A REMARK ON A CONVERSE THEOREM OF COGDELL AND PIATETSKI-SHAPIRO

HERVÉ JACQUET AND BAIYING LIU

Abstract. In this paper, we reprove a global converse theorem of Cogdell and Piatetski-Shapiro using purely global methods.

1. Introduction

Let F be a number field or a function field. Denote by $\mathbb{A}$ the ring of adeles of F and by ψ a non-trivial additive character of $F\backslash\mathbb{A}$. Let $n \geq 4$. Let π be an irreducible generic representation of $GL_n(\mathbb{A})$. We assume that the central character ω_π of π is automorphic (condition $\mathcal{A}(n,0)$). We also assume that if τ is a cuspidal automorphic representation of $GL_m(\mathbb{A})$ the complete $L-$function $L(s,\pi\times\tau)$ converges for Res large enough. We denote by $\mathcal{A}(n,m)$ the condition that, for every such τ, the $L-$function $L(s,\pi\times\tau)$ has the standard analytic properties (is nice in the terminology of Cogdell and Piatetski-Shapiro [CPS94, CPS96, CPS99, Cog02], see p. 4 for details).

Following them, for every ξ in the space V_π of π, we let W_ξ be the corresponding element of the Whittaker model $\mathcal{W}(\pi,\psi)$ of π. We denote by U_n the group of upper triangular matrices in GL_n with unit diagonal. We set

$$U_\xi(g) = \sum_{\gamma\in U_{n-1}(F)\backslash GL_{n-1}(F)} W_\xi\left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right],$$

$$V_\xi(g) = \sum_{\gamma\in U_{n-1}(F)\backslash GL_{n-1}(F)} W_\xi\left[\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} g\right].$$

If π is automorphic cuspidal then $U_\xi = V_\xi$ for all $\xi\in V_\pi$. Conversely, if $U_\xi = V_\xi$ for all $\xi \in V_\pi$ or, what amounts to the same, $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi\in V_\pi$, then π is automorphic.

Date: August 1, 2017.

2000 *Mathematics Subject Classification.* Primary 11F70; Secondary 22E55.

Key words and phrases. Global Converse Theorem, Generic Representations.

The second mentioned author was supported in part by NSF Grant DMS-1620329, DMS-1702218, and start-up funds from the Department of Mathematics at Purdue University.

Let Z_{U_n} be the center of the group U_n. Cogdell and Piatetski-Shapiro ([PS76], [CPS96], [CPS99]) prove that the conditions $\mathcal{A}(n,m)$ with $0 \le m \le n-2$ imply that

$$\int_{Z_{U_n}(F)\backslash Z_{U_n}(\mathbb{A})} (U_\xi - V_\xi)(z)\theta(z)dz = 0$$

for all non-trivial characters θ of $Z_{U_n}(F)\backslash Z_{U_n}(\mathbb{A})$. They do not have the same relation for the trivial character which would then imply that $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\pi$. Nonetheless, they prove that π is automorphic by using an ingenious local construction.

Our goal in this paper is to prove that conditions $\mathcal{A}(n,m)$, $0 \le m \le n-3$, imply that

$$\int_{Z_{U_n}(F)\backslash Z_{U_n}(\mathbb{A})} (U_\xi - V_\xi)(z)dz = 0\,,\ \forall \xi \in V_\pi\,.$$

This proves directly that the conditions $\mathcal{A}(n,m)$ with $0 \le m \le n-2$ imply $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\pi$ and, in turn, imply π is automorphic (and cuspidal).

While our result is not needed, it gives a purely global proof of the Theorem of Cogdell and Piatetski-Shapiro. It is also germane to the conjecture that the conditions $\mathcal{A}(n,m)$ with $0 \le m \le [\frac{n}{2}]$ imply that π is automorphic (and cuspidal). Of course, the conjecture is true for $n = 2, 3, 4$ (see [JL70], [JPSS79], [PS76] and [CPS96]).

The material is arranged as follows. In the next section, for the convenience of the reader, we review the work of Cogdell and Piatetski-Shapiro and state our result. In section 3 we provide preliminary material of an elementary nature. In section 4 we prove our result.

Acknowledgements. The second mentioned author would like to thank James Cogdell for helpful conversation on their previous results when he was visiting Ohio State University. The authors also would like to thank him for helpful comments and suggestions on an earlier version of the paper. This material is based upon work supported by the National Science Foundation under agreement No. DMS-1128155. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. We also would like to thank the referee for a very careful reading of the paper and for many helpful comments and suggestions.

2. Preliminaries and the main result

In $G_n = GL_n$ we let U_n be the subgroup of upper triangular matrices with unit diagonal. We let A_n be the group of diagonal matrices and

Z_n the center of G_n. We define a character ψ_{U_n} of $U_n(\mathbb{A})$ which is trivial on $U_n(F)$ by

$$\psi_{U_n}(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n}) .$$

We let π be an irreducible generic representation of $G_n(\mathbb{A})$. As usual, this means that π is a restricted tensor product of local irreducible representations π_v. For a finite place v, π_v is an irreducible admissible representation of $G_n(F_v)$ on a complex vector space V_v. We assume that π_v is generic, that is, there is a non-zero linear form

$$\lambda_v : V_v \to \mathbb{C}$$

such that

$$\lambda_v(\pi_v(u)e) = \psi_{U_n,v}(u)\lambda_v(e)$$

for all vectors $e \in V_v$ and all $u \in U_n(F_v)$. We denote by $\mathcal{W}(\pi_v, \psi_{U_n,v})$ the space of functions

$$g \mapsto \lambda_v(\pi_v(g)e) \, , \, e \in V_v$$

on $G_n(F_v)$. It is the Whittaker model of π_v noted $\mathcal{W}(\pi_v, \psi_{U_n,v})$. For all finite v not in a finite set S, the space contains a unique vector $W_{v,0}$ fixed under $G_n(\mathcal{O}_v)$ and taking the value 1 at I_n. The representation π_v is then determined by its Langlands semi-simple conjugacy class $A_v \in G_n(\mathbb{C})$. We assume that there is an integer $m \geq 0$ such that for all finite $v \notin S$, any eigenvalue α of A_v verifies $q_v^{-m} \leq |\alpha| \leq q_v^m$.

For an infinite place v, the representation π_v is really an irreducible admissible Harish-Chandra module. We denote by (π_v, V_v) its canonical completion of slow growth in the sense of Casselman and Wallach. We assume that there is a non-zero continuous linear form

$$\lambda_v : V_v \to \mathbb{C}$$

satisfying the same condition as before. We also define $\mathcal{W}(\pi_v, \psi_{U_n,v})$ as before.

Finally, let ∞ be the set of infinite places of F and

$$G_{n,\infty} = \prod_{v \in \infty} G_n(F_v)$$

We let (π_∞, V_∞) be the topological tensor product of the representations (π_v, V_v), $v \in \infty$. Let λ be the tensor product of the linear forms λ_v, $v \in \infty$. We can define the space $\mathcal{W}(\pi_\infty, \psi_{U_n,\infty})$.

We denote by V the restricted tensor product $\bigotimes_v' V_v$ and π the natural representation of $G_n(\mathbb{A})$ on $V_\pi = V$. The Whittaker model

$\mathcal{W}(\pi, \psi_{U_n})$ of π is the space spanned by the functions

$$W_\infty \prod_{v \notin \infty} W_v$$

with $W_\infty \in \mathcal{W}(\pi_\infty, \psi_{U_n,\infty})$, $W_v \in \mathcal{W}(\pi_v, \psi_{U_n,v})$ and $W_v = W_{v,0}$ for almost all $v \notin S$. For every $\xi \in V_\pi$ we denote by W_ξ the corresponding element of $\mathcal{W}(\pi, \psi_{U_n})$.

We assume that the central character of π is automorphic. It is convenient to refer to this condition as condition $\mathcal{A}(n, 0)$. In view of our assumptions, for any cuspidal automorphic representation τ of $G_m(\mathbb{A})$, $1 \leq m \leq n-1$, the complete $L-$function $L(s, \pi \times \tau)$ is defined by a convergent product for Res large enough. *Condition* $\mathcal{A}(n, m)$ is that, for any such τ, the function $L(s, \pi \times \tau)$ extends to an entire function of s, bounded in vertical strips and satisfies the functional equation

$$L(s, \pi \times \tau) = \epsilon(s, \pi \times \tau, \psi) L(1 - s, \widetilde{\pi} \times \widetilde{\tau})\,.$$

In G_n let $Y_{n,m}$ be the unipotent radical of the standard parabolic subgroup of type $(m + 1, 1, 1, \ldots, 1)$. For instance:

$$Y_{3,1} = \left\{ \begin{pmatrix} 1 & 0 & \bullet \\ 0 & 1 & \bullet \\ 0 & 0 & 1 \end{pmatrix} \right\}, \; Y_{5,2} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \bullet & \bullet \\ 0 & 1 & 0 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet & \bullet \\ 0 & 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

If a function ϕ on $G_n(\mathbb{A})$ is invariant on the left under $Y_{n,m}(F)$ we set

$$\mathbb{P}^n_m(\phi)(g) = \int_{Y_{n,m}(F) \backslash Y_{n,m}(\mathbb{A})} \phi(yg) \overline{\psi}_{U_n}(y) dy\,.$$

Here dy is the Haar measure on $Y_{n,m}(\mathbb{A})$ normalized by the condition that the quotient $Y_{n,m}(F) \backslash Y_{n,m}(\mathbb{A})$ has measure 1. Our notation differs slightly from the notations of Cogdell and Piatetski-Shapiro ([Cog02]). Here $\mathbb{P}^n_m(\phi)$ is a function on $G_n(\mathbb{A})$, while in [Cog02], it is a function on $P_{m+1}(\mathbb{A})$, the mirabolic subgroup of $G_{m+1}(\mathbb{A})$, embedded in $G_n(\mathbb{A})$. Note that $Y_{n,n-1} = \{I_n\}$ and $\mathbb{P}^n_{n-1}$ is the identity.

Suppose π is as above. For each ξ in the space V_π of π we set

$$U_\xi(g) = \sum_{\gamma \in U_n(F) \backslash P_n(F)} W_\xi(\gamma g),$$

where P_n is the subgroup of matrices of G_n whose last row has the form

$$(0, 0, \ldots, 0, 1)\,.$$

This is also

$$U_\xi(g) = \sum_{\gamma \in U_{n-1}(F)\backslash G_{n-1}(F)} W_\xi\left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right].$$

Then we have the following result.

Lemma 2.1. *With the previous notations,*

$$\mathbb{P}^n_m(U_\xi)(g) = \sum_{\gamma \in U_{m+1}(F)\backslash P_{m+1}(F)} W_\xi\left[\begin{pmatrix} \gamma & 0 \\ 0 & I_{n-m-1} \end{pmatrix} g\right],$$

or equivalently

$$\mathbb{P}^n_m(U_\xi)(g) = \sum_{\gamma \in U_m(F)\backslash G_m(F)} W_\xi\left[\begin{pmatrix} \gamma & 0 \\ 0 & I_{n-m} \end{pmatrix} g\right].$$

Likewise, let R_n be the subgroup of matrices of G_n whose first column is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We can consider the function

$$V_\xi(g) = \sum_{\xi \in U_n(F)\backslash R_n(F)} W_\xi(\gamma g).$$

Let ω_n be the permutation matrix defined by

$$\omega_1 = 1\,,\ \omega_n = \begin{pmatrix} 0 & \omega_{n-1} \\ 1 & 0 \end{pmatrix}.$$

Then

$$R_n = \omega_n {}^t P_n^{-1} \omega_n\,,\ U_n = \omega_n {}^t U_n^{-1} \omega_n\,.$$

Moreover the automorphism $u \mapsto \omega_n {}^t u^{-1} \omega_n$ changes ψ_{U_n} into $\overline{\psi}_{U_n}$.

If π is automorphic cuspidal then for the cusp form ϕ_ξ corresponding to $\xi \in V_\pi$ we have

$$W_\xi(g) = \int_{U_n(F)\backslash U_n(\mathbb{A})} \phi_\xi(ug)\overline{\psi}_{U_n}(u)du$$

and

$$\phi_\xi(g) = U_\xi(g)\,.$$

By the previous observation relative to R_n, we also have

$$\phi_\xi(g) = V_\xi(g)\,.$$

Thus we have

$$U_\xi = V_\xi \,,\ \forall \xi \in V_\pi \,.$$

Conversely if π is given and

$$U_\xi = V_\xi \,,\ \forall \xi \in V_\pi \,,$$

then U_ξ is invariant on the left under $Z_n(F), P_n(F), R_n(F)$. Since these groups generate $G_n(F)$, for every $\xi \in V_\pi$ the function U_ξ is invariant on the left under $G_n(F)$ and hence π is automorphic.

In general, U_ξ and V_ξ are invariant on the left under $P_n(F) \cap R_n(F)$ and $A_n(F)$. In other words, they are invariant under $S_n(F)$ where S_n is the standard parabolic subgroup of type $(1, n-2, 1)$. The notations here differ slightly from those of Cogdell and Piatetski-Shapiro ([CPS99]). Moreover, we have, for all $g, h \in G_(\mathbb{A})$,

$$W_{\pi(h)\xi}(g) = W_\xi(gh)$$

and similar formulae for U_ξ and V_ξ. As a consequence, if an identity involving W_ξ, U_ξ or V_ξ is true for all $\xi \in V_\pi$, the identity obtained by translating the function on the right by an arbitrary element of $G_n(\mathbb{A})$ is also true for all $\xi \in V_\pi$, and conversely. For instance, the relation $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\pi$ is equivalent to the relation $U_\xi(g) = V_\xi(g)$ for all $\xi \in V_\pi$ and for all $g \in G_n(\mathbb{A})$. We appeal repeatedly to this principle.

Following Cogdell and Piatetski-Shapiro ([Cog02, Section 5.2]), for $1 \leq m \leq n-2$, we define

$$\alpha_m = \begin{pmatrix} 0 & 1 & 0 \\ I_m & 0 & 0 \\ 0 & 0 & I_{n-m-1} \end{pmatrix} .$$

We note that $\alpha_m \in P_n$. We also define

$$V_\xi^m(g) = V_\xi(\alpha_m g) \,, \forall g \in G_n(\mathbb{A}) \,.$$

Thus V_ξ^m is invariant under $Q_m = \alpha_m^{-1} R_n \alpha_m$. This is the subgroup of matrices of G_n whose $(m+1)$-th column has the form

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} ,$$

with 1 in the $(m+1)$-th row. Note that Q_m contains the group $Y_{n,m}$. Thus we may consider $\mathbb{P}^n_m(V^m_\xi)$.

Theorem 2.2 (Cogdell, Piatetski-Shapiro, Section 5 of [Cog02], [CPS99])**.** *Suppose conditions $\mathcal{A}(n,j)$, $0 \le j \le m$, are satisfied. Then, for all $\xi \in V_\pi$,*

$$\mathbb{P}^n_m(U_\xi) = \mathbb{P}^n_m(V^m_\xi).$$

We denote by $\mathcal{E}(n,m)$ the condition that

$$\mathbb{P}^n_m(U_\xi) = \mathbb{P}^n_m(V^m_\xi)\,,\ \forall \xi \in V_\pi\,.$$

This condition can be simplified. Indeed, we have the following result.

Proposition 2.3 (Cogdell, Piatetski-Shapiro, Section 5 of [Cog02], [CPS99])**.** *Let $k, 1 \le k \le n-m$, be an integer. The condition $\mathcal{E}(n,m)$ is equivalent to the condition*

$$\begin{aligned}(2.1)\qquad &\int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int U_\xi\left[\begin{pmatrix} I_m & x \\ 0 & u\end{pmatrix} g\right]\overline{\psi}_{U_{n-m}}(u)dxdu \\ &= \int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int V^m_\xi\left[\begin{pmatrix} I_m & x \\ 0 & u\end{pmatrix} g\right]\overline{\psi}_{U_{n-m}}(u)dxdu\,,\end{aligned}$$

for all $\xi \in V_\pi$ and for all $g \in G_n(\mathbb{A})$, where $x \in M_{m\times(n-m)}(F)\backslash M_{m\times(n-m)}(\mathbb{A})$ with zero first k columns.

Proof: For the convenience of the reader, we review the proof. For $k=1$ our conclusion is just the hypothesis. Thus we may assume our assertion true for k, $1 \le k \le n-m-1$ and prove it for $k+1$. In the integral we write $x = (0,0,\ldots,0,x_{k+1},x_{k+2},\ldots x_{n-m})$ where the x_i are column vectors of length m. We also introduce

$$\gamma_\beta = \begin{pmatrix} I_m & 0 & 0 \\ X_\beta & I_k & 0 \\ 0 & 0 & I_{n-m-k}\end{pmatrix}\,,\ X_\beta = \left.\overbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta\end{pmatrix}}^{m}\right\}k\,,\ \beta \in F^m\,.$$

Since γ_β is in $P_n(F) \cap Q_m(F)$, in the identity, we can conjugate the matrices by γ_β. We note that

$$\gamma_\beta \begin{pmatrix} I_m & x \\ 0 & u\end{pmatrix}\gamma_\beta^{-1} = \begin{pmatrix} I_m & x \\ 0 & u'\end{pmatrix}$$

where

$$u'_{m+k,m+k+1} = u_{m+k,m+k+1} + \beta x_{k+1}\,.$$

Hence the equality becomes

$$\begin{aligned}&\int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int U_\xi\left[\begin{pmatrix} I_m & x\\ 0 & u\end{pmatrix}g\right]\overline{\psi}_{U_{n-m}}(u)\psi(-\beta x_{k+1})dxdu\\ =&\int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int V_\xi^m\left[\begin{pmatrix} I_m & x\\ 0 & u\end{pmatrix}g\right]\overline{\psi}_{U_{n-m}}(u)\psi(-\beta x_{k+1})dxdu\,,\end{aligned}$$

where $x \in M_{m\times(n-m)}(F)\backslash M_{m\times(n-m)}(\mathbb{A})$, with zero first k columns. Summing over all $\beta \in F^m$ and applying the theory of Fourier series, we get

$$\begin{aligned}&\int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int U_\xi\left[\begin{pmatrix} I_m & x\\ 0 & u\end{pmatrix}g\right]\overline{\psi}_{U_{n-m}}(u)dxdu\\ =&\int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int V_\xi^m\left[\begin{pmatrix} I_m & x\\ 0 & u\end{pmatrix}g\right]\overline{\psi}_{U_{n-m}}(u)dxdu\,,\end{aligned}$$

where $x \in M_{m\times(n-m)}(F)\backslash M_{m\times(n-m)}(\mathbb{A})$, with zero first $k+1$ columns.

The other direction is obvious, since the if the condition (2.1) holds for $k+1$, then integrating both sides with respect to

$$x \in M_{m\times(n-m)}(F)\backslash M_{m\times(n-m)}(\mathbb{A})$$

with only $(k+1)$-th column being possibly nonzero, we obtain the condition (2.1) for k, which is equivalent to the condition $\mathcal{E}(n, n-m)$ by induction assumption. This concludes the proof of the proposition. $\square$

Since $\alpha_m \in P_n$ and U_ξ is $P_n(F)$ invariant on the left, we can apply the condition (2.1) with g replaced by α_m^{-1}. Thus the condition (2.1) can be written also as

$$\begin{aligned}&\int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int U_\xi\left[\alpha_m\begin{pmatrix} I_m & x\\ 0 & u\end{pmatrix}\alpha_m^{-1}\right]\overline{\psi}_{U_{n-m}}(u)dxdu\\ =&\int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int V_\xi\left[\alpha_m\begin{pmatrix} I_m & x\\ 0 & u\end{pmatrix}\alpha_m^{-1}\right]\overline{\psi}_{U_{n-m}}(u)dxdu\,,\end{aligned}\tag{2.2}$$

for all $\xi \in V_\pi$, where $x \in M_{m\times(n-m)}(F)\backslash M_{m\times(n-m)}(\mathbb{A})$ with zero first k columns. In this paper, we are mainly interested in the case $k = n-m$ of condition (2.2). We make it explicit.

Taking $k = n-m$ and $m = n-2$, then condition (2.2) leads to the condition of Cogdell and Piatetski-Shapiro

$$\int_{F\backslash\mathbb{A}}(U_\xi - V_\xi)\begin{pmatrix} 1 & 0 & z\\ 0 & I_{n-2} & 0\\ 0 & 0 & 1\end{pmatrix}\psi(-z)dz = 0$$

for all $\xi \in V_\pi$. Since U_ξ and V_ξ are invariant on the left under $A_n(F)$ and this relation is true for any right translate of U_ξ and V_ξ we can conjugate by an element of $A_n(F)$ and obtain the condition

$$\int_{F\backslash\mathbb{A}} (U_\xi - V_\xi)\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \psi(-\alpha z)dz = 0\,,$$

for all $\xi \in V_\pi$ and all $\alpha \in F^\times$.

Now let us take $k = n - m$ and $m = n - 3$. Let e_0 and f_0 be respectively the following row and column of size $n-2$:

$$e_0 = (0, 0, \dots, 0, 1)\,,\ f_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\,.$$

Condition (2.2) reads

$$\int_{(F\backslash\mathbb{A})^2}\int_{F\backslash\mathbb{A}} (U_\xi - V_\xi)\begin{pmatrix} 1 & xe_0 & z \\ 0 & I_{n-2} & yf_0 \\ 0 & 0 & 1 \end{pmatrix} \psi(-x-y)dzdxdy = 0\,,$$

for all $\xi \in V_\pi$. Abusing notation, we write the above integral as

$$\int_{(F\backslash\mathbb{A})^3} (U_\xi - V_\xi)\begin{pmatrix} 1 & xe_0 & z \\ 0 & I_{n-2} & yf_0 \\ 0 & 0 & 1 \end{pmatrix} \psi(-x-y)dzdxdy = 0\,,$$

for all $\xi \in V_\pi$. For instance, for $n = 4$ the condition reads

$$\int_{(F\backslash\mathbb{A})^3} (U_\xi - V_\xi)\begin{pmatrix} 1 & 0 & x & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \psi(-x-y)dzdxdy = 0\,,$$

for all $\xi \in V_\pi$. Again we can conjugate by an element of $A_n(F)$ to obtain

$$\int_{(F\backslash\mathbb{A})^3} (U_\xi - V_\xi)\begin{pmatrix} 1 & xe_0 & z \\ 0 & I_{n-2} & yf_0 \\ 0 & 0 & 1 \end{pmatrix} \psi(-\alpha x-\beta y)dzdxdy = 0\,, \tag{2.3}$$

for all $\xi \in V_\pi, \alpha \in F^\times, \beta \in F^\times$. Our own contribution is the following.

Theorem 2.4. *Suppose $n > 3$. Then condition $\mathcal{E}(n, n-3)$ is equivalent to the condition*

$$\int_{F\backslash\mathbb{A}} (U_\xi - V_\xi)\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} dz = 0$$

for all $\xi \in V_\pi$.

Combining our Theorem 2.4 with the results of Cogdell and Piatetski-Shapiro ([Cog02]) we arrive at the following result.

Corollary 2.5. *Suppose conditions* $\mathcal{A}(n,i)$, $0 \le i \le n-3$, *are satisfied. Then for every* $\xi \in V_\pi$

$$\int_{F\backslash\mathbb{A}} (U_\xi - V_\xi)\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} dz = 0\,.$$

3. Separable functions

Let U and V be vector spaces over F. A function

$$\Phi : U(F)\backslash U(\mathbb{A}) \times V(F)\backslash V(\mathbb{A}) \to \mathbb{C}$$

is said to be (additively) *separable*, if there exist two functions $\Phi_1 : U(F)\backslash U(\mathbb{A}) \to \mathbb{C}$ and $\Phi_2 : V(F)\backslash V(\mathbb{A}) \to \mathbb{C}$ such that, for all $(u,v) \in U(\mathbb{A}) \times V(\mathbb{A})$,

$$\Phi(u,v) = \Phi_1(u) + \Phi_2(v)\,.$$

It amounts to the same to demand that, for all (u,v),

$$\Phi(u,v) = \Phi(u,0) + \Phi(0,v) - \Phi(0,0)\,.$$

In what follows, if $\Phi : U(F)\backslash U(\mathbb{A}) \times V(F)\backslash V(\mathbb{A}) \to \mathbb{C}$ is a function, an integral

$$\int\int \Phi(u,v)dudv$$

means that the integral is over the product $U(F)\backslash U(\mathbb{A}) \times V(F)\backslash V(\mathbb{A})$ and the Haar measure du (resp. dv) is normalized by demanding that the quotients have volume 1. This convention remains in force for the rest of this paper.

Lemma 3.1. *Suppose that*

$$\Phi : F\backslash\mathbb{A} \times F\backslash\mathbb{A} \to \mathbb{C}$$

is a smooth function such that, for all $\alpha \in F^\times$, $\beta \in F^\times$,

$$\int\int \Phi(u,v)\psi(\alpha u + \beta v)dudv = 0\,.$$

Then Φ *is separable.*

PROOF: We write the Fourier expansion of Φ,

$$\begin{aligned}&\Phi(x,y)\\ =&\sum_{\alpha\in F,\beta\in F}\psi(\alpha x+\beta y)\left(\int\int\Phi(u,v)\psi(-\alpha u-\beta v)dudv\right).\end{aligned}$$

In view of the assumptions, we have

$$\begin{aligned}&\Phi(x,y)\\ =&\int\int\Phi(u,v)dudv\\ +&\sum_{\alpha\in F^\times}\psi(\alpha x)\left(\int\int\Phi(u,v)\psi(-\alpha u)dudv\right)\\ +&\sum_{\beta\in F^\times}\psi(\beta y)\left(\int\int\Phi(u,v)\psi(-\beta v)dudv\right).\end{aligned}$$

The function on the right hand side of the equation is indeed separable. $\square$

Proposition 3.2. *Suppose U and V are finite dimensional spaces over F in duality by the bi-linear form $(u,v)\mapsto\langle u,v\rangle$. Suppose*

$$\Phi : U(F)\backslash U(\mathbb{A})\times V(F)\backslash V(\mathbb{A})\to\mathbb{C}$$

is a smooth function with the following property: for all pairs $(e,f)\in U(F)\times V(F)$ with $\langle e,f\rangle=1$ we have

$$\int_{(F\backslash\mathbb{A})^2}\Phi(u+xe,v+yf)\psi(\alpha x+\beta y)dxdy=0$$

for all $u\in U(\mathbb{A})$, all $v\in V(\mathbb{A})$, all $\alpha\in F^\times$, and all $\beta\in F^\times$. Then Φ is separable.

PROOF: If $\dim(U)=\dim(V)=1$, our assertion follows from the previous lemma. Thus we may assume that $\dim(U)=\dim(V)=n+1$, $n>0$ and our assertion is true for dimension n. Let $e\in U(F), f\in V(F)$ with $\langle e,f\rangle=1$. Let U_1 be the subspace of U orthogonal to f and V_1 the subspace of V orthogonal to e. By Lemma 3.1, for $u\in U_1(\mathbb{A}), v\in V_1(\mathbb{A})$ we have

$$\Phi[u+se,v+tf]=\Phi[u+se,v]+\Phi[u,v+tf]-\Phi[u,v].$$

Each one of the functions

$$(u,v)\mapsto\Phi[u+se,v]\,,\;(u,v)\mapsto\Phi[u,v+tf]\,,\;(u,v)\mapsto\Phi[u,v]$$

satisfies the assumptions of the proposition. By the induction hypothesis, the right hand side is equal to

$$\Phi[u+se,0]+\Phi[se,v]-\Phi[se,0]+\Phi[u,tf]+\Phi[0,v+tf]-\Phi[0,tf]$$
$$-\Phi[u,0]-\Phi[0,v]+\Phi[0,0]\,.$$

Thus it suffices to show that $(u,t)\mapsto\Phi[u,tf]$ and $(s,v)\mapsto\Phi[se,v]$ are separable functions. Let us show this is the case for the first function. Let $e_1,e_2,\dots,e_n$ be a basis of $U_1(F)$. Write

$$u=s_1e_1+s_2e_2+\cdots+s_ne_n\,.$$

Now $\langle e_1+e,f\rangle=1$. Thus,

$$\Phi[s_1e_1+s_2e_2+\cdots s_ne_n+s_1e,v+tf]$$

must be separable as a function of (s_1,t). All the terms on the right hand side (with $s=s_1$) have this property, except possibly the term

$$\Phi[u,tf]\,.$$

Thus this term must have this property as well. Hence

$$\Phi[s_1e_1+s_2e_2+\cdots+s_ne_n,tf]$$

is a separable function of the pair (s_1,t). Likewise it is a separable function of each pair (s_j,t), $1\le j\le n$. By the lemma below it is a separable function of $((s_1,s_2,\dots,s_n),t)$, that is, $\Phi[u,tf]$ is a separable function of (u,t). □

Lemma 3.3. *Suppose*

$$\Phi((s_1,s_2,\dots,s_n),t)$$

is a function with the property that for each index j it is a separable function of (s_j,t). Then it is a separable function of the pair

$$((s_1,s_2,\dots,s_n),t)\,.$$

PROOF: Our assertion is obvious if $n=1$. So we may assume $n>1$ and our assertion true for $n-1$. We have, by separability in (s_1,t),

$$\Phi((s_1,s_2,\dots s_n),t)=$$
$$\Phi((s_1,s_2,\dots s_n),0)+\Phi((0,s_2,\dots s_n),t)-\Phi((0,s_2,\dots s_n),0)\,.$$

By the induction hypothesis the term $\Phi((0,s_2,\dots s_n),t)$ is a separable function of the pair $((s_2,\dots s_n),t)$. Thus the right hand side is a separable function of the pair $((s_1,s_2,\dots s_n),t)$. □

Finally, we have a simple criterion to decide whether a separable function vanishes.

Proposition 3.4. *Suppose Φ is a separable smooth function on*

$$U(F)\backslash U(\mathbb{A}) \times V(F)\backslash V(\mathbb{A}),$$

where (U, V) is a pair of vector spaces over F in duality by the bilinear form $(u, v) \mapsto \langle u, v\rangle$. Suppose that

$$\int\int \Phi(u, v)\psi(\langle u, f\rangle + \langle e, v\rangle)dudv = 0,$$

if $e \in U(F)$, $f \in V(F)$, and either $e = 0$ or $f = 0$ (or both $e = 0$ and $f = 0$). Then $\Phi = 0$.

Proof: By assumption,

$$\Phi(u, v) = \Phi(u, 0) + \Phi(0, v) - \Phi(0, 0).$$

Taking $e = 0$ and $f \neq 0$ we get

$$\int \Phi(u, 0)\psi(\langle u, f\rangle)du + \left(\int \Phi(0, v)dv - \Phi(0, 0)\right) \int \psi(\langle u, f\rangle)du = 0.$$

Since $f \neq 0$, there exists $u_0 \in U(F)\backslash U(\mathbb{A})$, such that $\psi(\langle u_0, f\rangle) \neq 1$. By changing of variables,

$$\int \psi(\langle u, f\rangle)du = \int \psi(\langle u + u_0, f\rangle)du = \psi(\langle u_0, f\rangle) \cdot \int \psi(\langle u, f\rangle)du,$$

which implies that $\int \psi(\langle u, f\rangle)du = 0$. Hence we get

$$\int \Phi(u, 0)\psi(\langle u, f\rangle)du = 0.$$

This shows that $u \mapsto \Phi(u, 0)$ is a constant function. Likewise $v \mapsto \Phi(0, v)$ is a constant function. By the above formula from assumption, $(u, v) \mapsto \Phi(u, v)$ is a constant function. Now if we take $e = 0$ and $f = 0$ we get

$$\int\int \Phi(u, v)dudv = 0.$$

But since Φ is constant, this integral is just the constant value of Φ. Hence Φ is 0 as claimed. □

4. Proof of Theorem 2.4

It will be convenient to introduce for every $\xi \in V_\pi$ the function

$$\Phi_\xi(u, v) = \int_{F\backslash\mathbb{A}} (U_\xi - V_\xi) \begin{pmatrix} 1 & u & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} dz.$$

Here u is a row of size $n-2$ and v a column of size $n-2$. The scalar product of u and v is denoted by $\langle u, v\rangle$. Thus Φ_ξ is a smooth function on $(F\backslash\mathbb{A})^{n-2} \times (F\backslash\mathbb{A})^{n-2}$.

Proposition 4.1. *Let A be a rational column of size $n-2$ and B a rational row of size $n-2$. Suppose $A = 0$ or $B = 0$ (or both $A = 0$ and $B = 0$). Then, for every $\xi \in V_\pi$, the integral*

$$\int\int \Phi_\xi(u, v)\psi(\langle u, A\rangle + \langle B, v\rangle)dudv$$

is 0.

Proof: It suffices to prove the integral

$$\int\int\int U_\xi \begin{pmatrix} 1 & u & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \psi(\langle u, A\rangle + \langle B, v\rangle)dudvdz$$

has the properties described in the proposition.

Indeed, the automorphism

$$g \mapsto \omega_n{}^t g^{-1}\omega_n$$

changes the function U_ξ relative to the character ψ into the function V_ξ relative to the character ψ^{-1} and leaves invariant the group over which we integrate. Thus the integral

$$\int\int\int V_\xi \begin{pmatrix} 1 & u & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \psi(uA + Bv)dudvdz$$

has the same properties and so does the integral of the difference $U_\xi - V_\xi$. Note that $\langle u, A\rangle = uA$, $\langle B, v\rangle = Bv$.

Now consider the integral for U_ξ. If $B = 0$ it suffices to show that, for every ξ,

$$\int\int U_\xi \begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} dvdz = 0\,.$$

Indeed, as we have remarked, if this identity is true for all ξ, then the identity obtained by translating on the right by an element of G_n is still true for every ξ. In particular, then, for all ξ and all u,

$$\int\int U_\xi \left[\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & u & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] dvdz = 0\,.$$

Integrating over u with respect to the character $\psi(uA)$ we find

$$\int du \int\int U_\xi \left[\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] dvdz\psi(uA)du = 0\,,$$

or

$$\int\int\int U_\xi \begin{pmatrix} 1 & u & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \psi(uA)dudvdz = 0\,,$$

as claimed.

Replacing U_ξ by its definition, we find

$$\int\int U_\xi \begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} dvdz =$$

$$\int\int \sum_{\gamma \in U_{n-1}(F)\backslash G_{n-1}(F)} W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \right] dvdz\,.$$

Exchanging summation and integration we find

$$\int\int \sum_{\gamma \in U_{n-1}(F)\backslash G_{n-1}(F)} W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \right] dvdz\,.$$

$$= \sum_{\gamma \in U_{n-1}(F)\backslash G_{n-1}(F)} \int\int W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \right] dvdz\,.$$

After a change of variables this becomes

$$= \sum_{\gamma \in U_{n-1}(F)\backslash G_{n-1}(F)} \int\int W_\xi \left[\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right] dvdz$$

which is 0.

Now suppose $B \neq 0$ (and thus $A = 0$). To prove that the integral vanishes we may conjugate by a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\,,\ \gamma \in G_{n-2}(F)\,.$$

This amounts to replacing B by $B\gamma^{-1}$. Thus we may assume $B = (0,0,\dots,0,-1)$. Then the integral takes the form

$$\int \mathbb{P}^n_{n-2} U_\xi \begin{pmatrix} 1 & u & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} du\,.$$

Replacing $\mathbb{P}^n_{n-2} U_\xi$ by its expression in terms of W_ξ we find

$$\int \sum_{\gamma \in U_{n-2}(F)\backslash G_{n-2}(F)} W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} 1 & u & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] du\,.$$

Let us write this as

$$\int \left(\int \sum_\gamma W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & u'' & 0 \\ 0 & I_{n-3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u' & 0 \\ 0 & I_{n-3} & 0 \\ 0 & 0 & I_2 \end{pmatrix} \right] du'' \right) du'\,,$$

with $\gamma \in U_{n-2}(F)\backslash G_{n-2}(F)$. The inner integral is in fact a multiple of the integral

$$\int \psi\left(\langle \gamma_{n-2}, \begin{pmatrix} u'' \\ 0 \end{pmatrix} \rangle \right) du'\,,$$

where γ_{n-2} is the last row of γ. This integral is then 0 unless the last row has the form

$$(0, \bullet, \bullet, \dots, \bullet)\,,$$

in which case the integral is 1. Thus our expression is

$$\int \sum_\gamma W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} 1 & u' & 0 \\ 0 & I_{n-3} & 0 \\ 0 & 0 & I_2 \end{pmatrix} \right] du'\,,$$

where $\gamma \in U_{n-2}(F)\backslash G_{n-2}(F)$ and the last row of γ has the form $(0, \bullet, \bullet, \dots, \bullet)$.

Now let us write the Bruhat decomposition of $\gamma \in U_{n-2}(F)\backslash G_{n-2}(F)$:

$$\gamma = w\nu\alpha$$

with w a permutation matrix, α a diagonal matrix and $\nu \in U_{n-2}(F)$. The last row of w cannot have the form $(1,0,\dots,0)$ otherwise the last row of γ would have the form $(x, \bullet, \bullet, \dots, \bullet)$, $x \neq 0$. Thus the last row of w has the form $(0, \bullet, \bullet, \dots, \bullet)$. Let us write down the contribution of such a w to the above expression and show it is 0. We introduce the abelian group

$$X = \left\{ \begin{pmatrix} 1 & u' & 0 \\ 0 & I_{n-3} & 0 \\ 0 & 0 & I_2 \end{pmatrix} \right\}\,.$$

The contribution of w has the form

$$\int_{X(F)\backslash X(\mathbb{A})} \sum_{\gamma} W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & I_2 \end{pmatrix} x \right] dx \,,$$

where $\gamma = w\nu\alpha$, $\alpha \in A_{n-2}(F)$ and ν is in a set of representatives for the cosets $w^{-1}U_{n-2}(F) \cap U_{n-2}(F)\backslash U_{n-2}(F)$. For a set of representatives, we will take the group

$$S = w^{-1}\overline{U_{n-2}}(F)w \cap U_{n-2}(F)\,,$$

where as usual $\overline{U_{n-2}}$ is the subgroup opposite to U_{n-2} (that is, its transpose).

Now viewed as a subgroup of $G_{n-2}(F)$ the group $X(F)$ is the unipotent radical of the standard parabolic subgroup of type $(1, n-3)$ and $U_{n-2}(F)$ is contained in this parabolic subgroup. In particular, X is normalized by $U_{n-2}(F)$ and by S.

We keep in mind that X is an abelian group. Let us write X as the product X_1X_2 where

$$X_1(F) = w^{-1}U_{n-2}(F)w \cap X(F)\,,\ X_2(F) = w^{-1}\overline{U_{n-2}}(F)w \cap X(F)\,.$$

Thus S contains $X_2(F)$ and is the product of $X_2(F)$ and the group T,

$$T = S \cap \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & I_2 \end{pmatrix} : \mu \in U_{n-3}(F) \right\}.$$

Moreover the group S normalizes X_2. In particular it normalizes the groups

$$X(F) = X_1(F)X_2(F)\,,\ X_2(\mathbb{A})\,,$$

hence also the closed subgroup

$$X_1(F)X_2(\mathbb{A})\,.$$

Then our expression becomes

$$\int\int \sum W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \nu\alpha & 0 \\ 0 & I_2 \end{pmatrix} x_1x_2 \right] dx_1dx_2\,.$$

Here x_1 and x_2 are integrated over $X_1(F)\backslash X_1(\mathbb{A})$ and $X_2(F)\backslash X_2(\mathbb{A})$ respectively. We sum for $\alpha \in A_{n-2}(F)$ and $\nu \in S$. We can take the sum over α outside as follows:

$$\sum_\alpha \int\int \sum_\nu W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \nu & 0 \\ 0 & I_2 \end{pmatrix} x_1x_2 \begin{pmatrix} \alpha & 0 \\ 0 & I_2 \end{pmatrix} \right] dx_1dx_2\,.$$

We now show that each term of the α sum is 0 for all ξ. As usual, we may take $\alpha = I_{n-2}$. Now we write

$$\nu = \tau\sigma$$

with $\sigma \in X_2(F)$ and τ in T. We combine the integration over x_2 and the sum over σ to obtain an integral over $X_2(\mathbb{A})$. We arrive at

$$\int dx_1 \left(\sum_\tau \int W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_1 x_2 \right] dx_2 \right).$$

Here the integral is for $x_1 \in X_1(F)\backslash X_1(\mathbb{A})$ and $x_2 \in X_2(\mathbb{A})$. We will show that for every τ and every ξ the following integral is 0:

$$\int_{X_1(F)\backslash X_1(\mathbb{A})} dx_1 \left(\int_{X_2(\mathbb{A})} W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_1 x_2 \right] dx_2 \right).$$

In order for this expression to even make sense we better show first that, on $X(\mathbb{A})$, the function

$$x \mapsto \int_{X_2(\mathbb{A})} W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x x_2 \right] dx_2$$

is invariant under $X_1(F)$. Recall it is invariant under $X_2(\mathbb{A})$. So it amounts to the same to prove it is invariant under $X_1(F)X_2(\mathbb{A})$. Since τ normalizes the groups $X(\mathbb{A})$ and $X_1(F)X_2(\mathbb{A})$ it amounts to the same to prove that on $X(\mathbb{A})$ the function

$$x \mapsto \int W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_2 \right] dx_2$$

is invariant under $X_1(F)X_2(\mathbb{A})$. The invariance under $X_2(\mathbb{A})$ being clear we check the invariance under $X_1(F)$. But if $x_1 \in X_1(F)$ we have

$$\int W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x_1 x \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_2 \right] dx_2$$
$$= \int W_\xi \left[y_1 \begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_2 \right] dx_2$$

with

$$y_1 = \begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x_1 \begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix}^{-1}.$$

Since y_1 is in $U_{n-2}(F)$, this expression does not depend on x_1.

At this point we can reformulate our goal as follows: we have to prove that for every τ and every ξ, the integral of the function

$$x \mapsto \int W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x x_2 \right] dx_2$$

over the quotient

$$X_1(F)X_2(\mathbb{A})\backslash X(\mathbb{A})$$

is 0. Conjugation by τ defines an automorphism of this quotient which preserves the Haar measure. Hence it suffices to prove that the integral of the function

$$x \mapsto \int W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_2 \right] dx_2$$

over the same quotient vanishes. Equivalently, we want to prove that

$$\int_{X_1(F)\backslash X_1(\mathbb{A})} dx_1 \int_{X_2(\mathbb{A})} W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x_1 \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_2 \right] dx_2 = 0$$

for all τ and all ξ. At this point we may exchange the order of integration. So it will be enough to prove that

$$\int_{X_1(F)\backslash X_1(\mathbb{A})} dx_1 W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x_1 \right] g = 0$$

for all ξ and g. Now let Y be the subgroup defined by

$$Y(F) = U_{n-2}(F) \cap w X_1(F) w^{-1} \, .$$

The integral can be written as

$$\int_{Y(F)\backslash Y(\mathbb{A})} \psi_{U_n}(y) dy \; W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} g \right] .$$

Now we claim that the subgroup Y contains a root subgroup for a positive simple root. Thus the character ψ_{U_n} is non-trivial on the subgroup $Y(\mathbb{A})$ and the integral vanishes which concludes the proof.

It remains to prove the claim. Since the proof requires an inductive argument we state the claim as a separate lemma. $\square$

Lemma 4.2. *Let X be the unipotent radical of the standard parabolic subgroup of type $(1, m-1)$ in GL_m. Let $w \in GL_m$ be a permutation matrix whose last row has the form $(0, \bullet, \bullet, \dots, \bullet)$. Then conjugation by w changes one of the root subgroups of X into the root subgroup associated to a positive simple root.*

PROOF: Our assertion is trivial if $m = 2$ because then $w = I_2$. So we may assume $m > 2$ and our assertion true for $m - 1$. The matrix w has the form

$$w = \begin{pmatrix} w_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w_2 \end{pmatrix}$$

where w_1 and w_2 are permutation matrices of size $m - 1$. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & w_2 \end{pmatrix}$$

normalizes X, it suffices to prove our assertion for

$$w = \begin{pmatrix} w_1 & 0 \\ 0 & 1 \end{pmatrix} .$$

If the last row of w_1 has the form

$$(1, 0, \dots, 0) \,,$$

the root group corresponding to $e_1 - e_m$ is conjugated to the root group corresponding to $e_{m-1} - e_m$. So our assertion is true in this case. If the last row of w_1 has the form

$$(0, \bullet, \dots, \bullet) \,,$$

we can apply the induction hypothesis to w_1 and obtain again our assertion. $\square$

Proof of Theorem 2.4: Recall from (2.3) that

$$\int_{(F\backslash\mathbb{A})^3} (U_\xi - V_\xi) \begin{pmatrix} 1 & xe_0 & z \\ 0 & I_{n-2} & yf_0 \\ 0 & 0 & 1 \end{pmatrix} \psi(\alpha x + \beta y) dz dx dy = 0$$

for all $\xi \in V_\pi$, $\alpha \in F^\times, \beta \in F^\times$. Here e_0 and f_0 are of size $n-2$ and

$$e_0 = (0, 0, \dots, 0, 1) \,,\; f_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} .$$

Note that if $\Phi_\xi = 0$ for all $\xi \in V_\pi$, then the condition $\mathcal{E}(n, n-3)$ holds, since the integral over z is just an inner integral of the above integral. Hence, in the following, we assume that the condition $\mathcal{E}(n, n-3)$ holds, that is, the above integral equals to 0.

We can conjugate by a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} ,\; \gamma \in G_{n-2}(F)$$

to obtain

$$\int_{(F\backslash\mathbb{A})^3} (U_\xi - V_\xi) \begin{pmatrix} 1 & xe_0\gamma^{-1} & z \\ 0 & I_{n-2} & y\gamma f_0 \\ 0 & 0 & 1 \end{pmatrix} \psi(\alpha x + \beta y) dz dx dy = 0 \,,$$

for all $\xi \in V_\pi$, $\alpha \in F^\times, \beta \in F^\times$. Thus

$$\int_{(F\backslash\mathbb{A})^3} (U_\xi - V_\xi) \begin{pmatrix} 1 & xe & z \\ 0 & I_{n-2} & yf \\ 0 & 0 & 1 \end{pmatrix} \psi(\alpha x + \beta y) dz dx dy = 0\,,$$

for e (resp. f) an F-row (resp. column) of size $n-2$ and $\langle e, f\rangle = 1$ and for all $\xi \in V_\pi$, $\alpha \in F^\times, \beta \in F^\times$. Moreover, right translating by an adelic matrix in the unipotent radical of the parabolic subgroup of type $(1, n-2, 1)$ we obtain

$$\int_{(F\backslash\mathbb{A})^3} \Phi_\xi(u + xe, v + yf)\psi(\alpha x + \beta y) dz dx dy = 0\,,$$

for all $(u, v) \in \mathbb{A}^{n-2} \times \mathbb{A}^{n-2}$, and for all $\xi \in V_\pi$, $\alpha \in F^\times$, $\beta \in F^\times$. Thus by Proposition 3.2, the function Φ_ξ is separable, for all $\xi \in V_\pi$. By Propositions 4.1 and 3.4, it is in fact 0 and we are done. $\square$

References

[Cog02] J. Cogdell, *L-functions and Converse Theorems for GL_n*, Automorphic forms and applications, 97–177, IAS/Park City Math. Ser., **12**, Amer. Math. Soc., Providence, RI, 2007.

[CPS94] J. Cogdell and I. Piatetski-Shapiro, *Converse theorems for GL_n*, Inst. Hautes Études Sci. Publ. Math. No. **79** (1994), 157–214.

[CPS96] J. Cogdell and I. Piatetski-Shapiro, *A Converse Theorem for GL_4*, Math. Res. Letters **3** (1996), 67–76.

[CPS99] J. Cogdell and I. Piatetski-Shapiro, *Converse theorems for GL_n, II*, J. Reine Angew. Math. **507** (1999), 165–188.

[JPSS79] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Automorphic forms on $GL(3)$*. Ann. of Math. (2) **109** (1979), no. 1–2, 169–258.

[JL70] H. Jacquet and R. Langlands, *Automorphic forms on $GL(2)$*. Lecture Notes in Mathematics, Vol. **114**. Springer-Verlag, Berlin-New York, 1970.

[PS76] I. Piatetski-Shapiro, *Zeta functions of $GL(n)$*. Univ. Maryland Techn. Rep. TR 76–46 (1976).

Department of Mathematics, Columbia University, Rm 615, MC 4408 2990 Broadway, New York, NY 10027 USA

E-mail address: hj@math.columbia.edu

Department of Mathematics, Purdue University, 150 N. University St, West Lafayette, IN, 47907 USA

E-mail address: liu2053@purdue.edu