The Jacquet–Langlands Correspondence via Twisted Descent

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The existence of the well-known Jacquet–Langlands correspondence was established by
Jacquet and Langlands via the trace formula method in 1970 [13]. An explicit construc-
tion of such a correspondence was obtained by Shimizu via theta series in 1972 [30]. In
this paper, we extend the automorphic descent method of Ginzburg–Rallis–Soudry [10]
to a new setting. As a consequence, we recover the classical Jacquet–Langlands corre-
spondence for PGL(2) via a new explicit construction.

1 Introduction

The classical Jacquet–Langlands correspondence between automorphic forms on GL(2)
and $D^{\times}$, with a quaternion division algebra D, is one of the first established instances of
Langlands functorial transfers. The existence of such a correspondence was established
by Jacquet and Langlands via the trace formula method in 1970 [13]. An explicit con-
struction of the Jacquet–Langlands correspondence was obtained by Shimizu via theta
series in 1972 [30]. Shimizu's construction was extended in the general framework of

theta correspondences for reductive dual pairs in the sense of Howe [12], and has been important to many arithmetic applications, including the famous Waldspurger formula for the central value of the standard L-function of GL(2) [33], and the work of Harris and Kudla on Jacquet's conjecture for the central value of the triple product L-function [11], for instance.

The automorphic descent method developed by Ginzburg *et al.* in [10] constructs a map which is backward to the Langlands functorial transfer from quasi-split classical groups to the general linear group. The starting point of their method is the symmetry of the irreducible cuspidal automorphic representation on GL(n). However, their theory is not able to cover the classical groups which are not quasi-split. In this paper, we extend their method by considering more invariance properties of irreducible cuspidal auto-morphic representations of GL(n), so that this extended descent is able to reach certain classical groups which are not quasi-split. Owing to the nature of the newly introduced invariance property and the new setting of the construction, we call the method intro-duced below *twisted automorphic descent*.

Let τ be an irreducible unitary cuspidal automorphic representation of GL$_{2n}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of a number field F. Assume that the partial exterior square L-function $L^S(s, \tau, \wedge^2)$ has a simple pole at $s = 1$. It is well known now that τ is the Lang-lands functorial transfer from an irreducible generic cuspidal automorphic representa-tion π_0 of the F-split odd special orthogonal group $G_0(\mathbb{A}) = \mathrm{SO}_{2n+1}(\mathbb{A})$. The automorphic descent method of Ginzburg *et al.* [10] and the irreducibility of the descent [19] show that this π_0 is unique and can be explicitly constructed by means of a certain Bessel–Fourier coefficient of the residual representation of $\mathrm{SO}_{4n}(\mathbb{A})$ with cuspidal datum (GL$_{2n}, \tau$).

The objective of this paper is to extend the descent method of [10] to construct more general cuspidal representations and more general groups which are pure inner forms of the F-split odd special orthogonal group $G_0(\mathbb{A})$. To this end, we take σ to be an irreducible cuspidal automorphic representation of an F-anisotropic SO_2^δ asso-ciated with a non-square class δ of $F^\times$, and assume that the central value $L(\frac{1}{2}, \tau \times \sigma)$ is non-zero. The main idea is to make conditions such as $L(\frac{1}{2}, \tau \times \sigma) \neq 0$ into play in the construction of more general cuspidal automorphic representations of classical groups using the irreducible unitary cuspidal automorphic representation τ of GL and the irre-ducible cuspidal representation σ of SO_2^δ. We refer the reader to [22] for a more general framework of such constructions, which are technically much more involved than the current case in this paper. Hence, we leave to [22] the detailed discussions for general construction. We give below a more detailed description of the construction and the main results of the paper.

1.1   A residual representation of $\mathrm{SO}_{4n+2}^{\delta}(\mathbb{A})$

Let V be a quadratic space of dimension $4n+2$ defined over F with a non-degenerate quadratic form $\langle \cdot, \cdot \rangle$. We assume that the Witt index of V is $2n$ with a polar decomposition

$$V = V^+ \oplus V_0 \oplus V^-,$$

where V^+ is a maximal isotropic subspace of V, and V_0 is an anisotropic subspace of dimension 2. We may take the quadratic form on V_0 to be associated with $J_\delta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix} \right)$, where $-\delta \notin F^{\times 2}$, and the quadratic form of V is taken to be associated with

$$\begin{pmatrix} & & w_{2n} \\ & J_\delta & \\ w_{2n} & & \end{pmatrix},$$

where w_r is the $(r \times r)$-matrix with 1's on its anti-diagonal and zero elsewhere. Denote by $H^\delta = \mathrm{SO}_{4n+2}^{\delta}$ the corresponding F-quasi-split special even orthogonal group. We fix a maximal flag

$$\mathcal{F} : 0 \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_{2n}^+ = V^+$$

in V^+, and choose a basis $\{e_1, \ldots, e_{2n}\}$ of V^+ over F such that $V_i^+ = \mathrm{Span}\{e_1, \ldots, e_i\}$. We also let $\{e_{-1}, \ldots, e_{-2n}\}$ be a basis for V^-, which is dual to $\{e_1, \ldots, e_{2n}\}$ in the sense that

$$\langle e_i, e_{-j} \rangle = \delta_{i,j} \text{ for } 1 \le i, j \le 2n,$$

and let $V_\ell^- = \mathrm{Span}\{e_{-1}, \ldots, e_{-\ell}\}$.

Let P be the parabolic subgroup fixing V^+. Then P has a Levi decomposition $P = MU$ such that $M \simeq \mathrm{GL}_{2n} \times \mathrm{SO}_2^\delta$, where SO_2^δ is the F-quasi-split special orthogonal group of (V_0, J_δ). Let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$, and σ be an irreducible unitary (cuspidal) representation of $\mathrm{SO}_2^\delta(\mathbb{A})$. Then $\tau \otimes \sigma$ is an irreducible unitary cuspidal representation of $M(\mathbb{A})$. For $s \in \mathbb{C}$ and an automorphic function

$$\phi_{\tau \otimes \sigma} \in \mathcal{A} \left(M(F) U(\mathbb{A}) \backslash H^\delta(\mathbb{A}) \right)_{\tau \otimes \sigma},$$

following [26, Section II.1], one defines $\lambda_s \phi_{\tau \otimes \sigma}$ to be $(\lambda_s \circ m_P) \phi_{\tau \otimes \sigma}$, where $\lambda_s \in X_M^{H^\delta} \simeq \mathbb{C}$ (see [26, Section I.1] for the definition of $X_M^{H^\delta}$ and the map m_P), and defines the corresponding Eisenstein series by

$$E(h, s, \phi_{\tau \otimes \sigma}) = \sum_{\gamma \in P(F) \backslash H^\delta(F)} \lambda_s \phi_{\tau \otimes \sigma}(\gamma h),$$

which converges absolutely for $\text{Re}(s) \gg 0$ and has meromorphic continuation to the whole complex plane [26, Section IV]. We note here that under the normalization of Shahidi, we can take $\lambda_s = s\tilde{\alpha}$ (see [28]), where α is the unique reduced root of the maximal F-split torus of H^δ in U.

As in [18], define

$$\beta(s) = L(s+1, \tau \times \sigma) \cdot L(2s+1, \tau, \wedge^2),$$

where the L-functions are defined through the Langlands–Shahidi method [29]. Then define the normalized Eisenstein series

$$E^*(h, s, \phi_{\tau \otimes \sigma}) = \beta(s) E(h, s, \phi_{\tau \otimes \sigma}),$$

which has the following properties.

Proposition 1.1 ([18, Proposition 4.1]). Let τ and σ be as above. Then the normalized Eisenstein series $E^*(h, s, \phi_{\tau \otimes \sigma})$ has a simple pole at $s = \frac{1}{2}$ if and only if $L(s, \tau, \wedge^2)$ has a pole at $s = 1$ and $L(\frac{1}{2}, \tau \times \sigma) \neq 0$.   □

Let $\mathcal{E}_{\tau \otimes \sigma}$ denote the automorphic representation of $H^\delta(\mathbb{A})$ generated by the residues at $s = \frac{1}{2}$ of $E(h, s, \phi_{\tau \otimes \sigma})$ for all $\phi_{\tau \otimes \sigma} \in \mathcal{A}(M(F)U(\mathbb{A})\backslash H^\delta(\mathbb{A}))_{\tau \otimes \sigma}$. From now on, we assume that $L(s, \tau, \wedge^2)$ has a pole at $s = 1$ and $L(\frac{1}{2}, \tau \times \sigma) \neq 0$. In this case, τ has trivial central character [15]. By Proposition 1.1, the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ is non-zero. By the L^2-criterion in [26; 18, Theorem 6.1], the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ is square-integrable. Moreover, the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ is irreducible, following Theorem A of Moeglin in [24], for instance. Note that the global Arthur parameter (see [2]) for $\mathcal{E}_{\tau \otimes \sigma}$ is $(\tau, 2) \boxplus \psi_\sigma$ [18, Section 6].

1.2 Fourier coefficients attached to partition $[(2\ell+1)1^{4n-2\ell+1}]$

Following [16], one defines Fourier coefficients of automorphic forms of classical groups attached to nilpotent orbits and hence to a partition. For $H^\delta = \text{SO}_{4n+2}^\delta$, we consider here the Fourier coefficients attached to the partition $[(2\ell+1)1^{4n-2\ell+1}]$ of $4n+2$ with $1 \leq \ell \leq 2n$, which is also known as Bessel–Fourier coefficients. More precisely, for $1 \leq \ell \leq 2n$, consider the following partial flag

$$\mathcal{F}_\ell : 0 \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_\ell^+,$$

which defines the standard parabolic subgroup P_ℓ of H^δ, with the Levi decomposition $P_\ell = M_\ell \cdot N_\ell$. The Levi part $M_\ell \simeq (\mathrm{GL}_1)^\ell \times \mathrm{SO}(W_\ell)$, with

$$W_\ell = \left(V_\ell^+ \oplus V_\ell^-\right)^\perp.$$

Following [35], the F-rational nilpotent orbits in the F-stable nilpotent orbits in the Lie algebra of $\mathrm{SO}_{4n+2}^\delta$ associated with the partition $[(2\ell+1)1^{4n-2\ell+1}]$ are parameterized by F-rational orbits in the F-anisotropic vectors in W_ℓ under the action of $\mathrm{GL}_1(F) \times \mathrm{SO}(W_\ell)$. The action of $\mathrm{GL}_1(F) \times \mathrm{SO}(W_\ell)$ on W_ℓ is induced naturally from the adjoint action of M_ℓ on the unipotent radical N_ℓ.

Take an anisotropic vector $w_0 \in W_\ell$ with $\langle w_0, w_0 \rangle$ in a given square class of $F^\times$, and define a homomorphism $\chi_{\ell, w_0} : N_\ell \longrightarrow \mathbb{G}_a$ by

$$\chi_{\ell, w_0}(u) = \sum_{i=2}^{\ell} \langle u \cdot e_i, e_{-(i-1)} \rangle + \langle u \cdot w_0, e_{-\ell} \rangle.$$

Recall that here $\langle \cdot, \cdot \rangle$ denotes the quadratic form on V. Define also a character

$$\psi_{\ell, w_0} = \psi \circ \chi_{\ell, w_0} : N_\ell(\mathbb{A}) \longrightarrow \mathbb{C}^\times,$$

where $\psi : F \backslash \mathbb{A} \longrightarrow \mathbb{C}^\times$ is a fixed non-trivial additive character. Hence, the character ψ_{ℓ, w_0} is trivial on $N_\ell(F)$. Now the adjoint action of M_ℓ on N_ℓ induces an action of $\mathrm{SO}(W_\ell)$ on the set of all such characters ψ_{ℓ, w_0}. The stabilizer L_{ℓ, w_0} of χ_{ℓ, w_0} in $\mathrm{SO}(W_\ell)$ equals $\mathrm{SO}(w_0^\perp \cap W_\ell)$.

Let Π be an automorphic representation of $H^\delta(\mathbb{A})$ (see [3, Section 4.6]), occurring in the discrete spectrum. For $f \in V_\Pi$ and $h \in H^\delta(\mathbb{A})$, we define the ψ_{ℓ, w_0}-Fourier coefficients of f by

$$f^{\psi_{\ell, w_0}}(h) = \int_{[N_\ell]} f(vh)\, \psi_{\ell, w_0}^{-1}(v)\, \mathrm{dv}; \tag{1.1}$$

here $[N_\ell]$ denotes the quotient $N_\ell(F) \backslash N_\ell(\mathbb{A})$. This is one of the Fourier coefficients of f associated with the partition $[(2\ell+1)1^{4n-2\ell+1}]$. It is clear that $f^{\psi_{\ell, w_0}}(h)$ is left $L_{\ell, w_0}(F)$-invariant, with $L_{\ell, w_0} = \mathrm{SO}(w_0^\perp \cap W_\ell)$. Following [10, Section 3.1], we define the space

$$\sigma_{\psi_{\ell, w_0}}(\Pi) = L_{\ell, w_0}(\mathbb{A}) - \mathrm{Span}\left\{ f^{\psi_{\ell, w_0}}|_{L_{\ell, w_0}(\mathbb{A})} \mid f \in V_\Pi \right\},$$

which is a representation of $L_{\ell, w_0}(\mathbb{A})$, with right translation action. The situation that is the main concern of the paper is the case of the so-called first occurrence, as described in Theorem 1.2. In such a situation, the function $f^{\psi_{\ell, w_0}}(h)$ will be cuspidal for all $f \in V_\Pi$, in particular, square-integrable. Hence, we may identify the space $\sigma_{\psi_{\ell, w_0}}(\Pi)$ as a subspace of the discrete spectrum of the space of square-integrable automorphic functions

on $L_{\ell,w_0}(\mathbb{A})$, and $\sigma_{\psi_{\ell,w_0}}(\Pi)$ becomes an automorphic $L_{\ell,w_0}(\mathbb{A})$-module in the sense of [3, Section 4.6].

To make a precise choice of the anisotropic vector w_0 for the cases $1 \leq \ell < 2n$, we take

$$w_0 = y_\alpha = e_{2n} - \frac{\alpha}{2}e_{-2n}$$

for $\alpha \in F^\times$, and hence we have $\langle w_0, w_0 \rangle = -\alpha$. For such an α, we consider the three-dimensional quadratic form given by $J_{\delta,\alpha} = \begin{pmatrix} & & 1 \\ & \delta & \\ \alpha & & \end{pmatrix}$, which can be split or non-split over F, depending on the Hilbert symbol (δ, α). For the cases $\ell = 2n$, we take any non-zero $w_0 \in V_0$. Hence, for $1 \leq \ell < 2n$ and $w_0 = y_\alpha$, we have $L_{\ell,\alpha} = \mathrm{SO}_{4n-2\ell+1}^{\delta,\alpha}$, which is a special odd orthogonal group defined by the form

$$J = \begin{pmatrix} & & w_{2n-\ell-1} \\ & J_{\delta,\alpha} & \\ w_{2n-\ell-1} & & \end{pmatrix}.$$

We denote ψ_{ℓ,y_α} by $\psi_{\ell,\alpha}$, and $\sigma_{\psi_{\ell,y_\alpha}}(\Pi)$ by $\sigma_{\psi_{\ell,\alpha}}(\Pi)$.

1.3 The twisted automorphic descent

Now, we apply the ψ_{ℓ,w_0}-Fourier coefficients to the residual representation $\mathcal{E}_{\tau\otimes\sigma}$ of $\mathrm{SO}_{4n+2}^\delta(\mathbb{A})$ and investigate more carefully its properties depending on the integer ℓ. One of the main results in this paper is the following theorem.

Theorem 1.2. Assume that an irreducible unitary cuspidal automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$ has the property that $L(s, \tau, \wedge^2)$ has a pole at $s = 1$ and there exists an irreducible unitary (cuspidal) representation σ of $\mathrm{SO}_2^\delta(\mathbb{A})$ such that $L(\frac{1}{2}, \tau \times \sigma) \neq 0$. Then the following hold.

(1) The representation $\sigma_{\psi_{\ell,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ of $\mathrm{SO}_{4n-2\ell+1}^{\delta,\alpha}(\mathbb{A})$ is zero for all $n < \ell \leq 2n$.

(2) For any square class α in $F^\times$, the representation $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ of $\mathrm{SO}_{2n+1}^{\delta,\alpha}(\mathbb{A})$ is cuspidal automorphic.

(3) There exists a square class α in $F^\times$ such that the representation $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ of $\mathrm{SO}_{2n+1}^{\delta,\alpha}(\mathbb{A})$ is non-zero, and in this case

$$\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau\otimes\sigma}) = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_r \oplus \cdots,$$

where π_i are irreducible cuspidal automorphic representations of $\mathrm{SO}_{2n+1}^{\delta,\alpha}(\mathbb{A})$, which are nearly equivalent, but are not globally equivalent, that is, the decomposition is multiplicity-free.

(4) When $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ is non-zero, any direct summand π_i of $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ has a weak Langlands functorial transfer to τ in the sense that the Satake parameter of the local unramified component τ_v of τ is the local functorial transfer of that of the local unramified component π_v of π for almost all unramified local places v of F.

(5) When $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ is non-zero, every irreducible direct summand of $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ has a non-zero Fourier coefficient attached to the partition $[(2n-1)1^2]$.

(6) The residual representation $\mathcal{E}_{\tau\otimes\sigma}$ has a non-zero Fourier coefficient attached to the partition $[(2n+1)(2n-1)1^2]$. □

For these square classes α in $F^\times$ such that $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ is non-zero, we call $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ the *twisted automorphic descent* of τ to $\mathrm{SO}_{2n+1}^{\delta,\alpha}(\mathbb{A})$. It is clear from the notation and more importantly the explicit construction that the twisted automorphic descent $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ depends on the data $(\tau; \delta, \sigma; \alpha)$. The $\mathrm{GL}_1(F) \times \mathrm{SO}(W_\ell)(F)$-orbits of anisotropic vectors in W_ℓ labeled by the pair (δ, α) determine more refined properties of the twisted automorphic descent $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$, which will be briefly discussed at the end of Section 3 in connection to Parts (5) and (6) of Theorem 1.2 and to the structure of the relevant global Vogan packets. We refer to the reader [22] for more discussion on the general framework of the twisted automorphic descent constructions for classical groups with connections to explicit structures of global Arthur packets and global Vogan packets. In order to show the potential of the theory of twisted automorphic descents, we provide in Section 5 a complete description of the method and result for the case of $n=1$, which recovers the classical Jacquet–Langlands correspondence. Hence, the twisted automorphic descent discussed in this paper provides a new method to establish the Jacquet–Langlands correspondence for PGL(2).

Parts (1) and (2) of the theorem are usually called the *tower property* and *cuspidality*. They will be proved in Section 2 after establishing the vanishing property of the corresponding local Jacquet module at one unramified local place, following the idea of [10]. In order to establish the *first occurrence* at $\ell = n$, which is the first assertion in Part (3) of the theorem, we have to understand the structure of the Fourier coefficients of the residual representation $\mathcal{E}_{\tau\otimes\sigma}$ following the general framework of [16]. We first prove that the residual representation $\mathcal{E}_{\tau\otimes\sigma}$ has a non-zero Fourier coefficient attached to the partition $[(2n)^2 1^2]$ in Section 3 (Proposition 3.1), which is based on the cuspidal

support of $\mathcal{E}_{\tau \otimes \sigma}$. After that, we use the argument similar to that in [6, Section 5], but with full details, to prove that there exists at least one square class α in $F^\times$ such that the twisted automorphic descent $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is non-zero (Proposition 3.3), which yields the first assertion in Part (3) of the theorem. It is clear that the second assertion in Part (3) of the theorem follows from Part (4) for the assertion of nearly equivalence, while the multiplicity-free decomposition follows from the local uniqueness of Bessel models [1, 5, 20]. If we assume that the global Gan–Gross–Prasad conjecture holds for this case, then we may deduce that $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is irreducible. We will not discuss this issue in such generality in this paper, but we will discuss the case of $n = 1$ in Section 5. Next, we continue in Section 3 to obtain Part (5) of the theorem by showing that the irreducible summands in the twisted automorphic descent $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ have non-zero Fourier coefficients attached to the partition $[(2n-1)1^2]$. Note that if $n = 1$, this partition is trivial, and the corresponding Fourier coefficient is the identity map. It is clear that if the group $\mathrm{SO}_3^{\delta,\alpha}$ is F-anisotropic, then its only nilpotent orbit is the trivial one, corresponding to the trivial partition. However, if the group $\mathrm{SO}_3^{\delta,\alpha}$ is F-split, because of Part (4), we expect that the twisted automorphic descent has a non-zero Whittaker–Fourier coefficient. There will be a more detailed discussion for this case in Section 5. In Section 4, we are going to prove that the Satake parameter of the unramified local component of any of the direct summand π_i of $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ transfers canonically to the Satake parameter of the unramified local component of τ, and hence the direct summands of $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ are nearly equivalent. This proves Parts (3) and (4) of the theorem. The last part (Part (6) of the theorem) follows from Part (5) as a consequence of the Fourier coefficients associated with a composite of two partitions as discussed in [9, 17], which will be briefly discussed before the end of Section 3.

2   On Vanishing and Cuspidality

We are going to show that the ψ_{ℓ, w_0}-Fourier coefficients of the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ of $\mathrm{SO}_{4n+2}^\delta(\mathbb{A})$ vanishes for all $n < \ell \leq 2n$. Following the tower property proved in [10], the twisted automorphic descent $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is cuspidal, which might also vanish. Such vanishing property of the family of the Fourier coefficients $\mathcal{E}_{\tau \otimes \sigma}^{\psi_{\ell, w_0}}$ should be determined by the global Arthur parameter of the square-integrable residual representation $\mathcal{E}_{\tau \otimes \sigma}$ and the structure of its Fourier coefficients as conjectured in [16]. Through the local–global relation, one may take the local approach to prove such vanishing property as shown in [10]. However, a purely global argument should be interesting and expected [8]. We will come back to this issue in future.

In the following, we follow [10] to take the local approach, which is a calculation of twisted Jacquet module at one unramified local place. This proves Part (1) of Theorem 1.2. Following the argument of the tower property in [10], we deduce in the second section the cuspidality of the twisted automorphic descent, which is Part (2) of Theorem 1.2.

2.1 Calculation of certain Jacquet modules

Most of the results are deduced from the general results of [10, Chapter 5].

Let k be a non-Archimedean local field and V be a non-degenerate quadratic space of dimension $4n + 2$ over k. We have a polar decomposition

$$V = V^+ \oplus V_0 \oplus V^-,$$

where V^+ is a maximal isotropic subspace of V and V_0 is anisotropic. There are two cases to be considered

(1) $\dim V_0 = 0$, if $SO(V)$ is split;
(2) $\dim V_0 = 2$, if $SO(V)$ is quasi-split.

We denote the Witt index of V by $\tilde{m}$, which can equal to $2n + 1$ or $2n$ in case (1) or (2), respectively.

Let $H = SO(V)$ be the special orthogonal group on V over k. Fix a basis $\{e_1, \ldots, e_{2n}\}$ of V^+, and if $V_0 \neq 0$, we take a basis $\{e_0^{(1)}, e_0^{(2)}\}$ for V_0 such that $\langle e_0^{(1)}, e_0^{(1)} \rangle = 1$ and $\langle e_0^{(2)}, e_0^{(2)} \rangle = \delta$ for some $\delta \in k^\times$. Otherwise we fix a basis $\{e_1, \ldots, e_{2n+1}\}$ for V^+. As in the global case (Section 2.2), for $0 < \ell \leq 2n$, let P_ℓ be the parabolic subgroup of H preserving the partial flag

$$\mathcal{F}_\ell : 0 \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_\ell^+.$$

Then $P_\ell = M_\ell \cdot N_\ell$ with $M_\ell \simeq \mathbb{G}_m^\ell \times SO(W_\ell)$, where W_ℓ is the same as in Section 1.2. For $\alpha \in k^\times$, we take an anisotropic vector $w_0 \in V$ as follows:

$$w_0 = \begin{cases} y_\alpha = e_{2n} - \dfrac{\alpha}{2} e_{-2n} & \text{if } \ell < 2n, \\[2mm] \alpha e_0^{(1)} \text{ or } \alpha e_0^{(2)} \text{ in } V_0 & \text{if } \ell = 2n \text{ and } V_0 \neq 0. \end{cases} \tag{2.1}$$

Remark 2.1. By a change of basis if necessary, we see that each anisotropic vector in V is of the form $\alpha e_i + \beta e_{-i}$ or $\alpha e_0^{(j)}$ (this occurs only if $V_0 \neq 0$) for some $\alpha, \beta \in k^\times$. Then the above choices of w_0 essentially represent any anisotropic vector in V. $\qquad\square$

Similar to the global case (Section 2.2), one defines a character ψ_ℓ on $N_\ell(k)$ by $\psi_\ell = \psi \circ \chi_{\ell,w_0}$, where ψ is a non-trivial additive character on k. Let $L_{\ell,w_0} = \mathrm{SO}(W_\ell \cap w_0^\perp)$ be the stabilizer of χ_{ℓ,w_0} in $\mathrm{SO}(W_\ell)$.

For any smooth representation (Π, V_Π) of $H(k)$, we define the (twisted) Jacquet module

$$J_{\psi_{\ell,w_0}}(V_\Pi) = V_\Pi / \mathrm{Span}\left\{ \Pi(u)\xi - \psi_{\ell,w_0}(u)\xi \mid u \in N_\ell(k),\ \xi \in V_\Pi \right\},$$

which is an L_{ℓ,w_0}-module. To simplify the notation, we will use $\chi_{\ell,\alpha}$, $L_{\ell,\alpha}$ and $J_{\psi_\ell,\alpha}$ if $w_0 = y_\alpha$. Note that with the notation of the previous section, for $0 \le \ell < 2n$, the form of the odd orthogonal group $L_{\ell,\alpha}$ is

$$J = \begin{pmatrix} & & w_{2n-\ell-1} \\ & J_{\delta,\alpha} & \\ w_{2n-\ell-1} & & \end{pmatrix},$$

and hence we have the following cases:

(i)   if $J_{\delta,\alpha}$ is non-split over k, then $L_{\ell,\alpha}$ is non-split, and $\mathrm{Witt}(k \cdot y_{-\alpha} + V_0) = 0$;

(ii) otherwise, the group $L_{\ell,\alpha}$ is k-split. In this case, $\mathrm{Witt}(k \cdot y_{-\alpha} + V_0) = 1$ if $V_0 \ne 0$ (i.e. $-\delta \notin k^{\times 2}$), and $\mathrm{Witt}(k \cdot y_{-\alpha} + V_0) = 0$ if $V_0 = 0$.

Moreover, if $\ell = 2n$, this will be a degenerate case, and L_{2n,w_0} is a trivial group.

For any $0 < j \le 2n$, let $V_j^+ = \mathrm{Span}\{e_1, \ldots, e_j\}$, and let Q_j be the standard parabolic subgroup of H which preserves V_j^+. The group Q_j has a Levi decomposition $Q_j = D_j \cdot U_j$ with $D_j \simeq \mathrm{GL}_j(k) \times \mathrm{SO}(W_j)$. For $0 \le t < j$, let $\tau^{(t)}$ be the tth Bernstein–Zelevinsky derivative of τ along the subgroup

$$Z_t' = \left\{ \begin{pmatrix} I_s & Y \\ 0 & z \end{pmatrix} \in \mathrm{GL}_j(k) \mid z \in Z_t(k) \right\}$$

and corresponding to the character

$$\psi_t'\left(\begin{pmatrix} I_s & Y \\ 0 & z \end{pmatrix} \right) = \psi^{-1}\left(z_{1,2} + \cdots + z_{t-1,t} \right).$$

Then $\tau^{(t)}$ is the representation of $\mathrm{GL}_s(k)$ with $s = j - t$, acting on the Jacquet module $J_{Z_t', \psi_t'}(V_\tau)$ via the embedding

$$d \longmapsto \mathrm{diag}(d, I_t) \in \mathrm{GL}_j(k).$$

For $a \in k^\times$, we also consider the character $\psi''_{t,a}$ on Z'_t defined by

$$\psi''_{t,a}\left(\begin{pmatrix} I_s & Y \\ 0 & z \end{pmatrix}\right) = \psi^{-1}\left(z_{1,2} + \cdots + z_{t-1,t} + ay_{s,1}\right).$$

Denote the corresponding Jacquet module $J_{Z'_t,\psi''_{t,a}}(V_\tau)$ by $\tau_{(t),a}$, which is a representation of the mirabolic subgroup P^1_{s-1} of $\mathrm{GL}_s(k)$. Since $\tau_{(t),a} \simeq \tau_{(t),a'}$ for any $a, a' \in k^\times$, according to [10, Lemma 5.2], we sometimes denote by $\tau_{(t)}$ any of such representations $\tau_{(t),a}$.

Let τ and σ be smooth representations of $\mathrm{GL}_j(k)$ and $\mathrm{SO}(W_j)$, respectively. The Jacquet module $J_{\psi_{\ell,\alpha}}(\mathrm{Ind}^H_{Q_j} \tau \otimes \sigma)$ has been studied in [10]. We state it here for completeness.

Proposition 2.2 ([10, Theorem 5.1]). Set $|\cdot| := |\det(\cdot)|$. The following hold.

(1) Assume that $0 \le \ell < \tilde{m}$ and $1 \le j < \tilde{m}$; then

$$
\begin{aligned}
& J_{\psi_{\ell,\alpha}}\left(\mathrm{Ind}^H_{Q_j} \tau \otimes \sigma\right) \\
& \equiv \bigoplus_{\ell+j-\tilde{m}<t\le\ell,\ 0\le t\le j} \mathrm{ind}^{L_{\ell,\alpha}}_{Q'_{j-t}} |\cdot|^{\frac{1-t}{2}} \tau^{(t)} \otimes J_{\psi'_{\ell-t,\alpha}}\left(\sigma^{\omega^t_b}\right) \\
& \oplus \begin{cases} \mathrm{ind}^{L_{\ell,\alpha}}_{Q'_{j-\ell,\ell}} |\cdot|^{-\frac{\ell}{2}} \tau_{(\ell)} \otimes \sigma^{\omega^\ell_b} & \text{if } \ell < j, \\ 0 & \text{otherwise;} \end{cases} \\
& \oplus \delta_\alpha \cdot \begin{cases} \mathrm{ind}^{L_{\ell,\alpha}}_{Q'_\alpha} |\cdot|^{\frac{1+2n-\ell-j}{2}} \tau^{(\ell+j-\tilde{m})} \otimes J_{\psi'_{\tilde{m}-j,v_\alpha}}\left(\sigma^{\omega^{\ell+j-2n}_b}\right) & \text{if } 0 < 2n-\ell \le j, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

(2) Assume that $0 \le \ell \le \tilde{m}$ and $j = \tilde{m}$; then

$$J_{\psi_{\ell,\alpha}}\left(\mathrm{Ind}^H_{Q_j} \tau \otimes \sigma\right) \equiv \mathrm{ind}^{L_{\ell,\alpha}}_{Q'_{\tilde{m}-\ell}} |\cdot|^{-\frac{\ell}{2}} \tau_{(\ell)} \otimes \sigma^{\omega^\ell_b} \oplus \delta_\alpha \cdot \mathrm{ind}^{L_{\ell,\alpha}}_{Q'_\alpha} |\cdot|^{\frac{1-\ell}{2}} \tau^{(\ell)} \otimes J_{\psi'_{0,v_\alpha}}\left(\sigma^{\omega^\ell_b}\right).$$

(3) Assume that $\ell = \tilde{m}$ and $w_0 \in V_0$; then

$$J_{\psi_{\ell,w_0}}\left(\mathrm{Ind}^H_{Q_j} \tau \otimes \sigma\right) \equiv d_\tau \cdot J_{\psi'_{\ell-j,w_0}}\left(\sigma^{\omega^j_b}\right),$$

here $d_\tau = \dim \tau^{(j)}$.

Here, $Q'_s = L_{\ell,\alpha} \cap \eta_t^{-1} Q_{\ell,j}^{(w)} \eta_t$ with w being the representative of $Q_j \backslash H / Q_\ell$ corresponding to the pair $(0, s)$ as in [10, Chapter 4], and with $t = j - s$,

$$Q_{\ell,j}^{(w)} = Q_\ell \cap w^{-1} Q_j w, \quad \eta_t = \begin{pmatrix} \epsilon & & \\ & I_{4n+2-2\ell} & \\ & & \epsilon^* \end{pmatrix}, \quad \text{and} \quad \epsilon = \begin{pmatrix} & I_{\ell-t} \\ I_t & \end{pmatrix}.$$

Also $Q'_{s,t} = L_{\ell,\alpha} \cap \eta_{s,t}^{-1} Q_{\ell,j}^{(w)} \eta_{s,t}$ with t being the same as above, and

$$\eta_{s,t} = \begin{pmatrix} \epsilon & & \\ & \gamma_s & \\ & & \epsilon^* \end{pmatrix}, \quad \gamma_s = \begin{pmatrix} 0 & I_s & & & \\ I_{2n-\ell-s} & 0 & & & \\ & & I_{V_0} & & \\ & & & 0 & I_{2n-\ell-s} \\ & & & I_s & 0 \end{pmatrix}.$$

Set $\omega_b = \mathrm{diag}(I_{\tilde{m}}, \omega_b^0, I_{\tilde{m}})$, where $\omega_b^0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ if $\tilde{m} = 2n$, and $\omega_b^0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ if $\tilde{m} = 2n+1$. Moreover, $\delta_\alpha = 0$ unless $\mathrm{Witt}(k \cdot y_{-\alpha} + V_0) = 1$, and in such cases, set $v_\alpha \in V_0$ such that $\langle v_\alpha, v_\alpha \rangle = -\alpha$,

$$\gamma_\alpha = \begin{pmatrix} I_{2n-\ell-1} & & & & \\ & 1 & & & \\ & -v_\alpha & I_{V_0} & & \\ & \dfrac{\alpha}{2} & v'_\alpha & 1 & \\ & & & & I_{2n-\ell-1} \end{pmatrix},$$

$\gamma = \mathrm{diag}(I_{s-1}, \gamma_\alpha, I_{s-1})$, and $\eta_{\gamma_\alpha,t}$ is of the same form as $\eta_{s,t}$ where γ_s is replaced by γ_α. Finally, $Q'_\alpha = L_{\ell,\alpha} \cap \eta_{\gamma_\alpha,t}^{-1} Q_{\ell,j}^{(w)} \eta_{\gamma_\alpha,t}$, and $\psi'_{l,\alpha}$ (or $\psi'_{l.v_\alpha}$) denotes the corresponding character like $\psi_{\ell,\alpha}$ but on the groups of smaller rank. Note that "ind" denotes the compact induction, and "$\equiv$" denotes isomorphism of representations, up to semi-simplification.   □

Remark 2.3. Define

$$V_{\ell,s}^\pm = \mathrm{Span}_k \left\{ e_{\pm(\ell+1)}, \ldots, e_{\pm(\ell+s)} \right\} \subset W_\ell.$$

When $w_0 \in W_{\ell+s}$ or $H(k)$ is split, Q'_s is the maximal parabolic subgroup of L_{ℓ,w_0} which preserves the isotropic subspace $\omega_b^t V_{\ell,s}^+ \cap w_0^\perp$. Otherwise, it is a proper subgroup of the parabolic subgroup. Moreover, $Q'_{s,t}$ is a subgroup of the maximal parabolic subgroup of $L_{\ell,\alpha}$, which preserves the isotropic subspace $\eta_{s,t}^{-1} V_{\ell,s}^+ \cap Y_\alpha^\perp$.   □

With the above preparation, we turn to the calculation of the twisted Jacquet module for any unramified local component of the residual representation $\mathcal{E}_{\tau \otimes \sigma}$.

Let τ be an irreducible unramified representation of $\mathrm{GL}_{2n}(k)$ with central character $\omega_\tau = \mathbf{1}$. Since τ is generic and self-dual, we may write τ as the full induced representation from the Borel subgroup as follows:

$$\tau = \mu_1 \times \cdots \times \mu_n \times \mu_n^{-1} \times \cdots \times \mu_1^{-1},$$

where μ_i's are unramified characters on $k^\times$. Also, let σ be an irreducible unramified representation of $\mathrm{SO}(W_{2n})$. When $H = \mathrm{SO}(V)$ is k-split, the representation σ is given by a unramified character μ_0 of $k^\times$; and when H is k-quasisplit, $\mathrm{SO}(W_{2n})$ is a compact torus, and hence σ is the trivial representation.

Let $\pi_{\tau \otimes \sigma}$ be the unramified constituent of $\mathrm{Ind}_{Q_{2n}}^H \tau \cdot |\det|^{\frac{1}{2}} \otimes \sigma$. In the following proposition, we will consider the vanishing of $J_{\psi_{\ell,\alpha}}(\pi_{\tau \otimes \sigma})$ by explicitly calculating this Jacquet module using Proposition 2.2.

Proposition 2.4. Let τ be an irreducible unramified representation of $\mathrm{GL}_{2n}(k)$ with central character $\omega_\tau = \mathbf{1}$, and σ an irreducible unramified representation of $\mathrm{SO}(W_{2n})$. The following hold.

    (i) Assume that J_δ is non-split over k.

        (1) If $w_0 = y_\alpha$ for $\alpha \in k^\times$ such that $J_{\delta,\alpha}$ is split, then $J_{\psi_{\ell,\alpha}}(\pi_{\tau \otimes \sigma}) = 0$ for $\ell \geq n+1$.

        (2) If $w_0 \in V_0$, then $J_{\psi_{2n,w_0}}(\pi_{\tau \otimes \sigma}) = 0$.

    (ii) Assume that J_δ splits over k. For any choice of $\alpha \in k^\times$, $J_{\psi_{\ell,\alpha}}(\pi_{\tau \otimes \sigma}) = 0$ for $\ell \geq n+1$. $\square$

Proof. We suppose first that H is non-split, that is, J_δ is non-split over k.

In this case, $\pi_{\tau \otimes \sigma}$ is the unramified constituent of the representation of H induced from the character (here $\sigma = \mathbf{1}$)

$$\mu_1 |\cdot|^{\frac{1}{2}} \otimes \cdots \otimes \mu_n |\cdot|^{\frac{1}{2}} \otimes \mu_n^{-1} |\cdot|^{\frac{1}{2}} \otimes \cdots \otimes \mu_1^{-1} |\cdot|^{\frac{1}{2}} \otimes \mathbf{1}. \tag{2.2}$$

Moreover, one can find a Weyl element of $\mathrm{SO}(V)$ which conjugates above character to

$$\mu_1 |\cdot|^{\frac{1}{2}} \otimes \mu_1 |\cdot|^{-\frac{1}{2}} \otimes \cdots \otimes \mu_n |\cdot|^{\frac{1}{2}} \otimes \mu_n |\cdot|^{-\frac{1}{2}} \otimes \mathbf{1}.$$

Then, on induction by stages, one sees that $\pi_{\tau \otimes \sigma}$ is the unramified constituent of $\mathrm{Ind}_{Q_{2n}}^H \tau' \otimes \sigma$, where

$$\tau' = \mathrm{Ind}_{P_{2,\ldots,2}}^{\mathrm{GL}_{2n}(k)} \mu_1 \left(\det{}_{\mathrm{GL}_2} \right) \otimes \cdots \otimes \mu_n \left(\det{}_{\mathrm{GL}_2} \right).$$

Then since the twisted Jacquet functor corresponding to the descent is exact, we get (i) (1) by Proposition 2.2(2) if we take $w_0 = y_\alpha$ as in (2.1) (here $j = \tilde{m} = 2n$, and $\delta_\alpha = 1$ by our assumption on $\alpha \in k^\times$). Note that we have used the fact that $\tau'_{(n)} = 0$.

If $w_0 \in V_0$, then by the arguments above we see that we also need to consider the Jacquet module $J_{\psi_{2n,w_0}}(\mathrm{Ind}_{Q_{2n}}^H \tau' \otimes \sigma)$. By Proposition 2.2(3), we have (see the notation in Proposition 2.2)

$$J_{\psi_{2n,w_0}}\left(\mathrm{Ind}_{Q_{2n}}^H \tau' \otimes \sigma\right) \equiv d_{\tau'} \cdot J_{\psi'_{0,w_0}}\left(\sigma^{\omega_b^{2n}}\right).$$

Note that $d_{\tau'}$ is the dimension of the space of ψ-Whittaker functionals on τ', hence is zero by the construction of τ'. Then the above Jacquet module is zero, and this finishes part (i) of the proposition.

We turn to consider the case where H is split with Witt index $2n+1$ over k. By our assumption, $\pi_{\tau \otimes \sigma}$ is the unramified constituent of a representation induced from the character

$$\mu_1 |\cdot|^{\frac{1}{2}} \otimes \cdots \otimes \mu_n |\cdot|^{\frac{1}{2}} \otimes \mu_n^{-1} |\cdot|^{\frac{1}{2}} \otimes \cdots \otimes \mu_1^{-1} |\cdot|^{\frac{1}{2}} \otimes \mu_0.$$

If n is even, $\pi_{\tau \otimes \sigma}$ is the unramified constituent of $\mathrm{Ind}_{Q_{2n}}^H \tau' \otimes \sigma$, where

$$\tau' = \mathrm{Ind}_{P_{2,\ldots,2}}^{\mathrm{GL}_{2n}(k)} \mu_1 \left(\det_{\mathrm{GL}_2}\right) \otimes \cdots \otimes \mu_n \left(\det_{\mathrm{GL}_2}\right).$$

And if n is odd, $\pi_{\tau \otimes \sigma}$ is the unramified constituent of $\mathrm{Ind}_{Q_{2n}}^H \tau' \otimes \sigma^{\omega_b^0}$ with $\omega_b^0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus, applying Proposition 2.2(1) with $\tilde{m} = 2n+1$ and $j = 2n$, one sees that $J_{\psi_{\ell,\alpha}}(\pi_{\tau \otimes \sigma}) = 0$ for $\ell \geq n+1$. Note that in this case we have that $\delta_\alpha = 0$ for any choice of $\alpha \in k^\times$. Note that we also use the fact $\tau'^{(\ell)} = 0$ for $\ell > n$ and $\tau'_{(\ell)} = 0$ for $\ell > n-1$. ∎

For later use, we write out the Jacquet module $J_{\psi_{\ell,\alpha}}(\pi_{\tau \otimes \sigma})$ for the above unramified τ and σ, in the case that $\ell = n$ and the form $J_{\delta,\alpha}$ is split. Note that the last condition means that the quadratic form $\delta_1 x^2 + \delta_2 y^2 + \alpha z^2$ represents 0 over k.

Proposition 2.5. Assume that τ and σ are unramified representations as above, and the form $J_{\delta,\alpha}$ is split. Then the Jacquet module can be realized as follows:

$$J_{\psi_{n,\alpha}}\left(\pi_{\tau \otimes \sigma}\right) \prec \mathrm{Ind}_{B_{\mathrm{SO}(2n+1)}}^{\mathrm{SO}(2n+1)} \mu_1 \otimes \cdots \otimes \mu_n.$$

Here, "$\pi_1 \prec \pi_2$" denotes that π_1 is a subquotient of π_2. □

Proof. Suppose first that J_δ is non-split. We have that $\mathrm{Witt}(V) = 2n$, $\mathrm{Witt}(k \cdot y_{-\alpha} + V_0) = 1$ ($J_{\delta,\alpha}$ is split), and $\sigma = 1$. Conjugating by a Weyl element as in the proof of

Proposition 2.4, one sees that $\pi_{\tau \otimes \sigma}$ is the unramified constituent of $\mathrm{Ind}_{Q_{2n-2}}^{H} \tau' \otimes \sigma$, where

$$\tau' = \mathrm{Ind}_{P_{2,\ldots,2}}^{\mathrm{GL}_{2n}(k)} \mu_1 \left(\det{}_{\mathrm{GL}_2} \right) \otimes \cdots \otimes \mu_n \left(\det{}_{\mathrm{GL}_2} \right).$$

Now applying Proposition 2.2(2) for $\ell = n$, $j = 2n = \tilde{m}$, and using the fact that $\tau'_{(n)} = 0$, we have

$$J_{\psi_{n,\alpha}} \left(\pi_{\tau \otimes \sigma} \right) \prec \mathrm{ind}_{Q'_\alpha}^{L_{n,\alpha}} | \det (\cdot) |^{\frac{1-n}{2}} \tau'^{(n)} \otimes J_{\psi'_{0,v_\alpha}} \left(\sigma^{\omega_b^n} \right).$$

For the case that J_δ splits, we note that $\tilde{m} = \mathrm{Witt}(V) = 2n + 1$, and $\mathrm{Witt}(k \cdot y_{-\alpha} + V_0) = 0$. Applying Proposition 2.2(1) for $j = 2n$ and $\ell = n$, we see that

$$J_{\psi_{n,\alpha}} \left(\pi_{\tau \otimes \sigma} \right) \prec \mathrm{ind}_{Q'_n}^{L_{n,\alpha}} | \det (\cdot) |^{\frac{1-n}{2}} \tau'^{(n)} \otimes J_{\psi'_{0,\alpha}} \left(\sigma^{\omega_b^n} \right).$$

Note that $| \det(\cdot) |^{\frac{1-n}{2}} \tau'^{(n)} = \mathrm{Ind}_{B_{\mathrm{GL}_n}}^{\mathrm{GL}_n} \mu_1 \otimes \cdots \otimes \mu_n$, and the two Jacquet modules $J_{\psi'_{0,v_\alpha}} (\sigma^{\omega_b^n})$ and $J_{\psi'_{0,\alpha}} (\sigma^{\omega_b^n})$ are just the restriction of $\sigma^{\omega_b^n}$ of $\mathrm{SO}_2^\delta(k)$ to the trivial group. Therefore, the result follows. ∎

2.2 Cuspidality of the twisted automorphic descent

Now we go back to the global case and show that the twisted automorphic descent $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is cuspidal (but it may be zero). This result is a combination of the tower property of descent [10, Chapter 5] and the local result of the previous subsection.

Proposition 2.6. For all $\ell > n$, the ψ_{ℓ, w_0}-Fourier coefficients of the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ are zero for all anisotropic vectors $w_0 \in W_\ell$. In particular, the twisted automorphic descent $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is cuspidal. □

Proof. We will show this proposition using the vanishing properties of the corresponding twisted Jacquet modules we have studied in the previous section. By Remark 2.1, we see that up to conjugation, we only need to consider the ψ_{ℓ, w_0}-coefficients for $w_0 = y_\alpha$ ($\alpha \in F^\times$) or $w_0 \in V_0$. Let v be a finite local place of the number field F at which all data involved in the ψ_{ℓ, w_0}-Fourier coefficients of $\mathcal{E}_{\tau \otimes \sigma}$ are unramified. The unramified local component at v of the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ is denoted by $\mathcal{I}_{\tau_v \otimes \sigma_v}$. For any integer $\ell > n$, if $\mathcal{E}_{\tau \otimes \sigma}$ has non-zero ψ_{ℓ, w_0}-Fourier coefficients, then the corresponding local Jacquet module $J_{\psi_{\ell, w_0}}(\mathcal{I}_{\tau_v \otimes \sigma_v})$ is non-zero. But the latter is zero according to Proposition 2.4.

Now, we use the tower property as in [10] to prove the cuspidality of the twisted automorphic descent $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ (here $\ell = n$, and $w_0 = y_\alpha$ by our choice). The idea is to calculate all constant terms along the unipotent radical of maximal parabolic subgroups

of $\mathrm{SO}_{2n+1}^{\delta,\alpha}$. Since the argument is inductive, we consider a more general situation as the book [10] did for the tower property.

We denote the Witt index of the space $W_\ell \cap y_\alpha^\perp$ by $\tilde{m}_{\ell,\alpha}$. For $1 \le p \le \tilde{m}_{\ell,\alpha}$, let Q_p^* be the standard maximal parabolic subgroup of $L_{\ell,\alpha}$, which preserves the totally isotropic subspace $V_{\ell,p}^+ = \mathrm{Span}\{e_{\ell+1}, \ldots, e_{\ell+p}\} \cap y_\alpha^\perp$. Denote by U_p^* the unipotent radical of Q_p^*. For the $\psi_{\ell,\alpha}$-Fourier coefficients $f^{\psi_{\ell,\alpha}}$ for $f \in \mathcal{E}_{\tau \otimes \sigma}$, we denote by

$$c_p\left(f^{\psi_{\ell,\alpha}}\right) = \int_{U_p^*(F)\backslash U_p^*(\mathbb{A})} f^{\psi_{\ell,\alpha}}(u)\, \mathrm{d}u$$

its constant term along U_p^*. By [10, Theorem 7.3], the constant term $c_p(f^{\psi_{n,\alpha}})$ is expressed as a sum of integrals of $f^{\psi_{n+p,\alpha}}$ if we have

$$\left(f^{U_{p-i}}\right)^{\psi_{n+i,\alpha}} = 0; \quad (0 \le i \le p-1) \tag{2.3}$$

Recall that U_j is the unipotent subgroup of the parabolic subgroup Q_j of H, which preserves the isomorphic subspace $V_j^+ = \mathrm{Span}\{e_1, \ldots, e_j\}$ (see Section 2.1).

We claim that the condition (2.3) holds. In fact, by the cuspidal support of the Eisenstein series $E(h, s, \phi_{\tau \otimes \sigma})$, which produces the residual representation $\mathcal{E}_{\tau \otimes \sigma}$, the constant term $E^{U_j}(h, s, \phi_\pi)$ is always zero for any $1 \le j \le p$ and $1 \le p \le n$. This implies the condition (2.3).

It follows that the constant term $c_p(f^{\psi_{n,\alpha}})$ is a sum of integrals of $f^{\psi_{n+p,\alpha}}$, which are always zero by the discussion at the beginning of this proof. Therefore, all the constant terms $c_p(f^{\psi_{n,\alpha}})$ are zero, which implies that the twisted automorphic descent $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is cuspidal. ∎

3   Certain Fourier Coefficients of the Residual Representation

We prove Parts (3), (5), and (6) of Theorem 1.2 in this section. Before that, we need to establish certain properties of the Fourier coefficients of the residual representation $\mathcal{E}_{\tau \otimes \sigma}$. We are going to adapt the notation and the arguments used in [17] in the proofs given in this section. In particular, [17, Lemma 2.5] (see also [10, Corollary 7.1]) is a technical key, which will be used many times in the proofs.

In this section, we will let $E_{i,j}$ be the unipotent subgroup of $\mathrm{SO}_{4n+2}^\delta$ consisting of elements u with 1's on the diagonal, and for $k \ne \ell$, $u_{k,\ell} = 0$ unless $(k, \ell) = (i, j)$ or $(k, \ell) = (4n+3-j, 4n+3-i)$. For $x \in F$, let $E_{i,j}(x)$ be the element $u \in E_{i,j}$ with $u_{i,j} = x$.

3.1 Fourier coefficients associated with the partition $[(2n)^2 1^2]$

We first show that the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ has a non-zero Fourier coefficient associated with the partition $[(2n)^2 1^2]$, based on the cuspidal support of $\mathcal{E}_{\tau \otimes \sigma}$, which serves as a base for Parts (3), (5), and (6) of Theorem 1.2.

Proposition 3.1. The residual representation $\mathcal{E}_{\tau \otimes \sigma}$ has a non-zero Fourier coefficient attached to the partition $[(2n)^2 1^2]$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Proof.   By [35, I.6], there is only one F-rational nilpotent orbit in each F-stable nilpotent orbit in the Lie algebra of $\mathrm{SO}_{4n+2}^{\delta}(F)$ attached to the partition $[(2n)^2 1^2]$. Moreover, a nilpotent element in the only F-rational nilpotent orbit can be chosen as follows.

Let $X_{[(2n)^2 1^2]} = \sum_{i=1}^{n-1} E_{i+1,i}(\frac{1}{2})$. Then $X_{[(2n)^2 1^2]}$ is a representative of the F-rational orbit corresponding to the partition $[(2n)^2 1^2]$. To define the Fourier coefficients attached to the partition $[(2n)^2 1^2]$, we introduce a one-dimensional toric subgroup $\mathcal{H}_{[(2n)^2 1^2]}$ as follows: for $t \in F^{\times}$,

$$\mathcal{H}_{[(2n)^2 1^2]}(t) := \mathrm{diag}\left(t^{2n-1}, t^{2n-3}, \ldots, t^{1-2n}, 1, 1, t^{2n-1}, t^{2n-3}, \ldots, t^{1-2n}\right). \tag{3.1}$$

It is easy to see that under the adjoint action,

$$\mathrm{Ad}\left(\mathcal{H}_{[(2n)^2 1^2]}(t)\right)\left(X_{[(2n)^2 1^2]}\right) = t^{-2} X_{[(2n)^2 1^2]} \quad \forall t \in F^{\times}.$$

Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{SO}_{4n+2}^{\delta}(F)$. Under the adjoint action of $\mathcal{H}_{[(2n)^2 1^2]}$, $\mathfrak{g}$ has the following direct sum decomposition into $\mathcal{H}_{[(2n)^2 1^2]}$-eigenspaces:

$$\mathfrak{g} = \mathfrak{g}_{-m} \oplus \cdots \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m, \tag{3.2}$$

for some positive integer m, where $\mathfrak{g}_l := \{X \in \mathfrak{g} \mid \mathrm{Ad}(\mathcal{H}_{[(2n)^2 1^2]}(t))(X) = t^l X\}$. Let $V_{[(2n)^2 1^2], j}$ $(j = 1, \ldots, m)$ be the unipotent subgroup of $\mathrm{SO}_{4n+2}^{\delta}(F)$ with Lie algebra $\oplus_{l=j}^{m} \mathfrak{g}_l$. Let $L_{[(2n)^2 1^2]}$ be the algebraic subgroup of $\mathrm{SO}_{4n+2}^{\delta}(F)$ with Lie algebra $\mathfrak{g}_0$. Then we define a character of $V_{[(2n)^2 1^2], 2}$ as follows: for $v \in V_{[(2n)^2 1^2], 2}(\mathbb{A})$ and for a non-trivial character ψ of $F \backslash \mathbb{A}$,

$$\psi_{[(2n)^2 1^2]}(v) := \psi\left(\mathrm{tr}\left(X_{[(2n)^2 1^2]} \log(v)\right)\right)$$

$$= \psi\left(v_{1,2} + v_{2,3} + \cdots + v_{2n-1,2n}\right). \tag{3.3}$$

For an arbitrary automorphic form φ on $\mathrm{SO}_{4n+2}^{\delta}(\mathbb{A})$, the $\psi_{[(2n)^2 1^2]}$-Fourier coefficient of φ is defined by

$$\varphi^{\psi_{[(2n)^2 1^2]}}(g) := \int_{[V_{[(2n)^2 1^2], 2}]} \varphi(vg)\, \psi_{[(2n)^2 1^2]}^{-1}(v)\, \mathrm{d}v, \tag{3.4}$$

where, for a group G, we denote the quotient $G(F)\backslash G(\mathbb{A})$ simply by $[G]$. When an irreducible automorphic representation π of $\mathrm{SO}_{4n+2}^\delta(\mathbb{A})$ is generated by automorphic forms φ, we say that π has a non-zero $\psi_{[(2n)^2 1^2]}$-Fourier coefficient or a non-zero Fourier coefficient attached to $[(2n)^2 1^2]$ if there exists an automorphic form φ in the space of π with a non-zero $\psi_{[(2n)^2 1^2]}$-Fourier coefficient $\varphi^{\psi_{[(2n)^2 1^2]}}(g)$.

Note that by [25, Section 1.7], $V_{[(2n)^2 1^2],1}/\ker_{V_{[(2n)^2 1^2],2}}(\psi_{[(2n)^2 1^2]})$ is a Heisenberg group $V_{[(2n)^2 1^2],1}/V_{[(2n)^2 1^2],2} \oplus Z$, with center

$$Z = V_{[(2n)^2 1^2],2}/\ker_{V_{[(2n)^2 1^2],2}}\left(\psi_{[(2n)^2 1^2]}\right).$$

And, one can see that $(E_{2n+1,n+1}E_{n,2n+2}) \oplus (E_{n,2n+1}E_{2n+2,n+1})$ is a complete polarization of

$$V_{[(2n)^2 1^2],1}/V_{[(2n)^2 1^2],2}.$$

Let $Y = E_{2n+1,n+1}E_{n,2n+2}$.

To show that the residual representation $\mathcal{E}_{\tau\otimes\sigma}$ has a non-zero Fourier coefficient attached to the partition $[(2n)^2 1^2]$, we need to show that there is a $\varphi \in \mathcal{E}_{\tau\otimes\sigma}$, such that $\varphi^{\psi_{[(2n)^2 1^2]}}$ is non-vanishing. By a similar argument as in [9, Lemma 1.1], $\varphi^{\psi_{[(2n)^2 1^2]}}$ is non-vanishing if and only if the following integral is non-vanishing:

$$\int_{[V_{[(2n)^2 1^2],2}Y]} \varphi\,(vyg)\,\psi_{[(2n)^2 1^2]}^{-1}(v)\ \mathrm{d}v\mathrm{d}y. \tag{3.5}$$

First, one can see that the following quadruple satisfies all the conditions for [10, Corollary 7.1] (which is still true for the case of $\mathrm{SO}_{4n+2}^\delta(\mathbb{A})$):

$$\left(V_{[(2n)^2 1^2],2}X_{E_{n,2n+2}},\,\psi_{[(2n)^2 1^2]},\,E_{2n+1,n+1},\,E_{n,2n+1}\right). \tag{3.6}$$

Applying [10, Corollary 7.1], the integral in (3.5) is non-vanishing if and only the following integral is non-vanishing

$$\int_{[V_{[(2n)^2 1^2],2}Y']} \varphi\,(vyg)\,\psi_{[(2n)^2 1^2]}^{-1}(v)\,\mathrm{d}v\,\mathrm{d}y, \tag{3.7}$$

with $Y' = E_{n,2n+1}E_{n,2n+2}$.

Let $W = V_{[(2n)^2 1^2],2}Y'$, in which the elements have the following form:

$$w = \begin{pmatrix} z & q_1 & q_2 \\ 0 & I_2 & q_1^* \\ 0 & 0 & z^* \end{pmatrix} \begin{pmatrix} I_{2n} & 0 & 0 \\ p_1 & I_2 & 0 \\ p_2 & p_1^* & I_{2n} \end{pmatrix} \in \mathrm{SO}_{4n+2}^\delta, \tag{3.8}$$

where $z \in Z_{2n}$, the standard maximal unipotent subgroup of GL_{2n}; $q_1 \in \mathrm{Mat}_{2n \times 2}$, with $q_1(i, j) = 0$, for $n + 1 \leq i \leq 2n$ and $1 \leq j \leq 2$; $q_2 \in \mathrm{Mat}_{2n \times 2n}$, with $q_2(i, j) = 0$, for $i \geq j$; $p_1 \in \mathrm{Mat}_{2 \times 2n}$, with $p_1(i, j) = 0$ for $1 \leq i \leq 2$ and $1 \leq j \leq n + 1$; $p_2 \in \mathrm{Mat}_{2n \times 2n}$ (with certain additional property we do not specify here), with $p_2(i, j) = 0$, for $i \geq j$; all entries marked with a * are determined by the other entries and symmetry. Let $\psi_W(w) := \psi_{[(2n)^2 1^2]}(v)$, for $w = vy \in W$, where $v \in V_{[(2n)^2 1^2], 2}$ and $y \in Y'$. For $w \in W$ of form in (3.8), $\psi_W(w) = \psi(\sum_{i=1}^{2n-1} z_{i,i+1})$.

The idea of proving the non-vanishing property of the integral in (3.7) is to express the Fourier coefficient in (3.7) in terms of the coefficient in (3.20) (which is non-vanishing) and the coefficients in Lemma 3.2 (which are identically zero) using exchange of unipotent subgroups and Fourier expansion. More explicitly, we use the lower triangular part of elements w in (3.8) to fill the zeros in the upper triangular part. We proceed one row/column at a time, from top to bottom in the upper triangular part, and from left to right in the lower triangular part. For the first $n - 1$ rows, the number of zeros in the upper triangular part is exactly equal to the number of non-zero entries in the corresponding column in the lower triangular part. These rows will be treated via exchange of unipotent subgroups, using [10, Corollary 7.1]. For the $i = n, \ldots, 2n - 1$ rows, the number of zeros in the ith row in the upper triangular part exceeds the number of non-zero entries in the corresponding column in the lower triangular part by one. For the nth row, we first add the constant integral over the one-dimensional subgroup $E_{n, 3n+2}$, then do exchange of unipotent subgroups. For the row $i = n + 1, \ldots, 2n - 1$, we treat each row/column by a two-step process, that is, before exchanging unipotent subgroups for each row/column, one needs to take Fourier expansion along a one-dimensional subgroup $E_{i, 4n+2-i}$, producing Fourier coefficients in Lemma 3.2.

To continue, we define a sequence of unipotent subgroups as follows.

For $1 \leq i \leq n - 1$, $1 \leq j \leq i$, let $X_j^i = E_{i, 2n+3+i-j}$ and $Y_j^i = E_{2n+3+i-j, i+1}$. For $n \leq i \leq 2n - 2$, $1 \leq j \leq 2n - i - 1$, let $X_j^i = E_{i, 4n+2-i-j}$ and $Y_j^i = E_{4n+2-i-j, i+1}$. For $n + 1 \leq i \leq 2n - 1$, let $X_i = E_{i, 2n+1} E_{i, 2n+2}$ and $Y_i = E_{2n+1, i+1} E_{2n+2, i+1}$.

Let $\tilde{W}$ be the subgroup of W with elements of the form as in (3.8), but with the p_1 and p_2 parts zero. Let $\psi_{\tilde{W}} = \psi_W |_{\tilde{W}}$. For any subgroup of W containing $\tilde{W}$, we automatically extend $\psi_{\tilde{W}}$ trivially to this subgroup and still denote the character by $\psi_{\tilde{W}}$.

Next, we apply [10, Corollary 7.1] (exchanging unipotent subgroups) to a sequence of quadruples. For i going from 1 to $n - 1$, the following sequence of

quadruples satisfies all the conditions for [10, Corollary 7.1]:

$$\left(\tilde{W}_i \prod_{j=2}^{i} Y_j^i, \psi_{\tilde{W}}, X_1^i, Y_1^i \right),$$

$$\left(X_1^i \tilde{W}_i \prod_{j=3}^{i} Y_j^i, \psi_{\tilde{W}}, X_2^i, Y_2^i \right),$$

$$\cdots,$$

$$\left(\prod_{j=1}^{k-1} X_j^i \tilde{W}_i \prod_{j=k+1}^{i} Y_j^i, \psi_{\tilde{W}}, X_k^i, Y_k^i \right),$$

$$\cdots,$$

$$\left(\prod_{j=1}^{i-1} X_j^i \tilde{W}_i, \psi_{\tilde{W}}, X_i^i, Y_i^i \right),$$

(3.9)

where

$$\tilde{W}_i = \prod_{s=1}^{i-1} \prod_{j=1}^{s} X_j^s \tilde{W} \prod_{t=n+1}^{2n-1} Y_t \prod_{j=1}^{2n-t-1} Y_j^t \prod_{k=i+1}^{n} \prod_{j=1}^{k} Y_j^k.$$

Applying [10, Corollary 7.1] repeatedly to the above sequence of quadruples, one obtains that the integral in (3.7) is non-vanishing if and only if the following integral is non-vanishing:

$$\int_{[\tilde{W}_i']} \varphi(wg) \psi_{\tilde{W}_i'}^{-1}(w) \, dw,$$

(3.10)

where

$$\tilde{W}_i' = \prod_{s=1}^{i} \prod_{j=1}^{s} X_j^s \tilde{W} \prod_{t=n+1}^{2n-1} Y_t \prod_{j=1}^{2n-t-1} Y_j^t \prod_{k=i+1}^{n} \prod_{j=1}^{k} Y_j^k,$$

(3.11)

and $\psi_{\tilde{W}_i'}$ is extended from $\psi_{\tilde{W}}$ trivially.

To show that the integral over $\tilde{W}_{n-1}'$ in (3.10) is non-vanishing, it suffices to show that the following integral is non-vanishing:

$$\int_{[E_{n,3n+2}]} \int_{[\tilde{W}_{n-1}']} \varphi(wng) \psi_{\tilde{W}_{n-1}'}^{-1}(w) \, dw \, dn,$$

(3.12)

since it factors though the integral over $\tilde{W}'_{n-1}$ in (3.10). One can see that the following sequence of quadruples satisfies all the conditions for [10, Corollary 7.1]:

$$\left(E_{n,3n+2} \tilde{W}_n \prod_{j=2}^{n-1} Y_j^n, \psi_{\tilde{W}}, X_1^n, Y_1^n \right),$$

$$\left(X_1^n E_{n,3n+2} \tilde{W}_n \prod_{j=3}^{n-1} Y_j^n, \psi_{\tilde{W}}, X_2^n, Y_2^n \right),$$

$$\cdots,$$

$$\left(\prod_{j=1}^{k-1} X_j^n E_{n,3n+2} \tilde{W}_n \prod_{j=k+1}^{n-1} Y_j^n, \psi_{\tilde{W}}, X_k^n, Y_k^n \right),$$

$$\cdots,$$

$$\left(\prod_{j=1}^{n-2} X_j^n E_{n,3n+2} \tilde{W}_n, \psi_{\tilde{W}}, X_{n-1}^n, Y_{n-1}^n \right),$$

(3.13)

where

$$\tilde{W}_n = \prod_{t=1}^{n-1} \prod_{j=1}^{t} X_j^t \tilde{W} \prod_{k=n+1}^{2n-1} Y_k \prod_{j=1}^{2n-k-1} Y_j^k.$$

Applying [10, Corollary 7.1] repeatedly to the above sequence of quadruples, the integral in (3.12) is non-vanishing if and only if the following integral is non-vanishing:

$$\int_{[\tilde{W}'_n]} \varphi(wg) \psi_{\tilde{W}'_n}^{-1}(w) \, dw,$$

(3.14)

where

$$\tilde{W}'_n = E_{n,3n+2} \prod_{j=1}^{n-1} X_j^n \prod_{t=1}^{n-1} \prod_{j=1}^{t} X_j^t \tilde{W} \prod_{k=n+1}^{2n-1} Y_k \prod_{j=1}^{2n-k-1} Y_j^k,$$

(3.15)

and $\psi_{\tilde{W}'_n}$ is extended from $\psi_{\tilde{W}}$ trivially.

For i going from $n+1$ to $2n-1$, define

$$\tilde{W}'_i = \prod_{s=n+1}^{i} X_s \prod_{j=1}^{2n-s-1} X_j^s \prod_{\ell=n}^{i} E_{\ell,4n+3-\ell} \prod_{t=1}^{n-1} \prod_{j=1}^{t} X_j^t \tilde{W} \prod_{k=i+1}^{2n-1} Y_k \prod_{j=1}^{2n-k-1} Y_j^k,$$

(3.16)

and $\psi_{\tilde{W}_i'}$ is extended from $\psi_{\tilde{W}}$ trivially. We claim that the following integral is non-vanishing:

$$\int_{[\tilde{W}_{i-1}']} \varphi(wg)\, \psi_{\tilde{W}_{i-1}'}^{-1}(w)\, \mathrm{d}w, \tag{3.17}$$

if and only if the following integral is non-vanishing:

$$\int_{[\tilde{W}_i']} \varphi(wg)\, \psi_{\tilde{W}_i'}^{-1}(w)\, \mathrm{d}w, \tag{3.18}$$

Indeed, first, we take the Fourier expansion of the integral in (3.17) along $E_{i,4n+2-i}$. Under the action of GL_1, we get two kinds of Fourier coefficients corresponding to the two orbits of the dual of $[E_{i,4n+2-i}]$: the trivial one and the non-trivial one. By Lemma 3.2, which will be stated and proved right after the proof of the theorem, all the Fourier coefficients corresponding to the non-trivial orbit are identically zero. Therefore, only the Fourier coefficient attached to the trivial orbit survives. When $i = 2n-1$, the Fourier coefficient attached to the trivial orbit is exactly the integral in (3.18), that is, the claim is proved. When $n+1 \le i \le 2n-2$, one can see that the following sequence of quadruples satisfies all the conditions for [10, Corollary 7.1]:

$$\left(E_{i,4n+2-i}\tilde{W}_i Y_i \prod_{j=2}^{2n-i-1} Y_j^i,\ \psi_{\tilde{W}},\ X_1^i,\ Y_1^i \right),$$

$$\left(X_1^i E_{i,4n+2-i}\tilde{W}_i Y_i \prod_{j=3}^{2n-i-1} Y_j^i,\ \psi_{\tilde{W}},\ X_2^i,\ Y_2^i \right),$$

$$\cdots,$$

$$\left(\prod_{j=1}^{k-1} X_j^i E_{i,4n+2-i}\tilde{W}_i Y_i \prod_{j=k+1}^{2n-i-1} Y_j^i,\ \psi_{\tilde{W}},\ X_k^i,\ Y_k^i \right),$$

$$\cdots,$$

$$\left(\prod_{j=1}^{2n-i-2} X_j^i E_{i,4n+2-i}\tilde{W}_i Y_i,\ \psi_{\tilde{W}},\ X_{2n-i-1}^i,\ Y_{2n-i-1}^i \right),$$

$$\left(\prod_{j=1}^{2n-i-1} X_j^i E_{i,4n+2-i}\tilde{W}_i,\ \psi_{\tilde{W}},\ X_i,\ Y_i \right),$$

where

$$\tilde{W}_i = \prod_{s=n+1}^{i-1} X_s \prod_{j=1}^{2n-s-1} X_j^s \prod_{\ell=n}^{i-1} E_{\ell,4n+2-\ell} \prod_{t=1}^{n-1} \prod_{j=1}^{t} X_j^t \tilde{W} \prod_{k=i+1}^{2n-1} Y_k \prod_{j=1}^{2n-k-1} Y_j^k.$$

Applying [10, Corollary 7.1] repeatedly to the above sequence of quadruples, we deduce that the Fourier coefficient attached to the trivial orbit above is non-vanishing if and only if the integral in (3.18) is non-vanishing. This proves the claim.

It is easy to see that elements of $\tilde{W}'_{2n-1}$ have the following form:

$$w = \begin{pmatrix} z & q_1 & q_2 \\ 0 & I_2 & q_1^* \\ 0 & 0 & z^* \end{pmatrix}, \tag{3.19}$$

where $z \in Z_{2n}$, the standard maximal unipotent subgroup of GL_{2n}; $q_1 \in \mathrm{Mat}_{2n \times 2}$, with $q_1(2n, j) = 0$, for $1 \le j \le 2$; $q_2 \in \mathrm{Mat}_{2n \times 2n}$, with certain symmetry property we do not specify here; all entries marked with a $*$ are determined by the other entries and symmetry. For $w \in \tilde{W}'_{2n-1}$ of form in (3.19), $\psi_{\tilde{W}'_{2n-1}}(w) = \psi(\sum_{i=1}^{2n-1} z_{i,i+1})$.

Now, we need to take the Fourier expansion of the integral over $\tilde{W}'_{2n-1}$ in (3.18) along

$$E_{2n,2n+1} E_{2n,2n+2},$$

which can be identified with a certain quadratic extension E of F, and a torus which can be identified with $F^\times \times (E^\times)^1$ acting on it. Here, $(E^\times)^1 \subset E^\times$ is the kernel of the norm map to $F^\times$. This Fourier expansion gives us two kinds of Fourier coefficients corresponding to the two orbits of characters: the trivial one and the non-trivial one, and the Fourier coefficients corresponding to the non-trivial orbit are generic Fourier coefficients. Since $\mathcal{E}_{\tau \otimes \sigma}$ is not generic, only the Fourier coefficient corresponding to the trivial orbit survives. Therefore, the integral over $\tilde{W}'_{2n-1}$ in (3.18) becomes

$$\int_{[E_{2n,2n+1} E_{2n,2n+2}]} \int_{[\tilde{W}'_{2n-1}]} \varphi(wxg) \, \psi_{\tilde{W}'_{2n-1}}^{-1}(w) \, \mathrm{d}w \mathrm{d}x$$

$$= \int_{N_{2n}} \varphi(ng) \, \psi_{N_{2n}}^{-1}(n) \, \mathrm{d}n, \tag{3.20}$$

where N_{2n} is the unipotent radical of the parabolic subgroup with Levi isomorphic to $\mathrm{GL}_1^{2n} \times \mathrm{SO}_2^\delta$, and $\psi_{N_{2n}}(n) = \psi(\sum_{i=1}^{2n-1} n_{i,i+1})$.

Let U be the unipotent radical of the parabolic subgroup $P = MU$ with Levi $M \cong \mathrm{GL}_{2n} \times \mathrm{SO}_2^\delta$. Write $N_{2n} = U N'_{2n}$ with $N'_{2n} = M \cap N_{2n}$. Then the integral in (3.20)

becomes

$$\int_{[N'_{2n}]} \varphi^U (ng) \, \psi_{N'_{2n}}^{-1} (n) \, dn, \tag{3.21}$$

where φ^U is the constant term of φ along U, and $\psi_{N'_{2n}} = \psi_{N_{2n}}|_{N'_{2n}}$. By a similar calculation as in [18], one can see that $\varphi^U \in A(U(\mathbb{A})M(F)\backslash SO_{4n+2}^\delta(\mathbb{A}))_{|\cdot|^{-\frac{1}{2}} \tau \otimes \sigma}$. Since τ is generic, it follows that the integral in (3.21) is non-vanishing. Therefore, the integral in (3.5), hence the Fourier coefficient attached to $[(2n)^2 1^2]$ in (3.4) is non-vanishing.

This completes the proof of the proposition.   ∎

Now, we prove the lemma used in the above proof.

Lemma 3.2. For any $\varphi \in \mathcal{E}_{\tau \otimes \sigma}$, the following integral is identically zero:

$$\int_{[E_{i+1,4n+1-i}]} \int_{[\tilde{W}'_i]} \varphi (wxg) \, \psi_{\tilde{W}'_i}^{-1} (w) \, \psi^{-1} (ax) \, dw \, dx, \tag{3.22}$$

where $n \le i \le 2n - 2$, $a \in F^\times$, $\tilde{W}'_n$ is as in (3.15) with $i = n$ there, and $\tilde{W}'_i$ is as in (3.16) for $n + 1 \le i \le 2n - 2$.   □

Proof.   We continue to use notation introduced in Proposition 3.1. First note that elements in $\tilde{W}'_i$ have the following form:

$$w = \begin{pmatrix} z & q_1 & q_2 \\ 0 & I_2 & q_1^* \\ 0 & 0 & z^* \end{pmatrix} \begin{pmatrix} I_{2n} & 0 & 0 \\ p_1 & I_2 & 0 \\ p_2 & p_1^* & I_{2n} \end{pmatrix} \in SO_{4n+2}^\delta, \tag{3.23}$$

where $z \in Z_{2n}$, the standard maximal unipotent subgroup of GL_{2n}; $q_1 \in \text{Mat}_{2n \times 2}$, with $q_1(k, j) = 0$, for $i + 1 \le k \le 2n$ and $1 \le j \le 2$; $q_2 \in \text{Mat}_{2n \times 2n}$, with $q_2(k, j) = 0$, for $i + 1 \le k \le 2n$ and $1 \le j \le 2n - i - 1$; $p_1 \in \text{Mat}_{2 \times 2n}$, with $p_1(k, j) = 0$ for $1 \le k \le 2$ and $1 \le j \le i + 1$; $p_2 \in \text{Mat}_{2n \times 2n}$, with $p_2(k, j) = 0$, for $2n - i \le k \le 2n$ or $1 \le j \le i + 1$; all entries marked with a $*$ are determined by the other entries and symmetry. For $w \in \tilde{W}'_i$ of form in (3.23), $\psi_{\tilde{W}'_i}(w) = \psi(\sum_{i=1}^{2n-1} z_{i,i+1})$.

In the rest of this proof, we set $\tilde{W}_i^{(1)} := \tilde{W}'_i$, and denote by $\tilde{W}_i^{(2)}$ the subgroup of $\tilde{W}_i^{(1)}$ consisting of the elements of the following form:

$$\begin{pmatrix} I_{i+1} & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & z^* & 0 \\ 0 & 0 & 0 & 0 & I_{i+1} \end{pmatrix},$$

where $z \in Z_{2n-i-1}$, the standard maximal unipotent subgroup of GL_{2n-i-1}. Write

$$\tilde{W}_i^{(1)} = \tilde{W}_i^{(2)} \tilde{W}_i^{(3)},$$

where $\tilde{W}_i^{(3)}$ is a subgroup of $\tilde{W}_i^{(1)}$ consisting of elements with $\tilde{W}_i^{(2)}$-part being trivial. Let $\psi_{\tilde{W}_i^{(2)}} = \psi_{\tilde{W}_i^{(1)}}|_{\tilde{W}_i^{(2)}}$ and $\psi_{\tilde{W}_i^{(3)}} = \psi_{\tilde{W}_i^{(1)}}|_{\tilde{W}_i^{(3)}}$. Then the integral in (3.22) can be written as

$$\int_{[\tilde{W}_i^{(2)}]} \int_{[E_{i+1,4n+1-i}]} \int_{[\tilde{W}_i^{(3)}]} \varphi(w_1 x w_2 g) \psi_{\tilde{W}_i^{(2)}}^{-1}(w_2) \psi^{-1}(ax) \psi_{\tilde{W}_i^{(3)}}^{-1}(w_1) \, \mathrm{d}w_1 \, \mathrm{d}x \, \mathrm{d}w_2. \tag{3.24}$$

Therefore, to show that the integral in (3.22) is identically zero, it is suffices to show that the following integral is identically zero

$$\int_{[E_{i+1,4n+1-i}]} \int_{[\tilde{W}_i^{(3)}]} \varphi(w x g) \psi^{-1}(ax) \psi_{\tilde{W}_i^{(3)}}^{-1}(w) \, \mathrm{d}w \, \mathrm{d}x. \tag{3.25}$$

To continue, we define a sequence of unipotent subgroups as follows. For $1 \le j \le 2n - i - 2$, let $X_j^{i+1} = E_{i+1,4n+1-i-j}$ and $Y_j^{i+1} = E_{4n+1-i-j,i+1}$. Let $X_{i+1} = E_{i+1,2n+1} E_{i+1,2n+2}$ and $Y_{i+1} = E_{2n+1,i+2} E_{2n+2,i+2}$. Write $\tilde{W}_i^{(3)} = \tilde{W}_i^{(4)} \prod_{j=1}^{2n-i-2} Y_j^{i+1} Y_{i+1}$, where $\tilde{W}_i^{(4)}$ is a subgroup of $\tilde{W}_i^{(3)}$ consisting of elements with $\prod_{j=1}^{2n-i-2} Y_j^{i+1} Y_{i+1}$-part being trivial. Let $\psi_{\tilde{W}_i^{(4)}} = \psi_{\tilde{W}_i^{(3)}}|_{\tilde{W}_i^{(4)}}$. Then, one can see that the following sequence of quadruples satisfies the conditions in [10, Corollary 7.1]:

$$\left(E_{i+1,4n+1-i} \tilde{W}_i^{(4)} Y_{i+1} \prod_{j=2}^{2n-i-2} Y_j^{i+1}, \psi_{\tilde{W}^{(4)}}, X_1^{i+1}, Y_1^{i+1} \right),$$

$$\left(X_1^{i+1} E_{i+1,4n+1-i} \tilde{W}_i^{(4)} Y_{i+1} \prod_{j=3}^{2n-i-2} Y_j^{i+1}, \psi_{\tilde{W}^{(4)}}, X_2^{i+1}, Y_2^{i+1} \right),$$

$$\cdots,$$

$$\left(\prod_{j=1}^{k-1} X_j^{i+1} E_{i+1,4n+1-i} \tilde{W}_i^{(4)} Y_{i+1} \prod_{j=k+1}^{2n-i-2} Y_j^{i+1}, \psi_{\tilde{W}^{(4)}}, X_k^{i+1}, Y_k^{i+1} \right),$$

$$\cdots,$$

$$\left(\prod_{j=1}^{2n-i-3} X_j^{i+1} E_{i+1,4n+1-i} \tilde{W}_i^{(4)} Y_{i+1}, \psi_{\tilde{W}^{(4)}}, X_{2n-i-2}^{i+1}, Y_{2n-i-2}^{i+1} \right),$$

$$\left(\prod_{j=1}^{2n-i-2} X_j^{i+1} E_{i+1,4n+1-i} \tilde{W}_i^{(4)}, \psi_{\tilde{W}^{(4)}}, X_{i+1}, Y_{i+1} \right).$$

After applying [10, Corollary 7.1] to the above sequence of quadruples, one can see that the integral in (3.25) is identically zero if and only if the following integral is identically zero:

$$\int_{[\prod_{j=1}^{2n-i-2} X_j^{i+1} E_{i+1,4n+1-i}]} \int_{[\tilde{W}_i^{(4)}]} \varphi\,(wxg)\,\psi^{-1}\,(ax)\,\psi_{\tilde{W}_i^{(4)}}^{-1}\,(w)\,\mathrm{d}w\,\mathrm{d}x. \tag{3.26}$$

Finally, the integral in (3.26) contains an inner integral which is exactly a Fourier coefficient attached to the partition $[(2i+3)1^{4n-2i-1}]$. Since $n \leq i \leq 2n-2$, by Proposition 2.6, any Fourier coefficient attached to the partition $[(2i+3)1^{4n-2i-1}]$ will be identically zero. Therefore, the integral in (3.26) is identically zero, and hence the integral in (3.22) is identically zero. This completes the proof of the lemma. ∎

3.2   **Parts (3), (5), and (6) of Theorem** 1.2

We are ready to prove Parts (3), (5), and (6) of Theorem 1.2 based on the preparation in Section 3.1.

Proposition 3.3 (Part (3) of Theorem 1.2). *The residual representation $\mathcal{E}_{\tau\otimes\sigma}$ has a non-zero Fourier coefficient attached to the partition $[(2n+1)1^{2n+1}]$.* ☐

Proof.   We prove this theorem by contradiction. Assume that the residual representation $\mathcal{E}_{\tau\otimes\sigma}$ has no non-zero Fourier coefficients attached to the partition $[(2n+1)1^{2n+1}]$. In the following, we will continue to use the notation introduced in Proposition 3.1.

In Proposition 3.1, we have proved that the integral in (3.5) is non-vanishing. We name this integral as follows:

$$f(g) := \int_{[V_{[(2n)^2 1^2],2} Y]} \varphi\,(vyg)\,\psi_{[(2n)^2 1^2]}^{-1}\,(v)\,\mathrm{d}v\,\mathrm{d}y, \quad g \in \mathrm{SO}_{4n+2}^{\delta}\,(\mathbb{A})\,,\ \varphi \in \mathcal{E}_{\tau\otimes\sigma}. \tag{3.27}$$

Let

$$\omega = \begin{pmatrix} 0 & 0 & I_{2n} \\ 0 & I_2 & 0 \\ I_{2n} & 0 & 0 \end{pmatrix}. \tag{3.28}$$

Then it is easy to see that $f(g) = f(\omega g)$.

Applying the equality in [10, Lemma 7.1] to the quadruple in (3.6), one can see that

$$f(g) = \int_{E_{2n+1,n+1}(\mathbb{A})} \int_{[V_{[(2n)^2 1^2],2} Y']} \varphi\,(vyxg)\,\psi_{[(2n)^2 1^2]}^{-1}\,(v)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}x. \tag{3.29}$$

Applying the equality in [10, Lemma 7.1] repeatedly to the sequence of quadruples in (3.9) for i going from 1 to $n-1$, we obtain that

$$f(g) = \int_{\prod_{s=1}^{n-1} \prod_{j=1}^{s} Y_j^s E_{2n+1,n+1}(\mathbb{A})} \int_{[\tilde{W}'_{n-1}]} \varphi(wxg) \psi_{\tilde{W}'_{n-1}}^{-1}(w) \, dw \, dx. \tag{3.30}$$

Next, take the Fourier expansion of f along $E_{n,3n+2}$. Under the action of GL_1, we get two kinds of Fourier coefficients corresponding to the two orbits of the dual of $[E_{n,3n+2}]$: the trivial one and the non-trivial one. Since we have assumed that $\mathcal{E}_{\tau\otimes\sigma}$ has no non-zero Fourier coefficients attached to the partition $[(2n+1)1^{2n+1}]$, by a similar argument to that in the proof of Lemma 3.2, all the Fourier coefficients corresponding to the non-trivial orbit are identically zero. Hence, we obtain that

$$f(g) = \int_{\prod_{s=1}^{n-1} \prod_{j=1}^{s} Y_j^s E_{2n+1,n+1}(\mathbb{A})} \int_{[E_{n,3n+2}]} \int_{[\tilde{W}'_{n-1}]} \varphi(wxyg) \psi_{\tilde{W}'_{n-1}}^{-1}(w) \, dw \, dx \, dy. \tag{3.31}$$

Then, applying the equality in [10, Lemma 7.1] repeatedly to the sequence of quadruples in (3.13), one can see that

$$f(g) = \int_{\prod_{j=1}^{n-1} Y_j^n \prod_{s=1}^{n-1} \prod_{j=1}^{s} Y_j^s E_{2n+1,n+1}(\mathbb{A})} \int_{[\tilde{W}'_n]} \varphi(wxg) \psi_{\tilde{W}'_n}^{-1}(w) \, dw \, dx. \tag{3.32}$$

Carrying out the steps from (3.17) to (3.21), applying the equality in [10, Lemma 7.1] instead of [10, Corollary 7.1], we obtain that

$$f(g) = \int_{\prod_{s=n+1}^{2n-1} Y_s \prod_{j=1}^{2n-s-1} Y_j^s \prod_{k=1}^{n-1} \prod_{\ell=1}^{k} Y_\ell^k E_{2n+1,n+1}(\mathbb{A})} \int_{[N'_{2n}]} \varphi^U(nxg) \psi_{N'_{2n}}^{-1}(n) \, dn \, dx, \tag{3.33}$$

where N'_{2n}, U, $\psi_{N'_{2n}}^{-1}$ are defined to that in the proof of Proposition 3.1, φ^U is the constant term of φ along U. Note that in (3.33), $\prod_{s=n+1}^{2n-1} Y_s \prod_{j=1}^{2n-s-1} Y_j^s \prod_{k=1}^{n-1} \prod_{\ell=1}^{k} Y_\ell^k E_{2n+1,n+1}$ is equal to $V_{[(2n)^2 1^2, 2]} Y \cap U^-$, where U^- is the unipotent radical of the parabolic subgroup opposite to $P = MU$. By a similar calculation to that in [18], we deduce that

$$\varphi^U \in \mathcal{A}\left(U(\mathbb{A}) M(F) \backslash \mathrm{SO}_{4n+2}^\delta(\mathbb{A})\right)_{|\cdot|^{-\frac{1}{2}} \tau\otimes\sigma}.$$

For $t \in \mathbb{A}^\times$, let $D(t) = \begin{pmatrix} tI_{2n} & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & t^{-1}I_{2n} \end{pmatrix}$. Then it is easy to see that $\omega D(t)\omega^{-1} = D(t^{-1})$, where ω is the Weyl element defined in (3.28). Consider $f(D(t)g)$. Note that $f(g)$

has the form in (3.33). Conjugating $D(t)$ to the left, by changing of variables on

$$\prod_{s=n+1}^{2n-1} Y_s \prod_{j=1}^{2n-s-1} Y_j^s \prod_{k=1}^{n-1} \prod_{\ell=1}^{k} Y_\ell^k E_{2n+1,n+1}(\mathbb{A}),$$

we get a factor $|t|_{\mathbb{A}}^{-2n^2+1}$. Since $\varphi^U \in \mathcal{A}(U(\mathbb{A})M(F)\backslash \mathrm{SO}_{4n+2}^{\delta}(\mathbb{A}))_{|\cdot|^{-\frac{1}{2}}\tau\otimes\sigma}$, we have

$$\varphi^U (D(t)\, nxg) = \delta_P (D(t))^{\frac{1}{2}} |D(t)|^{-\frac{1}{2}} \varphi^U (nxg) = |t|_{\mathbb{A}}^{2n^2} \varphi^U (nxg).$$

Note that $P = MU$ is the parabolic subgroup with Levi $M \cong \mathrm{GL}_{2n} \times \mathrm{SO}_2^{\delta}$. Therefore, one can get $f(D(t)g) = |t|_{\mathbb{A}} f(g)$. On the other hand,

$$f(D(t)\, g) = f(\omega D(t)\, g) = f\left(D\left(t^{-1}\right)\omega g\right) = |t^{-1}|_{\mathbb{A}} f(\omega g) = |t|_{\mathbb{A}}^{-1} f(g).$$

Hence, $|t|_{\mathbb{A}} f(g) = |t|_{\mathbb{A}}^{-1} f(g)$. Since $t \in \mathbb{A}^{\times}$, we get $f(g) \equiv 0$, which is a contradiction.

Therefore, the residual representation $\mathcal{E}_{\tau\otimes\sigma}$ has a non-zero Fourier coefficient attached to the partition $[(2n+1)1^{2n+1}]$. This completes the proof of the proposition. ∎

We remark that the above fleshes out a sketch given in [6, Section 5], with full details for each step and precise references as needed.

Proposition 3.4 (Part (5) of Theorem 1.2). *Every irreducible component of the twisted automorphic descent $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ has a non-zero Fourier coefficient attached to the partition $[(2n-1)1^2]$.* □

Proof.   Recall that $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ is the completion space spanned by the cuspidal automorphic functions produced by the twisted descent, by Part (2) of Theorem 1.2. This twisted descent $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ projects onto the decomposition of the cuspidal spectrum of $\mathrm{SO}_{2n+1}^{\delta,\alpha}$. Let π be any irreducible constituent in this decomposition. Consider the following integral:

$$\langle \varphi_{\pi}, \xi^{\psi_{n,\alpha}} \rangle = \int_{[\mathrm{SO}_{2n+1}^{\delta,\alpha}]} \varphi_{\pi}(h)\, \overline{\xi^{\psi_{n,\alpha}}}(h)\, \mathrm{d}h, \tag{3.34}$$

which is non-zero for some data $\varphi_{\pi} \in \pi$, $\xi \in \mathcal{E}_{\tau\otimes\sigma}$, since π is an irreducible component of $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$.

Assume that $\xi = \mathrm{Res}_{s=\frac{1}{2}} E(\cdot, s, \phi_{\tau\otimes\sigma})$; then from (3.34) we know that the following integral is also non-zero for some choice of data:

$$\langle \varphi_{\pi}, E(\cdot, s, \phi_{\tau\otimes\sigma})^{\psi_{n,\alpha}} \rangle = \int_{[\mathrm{SO}_{2n+1}^{\delta,\alpha}]} \varphi_{\pi}(h)\, \overline{E(h, s, \phi_{\tau\otimes\sigma})^{\psi_{n,\alpha}}}\, \mathrm{d}h. \tag{3.35}$$

Then, by the unfolding in [21, Proposition 3.7] (take $j = 2n$, $\ell = n$ and $\beta = j - \ell = n$ there), the non-vanishing of the integral in (3.35) implies the non-vanishing of $\varphi_\pi{}^{\psi_{n-1,\alpha'}}$ for some $\alpha' \in F^\times$. Therefore, π has a non-vanishing Fourier coefficient attached to the partition $[(2n-1)1^2]$. This completes the proof of the proposition.   ∎

Finally, Propositions 3.3 and 3.4 imply that the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ has a non-zero Fourier coefficient attached to the partition $[(2n+1)(2n-1)1^2]$ by a similar argument as in the proofs of [9, Lemma 2.6; 17, Lemma 3.1 and Proposition 3.3]. This proves Part (6) of Theorem 1.2. The analogues for orthogonal groups of [17, Lemma 3.1 and Proposition 3.3] will be discussed in [23].

It is natural to ask about the relation of Part (6) of Theorem 1.2 with Conjecture 4.2 in [16]. More precisely, one may note that, for suitable pairs (α, δ), the group $\mathrm{SO}_{2n+1}^{\delta,\alpha}$ is split, and one may then ask for which such pairs the twisted automorphic descent $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\sigma})$ has no generic irreducible summands. As we remarked in the Introduction, this is one of the technical key points to connect the construction of twisted automorphic descents to the structure of more general global packets. In Section 5, we will provide the complete theory for the case of $n = 1$, which recovers the Jacquet–Langlands correspondence for PGL_2.

4 Relations with the Langlands Functorial Transfers

We first give a proof for Part (4) of Theorem 1.2 and then discuss the relation between the construction of the twisted automorphic descent given in this paper and the automorphic descent given by Ginzburg *et al.* in [10], which is the relation with the corresponding Langlands functorial transfer.

4.1 Part (4) of Theorem 1.2

Assume that the twisted automorphic descent $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ is non-zero. By Part (2) of Theorem 1.2, $\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ is cuspidal on $\mathrm{SO}_{2n+1}^{\delta,\alpha}(\mathbb{A})$. We write

$$\sigma_{\psi_{n_\alpha}}(\mathcal{E}_{\tau\otimes\sigma}) = \oplus_i \pi_i,$$

where π_i are irreducible cuspidal automorphic representations of $\mathrm{SO}_{2n+1}^{\delta,\alpha}(\mathbb{A})$. We have to show that

   (1)   the irreducible summands are nearly equivalent; that is, their local components at almost all unramified local places are equivalent; and

(2) the weak Langlands functorial transfer from $\mathrm{SO}_{2n+1}^{\delta,\alpha}$ to GL_{2n} takes π_i to the given τ.

It is clear that (2) implies (1). As discussed in [5], when α varies in the square classes of $F^\times$, the family $\mathrm{SO}_{2n+1}^{\delta,\alpha}$ are pure inner F-rational forms of each other, and hence they share the Langlands dual group, which is $\mathrm{Sp}_{2n}(\mathbb{C})$. The Langlands functorial transfer considered here is the one determined by the natural embedding of $\mathrm{Sp}_{2n}(\mathbb{C})$ into $\mathrm{GL}_{2n}(\mathbb{C})$.

Let π be one of the irreducible summands in the decomposition of $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$. Write $\pi = \otimes_v \pi_v$. Take v to be a finite local place of F where π_v, τ_v, σ_v, and the other relevant data in the construction of the descent are unramified. Hence, π_v is an irreducible unramified representation of the F_v-split $\mathrm{SO}_{2n+1}(F_v)$, occurring as the unramified constituent in the Jacquet module $J_{\psi_{n,\alpha}}(\mathrm{Ind}_{Q_{2n}(F_v)}^{L_{n,\alpha}(F_v)}\tau_v|\det(\cdot)|_v^{\frac{1}{2}} \otimes \sigma_v)$ by the local–global relation of the construction of the descent.

On the other hand, since τ_v is unramified, generic, and self-dual, it follows that τ_v is the fully induced representation

$$\mu_1 \times \cdots \times \mu_n \times \mu_n^{-1} \times \cdots \times \mu_1^{-1},$$

where μ_is are unramified characters. Also, σ_v is an unramified character μ_0 on $F_v^\times$ or a trivial representation on $\mathrm{SO}_2^\delta(F_v)$, depending on whether J_δ is split or not. Now using Proposition 2.5, we obtain that the Satake parameter of π_v is transferred to the Satake parameter of τ_v. This completes the proof of Part (4) of Theorem 1.2.

4.2 On functorial relations

When an irreducible unitary cuspidal representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$ has the property that the exterior square L-function $L(s, \tau, \wedge^2)$ has a pole at $s = 1$, the automorphic descent of Ginzburg *et al.* in [10] constructs an irreducible generic cuspidal automorphic representation π_0 of F-split $\mathrm{SO}_{2n+1}(\mathbb{A})$. The irreducibility of their descent was proved in [19].

On the other hand, when τ has the additional property that the central value $L(\frac{1}{2}, \tau \otimes \sigma)$ is non-zero, the theory of the twisted automorphic descents produces additional irreducible cuspidal automorphic representations π of the split form of SO_{2n+1}. These representations arise at values of α where $\mathrm{SO}^{\delta,\alpha}$ happens to be split, and are nearly equivalent to π_0 by Part (4) of Theorem 1.2. Also, they belong to the same global L-packet with the global Arthur parameter $(\tau, 1)$ [2].

Moreover, for the α's, which make $\mathrm{SO}_{2n+1}^{\delta,\alpha}$ non-split over F, the twisted automorphic descents $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ produce irreducible cuspidal automorphic

representations π of $SO_{2n+1}^{\delta,\alpha}(\mathbb{A})$, which are still nearly equivalent to π_0 and hence are expected to be in the same global Vogan packet as π_0. By a careful investigation one ought to be able to verify the expectation regarding global Vogan packets, and we refer to the reader [22] for more detailed discussion on the general theory related to this issue. A complete story is discussed in Section 5 for the special case of $n = 1$.

5 Explicit Theory of Twisted Automorphic Descents for the Case of $n = 1$

In this section, we will give an explicit description on the twisted automorphic descent $\sigma_{\psi_{n,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ in the case $n = 1$, which recovers the global Jacquet–Langlands correspondence between GL_2 and its inner forms.

In this case, we have that $H^\delta = SO_6^\delta$ and we take $\ell = 1$. The quadratic subspace W_1, as defined in Section 1.2, has a basis $\{e_2, e_0^{(1)}, e_0^{(2)}, e_{-2}\}$. Recall that $P_1 = M_1 N_1$ is the standard parabolic subgroup with the Levi $M_1 = GL_1 \times SO(W_1)$, and

$$N_1 = \left\{ n_1(x) = \begin{pmatrix} 1 & x & -\frac{1}{2}q(x,x) \\ & I_4 & x' \\ & & 1 \end{pmatrix} \mid x \in W_1 \right\},$$

where $q(\cdot,\cdot)$ is the quadratic form of W_1. Since the Witt index of W_1 is greater than zero, $SO(W_1)$ acts transitively on the set of vectors of the same length. Thus, for every anisotropic vector w_0 in W_1, we may choose $y_\alpha = e_2 - \frac{\alpha}{2}e_{-2}$ as a standard anisotropic vector for some $\alpha \in F^\times$. Then we focus on the character $\psi_{1,\alpha}$ of $N_1(F)\backslash N_1(\mathbb{A})$ and the stabilizer $L_{1,\alpha} = SO(J_{\delta,\alpha})$. If $J_{\delta,\alpha}$ is non-split, then $L_{1,\alpha}$ is isomorphic to $PD_{\delta,\alpha}^\times$, where $D_{\delta,\alpha}$ is the quaternion algebra $(\frac{-\delta,-\alpha}{F})$. Note that $D_{\delta,\alpha}$ is uniquely determined by the coset of α in $F^\times/\mathrm{Nm}(F(\sqrt{-\delta})^\times)$.

Recall that σ is a character of $SO_2^\delta(\mathbb{A})$ and τ is an irreducible unitary cuspidal automorphic representation of $GL_2(\mathbb{A})$. By Proposition 1.1, the residual representation $\mathcal{E}_{\tau\otimes\sigma}$ is non-zero if and only if $L(s,\tau,\wedge^2)$ has a pole at $s = 1$ and $L(\frac{1}{2},\tau\times\sigma) \neq 0$. Note that $L(s,\tau,\wedge^2)$ has a pole at $s = 1$ if and only if τ has a trivial central character. Throughout this section, we assume that τ has a trivial central character and hence τ can be regarded as a representation of $PGL_2(\mathbb{A})$.

Let π be an irreducible cuspidal automorphic representation of $L_{1,\alpha}(\mathbb{A})$ and φ be an automorphic form in π. Define the global zeta integral by

$$\mathcal{Z}\left(s, \phi_{\tau\otimes\sigma}, \varphi, \psi_{1,\alpha}\right) = \int_{[L_{1,\alpha}]} \varphi(h)\, E^{\psi_{1,\alpha}}(h, s, \phi_{\tau\otimes\sigma})\, dh, \tag{5.1}$$

where $E^{\psi_{1,\alpha}}(\cdot)$ is the $\psi_{1,\alpha}$-Fourier coefficient of the Eisenstein series $E(h, s, \phi_{\tau\otimes\sigma})$ defined in (1.1). This global zeta integral is a special case of those considered in [7] and [21].

We recall some notation in [21] before evaluating this global zeta integral. Let ϵ be the minimal representative in the Weyl group of SO_6^δ corresponding to the open cell in $P_2\backslash \mathrm{SO}_6^\delta/P_1$, that is,

$$
\epsilon = \begin{pmatrix}
0 & 1 & & & 0 & 0 \\
0 & 0 & & & 0 & 1 \\
& & 1 & 0 & & \\
& & 0 & -1 & & \\
1 & 0 & & & 0 & 0 \\
0 & 0 & & & 1 & 0
\end{pmatrix} \in \mathrm{SO}_6^\delta;
$$

then the complement of $N_1^\epsilon = N_1 \cap \epsilon^{-1} P_2 \epsilon$ in N_1 is

$$
\bar{N}_1^\epsilon := \{n_1\left((x, y, z, 0)\right) \mid (x, y, z, 0) \in W_1\}.
$$

Define

$$
\mathcal{J}\left(s, \phi_\alpha^W\right)(h) = \int_{N_1^\epsilon(\mathbb{A})\backslash N_1(\mathbb{A})} \lambda_s \phi_\alpha^W\left(\epsilon n h\right) \psi_{1,\alpha}^{-1}(n)\ \mathrm{d}n,
$$

with

$$
\phi_\alpha^W(h) = \int_{[N_{1,2}]} \phi_{\tau\otimes\sigma}\left(u_{1,2}(x)h\right) \psi^{-1}\left(\frac{\alpha}{2}x\right)\ \mathrm{d}x.
$$

where $N_{1,2} = \left\{ u_{1,2}(x) = \begin{pmatrix} \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} & & \\ & \begin{smallmatrix} 1 \\ & 1 \end{smallmatrix} & \begin{smallmatrix} -x \\ 1 \end{smallmatrix} \\ \end{pmatrix} \right\}$. Since the adjoint action of ϵ on the subgroup SO_2^δ of the Levi subgroup of P_2 is the inverse on SO_2^δ, we obtain that

$$
\mathcal{J}\left(s, \phi_\alpha^W\right)(xh) = \sigma^{-1}(x)\, \mathcal{J}\left(s, \phi_\alpha^W\right)(h),
$$

where $x \in \mathrm{SO}_2^\delta$ is embedded into SO_6^δ via the Levi subgroup.

Following the unfolding of the global zeta integral for the general case considered in [7] and [21], we obtain

$$
\mathcal{Z}\left(s, \phi_{\tau\otimes\sigma}, \varphi, \psi_{1,\alpha}\right) = \int_{\mathrm{SO}_2^\delta(\mathbb{A})\backslash \mathrm{SO}_3^{\delta,\alpha}(\mathbb{A})} \mathcal{J}\left(s, \phi_\alpha^W\right)(\epsilon h)\, \mathcal{P}_{\sigma^{-1}}(\varphi)(h)\ \mathrm{d}h, \tag{5.2}
$$

which has an eulerian product decomposition, where

$$
\mathcal{P}_{\sigma^{-1}}(\varphi)(h) := \int_{[\mathrm{SO}_2^\delta]} \varphi(xh)\, \sigma^{-1}(x)\ \mathrm{d}x.
$$

We denote by JL the Jacquet-Langlands correspondence [13] between the set $\mathcal{A}_{\text{cusp}}(L_{1,\alpha})$ and the set $\mathcal{A}_{\text{cusp}}(\text{PGL}_2)$, where $\mathcal{A}_{\text{cusp}}(L_{1,\alpha})$ is the set of irreducible automorphic representations of $L_{1,\alpha}(\mathbb{A})$ of infinite dimension and $\mathcal{A}_{\text{cusp}}(\text{PGL}_2)$ is the set of irreducible cuspidal automorphic representations of $\text{PGL}_2(\mathbb{A})$. Referring to Theorem 2 [33], the period integral $\mathcal{P}_{\sigma^{-1}}(\varphi)$ is non-zero on π if and only if

$$L\left(\frac{1}{2}, \text{JL}(\pi) \times \sigma^{-1}\right) \neq 0 \quad \text{and} \quad \text{Hom}_{\text{SO}_2^\delta(F_v)}(\pi_v, \sigma_v) \neq 0 \text{ for all } v.$$

Note that the dimension of the Hom space is at most one. In addition, by the theorem of Tunnell–Saito in [27, 32], an equivalent statement (see [34, Section V]) for the local period condition is that

$$\text{SO}_3^{\delta,\alpha} \text{ is ramified at a place } v \text{ of } F \text{ if and only if } \varepsilon\left(\frac{1}{2}, \text{JL}(\pi)_v \times \sigma_v^{-1}, \psi_v\right)\eta_{E,v}(-1) = -1,$$

where η_E is a quadratic character of $F^\times \backslash \mathbb{A}^\times$ associated with the quadratic extension $E := F(\sqrt{-\delta})$ via the class field theory. Note that $\text{SO}_3^{\delta,\alpha} \cong \text{PD}_{\delta,\alpha}^\times$ is ramified at a local place v, meaning that $\text{SO}_3^{\delta,\alpha}$ is not split at v. Therefore, for a pure tensor product vector $\varphi = \otimes_v \varphi_v$ in $\pi = \otimes_v \pi_v$

$$\mathcal{P}_{\sigma^{-1}}(\varphi)(h) = C_0 \prod_v \ell_v(\pi_v(h_v)\varphi_v), \tag{5.3}$$

where ℓ_v is a non-zero element in $\text{Hom}_{\text{SO}_2^\delta(F_v)}(\pi_v, \sigma_v)$, and C_0 is 1 if $L(\frac{1}{2}, \text{JL}(\pi) \times \sigma^{-1}) \neq 0$ and it is zero otherwise.

Let S be a finite set of places containing all archimedean places, such that all data are unramified over places outside S. Then,

$$\mathcal{Z}\left(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{1,\alpha}\right) = C_0 \frac{L^S\left(s + \frac{1}{2}, \pi \times \tau\right)}{L^S(s+1, \tau \times \sigma) L^S(2s+1, \tau, \wedge^2)} \prod_{v \in S} \mathcal{Z}_v\left(s, \phi_v, \varphi_v, \psi_{1,\alpha}\right), \tag{5.4}$$

where the local zeta integrals at the ramified local places are given by

$$\mathcal{Z}_v\left(s, \phi_v, \varphi_v, \psi_{1,\alpha}\right) = \int_{\tilde{N}_1^\epsilon(F_v)} \mathcal{J}_v\left(s, \phi_{\alpha,v}^W\right)(h)\, \ell_v(\pi_v(h)\varphi_v)\, \mathrm{d}h.$$

The first main result in the case $n=1$ of the general theory of twisted automorphic descents (as displayed in the previous sections and also in [22]) can be formulated as follows.

Theorem 5.1. With notation as above, assume that τ is of trivial central character. Given δ and σ, if the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ is not identically zero, then the following hold.

(1) The set of $\alpha \in F^\times$ such that the $\psi_{1,\alpha}$-Fourier coefficient $\mathcal{E}_{\tau\otimes\sigma}^{\psi_{1,\alpha}}$ is not identically zero is a single coset $\alpha_0 \cdot \mathrm{Nm}(E^\times)$.

(2) The twisted automorphic descent $\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})$ is irreducible and has the property that

$$\mathrm{JL}\left(\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{\tau\otimes\sigma})\right) \cong \tau.$$

(3) The norm class α_0 is determined by the property that $\mathrm{D}_{\delta,\alpha_0}$ is ramified at a place v of F if and only if $\varepsilon(\frac{1}{2}, \tau_v \times \sigma_v, \psi_v)\eta_{E,v}(-1) = -1$.   □

Proof.   By the definition of the $\psi_{1,\alpha}$-Fourier coefficients and Proposition 2.5 and 2.6, the $\psi_{1,\alpha}$-Fourier coefficient $\mathcal{E}_{\tau\otimes\sigma}^{\psi_{1,\alpha}}(\phi)(h)$ on $L_{1,\alpha}(\mathbb{A})$ is in $L_{\mathrm{cusp}}^2(L_{1,\alpha}(F)\backslash L_{1,\alpha}(\mathbb{A}))$. Applying the spectral decomposition on $L_{\mathrm{cusp}}^2(L_{1,\alpha}(F)\backslash L_{1,\alpha}(\mathbb{A}))$, we have

$$\mathcal{E}_{\tau\otimes\sigma}^{\psi_{1,\alpha}}(\phi) = \sum_{\pi \in \mathcal{A}_{\mathrm{cusp}}(L_{1,\alpha})} \sum_{\varphi_\pi \in B(\pi)} \langle \mathcal{E}_{\tau\otimes\sigma}^{\psi_{1,\alpha}}(\phi), \varphi_\pi \rangle \varphi_\pi, \tag{5.5}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L_{\mathrm{cusp}}^2(L_{1,\alpha}(F)\backslash L_{1,\alpha}(\mathbb{A}))$ and $B(\pi)$ is an orthonormal basis of the space π.

Let us compute the spectral coefficient $\langle \mathcal{E}_{\tau\otimes\sigma}^{\psi_{1,\alpha}}(\phi), \varphi_\pi \rangle$. It is clear that the integration domain $N_1(F)\backslash N_1(\mathbb{A})$ is compact and φ_π is rapidly decreasing, so, we can switch the order of taking the residue and taking the integration, and obtain

$$\mathrm{Res}_{s=\frac{1}{2}} \mathcal{Z}\left(s, \phi_{\tau\otimes\sigma}, \bar{\varphi}_\pi, \psi_{1,\alpha}\right) = \mathrm{Res}_{s=\frac{1}{2}} \int_{[L_{1,\alpha}]} E^{\psi_{1,\alpha}}\left(\phi_{\tau\otimes\sigma}, s\right)(h)\,\bar{\varphi}_\pi(h)\,\mathrm{d}h$$

$$= \int_{[L_{1,\alpha}]} \mathcal{E}_{\tau\otimes\sigma}^{\psi_{1,\alpha}}(\phi)(h)\,\bar{\varphi}_\pi(h)\,\mathrm{d}h$$

$$= \langle \mathcal{E}_{\tau\otimes\sigma}^{\psi_{1,\alpha}}(\phi), \varphi_\pi \rangle.$$

It follows that the Fourier coefficient $\mathcal{E}_{\tau\otimes\sigma}^{\psi_{1,\alpha}}(\phi)$ is not identically zero if and only if the residue $\mathrm{Res}_{s=\frac{1}{2}} \mathcal{Z}(s, \cdot)$ is not zero for some choice of π and some choice of $\varphi_\pi \in \pi$.

By Part (3) of Theorem 1.2, $\mathrm{Res}_{s=\frac{1}{2}} \mathcal{Z}(s, \cdot)$ is zero unless taking π such that $\mathrm{JL}(\pi) \cong \tau$. Let us consider π with $\mathrm{JL}(\pi) \cong \tau$. Recall that the infinite eulerian products for $L^S(s, \tau \times \sigma)$ and $L^S(s, \tau, \wedge^2)$ converge absolutely for $\mathrm{Re}(s) > 1$ following by Jacquet and Shalika in [14, 15], and then are non-vanishing at $\mathrm{Re}(s) > 1$. Since $\mathcal{Z}(s, \cdot)$ has at most a simple pole at $s = \frac{1}{2}$ and $L^S(s + \frac{1}{2}, \pi \times \tau)$ has a pole at $s = \frac{1}{2}$, by Equation 5.4, $\mathcal{Z}_v(s, \phi_{\tau_v \otimes \sigma_v}, \bar{\varphi}_v, \psi_{1,\alpha})$ for $s \in S$ is holomorphic at $s = \frac{1}{2}$. If $\mathrm{Hom}_{\mathrm{SO}_2^\delta(F_v)}(\pi_v, \sigma_v^{-1}) \neq 0$, then there exists a choice of data $\phi_{\tau_v \otimes \sigma_v}$ and φ_v such that the local zeta integral $\mathcal{Z}(s,)$ is not zero at $\mathrm{Re}(s) = \frac{1}{2}$. The argument is similar to the one in [31, Sections 6 and 7]. We omit the

details here. Also a more general situation of the same nature has been discussed in [22, Appendix A].

Hence, according to Equation (5.4), $\mathrm{Res}_{s=\frac{1}{2}} \mathcal{Z}(s, \cdot)$ is non-zero if and only if

(i) $L^S(s + \frac{1}{2}, \pi \times \tau)$ has a pole at $s = \frac{1}{2}$;

(ii) $L(\frac{1}{2}, \mathrm{JL}(\pi) \times \sigma) \neq 0$;

(iii) $L_{1,\alpha}$ is ramified at v if and only if $\varepsilon(\frac{1}{2}, \mathrm{JL}(\pi)_v \times \sigma_v, \psi_v)\eta_{E,v}(-1) = -1$.

By the equality of L-functions:

$$L^S(s, \pi \times \tau) = L^S(s, \mathrm{JL}(\pi) \times \tau),$$

it is clear that $L^S(s + \frac{1}{2}, \mathrm{JL}(\pi) \times \tau)$ has a pole at $s = \frac{1}{2}$ if and only if $\mathrm{JL}(\pi) = \tau$. Hence Condition (i) holds if and only if $\mathrm{JL}(\pi) = \tau$.

Further, by Proposition 1.1, $\mathcal{E}_{\tau \otimes \sigma} \neq 0$ implies $L(\frac{1}{2}, \tau \times \sigma) \neq 0$. Hence when $\mathrm{JL}(\pi) = \tau$, Condition (ii) holds if $\mathcal{E}_{\tau \otimes \sigma} \neq 0$. Finally, the twisted automorphic descent $\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is zero unless the group $L_{1,\alpha}$ is determined by Condition (iii). In such a case, by the uniqueness of the local Bessel model, if $\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is non-zero, there is a unique cuspidal automorphic representation π such that $\mathrm{Res}_{s=\frac{1}{2}} \mathcal{Z}(s, \cdot)$ is non-zero and then $\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is irreducible. By Proposition 3.3, the twisted automorphic descent $\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is non-zero for some α. On the other hand, if α_0 are congruent to α modulo $\mathrm{Nm}(E^\times)$, then L_{1,α_0} is isomorphic to $L_{1,\alpha}$ and $\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is also an automorphic representation of $L_{1,\alpha_0}(\mathbb{A})$, which satisfies Conditions (i)–(iii). Thus the twisted descent $\sigma_{\psi_{1,\alpha_0}}(\mathcal{E}_{\tau \otimes \sigma})$ is non-zero. Therefore, there is a unique α_0 modulo $\mathrm{Nm}(E^\times)$ such that $\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{\tau \otimes \sigma})$ is not identically zero and $\mathrm{JL}(\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{\tau \otimes \sigma})) = \tau$ by the spectral decomposition (5.5).   ∎

Remark 5.2. It is clear from the above theorem that the projective quaternion group $\mathrm{SO}_3^{\delta,\alpha}$ is unique modulo $\mathrm{Nm}(E^\times)$. However, the F-rational orbits of the characters $\psi_{1,\alpha}$ with the stabilizer $\mathrm{PD}_{\delta,\alpha}^\times$ are in one-to-one correspondence with the square classes

$$\mathrm{Nm}\left(E^\times\right)/F^{\times 2}. \qquad \Box$$

The following theorem asserts that by choosing suitable δ and σ involved in the construction of the twisted automorphic descents, we are able to obtain all infinite-dimensional cuspidal automorphic representations of $\mathrm{PD}^\times(\mathbb{A})$, which is the second main result for this special case. We refer to the reader [22] for a treatment of the general situation.

Theorem 5.3. Let D be a quaternion algebra containing a quadratic extension $F(\sqrt{-\delta})$ of F. For any given infinite-dimensional irreducible cuspidal automorphic representation

π of $PD^\times(\mathbb{A})$, there exists a character σ of $SO_2^\delta(\mathbb{A})$ such that the residual representation $\mathcal{E}_{JL(\pi)\otimes\sigma}$ is non-zero and an element $\alpha \in F^\times$ such that $D_{\delta,\alpha} \cong D$ and $\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{JL(\pi)\otimes\sigma}) \cong \pi$. Moreover, any such σ satisfies the period condition that $\mathcal{P}_\sigma(\pi) \neq 0$ or equivalently the L-function condition and the local period condition that

$$L\left(\frac{1}{2}, JL(\pi) \times \sigma\right) \neq 0 \quad \text{and} \quad \text{Hom}_{SO_2^\delta(F_v)}\left(\pi_v, \sigma_v^{-1}\right) \neq 0 \quad \text{for all } v. \qquad (5.6)$$

$\square$

Proof. Consider $SO_2^\delta(\mathbb{A})$ as a subgroup of $PD^\times(\mathbb{A})$. Since $SO_2^\delta(F)\backslash SO_2^\delta(\mathbb{A})$ is compact and the space π is an L^2-space, the restriction of π onto the subgroup $SO_2^\delta(F)\backslash SO_2^\delta(\mathbb{A})$ is semi-simple and non-zero. By the spectral decomposition, we can choose a character σ of $SO_2^\delta(\mathbb{A})$ such that $\mathcal{P}_\sigma(\pi) \neq 0$. By [33], it is equivalent that $L(\frac{1}{2}, \pi \times \sigma) \neq 0$ and $\text{Hom}_{SO_2^\delta(F_v)}(\pi_v, \sigma_v^{-1}) \neq 0$ for all places v.

Since $L(\frac{1}{2}, JL(\pi) \times \sigma) = L(\frac{1}{2}, \pi \times \sigma) \neq 0$ and $JL(\pi)$ is of trivial central character, the residual representation $\mathcal{E}_{JL(\pi)\otimes\sigma}$ is not identically zero. By Theorem 5.1, there exists a unique $SO_3^{\delta,\alpha}$ such that $JL(\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{JL(\pi)\otimes\sigma})) = JL(\pi)$.

It is enough to show that $SO_3^{\delta,\alpha}$ is isomorphic to $PD^\times$. By the proof of Theorem 5.1, $\mathcal{P}_\sigma(\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{JL(\pi)\otimes\sigma})) \neq 0$. Referring to Condition (iii) in the proof of Theorem 5.1, both $SO_3^{\delta,\alpha}$ and $PD^\times$ are ramified at v if and only if $\varepsilon(\frac{1}{2}, JL(\pi)_v \times \sigma_v, \psi_v)\eta_{E,v}(-1) = -1$. Then $SO_3^{\delta,\alpha} \cong PD^\times$ and $\sigma_{\psi_{1,\alpha}}(\mathcal{E}_{JL(\pi)\otimes\sigma}) = \pi$. $\blacksquare$

Remark 5.4. By the uniqueness of this model, the restriction of π onto $SO_2^\delta(F)\backslash SO_2^\delta(\mathbb{A})$ is multiplicity-free. Since π is of infinite dimension and σ is of dimension 1, by the spectral decomposition as in the proof of Theorem 5.3, there are infinitely many choices of σ satisfying (5.6). $\square$

Foundation. We also would like to thank the referee for very helpful comments on the previous version of the paper, which make the paper more readable.

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