.$$

Then the elements of P'^{η}_w are of form:

$$\begin{pmatrix} I_\ell &&&&&& \\ & d & d_1 & u & v_1 & v & \\ && 1 & 0 & 0 & v'_1 & \\ &&& e & 0 & u' & \\ &&&& 1 & d'_1 & \\ &&&&& d^* & \\ &&&&&& I_\ell \end{pmatrix} \tag{3.31}$$

with $d_1+(-1)^{m+1}\frac{\kappa}{2}v_1=0$, where d_1 and v_1 are column vectors of size $\beta-1$; d,d^* are of size $(\beta-1)\times(\beta-1)$; and e belongs to G_{n-j}. Note that P'^{η}_w is the stabilizer of γy_κ in P'_w. Hence we have

$$P'^{\eta}_w = (\mathrm{GL}(V^+_{\ell,\beta-1})\times G_{n-j})\ltimes U^\eta(V^+_{\ell,\beta-1}), \tag{3.32}$$

where $U^\eta(V^+_{\ell,\beta-1})$ is the subgroup of $U(V^+_{\ell,\beta-1})$ consisting elements which fix the vector γy_κ. Here $U(V^+_{\ell,\beta-1})$ is the unipotent radical of the parabolic subgroup $P(V^+_{\ell,\beta-1})$ of $G_{n-\ell}$ preserving the totally isotropic subspace $V^+_{\ell,\beta-1}$.

In order to precede our calculation, we need to know the structure of stabilizer in $H_{n-\ell}$. Let $Q_{\beta-1,\eta}$ be the parabolic subgroup of $H_{n-\ell}$, which preserves the totally isotropic subspace $(\eta^{-1}V^+_{\ell,\beta})\cap y_\kappa^\perp$ of $W_\ell\cap y_\kappa^\perp$ and has the Levi decomposition

$$Q_{\beta-1,\eta}=L_{\beta-1,\eta}V_{\beta-1,\eta}.$$

Recall that the space $W_\ell\cap y_\kappa^\perp$ has the polar decomposition

$$W_\ell\cap y_\kappa^\perp = V^+_{\ell,\tilde{m}-\ell-1}\oplus W_0\oplus V^-_{\ell,\tilde{m}-\ell-1},$$

where W_0 is a non-degenerate subspace of $W_\ell \cap y_\kappa^\perp$ with the same anisotropic kernel as $W_\ell \cap y_\kappa^\perp$ and with $\dim_E W_0 = \dim_E V_0 + 1 \leq 3$. By taking as before that $w_0 = y_\kappa = e_{\tilde{m}} + (-1)^{m+1}\frac{\kappa}{2}e_{-\tilde{m}}$, we obtain that $W_0 = \mathrm{Span}\,\{y_{-\kappa}\} \oplus V_0$. Then it is easy to check that

$$(\eta^{-1}V^+_{\ell,\beta}) \cap y_\kappa^\perp = \mathrm{Span}\,\{e_{\tilde{m}-j+\ell+1}, \ldots, e_{\tilde{m}-1}\} = V^+_{\tilde{m}-\beta,\beta-1},$$

and

$$L_{\beta-1,\eta} = \mathrm{GL}(V^+_{\tilde{m}-\beta,\beta-1}) \times H_{n-j+1},$$

where $H_{n-j+1} := \mathrm{U}(q_{W_{j-1}\cap y_\kappa^\perp})$ with

$$W_{j-1} \cap y_\kappa^\perp = V^+_{\ell,\tilde{m}-j} \oplus W_0 \oplus V^-_{\ell,\tilde{m}-j}.$$

It follows that

$$\mathrm{GL}(V^+_{\tilde{m}-\beta,\beta-1}) = \mathrm{GL}((\eta^{-1}V^+_{\ell,\beta}) \cap y_\kappa^\perp) = \eta^{-1}\mathrm{GL}(V^+_{\ell,\beta-1})\eta \subset H^\eta_{n-\ell},$$

and

$$V_{\beta-1,\eta} = \eta^{-1}U^\eta(V^+_{\ell,\beta-1})\eta \subset H^\eta_{n-\ell}.$$

It is easy to check that

$$\eta^{-1}W_j = V^+_{\ell,\tilde{m}-j} \oplus V_0 \oplus V^-_{\ell,\tilde{m}-j} = y^\perp_{-\kappa} \cap (W_{j-1} \cap y_\kappa^\perp).$$

Hence we have

$$\mathrm{U}(q_{\eta^{-1}W_j}) = \eta^{-1}\mathrm{U}(q_{W_j})\eta = \eta^{-1}G_{n-j}\eta \subset H^\eta_{n-\ell}.$$

Putting together all these subgroups, we obtain the structure of $H^\eta_{n-\ell}$:

$$H^\eta_{n-\ell} = (\mathrm{GL}(V^+_{\tilde{m}-\beta,\beta-1}) \times \mathrm{U}(q_{\eta^{-1}W_j})) \rtimes V_{\beta-1,\eta}. \tag{3.33}$$

Finally, we are ready to consider the partial Fourier expansion of cuspidal automorphic forms φ_π on $H_{n-\ell}(\mathbb{A})$. Let $Z^\eta_{\ell,\beta-1}$ be the maximal unipotent subgroup of $\mathrm{GL}(V^+_{\tilde{m}-\beta,\beta-1})$ consisting of elements of following type:

$$\eta^{-1}\begin{pmatrix} I_\ell & & & & \\ & d & & & \\ & & I_{m-2j+2} & & \\ & & & d^* & \\ & & & & I_\ell \end{pmatrix}\eta$$

with $d \in Z_{\beta-1}$. Then $N^\eta_{\ell,\beta-1} := Z^\eta_{\ell,\beta-1}V_{\beta-1,\eta}$ is a unipotent subgroup of $H_{n-\ell}$ of the type as defined in (2.4) with the corresponding character defined as in (2.6) with $y_{-\kappa}$. Then, it is easy to check that the corresponding stabilizer $H^{y_{-\kappa}}_{n-j+1}$ is equal to $\mathrm{U}(q_{\eta^{-1}W_j})$, which is isomorphic to G_{n-j}.

Define $C_{\beta-1,\eta} := V_{\beta-1,\eta} \cap V_{\beta,\eta}$, which is also equal to

$$\{u \in V_{\beta-1,\eta} \mid u \cdot e_{\tilde{m}} = e_{\tilde{m}}\}$$

and is a normal subgroup of $H^{\eta}_{n-\ell}$. It follows that

$$C_{\beta-1,\eta} \backslash H^{\eta}_{n-\ell} \cong P^1_{\beta} \times H^{y_{-\kappa}}_{n-j+1},$$

where P^1_{β} is the mirabolic subgroup of $\mathrm{Res}_{E/F}(\mathrm{GL}_{\beta})$ given by

$$P^1_{\beta} = \left\{ \begin{pmatrix} d & d_1 \\ 0 & 1 \end{pmatrix} \in \mathrm{Res}_{E/F}(\mathrm{GL}_{\beta}) \right\}.$$

Going back to the expression (3.30) of $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_{\pi}, \psi_{\ell,w_0})$, the inner integral

$$\Phi(h) := \int_{N^{\eta}_{\ell}(\mathbb{A}) \backslash N_{\ell}(\mathbb{A})} \phi_{\lambda}^{\psi_{Z'_{\ell},\kappa}}(\epsilon_{0,\beta}\eta n h)\psi^{-1}_{\ell,w_0}(n)\,\mathrm{d}n \tag{3.34}$$

as function in h, is left $C_{\beta-1,\eta}(\mathbb{A})$-invariant. We recall that N_{ℓ} consists of elements of the form

$$\begin{pmatrix} c & x_1 & x_2 & x_3 & y_6 & x_4 & x_5 \\ & I_{\tilde{m}-j} & & & & & x'_4 \\ & & I_{j-\ell} & & & & y'_6 \\ & & & I_{m-2\tilde{m}} & & & x'_3 \\ & & & & I_{j-\ell} & & x'_2 \\ & & & & & I_{\tilde{m}-j} & x'_1 \\ & & & & & & c^* \end{pmatrix}$$

where $c \in Z_{\ell}$ and the stabilizer N^{η}_{ℓ} consists element of the form

$$\begin{pmatrix} c & 0 & 0 & 0 & y_6 & 0 & 0 \\ & I_{\tilde{m}-j} & & & & & 0 \\ & & I_{j-\ell} & & & & y'_6 \\ & & & I_{m-2\tilde{m}} & & & 0 \\ & & & & I_{j-\ell} & & 0 \\ & & & & & I_{\tilde{m}-j} & 0 \\ & & & & & & c^* \end{pmatrix}.$$

Then $\eta(N^{\eta}_{\ell} \backslash N_{\ell})\eta^{-1}$ is isomorphic to a complementary subgroup consisting of elements of the form

$$n_0(x_1, x_2, x_3) := \begin{pmatrix} I_{\ell} & x_1 & x_2 & 0 & x_3 \\ & I_{j-\ell} & & & 0 \\ & & I_{m-2j} & & x'_2 \\ & & & I_{j-\ell} & x'_1 \\ & & & & I_{\ell} \end{pmatrix},$$

and $\psi_{\ell,\kappa}(\eta^{-1}n_0(x_1,x_2,x_3)\eta)$ is not trivial on x_1. In detail,

$$\psi_{\ell,\kappa}\circ \mathrm{Int}_{\eta^{-1}}(n_0(x_1,x_2,x_3)) = \psi((x_1)_{\ell,j-\ell}).$$

The stabilizer $(\epsilon_{0,\beta}\eta)N_\ell^\eta(\epsilon_{0,\beta}\eta)^{-1}$ in P_j consists of elements of the form

$$\begin{pmatrix} I_{j-\ell} & y_6' & & & \\ & c^* & & & \\ & & e & & \\ & & & c & y_6 \\ & & & & I_{j-\ell} \end{pmatrix}.$$

The image of the domain of integration $N_\ell^\eta\backslash N_\ell$ under the adjoint action of $\epsilon_{0,\beta}\eta$ is a subgroup $U_{j,\eta}^-$ of U_j^- (the unipotent radical of the parabolic subgroup opposite P_j), consisting of elements of the form

$$\begin{pmatrix} I_{j-\ell} & & & & \\ & I_\ell & & & \\ & x_2' & I_{m-2j} & & \\ x_1 & x_3 & x_2 & I_\ell & \\ & x_1' & & & I_{j-\ell} \end{pmatrix}. \tag{3.35}$$

Denote by $\psi_{(m-j+\ell,j-\ell)}$ the character over $(\epsilon_{0,\beta}\eta)N_\ell^\eta\backslash N_\ell(\epsilon_{0,\beta}\eta)^{-1}$, given by $\psi_{(m-j+\ell,j-\ell)}(n) = \psi(n_{m-j+\ell,j-\ell})$ where $n_{m-j+\ell,j-\ell} = (x_1)_{\ell,j-\ell}$.

Recall that $\eta^{-1}C_{\beta-1,\eta}\eta$ consists of elements of the form

$$\begin{pmatrix} I_\ell & & & & & & \\ & I_{\beta-1} & 0 & u & 0 & 0 & \\ & & 1 & 0 & 0 & 0 & \\ & & & I_{m-2j} & 0 & u' & \\ & & & & 1 & 0 & \\ & & & & & I_{\beta-1} & \\ & & & & & & I_\ell \end{pmatrix}.$$

It follows that $N_{\ell,\beta-1}^\eta = Z_{\ell,\beta-1}^\eta V_{\beta-1,\eta} = Z_\beta C_{\beta-1,\eta}$. As a subgroup of P_j, the stabilizer $(\epsilon_{0,\beta}\eta)N_{\ell,\beta-1}^\eta(\epsilon_{0,\beta}\eta)^{-1}$ consists of elements of the form

$$\begin{pmatrix} d & d_1 & 0 & u & 0 & v_1 & v \\ & 1 & & & & & v_1' \\ & & I_\ell & & & & 0 \\ & & & I_{m-2j} & & & u' \\ & & & & I_\ell & & 0 \\ & & & & & 1 & d_1' \\ & & & & & & d^* \end{pmatrix},$$

where $d \in Z_{\beta-1}$. Note that $(\epsilon_{0,\beta}\eta)Z_\beta(\epsilon_{0,\beta}\eta)^{-1}$ consists of elements of the above form with all matrices being zero except d and d_1 and $(\epsilon_{0,\beta}\eta)C_{\beta-1,\eta}(\epsilon_{0,\beta}\eta)^{-1}$ is equal to the subgroup where $d = I_{\beta-1}$ and $d_1 = 0$.

It follows that the expression in (3.30) of the global zeta integral $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ is equal to

$$\int_{H^\eta_{n-\ell}(F)C_{\beta-1,\eta}(\mathbb{A})\backslash H_{n-\ell}(\mathbb{A})} \Phi(h) \int_{[C_{\beta-1,\eta}]} \varphi_\pi(ch)\, \mathrm{d}c\, \mathrm{d}h, \tag{3.36}$$

where $[C_{\beta-1,\eta}] := C_{\beta-1,\eta}(F)\backslash C_{\beta-1,\eta}(\mathbb{A})$, as before. Note that Φ is $C_{\beta-1,\eta}(\mathbb{A})$-invariant.

We denote the inner integration $\int_c$ by

$$\varphi_\pi^{C_{\beta-1,\eta}}(h) = \int_{[C_{\beta-1,\eta}]} \varphi_\pi(ch)\, \mathrm{d}c.$$

The integral in (3.36) becomes

$$\int_{H^\eta_{n-\ell}(F)C_{\beta-1,\eta}(\mathbb{A})\backslash H_{n-\ell}(\mathbb{A})} \Phi(h)\varphi_\pi^{C_{\beta-1,\eta}}(h)\, \mathrm{d}h. \tag{3.37}$$

Now we are in the standard step in the global unfolding process using partial Fourier expansion along the mirabolic subgroup P^1_β. Both functions $\Phi(h)$ and $\varphi^{C_{\beta-1,\eta}}(h)$ are automorphic on $P^1_\beta(\mathbb{A})$ and $\varphi_\pi^{C_{\beta-1,\eta}}(h)$ is cuspidal because of the cuspidality of $\varphi_\pi(h)$. Following the standard Fourier expansion of cuspidal automorphic forms on general linear group [Sha74] and [Pia79], see also [JL12], we have

$$\varphi_\pi^{C_{\beta-1,\eta}}(h) = \sum_d \mathcal{B}^{\psi^{-1}_{\beta-1,y_{-\kappa}}}(\varphi_\pi)\left(\eta^{-1}\begin{pmatrix} I_\ell & & \\ & d & \\ & & 1\end{pmatrix}^{\wedge} \eta h\right) \tag{3.38}$$

with $d \in Z_{\beta-1}(F)\backslash \mathrm{Res}_{E/F}\mathrm{GL}_{\beta-1}(F)$, which converges absolutely and uniformly in g varying in compact subsets. Note that the choice of the character $\psi_{\beta-1,y_{-\kappa}}$ is given by the previous integration. Recall that the Bessel-Fourier coefficient with respect to $\psi_{\beta-1,y_{-\kappa}}$ is defined as in (2.11) by

$$\mathcal{B}^{\psi^{-1}_{\beta-1,y_{-\kappa}}}(\varphi_\pi)(h) = \int_{N^\eta_{\ell,\beta-1}(F)\backslash N^\eta_{\ell,\beta-1}(\mathbb{A})} \varphi_\pi(nh)\psi_{\beta-1,y_{-\kappa}}(n)\, \mathrm{d}n.$$

By using (3.38), the expression (3.37) of $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ is equal to

$$\int_{Z_\beta(F)H^{y_{-\kappa}}_{n-j+1}(F)C_{\beta-1,\eta}(\mathbb{A})\backslash H_{n-\ell}(\mathbb{A})} \Phi(h)\mathcal{B}^{\psi^{-1}_{\beta-1,y_{-\kappa}}}(\varphi_\pi)(h)\, \mathrm{d}h. \tag{3.39}$$

By pulling out the integration on $Z_\beta(F)\backslash Z_\beta(\mathbb{A})$ and using the fact that $\mathcal{B}^{\psi^{-1}_{\beta-1,y_{-\kappa}}}(\varphi_\pi)(h)$ is left $(Z_\beta(\mathbb{A}),\psi^{-1}_{\beta-1,y_{-\kappa}})$-equivariant, the integral in (3.39) is equal to

$$\int_h \mathcal{B}^{\psi^{-1}_{\beta-1,y_{-\kappa}}}(\varphi_\pi)(h)\int_{[Z_\beta]}\Phi(zh)\psi^{-1}_{\beta-1,y_{-\kappa}}(z)\,\mathrm{d}z\,\mathrm{d}h \tag{3.40}$$

where $h\in H^{y_{-\kappa}}_{n-j+1}(F)N^{\eta}_{\ell,\beta-1}(\mathbb{A})\backslash H_{n-\ell}(\mathbb{A})$, $N^{\eta}_{\ell,\beta-1}=Z_\beta C_{\beta-1,\eta}$ defined as before and $[Z_\beta]:=Z_\beta(F)\backslash Z_\beta(\mathbb{A})$.

The inner integration

$$\int_{[Z_\beta]}\Phi(zh)\psi^{-1}_{\beta-1,y_{-\kappa}}(z)\,\mathrm{d}z$$

can be calculated more explicitly. By (3.34), it is equal to

$$\int_{[Z_\beta]}\int_{N^{\eta}_{\ell}(\mathbb{A})\backslash N_\ell(\mathbb{A})}\phi_\lambda^{\psi_{Z'_\ell,\kappa}}(\epsilon_{0,\beta}\eta nzh)\psi^{-1}_{\ell,w_0}(n)\,\mathrm{d}n\psi^{-1}_{\beta-1,y_{-\kappa}}(z)\,\mathrm{d}z. \tag{3.41}$$

The element $(\epsilon_{0,\beta}\eta)z(\epsilon_{0,\beta}\eta)^{-1}$ is given as above. Combining this subgroup with N^{η}_{ℓ}, one obtains a subgroup $(\epsilon_{0,\beta}\eta)N^{\eta}_{\ell}Z_\beta(\epsilon_{0,\beta}\eta)^{-1}$ of P_j consisting of elements of form

$$\begin{pmatrix} d & d_1 & (y_6)'_{*,*} & & & & \\ & 1 & (y_6)'_{\beta,*} & & & & \\ & & c^* & & & & \\ & & & I_{m-2j} & & & \\ & & & & c & (y_6)_{*,\beta} & (y_6)_{*,*} \\ & & & & & 1 & d_1' \\ & & & & & & d \end{pmatrix}, \tag{3.42}$$

where $(y_6)_{*,\beta}$ consists of the first column of y_6, and $(y_6)_{*,*}$ is the rest, so that $y_6=((y_6)_{*,\beta},(y_6)_{*,*})$. Define

$$\phi^{Z_j,\kappa}(h)=\int_{[Z_\beta]}\phi_\lambda^{\psi_{Z'_\ell,\kappa}}(\epsilon_{0,\beta}\eta zh)\psi^{-1}_{\beta-1,y_{-\kappa}}\,\mathrm{d}z.$$

Then,

$$\phi^{Z_j,\kappa}(h)=\int_{[Z_j]}\phi_\lambda(zh)\psi_{Z_j,\kappa}\,\mathrm{d}z,$$

where $\psi_{Z_j,\kappa}(z)$ is given by

$$\psi(-z_{1,2}-\cdots-z_{\beta-1,\beta}+(-1)^{m+1}\frac{\kappa}{2}z_{\beta,\beta+1}+z_{\beta+1,\beta+2}+\cdots+z_{j-1,j}) \tag{3.43}$$

with $\beta = j - \ell$. Hence,

$$\int_{[Z_\beta]} \Phi(zg)\,\mathrm{d}z = \int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \phi^{Z_j,\kappa}(\epsilon_{0,\beta}\eta ng)\psi_{\ell,\kappa}^{-1}(n)\,\mathrm{d}n$$
$$= \int_{U_{j,\eta}^-(\mathbb{A})} \phi^{Z_j,\kappa}(n\epsilon_{0,\beta}\eta g)\psi_{(m-j+\ell,j-\ell)}(n)\,\mathrm{d}n.$$

Denote the last integral by

$$\mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}})(g) = \int_{U_{j,\eta}^-(\mathbb{A})} \phi^{Z_j,\kappa}(ng)\psi_{(m-j+\ell,j-\ell)}(n)\,\mathrm{d}n.$$

Recall that the group $U_{j,\eta}^-$ consists of elements of form (3.35).

Therefore, we obtain, from (3.36) and (3.37), that the global zeta integral $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ equals

$$\int_{R_{\ell,\beta-1}^\eta(\mathbb{A})\backslash H_{n-\ell}(\mathbb{A})} \int_{[H_{n-\ell}^\eta]} \mathcal{B}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi)(xh)\mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}})(\epsilon_{0,\beta}\eta xh)\,\mathrm{d}x\,\mathrm{d}h,$$

where $[H_{n-\ell}^\eta] := H_{n-\ell}^\eta(F)\backslash H_{n-\ell}^\eta(\mathbb{A})$.

Proposition 3.7. (Case $(j>\ell)$) *Let $E(\phi_{\tau\otimes\sigma},s)$ be the Eisenstein series on $G_n(\mathbb{A})$ as in* (2.15) *and π be an irreducible cuspidal automorphic representation of $H_{n-\ell}(\mathbb{A})$. The global zeta integral $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ as in* (3.1) *is equal to*

$$\int_{R_{\ell,\beta-1}^\eta(\mathbb{A})\backslash H_{n-\ell}(\mathbb{A})} \int_{[H_{n-\ell}^\eta]} \mathcal{B}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi)(xh)\mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}})(\epsilon_{0,\beta}\eta xh)\,\mathrm{d}x\,\mathrm{d}h$$

with $[H_{n-\ell}^\eta] := H_{n-\ell}^\eta(F)\backslash H_{n-\ell}^\eta(\mathbb{A})$.

In order to show that the integral expression for the global zeta integral $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ as in Proposition 3.7 is eulerian, it is enough to show that the inner integral

$$\int_{[H_{n-\ell}^\eta]} \mathcal{B}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi)(xh)\mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}})(\epsilon_{0,\beta}\eta xh)\,\mathrm{d}x \tag{3.44}$$

is an eulerian product. In fact, for a fixed h, as a function of x, $\mathcal{J}_{\ell,\kappa}(R(\epsilon_{0,\beta}\eta h)\cdot\phi^{Z_{j,\kappa}})(\epsilon_{0,\beta}\eta x(\epsilon_{0,\beta}\eta)^{-1})$ belongs to the space of the automorphic representation $\sigma^{w_q^\ell}$ of $G_{n-j}(\mathbb{A})$, where R denote the right translation. Hence, for a fixed h, this above inner integral is the Bessel period for the pair $(\pi,\sigma^{w_q^\ell})$ as defined in (2.12). By the local uniqueness of the Bessel models [AGRS10], [SZ12], [JBZ11] and also [GGP12], integral (3.44) can be written as an eulerian product:

$$\prod_{\nu}\int_{h_\nu} < \mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}}_{\tau\otimes\sigma,\nu})(\epsilon_{0,\beta}\eta h_\nu), \mathcal{B}_\nu^{\psi^{-1}_{\beta-1,y_{-\kappa}}}(\varphi_{\pi,\nu})(h_\nu) >_{G_{n-j}} \mathrm{d}h_\nu. \tag{3.45}$$

Here the domain of the integration $\int_{h_\nu}$ is $R^\eta_{\ell,\beta-1}(F_\nu)\backslash H_{n-\ell}(F_\nu)$, the linear functional $\mathcal{B}_\nu^{\psi^{-1}_{\beta-1,y_{-\kappa}}}$ is an element in

$$\mathrm{Hom}_{H_{n-\ell}(F_\nu)}(\pi, \mathrm{Ind}^{H_{n-\ell}(F_\nu)}_{R^\eta_{\ell,\beta-1}(F_\nu)}(\psi_{\beta-1,y_{-\kappa}}\otimes\tilde{\sigma}^{w^\ell_q})),$$

and $\langle\cdot,\cdot\rangle_{G_{n-j}}$ is an invariant pairing of $\sigma^{w^\ell_q}$ and $\tilde{\sigma}^{w^\ell_q}$. The local uniqueness of the Bessel models [AGRS10], [SZ12], [JBZ11] and also [GGP12] asserts that the above Hom-space is at most one-dimensional. One can normalize the local pairing suitably at unramified local places with explicit normalization given in Section 4, so that the eulerian product makes sense. Hence we obtain the following theorem.

Theorem 3.8. *Let $E(\phi_{\tau\otimes\sigma}, s)$ be the Eisenstein series on $G_n(\mathbb{A})$ as in (2.15) and let π be an irreducible cuspidal automorphic representation of $H_{n-\ell}(\mathbb{A})$. Assume that the following hold:*

(1) *the real part of s, $\Re(s)$, is large;*
(2) *the automorphic form φ_π is factorizable, and $\phi_{\tau\otimes\sigma}$ and φ_σ are compatibly factorizable;*
(3) *π and σ have a non-zero Bessel period, i.e. $\mathcal{P}^{\psi^{-1}_{\beta-1,y_{-\kappa}}}(\varphi_\pi,\varphi_\sigma)$ is nonzero for a some choice of data; and*
(4) *the relevant local Bessel vectors are suitably normalized at all unramified local places, so that the eulerian product below makes sense (the detail of the normalization will be given in Sections. 4.3, 4.4, and 4.5).*

Then the global zeta integral $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ is eulerian. More precisely, it is equal to

$$\prod_{\nu}\int_{h_\nu} < \mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}}_{\tau\otimes\sigma,\nu})(\epsilon_{0,\beta}\eta h_\nu), \mathcal{B}_\nu^{\psi^{-1}_{\beta-1,y_{-\kappa}}}(\varphi_{\pi,\nu})(h_\nu) >_{G_{n-j}} \mathrm{d}h_\nu,$$

where the integration is taken over $R^\eta_{\ell,\beta-1}(F_\nu)\backslash H_{n-\ell}(F_\nu)$, and the product is taken over all local places.

The main local result of this paper is to calculate the unramified local integral explicitly in terms of the local L-functions. For the purpose of our investigation of the global tensor product L-functions $L(s,\pi\times\tau)$, it is enough to consider the case when $j=\ell+1$. We define the local zeta integral $\mathcal{Z}_\nu(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ to be the integral in the Euler product in Theorem 3.8, which is

$$\int_{h} < \mathcal{J}_{\ell,\kappa}(\phi^{Z_{j,\kappa}}_{\tau\otimes\sigma,\nu})(\epsilon_{0,\beta}\eta h_\nu), \mathcal{B}_\nu^{\psi^{-1}_{\beta-1,y_{-\kappa}}}(\varphi_{\pi,\nu})(h_\nu) >_{G_{n-j}} \mathrm{d}h_\nu, \tag{3.46}$$

where the integration is taken over $R^\eta_{\ell,\beta-1}(F_\nu)\backslash H_{n-\ell}(F_\nu)$.

Theorem 3.9 (L-function for case $j = \ell+1$). *With all data being unramified, the local unramified zeta integral $\mathcal{Z}_\nu(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ is equal to the following product*

$$\prod_{i=1}^{r} \frac{L(s+\frac{1}{2}, \tau_{i,\nu}\otimes\pi_\nu)}{L(s+1, \tau_{i,\nu}\times\sigma_\nu)L(2s_i+1, \tau_{i,\nu}, Asai\otimes\xi^m)} \times \prod_{1\le i<j\le r} \frac{1}{L(2s+1, \tau_{i,\nu}\times\tau_{j,\nu})} \langle f_\pi, f_\sigma\rangle_{G_{n-j}(F_\nu)}, \tag{3.47}$$

where $\langle f_\pi, f_\sigma\rangle_{G_{n-j}(F_\nu)}$ is independent with s.

This theorem will be proved in Section 4. It is also of interest to understand the local zeta integrals when $j > \ell+1$. We will come back to this issue in our future considerations.

3.4 Eulerian product: $j \le \ell$ case. In this section, we consider the case $j \le \ell < \tilde{m}$. By Proposition 3.6, we only need to consider the representative $\epsilon_{0,0}$ and $\eta_{\epsilon,I_{m-2\ell}}$, where ϵ is defined in (3.15). For simplicity, we denote by $\eta = \eta_{\epsilon,I_{m-2\ell}}$.

Recall that N_ℓ^η is the stabilizer in N_ℓ. By (3.18) and (3.19), it is easy to see that $(\epsilon_{0,0}\eta)N_\ell^\eta(\epsilon_{0,0}\eta)^{-1} = N_\ell^\eta$. In more detail, by (3.18) we have

$$N_\ell^\eta = \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ & b & y_4 & z_4 & 0 \\ & & I_{m-2\ell} & y_4' & 0 \\ & & & b^* & 0 \\ & & & & c^* \end{pmatrix},$$

and decompose N_ℓ^η as $Z_j N_{j,\ell-j}$ (which is different with the case $j > \ell$), where Z_j is identified as a subgroup of G_n, which is the maximal unipotent subgroup of $\mathrm{GL}(V_j^+)$, and

$$N_{j,\ell-j} = \left\{ \begin{pmatrix} I_j & & & & \\ & b & y_4 & z_4 & \\ & & I_{m-2\ell} & y_4' & \\ & & & b^* & \\ & & & & I_j \end{pmatrix} \mid b \in Z_{\ell-j} \right\}.$$

Note that $N_{j,\ell-j}$ is the unipotent subgroup of G_{n-j} as defined in (2.4) and the character $\psi_{\ell,\kappa}$ restricted on $N_{j,\ell-j}$ is the character $\psi_{\ell-j,\kappa}$ of the subgroup $N_{\ell-j}$ (of G_{n-j}) as defined in (2.6), which is denoted by $\psi_{n-j,\ell-j;\kappa}$.

$$\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) = \int_{[H_{n-\ell}]} \varphi(h) \int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \int_{[N_\ell^\eta]} \lambda\phi(\epsilon_{0,0}\eta u n h)\psi_{\ell,w_0}^{-1}(un)\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}h. \tag{3.48}$$

where $[H_{n-\ell}] := H_{n-\ell}(F)\backslash H_{n-\ell}(\mathbb{A})$ and $[N_\ell^\eta] := N_\ell^\eta(F)\backslash N_\ell^\eta(\mathbb{A})$. The inner integral

$$\int_{[N_\ell^\eta]} \lambda\phi(\epsilon_{0,0}\eta unh)\psi_{\ell,w_0}^{-1}(u)\,\mathrm{d}u \tag{3.49}$$

can be written as the following integral

$$\int_{[N_{j,\ell-j}]}\int_{[Z_j]} \lambda\phi(\epsilon_{0,0}\eta cunh)\psi_{\ell,\kappa}^{-1}(cu)\,\mathrm{d}c\,\mathrm{d}u.$$

Since τ is generic, we have a nonzero Whittaker function

$$\phi_\lambda^{\psi_{Z_j,\kappa}}(h) = \int_{[Z_j]} \lambda\phi(\hat{z}h)\psi_{Z_j,\kappa}(z)\,\mathrm{d}z,$$

where $\psi_{Z_j,\kappa}$ is the restriction of $\psi_{\ell,\kappa}$ to Z_j. Hence the inner integral (3.49) can be written as

$$\int_{[N_\ell^\eta]} \lambda\phi(\epsilon_{0,0}\eta unh)\psi_{\ell,w_0}^{-1}(u)\,\mathrm{d}u = \mathcal{B}^{\psi^{-1}_{n-j,\ell-j,\kappa}}(\phi_\lambda^{\psi_{Z_j,\kappa}})(\epsilon_{0,0}\eta nh), \tag{3.50}$$

where $\mathcal{B}^{\psi^{-1}_{n-j,\ell-j,\kappa}}$ is the Bessel period on the group $G_{n-j}(\mathbb{A})$ with respect to the subgroup $N_{j,\ell-j}$ and the character $\psi_{n-j,\ell-j,\kappa}$.

Therefore, the global zeta integral has the expression:

$$\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0}) = \int_{[H_{n-\ell}]}\varphi(h)\int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \mathcal{B}^{\psi^{-1}_{n-j,\ell-j,\kappa}}(\phi_\lambda^{\psi_{Z_j,\kappa}})(\epsilon_{0,0}\eta nh)\psi_{\ell,w_0}^{-1}(n)\,\mathrm{d}n\,\mathrm{d}h.$$

Proposition 3.10. (Case ($j \le \ell$)) *Let $E(\phi_{\tau\otimes\sigma},s)$ be the Eisenstein series on $G_n(\mathbb{A})$ as in (2.15) and π be an irreducible cuspidal automorphic representation of $H_{n-\ell}(\mathbb{A})$. The global zeta integral $\mathcal{Z}(s,\phi_{\tau\otimes\sigma},\varphi_\pi,\psi_{\ell,w_0})$ as in (3.1) is equal to*

$$\int_{[H_{n-\ell}]}\varphi(h)\int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \mathcal{B}^{\psi^{-1}_{n-j,\ell-j,\kappa}}(\phi_\lambda^{\psi_{Z_j,\kappa}})(\epsilon_{0,0}\eta nh)\psi_{\ell,w_0}^{-1}(n)\,\mathrm{d}n\,\mathrm{d}h.$$

It remains to show that the global zeta integral in Proposition 3.10 is eulerian. To this end, we need to reverse the order of the integration in

$$\int_{[H_{n-\ell}]}\varphi(h)\int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \mathcal{B}^{\psi^{-1}_{n-j,\ell-j,\kappa}}(\phi_\lambda^{\psi_{Z_j,\kappa}})(\epsilon_{0,0}\eta nh)\psi_{\ell,w_0}^{-1}(n)\,\mathrm{d}n\,\mathrm{d}h.$$

This can be deduced from the following lemma.

Lemma 3.11. *The automorphic function*

$$\Psi(h)=\int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})}\mathcal{B}^{\psi^{-1}_{n-j,\ell-j,\kappa}}(\phi_\lambda^{\psi_{Z_j,\kappa}})(\epsilon_{0,0}\eta nh)\psi^{-1}_{\ell,w_0}(n)\,\mathrm{d}n$$

is uniformly moderate growth on $H_{n-\ell}(\mathbb{A})$*.*

Proof. The proof is similar to the orthogonal case in Appendix 2 to §5 [GPR97]. □

Since φ_π is of rapid decay, after replacing n by hnh^{-1}, the global zeta integral is equal to

$$\int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})}\int_{[H_{n-\ell}]}\varphi(h)\mathcal{B}^{\psi^{-1}_{n-j,\ell-j,\kappa}}(\phi_\lambda^{\psi_{Z_j,\kappa}})(h\epsilon_{0,0}\eta n)\psi^{-1}_{\ell,w_0}(n)\,\mathrm{d}h\,\mathrm{d}n. \tag{3.51}$$

By the local uniqueness of the Bessel models and a suitable normalization at unramified local places, we can factorize (3.51) as follows

$$\prod_\nu\int_{N_\ell^\eta(F_\nu)\backslash N_\ell(F_\nu)}\left\langle\varphi_\nu,\mathcal{B}_\nu^{\psi^{-1}_{n-j,\ell-j,\kappa}}(R(\epsilon_{0,0}\eta n_\nu)\cdot\phi_{\lambda,\nu}^{\psi_{Z_j,\kappa}})\right\rangle_{H_{n-\ell}}\psi^{-1}_{\ell,w_0}(n_\nu)\,\mathrm{d}n_\nu.$$

Here the linear functional $\mathcal{B}_\nu^{\psi^{-1}_{n-j,\ell-j,\kappa}}$ is an element in

$$\mathrm{Hom}_{H_{n-\ell}(F_\nu)}(\sigma,\mathrm{Ind}^{G_{n-j}(F_\nu)}_{N_{j,\ell-j}H_{n-\ell}(F_\nu)}\psi_{n-j,\ell-j,\kappa}\otimes\tilde{\pi}),$$

and $\langle\cdot,\cdot\rangle_{H_{n-\ell}}$ is an invariant pairing of π and $\tilde{\pi}$.

Note that in this case, $N_\ell^\eta\backslash N_\ell$ is isomorphic to a subgroup of N_ℓ consisting of elements of the form

$$\begin{pmatrix} I_j & x_1 & x_2 & x_3 & x_4\\ & I_{\ell-j} & & & x_3'\\ & & I_{m-2\ell} & & x_2'\\ & & & I_{\ell-j} & x_1'\\ & & & & I_j\end{pmatrix}.$$

The restriction of $\psi_{\ell,\kappa}$ on $N_\ell^\eta\backslash N_\ell$ is $\psi((x_1)_{j,1})$. Under the adjoint action of $\epsilon_{0,0}\eta$, the image of the domain of integration $N_\ell^\eta\backslash N_\ell$ is also denoted by $U^-_{j,\eta}$, which is a subgroup of the unipotent radical U_j^- of the parabolic subgroup opposite to P_j, consisting of elements of the form

$$\begin{pmatrix} I_j & & & & \\ x_3' & I_{\ell-j} & & & \\ x_2' & & I_{m-2\ell} & & \\ x_1' & & & I_{\ell-j} & \\ x_4 & x_1 & x_2 & x_3 & I_j\end{pmatrix}.$$

Moreover, the induced character on $U^-_{j,\ell}$ is $\psi^{-1}(n_{m-j,1})$.

As remarked in the introduction of this paper, for a general pair (j, ℓ) of integers, the global zeta integrals considered here are eulerian and have potential applications to the explicit constructions of endoscopy correspondences discussed in [Jia13]. However, for the moment, the unramified calculation of the local zeta integrals is better understood only for the case where $j = \ell + 1$. Fortunately, this case is enough to catch the product of local tensor product L-factors as expected, which is carried out in Sections. 4.4 and 4.5. The general case remains to be fully developed.

4 Unramified calculation and local L-functions

We start here to develop the local theory for the family of global zeta integrals discussed in previous sections. The quasi-split orthogonal group cases were done in [GPR97]. In the following, we extend the idea and the method in [GPR97] to the quasi-split unitary group cases. It turns out that the argument in this case is much more technically involved, when the place ν splits in the quadratic extension E.

To achieve the goal of this section, we reformulate the local zeta integrals through the pairing of Bessel models in Section 4.1, including some general statements on twisted Jacquet modules, which we recall from [GRS11, Chapter 5]. In Section 4.2, we discuss unramified representations considered in the local zeta integrals and their Satake parameters, with which, we define the unramified local L-functions we need. In Section 4.3, we specify the local zeta integrals for unramified data by considering the cases when the unramified local place ν of F is split or not in E. By the Bernstein rationality, the unramified local zeta integrals are expressed as a rational function with respect to the parameters coming from the relevant representations. This rational function is explicitly calculated in Sections. 4.4 and 4.5, and identified with the expected local L-functions. Hence we carry out the complete proof of Theorem 3.9. Note that the condition $j = \ell + 1$ is used only from Sections. 4.4 and 4.5.

Throughout this section, denote by ν the local place of F. If ν is inert, then E_ν is the unramified quadratic extension of F_ν. If ν splits in E, then $E_\nu \cong F_\nu \times F_\nu$.

Let $\mathfrak{o}$ be the ring of integers of F_ν, and fix a prime element ϖ of $\mathfrak{o}$. Let q_{F_ν} be the cardinality of the residue fields of F_ν. If ν is non-split, let q_{E_ν} be the cardinality of the residue field of E_ν. When ν is inert, one has that $q_{E_\nu} = q_{F_\nu}^2$; and when ν is ramified, one have that $q_{E_\nu} = q_{F_\nu}$. We fix the normalized absolute values $|x|_{F_\nu} = |x|_\nu$ for $x \in F_\nu$, $|x|_{E_\nu} = |x\bar{x}|_{F_\nu}$ for $x \in E_\nu$ if ν is inert, and $|x|_{E_\nu} = |x\bar{x}|_{F_\nu}^{1/2}$ for $x \in E_\nu$ if ν is ramified.

When ν splits in E, we need to write down the structure of the unitary group $G_n(F_\nu)$ more explicitly, which are needed for the unramified calculation of the local integrals. In this case, $E_\nu = F_\nu \otimes_F E$ and one may take that $\rho = d^2$ or $\sqrt{\rho} = d$ for some $d \in F_\nu^\times$, and hence has that $E_\nu \cong F_\nu \oplus F_\nu$. This isomorphism is explicitly given by the following mapping: for $x, y \in F_\nu$,

$$x \otimes 1 + y \otimes \sqrt{\rho} \mapsto (x + yd, x - yd).$$

Here we consider elements of $\oplus_{\omega|\nu}E_\omega$ as elements of $F_\nu \otimes_F E$. When $x \in E_\nu$ is taken to $(x_1, x_2) \in F_\nu \times F_\nu$, the corresponding absolute values are normalized so that $|x|_{E_\nu} = |x_1 x_2|_{F_\nu}$. It follows that

$$\mathrm{GL}_m(E_\nu) \cong \mathrm{GL}_m(F_\nu) \times \mathrm{GL}_m(F_\nu)$$

given by

$$g_1 \otimes 1 + g_2 \otimes \sqrt{\rho} \mapsto (g_1 + dg_2, g_1 - dg_2).$$

Then the unitary group $G_n(F_\nu)$ consists of all elements

$$g = g_1 \otimes 1 + g_2 \otimes \sqrt{\rho} \in \mathrm{GL}_m(E_\nu)$$

satisfying

$$(g_1 + dg_2) J_m {}^t(g_1 - dg_2) = J_m.$$

The restriction of the above isomorphism to $G_n(F_\nu)$ gives the isomorphism: $G_n(F_\nu) \cong \mathrm{GL}_m(F_\nu)$, given explicitly by

$$g_1 \otimes 1 + g_2 \otimes \sqrt{\rho} \mapsto (g_1 + dg_2, g_1 - dg_2) \mapsto g_1 + dg_2. \tag{4.1}$$

Next, we explain the data in the local integral as needed for Theorem 3.8. We take a normalized parabolically induced representation

$$\Pi(\tau, \sigma, s) = \mathrm{Ind}_{P_j(F_\nu)}^{G_n(F_\nu)}(|\det|_{E_\nu}^s \tau \otimes \sigma),$$

where τ and σ are irreducible admissible representations of $\mathrm{GL}_j(E_\nu)$ and $G_{n-j}(F_\nu)$, respectively. Assume that τ is generic. Let π be an irreducible admissible representation of $H_{n-\ell}(F_\nu)$. Recall that the unitary group $H_{n-\ell}$ is defined in (2.9).

When ν splits in E, the induced representation $\Pi(\tau, \sigma, s)$ can be made more specific. In this case, the representation τ can be expressed as $\tau_1 \otimes \tau_2$, where τ_i are irreducible representations of $\mathrm{GL}_j(F_\nu)$. The representation σ is an irreducible representation of $\mathrm{GL}_{m-2j}(F_\nu)$. The representation $\Pi(\tau, \sigma, s)$ can be realized as the representation of $\mathrm{GL}_m(F_\nu)$, induced from the standard parabolic subgroup $P_{j,m-2j,j}(F_\nu)$ with the following representation

$$\begin{pmatrix} g_1 & x & y \\ & h & z \\ & & g_2 \end{pmatrix} \mapsto \left|\frac{\det(g_1)}{\det(g_2)}\right|^s \tau_1(g_1) \otimes \sigma(h) \otimes \tau_2(g_2^*),$$

where $g_1, g_2 \in \mathrm{GL}_j(F_\nu)$ and $g_2^* = J_j {}^t g^{-1} J_j^{-1}$. For the simplicity of notation, most of the time, we will omit the subscript ν from the corresponding notation. For instance, we may use F for the local field F_ν and use π for π_ν and so on, when no confusion will result.

4.1 Local zeta integrals and twisted Jacquet modules. We reformulate the local zeta integrals at any finite local place in terms of the uniqueness of local Bessel functionals, and relate them to the corresponding twisted Jacquet modules. This general formulation is better for the development of the complete local theory, although only unramified case will contribute to the proof of Theorem 3.9.

Let W_j be a nonzero member in the space

$$\mathrm{Hom}_{\mathrm{GL}_j(F)}(\tau, \mathrm{Ind}_{Z_j(E)}^{\mathrm{GL}_j(E)}(\psi_{Z_j,\kappa})).$$

This can be canonically extended to a partial Whittaker function

$$W_j(f) \in \mathrm{Ind}_{Z_j(E)\times G_{n-j}(F)\ltimes U_j(F)}^{G_n(F)}(\psi_{Z_j,\kappa}\otimes\sigma)$$

for $f \in \Pi(\tau,\sigma,s)$. As suggested by the global calculation in Section 3, we can formally define the following function

$$\mathcal{J}(f)(g) := \int_{N_\ell^\eta(F)\backslash U_\ell(F)} W_j(f)(\epsilon_{0,j-\ell}\eta u g)\psi_{\ell,\kappa}^{-1}(u)\,\mathrm{d}u.$$

Following the same argument in Appendix 2 to §5 [GPR97], the integral defining $\mathcal{J}(f)$ is convergent for $\Re(s)$ sufficiently large and is analytic in s. In addition, as a function on $H_{n-\ell}(F)$, $\mathcal{J}(f)$ belongs to the space

$$\mathrm{Ind}_{R_{\ell,\beta-1}^\eta(F)}^{H_{n-\ell}(F)}(\psi_{\beta-1,y_{-\kappa}}^{-1}\otimes\sigma^{w_q^\ell}),$$

where $\sigma^{w_q^\ell} := \sigma\circ\mathrm{Int}(w_q^\ell)$ is a representation of $G_{n-j}(F)$ conjugate by w_q^ℓ. In fact, in the unitary group case, w_q^ℓ is the identity.

Let $\mathcal{B}_{\beta-1}$ be a non-trivial member in the Hom-space

$$\mathrm{Hom}_{H_{n-\ell}(F)}(\pi, \mathrm{Ind}_{R_{\ell,\beta-1}^\eta(F)}^{H_{n-\ell}(F)}(\psi_{\beta-1,y_{-\kappa}}\otimes\tilde\sigma^{w_q^\ell})),$$

where $\tilde\sigma$ is the dual of σ. Let $\langle\cdot,\cdot\rangle_\sigma$ be an invariant pairing of σ and $\tilde\sigma$. By the uniqueness of local Bessel models [AGRS10], [GGP12], [SZ12] and [JBZ11], $\mathcal{B}_{\beta-1}$ is unique up to a constant. We may define a pairing

$$\langle\mathcal{J}(f),\mathcal{B}_{\beta-1}(v)\rangle = \int_{R_{\ell,\beta-1,y_{-\kappa}}^\eta(F)\backslash H_{n-\ell}(F)} \langle\mathcal{J}(f)(h),\mathcal{B}_{\beta-1}(v)(h)\rangle_\sigma\,\mathrm{d}h.$$

LEMMA 4.1. *For any τ and σ as above, the pairing $\langle\mathcal{J}(f),\mathcal{B}_{\beta-1}(v)\rangle$ is absolutely convergent for $\Re(s)$ sufficiently large.*

Proof. The proof is similar to Theorem A of Appendix (I) to §5 in [GPR97]. □

It is easy to check that this pairing, where it exists, defines a linear functional of Gross-Prasad type in the Hom-space

$$\mathrm{Hom}_{N_\ell \times H^{\triangle}_{n-\ell}}(\Pi(\tau,\sigma,s)\otimes\pi, \psi_{\ell,\kappa}). \tag{4.2}$$

Again, by the uniqueness of local Bessel functionals, the dimension of this Hom-space is at most one, when $\Pi(\tau,\sigma,s)$ is irreducible. Therefore, the local zeta integral is defined by

$$\mathcal{Z}(s,f,v,\psi_{\ell,\kappa}) := \langle \mathcal{J}(f), \mathcal{B}_{\beta-1}(v)\rangle\,, \tag{4.3}$$

for $f \in \Pi(\tau,\sigma,s)$ and $v \in \pi$, which is proportional to the local zeta integral defined as an eulerian factor of the global zeta integral in Section 3. For the unramified data, we may normalize the pairing, so that this proportional constant is one.

In order to proceed with the explicit calculation of the local integrals, we have to understand those Bessel models involved in the local zeta integrals from the representation-theoretic point of view. This means to see more precisely the structures of those twisted Jacquet models. We recall relevant results from [GRS11, Chapter 5].

Let (Π, V_Π) be a smooth representation of $G_n(F)$. Let $J_{\psi_{\ell,\kappa}}(\Pi)$ be the twisted Jacquet module of Π with respect to $N_\ell(F)$ and its character $\psi_{\ell,\kappa}$, the space of which is defined by

$$V_\Pi/\mathrm{Span}\left\{\Pi(n)v - \psi_{\ell,\kappa}(n)v \mid n \in N_\ell(F), v \in V_\Pi\right\}. \tag{4.4}$$

Note that $J_{\psi_{\ell,\kappa}}(\Pi)$ is a smooth representation of $H_{n-\ell}(F)$. Twisted Jacquet modules for other unipotent groups will be considered throughout the section. They are defined analogously.

Next, we study the twisted Jacquet module $J_{\psi_{\ell,\kappa}}(\Pi)$ for the induced representation $\Pi = \Pi(\tau,\sigma,s)$. To do so, we consider the structure of the restriction of the induced representation Π to the standard parabolic subgroup P_ℓ, which is denoted by $\mathrm{Res}_{P_\ell}(\Pi)$. This can be described in terms of the generalized Bruhat decomposition $P_j\backslash G_n/P_\ell$, which was discussed in Section 3. Hence, as a representation of P_ℓ, $\mathrm{Res}_{P_\ell}(\Pi)$ can be expressed (up to semi-simplification) as a finite direct sum $\oplus_{\alpha,\beta}\Pi_{\epsilon_{\alpha,\beta}}$ parameterized by the set of representatives $\{\epsilon_{\alpha,\beta}\}$ as discussed in Sect. 3.1 and [GRS11, §5.1].

Let $\tau^{(t)}$ denote the t-th Bernstein-Zelevinsky derivative of τ along the subgroup Z'_t defined in (3.25) with the character

$$\psi'_t\begin{pmatrix} I_\beta & y \\ 0 & z\end{pmatrix} = \psi^{-1}(z_{1,2}+z_{2,3}+\cdots+z_{t-1,t}).$$

We embed GL_β into GL_j through the map $g \in \mathrm{GL}_\beta \mapsto \mathrm{diag}(g, I_t) \in \mathrm{GL}_j$. The image, which is still denoted by GL_β, normalizes the character ψ'_t. Hence $\tau^{(t)}$ is the representation of GL_β via the twisted Jacquet module $J_{\psi'_t}(\tau)$. We also define the following character of Z'_t,

$$\psi''_t \begin{pmatrix} I_\beta & y \\ 0 & z \end{pmatrix} = \psi^{-1}(z_{1,2} + z_{2,3} + \cdots + z_{t-1,t} + y_{\beta,1}),$$

which is conjugate to the character $\psi_{Z'_\ell,\kappa}$ as defined in (3.28) for any nonzero κ, by an element in the subgroup GL_β. Denote the corresponding Jacquet module $J_{\psi''_t}(\tau)$ by $\tau_{(t)}$, which is a representation of the mirabolic subgroup of GL_β.

Recall that $P'_\beta = H_{n-\ell}^{\eta_\epsilon, I_{m-2\ell}}$ is as defined in (3.9). By the discussion in Page 26, when $\ell + \beta < \tilde{m}$, P'_β is a maximal parabolic subgroup of $H_{n-\ell}$. For the proof of Theorem 3.9, which only concerns the case of $j = \ell + 1$, we may assume that $\ell < j$ in the following discussion. Put $P''_{j-\ell} = H_{n-\ell}^{\eta_\epsilon,\gamma}$ for γ as defined in (3.16). Note that $P'_w \gamma H_{n-\ell}$ is the open double coset discussed in Page 569, and $P''_{j-\ell}$ is not a proper maximal parabolic subgroup. Although we only need in this paper the case when $\ell < j$, we recall from [GRS11] the following general result.

Proposition 4.2. ([GRS11, Theorem 5.1]) *Assume that $0 \le \ell < \tilde{m}$ and $1 \le j < m$. If ν is inert, then, up to semi-simplification, the following isomorphism holds*

$$J_{\psi_{\ell,\kappa}}(\mathrm{Ind}_{P_j}^{G_n}(\tau \otimes \sigma)) \equiv \Upsilon_1 \oplus \Upsilon_2 \oplus \Upsilon_3$$

where

$$\Upsilon_1 = \oplus_{\substack{j-\ell \le \beta < \tilde{m}-\ell \\ 0 \le \beta \le j}} \mathrm{ind}_{P'_\beta}^{H_{n-\ell}}(|\det|_E^{\frac{1-t}{2}} \tau^{(t)} \otimes J_{\psi'_{\ell-t,\kappa}}(\sigma^{w_q^t})),$$

$$\Upsilon_2 = \begin{cases} \mathrm{ind}_{P''_{j-\ell}}^{H_{n-\ell}}(|\det|_E^{-\frac{\ell}{2}} \tau_{(\ell)} \otimes \sigma^{w_q^\ell}), & \ell < j, \\ 0, & \ell \not< j. \end{cases}$$

and Υ_3 is the remaining representation in the above semi-simplification of $J_{\psi_{\ell,\kappa}}(\mathrm{Ind}_{P_j}^{G_n}(\tau \otimes \sigma))$, whose detail can be found in [GRS11, Theorem 5.1].

We note that the detailed description of Υ_3 is not needed in the following explicit unramified calculation, and hence we omit it here.

If ν is split, let $\underline{\ell} = [\ell_1, \ell_2, \ell_3]$ be a partition of a positive integer N and consider the twisted Jacquet module $J_{\tilde{\psi}}(\mathrm{Ind}_{P_j,N-j}^{\mathrm{GL}_N} \tau_1 \times \tau_2)$ in [GRS11, Section 3.6]. In order to simplify our calculation, up to a suitable conjugation, we will use the Gelfand-Graev character defined in [GRS11, Section 3.6]. Let N_ℓ consist of elements of form

$$n = \begin{pmatrix} z^{(1)} & y^{(1)} & x \\ & I_{m-2\ell} & y^{(2)} \\ & & z^{(2)} \end{pmatrix} \in \mathrm{GL}_m(F),$$

where $z^{(1)}, z^{(2)} \in Z_\ell(F)$. We will take the character $\psi_{\ell,\kappa}$ to be the following character

$$\tilde{\psi}(n) = \psi(\sum_{i=1}^{\ell-1}(z^{(1)}_{i,i+1} + z^{(2)}_{i,i+1}) + y^{(1)}_{\ell,1} + y^{(2)}_{1,1}).$$

The stabilizer of the character $\tilde{\psi}(n)$ inside $G_{n-\ell}(E_\nu) \cong \mathrm{GL}_{m-2\ell}(F)$ is

$$\tilde{L}_\ell = \left\{\mathrm{diag}\,\{I_\ell, \gamma, I_\ell\} \in \mathrm{GL}_m(F) \mid \gamma = \begin{pmatrix} 1 & \\ & g \end{pmatrix},\ g \in \mathrm{GL}_{m-2\ell-1}(F)\right\}.$$

Define

$$\tau_2^{[\ell_1-\alpha]} := [(\tau_2^*)^{\ell_1-\alpha}]^* \text{ and } (\tau_2)_{[\ell_1]} := [(\tau_2^*)_{(\ell_1)}]^*.$$

which are representations of $\mathrm{GL}_{\ell_2-\beta+\alpha}(F)$ and the mirabolic subgroup of $\mathrm{GL}_{N-j-\ell_1}(F)$, respectively, where the inner $*$ denotes composition with the map

$$g \to J'_{\ell_1-\alpha}\,{}^t g^{-1} J'^{-1}_{\ell_1-\alpha},$$

where $J'_{\ell_1-\alpha} = \mathrm{diag}(J_{\ell_1-\alpha}, J_{\ell_2-\beta+\alpha})$, and the outer $*$ denotes composition with the map

$$g \to J_{\ell_2-\beta+\alpha} \cdot {}^t g^{-1} J^{-1}_{\ell_2-\beta+\alpha}.$$

More information about $\tau_2^{[\ell_1-\alpha]}$ and $(\tau_2)_{[\ell_1]}$ can be found in [GRS11, Pages 113 and 115].

Proposition 4.3. ([GRS11, Theorem 5.7]) *Up to semi-simplification, the following isomorphism holds*

$$J_{\tilde{\psi}}(\mathrm{Ind}^{\mathrm{GL}_N(F)}_{P_{j,N-j}} \tau_1 \times \tau_2) \equiv \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \oplus \mathcal{L}_5$$

where $\mathcal{L}_1$ *is given by the following direct sum*

$$\oplus_{\substack{j-\ell_3<\beta<\ell_2 \\ 0\le\beta\le j}} \mathrm{Ind}^{\mathrm{GL}_{\ell_2-1}}_{P_{\beta,\ell_2-\beta-1}}(|\cdot|^{\frac{1-(j-\beta)+\ell_3-\ell_1}{2}}\tau_1^{(j-\beta)}) \otimes |\cdot|^{\frac{j-\beta}{2}} J_{\psi_{(\ell_1,\ell_2-\beta,\ell_3-j+\beta)}}(\tau_2);$$

$\mathcal{L}_2$ *is given by the following direct sum*

$$\oplus_{\substack{0<r<j-\ell_3 \\ j-\ell_2-\ell_3\le r\le\ell_1}} \mathrm{Ind}^{\mathrm{GL}_{\ell_2-1}}_{P_{j_{\ell_3-r-1},\ell_2+\ell_3-j+r}}(|\cdot|^{\frac{r-\ell_1}{2}} J_{\psi_{(r,j-\ell_3-r,\ell_3)}}(\tau_1)) \otimes |\cdot|^{\frac{\ell_3-r-1}{2}} \tau_2^{[\ell_1-r]};$$

$\mathcal{L}_3$ *is given by the following representation*

$$\begin{cases} \mathrm{Ind}^{\mathrm{GL}_{\ell_2-1}}_{P_{j-\ell_3-1,\ell_2+\ell_3-j}}(|\cdot|^{-\frac{\ell_1}{2}}(\tau_1)_{(\ell_3)}) \otimes |\cdot|^{\frac{\ell_3-1}{2}}\tau_2^{[\ell_1]}, & \textit{if } 0 < j-\ell_3 \le \ell_2, \\ \tau_1^{(j)} \otimes |\det|^{-\frac{\ell_3}{2}}\tau_{2[\ell_1]}, & \textit{if } \ell_3 = j, \\ 0, & \textit{otherwise}; \end{cases}$$

$\mathcal{L}_4$ is given by the following representation

$$\begin{cases}\mathrm{Ind}^{\mathrm{GL}_{\ell_2-1}}_{P_{j-\ell_3,\ell_2+\ell_3-j-1}}(|\cdot|^{\frac{1-\ell_1}{2}}(\tau_1)^{(\ell_3)})\otimes|\cdot|^{\frac{\ell_3}{2}}\tau_{2[\ell_1]}, & \text{if } 0<j-\ell_3<\ell_2,\\ 0, & \text{otherwise};\end{cases}$$

and $\mathcal{L}_5$ is given by the following representation

$$\begin{cases}\mathrm{ind}^{\mathrm{GL}_{\ell_2-1}}_{P'_{j-\ell_3-1,1,\ell_2+\ell_3-j-1}}(|\cdot|^{-\frac{\ell_1}{2}}(\tau_1)_{(\ell_3)})\otimes|\cdot|^{\frac{\ell_3}{2}}\tau_{2[\ell_1]}, & \text{if } 0<j-\ell_3<\ell_2,\\ 0, & \text{otherwise.}\end{cases}$$

The other notation in this proposition is referred to [GRS11, Section 5.2]. We are going to apply the case of $\underline{\ell}=[\ell,m-2j,\ell]$ to the unramified calculation.

4.2 Unramified representations and local L-functions of unitary groups. Let $B_H=T_HN_H$ be a Borel subgroup of $H_{n-\ell}$ with the maximal F-torus T_H and the unipotent radical N_H. Let $K_G=G_n(\mathfrak{o}_F)$ (resp. $K_H=H_{n-\ell}(\mathfrak{o}_F)$) be the standard maximal open compact subgroup of G_n (resp. $H_{n-\ell}$). Denote by $W(G_n)=N(T)/T$ the Weyl group of G_n. When ν is inert over E, $W(G_n)$ is the Weyl group associated to a root system of type B. When ν is split over E, $W(G_n)$ is the Weyl group associated to a root system of type A.

From now on, we assume that the representations τ, σ, and π are unramified. Let χ_τ and χ_σ be the unramified characters corresponding to the spherical representations τ and σ. Then $\chi_\tau=\otimes_{i=1}^{j}\chi_i$ and $\chi_\sigma=\otimes_{i=j+1}^{\tilde{m}}\chi_i$. Define $\chi_s:=|\cdot|^s\chi_\tau\otimes\chi_\sigma$. Let $\Pi_s:=\Pi(\chi_s)$ and $\pi:=\pi(\mu)$ be the unramified constituents of the normalized induced representations

$$\mathrm{Ind}^{G_n(F)}_{P_j(F)}(|\det|^s\tau\otimes\sigma)\quad\text{and}\quad\mathrm{Ind}^{H_{n-\ell}(F)}_{B_H(F)}(\mu),$$

respectively.

If ν is inert over E, χ_i and μ_i are unramified characters of $E^\times=F(\sqrt{\rho})^\times$.

If ν is split over E, $H_{n-\ell}(F)\cong\mathrm{GL}_{m-2\ell-1}(F)$ and μ_i splits into a product $\theta_i\vartheta_i$ of two unramified characters of $F^\times$. Moreover, if $m-2\ell-1$ is odd, μ splits as $\otimes_{i=1}^{(m-2\ell-2)/2}\theta_i\otimes\vartheta_i\otimes\mu_0$. Here μ_0 is also an unramified character of $F^\times$. In particular, $\pi(\mu)$ is the unramified constituent of the following induced representation

$$\mathrm{Ind}^{H_{n-\ell}(F)}_{B_H}((\otimes_{i=1}^{\widetilde{m}_H}\theta_i)\otimes(\otimes_{i=1}^{\widetilde{m}_H}\vartheta^{-1}_{\widetilde{m}_H+1-i}))$$

if m is odd, and of the following induced representation

$$\mathrm{Ind}^{H_{n-\ell}(F)}_{B_H}((\otimes_{i=1}^{\widetilde{m}_H}\theta_i)\otimes\mu_0\otimes(\otimes_{i=1}^{\widetilde{m}_H}\vartheta^{-1}_{\widetilde{m}_H+1-i}))$$

if m is even, where $\tilde{m}_H$ is the Witt index of the hermitian vector subspace $(W_\ell\cap w_0^\perp,q_{W_\ell\cap w_0^\perp})$, which defines $H_{n-\ell}$. Since $E\cong F\times F$, we must have

$$\mathrm{GL}_j(E)\cong\mathrm{GL}_j(F)\times\mathrm{GL}_j(F)$$

and χ_τ splits as a product $\Xi_\tau \Theta_\tau$ of unramified characters with

$$\Xi_\tau = \otimes_{i=1}^{j} \Xi_i,$$
$$\Theta_\tau = \otimes_{i=1}^{j} \Theta_i.$$

The representation $\tau = \tau_1 \otimes \tau_2$ is the unramified constituent of the induced representation

$$\mathrm{Ind}_{B_{\mathrm{GL}_j}(F)}^{\mathrm{GL}_j(F)}(\Xi_\tau) \otimes \mathrm{Ind}_{B_{\mathrm{GL}_j}(F)}^{\mathrm{GL}_j(F)}(\Theta_\tau),$$

where τ_1 is induced from Ξ_τ and τ_2 is induced from Θ_τ. Also, if we set $\chi_i = |\cdot|^s \Theta_i$ and $\chi_{m+1-i} = |\cdot|^{-s}\Xi_i^{-1}$ for $1 \le i \le j$, then the representation $\Pi(\chi_s)$ of $G_n(F)$ is regarded as the representation of $\mathrm{GL}_m(F)$ as defined in Page 585, via the isomorphism $G_n(F) \cong \mathrm{GL}_m(F)$. It follows that $\mathrm{Res}_{E/F}(\mathrm{GL}_j)(F) \cong \mathrm{GL}_j(F) \times \mathrm{GL}_j(F)$, in particular.

In the following, we write down the Satake parameters for the unramified representations discussed above and write the relevant unramified local L-functions, following the arguments in [BS09] or [KK11] for instance.

The Langlands dual group ${}^L\mathrm{U}_m$ of U_m is $\mathrm{GL}_m(\mathbb{C}) \rtimes \Gamma(E/F)$, where $\Gamma(E/F)$ is the Galois group on E and the nontrivial element ι acts on $\mathrm{GL}_m(\mathbb{C})$ via $\iota(g) = J_m {}^t g^{-1} J_m^{-1}$. A $2m$-dimensional complex representation ρ_{2m} of the Langlands dual group ${}^L\mathrm{U}_m$ is given by

$$(g;1) \mapsto \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \text{ and } (I_m;\iota) \mapsto \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix},$$

for any $g \in \mathrm{GL}_m(\mathbb{C})$. The Langlands dual group ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_j$ of $\mathrm{Res}_{E/F}\mathrm{GL}_j$ is $(\mathrm{GL}_j(\mathbb{C}) \times \mathrm{GL}_j(\mathbb{C})) \rtimes \Gamma(E/F)$. The element ι acts on $\mathrm{GL}_j(\mathbb{C}) \times \mathrm{GL}_j(\mathbb{C})$ by $\iota(g_1, g_2) = (g_2, g_1)$. We consider a j^2 dimensional representation of ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_j$, which is realized in the space of all $j \times j$ matrices by

$$(g_1, g_2; 1) \cdot x \mapsto g_1 \cdot x \cdot {}^t g_2,$$
$$(I_j, I_j; \iota) \cdot x \mapsto {}^t x$$

where $x \in M_{j\times j}$, and is called the *Asai* representation of ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_j$.

In addition, the Langlands dual group ${}^L(\mathrm{U}_m \times \mathrm{Res}_{E/F}\mathrm{GL}_j)$ is

$$(\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_j(\mathbb{C}) \times \mathrm{GL}_j(\mathbb{C})) \rtimes \Gamma(E/F).$$

The element ι acts on it by $\iota(g, g_1, g_2) = (g^*, g_2, g_1)$. A $2mj$-dimensional complex representation ρ_{2mj} of ${}^L(\mathrm{U}_m \times \mathrm{Res}_{E/F}\mathrm{GL}_j)$ is given by

$$(g, g_1, g_2, 1) \mapsto \begin{pmatrix} g \otimes g_1 & 0 \\ 0 & g^* \otimes g_2 \end{pmatrix},$$
$$(I_m, I_j, I_j, \iota) \mapsto \begin{pmatrix} 0 & I_{mj} \\ I_{mj} & 0 \end{pmatrix},$$

where $g \otimes g_i$ is the Kronecker product.

We first consider the irreducible unramified representation $\pi(\mu)$ of $H_{n-\ell}(F)$. When ν is inert over E, the Satake parameter of $\pi(\mu)$ is the semi-simple conjugacy class in ${}^LH_{n-\ell}$ of type

$$c(\pi(\mu)) = (\mathrm{diag}(\mu_1(\varpi_E), \mu_2(\varpi_E), \ldots, \mu_{\tilde{m}_H}(\varpi_E), 1, \ldots, 1); \iota),$$

where ϖ_E is the ν-uniformizer of E. To simplify the notation, we may use μ_i for $\mu_i(\varpi_E)$ in the following, if it does not cause any confusion.

When ν is split over E, the Satake parameter of $\pi(\mu)$ is the semi-simple conjugacy class in ${}^LH_{n-\ell}$ of type

$$c(\pi(\mu)) = (\mathrm{diag}(\theta_1(\varpi), \ldots, \theta_{\tilde{m}_H}(\varpi), \vartheta_1^{-1}(\varpi), \ldots \vartheta_{\tilde{m}_H}^{-1}(\varpi)); 1)$$

if m is odd, and of type

$$c(\pi(\mu)) = (\mathrm{diag}(\theta_1(\varpi), \ldots, \theta_{\tilde{m}_H}(\varpi), \mu_0(\varpi), \vartheta_1^{-1}(\varpi), \ldots \vartheta_{\tilde{m}_H}^{-1}(\varpi)); 1)$$

if m is even, where ϖ is the ν-uniformizer of F.

Next, we consider the irreducible unramified representation τ of $\mathrm{Res}_{E/F}(\mathrm{GL}_j)(F)$. When ν is inert over E, the Satake parameter of τ is the semi-simple conjugacy class in ${}^L\mathrm{Res}_{E/F}(\mathrm{GL}_j)$ of type

$$c(\tau) = (\mathrm{diag}(\chi_1(\varpi_E), \chi_2(\varpi_E), \ldots, \chi_j(\varpi_E)), I_j; \iota).$$

Again, we use χ_i for $\chi_i(\varpi_E)$ if it does not cause any confusion.

When ν is split over E, the Satake parameter of τ is the semi-simple conjugacy class in ${}^L\mathrm{Res}_{E/F}(\mathrm{GL}_j)$ of type

$$c(\tau) = (\mathrm{diag}(\Theta_1, \ldots, \Theta_j), \mathrm{diag}(\Xi_1, \ldots, \Xi_j); 1),$$

where Θ_i is used for $\Theta_i(\varpi)$ and Ξ_i is used for $\Xi_i(\varpi)$, to simplify the notation (we use similar notation for $\chi_i(\varpi)$ and $\mu_i(\varpi)$).

Therefore, if ν is inert over E and E is the unramified quadratic field extension of F, the unramified tensor product local L-function $L(s, \pi \times \tau)$ is defined to be

$$\prod_{\substack{1\leq i\leq j\\ 1\leq i'\leq \tilde{m}_H}} (1 - \chi_i\mu_{i'}q_F^{-2s})^{-1}(1 - \chi_i\mu_{i'}^{-1}q_F^{-2s})^{-1} \prod_{1\leq k\leq n} (1 - \chi_k q_F^{-2s})^{-1}, \tag{4.5}$$

if m is even; and to be

$$\prod_{\substack{1\leq i\leq j\\ 1\leq i'\leq \tilde{m}_H}} (1 - \chi_i\mu_{i'}q_F^{-2s})^{-1}(1 - \chi_i\mu_{i'}^{-1}q_F^{-2s})^{-1}, \tag{4.6}$$

if m is odd. When ν is split over E, the unramified tensor product local L-function $L(s, \pi \times \tau)$ is defined to be

$$L(s, \pi \times \tau) = L(s, \pi \times \tau_1)L(s, \tilde{\pi} \times \tau_2), \tag{4.7}$$

where τ_1 and τ_2 are defined according to $\Theta_1, \ldots, \Theta_j$ and $\Xi_1, \ldots, \Xi_j$, respectively.

Moreover, the unramified local *Asai* L-function of τ is defined as, when ν is inert,

$$L(s, \tau, Asai) = \prod_{1 \le i_1 < i_2 \le j} (1 - \chi_{i_1} \chi_{i_2} q_F^{-2s})^{-1} \prod_{1 \le i \le j} (1 - \chi_i q_F^{-s})^{-1}; \tag{4.8}$$

and when ν is split

$$L(s, \tau, Asai) = L(s, \tau_1 \times \tau_2) = \prod_{1 \le i,k \le j} (1 - \Theta_i \Xi_k q_F^{-s})^{-1}. \tag{4.9}$$

The unramified tensor product local L-function $L(s, \sigma \times \tau)$ can be defined in the same way.

4.3 Unramified local zeta integrals. Let f_{χ_s} and f_μ be the spherical vectors in $\Pi(\chi_s)$ and $\pi(\mu)$, normalized by $f_\chi(e_G) = f_\mu(e_H) = 1$. Denote by f_τ and f_σ the unramified function in τ and σ accordingly. We are going to calculate explicitly the unramified local zeta integral $\mathcal{Z}_\nu(s, f_{\chi_s}, f_\mu, \psi_{\ell,\kappa})$.

By the Bernstein rationality theorem ([GPR87] and see also [Ban98]), $\mathcal{Z}_\nu(s, f_{\chi_s}, f_\mu, \psi_{\ell,\kappa})$ is a rational function of the parameters χ_s and μ. Thus, we can write

$$\mathcal{Z}_\nu(s, f_{\chi_s}, f_\mu, \psi_{\ell,\kappa}) = \frac{P(\chi_s, \mu)}{Q(\chi_s, \mu)} \tag{4.10}$$

where $P(\chi_s, \mu)$ and $Q(\chi_s, \mu)$ are polynomials of variables in χ_i, μ_i and q_E^{-s}. Although the polynomials $P(\chi_s, \mu)$ and $Q(\chi_s, \mu)$ may not be unique, in the next two subsections, we try to produce explicitly a pair of polynomials $P(\chi_s, \mu)$ and $Q(\chi_s, \mu)$ satisfying (4.10).

4.4 Polynomial $\boldsymbol{Q(\chi_s, \mu)}$. For a technical reason, which will be mentioned in the argument below, we assume that $j = \ell + 1$. This is enough to produce the unramified local L-functions as needed. The method used here is an extension of that in [GPR97] to the unitary group case. In order to define a polynomial $Q(\chi_s, \mu)$ which serves a candidate for (4.10), we first introduce a proper Hecke algebra element Φ_0 in the extended spherical Hecke algebra of $H_{n-\ell}$ as defined below, so that for any section f_{χ_s} in the unramified induced representation

$$\mathrm{Ind}_{P_j(F)}^{G_n(F)}(|\det|^s \tau \otimes \sigma),$$

the convolution $\mathcal{J}(f_{\chi_s} * \Phi_0)$ is supported in the Zariski open orbit $P_j \epsilon_{0,1} \eta R_{\ell,w_0}$, and

$$\mathcal{Z}_\nu(s, f_{\chi_s} * \Phi_0, f_\mu, \psi_{\ell,\kappa}) \in \mathbb{C}[\chi_s, \mu^{\pm 1}] \cdot \mathcal{Z}_\nu(s, f_{\chi_s}, f_\mu, \psi_{\ell,\kappa}).$$

Since $\mathcal{J}(f_{\chi_s} * \Phi_0)$ is supported in the Zariski open orbit, the local zeta integral $\mathcal{Z}_\nu(s, f_{\chi_s} * \Phi_0, f_\mu, \psi_{\ell,\kappa})$ is in $\mathbb{C}[\chi_s, \mu^{\pm 1}]$, which is taken to be a candidate for $P(\chi_s, \mu)$ in (4.10). Then take $Q(\chi_s, \mu)$ to be an element in $\mathbb{C}[\chi_s, \mu^{\pm 1}]$ satisfying

$$\mathcal{Z}_\nu(s, f_{\chi_s} * \Phi_0, f_\mu, \psi_{\ell,\kappa}) = Q(\chi_s, \mu) \cdot \mathcal{Z}_\nu(s, f_{\chi_s}, f_\mu, \psi_{\ell,\kappa}).$$

This makes our choice of a pair of polynomials $P(\chi_s,\mu)$ and $Q(\chi_s,\mu)$ for the expression in (4.10). The polynomial $Q(\chi_s,\mu)$ is calculated here and the polynomial $P(\chi_s,\mu)$ will be done in the next section.

Let $\mathcal{H}(H_{n-\ell},K_H)$ be the spherical Hecke algebra with convolution $\circ$ of all K_H-bi-invariant (smooth) functions with compact support on $H_{n-\ell}$. We choose generators X_i for all $1\le i\le \tilde{m}_H$ of the Hecke algebra $\mathcal{H}(H_{n-\ell},K_H)$ such that the following isomorphism holds. By the Satake isomorphism, if ν is inert over E and E is the unramified quadratic field extension of F, the Hecke algebra can be realized as follows:

$$\mathcal{H}(H_{n-\ell},K_H)\simeq\mathbb{C}\left[X_1,X_1^{-1},\dots,X_{\tilde{m}_H},X_{\tilde{m}_H}^{-1}\right]^{W(H_{n-\ell})};$$

and if ν is split over E, the Hecke algebra can be realized as follows:

$$\mathcal{H}(H_{n-\ell},K_H)\simeq\mathbb{C}\left[X_1^{\pm1},X_2^{\pm1},\dots,X_{m-2\ell-1}^{\pm1}\right]^{S_{m-2\ell-1}}.$$

Here $S_{m-2\ell-1}$ is the symmetric group on the sets $\{X_1,\dots,X_{m-2\ell-1}\}$ and $\left\{X_1^{-1},\dots,X_{m-2\ell-1}^{-1}\right\}$. We will specify the action of the generators X_i later.

Define an extended Hecke algebra as in [GPR97, §2]:

$$\mathcal{A}_{H_{n-\ell}}:=\mathbb{C}\left[X,X^{-1}\right]\otimes\mathcal{H}(H_{n-\ell},K_H).$$

Let $\Pi(\chi_s)$ be the unramified representation of $G_n(F)$ as defined in §4.2. We consider the subspace of all K_H-invariant vectors

$$J^*_{\psi_{\ell,\kappa}}(\chi_s):=(J_{\psi_{\ell,\kappa}}(\chi_s))^{K_H}$$

of the twisted Jacquet module $J_{\psi_{\ell,\kappa}}(\chi_s):=J_{\psi_{\ell,\kappa}}(\Pi(\chi_s))$. Although it is naturally a module of the Hecke algebra $\mathcal{H}(H_{n-\ell},K_H)$, we may extend it to be a module of the extended Hecke algebra $\mathcal{A}_{H_{n-\ell}}$ as follows: for $\phi\in J^*_{\psi_{\ell,\kappa}}(\chi_s)$ and $X\otimes\Phi\in\mathcal{A}_{H_{n-\ell}}$,

$$\phi*(X\otimes\Phi)=q_E^{-s}(\phi\circ\Phi),$$

where $\phi\circ\Phi$ is the left action on ϕ via convolution. Define, as in [GPR97, § 2], the *support ideal* as follows:

$$\mathcal{I}_{supp}(\chi_s)=\left\{\Phi\in\mathcal{A}_{H_{n-\ell}}\mid J^*_{\psi_{\ell,\kappa}}(\chi_s)*\Phi\subseteq\Lambda\right\},$$

where Λ is the smooth representation of $H_{n-\ell}(F)$ consisting of functions in $\Pi(\chi_s)$ supported in the open double coset $P_j\epsilon_{0,1}\eta R_{\ell,w_0}$. More precisely, by Proposition 4.2 and 4.3, the smooth representation Λ can be realized via the following isomorphisms:

$$\Lambda\cong\mathrm{ind}_{P'_{1,\ell}(F)}^{H_{n-\ell}(F)}(|\det|_E^{-\frac{\ell}{2}+s}\tau_{(\ell)}\otimes\sigma^{w_b^\ell})$$

if ν is inert over E; and

$$\Lambda\cong\mathrm{ind}_{\mathrm{GL}_{m-2j}(F)}^{\mathrm{GL}_{m-2j+1}(F)}(\sigma)$$

if ν is split over E. Here we use the assumption that $j=\ell+1$.

First, consider the case when $\ell = 0$, which implies that $j = \ell + 1 = 1$. In the case, the twisted Jacquet functor is just the restriction to the subgroup $H_n(F)$ of $G_n(F)$. By restricting to the subgroup $H_n(F)$, the induced representation

$$\Pi = \mathrm{Ind}_{P_1}^{G_n}(|\cdot|_E^s \chi \otimes \sigma)$$

decomposes via an exact sequence of $H_n(F)$-modules, according to Proposition 4.2.

If ν is inert over E, the case is similar to [GPR97, §2] and we have

$$0 \to \mathrm{Ind}_{G_{n-1}}^{H_n}(\sigma) \to J_{\psi_{0,\kappa}}(\Pi) \to \mathrm{Ind}_{P_1'}^{H_n}(|t|_E^{\frac{1}{2}+s}\chi \otimes J_{\psi'_{0,\kappa}}(\sigma)) \to 0.$$

If ν is split over E, more explanation is needed. The double coset decomposition

$$P_{1,m-2,1}\backslash \mathrm{GL}_m/H_n$$

has 6 representatives for $m > 2$, which are denoted by γ_i for $1 \leq i \leq 6$. Let $P_{1,m-2,1}\gamma_1 H_n$ be the open orbit, and $P_{1,m-2,1}\gamma_i H_n$ for $i = 2$ or $i = 3$ be the orbits with the greatest dimension in those orbits except the open orbit. Using Proposition 4.3 repeatedly, we have

$$0 \to \mathrm{ind}_{\mathrm{GL}_{m-2}}^{\mathrm{GL}_{m-1}}(\sigma) \to \Omega \to \Sigma \to 0,$$

where

$$\Omega := \{f \in \Pi \mid supp(f) \subseteq \cup_{i=1}^{3} P_{1,m-2,1}\gamma_i H_n\}, \tag{4.11}$$

and

$$\Sigma := \mathrm{Ind}_{P_{1,m-2}}^{H_n}(|\cdot|^{\frac{1}{2}+s}\Theta \otimes \sigma_{[0]}) \oplus \mathrm{Ind}_{P_{m-2,1}}^{H_n}(\sigma_{[0]} \otimes |\cdot|^{-\frac{1}{2}-s}\Xi^{-1}).$$

Lemma 4.4. *Assume that $\ell = 0$ and $j = \ell + 1 = 1$. The support ideal $\mathcal{I}_{supp}(\chi_s)$ contains*

$$\Phi_0 = \prod_i (1 - q_E^{-\frac{1}{2}}\chi(\varpi)XX_i)(1 - q_E^{-\frac{1}{2}}\chi(\varpi)XX_i^{-1})$$

if ν is inert over E, and

$$\Phi_0 = \prod_i (1 - q_E^{-\frac{1}{2}}\Theta(\varpi)XX_i)(1 - q_E^{-\frac{1}{2}}\Xi(\varpi)XX_i^{-1})$$

if ν is split over E.

Proof. The proof follows the same argument used in [GPR97, § 2, Lemma 2.1], which uses the Satake Isomorphism for F-quasisplit classical groups and the definition of the support ideal $\mathcal{I}_{supp}(\chi_s)$. We omit the details here. □

Next, we deal with the general case with $j=\ell+1$ for the relation between $H_{n-\ell}$ and $G_{n-\ell}$.

If ν is inert over E, by Proposition 4.2, we have the exact sequence of $H_{n-\ell}(E)$ modules for $j=\ell+1$,

$$0\to \mathrm{ind}_{P_1''}^{H_{n-\ell}}|\cdot|_E^{s-\frac{\ell}{2}}\tau_{(\ell)}\otimes\sigma^{w_q^\ell}\to \Pi_{\epsilon_{0,1}\eta_{\epsilon,I_{m-2\ell}}}\to\mathcal{Y}\to 0$$

where

$$\mathcal{Y}:=\mathrm{ind}_{P_1'}^{H_{n-\ell}}|\cdot|_E^{\frac{1-\ell}{2}+s}\tau^{(\ell)}\otimes J_{\psi'_{0,\kappa}}(\sigma^{w_q^\ell}),$$

and $\Pi_{\epsilon_{0,1}\eta_{\epsilon,I_{m-2\ell}}}$ is the smooth representation of $H_{n-\ell}$ consisting of functions in $\pi(\chi_s)$ which are supported in $P_j\epsilon_{0,1}\eta_{\epsilon,I_{m-2\ell}}N_\ell G_{n-\ell}$. Recall that $\tau^{(\ell)}$ is the ℓ-th Bernstein-Zelevinsky derivative of τ, which is a representation of $\mathrm{GL}_1(E)$. Up to semi-simplification,

$$\tau^{(\ell)}=\oplus_{i=1}^j\chi_i\otimes|\cdot|_E^{\frac{\ell}{2}},$$

and then $\mathcal{Y}\equiv\oplus_{i=1}^j\mathrm{ind}_{P_1'}^{H_{n-\ell}}|\det|_E^{\frac{1}{2}+s}\chi_i\otimes J_{\psi'_{0,\kappa}}(\sigma^{w_q^\ell})$.

If ν is split, we apply Proposition 4.3 repeatedly and obtain the exact sequence

$$0\to\mathrm{ind}_{\mathrm{GL}_{m-2j}}^{\mathrm{GL}_{m-2\ell-1}}\sigma_{(0)}\to\Omega\to\mathcal{U}\to 0,$$

where $\mathcal{U}$ is defined to be the following representation

$$\mathrm{Ind}_{P_{1,m-2j}}^{\mathrm{GL}_{m-2\ell-1}}(|\det|^{\frac{1-\ell}{2}+s}(\tau_1)^{(\ell)}\otimes\sigma)\oplus\mathrm{Ind}_{P_{1,m-2j}}^{\mathrm{GL}_{m-2\ell-1}}(\sigma\otimes|\det|^{\frac{\ell-1}{2}-s}(\tau_2^*)^{[\ell]})$$

and Ω is defined in (4.11) consisting of functions supported in the first greatest orbits. In this case, we have, up to semi-simplification,

$$\tau_1^{(\ell)}=\oplus_{i=1}^j\Theta_i\otimes|\cdot|^{\frac{\ell}{2}}\text{ and }(\tau_2^*)^{[\ell]}=\oplus_{i=1}^j\Xi_i^{-1}\otimes|\cdot|^{-\frac{\ell}{2}}.$$

Note that $\Phi\in\mathcal{I}_{supp}(\chi_s)$ if and only if Φ annihilates all the boundary components of $J_{\psi_{\ell,\kappa}}(\Pi)$, that is, all the summands in Proposition 4.2 and Proposition 4.3 except the space $\mathrm{ind}_{P_{1,\ell}'}^{H_{n-\ell}}(|\det|_E^{-\frac{\ell}{2}+s}\tau_{(\ell)}\otimes\sigma^{w_b^\ell})$ and the space $\mathrm{ind}_{\mathrm{GL}_{m-2j}}^{\mathrm{GL}_{m-2\ell-1}}\sigma$, respectively. It is sufficient to annihilate the quotients in $\Pi_{\epsilon_{0,1}}$ and Ω.

In order to annihilate K_H-fixed vectors in the space

$$\oplus_{i=1}^j\mathrm{ind}_{P_1'}^{H_{n-\ell}}(|\det|_E^{\frac{1}{2}+s}\chi_i\otimes J_{\psi'_{0,\kappa}}(\sigma^{w_b^\ell}))$$

if ν is inert, and in the space

$$\oplus_{i=1}^j\mathrm{Ind}_{P_{1,m-2j}}^{\mathrm{GL}_{m-2\ell-1}}(|\cdot|^{\frac{1}{2}+s}\Theta_i\oplus|\cdot|^{-\frac{1}{2}-s}\Xi_i^{-1})\otimes\sigma$$

(up to isomorphism) if ν is split, as in Lemma 4.4, we may take the following specific element in $\mathcal{A}_{H_{n-\ell}}$,

$$\Phi_0 = \begin{cases} \prod_{i=1}^{j}\prod_{i'=1}^{\tilde{m}_H}\left(1-q_E^{-\frac{1}{2}}\chi_i XX_{i'}\right)\left(1-q_E^{-\frac{1}{2}}\chi_i XX_{i'}^{-1}\right), \text{ if } \nu \text{ is inert,} \\ \prod_{i=1}^{j}\prod_{i'=1}^{m-2\ell-1}\left(1-q_E^{-\frac{1}{2}}\Theta_i XX_{i'}\right)\left(1-q_E^{-\frac{1}{2}}\Xi_i XX_{i'}^{-1}\right), \text{ if } \nu \text{ is split,} \end{cases} \tag{4.12}$$

which is an element of the support ideal $\mathcal{I}_{supp}(\chi_s)$. In addition, all the other boundary components of the Jacquet module $J_{\psi_{\ell,\kappa}}(\chi_s)$ are of form

$$\mathrm{ind}_{P'_\beta}^{H_{n-\ell}}\left(|\det|_E^{\frac{1-t}{2}+s}\tau^{(t)}\otimes\sigma'\right)$$

if ν is inert, and of form

$$\mathrm{ind}_{P_{\beta,m-2\ell-1-\beta}}^{\mathrm{GL}_{m-2\ell-1}}\left(|\det|^{\frac{1-t}{2}+s}\Xi_\tau\otimes\sigma'\right)$$

or

$$\mathrm{ind}_{P_{m-2\ell-1-\beta,\beta}}^{\mathrm{GL}_{m-2\ell-1}}\left(\sigma'\otimes|\det|^{-\frac{1-t}{2}-s}\Theta_\tau\right)$$

if ν is split, where σ' is a suitable representation independent of s, more details of which can be found in Proposition 4.2 and Proposition 4.3. It is easy to check that Φ_0 also annihilates K_H-fixed vectors in those boundary components. In addition, we specify the action of the generators X_i such that the adjoint operator Φ_0^* across in the zeta integral $\mathcal{Z}_\nu(s, f_\chi * \Phi_0, f_\mu, \psi_{\ell,\kappa})$ acts on f_μ with the property that

$$f_\mu * \Phi_0^* = Q(\chi_s,\mu)f_\mu,$$

where

$$Q(\chi_s,\mu) = \prod_{i=1}^{j}\prod_{i'=1}^{\tilde{m}_H}\left(1-q_E^{-\frac{1}{2}-s}\chi_i\mu_{i'}\right)\left(1-q_E^{-\frac{1}{2}-s}\chi_i\mu_{i'}^{-1}\right)$$

if ν is inert; and

$$Q(\chi_s,\mu) = \prod_{i=1}^{j}\prod_{i'=1}^{m-2\ell-1}\left(1-q_E^{-\frac{1}{2}-s}\Theta_i\mu_{i'}\right)\left(1-q_E^{-\frac{1}{2}-s}\Xi_i\mu_{i'}^{-1}\right)$$

if ν is split.

Proposition 4.5. *With $\Phi_0 \in \mathcal{I}_{supp}(\chi_s)$ as chosen above, the following identity holds:*

$$\mathcal{Z}_\nu(s, f_\chi * \Phi_0, f_\mu, \psi_{\ell,\kappa}) = Q(\chi_s,\mu)\cdot\mathcal{Z}_\nu(s, f_\chi, f_\mu, \psi_{\ell,\kappa}). \tag{4.13}$$

Moreover, $\mathcal{Z}_\nu(s, f_{\chi_s} * \Phi_0, f_\mu, \psi_{\ell,\kappa})$ is a polynomial function in q_E^{-s} of parameters χ_τ and χ_σ.

Proof. The proof is similar to the proof of Theorem 5.1 in [GPR97]. In fact, $\mathcal{J}(f_{\chi_s} * \Phi_0)(h)$, as defined in Section 4.1, belongs to the space Λ, which is independent of the choice of σ. Also $\mathcal{J}(f_{\chi_s} * \Phi_0)(h)$ is analytic in s because of the support of $\mathcal{J}(f_{\chi_s} * \Phi_0)$. The local zeta integral is equal to the pairing the function $\mathcal{J}(f_{\chi_s} * \Phi_0)(h)$ with a Bessel function as in (4.3), and is absolutely convergent for all s. Hence the zeta function $\mathcal{Z}_\nu(s, f_{\chi_s} * \Phi_0, f_\mu, \psi_{\ell,\kappa})$ is a polynomial function of q_E^s and q_E^{-s} for all choice of π and all s. □

We remark that the proof of this proposition only uses the genericity of τ, which is true because τ is the unramified local component of the corresponding irreducible automorphic representation of $\mathrm{GL}_j(\mathbb{A}_E)$ as given in (2.13). Hence it holds for all choices χ_σ and μ, and therefore, for all irreducible unramified representations σ and π.

By the definition of the unramified local tensor product L-functions as in (4.5) and (4.6), and Proposition 4.5, one must have the following identity:

$$Q(\chi_s, \mu) = \begin{cases} L^{-1}(\frac{1}{2} + s, \tau \times \pi) d(\chi_\tau, s) & \text{if } \nu \text{ is inert,} \\ L^{-1}(\frac{1}{2} + s, \tau \times \pi) & \text{if } \nu \text{ is split,} \end{cases}$$

where

$$d(\chi_\tau, s) = \begin{cases} \prod_{i=1}^{j} (1 - q_E^{-\frac{1}{2} - s} \chi_i)^{-1} & \text{if } m \text{ is even and } \nu \text{ is inert;} \\ 1 & \text{otherwise.} \end{cases}$$

Note that $d(\chi_\tau, s) = L(s + \frac{1}{2}, \tau)$. Let $d(\chi_i, s) = (1 - q_E^{-\frac{1}{2} - s} \chi_i)^{-1}$. Thus, based on the calculation of $Q(\chi_s, \mu)$, the numerator $Q \cdot \mathcal{Z}_\nu$ is determined and we have a unique choice of $P(\chi_s, \mu)$.

4.5 Calculation of $P(\chi_s, \mu)$. In this section, we will first calculate the numerator $P(\chi_s, \mu)$ when Π and π are *spherical* and *generic*, and then extend the results to general case by *Density Principle* in Appendix IV to [GPR97, Section 5].

We define a linear functional in the Hom space (4.2),

$$T(f_{\chi_s}, f_\mu) := \int_{H_{n-\ell}(F)} \int_{N_\ell(F)} f_{\chi_s}(\epsilon_{0,1} \eta n m) f_\mu(m) \psi_{\ell,\kappa}(n) \, \mathrm{d}n \, \mathrm{d}m.$$

Note that properties of T are studied in [Kho08] when ν is inert.

Lemma 4.6.

$$\mathcal{Z}_\nu(s, f_{\chi_s}, f_\mu, \psi_{\ell,\kappa}) = T(f_{\chi_s}, f_\mu).$$

Proof. For all unramified places, the proof is similar to the orthogonal case as Theorem (A) in Appendix I to [GPR97, Chapter 5]. □

Case $\ell = 0$**:** First of all, we consider the case $\ell = 0$, and hence $j = 1$. The Bessel period is also studied by Gan et al. in [GGP12]. Referring to [Har12, Proposition 2.5], for any quasi-character χ_1, we have the following inductive formula.

LEMMA 4.7. *If ν is inert, then*

$$T(f_{\chi_1\otimes\chi_\sigma}, f_\mu) = \frac{L(\frac{1}{2}, \chi_1 \times \pi)}{L(1, \chi_1 \times \sigma)L(1, \xi_{E/F}^m \otimes \chi_1)} T(f_\mu, f_{\chi_\sigma});$$

and if ν is split, then

$$T(f_{\chi_1\otimes\chi_\sigma}, f_\mu) = \frac{L(\frac{1}{2}, \Theta_1 \times \pi)L(\frac{1}{2}, \Xi_1 \times \tilde{\pi})}{L(1, \Theta_1 \times \tilde{\sigma})L(1, \Xi_1 \times \sigma)L(1, \Theta_1\Xi_1)} T(f_\mu, f_{\chi_\sigma}).$$

Note that $T(f_\mu, f_{\chi_\sigma})$ is a bilinear form on $\pi(\mu)$ and σ. Correspondingly, one has

$$Q(\chi_1 \otimes \chi_\sigma, \mu) = \begin{cases} [L\left(\frac{1}{2}, \chi_1 \times \pi\right) d(\chi_1)]^{-1} & \text{if } \nu \text{ is inert,} \\ [L(\frac{1}{2}, \Theta_1 \times \pi)L(\frac{1}{2}, \Xi_1 \times \tilde{\pi})]^{-1} & \text{if } \nu \text{ is split.} \end{cases}$$

which is the same as the result of Proposition 4.5. Hence one has

$$P(\chi_1 \otimes \chi_\sigma, \mu) = \frac{d(\chi_1)}{L(1, \chi_1 \times \sigma)L(1, \chi_1 \otimes \xi_{E/F}^m)} T(\mu, \chi_\sigma), \tag{4.14}$$

where $T(f_\mu, f_{\chi_\sigma})$ is simply denoted by $T(\mu, \chi_\sigma)$. Note that $P(\chi_1 \otimes \chi_\sigma, \mu)$ is a polynomial function of the parameter χ_1, and

$$L(1, \chi_1 \otimes \xi_{E/F}^m) = L(1, \Theta_1 \otimes \Xi_1).$$

A comment on the notation χ_1 is in order. The above discussion holds for all quasi-characters χ_1 and hence the variable s is carried by this character χ_1 here.

General Case $\ell > 0$**:** In the discussion below, for technical reasons we also assume that χ is a general quasi-character, i.e. we take χ to be χ_s here, since the proof works for any quasi-character χ. Hence in the discussion, there will be no variable s. However, the variable s will be put back into the final formula.

Let ω be an element of Weyl group $W(H_{n-\ell})$ and I_ω be the intertwining operator mapping $\Pi(\chi)$ to $\Pi(\omega\chi)$. By the uniqueness of Bessel model, we have a local gamma factor $\gamma_\omega(\chi, \gamma)$ defined by

$$T(I_\omega(f_\chi), f_\mu) = \gamma_\omega(\chi, \mu) T(f_\chi, f_\mu).$$

In order to calculate $T(f_\chi, f_\mu)$ in the general case $\ell > 0$, we need to calculate the local gamma factor γ_ω.

When ν is inert, let $\{\beta_i \mid 1 \leq i \leq \tilde{m}\}$ be a set of simple roots of G_n. Then the sets $\{\beta_i \mid 1 \leq i \leq \ell\}$ and $\{\beta_i \mid \ell + 2 \leq i \leq \tilde{m}\}$ are also sets of simple roots of $\mathrm{GL}_{\ell+1}(E)$ and $H(W_{\ell+1})$ respectively.

When ν is split, let $\{\beta_i \mid 1 \leq i \leq m-1\}$ be a set of simple roots of GL_m. Recall that $P_{\ell+1,m-2\ell-2,\ell+1}$ is a standard parabolic subgroup of GL_m with the Levi subgroup $\mathrm{GL}_{\ell+1} \times \mathrm{GL}_{m-2\ell-2} \times \mathrm{GL}_{\ell+1}$. Then the set $\{\beta_i \mid 1 \leq i \leq \ell\}$ and $\{\beta_i \mid m-\ell \leq i \leq m-1\}$ are sets of simple roots of the general linear groups of the Levi subgroup, and $\{\beta_i \mid \ell+2 \leq i \leq m-\ell-2\}$ is the set of simple roots of the subgroup $\mathrm{GL}_{m-2\ell-2}$ of the Levi subgroup. Let ω_i be the simple reflection corresponding to the simple root β_i.

Lemma 4.8. *If ν is inert, then*

$$\gamma_{\omega_i}(\chi,\mu) = \begin{cases} \frac{1-\chi_{i+1}\chi_i^{-1}q_E^{-1}}{1-\chi_i\chi_{i+1}^{-1}} & \text{if } 1 \leq i \leq \ell, \\ \gamma_{\omega_i}(\chi_{\ell+1} \otimes \chi_\sigma, \mu) & \text{if } \ell+1 \leq i \leq \tilde{m}. \end{cases}$$

If ν is split, then the gamma factor $\gamma_{\omega_i}(\chi,\mu)$ is equal to

$$\begin{cases} \frac{1-\chi_{i+1}\chi_i^{-1}q_E^{-1}}{1-\chi_i\chi_{i+1}^{-1}} & \text{if } 1 \leq i \leq \ell \text{ or } m-\ell \leq i \leq m, \\ \gamma_{\omega_i}(\chi_{\ell+1} \otimes \chi_\sigma \otimes \chi_{m-\ell}, \mu) & \text{if } \ell+1 \leq i \leq m-\ell-1. \end{cases}$$

Proof. Khoury proved the inert case in [Kho08, Proposition 11.1]. For the split case, the proof is given in [Zha12]. □

Now, let

$$P^*(\chi,\mu) = \frac{\zeta(\chi,1)T(\mu,\chi_\sigma)}{P(\chi,\mu)}.$$

Note that T is a bilinear form on $\pi(\mu)$ and σ, which means that

$$T \in \mathrm{Hom}_{G_{n-j}^{\triangle}(F)}(\pi(\mu)\otimes\sigma,\mathbb{C}).$$

Following [CS80] and [Sha10, Section 3.5], the functions $\zeta(\chi,t)$ can be defined as follows. Write $q=q_E$ to simplify the notation. When ν is inert, if m is even, $\zeta(\chi,t)$ is defined by

$$\prod_{1\leq i_1<i_2\leq\tilde{m}}(1-\chi_{i_1}\chi_{i_2}^{-1}q^{-t})(1-\chi_{i_1}\chi_{i_2}q^{-t})\cdot\prod_{1\leq i\leq\tilde{m}}(1-\chi_i q_F^{-t});$$

and if m is odd, $\zeta(\chi,t)$ is defined by

$$\prod_{1\leq i_1<i_2\leq\tilde{m}}(1-\chi_{i_1}\chi_{i_2}^{-1}q^{-t})(1-\chi_{i_1}\chi_{i_2}q^{-t})\cdot\prod_{1\leq i\leq\tilde{m}}(1+\chi_i q_F^{-t})(1-\chi_i q^{-t}).$$

When ν is split, $\zeta(\chi,t)=\prod_{1\leq i_1<i_2\leq m}(1-\chi_{i_1}\chi_{i_2}^{-1}q^{-t})$. In addition, if $\tilde{m}=1$, $\zeta(\chi,t)=1$ for all cases. We remark that $\zeta(\chi,t)$ is the zeta polynomial function associated to G_n as in [GPR97, Page 157].

For the case $\ell=0$, according to (4.14), we have

$$P^*(\chi_1\otimes\chi_\sigma,\mu) = \frac{\zeta(\chi_\sigma,1)}{d(\chi_1)}, \tag{4.15}$$

where $\zeta(\chi_\sigma,1)$ is the zeta polynomial function associated to H_n, as in [GPR97, Page 157].

Corollary 4.9. *If $1 \le i \le \ell$, or $m-\ell \le i \le m-1$ when ν is split, then*

$$P^*(\omega_i\chi,\mu) = P^*(\chi,\mu).$$

If $i = \ell+1$ when ν is inert, or $i = \ell+1$ or $m-\ell-1$ when ν is split, then

$$\frac{P^*(\chi,\mu)}{P^*(\omega_i\chi,\mu)} = \frac{\zeta(\chi_\sigma,1)d(\chi_i)}{\zeta(\chi_{\sigma'},1)d(\chi_{i+1})}.$$

where $\chi_{\sigma'} = \chi_{\ell+1}\otimes\chi_{\ell+3}\otimes\cdots\otimes\chi_{\tilde m}$ when $i=\ell_1$ and ν is inert, and $\chi_{\sigma'} = \chi_{\ell+1}\otimes \chi_{\ell+3}\otimes\cdots\otimes\chi_{m-\ell-1}$ when $i=\ell$ and ν is split, and $\chi_{\sigma'} = \chi_{\ell+2}\otimes\cdots\otimes\chi_{m-\ell-2}\otimes\chi_{m-\ell}$.

If $\ell+1 < i \le \tilde m$ when ν is inert or $\ell+2 \le i \le m-\ell-2$ when ν is split, then

$$\frac{P^*(\chi,\mu)}{P^*(\omega_i\chi,\mu)} = \frac{\zeta(\chi_\sigma,1)}{\zeta((\omega\chi)_\sigma,1)}.$$

Proof. The proof is a straightforward calculation by Lemma 4.8. □

By Corollary 4.9,

$$\frac{P^*(\chi,\mu)d(\chi_\tau)}{\zeta(\chi_\sigma,1)}$$

is invariant under the action of the Weyl group $W(G_n)$ on χ. In the rest of this section, we will show that the quotient above is equal to one, i.e.

$$P^*(\chi,\mu) = \frac{\zeta(\chi_\sigma,1)}{d(\chi_\tau)}. \tag{4.16}$$

Let $T_0(\chi,\mu) = T(f^0_\chi, f_\mu)$, with

$$f^0_\chi(g) := \int_B 1_{B(\mathfrak{o})\omega_{G_n}B(\mathfrak{o})}(bg)\chi^{-1}\delta_B^{\frac12}(b)\,\mathrm{d}_l b,$$

where ω_{G_n} is the longest Weyl element in G_n, and $1_{B(\mathfrak{o})\omega_{G_n}B(\mathfrak{o})}$ is the characteristic function of $B(\mathfrak{o})\omega_{G_n}B(\mathfrak{o})$ and also is an Iwahori-fixed function. Denote by $\chi|_{H_{n-\ell}}$ the restriction character of χ into the torus of $H_{n-\ell}$.

Lemma 4.10.

$$T_0(\chi,\mu) = T_0(\chi|_{H_{n-\ell}},\mu).$$

Proof. The proof is similar to Proposition 8.1 in [GPR97]. □

As in the Appendix to §6 in [GPR97], we have the following expansion,

$$T(\chi,\mu) = \sum_{\omega\in W(G_n)} \frac{\gamma_\omega(\omega^{-1}\chi,\mu)}{c_\omega(\omega^{-1}\chi)} c_{\omega_{G_n}}(\omega^{-1}\chi)T_0(\omega^{-1}\chi,\mu),$$

where $c_\omega(\omega^{-1}\chi)$ is the Harish-Chandra c-function of the intertwining operator associated to the Weyl group element ω. In this formula, by replacing $\gamma_\omega(\omega^{-1}\chi,\mu)$ by the following expression:

$$\gamma_\omega(\omega^{-1}\chi,\mu)=\frac{T(\chi,\mu)c_\omega(\omega^{-1}\chi)}{T(\omega^{-1}\chi,\mu)},$$

canceling the factor $T(\chi,\mu)$ from both sides, and replacing $T(\omega^{-1}\chi,\mu)$ by

$$T(\omega^{-1}\chi,\mu)=\frac{P(\omega^{-1}\chi,\mu)}{Q(\omega^{-1}\chi,\mu)},$$

we obtain the following expression:

$$\begin{aligned}1&=\sum_{\omega\in W(G_n)}\frac{Q(\omega^{-1}\chi,\mu)}{P(\omega^{-1}\chi,\mu)}c_{\omega_{G_n}}(\omega^{-1}\chi)T_0(\omega^{-1}\chi,\mu)\\&=\sum_{\omega\in W(G_n)}\frac{c_{\omega_{G_n}}(\omega^{-1}\chi)}{\zeta(\omega^{-1}\chi,1)}Q(\omega^{-1}\chi,\mu)P^*(\omega^{-1}\chi,\mu)\frac{T_0(\omega^{-1}\chi,\mu)}{T(\mu,(\omega^{-1}\chi)_\sigma)}.\end{aligned}$$

Define

$$\Delta(\chi)=q^{\langle\varrho,\chi\rangle}\zeta(\chi,0)=\prod_{i=1}^{\tilde m}\chi_i^{-(\frac{m+1}{2}-i)}\zeta(\chi,0),$$

where ϱ is the half of the sum of all positive roots. Then it follows that $\Delta(\omega\chi)=\operatorname{sgn}(\omega)\Delta(\chi)$. Note that $c_{\omega_{G_n}}(\chi)=\zeta(\chi,1)\zeta^{-1}(\chi,0)$. It follows that $\Delta(\chi)$ can be expressed as follows:

$$\begin{aligned}&\sum_{\omega\in W(G_n)}\operatorname{sgn}(\omega)q^{\langle\varrho,\omega^{-1}\chi\rangle}Q(\omega^{-1}\chi,\mu)P^*(\omega^{-1}\chi,\mu)\frac{T_0(\omega^{-1}\chi,\mu)}{T(\mu,(\omega^{-1}\chi)_\sigma)}\\&\quad=\frac{P^*(\chi,\mu)d(\chi_\tau)}{\zeta(\chi_\sigma,1)}\sum_{\omega\in W(G_n)}\operatorname{sgn}(\omega)q^{\langle\varrho,\omega\chi\rangle}Q(\omega\chi,\mu)\frac{T_0(\omega\chi,\mu)\zeta((\omega\chi)_\sigma,1)}{T(\mu,(\omega\chi)_\sigma)d((\omega\chi)_\tau)}.\end{aligned}\tag{4.17}$$

In order to prove Equation (4.16), it is sufficient to show the following Lemma, which is similar to the orthogonal case ([GPR97, Lemma 6.3]).

LEMMA 4.11.

$$\Delta(\chi)=\sum_{\omega\in W(G_n)}\operatorname{sgn}(\omega)q^{\langle\varrho,\omega\chi\rangle}Q(\omega\chi,\mu)\frac{T_0((\omega\chi)|_{H_{n-\ell}},\mu)\zeta((\omega\chi)_\sigma,1)}{T(\mu,(\omega\chi)_\sigma)d((\omega\chi)_\tau)}.\tag{4.18}$$

Proof. We only give a proof for the inert case. For the split case, the proof is similar and we omit details here.

First, by Equation (4.15), this identity holds for $\ell=0$.

Next, we consider the general cases $\ell > 0$. Since the terms $\zeta(\chi_\sigma, 1)$, $Q(\chi, \mu)$, $d(\chi_\tau)$, $T_0(\chi|_{H_{n-\ell}}, \mu)$ and $T(\mu, (\omega\chi)_\sigma)$ are invariant under the action of the Weyl group $W(\mathrm{GL}_{\ell+1})$, we have the right hand side of (4.18),

$$RHS = \sum_{\omega \in W(GL_\ell) \times W(G_{n-\ell}) \backslash W(G_n)} \Sigma_{\omega_1}(\omega) \cdot \Sigma_{\omega_2}(\omega) \cdot q^{\left\langle \varrho_{U_\ell}, \omega\chi \right\rangle} \mathrm{sgn}(\omega).$$

where

$$\Sigma_{\omega_1}(\omega) := \sum_{\omega_1 \in W(GL_\ell)} \mathrm{sgn}(\omega_1) q^{\left\langle \varrho_{\mathrm{GL}_\ell}, \omega_1\omega\chi \right\rangle},$$

and

$$\Sigma_{\omega_2}(\omega) := \sum_{\omega_2 \in W(G_{n-\ell})} \mathrm{sgn}(\omega_2) q^{\left\langle \varrho_{G_{n-\ell}}, \omega_2\omega\chi \right\rangle} Q(\omega_2\omega\chi, \mu) \cdot \frac{T_0((\omega_2\omega\chi)|_{H_{n-\ell}}, \mu)\zeta((\omega_2\omega\chi)_\sigma, 1)}{T(\mu, (\omega_2\omega\chi)_\sigma) d((\omega_2\omega\chi)_\tau)}.$$

Decompose as $Q(\chi, \mu) = Q_1(\chi, \mu) Q(\chi_{\ell+1} \otimes \chi_\sigma, \mu)$, where

$$Q_1(\chi, \mu) = \prod_{i=1}^{\ell} \prod_{i'=1}^{\tilde{m}_L} (1 - q^{-\frac{1}{2}} \chi_i \mu_{i'})(1 - q^{-\frac{1}{2}} \chi_i \mu_{i'}^{-1})$$

and

$$Q(\chi_{\ell+1} \otimes \chi_\sigma, \mu) = \prod_{i'=1}^{\tilde{m}_L} (1 - q^{-\frac{1}{2}} \chi_{\ell+1} \mu_{i'})(1 - q^{-\frac{1}{2}} \chi_{\ell+1} \mu_{i'}^{-1}).$$

Thus, $Q(\omega_2\chi, \mu) = Q_1(\chi, \mu) Q(\omega_2(\chi_{\ell+1} \otimes \chi_\sigma), \mu)$ for $\omega_2 \in W(G_{n-\ell})$.

Define $\omega\chi = \chi^{(1)} \otimes \chi^{(2)}$ with $\chi^{(1)} = \omega\chi|_{\mathrm{GL}_\ell}$ and $\chi^{(2)} = \omega\chi|_{G_{n-\ell}}$. Note that

$$\left\langle \varrho_{H_{n-\ell}}, \omega_2\omega\chi \right\rangle = \left\langle \varrho_{H_{n-\ell}}, \omega_2\chi^{(2)} \right\rangle,$$
$$\zeta((\omega_2\omega\chi)_\sigma, 1) = \zeta((\omega_2\chi^{(2)})_\sigma, 1),$$

and

$$\begin{aligned} d((\omega_2\omega\chi)_\tau) &= d((\omega_2\chi^{(2)})_{\ell+1}) \prod_{i=1}^{\ell} (1 - q^{-1}\chi_i(\varpi_E)) \\ &= d((\omega_2\chi^{(2)})_{\ell+1}) d((\omega\chi)_\tau) d^{-1}((\omega\chi)_{\ell+1}). \end{aligned}$$

Consider the summation

$$\begin{aligned}\Sigma_{\omega_2}(\omega) =&Q_1(\omega\chi,\mu)\frac{d((\omega\chi)_{\ell+1})}{d((\omega\chi)_{\tau})}\\ &\cdot\sum_{\omega_2\in W(G_{n-\ell})}\operatorname{sgn}(\omega_2)q^{\left\langle \varrho_{G_{n-\ell}},\omega_2\chi^2\right\rangle}Q(\omega_2\chi^{(2)},\mu)\\ &\qquad\cdot\frac{T_0(\omega_2\chi^{(2)},\mu)\zeta((\omega_2\chi^{(2)})_\sigma,1)}{T(\mu,\omega_2\chi^{(2)})d((\omega_2\chi^{(2)})_{\ell+1})}\\ =&Q_1(\omega\chi,\mu)\frac{d((\omega\chi)_{\ell+1})}{d((\omega\chi)_{\tau})}\cdot\Delta_{H_{n-\ell}}(\chi^{(2)}).\end{aligned}$$

The last identity holds by the case $\ell=0$. Note that $\chi^{(2)}=\omega\chi|_{G_{n-\ell}}$.

Now, after replacing $\Sigma_{\omega_2}(\omega)$ by the expression above, the right hand side of (4.18) reduces to

$$\begin{aligned}RHS=&\sum_{\omega\in W(\mathrm{GL}_\ell)\times W(G_{n-\ell})\backslash W(G_n)}\Sigma_{\omega_1}(\omega)\\ &\cdot Q_1(\omega\chi,\mu)\frac{d((\omega\chi)_{\ell+1})}{d((\omega\chi)_{\tau})}\cdot\Delta_{H_{n-\ell}}(\chi^{(2)})\operatorname{sgn}(\omega)q^{\left\langle\varrho_{U_\ell},\omega\chi\right\rangle}.\end{aligned}$$

By using the definition of $\Sigma_{\omega_1}(\omega)$ and the definition of $\Delta_{H_{n-\ell}}(\chi^{(2)})$, and then by collapsing the three summations $\sum_\omega$, $\sum_{\omega_1}$ and $\sum_{\omega_2}$, we obtain that

$$RHS=\sum_{\omega\in W(G_n)}\operatorname{sgn}(\omega)q^{\langle\varrho,\omega\chi\rangle}Q_1(\omega\chi,\mu)\frac{d((\omega\chi)_{\ell+1})}{d((\omega\chi)_{\tau})}.$$

Recall that

$$Q_1(\chi,\mu)\frac{d((\chi)_{\ell+1})}{d((\chi)_{\tau})}=\prod_{i=1}^{\ell}\prod_{i'=1}^{\tilde m_H}(1-q^{-\frac12}\chi_i\mu_{i'})(1-q^{-\frac12}\chi_i\mu_{i'}^{-1}).$$

Then

$$\begin{aligned}RHS=&\sum_{\omega\in W(G_n)}\operatorname{sgn}(\omega)q^{\langle\varrho,\omega\chi\rangle}\\ &+\sum_{\vec n}c_{\vec n}\sum_{\omega\in W(G_n)}\operatorname{sgn}(\omega)q^{\langle\varrho,\omega\chi\rangle}\prod_{i=1}^{\ell}\chi_i^{n_i},\end{aligned}$$

where $\vec n=(n_1,n_2,\ldots,n_\ell)$ with $n_i\in\{0,1,2\}$ such that at least one n_i is nonzero, and $c_{\vec n}$ is the coefficient depending only on μ. Also note that

$$q^{\langle\varrho,\chi\rangle}\prod_{i=1}^{\ell}\chi_i^{n_i}=\prod_{i=1}^{\tilde m}\chi_i^{-(\frac{m+1}{2}-i-n_i)},$$

where $n_i = 0$ if $i > \ell$. Thus, it is sufficient to show that

$$\sum_{\omega \in W(G_n)} \mathrm{sgn}(\omega) q^{\langle \varrho, \omega\chi \rangle} \prod_{i=1}^{\ell} (\omega\chi)_i^{n_i} = 0.$$

Since $\sum_{i=1}^{\ell} n_i \neq 0$, $\ell > 0$ and $\tilde{m} - \ell - 1 \geq 1$, there exist at least two distinct integers i and i' with $i < i'$ such that

$$\frac{m+1}{2} - i - n_i = \frac{m+1}{2} - i' - n_{i'}.$$

Let i_0 be the maximal integer such that $i_0 + n_{i_0} = i_0' + n_{i_0'}$. Consider the Weyl group $W(G_n)$ as the subgroup of the permutation group on χ_i and χ_i^{-1} for $1 \leq i \leq \tilde{m}$. Then, define a Weyl element ω' by the following rules: ω' permutes χ_{i_0} and $\chi_{i_0'}$, and fixes χ_i for the rest i. It follows that $\mathrm{sgn}(\omega') = -1$ and ω' fixes $\prod_{i=1}^{\tilde{m}} \chi_i^{-(\frac{m+1}{2} - i - n_i)}$. Let $W(G_n)_{\vec{n}}$ be the stabilizer of $W(G_n)$ acting on $q^{\langle \varrho, \omega\chi \rangle} \prod_{i=1}^{\ell} \chi_i^{n_i}$. By the fact that $\mathrm{sgn}(\omega') = -1$ and $\omega' \in W(G_n)_{\vec{n}}$, we have the restriction of sgn on $W(G_n)_{\vec{n}}$ is not trivial. Therefore, we obtain

$$\begin{aligned} &\sum_{\omega \in W(G_n)} \mathrm{sgn}(\omega) q^{\langle \varrho, \omega\chi \rangle} \prod_{i}^{\ell} (\omega\chi)_i^{n_i} \\ &= \sum_{\omega \in W(G_n)} q^{\langle \varrho, \omega\chi \rangle} \prod_{i}^{\ell} (\omega\chi)_i^{n_i} \sum_{\omega' \in W(G_n)_{\vec{n}}} \mathrm{sgn}(\omega\omega') \\ &= 0. \end{aligned}$$

□

Comparing (4.17) and (4.18), we can get the identity (4.16). Hence, after replacing back χ_s for χ, we obtain the following formulas

$$P(\chi_s, \mu) = \frac{d((\chi_s)_\tau)\zeta((\chi_s)_\tau, 1)}{L(s+1, \tau \times \sigma) L(2s+1, \tau, Asai \otimes \xi_{E/F}^m)} T(\mu, \chi_\sigma) \tag{4.19}$$

if ν is inert, and

$$P(\chi_s, \mu) = \frac{\zeta((\chi_s)_{\tau_1}, 1)\zeta((\chi_s)_{\tau_2}, 1)}{L(s+1, \tau_1 \times \tilde{\sigma}) L(s+1, \tau_2 \times \sigma) L(2s+1, \tau_1 \times \tau_2)} \tag{4.20}$$

if ν is split. Note that $(\chi_s)_\tau$ denotes the quasi-character which is the restriction of the quasi-character χ_s to the torus of $\mathrm{Res}_{E/F}\mathrm{GL}_j$.

It is important to point out that from the beginning of this section up to this point, we assume that $\Pi(\chi_s)$ and $\pi(\mu)$ are generic and spherical. The following theorem extends the above results to general spherical $\Pi(\chi_s)$ and $\pi(\mu)$.

Theorem 4.12. *For all choices of χ and μ, the following identity holds:*

$$\begin{aligned}&\mathcal{Z}_\nu(s, f_\chi, f_\mu, \psi_{\ell,\kappa}) \\ &\quad= \frac{L(s+\frac{1}{2}, \tau\times\pi)}{L(s+1, \tau\times\sigma)L(2s+1, \tau, Asai\otimes\xi^m_{E/F})}\left\langle f_\mu, f_\sigma\right\rangle_\sigma \zeta(\chi_\tau, 1),\end{aligned} \tag{4.21}$$

where $\left\langle f_\mu, f_\sigma\right\rangle_\sigma$ and $\zeta(\chi_\tau, 1)$ are independent of s. Moreover, if we normalize W_j so that $W_j(f_\chi)(e) = 1$, then

$$\mathcal{Z}_\nu(s, f_\chi, f_\mu, \psi_{\ell,\kappa}) = \frac{L(s+\frac{1}{2}, \tau\times\pi)}{L(s+1, \tau\times\sigma)L(2s+1, \tau, Asai\otimes\xi^m_{E/F})}\left\langle f_\mu, f_\sigma\right\rangle_\sigma, \tag{4.22}$$

Proof. This proof is similar to Theorem 5.2 in [GPR97]. By Proposition 4.5, it is sufficient to show that

$$\mathcal{Z}_\nu(s, f_\chi * \Phi_0, f_\mu, \psi_{\ell,\kappa}) = P(\chi, \mu)$$

holds for all choices of χ and μ.

Define

$$f^*\left(\begin{pmatrix} g & & \\ & h & \\ & & g^*\end{pmatrix} u\epsilon_{0,1}\eta n k\right) = f_\tau(g)|\det g|_E^s \delta_{P_j}^{\frac{1}{2}} f_\sigma(h),$$

where $g \in \mathrm{Res}_{E/F}(\mathrm{GL}_j)$, $h \in G_{n-j}$, $u \in U_j$, $n \in U_\ell(\mathfrak{o})$ and $k \in K_H$. Recall that f_τ and f_σ are the unramified spherical vectors in τ and σ. In addition, we assume that $\mathrm{supp}(f^*) = P_j\epsilon_{0,1}\eta R_\ell(\mathfrak{o})$. Then f^* is in Λ and $\mathrm{supp}(f^*) \subseteq G_{n-j}K_H$. Since $\mathcal{J}(f^*)(e) = W_j(f^*)(\epsilon_{0,1}\eta) = \zeta(\chi_\tau, 1)f_\sigma$, we obtain

$$\mathcal{Z}(s, f^*, f_\mu, \psi_{\ell,\kappa}) = \zeta(\chi_\tau, 1)\left\langle f_\mu, f_\sigma\right\rangle_\sigma.$$

Define

$$f^\sharp = f_\chi * \Phi_0 - \frac{d(\chi_\tau)}{L(s+1, \tau_\nu\times\sigma_\nu)L(2s+1, \tau_\nu, Asai\otimes\xi^m_{E/F})} f^*.$$

By (4.19) and (4.20), if χ and μ are in general position and s is in a dense open set, then

$$\mathcal{Z}_\nu(s, f^\sharp, f_\mu, \psi_{\ell,\kappa}) = 0.$$

By the same argument, one can extend the *Density Principle* in Appendix IV to [GPR97, Section 5] to the unitary group case, which implies that $\mathcal{J}(f^\sharp)(g) = 0$ for all choices of χ, μ and s. Therefore, we obtain the following identity

$$\mathcal{Z}_\nu(s, f_\chi * \Phi_0, f_\mu, \psi_{\ell,\kappa}) = \mathcal{Z}_\nu(s, f^*, f_\mu, \psi_{\ell,\kappa}) = P(\chi, \mu),$$

for all choices of χ, μ and s. □

This completes the proof of Theorem 3.9, which is the key result for unramified local zeta integrals. With Theorems 3.8 and 4.12, we have the following main global result of this paper for $j = \ell + 1$. In this case, (H_{n-j+1}, G_{n-j}) is a spherical pair, and the Bessel period $\mathcal{P}^{\psi_{\beta^{-1},y_{-\kappa}}^{-1}}(\varphi_\pi, \varphi_\sigma)$ reduces to a spherical Bessel period.

Theorem 4.13 (Main). *Assume that $j = \ell + 1$. Let $E(\phi_{\tau\otimes\sigma}, s)$ be the Eisenstein series on $G_n(\mathbb{A})$ as in (2.15) and let π be an irreducible cuspidal automorphic representation of $H_{n-\ell}(\mathbb{A})$. Assume that the real part of s, $\Re(s)$, is large, and that π and σ have a non-zero spherical Bessel period. Then the global zeta integral $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ is eulerian, and is equal to*

$$c_{\pi,\sigma}\mathcal{Z}_S(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})\frac{L^S(s+\frac{1}{2}, \pi\times\tau)}{L^S(s+1, \sigma\times\tau)L^S(2s+1, \tau, Asai\otimes\xi^m_{E/F})},$$

where $c_{\pi,\sigma}$ is a constant depending on the Bessel period of π and σ and on other normalization constants, but independent of s, and

$$\mathcal{Z}_S(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) = \prod_{v\in S}\mathcal{Z}_v(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$$

is the finite product of ramified local zeta integrals.

It is clear that Theorem 4.13 extends the main result of [GPR97] to the generality considered in this paper. Note that when π is an irreducible cuspidal automorphic representation of $\mathrm{SO}_{2(n-\ell)+1}(\mathbb{A})$, one has to replace the complex representation $Asai\otimes\xi^m_{E/F}$ by the corresponding exterior square representation $\wedge^2$; and when π is an irreducible cuspidal automorphic representation of $\mathrm{SO}_{2(n-\ell)}(\mathbb{A})$, one has to replace the complex representation $Asai\otimes\xi^m_{E/F}$ by the corresponding symmetric square representation Sym^2.

There is a standard method to prove from this global identity that the partial L-functions $L^S(s+\frac{1}{2}, \pi\times\tau)$ has meromorphic continuation to the whole complex plane. It is more important to develop the local theory which extends the partial L-function to the complete L-function in this setting and hence to prove the functional equation and other analytic properties of the complete L-functions of this type. This is our on-going project and will be reported in our future work.

5 Final Remark

We expect that Theorem 4.13 holds if one replaces the single variable s by a multi-variable $(s_1, s_2, \ldots, s_r)$. This means that one replaces the representation τ by an isobaric sum of generic representations. Hence the resulting global zeta integral represents the following product of tensor product L-functions

$$L^S(s_1, \pi\times\tau_1)L^S(s_2, \pi\times\tau_2)\cdots L^S(s_r, \pi\times\tau_r).$$

We will come back to this in our future work.

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