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GLOBAL WELL-POSEDNESS OF CLASSICAL SOLUTIONS TO THE CAUCHY PROBLEM OF TWO-DIMENSIONAL BAROTROPIC COMPRESSIBLE NAVIER–STOKES SYSTEM WITH VACUUM AND LARGE INITIAL DATA*

XIANGDI HUANG† AND JING LI‡

Abstract. The Cauchy problem for the barotropic compressible Navier–Stokes equations on the whole two-dimensional space with vacuum as far field density is considered. When the shear viscosity is a positive constant and the bulk one is a power function of density with the power bigger than four-thirds, the global existence and uniqueness of strong and classical solutions is established. It should be remarked that there are no restrictions on the size of the data.

Key words. compressible Navier–Stokes equations, global classical solutions, large initial data, Cauchy problem, vacuum

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1. Introduction and main results.

We are concerned with the two-dimensional barotropic compressible Navier–Stokes equations which read as follows:

$$\begin{cases}\rho_t + \operatorname{div}(\rho u) = 0,\\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \mu\triangle u + \nabla((\mu+\lambda)\operatorname{div} u),\end{cases} \tag{1.1}$$

where $t \geq 0, x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \rho = \rho(x,t)$ and $u = (u_1(x,t), u_2(x,t))$ represent, respectively, the density and velocity, and the pressure P is given by

$$P(\rho) = R\rho^\gamma, \quad \gamma > 1. \tag{1.2}$$

The shear viscosity μ and the bulk one λ satisfy the following hypothesis:

$$0 < \mu = const, \quad \lambda(\rho) = b\rho^\beta,\ b > 0,\ \beta > 0. \tag{1.3}$$

In what follows, we set $R = b = 1$ without loss of any generality. Let $\Omega = \mathbb{R}^2$, and we consider the Cauchy problem with (ρ, u) vanishing at infinity (in some weak sense). For given initial data ρ_0 and u_0, we require that

$$\rho(x,0) = \rho_0(x), \quad \rho u(x,0) = \rho_0 u_0(x), \quad x \in \Omega = \mathbb{R}^2. \tag{1.4}$$

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†Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P. R. China (xdhuang@amss.ac.cn).

‡Department of Mathematics, and Institute of Mathematics and Interdisciplinary Sciences, Nanchang University, Nanchang 330031, P. R. China; Institute of Applied Mathematics, AMSS, and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, P. R. China; and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, P. R. China (ajingli@gmail.com).

When both the shear and bulk viscosities are positive constants, there is a huge literature on the global existence and large-time behavior of solutions to (1.1). The one-dimensional problem has been studied extensively by many people; see [14, 21, 30, 31] and the references therein. For the multidimensional case, when the initial data ρ_0 and m_0 are sufficiently regular and the initial density ρ_0 has a positive lower bound, the local existence and uniqueness of classical solutions are known in [26, 32]. Recently, for the case that the initial density need not be positive and may vanish in open sets, the existence and uniqueness of local strong and classical solutions were obtained by [3, 4, 29]. More recently, for the two-dimensional case and $\Omega = \mathbb{R}^2$, Li and Liang [22] obtained the existence and uniqueness of the local strong and classical solutions to (1.1)–(1.4) with a vacuum as far field density. The global classical solutions were first obtained by Matsumura and Nishida [25] for initial data close to a nonvacuum equilibrium in some Sobolev space H^s. Such a theory was later generalized to weak solutions by Hoff [15] and solutions in Besov spaces by Danchin [7]. For the existence of solutions for large data, the major breakthrough is due to Lions [24] (see also Feireisl [12, 13]), where he obtained global existence of weak solutions when the exponent γ is suitably large. The main restriction on initial data is that the initial energy is finite so that the density is allowed to vanish initially. Recently, Huang, Li, and Xin [18] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible Navier–Stokes equations in three-dimensional space with smooth initial data which are of small energy but possibly large oscillations; in particular, the initial density is allowed to vanish and even has compact support.

However, there are few results regarding global strong solvability for equations of multidimensional motions of viscous gas with no restrictions on the size of initial data. One of the first ever ones is due to Vaigant and Kazhikhov [34] who obtained that the two-dimensional system (1.1)–(1.4) admits a unique global strong solution for large initial data away from vacuum provided $\beta > 3$ and the domain Ω is bounded. Recently, under some additional compatibility conditions on the periodic initial data, Jiu, Wang, and Xin [19] considered periodic classical solutions and removed the condition that the initial density should be away from vacuum in Vaigant and Kazhikhov [34] but still under the same condition $\beta > 3$ as that in [34]. More recently, for periodic initial data with initial density allowed to vanish, we [16] not only relax the crucial condition $\beta > 3$ of [34] to the one that $\beta > 4/3$ but also obtain both the time-independent upper bound of the density and the large-time behavior of the strong and weak solutions. It should be noted that [16, 19, 34] only consider the periodic case or the case of bounded domains, and the global existence of strong and classical solutions to the Cauchy problem (1.1)–(1.4) in the whole space $\mathbb{R}^2$ remains open. In fact, this is the aim of this paper.

Before stating the main results, we explain the notations and conventions used throughout this paper. We denote

$$\int f dx = \int_{\mathbb{R}^2} f dx,$$
$$\frac{D}{Dt} f = \dot{f} = f_t + u \cdot \nabla f.$$

For $1 \le r \le \infty$, we also denote the standard Lebesgue and Sobolev spaces as follows:

$$L^r = L^r(\mathbb{R}^2), \quad W^{s,r} = W^{s,r}(\mathbb{R}^2), \quad H^s = W^{s,2}.$$

Then, we give the definition of strong solutions to (1.1).

Definition 1.1. *If all derivatives involved in* (1.1) *for* (ρ, u) *are regular distributions and* (1.1) *holds almost everywhere in* $\mathbb{R}^2 \times (0,T)$, *then* (ρ, u) *is called a strong solution to* (1.1).

Thus, the first main result concerning the global existence of strong solutions can be stated as follows.

Theorem 1.1. *Assume that*

$$\beta > 4/3, \quad \gamma > 1 \tag{1.5}$$

and that the initial data $(0 \le \rho_0, u_0)$ *satisfy that for some* $q > 2$ *and* $a \in (1,2)$

$$\bar{x}^a \rho_0 \in L^1 \cap H^1 \cap W^{1,q}, \quad \nabla u_0 \in L^2, \quad \rho_0^{1/2} u_0 \in L^2, \tag{1.6}$$

with

$$\bar{x} \triangleq (e + |x|^2)^{1/2} \log^{1+\eta_0}(e + |x|^2), \quad \eta_0 = \frac{3}{8} - \frac{1}{2\beta} > 0. \tag{1.7}$$

Then the problem (1.1)–(1.4) *has a unique global strong solution* (ρ, u) *satisfying that, for any* $0 < T < \infty$,

$$\begin{cases} \rho \in C([0,T]; L^1 \cap H^1 \cap W^{1,q}), \\ \bar{x}^a \rho \in L^\infty(0,T; L^1 \cap H^1 \cap W^{1,q}), \\ \sqrt{\rho} u,\ \nabla u,\ \bar{x}^{-1} u,\ \sqrt{t}\sqrt{\rho} u_t \in L^\infty(0,T; L^2), \\ \nabla u \in L^2(0,T; H^1) \cap L^{(q+1)/q}(0,T; W^{1,q}), \\ \sqrt{t} \nabla u \in L^2(0,T; W^{1,q}), \\ \sqrt{\rho} u_t,\ \sqrt{t} \nabla u_t,\ \sqrt{t} \bar{x}^{-1} u_t \in L^2(\mathbb{R}^2 \times (0,T)) \end{cases} \tag{1.8}$$

and that

$$\inf_{0 \le t \le T} \int_{B_N} \rho(x,t) dx \ge \frac{1}{4} \int_{\mathbb{R}^2} \rho_0(x) dx \tag{1.9}$$

for some constant $N > 0$ *and* $B_N \triangleq \left\{ x \in \mathbb{R}^2 \middle|\ |x| < N \right\}$.

If the initial data (ρ_0, m_0) satisfy some additional regularity and compatibility conditions, the global strong solutions become classical as stated by the following.

Theorem 1.2. *Suppose that* (1.5) *holds. In addition to* (1.6), *assume that* (ρ_0, u_0) *satisfies*

$$\begin{cases} \nabla^2 \rho_0,\ \nabla^2 \lambda(\rho_0),\ \nabla^2 P(\rho_0) \in L^2 \cap L^q, \\ \bar{x}^{\delta_0} \nabla^2 \rho_0,\ \bar{x}^{\delta_0} \nabla^2 \lambda(\rho_0),\ \bar{x}^{\delta_0} \nabla^2 P(\rho_0) \in L^2, \quad \nabla^2 u_0 \in L^2 \end{cases} \tag{1.10}$$

for some constant $\delta_0 \in (0,1)$ *and the following compatibility condition:*

$$-\mu \triangle u_0 - \nabla((\mu + \lambda(\rho_0)) \mathrm{div} u_0) + \nabla P(\rho_0) = \rho_0^{1/2} g, \tag{1.11}$$

with some $g \in L^2$. *Then, in addition to* (1.8) *and* (1.9), *the strong solution* (ρ, u) *obtained by Theorem* 1.1 *satisfies for any* $0 < T < \infty$,

$$\begin{cases} \nabla^2 \rho,\ \nabla^2 \lambda(\rho),\ \nabla^2 P(\rho) \in C([0,T]; L^2 \cap L^q), \\ \bar{x}^{\delta_0} \nabla^2 \rho,\ \bar{x}^{\delta_0} \nabla^2 \lambda(\rho),\ \bar{x}^{\delta_0} \nabla^2 P(\rho) \in L^\infty(0,T; L^2), \\ \nabla^2 u,\ \sqrt{\rho} u_t,\ \sqrt{t} \nabla u_t,\ \sqrt{t} \bar{x}^{-1} u_t,\ t\sqrt{\rho} u_{tt},\ t \nabla^2 u_t \in L^\infty(0,T; L^2), \\ t \nabla^3 u \in L^\infty(0,T; L^2 \cap L^q), \\ \nabla u_t,\ \bar{x}^{-1} u_t,\ t \nabla u_{tt},\ t \bar{x}^{-1} u_{tt} \in L^2(0,T; L^2), \\ t \nabla^2(\rho u) \in L^\infty(0,T; L^{(q+2)/2}). \end{cases} \tag{1.12}$$

A few remarks are in order.

Remark 1.1. As shown by [22, Remark 1.1], the solution (ρ, u) obtained in Theorem 1.2 is in fact a classical one to the Cauchy problem (1.1)–(1.4) in $\mathbb{R}^2 \times (0, \infty)$.

Remark 1.2. Theorems 1.1 and 1.2 generalize and improve the earlier results due to Vaigant and Kazhikhov [34], where they required that $\beta > 3$ and that the domain is bounded. Moreover, Theorems 1.1 and 1.2 also extend our previous result [16], where we consider the periodic case to the Cauchy problem in the whole space $\mathbb{R}^2$.

Remark 1.3. It is worth noting here that Zhang and Fang [35, Theorem 1.8] showed that if $(\rho, u) \in C^1([0,T]; H^k), k > 3$, is a spherically symmetric solution to the Cauchy problem (1.1)–(1.4) with the compact supported initial density $\rho_0 \not\equiv 0$, T must be finite provided $1 < \beta \le \gamma$. However, in our Theorem 1.2, for ρ, we have $\rho \in C([0,T]; H^2)$ but for u, only $\nabla u \in H^k$. Note that the function $u \in \{\nabla u \in H^k\}$ has no decay or decays much slower for large values of the spatial variable x than the one $u \in H^{k+1}$. Therefore, it seems that it is the slow decay of the velocity field for large values of the spatial variable x that leads to the global existence of smooth solutions.

Remark 1.4. It should be mentioned here that it seems that $\beta > 1$ is the extremal case for the system (1.1)–(1.3) (see [34]). Therefore, it would be interesting to study the problem (1.1)–(1.4) when $1 < \beta \le 4/3$. This is left for the future.

We now comment on the analysis of this paper. Note that, for initial data satisfying the conditions of Theorems 1.1 and 1.2, the local existence and uniqueness of strong and classical solutions to the Cauchy problem (1.1)–(1.4) have been established in [22]. Thus, to extend the strong and classical solutions globally in time, one needs global a priori estimates on smooth solutions to (1.1)–(1.4) in suitable higher norms. To do so, motivated by [16, 17], it turns out that the key issue in this paper is to derive the upper bound for the density. We then try to modify the analysis in [16, 34]. However, the methods in [16, 34] cannot be applied directly to our case since their arguments rely heavily on the fact that the domain is bounded. The key steps of this paper are as follows: We first obtain the spatial weighted mean estimate of the density (see (3.6)). Then, rewriting $(1.1)_2$ as (3.13) in terms of a sum of commutators of Riesz transforms and the operators of multiplication by u_i (see (3.12)) as in [16, 23, 28], we succeed in deriving the estimate of $L^\infty(0,T; L^p)$-norm of the density (see (3.9)) after using the spatial weighted mean estimate of the density we have just derived, the Hardy-type inequality (see (2.3)), and the L^p-estimate of the commutators due to Coifman, Rochberg, and Weiss [5](see (2.8)). Next, by energy-type estimates and the compensated compactness analysis [8, Theorem II.1], we show that $\log(1 + \|\nabla u\|_{L^2})$ does not exceed $\|\rho\|_{L^\infty}^{4/3}$ (see (3.25)). Then, after we establish a key estimate of $\|\rho u\|_{L^r}$ in terms of $\|\rho\|_{L^\infty}$, $\|\rho^{1/2} u\|_{L^2}$, and $\|\nabla u\|_{L^2}$ with the explicit expression of r (see (2.6) for details), we can use the $W^{1,p}$-estimate of the commutator due to Coifman and Meyer [6] (see (2.9)) to obtain an estimate on the $L^1(0,T; L^\infty)$-norm of the commutators in terms of $\|\rho\|_{L^\infty}$ (see (3.41)), which together with the Brezis–Wainger inequality (see (2.10)) leads to the key a priori estimate on $\|\rho\|_{L^\infty}$ provided $\beta > 4/3$. See Proposition 3.1 and its proof.

The next main step is to bound the gradients of the density. We first obtain the temporal weighted mean estimates on the material derivatives of the velocity by modifying the basic estimates on the material derivatives of the velocity due to Hoff [15]. Then, following [17], the L^p-bound of the gradient of the density can be obtained by solving a logarithm Gronwall inequality based on a Beale–Kato–Majda-type inequality (see Lemma 2.7), the a priori estimates we have just derived, and some careful initial layer analysis; and moreover, such a derivation also yields simultaneously the bound for the $L^1(0,T;L^\infty(\mathbb{R}^2))$-norm of the gradient of the velocity; see Lemma 4.2 and its proof.

The rest of the paper is organized as follows: In section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Section 3 is devoted to the derivation of the upper bound on the density which is the key to extend the local solution to all time. Based on the previous estimates, higher order ones are established in sections 4 and 5. Then finally, the main results, Theorems 1.1 and 1.2, are proved in section 6.

2. Preliminaries. The following local existence of strong and classical solutions can be found in [22].

LEMMA 2.1. *Let $\beta \geq 1$ and $\gamma > 1$. Assume that (ρ_0, u_0) satisfies* (1.6). *Then there exist a small time $T > 0$ and a unique strong solution (ρ, u) to the problem* (1.1)–(1.4) *in $\mathbb{R}^2 \times (0,T)$ satisfying* (1.8) *and* (1.9). *Moreover, if (ρ_0, u_0) satisfies* (1.10) *and* (1.11) *besides* (1.6), *(ρ, u) satisfies* (1.12) *also.*

The following Sobolev inequality will be used frequently.

LEMMA 2.2 (see [27, 33]). *There exists a universal positive constant C such that the following estimates hold for any $p \in (2,\infty)$:*

$$\|u\|_{L^p} \leq Cp^{1/2}\|\nabla u\|_{L^{2p/(p+2)}}, \quad \|v\|_{L^p} \leq Cp^{1/2}\|v\|_{L^2}^{2/p}\|\nabla v\|_{L^2}^{1-2/p} \tag{2.1}$$

for any function $u \in \left\{u \in L^p \,\middle|\, \nabla u \in L^{2p/(p+2)}\right\}$ and $v \in H^1$.

The following weighted L^p-bounds for elements of the Hilbert space $\tilde{D}^{1,2}(\mathbb{R}^2) \triangleq \{u \in H^1_{\rm loc}(\mathbb{R}^2) | \nabla u \in L^2(\mathbb{R}^2)\}$ can be found in [23, Theorem B.1].

LEMMA 2.3. *For $m \in [2,\infty)$ and $\theta \in (1+m/2,\infty)$, there exists a positive constant C such that we have, for all $v \in \tilde{D}^{1,2}(\mathbb{R}^2)$,*

$$\left(\int_{\mathbb{R}^2} \frac{|v|^m}{e+|x|^2}(\log(e+|x|^2))^{-\theta}dx\right)^{1/m} \leq C\|v\|_{L^2(B_1)} + C\|\nabla v\|_{L^2(\mathbb{R}^2)}, \tag{2.2}$$

while B_N denote a ball with radius N in $\mathbb{R}^2$, i.e, $B_N \triangleq \{x \in \mathbb{R}^2 | \, |x| < N\}$ for $N > 0$. Here we just simply take $N = 1$. Moreover, for general B_N, the constant C will depend on N as well.

The combination of Lemma 2.2 with Lemma 2.3 yields the following.

LEMMA 2.4. *For $\bar{x}$ and η_0 as in* (1.7), *there exists a positive constant C depending only on η_0 such that every function $v \in \tilde{D}^{1,2}(\mathbb{R}^2)$ satisfies for all $\delta \in (0,2)$*

$$\|v\bar{x}^{-\delta}\|_{L^{4/\delta}(\mathbb{R}^2)} \leq C\delta^{-\eta_0-1/2}\left(\|v\|_{L^2(B_1)} + \|\nabla v\|_{L^2(\mathbb{R}^2)}\right). \tag{2.3}$$

Proof. Noticing that

$$|\partial_i \bar{x}| \leq 6\log^{1+\eta_0}(e+|x|^2), \quad i = 1,2,$$

we obtain by direct calculations

$$\begin{aligned} &\|\nabla(v\bar{x}^{-\delta})\|_{L^{4/(2+\delta)}} \\ &= \|\bar{x}^{-\delta}\nabla v - \delta v\bar{x}^{-\delta-1}\nabla\bar{x}\|_{L^{4/(2+\delta)}} \\ &\leq \|\nabla v\|_{L^2}\|\bar{x}^{-\delta}\|_{L^{4/\delta}} + 6\delta\|v\bar{x}^{-1}\|_{L^2}\|\bar{x}^{-\delta}\log^{1+\eta_0}(e+|x|^2)\|_{L^{4/\delta}} \\ &\leq C\left(\|\nabla v\|_{L^2} + \delta^{-\eta_0}\|v\bar{x}^{-1}\|_{L^2}\right), \end{aligned} \tag{2.4}$$

where in the last inequality we have used the simple fact that

$$\begin{aligned}&\|\bar{x}^{-\delta}\log^{1+\eta_0}(e+|x|^2)\|_{L^{4/\delta}}\\&\le\|\bar{x}^{-\delta/2}\|_{L^{4/\delta}}\|(e+|x|^2)^{-\delta/(4(1+\eta_0))}\log(e+|x|^2)\|_{L^\infty}^{1+\eta_0}\\&\le C\delta^{-1-\eta_0},\end{aligned}$$

due to $(e+y)^{-\alpha}\log(e+y)\le\alpha^{-1}$ for $\alpha>0$ and any $y\ge0$. The desired estimate (2.3) thus directly follows from (2.1), (2.4), and (2.2). The proof of Lemma 2.4 is completed. □

A useful consequence of Lemma 2.4 is the following weighted bounds for elements of $\tilde{D}^{1,2}(\mathbb{R}^2)$ which are important for our analysis.

LEMMA 2.5. *Let $\bar{x}$ and η_0 be as in (1.7). For $\gamma>1$, assume that $\rho\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ is a nonnegative function such that*

$$\int_{B_{N_1}}\rho dx\ge M_1,\quad\int\rho^\gamma dx\le M_2,\quad\int\rho\bar{x}^\alpha dx\le M_3 \tag{2.5}$$

for positive constants $M_i(i=1,\dots,3),\alpha\in(1,2)$, and $N_1\ge1$. Then there is a positive constant C depending only on $M_i(i=1,\dots,3),N_1,\alpha,\gamma$, and η_0 such that every $v\in\tilde{D}^{1,2}(\mathbb{R}^2)$ satisfies

$$\|\rho v\|_{L^r}\le Cr^{\eta_0+1/2}(1+\|\rho\|_{L^\infty})\left(\|\rho^{1/2}v\|_{L^2(B_{N_1})}+\|\nabla v\|_{L^2}\right) \tag{2.6}$$

for any $r\in(1,\infty)$.

Proof. It follows from (2.5) and the Poincaré-type inequality [12, Lemma 3.2] that there exists a positive constant C depending only on M_1,M_2,N_1, and γ such that

$$\|v\|_{H^1(B_{N_1})}^2\le C\int_{B_{N_1}}\rho v^2dx+C\|\nabla v\|_{L^2(B_{N_1})}^2. \tag{2.7}$$

This combined with Holder's inequality, (2.3), and (2.5), also for $r\in(1,\infty)$ and $\sigma=4/(r(4+\alpha))$, noting that $\alpha\in(1,2)$ and $\alpha\sigma\in(0,2)$, we obtain

$$\begin{aligned}\|\rho v\|_{L^r}&\le\|(\rho\bar{x}^\alpha)^\sigma\|_{L^{1/\sigma}}\|v\bar{x}^{-\alpha\sigma}\|_{L^{4/(\alpha\sigma)}}\|\rho\|_{L^\infty}^{1-\sigma}\\&\le Cr^{\eta_0+1/2}\left(\|\rho^{1/2}v\|_{L^2(B_{N_1})}+\|\nabla v\|_{L^2}\right)(1+\|\rho\|_{L^\infty}),\end{aligned}$$

which shows (2.6) and finishes the proof of Lemma 2.5. □

Next, let $\mathcal{H}^1(\mathbb{R}^2)$ and BMO$(\mathbb{R}^2)$ stand for the usual HARDY and BMO space. Given a function b, define the linear operator

$$[b,R_iR_j](f)\triangleq bR_i\circ R_j(f)-R_i\circ R_j(bf),\ i,j=1,2,$$

where R_i is the usual RIESZ transform on $\mathbb{R}^2$: $R_i=(-\triangle)^{-1/2}\partial_i$. The following properties of the commutator $[b,R_iR_j](f)$ will be useful for our analysis.

LEMMA 2.6. *Let $b,f\in C_0^\infty(\mathbb{R}^2)$. Then for $p\in(1,\infty)$, there is $C(p)$ such that*

$$\|[b,R_iR_j](f)\|_{L^p}\le C(p)\|b\|_{\mathrm{BMO}}\|f\|_{L^p}. \tag{2.8}$$

Moreover, for $q_i\in(1,\infty)(i=1,2,3)$ with $q_1^{-1}=q_2^{-1}+q_3^{-1}$, there is a C depending only on $q_i(i=1,2,3)$ such that

$$\|\nabla[b,R_iR_j](f)\|_{L^{q_1}}\le C\|\nabla b\|_{L^{q_2}}\|f\|_{L^{q_3}}. \tag{2.9}$$

Remark 2.1. Properties (2.8) and (2.9) are due to Coifman, Rochberg, and Weiss [5] and Coifman and Meyer [6], respectively.

Next, we state the following Beale–Kato–Majda-type inequality which was proved in [1, 20] when $\mathrm{div} u \equiv 0$ and will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla \rho\|_{L^p}$.

LEMMA 2.7 (see [1, 20]). *For $2 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in L^2(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2)$:*

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \le C\left(\|\mathrm{div} u\|_{L^\infty(\mathbb{R}^2)} + \|\mathrm{rot} u\|_{L^\infty(\mathbb{R}^2)}\right)\log(e + \|\nabla^2 u\|_{L^q(\mathbb{R}^2)}) + C\|\nabla u\|_{L^2(\mathbb{R}^2)} + C.$$

Finally, the following Brezis–Wainger inequality will also be used.

LEMMA 2.8 (see [2, 10]). *For $q > 2$, there exists some positive constant C depending only on q such that every function $v \in \{v \in W^{1,q}(\mathbb{R}^2) | \nabla v \in L^2(\mathbb{R}^2)\}$ satisfies*

$$\|v\|_{L^\infty(\mathbb{R}^2)} \le C(\|v\|_{L^q(\mathbb{R}^2)} + \|\nabla v\|_{L^2(\mathbb{R}^2)}) \ln^{1/2}(e + \|v\|_{W^{1,q}(\mathbb{R}^2)}) + C. \tag{2.10}$$

3. A priori estimates (I): Upper bound of the density. In this section and the next, in addition to the conditions of Theorem 1.1, we will always assume that smooth (ρ_0, u_0) satisfies

$$\rho_0(x) > 0, \quad \frac{1}{2} \le \int_{B_{N_0}} \rho_0(x)dx \le \int_{\mathbb{R}^2} \rho_0(x)dx \le 2 \tag{3.1}$$

for some positive constant N_0. Moreover, suppose that (ρ, u) is the strong solution to (1.1)–(1.4) on $\mathbb{R}^2 \times (0, T]$ obtained by Lemma 2.1.

The following Proposition 3.1 will give an upper bound of the density which is the key to obtain higher order estimates.

PROPOSITION 3.1. *Under the conditions of Theorem* 1.1*, for*

$$E_0 \triangleq \|\rho_0 \bar{x}^a\|_{L^1} + \|\rho_0\|_{L^\infty} + \|\rho_0^{1/2} u_0\|_{L^2} + \|\nabla u_0\|_{L^2},$$

there is a positive constant C depending only on $\mu, \beta, \gamma, T, N_0, a,$ and E_0 such that

$$\sup_{0\le t\le T}(\|\rho\|_{L^\infty} + \|\nabla u\|_{L^2}) + \int_0^T \int \rho|u_t + u\cdot\nabla u|^2 dxdt \le C. \tag{3.2}$$

In the following, we will establish some a priori estimates and postpone the proof of Proposition 3.1 to the end of this section.

LEMMA 3.2. *There exist positive constants C and N_1 both depending only on a, γ, T, N_0, $\|\rho_0 \bar{x}^a\|_{L^1}, \|\rho_0\|_{L^\gamma}$, and $\|\rho_0^{1/2} u_0\|_{L^2}$ such that*

$$\sup_{0\le t\le T}\int \left(\rho|u|^2 + \rho^\gamma + \rho\bar{x}^a\right)dx + \int_0^T\int\left(\mu|\nabla u|^2 + \lambda(\rho)(\mathrm{div} u)^2\right)dxdt \le C \tag{3.3}$$

and

$$\inf_{0\le t\le T}\int_{B_{N_1}} \rho dx \ge 1/4. \tag{3.4}$$

Proof. First, the standard energy inequality reads

$$\sup_{0\le t\le T}\int\left(\rho|u|^2+\rho^\gamma\right)dx+\int_0^T\int\left(\mu|\nabla u|^2+(\mu+\lambda(\rho))(\mathrm{div}u)^2\right)dxdt\le\tilde C. \tag{3.5}$$

Next, multiplying $(1.1)_1$ by $\bar x^a$ and integrating the resulting equality over $\mathbb{R}^2$, we obtain after integration by parts and using (3.5) that

$$\begin{aligned}\frac{d}{dt}\int\rho\bar x^a dx&\le C\int\rho|u|\bar x^{a-1}\log^{1+\eta_0}(e+|x|^2)dx\\&\le C\left(\int\rho\bar x^{2a-2}\log^{2(1+\eta_0)}(e+|x|^2)dx\right)^{1/2}\left(\int\rho u^2dx\right)^{1/2}\\&\le C\left(\int\rho\bar x^a dx\right)^{1/2},\end{aligned}$$

which together with Gronwall's inequality gives

$$\sup_{0\le t\le T}\int\rho\bar x^a dx\le C. \tag{3.6}$$

This, along with (3.5), gives (3.3).

Finally, the mass conservation equation $(1.1)_1$ yields

$$\int\rho dx=\int\rho_0 dx. \tag{3.7}$$

For $N>1$, let φ_N be a smooth function such that

$$0\le\varphi_N(x)\le1,\quad \varphi_N=\begin{cases}1 & \text{if } |x|\le N,\\ 0 & \text{if } |x|\ge 2N,\end{cases}\quad |\nabla\varphi_N|\le 2N^{-1}. \tag{3.8}$$

It follows from $(1.1)_1$, (3.7), and (3.5) that

$$\begin{aligned}\frac{d}{dt}\int\rho\varphi_N dx&=\int\rho u\cdot\nabla\varphi_N dx\\&\ge -2N^{-1}\left(\int\rho dx\right)^{1/2}\left(\int\rho|u|^2dx\right)^{1/2}\ge -2\tilde C^{1/2}N^{-1},\end{aligned}$$

which gives

$$\inf_{0\le t\le T}\int\rho\varphi_N dx\ge\int\rho_0\varphi_N dx-2\tilde C^{1/2}N^{-1}T.$$

This combined with (3.1) yields that, for $N_1\triangleq 2(2+N_0+8\tilde C^{1/2}T)$,

$$\inf_{0\le t\le T}\int_{B_{N_1}}\rho dx\ge\int\rho\varphi_{N_1/2}dx\ge 1/4,$$

which shows (3.4). The proof of Lemma 3.2 is completed. ∎

Lemma 3.3. *Assume that* (1.5) *holds. Then there is a positive constant C depending only on $\mu,\beta,\gamma,T,N_0,a,$ and E_0 such that*

$$\sup_{0\le t\le T}\int\left(\rho+\rho^{2\beta\gamma+1}\right)dx\le C. \tag{3.9}$$

Proof. First, we denote

$$\nabla^{\perp} \triangleq (\partial_2, -\partial_1), \quad \frac{D}{Dt}f \triangleq \dot{f} \triangleq f_t + u \cdot \nabla f,$$

where $\dot{f}$ is the material derivative of f. Let G and ω denote the effective viscous flux and the vorticity, respectively, as follows:

$$G \triangleq (2\mu + \lambda(\rho))\mathrm{div}u - P, \quad \omega \triangleq \nabla^{\perp} \cdot u = \partial_2 u_1 - \partial_1 u_2.$$

We thus rewrite the momentum equations $(1.1)_2$ as

$$\rho\dot{u} = \nabla G + \mu\nabla^{\perp}\omega, \tag{3.10}$$

which shows that G solves

$$\triangle G = \mathrm{div}(\rho\dot{u}) = \partial_t(\mathrm{div}(\rho u)) + \mathrm{divdiv}(\rho u \otimes u).$$

This implies

$$G + \frac{D}{Dt}\left((-\triangle)^{-1}\mathrm{div}(\rho u)\right) = F, \tag{3.11}$$

with the commutator F defined by

$$F \triangleq \sum_{i,j=1}^{2}[u_i, R_iR_j](\rho u_j) = \sum_{i,j=1}^{2}\left(u_iR_i \circ R_j(\rho u_j) - R_i \circ R_j(\rho u_i u_j)\right). \tag{3.12}$$

Then, since $\rho > 0$ due to (3.1), the mass equation $(1.1)_1$ leads to

$$-\mathrm{div}u = \frac{1}{\rho}D_t\rho,$$

which combined with (3.11) gives that

$$\frac{D}{Dt}\theta(\rho) + P = \frac{D}{Dt}\psi - F, \tag{3.13}$$

with

$$\theta(\rho) \triangleq 2\mu\log\rho + \beta^{-1}\rho^{\beta}, \quad \psi \triangleq (-\triangle)^{-1}\mathrm{div}(\rho u). \tag{3.14}$$

Next, denoting $f \triangleq \max\{\theta(\rho) - \psi, 0\}$, multiplying (3.13) by $\rho f^{2\gamma-1}$, and integrating the resulting equality over $\mathbb{R}^2$ lead to

$$\begin{aligned}\frac{d}{dt}\int \rho f^{2\gamma}dx &\le C\int \rho f^{2\gamma-1}|F|dx \\ &\le C\|\rho^{1/(2\gamma)}f\|_{L^{2\gamma}}^{2\gamma-1}\|\rho\|_{L^{2\beta\gamma+1}}^{1/(2\gamma)}\|F\|_{L^{(2\beta\gamma+1)/\beta}} \\ &\le C\|\rho^{1/(2\gamma)}f\|_{L^{2\gamma}}^{2\gamma-1}\|\rho\|_{L^{2\beta\gamma+1}}^{1/(2\gamma)}\|\nabla u\|_{L^2}\|\rho u\|_{L^{(2\beta\gamma+1)/\beta}},\end{aligned} \tag{3.15}$$

where in the last inequality we have used the following simple fact that, for any $p \in (1, \infty)$,

$$\|F\|_{L^p} \le C(p)\|u\|_{\mathcal{BMO}}\|\rho u\|_{L^p} \le C(p)\|\nabla u\|_{L^2}\|\rho u\|_{L^p} \tag{3.16}$$

due to (2.8). It follows from the Holder inequality, (3.3), (2.3), (3.4), and (2.7) that

$$\|\rho u\|_{L^{(2\beta\gamma+1)/\beta}} \le \|(\rho\bar{x}^a)^\sigma\|_{L^{1/\sigma}}\|u\bar{x}^{-a\sigma}\|_{L^{4/(a\sigma)}}\|\rho^{1-\sigma}\|_{L^{(2\beta\gamma+1)/(1-\sigma)}} \\ \le C(1+\|\nabla u\|_{L^2})\,(1+\|\rho\|_{L^{2\beta\gamma+1}})\,, \tag{3.17}$$

where $\sigma = 4(\beta-1)/((4+a)(2\beta\gamma+1)-4)$. Substituting (3.17) into (3.15) gives

$$\frac{d}{dt}\int \rho f^{2\gamma}dx \le C\left(1+\int \rho f^{2\gamma}dx + \int \rho^{2\beta\gamma+1}dx\right)(1+\|\nabla u\|_{L^2}^2) \tag{3.18}$$

due to $\beta > 1$.

Next, we claim

$$\int \rho^{2\gamma\beta+1}dx \le C + C\int \rho f^{2\gamma}dx, \tag{3.19}$$

which together with (3.18), (3.3), and Gronwall's inequality yields

$$\sup_{0\le t\le T}\int \rho f^{2\gamma}dx \le C.$$

This combined with (3.19) and (3.7) directly gives (3.9).

Finally, it only remains to prove (3.19). In fact, for any $p\in(1,\infty)$, we have

$$\|\nabla\psi\|_{L^p} \le C(p)\|\rho u\|_{L^p}, \tag{3.20}$$

which, along with (2.1) and (3.3), gives that, for any $r\in(2,\infty)$,

$$\|\psi\|_{L^r} \le C(r)\|\nabla\psi\|_{L^{2r/(r+2)}} \\ \le C(r)\|\rho^{1/2}\|_{L^r}\|\rho^{1/2}u\|_{L^2} \le C(r)\|\rho\|_{L^{r/2}}^{1/2}. \tag{3.21}$$

It thus follows from (3.21), (3.7), and the fact that $\beta > 1$ that

$$\begin{aligned}\int \rho^{2\gamma\beta+1}dx &= \int_{(\rho\le 2)}\rho^{2\gamma\beta+1}dx + \int_{(\rho>2)}\rho^{2\gamma\beta+1}dx \\ &\le C\int_{(\rho\le 2)}\rho dx + C\int_{(\rho>2)}\rho f^{2\gamma}dx + C\int \rho|\psi|^{2\gamma}dx \\ &\le C + C\int \rho f^{2\gamma}dx + C\|\rho\|_{L^{(2\beta\gamma+1)/(2\beta\gamma+1-\gamma)}}\|\psi\|_{L^{2(2\beta\gamma+1)}}^{2\gamma} \\ &\le C + C\int \rho f^{2\gamma}dx + C(1+\|\rho\|_{L^{2\beta\gamma+1}})\|\rho\|_{L^{2\beta\gamma+1}}^{\gamma} \\ &\le C(\varepsilon) + C\int \rho f^{2\gamma}dx + \varepsilon\int \rho^{2\gamma\beta+1}dx.\end{aligned}$$

The proof of Lemma 3.3 is finished. ∎

The following L^p-estimate of the momentum will play an important role in the estimate of the upper bound of the density.

LEMMA 3.4. *Assume that* (1.5) *holds. Then, for any* $p > 4$, *there is a positive constant* $C(p)$ *depending only on* $p, \mu, \beta, \gamma, T, N_0, a$, *and* E_0 *such that*

$$\|\rho u\|_{L^p} \le C(p)R_T^{1+\beta/4+(\beta\eta_0)/2}(1+\|\nabla u\|_{L^2})^{1-2/p}, \tag{3.22}$$

with η_0 *as in* (1.7) *and*

$$R_T \triangleq 1 + \sup_{0\le t\le T}\|\rho\|_{L^\infty}.$$

Proof. First, take

$$\nu \triangleq \frac{\mu^{1/2}}{2(\mu+1)} R_T^{-\beta/2} \in (0, 1/4]$$

due to $R_T > 1$. Multiplying $(1.1)_2$ by $(2+\nu)|u|^\nu u$ and integrating the resulting equation over $\mathbb{R}^2$ lead to

$$\begin{aligned}
&\frac{d}{dt}\int_{R^2} \rho|u|^{2+\nu}dx + (2+\nu)\int_{R^2} |u|^\nu \left(\mu|\nabla u|^2 + (\mu+\rho^\beta)(\mathrm{div}u)^2\right)dx \\
&\le (2+\nu)\nu\int_{R^2}(\mu+\rho^\beta)|\mathrm{div}u||u|^\nu|\nabla u|dx + C\int_{R^2}\rho^\gamma|u|^\nu|\nabla u|dx \\
&\le \frac{2+\nu}{2}\int_{R^2}(\mu+\rho^\beta)(\mathrm{div}u)^2|u|^\nu dx + \frac{\nu^2(2+\nu)}{2}\int_{R^2}(\mu+\rho^\beta)|u|^\nu|\nabla u|^2dx \\
&+ C\int_{R^2}\rho^\gamma|u|^\nu|\nabla u|dx \\
&\le \frac{2+\nu}{2}\int_{R^2}(\mu+\rho^\beta)(\mathrm{div}u)^2|u|^\nu dx + \frac{(2+\nu)\mu}{8(\mu+1)}\int_{R^2}|u|^\nu|\nabla u|^2dx \\
&\quad + \mu\int_{R^2}|u|^\nu|\nabla u|^2dx + C\int_{R^2}\rho|u|^{2+\nu}dx + C\int_{R^2}\rho^{(2+\nu)\gamma-\nu/2}dx,
\end{aligned}$$

where we have used the fact $R_T \ge 1$ and

$$\frac{\nu^2(2+\nu)}{2}(\mu+\rho^\beta) \le \frac{(2+\nu)\mu}{8(\mu+1)} \tag{3.23}$$

which together with Gronwall's inequality and (3.9) thus gives

$$\sup_{0\le t\le T}\int \rho|u|^{2+\nu}dx \le C. \tag{3.24}$$

Then, it follows from Holder's inequality, (3.24), (3.3), (3.4), and (2.6) that, for $r = (p-2)(2+\nu)/\nu$,

$$\begin{aligned}
\|\rho u\|_{L^p} &\le \|\rho u\|_{L^{2+\nu}}^{2/p}\|\rho u\|_{L^r}^{1-2/p} \\
&\le CR_T^{(1+\nu)/p}\|\rho^{1/(2+\nu)}u\|_{L^{2+\nu}}^{2/p}\left(r^{\eta_0+1/2}R_T(1+\|\nabla u\|_{L^2})\right)^{1-2/p} \\
&\le C(p)R_T^{(1+\nu)/p}\left(R_T^{1+\beta/4+(\beta\eta_0)/2}(1+\|\nabla u\|_{L^2})\right)^{1-2/p} \\
&\le C(p)R_T^{1+\beta/4+(\beta\eta_0)/2}(1+\|\nabla u\|_{L^2})^{1-2/p},
\end{aligned}$$

which shows (3.22) and finishes the proof of Lemma 3.4. ∎

LEMMA 3.5. *Assume that* (1.5) *holds. Then there is a constant* C *depending only on* $\mu, \beta, \gamma, T, N_0, a,$ *and* E_0 *such that*

$$\sup_{0\le t\le T}\log(e+A^2(t)) + \int_0^T \frac{B^2(t)}{e+A^2(t)}dt \le CR_T^{4/3}, \tag{3.25}$$

where

$$A^2(t) \triangleq \int\left(\omega^2(t) + \frac{G^2(t)}{2\mu+\lambda(\rho(t))}\right)dx, \quad B^2(t) \triangleq \int \rho(t)|\dot u(t)|^2dx. \tag{3.26}$$

Proof. First, direct calculations show that

$$\nabla^\perp \cdot \dot u = \frac{D}{Dt}\omega - (\partial_1 u \cdot \nabla)u_2 + (\partial_2 u\cdot\nabla)u_1 = \frac{D}{Dt}\omega + \omega \mathrm{div} u \tag{3.27}$$

and that

$$\begin{aligned} \mathrm{div}\dot u &= \frac{D}{Dt}\mathrm{div}u + (\partial_1 u\cdot\nabla)u_1 + (\partial_2 u\cdot\nabla)u_2 \\ &= \frac{D}{Dt}\frac{G}{2\mu+\lambda} + \frac{D}{Dt}\frac{P}{2\mu+\lambda} - 2\nabla u_1\cdot\nabla^\perp u_2 + (\mathrm{div}u)^2. \end{aligned} \tag{3.28}$$

Then, multiplying (3.10) by $2\dot u$ and integrating the resulting equality over $\mathbb{R}^2$, we obtain after using (3.27) and (3.28) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}A^2 + 2B^2 = &-\int \omega^2 \mathrm{div}u dx + 4\int G\nabla u_1\cdot\nabla^\perp u_2 dx - 2\int G(\mathrm{div}u)^2 dx \\ &- \int \frac{(\beta-1)\lambda - 2\mu}{(2\mu+\lambda)^2}G^2\mathrm{div}u dx + 2\beta\int\frac{\lambda(\rho)P}{(2\mu+\lambda)^2}G\mathrm{div}u dx \\ &- 2\gamma\int\frac{P}{2\mu+\lambda}G\mathrm{div}u dx \triangleq \sum_{i=1}^6 I_i. \end{aligned} \tag{3.29}$$

Each I_i can be estimated as follows:

First, it follows from (3.10) that

$$\triangle G = \mathrm{div}(\rho\dot u), \quad \mu\triangle\omega = \nabla^\perp\cdot(\rho\dot u),$$

which together with the standard L^p-estimate of elliptic equations yield that, for $p\in(1,\infty)$,

$$\|\nabla G\|_{L^p} + \|\nabla\omega\|_{L^p} \le C(p,\mu)\|\rho\dot u\|_{L^p}. \tag{3.30}$$

In particular, we have

$$\|\nabla G\|_{L^2} + \|\nabla\omega\|_{L^2} \le C(\mu)R_T^{1/2}B. \tag{3.31}$$

This combined with (2.1) gives

$$\begin{aligned} \|\omega\|_{L^4} &\le C\|\omega\|_{L^2}^{1/2}\|\nabla\omega\|_{L^2}^{1/2} \\ &\le CR_T^{1/4}A^{1/2}B^{1/2}, \end{aligned} \tag{3.32}$$

which leads to

$$|I_1| \le C\|\omega\|_{L^4}^2\|\mathrm{div}u\|_{L^2} \le \varepsilon B^2 + C(\varepsilon)R_T\|\nabla u\|_{L^2}^2A^2. \tag{3.33}$$

Next, we will use an idea due to [9, 28] to estimate I_2. Noticing that

$$\mathrm{rot}\nabla u_1 = 0, \quad \mathrm{div}\nabla^\perp u_2 = 0,$$

one derives from [8, Theorem II.1] that

$$\|\nabla u_1\cdot\nabla^\perp u_2\|_{\mathcal{H}^1} \le C\|\nabla u\|_{L^2}^2.$$

This combined with the fact that BMO($\mathbb{R}^2$) is the dual space of $\mathcal{H}^1$ (see [11]) gives

$$\begin{aligned} |I_2| &\le C\|G\|_{\mathrm{BMO}}\|\nabla u_1\cdot\nabla^{\perp}u_2\|_{\mathcal{H}^1}\\ &\le C\|\nabla G\|_{L^2}\|\nabla u\|_{L^2}^2\\ &\le CR_T^{1/2}B\|\nabla u\|_{L^2}(1+A)\\ &\le \varepsilon B^2+C(\varepsilon)R_T\|\nabla u\|_{L^2}^2(1+A^2), \end{aligned} \tag{3.34}$$

where in the third inequality we have used (3.31) and the following simple fact that, for $t\in[0,T]$,

$$C^{-1}\|\nabla u(\cdot,t)\|_{L^2}^2-C\le A^2(t)\le CR_T^{\beta}\|\nabla u(\cdot,t)\|_{L^2}^2+C \tag{3.35}$$

due to (3.9).

Next, Holder's inequality yields that, for $\delta\in(0,2(\beta-1))$,

$$\begin{aligned} \sum_{i=3}^{6}|I_i| &\le C\int|\mathrm{div}u|\left(|G|\frac{|G|+P}{2\mu+\lambda}+\frac{G^2}{2\mu+\lambda}+\frac{P|G|}{2\mu+\lambda}\right)dx\\ &\le C\int\frac{G^2|\mathrm{div}u|}{2\mu+\lambda}dx+C\int\frac{P|G|}{2\mu+\lambda}|\mathrm{div}u|dx\\ &\le C\|\nabla u\|_{L^2}\left\|\frac{G^2}{2\mu+\lambda}\right\|_{L^2}+C\|\nabla u\|_{L^2}\|G\|_{L^{2(2+\delta)/\delta}}\|P\|_{L^{2+\delta}}\\ &\le C\|\nabla u\|_{L^2}\left\|\frac{G^2}{2\mu+\lambda}\right\|_{L^2}+C(\delta)\|\nabla u\|_{L^2}\|G\|_{L^2}^{\delta/(2+\delta)}\|\nabla G\|_{L^2}^{2/(2+\delta)}, \end{aligned} \tag{3.36}$$

where in the last inequality we have used (3.9) and (2.1).

Then, noticing that (3.26) gives

$$\|G\|_{L^2}\le CR_T^{\beta/2}A, \tag{3.37}$$

one deduces from the Holder inequality and (2.1) that for $0<\delta<1$,

$$\begin{aligned} \left\|\frac{G^2}{\sqrt{2\mu+\lambda}}\right\|_{L^2} &\le C\left\|\frac{G}{\sqrt{2\mu+\lambda}}\right\|_{L^2}^{1-\delta}\|G\|_{L^{2(1+\delta)/\delta}}^{1+\delta}\\ &\le C(\delta)A^{1-\delta}\|G\|_{L^2}^{\delta}\|\nabla G\|_{L^2}\\ &\le C(\delta)R_T^{(1+\delta\beta)/2}AB, \end{aligned} \tag{3.38}$$

where in the last inequality we have used (3.31). Putting (3.38), (3.37), and (3.31) into (3.36) yields

$$\begin{aligned} \sum_{i=3}^{6}|I_i| &\le C(\delta)R_T^{(1+\delta\beta)/2}\|\nabla u\|_{L^2}\left(AB+A^{\delta/(2+\delta)}B^{2/(2+\delta)}\right)\\ &\le C(\delta)R_T^{(1+\delta\beta)/2}\|\nabla u\|_{L^2}(AB+B+A)\\ &\le \varepsilon B^2+C(\varepsilon,\delta)R_T^{1+\delta\beta}(1+\|\nabla u\|_{L^2}^2)(1+A^2). \end{aligned} \tag{3.39}$$

Finally, substituting (3.33), (3.34), and (3.39) into (3.29), we obtain after choosing ε suitably small that for $\delta\in(0,\min\{1,2(\beta-1)\})$

$$\frac{d}{dt}A^2+B^2\le C(\delta)R_T^{1+\delta\beta}(1+\|\nabla u\|_{L^2}^2)(1+A^2). \tag{3.40}$$

Dividing (3.40) by $e + A^2$, choosing $\delta = 1/(3\beta)$, and using (3.3), we obtain (3.25) and finish the proof of Lemma 3.5. □

Next, the following lemma gives an estimate of the $L^1(0, T; L^\infty)$-norm of the commutator F defined by (3.12).

LEMMA 3.6. *Assume that* (1.5) *holds. Then there is a positive constant C depending only on $\mu, \beta, \gamma, T, N_0, a$, and E_0 such that*

$$\int_0^T \|F\|_{L^\infty} dt \le C R_T^{1+\beta/4+\beta\eta_0}. \tag{3.41}$$

Proof. First, it follows from the Gagliardo–Nirenberg inequality, (3.16), and (2.9) that for $p \in (8, \infty)$,

$$\begin{aligned}
\|F\|_{L^\infty} &\le C(p)\|F\|_{L^p}^{(p-4)/p}\|\nabla F\|_{L^{4p/(p+4)}}^{4/p} \\
&\le C(p)\left(\|\nabla u\|_{L^2}\|\rho u\|_{L^p}\right)^{(p-4)/p}\left(\|\nabla u\|_{L^4}\|\rho u\|_{L^p}\right)^{4/p} \\
&\le C(p)\|\nabla u\|_{L^2}^{(p-4)/p}\|\nabla u\|_{L^4}^{4/p}\|\rho u\|_{L^p} \\
&\le C(p)R_T^{1+\beta/4+(\beta\eta_0)/2}\left(1+\|\nabla u\|_{L^2}\right)^{2-6/p}\|\nabla u\|_{L^4}^{4/p},
\end{aligned} \tag{3.42}$$

where in the last inequality we have used (3.22).

Next, we obtain from (3.32), (3.38), (3.35), and (3.3) that

$$\begin{aligned}
\|\nabla u\|_{L^4} &\le C(\|\mathrm{div} u\|_4 + \|\omega\|_4) \\
&\le C\left\|\frac{G+P}{2\mu+\lambda}\right\|_{L^4} + CR_T^{1/4}A^{1/2}B^{1/2} \\
&\le C\left\|\frac{G^2}{\sqrt{2\mu+\lambda}}\right\|_{L^2}^{1/2} + C\left\|\frac{P}{2\mu+\lambda}\right\|_{L^4} + CR_T^{1/4}A^{1/2}B^{1/2} \\
&\le CR_T A^{1/2}B^{1/2} + CR_T^{\gamma} \\
&\le CR_T^{2\beta\gamma}(e+\|\nabla u\|_{L^2})\left(1+\frac{B^2}{e+A^2}\right)^{1/4}.
\end{aligned} \tag{3.43}$$

Substituting (3.43) into (3.42) yields that, for $p > 8$,

$$\begin{aligned}
\|F\|_{L^\infty} &\le C(p)R_T^{1+\beta/4+(\beta\eta_0)/2+8\beta\gamma/p}\left(e+\|\nabla u\|_{L^2}\right)^{2-2/p}\left(1+\frac{B^2}{e+A^2}\right)^{1/p} \\
&\le C(p)R_T^{1+(\beta/4+(\beta\eta_0)/2)p/(p-1)+9\beta\gamma/(p-1)}\left(e+\|\nabla u\|_{L^2}^2\right) + \frac{B^2}{e+A^2},
\end{aligned}$$

which together with (3.25) and (3.3) directly gives (3.41) after choosing p suitably large since $1 + \beta/4 > 4/3$ due to $\beta > 4/3$. The proof of Lemma 3.6 is completed. □

Now we are in a position to prove Proposition 3.1.

Proof of Proposition 3.1. For ψ as in (3.14), it follows from (3.21) and (3.3) that

$$\|\psi\|_{L^{2\gamma}} \le C,$$

which together with (2.10), (3.22), and (3.20) leads to

$$
\begin{aligned}
\|\psi\|_{L^\infty} &\le C\left(\|\psi\|_{L^{2\gamma}}+\|\nabla\psi\|_{L^2}\right)\log^{1/2}(e+\|\psi\|_{W^{1,2\gamma}})+C\\
&\le C\left(1+\|\rho u\|_{L^2}\right)\log^{1/2}(e+\|\rho u\|_{L^{2\gamma}})+C\\
&\le CR_T^{1/2}\log^{1/2}\left(R_T^{1+\beta/4+(\beta\eta_0)/2}(e+\|\nabla u\|_{L^2})\right)+C\\
&\le CR_T^{1/2}\log^{1/2}(e+A^2)+CR_T\\
&\le CR_T^{4/3},
\end{aligned}
\tag{3.44}
$$

where in the last inequality we have used (3.25). One thus derives from (3.13), (3.44), and (3.41) that

$$
R_T^\beta \le CR_T^{1+\beta/4+\beta\eta_0}.
$$

Because of (1.5) and (1.7), this directly gives

$$
\sup_{0\le t\le T}\|\rho\|_{L^\infty}\le C,
$$

which together with (3.40), (3.35), (3.3), and Gronwall's inequality yields (3.2). We complete the proof of Proposition 3.1. ∎

4. A priori estimates (II): Higher order estimates (I).

LEMMA 4.1. *Assume that* (1.5) *holds. Then there is a positive constant C depending only on $\mu,\beta,\gamma,T,N_0,a,$ and E_0 such that*

$$
\sup_{0\le t\le T} t\int\rho|\dot u|^2dx+\int_0^T t\|\nabla\dot u\|_{L^2}^2dt\le C.
\tag{4.1}
$$

Proof. We will adapt an idea due to [15] to prove (4.1). In fact, operating $\partial/\partial t+\operatorname{div}(u\cdot)$ to $(1.1)_2^j$ yields that

$$
\begin{aligned}
&(\rho\dot u_j)_t+\operatorname{div}(\rho u\dot u_j)-\mu\triangle\dot u_j-\partial_j((\mu+\lambda)\operatorname{div}\dot u)\\
&=\mu\partial_i(-\partial_i u\cdot\nabla u_j+\operatorname{div}u\partial_i u_j)-\mu\operatorname{div}(\partial_i u\partial_i u_j)\\
&\quad-\partial_j\left[(\mu+\lambda)\partial_i u\cdot\nabla u_i-(\mu+(1-\beta)\rho^\beta)(\operatorname{div}u)^2\right]\\
&\quad-\operatorname{div}(\partial_j u(\mu+\lambda)\operatorname{div}u)+(\gamma-1)\partial_j(P\operatorname{div}u)+\operatorname{div}(P\partial_j u).
\end{aligned}
\tag{4.2}
$$

Then, multiplying (4.2) by $\dot u$, we obtain after integration by parts that

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int\rho|\dot u|^2dx+\mu\int|\nabla\dot u|^2dx+\int(\mu+\lambda)(\operatorname{div}\dot u)^2dx\\
&\le\frac{\mu}{8}\int|\nabla\dot u|^2dx+C\|\nabla u\|_{L^4}^4+C\|\nabla u\|_{L^2}^2\\
&\le\frac{\mu}{8}\int|\nabla\dot u|^2dx+C\|\rho^{1/2}\dot u\|_{L^2}^2+C,
\end{aligned}
\tag{4.3}
$$

where in the second inequality we have used (3.43) and (3.2). Multiplying (4.3) by t and integrating the resulting inequality over $(0,T)$, we obtain (4.1) after using (3.2). We thus finish the proof of Lemma 4.1. ∎

LEMMA 4.2. *Assume that* (1.5) *holds, and let* $q > 2$ *be as in Theorem* 1.1. *Then there is a constant* C *depending only on* $\mu, \beta, \gamma, T, N_0, a, E_0, q$, *and* $\|\rho_0\|_{H^1\cap W^{1,q}}$ *such that*

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\rho\|_{H^1\cap W^{1,q}}+\|\nabla u\|_{L^2}+t\|\nabla^2 u\|_{L^2}^2\right)\\&+\int_0^T\left(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 u\|_{L^q}^{(q+1)/q}+t\|\nabla^2 u\|_{L^q}^2\right)dt\le C.\end{aligned}\tag{4.4}$$

Proof. Following [17], we will prove (4.4). First, denoting $\Phi \triangleq (2\mu+\lambda(\rho))\nabla\rho$, one deduces from $(1.1)_1$ that Φ satisfies

$$\Phi_t+(u\cdot\nabla)\Phi+(2\mu+\lambda(\rho))\nabla u\cdot\nabla\rho+\rho\nabla G+\rho\nabla P+\Phi\mathrm{div}u=0.\tag{4.5}$$

Multiplying (4.5) by $|\Phi|^{q-2}\Phi$ and integrating the resulting equation over $\mathbb{R}^2$, we obtain after integration by parts that

$$\begin{aligned}\frac{d}{dt}\|\Phi\|_{L^q}&\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^q}+C\|\nabla G\|_{L^q}\\&\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^q}+C\|\rho\dot u\|_{L^q},\end{aligned}\tag{4.6}$$

where in the second inequality we have used (3.30).

Next, noticing that the Gargliardo–Nirenberg inequality, (3.2), and (3.30) yield that

$$\begin{aligned}\|\mathrm{div}u\|_{L^\infty}+\|\omega\|_{L^\infty}&\le C\|G\|_{L^\infty}+C\|P\|_{L^\infty}+C\|\omega\|_{L^\infty}\\&\le C(q)+C(q)\|\nabla G\|_{L^q}^{q/(2(q-1))}+C(q)\|\nabla\omega\|_{L^q}^{q/(2(q-1))}\\&\le C(q)+C(q)\|\rho\dot u\|_{L^q}^{q/(2(q-1))},\end{aligned}\tag{4.7}$$

we deduce from the standard L^p-estimate for the elliptic system that

$$\begin{aligned}\|\nabla^2 u\|_{L^q}&\le C\|\nabla\mathrm{div}u\|_{L^q}+C\|\nabla\omega\|_{L^q}\\&\le C\|\nabla((2\mu+\lambda)\mathrm{div}u)\|_{L^q}+C\|\mathrm{div}u\|_{L^\infty}\|\nabla\rho\|_{L^q}+C\|\nabla\omega\|_{L^q}\\&\le C(\|\mathrm{div}u\|_{L^\infty}+1)\|\nabla\rho\|_{L^q}+C\|\nabla G\|_{L^q}+C\|\nabla\omega\|_{L^q}\\&\le C(\|\rho\dot u\|_{L^q}^{q/(2(q-1))}+1)\|\nabla\rho\|_{L^q}+C\|\rho\dot u\|_{L^q}\\&\le C\|\nabla\rho\|_{L^q}^{(2q-2)/(q-2)}+C\|\rho\dot u\|_{L^q}+C,\end{aligned}\tag{4.8}$$

where in the fourth inequality we have used (3.30). This together with Lemma 2.7 and (4.7) yields that

$$\begin{aligned}\|\nabla u\|_{L^\infty}&\le C\left(\|\mathrm{div}u\|_{L^\infty}+\|\omega\|_{L^\infty}\right)\log(e+\|\nabla^2 u\|_{L^q})+C\|\nabla u\|_{L^2}+C\\&\le C\left(1+\|\rho\dot u\|_{L^q}^{q/(2(q-1))}\right)\log(e+\|\rho\dot u\|_{L^q}+\|\nabla\rho\|_{L^q})+C\\&\le C\left(1+\|\rho\dot u\|_{L^q}\right)\log(e+\|\nabla\rho\|_{L^q}).\end{aligned}\tag{4.9}$$

Next, it follows from the Holder inequality, (3.3), (3.4), (2.6), and (3.2) that

$$\begin{aligned}\|\rho\dot u\|_{L^q}&\le\|\rho\dot u\|_{L^2}^{2(q-1)/(q^2-2)}\|\rho\dot u\|_{L^{q^2}}^{q(q-2)/(q^2-2)}\\&\le C\|\rho\dot u\|_{L^2}^{2(q-1)/(q^2-2)}\left(\|\rho^{1/2}\dot u\|_{L^2}+\|\nabla\dot u\|_{L^2}\right)^{q(q-2)/(q^2-2)}\\&\le C\|\rho^{1/2}\dot u\|_{L^2}+C\|\rho^{1/2}\dot u\|_{L^2}^{2(q-1)/(q^2-2)}\|\nabla\dot u\|_{L^2}^{q(q-2)/(q^2-2)},\end{aligned}\tag{4.10}$$

which together with (3.2) and (4.1) implies that

$$\begin{aligned}&\int_0^T \left(\|\rho\dot u\|_{L^q}^{1+1/q}+t\|\rho\dot u\|_{L^q}^2\right)dt\\ &\le C\int_0^T \left(\|\rho^{1/2}\dot u\|_{L^2}^2+t\|\nabla\dot u\|_{L^2}^2+t^{-(q^3-q^2-2q-1)/(q^3-q^2-2q)}\right)dt\\ &\le C.\end{aligned}\tag{4.11}$$

Then, substituting (4.9) into (4.6), we deduce from Gronwall's inequality and (4.11) that

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^q}\le C,\tag{4.12}$$

which, along with (4.8) and (4.11), shows

$$\int_0^T \left(\|\nabla^2 u\|_{L^q}^{(q+1)/q}+t\|\nabla^2 u\|_{L^q}^2\right)dt\le C.\tag{4.13}$$

Finally, it follows from $(1.1)_1$ that $\nabla\rho$ satisfies

$$\begin{aligned}(\|\nabla\rho\|_{L^2})'&\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^2}+C\|\nabla^2 u\|_{L^2}\\ &\le C(1+\|\nabla^2 u\|_{L^q})\|\nabla\rho\|_{L^2}+C\|\nabla^2 u\|_{L^2}.\end{aligned}\tag{4.14}$$

We obtain from (3.2), (3.30), and (4.12) that

$$\begin{aligned}\|\nabla^2 u\|_{L^2}&\le C\|\nabla\omega\|_{L^2}+C\|\nabla\mathrm{div}u\|_{L^2}\\ &\le C\|\nabla\omega\|_{L^2}+C\|\nabla((2\mu+\lambda)\mathrm{div}u)\|_{L^2}+C\|\mathrm{div}u\|_{L^{2q/(q-2)}}\|\nabla\rho\|_{L^q}\\ &\le C\|\nabla\omega\|_{L^2}+C\|\nabla G\|_{L^2}+C\|\nabla P\|_{L^2}+C\|\nabla u\|_{L^2}^{(q-2)/q}\|\nabla^2 u\|_{L^2}^{2/q}\\ &\le C\|\rho\dot u\|_{L^2}+C\|\nabla\rho\|_{L^2}+\frac12\|\nabla^2 u\|_{L^2}+C,\end{aligned}\tag{4.15}$$

which together with (4.14), (3.2), (4.13), and (4.1) gives

$$\sup_{0\le t\le T}\left(\|\nabla\rho\|_{L^2}+t\|\nabla^2 u\|_{L^2}^2\right)+\int_0^T\|\nabla^2 u\|_{L^2}^2dt\le C.\tag{4.16}$$

The combination of (4.12), (4.13), and (4.16) thus directly gives (4.4). We thus finish the proof of Lemma 4.2. ∎

LEMMA 4.3. *Under the conditions of Theorem* 1.1*, there is a constant C depending only on $\mu,\beta,\gamma,T,N_0,a,E_0,q,$ and $\|\nabla(\bar x^a\rho_0)\|_{L^2\cap L^q}$ such that*

$$\sup_{0\le t\le T}\|\bar x^a\rho\|_{L^1\cap H^1\cap W^{1,q}}\le C.\tag{4.17}$$

Proof. First, it follows from (2.2), (3.2)–(3.4), and (2.7) that, for any $\varepsilon\in(0,1)$ and any $s>2$,

$$\|u\bar x^{-\varepsilon}\|_{L^{s/\varepsilon}}\le C(\varepsilon,s).\tag{4.18}$$

Direct calculations show

$$\begin{aligned}\|\nabla(u\bar x^{-\varepsilon})\|_{L^q}&\le C\|\nabla u\|_{L^q}+C(\varepsilon)\|u\bar x^{-\varepsilon}\|_{L^\infty}\|(e+|x|^2)^{-1/2}\|_{L^q}\\ &\le C(\varepsilon)\|\nabla u\|_{L^q}+\frac12\|\nabla(u\bar x^{-\varepsilon})\|_{L^q}+C(\varepsilon)\|u\bar x^{-\varepsilon}\|_{L^{4/\varepsilon}},\end{aligned}$$

which combined with (4.18) implies

$$\|u\bar{x}^{-\varepsilon}\|_{L^\infty} \le C(\varepsilon) + C(\varepsilon)\|\nabla u\|_{L^q}. \tag{4.19}$$

Then, one derives from $(1.1)_1$ that $v \triangleq \rho\bar{x}^a$ satisfies

$$v_t + u\cdot\nabla v - avu\cdot\nabla\log\bar{x} + v\mathrm{div}u = 0,$$

which together with simple calculations gives that for any $p \in [2,q]$

$$\begin{aligned}(\|\nabla v\|_{L^p})_t &\le C(1+\|\nabla u\|_{L^\infty} + \|u\cdot\nabla\log\bar{x}\|_{L^\infty})\|\nabla v\|_{L^p} \\ &\quad + C\|v\|_{L^\infty}\left(\||\nabla u||\nabla\log\bar{x}|\|_{L^p} + \||u||\nabla^2\log\bar{x}|\|_{L^p} + \|\nabla^2 u\|_{L^p}\right) \\ &\le C(1+\|\nabla u\|_{W^{1,q}})\|\nabla v\|_{L^p} \\ &\quad + C\|v\|_{L^\infty}\left(\|\nabla u\|_{L^p} + \|u\bar{x}^{-1/4}\|_{L^\infty}\|\bar{x}^{-3/2}\|_{L^p} + \|\nabla^2 u\|_{L^p}\right) \\ &\le C(1+\|\nabla^2 u\|_{L^p} + \|\nabla u\|_{W^{1,q}})(1+\|\nabla v\|_{L^p} + \|\nabla v\|_{L^q}),\end{aligned} \tag{4.20}$$

where, in the second and the last inequalities, we have used (4.19) and (3.3). Choosing $p = q$ in (4.20) together with (4.4) thus shows

$$\sup_{0\le t\le T}\|\nabla(\rho\bar{x}^a)\|_{L^q} \le C. \tag{4.21}$$

Finally, setting $p = 2$ in (4.20), we deduce from (4.4) and (4.21) that

$$\sup_{0\le t\le T}\|\nabla(\rho\bar{x}^a)\|_{L^2} \le C,$$

which combined with (3.3) and (4.21) thus gives (4.17) and finishes the proof of Lemma 4.3. ∎

5. A priori estimates (III): Higher order estimates (II). In this section, in addition to the conditions of Theorem 1.2, we will always assume that (3.1) holds and that (ρ,u) is the classical solution to (1.1)–(1.4) on $\mathbb{R}^2\times(0,T]$ obtained by Lemma 2.1.

From now on, in addition to $\mu,\beta,\gamma,T,N_0,a,E_0,q$, and $\|\nabla(\bar{x}^a\rho_0)\|_{L^2\cap L^q}$, the positive constant C may depend on $\|\nabla^2 u_0\|_{L^2}$, $\|\bar{x}^{\delta_0}\nabla^2\rho_0\|_{L^2}$, $\|\bar{x}^{\delta_0}\nabla^2\lambda(\rho_0)\|_{L^2}$, $\|\bar{x}^{\delta_0}\nabla^2 P(\rho_0)\|_{L^2}$, and $\|g\|_{L^2}$, with g as in (1.11).

Lemma 5.1. *It holds that*

$$\sup_{0\le t\le T}\left(\|\rho^{1/2}u_t\|_{L^2} + \|\nabla u\|_{H^1}\right) + \int_0^T \|\nabla u_t\|_{L^2}^2 dt \le C. \tag{5.1}$$

Proof. First, taking into account on the compatibility condition (1.11), we define

$$\sqrt{\rho}\dot{u}(x,t=0) = g.$$

Then we deduce from (4.3) and Gronwall's inequality that

$$\sup_{0\le t\le T}\|\rho^{1/2}\dot{u}\|_{L^2} + \int_0^T \|\nabla\dot{u}\|_{L^2}^2 dt \le C,$$

which together with (4.15), (4.4), (3.2), (4.8), and (4.10) gives

$$\sup_{0\le t\le T}\left(\|\nabla u\|_{H^1} + \|\rho^{1/2}\dot{u}\|_{L^2}\right) + \int_0^T \left(\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla^2 u\|_{L^q}^2\right) dt \le C. \tag{5.2}$$

Then, it follows from (2.2), (2.7), (3.4), and (4.17) that, for $\varepsilon > 0$ and $\eta > 0$, every $v \in \tilde{D}^{1,2}(\mathbb{R}^2)$ satisfies

$$\|\rho^\eta v\|_{L^{(2+\varepsilon)/\tilde{\eta}}} + \|v\bar{x}^{-\eta}\|_{L^{(2+\varepsilon)/\tilde{\eta}}} \le C(\varepsilon,\eta)\|\rho^{1/2}v\|_{L^2} + C(\varepsilon,\eta)\|\nabla v\|_{L^2}, \tag{5.3}$$

with $\tilde{\eta} = \min\{1,\eta\}$. This combined with (4.19) and (5.2) yields that

$$\|\rho^\eta u\|_{L^{(2+\varepsilon)/\tilde{\eta}}\cap L^\infty} + \|u\bar{x}^{-\eta}\|_{L^{(2+\varepsilon)/\tilde{\eta}}\cap L^\infty} \le C(\varepsilon,\eta) \tag{5.4}$$

and that

$$\begin{aligned} \|\rho^{1/2}u_t\|_{L^2} &\le C\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\rho^{1/2}u\cdot\nabla u\|_{L^2} \\ &\le C + C\|\rho^{1/2}u\|_{L^\infty}\|\nabla u\|_{L^2} \le C. \end{aligned} \tag{5.5}$$

Next, differentiating $(1.1)_2$ with respect to t gives

$$\begin{aligned} &\rho u_{tt} + \rho u\cdot\nabla u_t - \nabla((2\mu+\lambda)\mathrm{div}u_t) - \mu\nabla^\perp\omega_t \\ &= -\rho_t(u_t + u\cdot\nabla u) - \rho u_t\cdot\nabla u + \nabla(\lambda_t\mathrm{div}u) - \nabla P_t. \end{aligned} \tag{5.6}$$

Multiplying (5.6) by u_t and integrating the resulting equation over $\mathbb{R}^2$, we obtain after using $(1.1)_1$ that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\rho|u_t|^2dx + \int\left((2\mu+\lambda)(\mathrm{div}u_t)^2 + \mu\omega_t^2\right)dx \\ &= -2\int\rho u\cdot\nabla u_t\cdot u_t dx - \int\rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)dx \\ &\quad - \int\rho u_t\cdot\nabla u\cdot u_t dx - \int\lambda_t\mathrm{div}u\mathrm{div}u_t dx + \int P_t\mathrm{div}u_t dx \\ &\le C\int\rho|u||u_t|\left(|\nabla u_t| + |\nabla u|^2 + |u||\nabla^2 u|\right)dx + C\int\rho|u|^2|\nabla u||\nabla u_t|dx \\ &\quad + C\int\rho|u_t|^2|\nabla u|dx + C\int|\lambda_t||\mathrm{div}u||\mathrm{div}u_t|dx + C\int|P_t||\mathrm{div}u_t|dx. \end{aligned} \tag{5.7}$$

We estimate each term on the right-hand side of (5.7) as follows:

First, the Holder inequality gives

$$\begin{aligned} &\int\rho|u||u_t|\left(|\nabla u_t| + |\nabla u|^2 + |u||\nabla^2 u|\right)dx + \int\rho|u|^2|\nabla u||\nabla u_t|dx \\ &\le C\|\rho^{1/2}u\|_{L^\infty}\|\rho^{1/2}u_t\|_{L^2}\left(\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^4}^2\right) \\ &\quad + C\|\rho^{1/4}u\|_{L^\infty}^2\|\rho^{1/2}u_t\|_{L^2}\|\nabla^2 u\|_{L^2} + C\|\rho^{1/2}u\|_{L^\infty}^2\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2} \\ &\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon), \end{aligned} \tag{5.8}$$

where in the second inequality we have used (5.4) and (5.5).

Then, the Holder inequality, (5.3), and (5.2) lead to

$$\begin{aligned} \int\rho|u_t|^2|\nabla u|dx &\le \|\nabla u\|_{L^2}\|\rho^{1/2}u_t\|_{L^6}^{3/2}\|\rho^{1/2}u_t\|_{L^2}^{1/2} \\ &\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon). \end{aligned} \tag{5.9}$$

Next, for $p \ge 1$, $(1.1)_1$ yields that ρ^p satisfies

$$(\rho^p)_t + u\cdot\nabla\rho^p + p\rho^p\mathrm{div}u = 0,$$

which together with (5.4) and (4.17) shows

$$\|\lambda_t\|_{L^2\cap L^q} \le C\|\bar{x}^{-a}u\|_{L^\infty}\|\rho\|_{L^\infty}^{\beta-1}\|\bar{x}^a\nabla\rho\|_{L^2\cap L^q} + C\|\nabla u\|_{L^2\cap L^q} \le C. \tag{5.10}$$

Similarly, we have

$$\|P_t\|_{L^2\cap L^q} \le C,$$

which combined with (5.2) and (5.10) yields

$$\begin{aligned}&\int |\lambda_t||\mathrm{div}u||\mathrm{div}u_t|dx + \int |P_t||\mathrm{div}u_t|dx\\ &\le C\|\lambda_t\|_{L^q}\|\nabla u\|_{L^{2q/(q-2)}}\|\nabla u_t\|_{L^2} + C\|P_t\|_{L^2}\|\nabla u_t\|_{L^2}\\ &\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon).\end{aligned} \tag{5.11}$$

Finally, putting (5.8), (5.9), and (5.11) into (5.7) and choosing ε suitably small give

$$\frac{d}{dt}\int \rho|u_t|^2 dx + \int \left((2\mu+\lambda)(\mathrm{div}u_t)^2 + \mu\omega_t^2\right) dx \le C,$$

which together with (5.5) and (5.2) gives (5.1) and finishes the proof of Lemma 5.1. ∎

The following higher order estimates of the solutions which are needed to guarantee the extension of the local classical solution to be a global one are similar to those in [22], so we omit their proofs here.

LEMMA 5.2. *The following estimates hold:*

$$\sup_{0\le t\le T}\left(\|\bar{x}^{\delta_0}\nabla^2\rho\|_{L^2} + \|\bar{x}^{\delta_0}\nabla^2\lambda\|_{L^2} + \|\bar{x}^{\delta_0}\nabla^2 P\|_{L^2}\right) \le C, \tag{5.12}$$

$$\sup_{0\le t\le T} t\|\nabla u_t\|_{L^2}^2 + \int_0^T t\left(\|\rho^{1/2}u_{tt}\|_{L^2}^2 + \|\nabla^2 u_t\|_{L^2}^2\right) dt \le C, \tag{5.13}$$

$$\sup_{0\le t\le T}\left(\|\nabla^2\rho\|_{L^q} + \|\nabla^2\lambda\|_{L^q} + \|\nabla^2 P\|_{L^q}\right) \le C, \tag{5.14}$$

$$\begin{aligned}&\sup_{0\le t\le T} t\left(\|\nabla^3 u\|_{L^2\cap L^q} + \|\nabla u_t\|_{H^1} + \|\nabla^2(\rho u)\|_{L^{(q+2)/2}}\right)\\ &+ \int_0^T t^2\left(\|\nabla u_{tt}\|_{L^2}^2 + \|u_{tt}\bar{x}^{-1}\|_{L^2}^2\right) dt \le C.\end{aligned} \tag{5.15}$$

6. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Without loss of generality, assume that

$$\int_{\mathbb{R}^2} \rho_0 dx = 1, \tag{6.1}$$

which implies that there exists a positive constant N_0 such that

$$\int_{B_{N_0}} \rho_0 dx \ge \frac{3}{4}\int_{\mathbb{R}^2} \rho_0 dx = \frac{3}{4}. \tag{6.2}$$

For $\delta > 0$, we construct $\rho_0^\delta = \hat\rho_0^\delta + \delta e^{-|x|^2}$, where $0 \le \hat\rho_0^\delta \in C_0^\infty(\mathbb{R}^2)$ satisfies that

$$\frac{1}{2} \le \int_{B_{N_0}} \hat\rho_0^R dx \le \int_{\mathbb{R}^2} \hat\rho_0^R dx \le \frac{3}{2} \tag{6.3}$$

and that

$$\bar x^a \hat\rho_0^\delta \to \bar x^a \rho_0 \quad \text{in } L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2) \text{ as } \delta \to 0. \tag{6.4}$$

Then, we consider the unique smooth solution u_0^δ of the following elliptic problem:

$$\begin{cases} -\triangle u_0^\delta + \rho_0^\delta u_0^\delta = \sqrt{\rho_0^\delta}((\sqrt{\rho_0}u_0) * j_\delta) - \triangle(u_0 * j_\delta), \\ u_0^\delta \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where j_δ is the standard mollifying kernel of width δ. Standard arguments yield that

$$\lim_{\delta\to 0} \left(\|\nabla(u_0^\delta - u_0)\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\rho_0^\delta} u_0^\delta - \sqrt{\rho_0} u_0\|_{L^2(\mathbb{R}^2)} \right) = 0$$

and that $(\rho_0^\delta, \rho_0^\delta u_0^\delta)$ satisfy (1.6) and (3.1).

The local existence result, Lemma 2.1, applies to show that the problem (1.1)–(1.4) with initial data $(\rho_0^\delta, \rho_0^\delta u_0^\delta)$ has a unique local strong solution (ρ^δ, u^δ) defined up to a positive time T_0. Lemmas 4.2 and 4.3 together with Lemma 2.1 thus yield that (ρ^δ, u^δ) exists on $\mathbb{R}^2 \times (0, T]$ for any $T > 0$ and satisfies all those estimates listed in Lemmas 4.2 and 4.3 with C independent of δ. Then letting $\delta \to 0$, standard arguments (see [3, 28, 34]) thus show that the problem (1.1)–(1.4) has a global strong solution (ρ, u) satisfying the properties listed in Theorem 1.1. Since the proof of the uniqueness of (ρ, u) satisfying (1.8) and (1.9) is similar to that of [22], we finish the proof of Theorem 1.1. ∎

Proof of Theorem 1.2. Without loss of generality, assume that ρ_0 satisfies (6.1) and (6.2). We choose $0 \le \hat\rho_0^\delta \in C_0^\infty(\mathbb{R}^2)$ satisfying (6.3), (6.4), and

$$\begin{cases} \left(\nabla^2\hat\rho_0^\delta,\, \nabla^2\lambda(\hat\rho_0^\delta),\, \nabla^2 P(\hat\rho_0^\delta)\right) \to \left(\nabla^2\rho_0,\, \nabla^2\lambda(\rho_0),\, \nabla^2 P(\rho_0)\right) \quad \text{in } L^q(\mathbb{R}^2), \\ \bar x^{\delta_0}\left(\nabla^2\hat\rho_0^\delta, \nabla^2\lambda(\hat\rho_0^\delta), \nabla^2 P(\hat\rho_0^\delta)\right) \to \bar x^{\delta_0}\left(\nabla^2\rho_0, \nabla^2\lambda(\rho_0), \nabla^2 P(\rho_0)\right) \text{ in } L^2(\mathbb{R}^2) \end{cases}$$

as $\delta \to 0$. Setting $\rho_0^\delta = \rho_0 * j_\delta + \delta e^{-|x|^2}$, we consider the unique smooth solution u_0^δ of the following elliptic problem:

$$\begin{cases} -\mu\triangle u_0^\delta - \nabla\left((\mu + \lambda(\rho_0^\delta))\mathrm{div} u_0^\delta\right) + \nabla P(\rho_0^\delta) = -\rho_0^\delta u_0^\delta + \sqrt{\rho_0^\delta} h^\delta, \\ u_0^\delta \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $h^\delta = (\sqrt{\rho_0}u_0 + g) * j_\delta$ with j_δ being the standard mollifying kernel of width δ. It is easy to check that

$$\lim_{\delta\to 0} \left(\|\nabla(u_0^\delta - u_0)\|_{H^1(\mathbb{R}^2)} + \|\sqrt{\rho_0^\delta} u_0^\delta - \sqrt{\rho_0} u_0\|_{L^2(\mathbb{R}^2)} \right) = 0$$

and that $(\rho_0^\delta, \rho_0^\delta u_0^\delta)$ satisfy (1.6), (1.10), (1.11), and (3.1).

Lemmas 2.1, 4.2, 4.3, 5.1, and 5.2 thus yield that the problem (1.1)–(1.4) with initial data $(\rho_0^\delta, \rho_0^\delta u_0^\delta)$ has a unique strong solution (ρ^δ, u^δ) on $\mathbb{R}^2 \times (0, T]$ for any $T > 0$ satisfying all those estimates presented in Lemmas 4.2, 4.3, 5.1, and 5.2 with C independent of δ. Then letting $\delta \to 0$, standard arguments thus show that the limit function (ρ, u) is the unique strong solution to the problem (1.1)–(1.4) satisfying (1.8), (1.9), and (1.12). We finish the proof of Theorem 1.2. ∎

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