A TWO WEIGHT FRACTIONAL SINGULAR INTEGRAL THEOREM WITH SIDE CONDITIONS, ENERGY AND k-ENERGY DISPERSED

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Abstract. This paper is a sequel to our paper [SaShUr7]. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$ (possibly having common point masses), and let T^α be a standard α-fractional Calderón-Zygmund operator on $\mathbb{R}^n$ with $0 \leq \alpha < n$. Suppose that $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map, and refer to the images ΩQ of cubes Q as *quasicubes*. Furthermore, assume as side conditions the $\mathcal{A}_2^\alpha$ conditions, punctured A_2^α conditions, and certain α-*energy conditions* taken over quasicubes. Then we show that T^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if the quasicube testing conditions hold for T^α and its dual, and if the quasiweak boundedness property holds for T^α.

Conversely, if T^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$, then the quasitesting conditions hold, and the quasiweak boundedness condition holds. If the vector of α-fractional Riesz transforms $\mathbf{R}_\sigma^\alpha$ (or more generally a strongly elliptic vector of transforms) is bounded from $L^2(\sigma)$ to $L^2(\omega)$, then both the $\mathcal{A}_2^\alpha$ conditions and the punctured A_2^α conditions hold.

Our quasienergy conditions are not in general necessary for elliptic operators, but are known to hold for certain situations in which one of the measures is one-dimensional [LaSaShUrWi], [SaShUr8], and for certain side conditions placed on the measures such as doubling and k-energy dispersed, which when $k = n-1$ is similar to the condition of uniformly full dimension in [LaWi, versions 2 and 3].

Contents

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Dedication: This paper is dedicated to the memory of Cora Sadosky and her contributions to the theory of weighted inequalities and her promotion of women in mathematics.

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1. Introduction

The boundedness of the Hilbert transform $Hf(x) = \int_{\mathbb{R}} \frac{f(y)}{y-x} dy$ on the real line $\mathbb{R}$ in the Hilbert space $L^2(\mathbb{R})$ has been known for at least a century (perhaps dating back to A & E[1]):

$$\|Hf\|_{L^2(\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})}, \qquad f \in L^2(\mathbb{R}). \tag{1}$$

This inequality has been the subject of much generalization, to which we now turn.

1.1. A brief history of the $T1$ theorem.

The celebrated $T1$ theorem of David and Journé [DaJo] extends (1) to more general kernels by characterizing those singular integral operators T on $\mathbb{R}^n$ that are bounded on $L^2(\mathbb{R}^n)$, and does so in terms of a weak boundedness property, and the membership of the two functions $T\mathbf{1}$ and $T^*\mathbf{1}$ in the space of bounded mean oscillation,

$$\begin{aligned} \|T\mathbf{1}\|_{BMO(\mathbb{R}^n)} &\lesssim \|\mathbf{1}\|_{L^\infty(\mathbb{R}^n)} = 1, \\ \|T^*\mathbf{1}\|_{BMO(\mathbb{R}^n)} &\lesssim \|\mathbf{1}\|_{L^\infty(\mathbb{R}^n)} = 1. \end{aligned}$$

These latter conditions are actually the following *testing conditions* in disguise,

$$\begin{aligned} \|T\mathbf{1}_Q\|_{L^2(\mathbb{R}^n)} &\lesssim \|\mathbf{1}_Q\|_{L^2(\mathbb{R}^n)} = \sqrt{|Q|}, \\ \|T^*\mathbf{1}_Q\|_{L^2(\mathbb{R}^n)} &\lesssim \|\mathbf{1}_Q\|_{L^2(\mathbb{R}^n)} = \sqrt{|Q|}, \end{aligned}$$

tested uniformly over all indicators of cubes Q in $\mathbb{R}^n$ for both T and its dual operator T^*. This theorem was the culmination of decades of investigation into the nature of cancellation conditions required for boundedness of singular integrals[2].

A parallel thread of investigation had begun even earlier with the equally celebrated theorem of Hunt, Muckenhoupt and Wheeden [HuMuWh] that extended (1) to measures more general than Lebesgue's by characterizing boundedness of the Hilbert transform on weighted spaces $L^2(\mathbb{R};w)$. This thread culminated in the theorem of Coifman and Fefferman[3] [CoFe] that characterizes those nonnegative weights w on $\mathbb{R}^n$ for which all of the 'nicest' of the $L^2(\mathbb{R}^n)$ bounded singular integrals T above are bounded on weighted spaces $L^2(\mathbb{R}^n;w)$, and does so in terms of the A_2 condition of Muckenhoupt,

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{w(x)}dx\right) \lesssim 1,$$

[1] Peter Jones used A&E to stand for Adam and Eve.
[2] See e.g. chapter VII of Stein [Ste] and the references given there for a historical background.
[3] See e.g. chapter V of [Ste] and the references given there for the long history of this investigation.

taken uniformly over all cubes Q in $\mathbb{R}^n$. This condition is also a testing condition in disguise, in particular it follows from

$$\left\| T\left(\mathbf{s}_Q \frac{1}{w} \right) \right\|_{L^2(\mathbb{R}^n;w)} \lesssim \left\| \mathbf{s}_Q \frac{1}{w} \right\|_{L^2(\mathbb{R}^n;w)} ,$$

tested over all 'indicators with tails' $\mathbf{s}_Q(x) = \frac{\ell(Q)}{\ell(Q)+|x-c_Q|}$ of cubes Q in $\mathbb{R}^n$.

A natural synthesis of these two threads leads to the 'two weight' question of characterizing those pairs of weights (σ, ω) having the property that nice singular integrals are bounded from $L^2(\mathbb{R}^n;\sigma)$ to $L^2(\mathbb{R}^n;\omega)$. Returning to the simplest (nontrivial) singular integral of all, namely the Hilbert transform $Hf(x) = \int_{\mathbb{R}} \frac{f(y)}{y-x} dy$ on the real line, Cotlar and Sadosky gave a beautiful function theoretic characterization of the weight pairs (σ, ω) for which H is bounded from $L^2(\mathbb{R};\sigma)$ to $L^2(\mathbb{R};\omega)$, namely a two-weight extension of the Helson-Szegö theorem. This characterization illuminated a deep connection between two quite different function theoretic conditions, but failed to shed much light on when either of them held. On the other hand, the two weight inequality for positive fractional integrals, Poisson integrals and maximal functions were characterized using testing conditions by one of us in [Saw] (see also [Hyt2]) and [Saw1], but relying in a very strong way on the positivity of the kernel, something the Hilbert kernel lacks. In light of these considerations, Nazarov, Treil and Volberg formulated the two weight question for the Hilbert transform [Vol], that in turn led to the following NTV conjecture:

Conjecture 1. [Vol] *The Hilbert transform is bounded from $L^2(\mathbb{R}^n;\sigma)$ to $L^2(\mathbb{R}^n;\omega)$, i.e.*

$$\|H(f\sigma)\|_{L^2(\mathbb{R}^n;\omega)} \lesssim \|f\|_{L^2(\mathbb{R}^n;\sigma)} , \qquad f \in L^2(\mathbb{R}^n;\sigma) , \tag{2}$$

if and only if the two weight A_2 condition with two tails holds,

$$\left(\frac{1}{|Q|} \int_Q \mathbf{s}_Q^2 d\omega(x) \right) \left(\frac{1}{|Q|} \int_Q \mathbf{s}_Q^2 d\sigma(x) \right) \lesssim 1 ,$$

uniformly over all cubes Q, and the two testing conditions hold,

$$\begin{aligned} \|H\mathbf{1}_Q\sigma\|_{L^2(\mathbb{R}^n;\omega)} &\lesssim \|\mathbf{1}_Q\|_{L^2(\mathbb{R}^n;\sigma)} = \sqrt{|Q|_\sigma} , \\ \|H^*\mathbf{1}_Q\omega\|_{L^2(\mathbb{R}^n;\sigma)} &\lesssim \|\mathbf{1}_Q\|_{L^2(\mathbb{R}^n;\omega)} = \sqrt{|Q|_\omega} , \end{aligned}$$

uniformly over all cubes Q.

In a groundbreaking series of papers including [NTV1],[NTV2] and [NTV4], Nazarov, Treil and Volberg used weighted Haar decompositions with random grids, introduced their 'pivotal' condition, and proved the above conjecture under the side assumption that the pivotal condition held. Subsequently, in joint work of two of us, Sawyer and Uriarte-Tuero, with Lacey [LaSaUr2], it was shown that the pivotal condition was not necessary in general, a necessary 'energy' condition was introduced as a substitute, and a hybrid merging of these two conditions was shown to be sufficient for use as a side condition. Eventually, these three authors with Shen established the NTV conjecture in a two part paper; Lacey, Sawyer, Shen and Uriarte-Tuero [LaSaShUr3] and Lacey [Lac]. A key ingredient in the proof was an 'energy reversal' phenomenon enabled by the Hilbert transform kernel equality

$$\frac{1}{y-x} - \frac{1}{y-x'} = \frac{x-x'}{(y-x)(y-x')},$$

having the remarkable property that the denominator on the right hand side remains *positive* for all y outside the smallest interval containing both x and x'. This proof of the NTV conjecture was given in the special case that the weights σ and ω had no point masses in common, largely to avoid what were then thought to be technical issues. However, these issues turned out to be considerably more interesting, and this final assumption of no common point masses was removed shortly after by Hytönen [Hyt2], who also simplified some aspects of the proof.

At this juncture, attention naturally turned to the analogous two weight inequalities for higher dimensional singular integrals, as well as α-fractional singular integrals such as the Cauchy transform in the plane. In a long paper begun in [SaShUr5] on the *arXiv* in 2013, and subsequently appearing in [SaShUr7], the authors introduced the appropriate notions of Poisson kernel to deal with the A_2^α condition on the one hand, and the α-energy condition on the other hand (unlike for the Hilbert transform, these two Poisson kernels differ in general). The main result of that paper established the $T1$ theorem for 'elliptic' vectors of singular integrals under the side assumption that an energy condition and its dual held, thus identifying the *culprit* in higher dimensions as the energy conditions. A general $T1$ conjecture is this (see below for definitions).

Conjecture 2. *Let $\mathbf{T}^{\alpha,n}$ denote an elliptic vector of standard α-fractional singular integrals in $\mathbb{R}^n$. Then $\mathbf{T}^{\alpha,n}$ is bounded from $L^2(\mathbb{R}^n;\sigma)$ to $L^2(\mathbb{R}^n;\omega)$, i.e.*

$$\left\|\mathbf{T}^{\alpha,n}(f\sigma)\right\|_{L^2(\mathbb{R}^n;\omega)} \lesssim \|f\|_{L^2(\mathbb{R}^n;\sigma)}, \qquad f\in L^2(\mathbb{R}^n;\sigma), \tag{3}$$

if and only if the two one-tailed $\mathcal{A}_2^\alpha$ conditions with holes hold, the punctured $A_2^{\alpha,\mathrm{punct}}$ conditions hold, and the two testing conditions hold,

$$\begin{aligned}\left\|\mathbf{T}^{\alpha,n}\mathbf{1}_Q\sigma\right\|_{L^2(\mathbb{R}^n;\omega)} &\lesssim \left\|\mathbf{1}_Q\right\|_{L^2(\mathbb{R}^n;\sigma)}=\sqrt{|Q|_\sigma}\ ,\\ \left\|\mathbf{T}^{\alpha,n,\mathrm{dual}}\mathbf{1}_Q\omega\right\|_{L^2(\mathbb{R}^n;\sigma)} &\lesssim \left\|\mathbf{1}_Q\right\|_{L^2(\mathbb{R}^n;\omega)}=\sqrt{|Q|_\omega}\ ,\end{aligned}$$

for all cubes Q in $\mathbb{R}^n$ (whose sides need not be parallel to the coordinate axes).

In [SaShUr9], the authors have recently shown that the energy conditions are *not* necessary for boundedness of elliptic vectors of singular integrals in general, but have left open the following conjecture, which in view of the aforementioned main result in [SaShUr7], would yield the $T1$ theorem for gradient elliptic operators. An elliptic α-fractional singular integral vector $\mathbf{T}^{\alpha,n}$ in $\mathbb{R}^n$ is said to be *gradient elliptic* if both $|\nabla_x\mathbf{K}^\alpha(x,y)|\gtrsim|x-y|^{\alpha-n-1}$ and $|\nabla_y\mathbf{K}^\alpha(x,y)|\gtrsim|x-y|^{\alpha-n-1}$.

Conjecture 3. *Let $\mathbf{T}^{\alpha,n}$ denote a gradient elliptic vector of standard α-fractional singular integrals in $\mathbb{R}^n$. If $\mathbf{T}^{\alpha,n}$ is bounded from $L^2(\mathbb{R}^n;\sigma)$ to $L^2(\mathbb{R}^n;\omega)$, then the energy conditions hold as defined in Definition 5 below.*

While the energy conditions are not necessary for elliptic operators in general [SaShUr9], there are some cases in which they have been proved to hold. Of course, they hold for the Hilbert transform on the line [LaSaUr2], and in recent joint work with M. Lacey and B. Wick, the five of us have established that the energy conditions hold for the Cauchy transform in the plane in the special case where one of the measures is supported on either a straight line or a circle, thus proving the $T1$ theorem in this case. The key to this result was an extension of the energy reversal phenomenon for the Hilbert transform to the setting of the Cauchy transform, and here the one-dimensional nature of the line and circle played a critical role. In particular, a special decomposition of a 2-dimensional measure into 'end' and 'side' pieces played a crucial role, and was in fact discovered independently in both [SaShUr3] and [LaWi1]. A further instance of energy reversal occurs in our $T1$ theorem [SaShUr8] when one measure is compactly supported on a $C^{1,\delta}$ curve in $\mathbb{R}^n$.

The paper [LaWi, v3] by Lacey and Wick overlaps both our paper [SaShUr7] and this paper to some extent, and we refer the reader to [SaShUr7] for a more detailed discussion.

Finally, we mention an entirely different approach to investigating the two weight problem that has attracted even more attention than the $T1$ approach we just described. Nazarov has shown that the two-tailed $\mathcal{A}_2^\alpha$ condition of Muckenhoupt (see below) is insufficient for (3), and this begs the question of strengthening the Muckenhoupt condition enough to make it sufficient for (3). The great advantage of this approach is that strengthened Muckenhoupt conditions are generally 'easy' to check as compared to the highly unstable testing conditions. The disadvantage of course is that such conditions have never been shown to characterize (3). The literature devoted to these issues, beginning with that of Pérez [Per], and continuing more recently with work of many groups involving, among others, D. Cruz-Uribe, M. Lacey, A. K. Lerner, J. M. Martell, F. Nazarov, C. Pérez, A. Reznikov and A. Volberg, is both too vast and too tangential to this paper to record here, and we encourage the reader to search the web for more on 'bumped-up' Muckenhoupt conditions[4].

This paper is concerned with the $T1$ approach and is a sequel to our first paper [SaShUr7]. We prove here a two weight inequality for standard α-fractional Calderón-Zygmund operators T^α in Euclidean space $\mathbb{R}^n$, where we assume n-dimensional $\mathcal{A}_2^\alpha$ conditions (with holes), punctured $A_2^{\alpha.\,\mathrm{punct}}$ conditions, and certain *α-energy conditions* as side conditions on the weights (in higher dimensions the Poisson kernels used in these two conditions differ). The two main differences in this theorem here are that we state and prove[5]

[4] starting e.g. with the recent articles [And] and [Lac3]

[5] Very detailed proofs of all of the results here can be found on the arXiv [SaShUr6].

our theorem in the more general setting of *quasicubes* (as in [SaShUr5]), and more notably, we now permit the weights, or measures, to have common point masses, something not permitted in [SaShUr7] (and only obtained for a partial range of α in [LaWi, version 3]). As a consequence, we use $\mathcal{A}_2^\alpha$ conditions with holes as in the one-dimensional setting of Hytönen [Hyt2], together with punctured $A_2^{\alpha,\mathrm{punct}}$ conditions, as the usual A_2^α '*without punctures*' fails whenever the measures have a common point mass. The extension to permitting common point masses uses the two weight Poisson inequality in [Saw] to derive functional energy, together with a delicate adaptation of arguments in [SaShUr5]. The key point here is the use of the (typically necessary) 'punctured' Muckenhoupt $A_2^{\alpha,\mathrm{punct}}$ conditions below. They turn out to be crucial in estimating the backward Poisson testing condition later in the paper. We remark that Hytönen's bilinear dyadic Poisson operator and shifted dyadic grids [Hyt2] in dimension $n=1$ can be extended to derive functional energy in higher dimensions, but at a significant cost of increased complexity. See the previous version of this paper on the *arXiv* for this approach[6], and also [LaWi] where Lacey and Wick use this approach. Finally, we point out that our use of punctured Muckenhoupt conditions provides a simpler alternative to Hytönen's method of extending to common point masses the NTV conjecture for the Hilbert transform [Hyt2]. The Muckenhoupt $\mathcal{A}_2^\alpha$ conditions (with holes) are also typically necessary for the norm inequality, but the proofs require extensive modification when quasicubes and common point masses are included.

On the other hand, the extension to quasicubes in the setting of *no* common point masses turns out to be, after checking all the details, mostly a cosmetic modification of the proof in [SaShUr7], as demonstrated in [SaShUr5]. The use of quasicubes is however crucial in our $T1$ theorem when one of the measures is compactly supported on a $C^{1,\delta}$ curve [SaShUr8], and this accounts for their inclusion here.

We also introduce a new side condition on a measure, that we call *k-energy dispersed*, which captures the notion that a measure is *not* supported too near a k-dimensional plane at any scale. When $0 \leq \alpha < n$ is appropriately related to k, we are able to obtain the necessity of the energy conditions for k-energy dispersed measures, and hence a $T1$ theorem for strongly elliptic operators $\mathbf{T}^\alpha$. The case $k = n-1$ is similar to the condition of uniformly full dimension introduced in [LaWi, versions 2 and 3].

We begin by recalling the notion of quasicube used in [SaShUr5] - a special case of the classical notion used in quasiconformal theory.

Definition 1. *We say that a homeomorphism $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map if*

$$\left\|\Omega\right\|_{Lip} \equiv \sup_{x,y\in\mathbb{R}^n} \frac{\left\|\Omega(x) - \Omega(y)\right\|}{\left\|x-y\right\|} < \infty, \tag{4}$$

and $\left\|\Omega^{-1}\right\|_{Lip} < \infty$.

Note that a globally biLipschitz map Ω is differentiable almost everywhere, and that there are constants $c, C > 0$ such that

$$c \leq J_\Omega(x) \equiv \left|\det D\Omega(x)\right| \leq C, \qquad x \in \mathbb{R}^n.$$

Example 1. *Quasicubes can be wildly shaped, as illustrated by the standard example of a logarithmic spiral in the plane $f_\varepsilon(z) = z\left|z\right|^{2\varepsilon i} = z e^{i\varepsilon \ln(z\overline{z})}$. Indeed, $f_\varepsilon : \mathbb{C} \to \mathbb{C}$ is a globally biLipschitz map with Lipschitz constant $1 + C\varepsilon$ since $f_\varepsilon^{-1}(w) = w\left|w\right|^{-2\varepsilon i}$ and*

$$\nabla f_\varepsilon = \left(\frac{\partial f_\varepsilon}{\partial z}, \frac{\partial f_\varepsilon}{\partial \overline{z}}\right) = \left(\left|z\right|^{2\varepsilon i} + i\varepsilon\left|z\right|^{2\varepsilon i}, i\varepsilon\frac{z}{\overline{z}}\left|z\right|^{2\varepsilon i}\right).$$

On the other hand, f_ε behaves wildly at the origin since the image of the closed unit interval on the real line under f_ε is an infinite logarithmic spiral.

Notation 1. *We define $\mathcal{P}^n$ to be the collection of half open, half closed cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. A half open, half closed cube Q in $\mathbb{R}^n$ has the form $Q = Q(c,\ell) \equiv \prod_{k=1}^n \left[c_k - \frac{\ell}{2}, c_k + \frac{\ell}{2}\right)$ for some $\ell > 0$ and $c = (c_1, ..., c_n) \in \mathbb{R}^n$. The cube $Q(c,\ell)$ is described as having center c and sidelength ℓ.*

We repeat the natural *quasi* definitions from [SaShUr5].

[6] Additional small arguments are needed to complete the shifted dyadic proof given there, but we omit them in favour of the simpler approach here resting on punctured Muckenhoupt conditions instead of holes. The authors can be contacted regarding completion of the shifted dyadic proof.

Definition 2. *Suppose that $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map.*

(1) *If E is a measurable subset of $\mathbb{R}^n$, we define $\Omega E \equiv \{\Omega(x) : x \in E\}$ to be the image of E under the homeomorphism Ω.*
   (a) *In the special case that $E = Q$ is a cube in $\mathbb{R}^n$, we will refer to ΩQ as a quasicube (or Ω-quasicube if Ω is not clear from the context).*
   (b) *We define the center $c_{\Omega Q} = c(\Omega Q)$ of the quasicube ΩQ to be the point Ωc_Q where $c_Q = c(Q)$ is the center of Q.*
   (c) *We define the side length $\ell(\Omega Q)$ of the quasicube ΩQ to be the sidelength $\ell(Q)$ of the cube Q.*
   (d) *For $r > 0$ we define the 'dilation' $r\Omega Q$ of a quasicube ΩQ to be ΩrQ where rQ is the usual 'dilation' of a cube in $\mathbb{R}^n$ that is concentric with Q and having side length $r\ell(Q)$.*

(2) *If $\mathcal{K}$ is a collection of cubes in $\mathbb{R}^n$, we define $\Omega\mathcal{K} \equiv \{\Omega Q : Q \in \mathcal{K}\}$ to be the collection of quasicubes ΩQ as Q ranges over $\mathcal{K}$.*

(3) *If $\mathcal{F}$ is a grid of cubes in $\mathbb{R}^n$, we define the inherited quasigrid structure on $\Omega\mathcal{F}$ by declaring that ΩQ is a child of $\Omega Q'$ in $\Omega\mathcal{F}$ if Q is a child of Q' in the grid $\mathcal{F}$.*

Note that if ΩQ is a quasicube, then $|\Omega Q|^{\frac{1}{n}} \approx |Q|^{\frac{1}{n}} = \ell(Q) = \ell(\Omega Q)$. For a quasicube $J = \Omega Q$, we will generally use the expression $|J|^{\frac{1}{n}}$ in the various estimates arising in the proofs below, but will often use $\ell(J)$ when defining collections of quasicubes. Moreover, there are constants R_{big} and R_{small} such that we have the comparability containments

$$Q + \Omega x_Q \subset R_{big}\Omega Q \text{ and } R_{small}\Omega Q \subset Q + \Omega x_Q \ .$$

Given a fixed globally biLipschitz map Ω on $\mathbb{R}^n$, we will define below the n-dimensional $\mathcal{A}_2^\alpha$ conditions (with holes), punctured Muckenhoupt conditions $A_2^{\alpha,\text{punct}}$, testing conditions, and energy conditions using Ω-quasicubes in place of cubes, and we will refer to these new conditions as quasi$\mathcal{A}_2^\alpha$, quasitesting and quasienergy conditions. We will then prove a $T1$ theorem with quasitesting and with quasi$\mathcal{A}_2^\alpha$ and quasienergy side conditions on the weights. Since quasi $\mathcal{A}_2^\alpha \cap$ quasi $A_2^{\alpha,\text{punct}} = \mathcal{A}_2^\alpha \cap A_2^{\alpha,\text{punct}}$ (see [SaShUr8]), we usually drop the prefix quasi from the various Muckenhoupt conditions (warning: quasi $\mathcal{A}_2^\alpha \neq \mathcal{A}_2^\alpha$).

Since the $\mathcal{A}_2^\alpha$ and punctured Muckenhoupt conditions typically hold, this identifies the culprit in higher dimensions as the pair of quasienergy conditions. We point out that these quasienergy conditions are implied by higher dimensional analogues of essentially all the other side conditions used previously in two weight theory, in particular doubling conditions and the Energy Hypothesis (1.16) in [LaSaUr2], as well as the condition of k-energy dispersed measures that is introduced below. This leads to our second theorem, which establishes the $T1$ theorem for strongly elliptic operators $\mathbf{T}^\alpha$ when both measures are k-energy dispered with k and α appropriately related.

It turns out that in higher dimensions, there are two natural 'Poisson integrals' P^α and $\mathcal{P}^\alpha$ that arise, the usual Poisson integral P^α that emerges in connection with energy considerations, and a different Poisson integral $\mathcal{P}^\alpha$ that emerges in connection with size considerations. The standard Poisson integral P^α appears in the energy conditions, and the reproducing Poisson integral $\mathcal{P}^\alpha$ appears in the $\mathcal{A}_2^\alpha$ condition. These two kernels coincide in dimension $n = 1$ for the case $\alpha = 0$ corresponding to the Hilbert transform.

2. Statements of results

Now we turn to a precise description of our main two weight theorem.

Assumption: We fix once and for all a globally biLipschitz map $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ for use in all of our quasi-notions.

We will prove a two weight inequality for standard α-fractional Calderón-Zygmund operators T^α in Euclidean space $\mathbb{R}^n$, where we assume the n-dimensional $\mathcal{A}_2^\alpha$ conditions, new punctured A_2^α conditions, and certain *α-quasienergy conditions* as side conditions on the weights. In particular, we show that for positive locally finite Borel measures σ and ω in $\mathbb{R}^n$, and assuming that both the *quasienergy condition* and its dual hold, a strongly elliptic vector of standard α-fractional Calderón-Zygmund operators $\mathbf{T}^\alpha$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ *if and only if* the $\mathcal{A}_2^\alpha$ condition and its dual hold (we assume a mild additional condition on the quasicubes for this), the punctured Muckenhoupt condition $A_2^{\alpha,\text{punct}}$ and its dual hold, the quasicube testing condition for $\mathbf{T}^\alpha$ and its dual hold, and the quasiweak boundedness property holds. In order to state our theorem precisely, we define these terms in the following subsections.

Remark 1. *It is possible to collect our various Muckenhoupt and quasienergy assumptions on the weight pair* (σ,ω) *into just* two *compact side conditions of Muckenhoupt and quasienergy type. We prefer however, to keep the individual conditions separate so that the interested reader can track their use below.*

2.1. **Standard fractional singular integrals and the norm inequality.** Let $0 \leq \alpha < n$. We define a standard α-fractional CZ kernel $K^\alpha(x,y)$ to be a function defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following fractional size and smoothness conditions of order $1+\delta$ for some $\delta > 0$,

$$\begin{aligned} |K^\alpha(x,y)| &\leq C_{CZ}|x-y|^{\alpha-n} \text{ and } |\nabla K^\alpha(x,y)| \leq C_{CZ}|x-y|^{\alpha-n-1}, \\ |\nabla K^\alpha(x,y) - \nabla K^\alpha(x',y)| &\leq C_{CZ}\left(\frac{|x-x'|}{|x-y|}\right)^\delta |x-y|^{\alpha-n-1}, \qquad \frac{|x-x'|}{|x-y|} \leq \frac{1}{2}, \end{aligned} \tag{5}$$

and the last inequality also holds for the adjoint kernel in which x and y are interchanged. We note that a more general definition of kernel has only order of smoothness $\delta > 0$, rather than $1+\delta$, but the use of the Monotonicity and Energy Lemmas below, which involve first order Taylor approximations to the kernel functions $K^\alpha(\cdot,y)$, requires order of smoothness more than 1.

2.1.1. *Defining the norm inequality.* We now turn to a precise definition of the weighted norm inequality

$$\|T^\alpha_\sigma f\|_{L^2(\omega)} \leq \mathfrak{N}_{T^\alpha_\sigma}\|f\|_{L^2(\sigma)}, \qquad f \in L^2(\sigma). \tag{6}$$

For this we introduce a family $\left\{\eta^\alpha_{\delta,R}\right\}_{0<\delta<R<\infty}$ of nonnegative functions on $[0,\infty)$ so that the truncated kernels $K^\alpha_{\delta,R}(x,y) = \eta^\alpha_{\delta,R}(|x-y|)K^\alpha(x,y)$ are bounded with compact support for fixed x or y. Then the truncated operators

$$T^\alpha_{\sigma,\delta,R}f(x) \equiv \int_{\mathbb{R}^n} K^\alpha_{\delta,R}(x,y) f(y)\, d\sigma(y), \qquad x \in \mathbb{R}^n,$$

are pointwise well-defined, and we will refer to the pair $\left(K^\alpha, \left\{\eta^\alpha_{\delta,R}\right\}_{0<\delta<R<\infty}\right)$ as an α-fractional singular integral operator, which we typically denote by T^α, suppressing the dependence on the truncations.

Definition 3. *We say that an* α*-fractional singular integral operator* $T^\alpha = \left(K^\alpha, \left\{\eta^\alpha_{\delta,R}\right\}_{0<\delta<R<\infty}\right)$ *satisfies the norm inequality (6) provided*

$$\left\|T^\alpha_{\sigma,\delta,R}f\right\|_{L^2(\omega)} \leq \mathfrak{N}_{T^\alpha_\sigma}\|f\|_{L^2(\sigma)}, \qquad f \in L^2(\sigma), 0<\delta<R<\infty.$$

It turns out that, in the presence of Muckenhoupt conditions, the norm inequality (6) is essentially independent of the choice of truncations used, and we now explain this in some detail. A *smooth truncation* of T^α has kernel $\eta_{\delta,R}(|x-y|)K^\alpha(x,y)$ for a smooth function $\eta_{\delta,R}$ compactly supported in (δ,R), $0 < \delta < R < \infty$, and satisfying standard CZ estimates. A typical example of an α-fractional transform is the α-fractional *Riesz* vector of operators

$$\mathbf{R}^{\alpha,n} = \left\{R^{\alpha,n}_\ell : 1 \leq \ell \leq n\right\}, \qquad 0 \leq \alpha < n.$$

The Riesz transforms $R^{n,\alpha}_\ell$ are convolution fractional singular integrals $R^{n,\alpha}_\ell f \equiv K^{n,\alpha}_\ell * f$ with odd kernel defined by

$$K^{\alpha,n}_\ell(w) \equiv \frac{w^\ell}{|w|^{n+1-\alpha}} \equiv \frac{\Omega_\ell(w)}{|w|^{n-\alpha}}, \qquad w = \left(w^1, ..., w^n\right).$$

However, in dealing with energy considerations, and in particular in the Monotonicity Lemma below where first order Taylor approximations are made on the truncated kernels, it is necessary to use the *tangent line truncation* of the Riesz transform $R^{\alpha,n}_\ell$ whose kernel is defined to be $\Omega_\ell(w)\psi^\alpha_{\delta,R}(|w|)$ where $\psi^\alpha_{\delta,R}$ is continuously differentiable on an interval $(0,S)$ with $0<\delta<R<S$, and where $\psi^\alpha_{\delta,R}(r) = r^{\alpha-n}$ if $\delta \leq r \leq R$, and has constant derivative on both $(0,\delta)$ and (R,S) where $\psi^\alpha_{\delta,R}(S) = 0$. Here S is uniquely determined by R and α. Finally we set $\psi^\alpha_{\delta,R}(0) = 0$ as well, so that the kernel vanishes on the diagonal and common point masses do not 'see' each other. Note also that the tangent line extension of a $C^{1,\delta}$ function on the line is again $C^{1,\delta}$ with no increase in the $C^{1,\delta}$ norm.

It was shown in the one dimensional case with no common point masses in [LaSaShUr3], that boundedness of the Hilbert transform H with one set of appropriate truncations together with the A^α_2 condition without holes, is equivalent to boundedness of H with any other set of appropriate truncations. We need to extend this to $\mathbf{R}^{\alpha,n}$ and more general operators in higher dimensions and to permit common point masses, so that we are free to use the tangent line truncations throughout the proof of our theorem. For this purpose, we note that the difference between the tangent line truncated kernel $\Omega_\ell(w)\,\psi^\alpha_{\delta,R}(|w|)$ and the corresponding cutoff kernel $\Omega_\ell(w)\,\mathbf{1}_{[\delta,R]}(|w|)\,|w|^{\alpha-n}$ satisfies (since both kernels vanish at the origin)

$$\begin{aligned} & \left|\Omega_\ell(w)\,\psi^\alpha_{\delta,R}(|w|)-\Omega_\ell(w)\,\mathbf{1}_{[\delta,R]}(|w|)\,|w|^{\alpha-n}\right| \\ \lesssim & \sum_{k=0}^{\infty}2^{-k(n-\alpha)}\left\{\left(2^{-k}\delta\right)^{\alpha-n}\mathbf{1}_{[2^{-k-1}\delta,2^{-k}\delta]}(|w|)\right\}+\sum_{k=1}^{\infty}2^{-k(n-\alpha)}\left\{\left(2^{k}R\right)^{\alpha-n}\mathbf{1}_{[2^{k-1}R,2^{k}R]}(|w|)\right\} \\ \equiv & \sum_{k=0}^{\infty}2^{-k(n-\alpha)}K_{2^{-k}\delta}(w)+\sum_{k=1}^{\infty}2^{-k(n-\alpha)}K_{2^{k}R}(w), \end{aligned}$$

where the kernels $K_\rho(w)\equiv\frac{1}{\rho^{n-\alpha}}\mathbf{1}_{[\rho,2\rho]}(|w|)$ are easily seen to satisfy, uniformly in ρ, the norm inequality (12) with constant controlled by the offset A_2^α condition (7) below. The equivalence of the norm inequality for these two families of truncations now follows from the summability of the series $\sum_{k=0}^{\infty}2^{-k(n-\alpha)}$ for $0\leq\alpha<n$. The case of more general families of truncations and operators is similar.

2.2. **Quasicube testing conditions.** The following 'dual' quasicube testing conditions are necessary for the boundedness of T^α from $L^2(\sigma)$ to $L^2(\omega)$,

$$\begin{aligned} \mathfrak{T}^2_{T^\alpha} &\equiv \sup_{Q\in\Omega\mathcal{P}^n}\frac{1}{|Q|_\sigma}\int_Q\left|T^\alpha(\mathbf{1}_Q\sigma)\right|^2\omega<\infty, \\ \left(\mathfrak{T}^*_{T^\alpha}\right)^2 &\equiv \sup_{Q\in\Omega\mathcal{P}^n}\frac{1}{|Q|_\omega}\int_Q\left|(T^\alpha)^*(\mathbf{1}_Q\omega)\right|^2\sigma<\infty, \end{aligned}$$

and where we interpret the right sides as holding uniformly over all tangent line truncations of T^α.

Remark 2. *We alert the reader that the symbols Q,I,J,K will all be used to denote either cubes or quasicubes, and the context will make clear which is the case. Throughout most of the proof of the main theorem only quasicubes are considered.*

2.3. **Quasiweak boundedness property.** The quasiweak boundedness property for T^α with constant C is given by

$$\begin{aligned} &\left|\int_Q T^\alpha(1_{Q'}\sigma)\,d\omega\right|\leq\mathcal{WBP}_{T^\alpha}\sqrt{|Q|_\omega\,|Q'|_\sigma}, \\ &\text{for all quasicubes } Q,Q' \text{ with } \frac{1}{C}\leq\frac{|Q|^{\frac{1}{n}}}{|Q'|^{\frac{1}{n}}}\leq C, \\ &\text{and either } Q\subset 3Q'\setminus Q' \text{ or } Q'\subset 3Q\setminus Q, \end{aligned}$$

and where we interpret the left side above as holding uniformly over all tangent line trucations of T^α. Note that the quasiweak boundedness property is implied by either the *tripled* quasicube testing condition,

$$\left\|\mathbf{1}_{3Q}\mathbf{T}^\alpha(\mathbf{1}_Q\sigma)\right\|_{L^2(\omega)}\leq\mathfrak{T}^{\text{triple}}_{\mathbf{T}^\alpha}\left\|\mathbf{1}_Q\right\|_{L^2(\sigma)},\qquad\text{for all quasicubes } Q \text{ in } \mathbb{R}^n,$$

or its dual defined with σ and ω interchanged and the dual operator $\mathbf{T}^{\alpha,*}$ in place of $\mathbf{T}^\alpha$. In turn, the tripled quasicube testing condition can be obtained from the quasicube testing condition for the truncated weight pairs $(\omega,\mathbf{1}_Q\sigma)$.

2.4. **Poisson integrals and $\mathcal{A}_2^\alpha$.** Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$, and suppose Q is an Ω-quasicube in $\mathbb{R}^n$. Recall that $|Q|^{\frac{1}{n}}\approx\ell(Q)$ for a quasicube Q. The two α-fractional Poisson integrals of μ on a quasicube Q are given by:

$$\mathrm{P}^{\alpha}(Q,\mu) \equiv \int_{\mathbb{R}^n} \frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}}+|x-x_Q|\right)^{n+1-\alpha}} d\mu(x),$$

$$\mathcal{P}^{\alpha}(Q,\mu) \equiv \int_{\mathbb{R}^n} \left(\frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}}+|x-x_Q|\right)^{2}}\right)^{n-\alpha} d\mu(x),$$

where we emphasize that $|x-x_Q|$ denotes Euclidean distance between x and x_Q and $|Q|$ denotes the Lebesgue measure of the quasicube Q. We refer to P^{α} as the *standard* Poisson integral and to $\mathcal{P}^{\alpha}$ as the *reproducing* Poisson integral.

We say that the pair K,K' in $\mathcal{P}^n$ are *neighbours* if K and K' live in a common dyadic grid and both $K\subset 3K'\setminus K'$ and $K'\subset 3K\setminus K$, and we denote by $\mathcal{N}^n$ the set of pairs (K,K') in $\mathcal{P}^n\times\mathcal{P}^n$ that are neighbours. Let

$$\Omega\mathcal{N}^n=\{(\Omega K,\Omega K'):(K,K')\in\mathcal{N}^n\}$$

be the corresponding collection of quasineighbour pairs of quasicubes. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$, possibly having common point masses, and suppose $0\leq\alpha<n$. Then we define the classical *offset* A_2^{α} *constants* by

$$A_2^{\alpha}(\sigma,\omega)\equiv\sup_{(Q,Q')\in\Omega\mathcal{N}^n}\frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q'|_{\omega}}{|Q'|^{1-\frac{\alpha}{n}}}. \tag{7}$$

Since the cubes in $\mathcal{P}^n$ are products of half open, half closed intervals $[a,b)$, the neighbouring quasicubes $(Q,Q')\in\Omega\mathcal{N}^n$ are disjoint, and the common point masses of σ and ω do not simultaneously appear in each factor.

We now define the *one-tailed* $\mathcal{A}_2^{\alpha}$ constant using $\mathcal{P}^{\alpha}$. The energy constants $\mathcal{E}_{\alpha}^{\mathrm{strong}}$ introduced in the next subsection will use the standard Poisson integral P^{α}.

Definition 4. *The one-tailed constants $\mathcal{A}_2^{\alpha}$ and $\mathcal{A}_2^{\alpha,*}$ for the weight pair (σ,ω) are given by*

$$\mathcal{A}_2^{\alpha} \equiv \sup_{Q\in\Omega\mathcal{P}^n}\mathcal{P}^{\alpha}(Q,\mathbf{1}_{Q^c}\sigma)\frac{|Q|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}}<\infty,$$

$$\mathcal{A}_2^{\alpha,*} \equiv \sup_{Q\in\Omega\mathcal{P}^n}\mathcal{P}^{\alpha}(Q,\mathbf{1}_{Q^c}\omega)\frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}}<\infty.$$

Note that these definitions are the analogues of the corresponding conditions with 'holes' introduced by Hytönen [Hyt] in dimension $n=1$ - the supports of the measures $\mathbf{1}_{Q^c}\sigma$ and $\mathbf{1}_{Q}\omega$ in the definition of $\mathcal{A}_2^{\alpha}$ are disjoint, and so the common point masses of σ and ω do not appear simultaneously in each factor. Note also that, unlike in [SaShUr5], where common point masses were not permitted, we can no longer assert the equivalence of $\mathcal{A}_2^{\alpha}$ with holes taken over *quasicubes* with $\mathcal{A}_2^{\alpha}$ with holes taken over *cubes*.

2.4.1. *Punctured A_2^{α} conditions.* As mentioned earlier, the *classical* A_2^{α} characteristic $\sup_{Q\in\Omega\mathcal{Q}^n}\frac{|Q|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}}$ fails to be finite when the measures σ and ω have a common point mass - simply let Q in the sup above shrink to a common mass point. But there is a substitute that is quite similar in character that is motivated by the fact that for large quasicubes Q, the sup above is problematic only if just *one* of the measures is *mostly* a point mass when restricted to Q. The one-dimensional version of the condition we are about to describe arose in Conjecture 1.12 of Lacey [Lac2], and it was pointed out in [Hyt2] that its necessity on the line follows from the proof of Proposition 2.1 in [LaSaUr2]. We now extend this condition to higher dimensions, where its necessity is more subtle.

Given an at most countable set $\mathfrak{P}=\{p_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$, a quasicube $Q\in\Omega\mathcal{P}^n$, and a positive locally finite Borel measure μ, define

$$\mu(Q,\mathfrak{P})\equiv|Q|_{\mu}-\sup\{\mu(p_k):p_k\in Q\cap\mathfrak{P}\},$$

where the supremum is actually achieved since $\sum_{p_k\in Q\cap\mathfrak{P}}\mu(p_k)<\infty$ as μ is locally finite. The quantity $\mu(Q,\mathfrak{P})$ is simply the $\widetilde{\mu}$ measure of Q where $\widetilde{\mu}$ is the measure μ with its largest point mass from $\mathfrak{P}$ in Q removed. Given a locally finite measure pair (σ,ω), let $\mathfrak{P}_{(\sigma,\omega)}=\{p_k\}_{k=1}^{\infty}$ be the at most countable set of common point masses of σ and ω. Then the weighted norm inequality (6) typically implies finiteness of the following *punctured* Muckenhoupt conditions:

$$\begin{aligned} A_2^{\alpha,\text{punct}}(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}, \\ A_2^{\alpha,*,\text{punct}}(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}}\frac{\sigma\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}. \end{aligned}$$

Lemma 1. *Let $\mathbf{T}^\alpha$ be an α-fractional singular integral operator as above, and suppose that there is a positive constant C_0 such that*

$$A_2^\alpha(\sigma,\omega) \le C_0\mathfrak{N}_{\mathbf{T}^\alpha}^2(\sigma,\omega),$$

*for all pairs (σ,ω) of positive locally finite measures **having no common point masses**. Now let σ and ω be positive locally finite Borel measures on $\mathbb{R}^n$ and let $\mathfrak{P}_{(\sigma,\omega)}$ be the possibly nonempty set of common point masses. Then we have*

$$A_2^{\alpha,\text{punct}}(\sigma,\omega)+A_2^{\alpha,*,\text{punct}}(\sigma,\omega) \le 4C_0\mathfrak{N}_{\mathbf{T}^\alpha}^2(\sigma,\omega).$$

Proof. Fix a quasicube $Q\in\Omega\mathcal{P}^n$. Suppose first that $\mathfrak{P}_{(\sigma,\omega)}\cap Q=\{p_k\}_{k=1}^{2N}$ is finite with an even number of points. Choose $k_1\in\mathbb{N}_{2N}=\{1,2,...,2N\}$ so that

$$\sigma(p_{k_1})=\max_{k\in\mathbb{N}_{2N}}\sigma(p_k).$$

Then choose $k_2\in\mathbb{N}_{2N}\setminus\{k_1\}$ such that

$$\omega(p_{k_2})=\max_{k\in\mathbb{N}_{2N}\setminus\{k_1\}}\omega(p_k).$$

Repeat this procedure so that

$$\begin{aligned} \sigma\left(p_{k_{2m+1}}\right) &= \max_{k\in\mathbb{N}_{2N}\setminus\{k_1,...k_{2m}\}}\sigma(p_k), \qquad k_{2m+1}\in\mathbb{N}_{2N}\setminus\{k_1,...k_{2m}\}, \\ \omega\left(p_{k_{2m+2}}\right) &= \max_{k\in\mathbb{N}_{2N}\setminus\{k_1,...k_{2m+1}\}}\omega(p_k), \qquad k_{2m+2}\in\mathbb{N}_{2N}\setminus\{k_1,...k_{2m+1}\}, \end{aligned}$$

for each $m\le N-1$. It is now clear that both

$$\sum_{i=0}^{N-1}\sigma\left(p_{k_{2i+1}}\right)\ge\frac{1}{2}\sigma\left(Q\cap\mathfrak{P}_{(\sigma,\omega)}\right) \text{ and } \sum_{i=0}^{N-1}\omega\left(p_{k_{2i+2}}\right)\ge\frac{1}{2}\left[\omega\left(Q\cap\mathfrak{P}_{(\sigma,\omega)}\right)-\omega(p_1)\right].$$

In the case of an odd number $2N-1$ of common point masses, the second inequality will have $N-1$ replaced with $N-2$.

Now, returning to the case of $2N$ common point masses , define new measures $\widetilde{\sigma}$ and $\widetilde{\omega}$ by

$$\widetilde{\sigma}\equiv\mathbf{1}_Q\sigma-\sum_{i=0}^{N-1}\sigma\left(p_{k_{2i+2}}\right)\delta_{p_{k_{2i+2}}} \text{ and } \widetilde{\omega}=\mathbf{1}_Q\omega-\sum_{i=0}^{N-1}\omega\left(p_{k_{2i+1}}\right)\delta_{p_{k_{2i+1}}}$$

so that

$$|Q|_{\widetilde{\sigma}}\ge\frac{1}{2}|Q|_\sigma \text{ and } |Q|_{\widetilde{\omega}}\ge\frac{1}{2}\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)$$

Now $\widetilde{\sigma}$ and $\widetilde{\omega}$ have no common point masses and $\mathfrak{N}_{\mathbf{T}^\alpha}(\sigma,\omega)$ is monotone in each measure separately, so we have

$$\frac{\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\le 4A_2^\alpha(\widetilde{\sigma},\widetilde{\omega})\le 4C_0\mathfrak{N}_{\mathbf{T}^\alpha}^2(\widetilde{\sigma},\widetilde{\omega})\le 4C_0\mathfrak{N}_{\mathbf{T}^\alpha}^2(\sigma,\omega).$$

Thus $A_2^{\alpha,\text{punct}}(\sigma,\omega)\le 4C_0\mathfrak{N}_{\mathbf{T}^\alpha}^2(\sigma,\omega)$ if the number of common point masses in Q is finite. A limiting argument proves the general case. The dual inequality $A_2^{\alpha,*,\text{punct}}(\sigma,\omega)\le 4C_0\mathfrak{N}_{\mathbf{T}^\alpha}^2(\sigma,\omega)$ now follows upon interchanging the measures σ and ω. □

Now we turn to the definition of a quasiHaar basis of $L^2(\mu)$.

2.5. **A weighted quasiHaar basis.** We will use a construction of a quasiHaar basis in $\mathbb{R}^n$ that is adapted to a measure μ (c.f. [NTV2] for the nonquasi case and [KaLiPeWa] for the geometrically doubling quasi-metric space case). Given a dyadic quasicube $Q \in \Omega\mathcal{D}$, where $\mathcal{D}$ is a dyadic grid of cubes from $\mathcal{P}^n$, let $\triangle_Q^\mu$ denote orthogonal projection onto the finite dimensional subspace $L_Q^2(\mu)$ of $L^2(\mu)$ that consists of linear combinations of the indicators of the children $\mathfrak{C}(Q)$ of Q that have μ-mean zero over Q:

$$L_Q^2(\mu) \equiv \left\{ f = \sum_{Q' \in \mathfrak{C}(Q)} a_{Q'} \mathbf{1}_{Q'} : a_{Q'} \in \mathbb{R}, \int_Q f d\mu = 0 \right\}.$$

Then we have the important telescoping property for dyadic quasicubes $Q_1 \subset Q_2$ (where $[Q_1, Q_2) \equiv \{Q \text{ dyadic} : Q_1 \subset Q \subsetneqq Q_2\}$):

$$\mathbf{1}_{Q_0}(x) \left(\sum_{Q \in [Q_1, Q_2)} \triangle_Q^\mu f(x) \right) = \mathbf{1}_{Q_0}(x) \left(\mathbb{E}_{Q_0}^\mu f - \mathbb{E}_{Q_2}^\mu f \right), \qquad Q_0 \in \mathfrak{C}(Q_1),\ f \in L^2(\mu). \tag{8}$$

We will at times find it convenient to use a fixed orthonormal basis $\left\{h_Q^{\mu,a}\right\}_{a \in \Gamma_n}$ of $L_Q^2(\mu)$ where $\Gamma_n \equiv \{0,1\}^n \setminus \{\mathbf{1}\}$ is a convenient index set with $\mathbf{1} = (1, 1, ..., 1)$. Then $\left\{h_Q^{\mu,a}\right\}_{a \in \Gamma_n \text{ and } Q \in \Omega\mathcal{D}}$ is an orthonormal basis for $L^2(\mu)$, with the understanding that we add the constant function $\mathbf{1}$ if μ is a finite measure. In particular we have

$$\|f\|_{L^2(\mu)}^2 = \sum_{Q \in \Omega\mathcal{D}} \left\| \triangle_Q^\mu f \right\|_{L^2(\mu)}^2 = \sum_{Q \in \Omega\mathcal{D}} \left| \widehat{f}(Q) \right|^2, \qquad \left| \widehat{f}(Q) \right|^2 \equiv \sum_{a \in \Gamma_n} \left| \left\langle f, h_Q^{\mu,a} \right\rangle_\mu \right|^2,$$

where the measure is suppressed in the notation $\widehat{f}$. Indeed, this follows from (8) and Lebesgue's differentiation theorem for quasicubes. We also record the following useful estimate. If I' is any of the 2^n $\Omega\mathcal{D}$-children of I, and $a \in \Gamma_n$, then

$$|\mathbb{E}_{I'}^\mu h_I^{\mu,a}| \leq \sqrt{\mathbb{E}_{I'}^\mu (h_I^{\mu,a})^2} \leq \frac{1}{\sqrt{|I'|_\mu}}. \tag{9}$$

2.6. **The strong quasienergy conditions.** Given a dyadic quasicube $K \in \Omega\mathcal{D}$ and a positive measure μ we define the quasiHaar projection $\mathsf{P}_K^\mu \equiv \sum_{J \in \Omega\mathcal{D}:\ J \subset K} \triangle_J^\mu$ on K by

$$\mathsf{P}_K^\mu f = \sum_{J \in \Omega\mathcal{D}:\ J \subset K} \sum_{a \in \Gamma_n} \langle f, h_J^{\mu,a} \rangle_\mu h_J^{\mu,a} \text{ and } \|\mathsf{P}_K^\mu f\|_{L^2(\mu)}^2 = \sum_{J \in \Omega\mathcal{D}:\ J \subset K} \sum_{a \in \Gamma_n} \left| \langle f, h_J^{\mu,a} \rangle_\mu \right|^2,$$

and where a quasiHaar basis $\{h_J^{\mu,a}\}_{a \in \Gamma_n \text{ and } J \in \Omega\mathcal{D}\Omega}$ adapted to the measure μ was defined in the subsubsection on a weighted quasiHaar basis above.

Now we define various notions for quasicubes which are inherited from the same notions for cubes. The main objective here is to use the familiar notation that one uses for cubes, but now extended to Ω-quasicubes. We have already introduced the notions of quasigrids $\Omega\mathcal{D}$, and center, sidelength and dyadic associated to quasicubes $Q \in \Omega\mathcal{D}$, as well as quasiHaar functions, and we will continue to extend to quasicubes the additional familiar notions related to cubes as we come across them. We begin with the notion of *deeply embedded*. Fix a quasigrid $\Omega\mathcal{D}$. We say that a dyadic quasicube J is $(\mathbf{r}, \varepsilon)$-*deeply embedded* in a (not necessarily dyadic) quasicube K, which we write as $J \Subset_{\mathbf{r},\varepsilon} K$, when $J \subset K$ and both

$$\begin{aligned} \ell(J) &\leq 2^{-\mathbf{r}} \ell(K), \\ \operatorname{qdist}(J, \partial K) &\geq \frac{1}{2} \ell(J)^\varepsilon \ell(K)^{1-\varepsilon}, \end{aligned} \tag{10}$$

where we define the quasidistance $\operatorname{qdist}(E, F)$ between two sets E and F to be the Euclidean distance $\operatorname{dist}(\Omega^{-1}E, \Omega^{-1}F)$ between the preimages $\Omega^{-1}E$ and $\Omega^{-1}F$ of E and F under the map Ω, and where we recall that $\ell(J) \approx |J|^{\frac{1}{n}}$. For the most part we will consider $J \Subset_{\mathbf{r},\varepsilon} K$ when J and K belong to a common quasigrid $\Omega\mathcal{D}$, but an exception is made when defining the strong energy constants below.

Recall that in dimension $n=1$, and for $\alpha=0$, the energy condition constant was defined by

$$\mathcal{E}^2 \equiv \sup_{I=\dot{\cup} I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \left(\frac{\mathrm{P}^\alpha\left(I_r, \mathbf{1}_I \sigma\right)}{|I_r|} \right)^2 \left\| \mathsf{P}_{I_r}^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^2 ,$$

where I and I_r are intervals in the real line, and $\dot{\cup}$ denotes a pairwise disjoint union. The extension to higher dimensions we use here is that of 'strong quasienergy condition' below. Later on, in the proof of the theorem, we will break down this strong quasienergy condition into various smaller quasienergy conditions, which are then used in different ways in the proof.

We define a quasicube K (not necessarily in $\Omega\mathcal{D}$) to be an *alternate* $\Omega\mathcal{D}$-quasicube if it is a union of 2^n $\Omega\mathcal{D}$-quasicubes K' with side length $\ell\left(K'\right) = \frac{1}{2}\ell\left(K\right)$ (such quasicubes were called shifted in [SaShUr5], but that terminology conflicts with the more familiar notion of shifted quasigrid). Thus for any $\Omega\mathcal{D}$-quasicube L there are exactly 2^n alternate $\Omega\mathcal{D}$-quasicubes of twice the side length that contain L, and one of them is of course the $\Omega\mathcal{D}$-parent of L. We denote the collection of alternate $\Omega\mathcal{D}$-quasicubes by $\mathcal{A}\Omega\mathcal{D}$.

The extension of the energy conditions to higher dimensions in [SaShUr5] used the collection

$$\mathcal{M}_{\mathbf{r},\varepsilon-\mathrm{deep}}\left(K\right) \equiv \left\{ \text{maximal } J \Subset_{\mathbf{r},\varepsilon} K \right\}$$

of *maximal* $(\mathbf{r},\varepsilon)$-deeply embedded dyadic subquasicubes of a quasicube K (a subquasicube J of K is a *dyadic* subquasicube of K if $J \in \Omega\mathcal{D}$ when $\Omega\mathcal{D}$ is a dyadic quasigrid containing K). This collection of dyadic subquasicubes of K is of course a pairwise disjoint decomposition of K.We also defined there a refinement and extension of the collection $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(K\right)$ for certain K and each $\ell \geq 1$. For an alternate quasicube $K \in \mathcal{A}\Omega\mathcal{D}$, define $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}\left(K\right)$ to consist of the *maximal* $\mathbf{r}$-deeply embedded $\Omega\mathcal{D}$-dyadic subquasicubes J of K. (In the special case that K itself belongs to $\Omega\mathcal{D}$, then $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}\left(K\right) = \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(K\right)$.) Then in [SaShUr5] for $\ell \geq 1$ we defined the refinement (where $\pi^\ell K'$ denotes the ℓ^{th} ancestor of K' in the grid):

$$\begin{aligned}\mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}\left(K\right) \equiv \Big\{ & J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}\left(\pi^\ell K'\right) \text{ for some } K' \in \mathfrak{C}_{\Omega\mathcal{D}}\left(K\right) : \\ & J \subset L \text{ for some } L \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(K\right) \Big\} ,\end{aligned}$$

where $\mathfrak{C}_{\Omega\mathcal{D}}\left(K\right)$ is the obvious extension to alternate quasicubes of the set of $\Omega\mathcal{D}$-dyadic children. Thus $\mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}\left(K\right)$ is the union, over all quasichildren K' of K, of those quasicubes in $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(\pi^\ell K'\right)$ that happen to be contained in some $L \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}\left(K\right)$. We then define the *strong* quasienergy condition as follows.

Definition 5. *Let $0 \leq \alpha < n$ and fix parameters $(\mathbf{r},\varepsilon)$. Suppose σ and ω are positive Borel measures on $\mathbb{R}^n$ possibly with common point masses. Then the* strong *quasienergy constant $\mathcal{E}_\alpha^{\mathrm{strong}}$ is defined by*[7]

$$\begin{aligned}\left(\mathcal{E}_\alpha^{\mathrm{strong}}\right)^2 \equiv & \sup_{\Omega\mathcal{D}} \sup_{\substack{I=\dot{\cup} I_r \\ I, I_r \in \Omega\mathcal{D}}} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\mathbf{r},\varepsilon-\mathrm{deep}}(I_r)} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_I \sigma\right)}{|J|^{\frac{1}{n}}} \right)^2 \left\| \mathsf{P}_J^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^2 \\ & + \sup_{\Omega\mathcal{D}} \sup_{I \in \mathcal{A}\Omega\mathcal{D}} \sup_{\ell \geq 0} \frac{1}{|I|_\sigma} \sum_{J \in \mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(I)} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_I \sigma\right)}{|J|^{\frac{1}{n}}} \right)^2 \left\| \mathsf{P}_J^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^2 .\end{aligned}$$

Similarly we have a dual version of $\mathcal{E}_\alpha^{\mathrm{strong}}$ denoted $\mathcal{E}_\alpha^{\mathrm{strong},*}$, and both depend on $\mathbf{r}$ and ε as well as on n and α. An important point in this definition is that the quasicube I in the second line is permitted to lie *outside* the quasigrid $\Omega\mathcal{D}$, but only as an alternate dyadic quasicube $I \in \mathcal{A}\Omega\mathcal{D}$. In the setting of quasicubes we continue to use the linear function $\mathbf{x}$ in the final factor $\left\| \mathsf{P}_J^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^2$ of each line, and not the pushforward of $\mathbf{x}$ by Ω. The reason of course is that this condition is used to capture the first order information in the Taylor expansion of a singular kernel. There is a logically weaker form of the quasienergy conditions that we discuss after stating our main theorem, but these refined quasienergy conditions are more complicated to state, and have as yet found no application - the strong energy conditions above suffice for use when one measure is compactly supported on a $C^{1,\delta}$ curve as in [SaShUr8].

[7]The first line in the display in Definition 5 in [SaShUr6] is missing notation that is corrected here.

2.7. **Statement of the Theorems.** We can now state our main quasicube two weight theorem for general measures allowing common point masses, as well as our application to energy dispersed measures. Recall that $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map, and that $\Omega\mathcal{P}^n$ denotes the collection of all quasicubes in $\mathbb{R}^n$ whose preimages under Ω are usual cubes with sides parallel to the coordinate axes. Denote by $\Omega\mathcal{D} \subset \Omega\mathcal{P}^n$ a dyadic quasigrid in $\mathbb{R}^n$. For the purpose of obtaining necessity of $\mathcal{A}_2^\alpha$ for $\frac{n}{2} \leq \alpha < n$, we adapt the notion of strong ellipticity from [SaShUr7].

Definition 6. *Fix a globally biLipschitz map Ω. Let $\mathbf{T}^\alpha = \left\{T_j^\alpha\right\}_{j=1}^J$ be a vector of singular integral operators with standard kernels $\left\{K_j^\alpha\right\}_{j=1}^J$. We say that $\mathbf{T}^\alpha$ is* strongly elliptic *with respect to Ω if for each $m \in \{1,-1\}^n$, there is a sequence of coefficients $\left\{\lambda_j^m\right\}_{j=1}^J$ such that*

$$\left|\sum_{j=1}^J \lambda_j^m K_j^\alpha\left(x, x+t\mathbf{u}\right)\right| \geq c t^{\alpha-n}, \qquad t \in \mathbb{R}, \tag{11}$$

holds for all *unit vectors $\mathbf{u}$ in the quasi-n-ant ΩV_m (i.e. an n-dimensional quasi-quadrant) where*

$$V_m = \left\{x \in \mathbb{R}^n : m_i x_i > 0 \text{ for } 1 \leq i \leq n\right\}, \qquad m \in \{1,-1\}^n .$$

Theorem 1. *Suppose that T^α is a standard α-fractional singular integral operator on $\mathbb{R}^n$, and that ω and σ are positive Borel measures on $\mathbb{R}^n$ (possibly having common point masses). Set $T_\sigma^\alpha f = T^\alpha(f\sigma)$ for any smooth truncation of T_σ^α. Let $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ be a globally biLipschitz map.*

(1) *Suppose $0 \leq \alpha < n$. Then the operator T_σ^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.*

$$\left\|T_\sigma^\alpha f\right\|_{L^2(\omega)} \leq \mathfrak{N}_{T_\sigma^\alpha} \left\|f\right\|_{L^2(\sigma)}, \tag{12}$$

uniformly in smooth truncations of T^α, and moreover

$$\mathfrak{N}_{T_\sigma^\alpha} \leq C_\alpha \left(\sqrt{\mathcal{A}_2^\alpha + \mathcal{A}_2^{\alpha,*} + A_2^{\alpha,\text{punct}} + A_2^{\alpha,*,\text{punct}}} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} + \mathcal{WBP}_{T^\alpha}\right),$$

provided that the two dual $\mathcal{A}_2^\alpha$ conditions and the two dual punctured Muckenhoupt conditions all hold, and the two dual quasitesting conditions for T^α hold, the quasiweak boundedness property for T^α holds for a sufficiently large constant C depending on the goodness parameter $\mathbf{r}$, and provided that the two dual strong quasienergy conditions hold uniformly over all dyadic quasigrids $\Omega\mathcal{D} \subset \Omega\mathcal{P}^n$, i.e. $\mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} < \infty$, and where the goodness parameters $\mathbf{r}$ and ε in the definition of the collections $\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(K)$ and $\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep},\Omega\mathcal{D}}^\ell(K)$ appearing in the strong energy conditions, are fixed sufficiently large and small respectively depending only on n and α.

(2) *Conversely, suppose $0 \leq \alpha < n$ and that $\mathbf{T}^\alpha = \left\{T_j^\alpha\right\}_{j=1}^J$ is a vector of Calderón-Zygmund operators with standard kernels $\left\{K_j^\alpha\right\}_{j=1}^J$. In the range $0 \leq \alpha < \frac{n}{2}$, we assume the* ellipticity *condition from (*[SaShUr7]*): there is $c > 0$ such that for* each *unit vector $\mathbf{u}$ there is j satisfying*

$$\left|K_j^\alpha\left(x, x+t\mathbf{u}\right)\right| \geq c t^{\alpha-n}, \qquad t \in \mathbb{R}. \tag{13}$$

For the range $\frac{n}{2} \leq \alpha < n$, we assume the strong ellipticity condition in Definition 6 above. Furthermore, assume that each operator T_j^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$,

$$\left\|\left(T_j^\alpha\right)_\sigma f\right\|_{L^2(\omega)} \leq \mathfrak{N}_{T_j^\alpha} \left\|f\right\|_{L^2(\sigma)}.$$

Then the fractional $\mathcal{A}_2^\alpha$ conditions (with 'holes') hold as well as the punctured Muckenhoupt conditions, and moreover,

$$\sqrt{\mathcal{A}_2^\alpha + \mathcal{A}_2^{\alpha,*} + A_2^{\alpha,\text{punct}} + A_2^{\alpha,*,\text{punct}}} \leq C\mathfrak{N}_{\mathbf{T}^\alpha}.$$

Problem 1. *Given any strongly elliptic vector $\mathbf{T}^\alpha$ of classical α-fractional Calderón-Zygmund operators, it is an open question whether or not the usual (quasi or not) energy conditions are necessary for boundedness of $\mathbf{T}^\alpha$. See* [SaShUr4] *for a failure of* energy reversal *in higher dimensions - such an energy reversal was used in dimension $n = 1$ to prove the necessity of the energy condition for the Hilbert transform, and also in* [SaShUr3] *and* [LaSaShUrWi] *for the Riesz transforms and Cauchy transforms respectively when one of the measures is supported on a line, and in* [SaShUr8] *when one of the measures is supported on a $C^{1,\delta}$ curve.*

Remark 3. *If Definition 6 holds for some $\mathbf{T}^{\alpha}$ and Ω, then Ω must be fairly tame, in particular the logarithmic spirals in Example 1 are ruled out! On the other hand, the vector of Riesz transforms $\mathbf{R}^{\alpha,n}$ is easily seen to be strongly elliptic with respect to Ω if Ω satisfies the following* sector separation property. *Given a hyperplane H and a perpendicular line L intersecting at point P, there exist spherical cones S_H and S_L intersecting only at the point $P' = \Omega(P)$, such that $H' \equiv \Omega H \subset S_H$ and $L' \equiv \Omega L \subset S_L$ and*

$$\operatorname{dist}(x, \partial S_H) \approx |x|, \ x \in H \text{ and } \operatorname{dist}(x, \partial S_L) \approx |x|, \ x \in L .$$

Examples of globally biLipshcitz maps Ω that satisfy the sector separation property include finite compositions of maps of the form

$$\Omega(x_1, x') = (x_1, x' + \psi(x_1)), \qquad (x_1, x') \in \mathbb{R}^n,$$

where $\psi : \mathbb{R} \to \mathbb{R}^{n-1}$ is a Lipschitz map with sufficiently small Lipschitz constant.

In order to state our application to energy dispersed measures, we introduce some notation and a definition. Fix a globally biLipschitz map $\Omega : \mathbb{R}^n \to \mathbb{R}^n$. For $0 \leq k \leq n-1$, denote by $\mathcal{L}_k^n$ the collection of all k-dimensional planes in $\mathbb{R}^n$. If in addition J is an Ω-quasicube in $\mathbb{R}^n$, denote by $M_k^n(J, \mu)$ the 'moments'

$$\mathsf{M}_k^n(J, \mu)^2 \equiv \inf_{L \in \mathcal{L}_k^n} \int_J \operatorname{dist}(x, L)^2 \, d\mu(x),$$

and note that $\mathsf{M}_0^n(J, \mu)$ is related to the energy $\mathsf{E}(J, \mu) \equiv \sqrt{\mathbb{E}_J^{\mu} \left| \frac{x - \mathbb{E}_J^{\mu} x}{|J|^{\frac{1}{n}}} \right|^2}$, $\mathbb{E}_J^{\mu} x = \frac{1}{|J|_{\mu}} \int_J x d\mu(x)$:

$$\mathsf{M}_0^n(J, \mu)^2 = \int_J |x - \mathbb{E}_J^{\mu} x|^2 \, d\mu(x) = |J|_{\mu} |J|^{\frac{2}{n}} \mathsf{E}(J, \mu)^2 .$$

Clearly the moments decrease in k and we now give a name to various reversals of this decrease.

Definition 7. *Suppose μ is a locally finite Borel measure on $\mathbb{R}^n$, and let k be an integer satisfying $0 \leq k \leq n-1$. We say that μ is k-*energy dispersed *if there is a positive constant $C = C_{k,n}$ such that for all Ω-quasicubes J,*

$$\mathsf{M}_0^n(J, \mu) \leq C \mathsf{M}_k^n(J, \mu) .$$

If both σ and ω are appropriately energy dispersed relative to the order $0 \leq \alpha < n$, then the $T1$ theorem holds for the α-fractional Riesz vector transform $\mathbf{R}^{\alpha,n}$.

Theorem 2. *Let $0 \leq \alpha < n$ and $0 \leq k \leq n-1$ sastisfy*

$$\begin{cases} n - k < \alpha < n, \ \alpha \neq n-1 & \text{if} \quad 1 \leq k \leq n-2 \\ 0 \leq \alpha < n, \ \alpha \neq 1, n-1 & \text{if} \quad k = n-1 \end{cases} .$$

*Suppose that $\mathbf{R}^{\alpha,n}$ is the α-fractional Riesz vector transform on $\mathbb{R}^n$, and that ω and σ are k-*energy dispersed *locally finite positive Borel measures on $\mathbb{R}^n$ (possibly having common point masses). Set $\mathbf{R}_{\sigma}^{\alpha,n} f = \mathbf{R}^{\alpha,n}(f\sigma)$ for any smooth truncation of $\mathbf{R}^{\alpha,n}$. Let $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ be a globally biLipschitz map. Then the operator $\mathbf{R}_{\sigma}^{\alpha,n}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.*

$$\|\mathbf{R}_{\sigma}^{\alpha} f\|_{L^2(\omega)} \leq \mathfrak{N}_{\mathbf{R}_{\sigma}^{\alpha}} \|f\|_{L^2(\sigma)},$$

uniformly in smooth truncations of $\mathbf{R}^{\alpha,n}$, if and only if the Muckenhoupt conditions hold, the testing conditions hold and the weak boundedness property holds. Moreover, we have the equivalence

$$\mathfrak{N}_{\mathbf{R}_{\sigma}^{\alpha,n}} \approx \sqrt{\mathcal{A}_2^{\alpha} + \mathcal{A}_2^{\alpha,*} + A_2^{\alpha,\mathrm{punct}} + A_2^{\alpha,*,\mathrm{punct}}} + \mathfrak{T}_{\mathbf{R}^{\alpha,n}} + \mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{*} + \mathcal{WBP}_{\mathbf{R}^{\alpha,n}} .$$

The case $k = n-1$ of k-energy dispersed is similar to the notion of uniformly full dimension introduced by Lacey and Wick in [LaWi, versions 2 and 3]. The proof of Theorem 2 shows that we can also take ω and σ to be k_1 and k_2 energy dispersed respectively, provided α satisfies the hypotheses with respect to *both* k_1 and k_2.

3. Proof of Theorem 1

We now give the proof of Theorem 1 in the following sections. Sections 5, 7 and 10 are largely taken verbatim from the corresponding sections of [SaShUr5], but are included here since their omission here would hinder the readability of an already complicated argument.

3.1. **Good quasicubes and energy Muckenhoupt conditions.** First we extend the notion of goodness to quasicubes.

Definition 8. *Let* $\mathbf{r} \in \mathbb{N}$ *and* $0 < \varepsilon < 1$. *Fix a quasigrid* $\Omega\mathcal{D}$. *A dyadic quasicube* J *is* $(\mathbf{r}, \varepsilon)$-good, *or simply* good, *if for* every *dyadic superquasicube* I, *it is the case that* ***either*** J *has side length greater than* $2^{-\mathbf{r}}$ *times that of* I, ***or*** $J \Subset_{\mathbf{r},\varepsilon} I$ *is* $(\mathbf{r}, \varepsilon)$*-deeply embedded in* I.

Note that this definition simply asserts that a dyadic quasicube $J = \Omega J'$ is $(\mathbf{r}, \varepsilon)$-good if and only if the cube J' is $(\mathbf{r}, \varepsilon)$-good in the usual sense. Finally, we say that J is $\mathbf{r}$-*nearby* in K when $J \subset K$ and

$$\ell(J) > 2^{-\mathbf{r}} \ell(K).$$

The parameters $\mathbf{r}, \varepsilon$ will be fixed sufficiently large and small respectively later in the proof, and we denote the set of such good dyadic quasicubes by $\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\text{good}}$, or simply $\Omega\mathcal{D}_{\text{good}}$ when the goodness parameters $(\mathbf{r}, \varepsilon)$ are understood. Note that if $J' \in \Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\text{good}}$ and if $J' \subset K \in \Omega\mathcal{D}$, then **either** J' is $\mathbf{r}$-*nearby* in K **or** $J' \subset J \Subset_{\mathbf{r},\varepsilon} K$.

Throughout the proof, it will be convenient to also consider pairs of quasicubes J, K where J is $(\boldsymbol{\rho}, \varepsilon)$-*deeply embedded* in K, written $J \Subset_{\boldsymbol{\rho},\varepsilon} K$ and meaning (10) holds with the same $\varepsilon > 0$ but with $\boldsymbol{\rho}$ in place of $\mathbf{r}$; as well as pairs of quasicubes J, K where J is $\boldsymbol{\rho}$-*nearby* in K, $\ell(J) > 2^{-\boldsymbol{\rho}} \ell(K)$, for a parameter $\boldsymbol{\rho} \gg \mathbf{r}$ that will be fixed later.

Notation 2. *We will typically use the side length* $\ell(J)$ *of a* Ω*-quasicube when we are describing collections of quasicubes, and when we want* $\ell(J)$ *to be a dyadic or related number; while we will typically use* $|J|^{\frac{1}{n}}$ *in estimates, and when we want to compare powers of volumes of quasicubes. We will continue to use the prefix 'quasi' when discussing quasicubes, quasiHaar, quasienergy and quasidistance in the text, but will not use the prefix 'quasi' when discussing other notions. In particular, since* quasi $\mathcal{A}_2^\alpha$ + quasi $A_2^{\alpha,\text{punct}} \approx \mathcal{A}_2^\alpha + A_2^{\alpha,\text{punct}}$ *(see e.g.* [SaShUr8] *for a proof) we do not use* quasi *as a prefix for the Muckenhoupt conditions, even though* quasi $\mathcal{A}_2^\alpha$ *alone is not comparable to* $\mathcal{A}_2^\alpha$. *Finally, we will not modify any mathematical symbols to reflect quasinotions, except for using* $\Omega\mathcal{D}$ *to denote a quasigrid, and* $\operatorname{qdist}(E, F) \equiv \operatorname{dist}\left(\Omega^{-1}E, \Omega^{-1}F\right)$ *to denote quasidistance between sets* E *and* F, *and using* $|x-y|_{\text{qdist}} \equiv \left|\Omega^{-1}x - \Omega^{-1}y\right|$ *to denote quasidistance between points* x *and* y. *This limited use of quasi in the text serves mainly to remind the reader we are working entirely in the 'quasiworld'.*

3.1.1. *Energy Muckenhoupt conditions.* We now show that the punctured Muckenhoupt conditions $A_2^{\alpha,\text{punct}}$ and $A_2^{\alpha,*,\text{punct}}$ control respectively the 'energy A_2^α conditions', denoted $A_2^{\alpha,\text{energy}}$ and $A_2^{\alpha,*,\text{energy}}$ where

$$A_2^{\alpha,\text{energy}}(\sigma, \omega) \equiv \sup_{Q \in \Omega\mathcal{P}^n} \frac{\left\| \mathsf{P}_Q^\omega \frac{\mathbf{x}}{\ell(Q)} \right\|_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}, \tag{14}$$

$$A_2^{\alpha,*,\text{energy}}(\sigma, \omega) \equiv \sup_{Q \in \Omega\mathcal{P}^n} \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} \frac{\left\| \mathsf{P}_Q^\sigma \frac{\mathbf{x}}{\ell(Q)} \right\|_{L^2(\sigma)}^2}{|Q|^{1-\frac{\alpha}{n}}}.$$

These energy A_2^α conditions play a critical role in controlling local parts of functional energy later in the paper, and it is a crucial requirement that they are necessary conditions, as shown by the next lemma.

Lemma 2. *For any positive locally finite Borel measures* σ, ω *we have*

$$\begin{aligned} A_2^{\alpha,\text{energy}}(\sigma, \omega) &\leq \max\{n, 3\} A_2^{\alpha,\text{punct}}(\sigma, \omega), \\ A_2^{\alpha,*,\text{energy}}(\sigma, \omega) &\leq \max\{n, 3\} A_2^{\alpha,*,\text{punct}}(\sigma, \omega). \end{aligned}$$

Proof. Fix a quasicube $Q \in \Omega\mathcal{D}$. If $\omega\left(Q, \mathfrak{P}_{(\sigma,\omega)}\right) \geq \frac{1}{2}|Q|_\omega$, then we trivially have

$$\begin{aligned} \frac{\left\| \mathsf{P}_Q^\omega \frac{\mathbf{x}}{\ell(Q)} \right\|_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} &\leq n \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} \\ &\leq 2n \frac{\omega\left(Q, \mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} \leq 2n A_2^{\alpha,\text{punct}}(\sigma, \omega). \end{aligned}$$

On the other hand, if $\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)<\frac{1}{2}\left|Q\right|_{\omega}$ then there is a point $p\in Q\cap\mathfrak{P}_{(\sigma,\omega)}$ such that

$$\omega\left(\{p\}\right)>\frac{1}{2}\left|Q\right|_{\omega}\ ,$$

and consequently, p is the largest ω-point mass in Q. Thus if we define $\widetilde{\omega}=\omega-\omega\left(\{p\}\right)\delta_p$, then we have

$$\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)=\left|Q\right|_{\widetilde{\omega}}\ .$$

Now we observe from the construction of Haar projections that

$$\triangle_J^{\widetilde{\omega}}=\triangle_J^{\omega},\qquad\text{for all } J\in\Omega\mathcal{D}\text{ with } p\notin J.$$

So for each $s\geq 0$ there is a unique quasicube $J_s\in\Omega\mathcal{D}$ with $\ell\left(J_s\right)=2^{-s}\ell\left(Q\right)$ that contains the point p. For this quasicube we have, if $\{h_J^{\omega,a}\}_{J\in\Omega\mathcal{D},\ a\in\Gamma_n}$ is a basis for $L^2\left(\omega\right)$,

$$\begin{aligned}\left\|\triangle_{J_s}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 &= \sum_{a\in\Gamma_n}\left|\left\langle h_{J_s}^{\omega,a},x\right\rangle_{\omega}\right|^2=\sum_{a\in\Gamma_n}\left|\left\langle h_{J_s}^{\omega,a},x-p\right\rangle_{\omega}\right|^2\\ &= \sum_{a\in\Gamma_n}\left|\int_{J_s}h_{J_s}^{\omega,a}\left(x\right)\left(x-p\right)d\omega\left(x\right)\right|^2=\sum_{a\in\Gamma_n}\left|\int_{J_s}h_{J_s}^{\omega,a}\left(x\right)\left(x-p\right)d\widetilde{\omega}\left(x\right)\right|^2\\ &\leq \sum_{a\in\Gamma_n}\left\|h_{J_s}^{\omega,a}\right\|_{L^2(\widetilde{\omega})}^2\left\|\mathbf{1}_{J_s}\left(x\right)\left(x-p\right)\right\|_{L^2(\widetilde{\omega})}^2\leq\sum_{a\in\Gamma_n}\left\|h_{J_s}^{\omega,a}\right\|_{L^2(\omega)}^2\left\|\mathbf{1}_{J_s}\left(x\right)\left(x-p\right)\right\|_{L^2(\widetilde{\omega})}^2\\ &\leq n2^n\ell\left(J_s\right)^2\left|J_s\right|_{\widetilde{\omega}}\leq 2^{-2s}\ell\left(Q\right)^2\left|Q\right|_{\widetilde{\omega}}.\end{aligned}$$

Thus we can estimate

$$\begin{aligned}\left\|\mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell\left(Q\right)}\right\|_{L^2(\omega)}^2 &= \frac{1}{\ell\left(Q\right)^2}\sum_{J\in\Omega\mathcal{D}:\ J\subset Q}\left\|\triangle_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\\ &= \frac{1}{\ell\left(Q\right)^2}\left(\sum_{J\in\Omega\mathcal{D}:\ p\notin J\subset Q}\left\|\triangle_J^{\widetilde{\omega}}\mathbf{x}\right\|_{L^2(\widetilde{\omega})}^2+\sum_{s=0}^{\infty}\left\|\triangle_{J_s}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\right)\\ &\leq \frac{1}{\ell\left(Q\right)^2}\left(\left\|\mathsf{P}_Q^{\widetilde{\omega}}\mathbf{x}\right\|_{L^2(\widetilde{\omega})}^2+\sum_{s=0}^{\infty}2^{-2s}\ell\left(Q\right)^2\left|Q\right|_{\widetilde{\omega}}\right)\\ &\leq \frac{1}{\ell\left(Q\right)^2}\left(\ell\left(Q\right)^2\left|Q\right|_{\widetilde{\omega}}+\sum_{s=0}^{\infty}2^{-2s}\ell\left(Q\right)^2\left|Q\right|_{\widetilde{\omega}}\right)\\ &\leq 3\left|Q\right|_{\widetilde{\omega}}\leq 3\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right),\end{aligned}$$

and so

$$\frac{\left\|\mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\omega)}^2}{\left|Q\right|^{1-\frac{\alpha}{n}}}\frac{\left|Q\right|_{\sigma}}{\left|Q\right|^{1-\frac{\alpha}{n}}}\leq\frac{3\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{\left|Q\right|^{1-\frac{\alpha}{n}}}\frac{\left|Q\right|_{\sigma}}{\left|Q\right|^{1-\frac{\alpha}{n}}}\leq 3A_2^{\alpha,\mathrm{punct}}\left(\sigma,\omega\right).$$

Now take the supremum over $Q\in\Omega\mathcal{D}$ to obtain $A_2^{\alpha,\mathrm{energy}}\left(\sigma,\omega\right)\leq\max\left\{n,3\right\}A_2^{\alpha,\mathrm{punct}}\left(\sigma,\omega\right)$. The dual inequality follows upon interchanging the measures σ and ω. □

3.1.2. *Plugged* $\mathcal{A}_2^{\alpha,\text{energy plug}}$ *conditions.* Using Lemma 2 we can control the 'plugged' energy $\mathcal{A}_2^{\alpha}$ conditions:

$$\begin{aligned}\mathcal{A}_2^{\alpha,\text{energy plug}}\left(\sigma,\omega\right) &\equiv \sup_{Q\in\Omega\mathcal{P}^n}\frac{\left\|\mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\omega)}^2}{\left|Q\right|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}\left(Q,\sigma\right),\\ \mathcal{A}_2^{\alpha,*,\text{energy plug}}\left(\sigma,\omega\right) &\equiv \sup_{Q\in\Omega\mathcal{P}^n}\mathcal{P}^{\alpha}\left(Q,\omega\right)\frac{\left\|\mathsf{P}_Q^{\sigma}\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\sigma)}^2}{\left|Q\right|^{1-\frac{\alpha}{n}}}.\end{aligned}$$

Lemma 3. *We have*

$$\begin{aligned}\mathcal{A}_2^{\alpha,\text{energy plug}}(\sigma,\omega) &\lesssim \mathcal{A}_2^{\alpha}(\sigma,\omega)+A_2^{\alpha,\text{energy}}(\sigma,\omega)\,,\\ \mathcal{A}_2^{\alpha,*,\text{energy plug}}(\sigma,\omega) &\lesssim \mathcal{A}_2^{\alpha,*}(\sigma,\omega)+A_2^{\alpha,*,\text{energy}}(\sigma,\omega)\,.\end{aligned}$$

Proof. We have

$$\begin{aligned}\frac{\left\Vert \mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\Vert_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}(Q,\sigma) &= \frac{\left\Vert \mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\Vert_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}(Q,\mathbf{1}_{Q^c}\sigma)+\frac{\left\Vert \mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\Vert_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}(Q,\mathbf{1}_{Q}\sigma)\\ &\lesssim \frac{|Q|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}(Q,\mathbf{1}_{Q^c}\sigma)+\frac{\left\Vert \mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\Vert_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}}\\ &\lesssim \mathcal{A}_2^{\alpha}(\sigma,\omega)+A_2^{\alpha,\text{energy}}(\sigma,\omega)\,.\end{aligned}$$

□

3.2. **Random grids and shifted grids.** Using the analogue for dyadic quasigrids of the good random grids of Nazarov, Treil and Volberg, a standard argument of NTV, see e.g. [Vol], reduces the two weight inequality (12) for T^{α} to proving boundedness of a bilinear form $\mathcal{T}^{\alpha}(f,g)$ with uniform constants over dyadic quasigrids, and where the quasiHaar supports $\operatorname{supp}\widehat{f}$ and $\operatorname{supp}\widehat{g}$ of the functions f and g are contained in the collection $\Omega\mathcal{D}^{\text{good}}$ of good quasicubes, whose children are all good as well, with goodness parameters $\mathbf{r}<\infty$ and $\varepsilon>0$ chosen sufficiently large and small respectively depending only on n and α. Here the quasiHaar support of f is $\operatorname{supp}\widehat{f}\equiv\{I\in\Omega\mathcal{D}:\triangle_I^{\sigma}f\neq 0\}$, and similarly for g. In fact we can assume even more, namely that the quasiHaar supports $\operatorname{supp}\widehat{f}$ and $\operatorname{supp}\widehat{g}$ of f and g are contained in the collection of $\boldsymbol{\tau}$-*good* quasicubes

$$\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\text{good}}^{\boldsymbol{\tau}}\equiv\left\{K\in\Omega\mathcal{D}:\mathfrak{C}_K\subset\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\text{good}}\text{ and }\pi_{\Omega\mathcal{D}}^{\ell}K\in\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\text{good}}\text{ for all }0\leq\ell\leq\boldsymbol{\tau}\right\}, \tag{15}$$

that are $(\mathbf{r},\varepsilon)$-good and whose children are also $(\mathbf{r},\varepsilon)$-good, and whose ℓ-parents up to level $\boldsymbol{\tau}$ are also $(\mathbf{r},\varepsilon)$-good. Here $\boldsymbol{\tau}>\mathbf{r}$ is a parameter to be fixed later. We may assume this restriction on the quasiHaar supports of f and g by the following lemma. See [SaShUr6] for a proof[8].

Lemma 4. *Given* $\mathbf{r}\geq 3$, $\boldsymbol{\tau}\geq 1$ *and* $\frac{1}{\mathbf{r}}<\varepsilon<1-\frac{1}{\mathbf{r}}$, *we have*

$$\Omega\mathcal{D}_{(\mathbf{r}-1,\delta)-\text{good}}\subset\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\text{good}}^{\boldsymbol{\tau}}\ ,$$

provided

$$0<\delta\leq\frac{\mathbf{r}\varepsilon-1}{\mathbf{r}+\boldsymbol{\tau}}. \tag{16}$$

For convenience in notation we will sometimes suppress the dependence on α in our nonlinear forms, but will retain it in the operators, Poisson integrals and constants. More precisely, let $\Omega\mathcal{D}^{\sigma}=\Omega\mathcal{D}^{\omega}$ be an $(\mathbf{r},\varepsilon)$-good quasigrid on $\mathbb{R}^n$, and let $\{h_I^{\sigma,a}\}_{I\in\Omega\mathcal{D}^{\sigma},\ a\in\Gamma_n}$ and $\left\{h_J^{\omega,b}\right\}_{J\in\Omega\mathcal{D}^{\omega},\ b\in\Gamma_n}$ be corresponding quasiHaar bases as described above, so that

$$\begin{aligned}f &= \sum_{I\in\Omega\mathcal{D}^{\sigma}}\triangle_I^{\sigma}f=\sum_{I\in\Omega\mathcal{D}^{\sigma},\ a\in\Gamma_n}\langle f,h_I^{\sigma,a}\rangle\ h_I^{\sigma,a}=\sum_{I\in\Omega\mathcal{D}^{\sigma},\ a\in\Gamma_n}\widehat{f}(I;a)\ h_I^{\sigma,a},\\ g &= \sum_{J\in\Omega\mathcal{D}^{\omega}}\triangle_J^{\omega}g=\sum_{J\in\Omega\mathcal{D}^{\omega},\ b\in\Gamma_n}\left\langle g,h_J^{\omega,b}\right\rangle\ h_J^{\omega,b}=\sum_{J\in\Omega\mathcal{D}^{\omega},\ b\in\Gamma_n}\widehat{g}(J;b)\ h_J^{\omega,b},\end{aligned}$$

where the appropriate measure is understood in the notation $\widehat{f}(I;a)$ and $\widehat{g}(J;b)$, and where these quasiHaar coefficients $\widehat{f}(I;a)$ and $\widehat{g}(J;b)$ vanish if the quasicubes I and J are not good. Inequality (12) is equivalent to boundedness of the bilinear form

$$\mathcal{T}^{\alpha}(f,g)\equiv\langle T_{\sigma}^{\alpha}(f),g\rangle_{\omega}=\sum_{I\in\Omega\mathcal{D}^{\sigma}\text{ and }J\in\Omega\mathcal{D}^{\omega}}\langle T_{\sigma}^{\alpha}(\triangle_I^{\sigma}f),\triangle_J^{\omega}g\rangle_{\omega}$$

[8] This lemma is misstated in [SaShUr7].

on $L^2(\sigma) \times L^2(\omega)$, i.e.

$$|\mathcal{T}^{\alpha}(f,g)| \leq \mathfrak{N}_{T^{\alpha}} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}, \tag{17}$$

uniformly over all quasigrids and appropriate truncations. We may assume the two quasigrids $\Omega\mathcal{D}^{\sigma}$ and $\Omega\mathcal{D}^{\omega}$ are equal here, and this we will do throughout the paper, although we sometimes continue to use the measure as a superscript on $\Omega\mathcal{D}$ for clarity of exposition. Roughly speaking, we analyze the form $\mathcal{T}^{\alpha}(f,g)$ by splitting it in a nonlinear way into three main pieces, following in part the approach in [LaSaShUr2] and [LaSaShUr3]. The first piece consists of quasicubes I and J that are either disjoint or of comparable side length, and this piece is handled using the section on preliminaries of NTV type. The second piece consists of quasicubes I and J that overlap, but are 'far apart' in a nonlinear way, and this piece is handled using the sections on the Intertwining Proposition and the control of the functional quasienergy condition by the quasienergy condition. Finally, the remaining local piece where the overlapping quasicubes are 'close' is handled by generalizing methods of NTV as in [LaSaShUr], and then splitting the stopping form into two sublinear stopping forms, one of which is handled using techniques of [LaSaUr2], and the other using the stopping time and recursion of M. Lacey [Lac]. See the schematic diagram in Subsection 7.4 below.

We summarize our assumptions on the Haar supports of f and g, and on the dyadic quasigrids $\Omega\mathcal{D}$.

Condition 1 (on Haar supports and quasigrids)**.** *We suppose the quasiHaar supports of the functions f and g satisfy $\operatorname{supp}\widehat{f}, \operatorname{supp}\widehat{g} \subset \Omega\mathcal{D}^{\boldsymbol{\tau}}_{(\mathbf{r},\varepsilon)-\mathrm{good}}$. We also assume that $|\partial Q|_{\sigma+\omega}=0$ for all dyadic quasicubes Q in the grids $\Omega\mathcal{D}$ (since this property holds with probability 1 for random grids $\Omega\mathcal{D}$).*

4. Necessity of the $\mathcal{A}_2^{\alpha}$ conditions

Here we consider in particular the necessity of the fractional $\mathcal{A}_2^{\alpha}$ condition (with holes) when $0 \leq \alpha < n$, for the boundedness from $L^2(\sigma)$ to $L^2(\omega)$ (where σ and ω may have common point masses) of the α-fractional Riesz vector transform $\mathbf{R}^{\alpha}$ defined by

$$\mathbf{R}^{\alpha}(f\sigma)(x) = \int_{\mathbb{R}^n} K_j^{\alpha}(x,y) f(y)\, d\sigma(y), \qquad K_j^{\alpha}(x,y) = \frac{x^j - y^j}{|x-y|^{n+1-\alpha}},$$

whose kernel $K_j^{\alpha}(x,y)$ satisfies (5) for $0 \leq \alpha < n$. More generally, necessity holds for elliptic operators as in the next lemma. See [SaShUr7] for the easier proof in the case without holes.

Lemma 5. *Suppose $0 \leq \alpha < n$. Let T^{α} be any collection of operators with α-standard fractional kernel satisfying the ellipticity condition (13), and in the case $\frac{n}{2} \leq \alpha < n$, we also assume the more restrictive condition (11). Then for $0 \leq \alpha < n$ we have*

$$\sqrt{\mathcal{A}_2^{\alpha}} \lesssim \mathfrak{N}_{\alpha}(T^{\alpha}).$$

Proof. First we give the proof for the case when T^{α} is the α-fractional Riesz transform $\mathbf{R}^{\alpha}$, whose kernel is $\mathbf{K}^{\alpha}(x,y) = \frac{x-y}{|x-y|^{n+1-\alpha}}$. Define the 2^n generalized n-ants $\mathcal{Q}_m$ for $m \in \{-1,1\}^n$, and their translates $\mathcal{Q}_m(w)$ for $w \in \mathbb{R}^n$ by

$$\mathcal{Q}_m = \{(x_1,...,x_n) : m_k x_k > 0\}, \quad \mathcal{Q}_m(w) = \{z : z-w \in \mathcal{Q}_m\}, \qquad w \in \mathbb{R}^n.$$

Fix $m \in \{-1,1\}^n$ and a quasicube I. For $a \in \mathbb{R}^n$ and $r>0$ let

$$s_I(x) = \frac{\ell(I)}{\ell(I) + |x-\zeta_I|}, \qquad f_{a,r}(y) = \mathbf{1}_{\mathcal{Q}_{-m}(a)\cap B(0,r)}(y)\, s_I(y)^{n-\alpha},$$

where ζ_I is the center of the cube I. Now

$$\ell(I)|x-y| \leq \ell(I)|x-\zeta_I| + \ell(I)|\zeta_I - y| \leq [\ell(I)+|x-\zeta_I|][\ell(I)+|\zeta_I-y|]$$

implies

$$\frac{1}{|x-y|} \geq \frac{1}{\ell(I)} s_I(x)\, s_I(y), \qquad x,y \in \mathbb{R}^n.$$

Now the key observation is that with $L\zeta \equiv m \cdot \zeta$, we have

$$L(x-y) = m\cdot(x-y) \geq |x-y|, \qquad x \in \mathcal{Q}_m(y),$$

which yields

$$L\left(\mathbf{K}^{\alpha}\left(x,y\right)\right)=\frac{L\left(x-y\right)}{\left|x-y\right|^{n+1-\alpha}}\geq\frac{1}{\left|x-y\right|^{n-\alpha}}\geq\ell\left(I\right)^{\alpha-n}s_{I}\left(x\right)^{n-\alpha}s_{I}\left(y\right)^{n-\alpha},\tag{18}$$

provided $x\in\mathcal{Q}_{m}\left(y\right)$. Now we note that $x\in\mathcal{Q}_{m}\left(y\right)$ when $x\in\mathcal{Q}_{m}\left(a\right)$ and $y\in\mathcal{Q}_{-m}\left(a\right)$ to obtain that for $x\in\mathcal{Q}_{m}\left(a\right)$,

$$\begin{aligned} L\left(T^{\alpha}\left(f_{a,r}\sigma\right)\left(x\right)\right) &= \int_{\mathcal{Q}_{-m}(a)\cap B(0,r)}\frac{L\left(x-y\right)}{\left|x-y\right|^{n+1-\alpha}}s_{I}\left(y\right)d\sigma\left(y\right) \\ &\geq \ell\left(I\right)^{\alpha-n}s_{I}\left(x\right)^{n-\alpha}\int_{\mathcal{Q}_{-m}(a)\cap B(0,r)}s_{I}\left(y\right)^{2n-2\alpha}d\sigma\left(y\right). \end{aligned}$$

Applying $\left|L\zeta\right|\leq\sqrt{n}\left|\zeta\right|$ and our assumed two weight inequality for the fractional Riesz transform, we see that for $r>0$ large,

$$\begin{aligned} &\ell\left(I\right)^{2\alpha-2n}\int_{\mathcal{Q}_{m}(a)}s_{I}\left(x\right)^{2n-2\alpha}\left(\int_{\mathcal{Q}_{-m}(a)\cap B(0,r)}s_{I}\left(y\right)^{2n-2\alpha}d\sigma\left(y\right)\right)^{2}d\omega\left(x\right) \\ &\leq\left\|LT(\sigma f_{a,r})\right\|_{L^{2}(\omega)}^{2}\lesssim\mathfrak{N}_{\alpha}\left(\mathbf{R}^{\alpha}\right)^{2}\left\|f_{a,r}\right\|_{L^{2}(\sigma)}^{2}=\mathfrak{N}_{\alpha}\left(\mathbf{R}^{\alpha}\right)^{2}\int_{\mathcal{Q}_{-m}(a)\cap B(0,r)}s_{I}\left(y\right)^{2n-2\alpha}d\sigma\left(y\right). \end{aligned}$$

Rearranging the last inequality, and upon letting $r\to\infty$, we obtain

$$\int_{\mathcal{Q}_{m}(a)}\frac{\ell\left(I\right)^{n-\alpha}}{\left(\ell\left(I\right)+\left|x-\zeta_{I}\right|\right)^{2n-2\alpha}}d\omega\left(x\right)\int_{\mathcal{Q}_{-m}(a)}\frac{\ell\left(I\right)^{n-\alpha}}{\left(\ell\left(I\right)+\left|y-\zeta_{I}\right|\right)^{2n-2\alpha}}d\sigma\left(y\right)\lesssim\mathfrak{N}_{\alpha}\left(\mathbf{R}^{\alpha}\right)^{2}.$$

Note that the ranges of integration above are pairs of opposing n-ants.

Fix a quasicube Q, which without loss of generality can be taken to be centered at the origin, $\zeta_{Q}=0$. Then choose $a=\left(2\ell\left(Q\right),2\ell\left(Q\right)\right)$ and $I=Q$ so that we have

$$\begin{aligned} &\left(\int_{\mathcal{Q}_{m}(a)}\frac{\ell\left(Q\right)^{n-\alpha}}{\left(\ell\left(Q\right)+\left|x\right|\right)^{2n-2\alpha}}d\omega\left(x\right)\right)\left(\ell\left(Q\right)^{\alpha-n}\int_{Q}d\sigma\right) \\ \leq\ &C_{\alpha}\int_{\mathcal{Q}_{m}(a)}\frac{\ell\left(Q\right)^{n-\alpha}}{\left(\ell\left(Q\right)+\left|x\right|\right)^{2n-2\alpha}}d\omega\left(x\right)\int_{\mathcal{Q}_{-m}(a)}\frac{\ell\left(Q\right)^{n-\alpha}}{\left(\ell\left(Q\right)+\left|y\right|\right)^{2n-2\alpha}}d\sigma\left(y\right)\lesssim\mathfrak{N}_{\alpha}\left(\mathbf{R}^{\alpha}\right)^{2}. \end{aligned}$$

Now fix $m=\left(1,1,...,1\right)$ and note that there is a fixed N (independent of $\ell\left(Q\right)$) and a fixed collection of rotations $\left\{\rho_{k}\right\}_{k=1}^{N}$, such that the rotates $\rho_{k}\mathcal{Q}_{m}\left(a\right)$, $1\leq k\leq N$, of the n-ant $\mathcal{Q}_{m}\left(a\right)$ cover the complement of the ball $B\left(0,4\sqrt{n}\ell\left(Q\right)\right)$:

$$B\left(0,4\sqrt{n}\ell\left(Q\right)\right)^{c}\subset\bigcup_{k=1}^{N}\rho_{k}\mathcal{Q}_{m}\left(a\right).$$

Then we obtain, upon applying the same argument to these rotated pairs of n-ants,

$$\left(\int_{B\left(0,4\sqrt{n}\ell(Q)\right)^{c}}\frac{\ell\left(Q\right)^{n-\alpha}}{\left(\ell\left(Q\right)+\left|x\right|\right)^{2n-2\alpha}}d\omega\left(x\right)\right)\left(\ell\left(Q\right)^{\alpha-n}\int_{Q}d\sigma\right)\lesssim\mathfrak{N}_{\alpha}\left(\mathbf{R}^{\alpha}\right)^{2}.\tag{19}$$

Now we assume for the moment the offset A_{2}^{α} condition

$$\ell\left(Q\right)^{2(\alpha-n)}\left(\int_{Q'}d\omega\right)\left(\int_{Q}d\sigma\right)\leq A_{2}^{\alpha},$$

where Q' and Q are neighbouring quasicubes, i.e. $\left(Q',Q\right)\in\Omega\mathcal{N}^{n}$. If we use this offset inequality with Q' ranging over $3Q\setminus Q$, and then use the separation of $B\left(0,4\sqrt{n}\ell\left(Q\right)\right)\setminus3Q$ and Q to obtain the inequality

$$\ell\left(Q\right)^{2(\alpha-n)}\left(\int_{B\left(0,4\sqrt{n}\ell(Q)\right)\setminus3Q}d\omega\right)\left(\int_{Q}d\sigma\right)\lesssim A_{2}^{\alpha},$$

together with (19), we obtain

$$\left(\int_{\mathbb{R}^{n}\setminus Q}\frac{\ell\left(Q\right)^{n-\alpha}}{\left(\ell\left(Q\right)+\left|x\right|\right)^{2n-2\alpha}}d\omega\left(x\right)\right)^{\frac{1}{2}}\left(\ell\left(Q\right)^{\alpha-n}\int_{Q}d\sigma\right)^{\frac{1}{2}}\lesssim\mathfrak{N}_{\alpha}\left(\mathbf{R}^{\alpha}\right)+\sqrt{A_{2}^{\alpha}}.$$

Clearly we can reverse the roles of the measures ω and σ and obtain

$$\sqrt{\mathcal{A}_2^{\alpha,*}} \lesssim \mathfrak{N}_\alpha\left(\mathbf{R}^\alpha\right) + \sqrt{A_2^\alpha}$$

for the kernels $\mathbf{K}^\alpha$, $0 \leq \alpha < n$.

More generally, to obtain the case when T^α is elliptic and the offset A_2^α condition holds, we note that the key estimate (18) above extends to the kernel $\sum_{j=1}^J \lambda_j^m K_j^\alpha$ of $\sum_{j=1}^J \lambda_j^m T_j^\alpha$ in (11) if the n-ants above are replaced by thin cones of sufficently small aperture, and there is in addition sufficient separation between opposing cones, which in turn may require a larger constant than $4\sqrt{n}$ in the choice of Q' above.

Finally, we turn to showing that the offset A_2^α condition is implied by the norm inequality, i.e.

$$\sqrt{A_2^\alpha} \equiv \sup_{(Q',Q)\in\Omega\mathcal{N}^n} \ell(Q)^\alpha \left(\frac{1}{|Q'|}\int_{Q'} d\omega\right)^{\frac{1}{2}} \left(\frac{1}{|Q|}\int_{Q} d\sigma\right)^{\frac{1}{2}} \lesssim \mathfrak{N}_\alpha\left(\mathbf{R}^\alpha\right);$$

$$\text{i.e. } \left(\int_{Q'} d\omega\right)\left(\int_{Q} d\sigma\right) \lesssim \mathfrak{N}_\alpha\left(\mathbf{R}^\alpha\right)^2 |Q|^{2-\frac{2\alpha}{n}}, \qquad (Q',Q)\in\Omega\mathcal{N}^n.$$

In the range $0 \leq \alpha < \frac{n}{2}$ where we only assume (13), we adapt a corresponding argument from [LaSaUr1].

The 'one weight' argument on page 211 of Stein [Ste] yields the *asymmetric* two weight A_2^α condition

$$|Q'|_\omega |Q|_\sigma \leq C\mathfrak{N}_\alpha\left(\mathbf{R}^\alpha\right)|Q|^{2\left(1-\frac{\alpha}{n}\right)}, \tag{20}$$

where Q and Q' are quasicubes of equal side length r and distance $C_0 r$ apart for some (fixed large) positive constant C_0 (for this argument we choose the unit vector $\mathbf{u}$ in (13) to point in the direction from Q to Q'). In the one weight case treated in [Ste] it is easy to obtain from this (even for a *single* direction $\mathbf{u}$) the usual (symmetric) A_2 condition. Here we will have to employ a different approach.

Now recall (see Sec 2 of [Saw] for the case of usual cubes, and the case of half open, half closed quasicubes here is no different) that given an open subset Φ of $\mathbb{R}^n$, we can choose $R \geq 3$ sufficiently large, depending only on the dimension, such that if $\left\{Q_j^k\right\}_j$ are the dyadic quasicubes maximal among those dyadic quasicubes Q satisfying $RQ \subset \Phi$, then the following properties hold:

$$\left\{\begin{array}{lll} \text{(disjoint cover)} & \Phi = \bigcup_j Q_j \text{ and } Q_j \cap Q_i = \emptyset \text{ if } i \neq j \\ \text{(Whitney condition)} & RQ_j \subset \Phi \text{ and } 3RQ_j \cap \Phi^c \neq \emptyset \text{ for all } j \\ \text{(finite overlap)} & \sum_j \chi_{3Q_j} \leq C\chi_\Phi \end{array}\right. . \tag{21}$$

So fix a pair of neighbouring quasicubes $(Q_0', Q_0) \in \Omega\mathcal{N}^n$, and let $\{\mathsf{Q}_i\}_i$ be a Whitney decomposition into quasicubes of the set $\Phi \equiv (Q_0' \times Q_0) \setminus \mathfrak{D}$ relative to the diagonal $\mathfrak{D}$ in $\mathbb{R}^n \times \mathbb{R}^n$. Of course, there are no common point masses of ω in Q_0' and σ in Q_0 since the quasicubes Q_0' and Q_0 are disjoint. Note that if $\mathsf{Q}_i = Q_i' \times Q_i$, then (20) can be written

$$|\mathsf{Q}_i|_{\omega\times\sigma} \leq C\mathfrak{N}_\alpha\left(\mathbf{R}^\alpha\right)|\mathsf{Q}_i|^{1-\frac{\alpha}{n}}, \tag{22}$$

where $\omega \times \sigma$ denotes product measure on $\mathbb{R}^n \times \mathbb{R}^n$. We choose R sufficiently large in the Whitney decomposition (21), depending on C_0, such that (22) holds for all the Whitney quasicubes Q_i. We have $\sum_i |\mathsf{Q}_i| = |Q' \times Q| = |Q|^2$.

Moreover, if $\mathsf{R} = Q' \times Q$ is a rectangle in $\mathbb{R}^n \times \mathbb{R}^n$ (i.e. Q', Q are quasicubes in $\mathbb{R}^n$), and if $\mathsf{R} = \dot{\cup}_i \mathsf{R}_i$ is a finite disjoint union of rectangles R_α, then by additivity of the product measure $\omega \times \sigma$,

$$|\mathsf{R}|_{\omega\times\sigma} = \sum_i |\mathsf{R}_i|_{\omega\times\sigma}.$$

Let $\mathsf{Q}_0 = Q_0' \times Q_0$ and set

$$\Lambda \equiv \left\{\mathsf{Q} = Q' \times Q : \mathsf{Q} \subset \mathsf{Q}_0, \ell(Q) = \ell(Q') \approx C_0^{-1}\,\mathrm{qdist}(Q,Q') \text{ and (20) holds}\right\}.$$

Divide Q_0 into $2n \times 2n = 4n^2$ congruent subquasicubes $\mathsf{Q}_0^1, ..., \mathsf{Q}_0^{4^n}$ of side length $\frac{1}{2}$, and set aside those $\mathsf{Q}_0^j \in \Lambda$ (those for which (20) holds) into a collection of stopping cubes Γ. Continue to divide the remaining $\mathsf{Q}_0^j \in \Lambda$ of side length $\frac{1}{4}$, and again, set aside those $\mathsf{Q}_0^{j,i} \in \Phi$ into Γ, and continue subdividing those that remain. We continue with such subdivisions for N generations so that all the cubes *not* set aside into Γ have side length 2^{-N} . The important property these latter cubes have is that they all lie within distance $r2^{-N}$ of the diagonal $\mathfrak{D} = \{(x,x) : (x,x) \in Q_0' \times Q_0\}$ in $\mathsf{Q}_0 = Q_0' \times Q_0$ since (20) holds for all pairs of cubes Q'

and Q of equal side length r having distance at least $C_0 r$ apart. Enumerate the cubes in Γ as $\{\mathsf{Q}_i\}_i$ and those remaining that are not in Γ as $\{\mathsf{P}_j\}_j$. Thus we have the pairwise disjoint decomposition

$$\mathsf{Q}_0 = \left(\bigcup_i \mathsf{Q}_i\right) \bigcup \left(\bigcup_j \mathsf{P}_j\right).$$

The countable additivity of the product measure $\omega \times \sigma$ shows that

$$|\mathsf{Q}_0|_{\omega\times\sigma} = \sum_i |\mathsf{Q}_i|_{\omega\times\sigma} + \sum_j |\mathsf{P}_j|_{\omega\times\sigma} \ .$$

Now we have

$$\sum_i |\mathsf{Q}_i|_{\omega\times\sigma} \lesssim \sum_i \mathfrak{N}_\alpha \left(\mathbf{R}^\alpha\right)^2 |\mathsf{Q}_i|^{1-\frac{\alpha}{n}} ,$$

and

$$\begin{aligned}
\sum_i |\mathsf{Q}_i|^{1-\frac{\alpha}{n}} &= \sum_{k\in\mathbb{Z}:\ 2^k\leq \ell(Q_0)} \ \sum_{i:\ \ell(Q_i)=2^k} \left(2^{2nk}\right)^{1-\frac{\alpha}{n}} \approx \sum_{k\in\mathbb{Z}:\ 2^k\leq \ell(Q_0)} \left(\frac{2^k}{\ell(Q_0)}\right)^{-n} \left(2^{2nk}\right)^{1-\frac{\alpha}{n}} \quad \text{(Whitney)}\\
&= \ell(Q_0)^n \sum_{k\in\mathbb{Z}:\ 2^k\leq \ell(Q_0)} 2^{nk\left(-1+2-\frac{2\alpha}{n}\right)} \leq C_\alpha \ell(Q_0)^n \ell(Q_0)^{n\left(1-\frac{2\alpha}{n}\right)} = C_\alpha |Q_0\times Q_0|^{2-\frac{2\alpha}{n}} = C_\alpha |\mathsf{Q}_0|^{1-\frac{\alpha}{n}} ,
\end{aligned}$$

provided $0\leq\alpha<\frac{n}{2}$. Using that the side length of $\mathsf{P}_j = P_j\times P_j'$ is 2^{-N} and $dist\left(\mathsf{P}_j,\mathfrak{D}\right)\leq C_r 2^{-N}$, we have the following limit,

$$\sum_j |\mathsf{P}_j|_{\omega\times\sigma} = \left|\bigcup_j \mathsf{P}_j\right|_{\omega\times\sigma} \to 0 \text{ as } N\to\infty,$$

since $\bigcup_j \mathsf{P}_j$ shrinks to the empty set as $N\to\infty$, and since locally finite measures such as $\omega\times\sigma$ are regular in Euclidean space. This completes the proof that $\sqrt{A_2^\alpha}\lesssim\mathfrak{N}_\alpha\left(\mathbf{R}^\alpha\right)$ for the range $0\leq\alpha<\frac{n}{2}$.

Now we turn to proving $\sqrt{A_2^\alpha}\lesssim\mathfrak{N}_\alpha\left(\mathbf{R}^\alpha\right)$ for the range $\frac{n}{2}\leq\alpha<n$, where we assume the stronger ellipticity condition (11). So fix a pair of neighbouring quasicubes $(K',K)\in\Omega\mathcal{N}^n$, and assume that $\sigma+\omega$ doesn't charge the intersection $\overline{K'}\cap\overline{K}$ of the closures of K' and K. It will be convenient to replace n by $n+1$, i.e to introduce an additional dimension, and work with the preimages $Q'=\Omega^{-1}K'$ and $Q=\Omega^{-1}K$ that are usual cubes, and with the corresponding pullbacks $\widetilde{\omega}=m_1\times\Omega^*\omega$ and $\widetilde{\sigma}=m_1\times\Omega^*\sigma$ of the measures ω and σ where m_1 is Lebesgue measure on the line. We may also assume that

$$Q'=[-1,0)\times\prod_{i=1}^n Q_i,\qquad Q=[0,1)\times\prod_{i=1}^n Q_i.$$

where $Q_i=[a_i,b_i]$ for $1\leq i\leq n$ (since the other cases are handled in similar fashion). It is important to note that we are considering the intervals Q_i here to be closed, and we will track this difference as we proceed.

Choose $\theta_1\in[a_1,b_1]$ so that both

$$\left|[-1,0)\times[a_1,\theta_1]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\omega}} ,\quad \left|[-1,0)\times[\theta_1,b_1]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\omega}} \geq \frac{1}{2}|Q'|_{\widetilde{\omega}} .$$

Now denote the two intervals $[a_1,\theta_1]$ and $[\theta_1,b_1]$ by $[a_1^*,b_1^*]$ and $[a_1^{**},b_1^{**}]$ where the order is chosen so that

$$\left|[0,1)\times[a_1^*,b_1^*]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\sigma}} \leq \left|[0,1)\times[a_1^{**},b_1^{**}]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\sigma}} .$$

Then we have both

$$\left|[-1,0)\times[a_1^*,b_1^*]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\omega}} \geq \frac{1}{2}|Q|_{\widetilde{\omega}} \text{ and } \left|[0,1)\times[a_1^{**},b_1^{**}]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\sigma}} \geq \frac{1}{2}|Q|_{\widetilde{\sigma}} \ .$$

Now choose $\theta_2 \in [a_2, b_2]$ so that both

$$\left|[-1,0) \times [a_1^*, b_1^*] \times [a_2, \theta_2] \times \prod_{i=3}^n Q_i\right|_{\widetilde{\omega}}, \quad \left|[-1,0) \times [a_1^*, b_1^*] \times [\theta_2, b_2] \times \prod_{i=3}^n Q_i\right|_{\widetilde{\omega}} \geq \frac{1}{4}|Q|_{\widetilde{\omega}},$$

and denote the two intervals $[a_2, \theta_2]$ and $[\theta_2, b_2]$ by $[a_2^*, b_2^*]$ and $[a_2^{**}, b_2^{**}]$ where the order is chosen so that

$$[0,1) \times \left|[a_1^{**}, b_1^{**}] \times [a_2^*, b_2^*] \times \prod_{i=2}^n Q_i\right|_{\widetilde{\sigma}} \leq \left|[0,1) \times [a_1^{**}, b_1^{**}] \times [a_2^{**}, b_2^{**}] \times \prod_{i=2}^n Q_i\right|_{\widetilde{\sigma}}.$$

Then we have both

$$\begin{aligned}\left|[-1,0) \times [a_1^*, b_1^*] \times [a_2^*, b_2^*] \times \prod_{i=3}^n Q_i\right|_{\widetilde{\omega}} &\geq \frac{1}{4}|Q|_{\widetilde{\omega}}, \\ \left|[0,1) \times [a_1^{**}, b_1^{**}] \times [a_2^{**}, b_2^{**}] \times \prod_{i=3}^n Q_i\right|_{\widetilde{\sigma}} &\geq \frac{1}{4}|Q|_{\widetilde{\sigma}},\end{aligned}$$

and continuing in this way we end up with two rectangles,

$$\begin{aligned} G &\equiv [-1,0) \times [a_1^*, b_1^*] \times [a_2^*, b_2^*] \times \ldots \times [a_n^*, b_n^*], \\ H &\equiv [0,1) \times [a_1^{**}, b_1^{**}] \times [a_2^{**}, b_2^{**}] \times \ldots \times [a_n^{**}, b_n^{**}], \end{aligned}$$

that satisfy

$$\begin{aligned} |G|_{\widetilde{\omega}} &= |[-1,0) \times [a_1^*, b_1^*] \times [a_2^*, b_2^*] \times \ldots [a_n^*, b_n^*]|_{\widetilde{\omega}} \geq \frac{1}{2^n}|Q|_{\widetilde{\omega}}, \\ |H|_{\widetilde{\sigma}} &= |[0,1) \times [a_1^{**}, b_1^{**}] \times [a_2^{**}, b_2^{**}] \times \ldots [a_n^{**}, b_n^{**}]|_{\widetilde{\sigma}} \geq \frac{1}{2^n}|Q|_{\widetilde{\sigma}}. \end{aligned}$$

However, the quasirectangles ΩG and ΩH lie in opposing quasi-n-ants at the vertex $\Omega\theta = \Omega(\theta_1, \theta_2, \ldots, \theta_n)$, and so we can apply (11) to obtain that for $x \in \Omega G$,

$$\left|\sum_{j=1}^J \lambda_j^m T_j^\alpha (\mathbf{1}_{\Omega H}\sigma)(x)\right| = \left|\int_{\Omega H} \sum_{j=1}^J \lambda_j^m K_j^\alpha(x,y)\, d\sigma(y)\right| \gtrsim \int_{\Omega H} |x-y|^{\alpha-n} d\sigma(y) \gtrsim |\Omega Q|^{\frac{\alpha}{n}-1} |\Omega H|_\sigma.$$

For the inequality above, we need to know that the distinguished point $\Omega\theta$ is not a common point mass of σ and ω, but this follows from our assumption that $\sigma+\omega$ doesn't charge the intersection $\overline{K'} \cap \overline{K}$ of the closures of K' and K. Then from the norm inequality we get

$$\begin{aligned} |\Omega G|_\omega \left(|\Omega Q|^{\frac{\alpha}{n}-1} |\Omega H|_\sigma\right)^2 &\lesssim \int_G \left|\sum_{j=1}^J \lambda_j^m T_j^\alpha (\mathbf{1}_{\Omega H}\sigma)\right|^2 d\omega \\ &\lesssim \mathfrak{N}^2_{\sum_{j=1}^J \lambda_j^m T_j^\alpha} \int \mathbf{1}_{\Omega H}^2 d\sigma = \mathfrak{N}^2_{\sum_{j=1}^J \lambda_j^m T_j^\alpha} |\Omega H|_\sigma, \end{aligned}$$

from which we deduce that

$$\begin{aligned} |\Omega Q|^{2\left(\frac{\alpha}{n}-1\right)} |\Omega Q'|_\omega |\Omega Q|_\sigma &\lesssim 2^{2n} |\Omega Q|^{2\left(\frac{\alpha}{n}-1\right)} |\Omega G|_\omega |\Omega H|_\sigma \lesssim 2^{2n} \mathfrak{N}^2_{\sum_{j=1}^J \lambda_j^m T_j^\alpha}; \\ |K|^{2\left(\frac{\alpha}{n}-1\right)} |K'|_\omega |K|_\sigma &\lesssim 2^{2n} \mathfrak{N}^2_{\sum_{j=1}^J \lambda_j^m T_j^\alpha}, \end{aligned}$$

and hence

$$A_2^\alpha \lesssim 2^{2n} \mathfrak{N}^2_{\sum_{j=1}^J \lambda_j^m T_j^\alpha}.$$

Thus we have obtained the offset A_2^α condition for pairs $(K', K) \in \Omega\mathcal{N}^n$ such that $\sigma+\omega$ doesn't charge the intersection $\overline{K'} \cap \overline{K}$ of the closures of K' and K. From this and the argument at the beginning of this proof, we obtain the one-tailed $\mathcal{A}_2^\alpha$ conditions. Indeed, we note that $|\partial(rQ)|_{\sigma+\omega} > 0$ for only a countable number of dilates $r > 1$, and so a limiting argument applies. This completes the proof of Lemma 5. □

5. Monotonicity Lemma and Energy lemma

The Monotonicity Lemma below will be used to prove the Energy Lemma, which is then used in several places in the proof of Theorem 1. The formulation of the Monotonicity Lemma with $m=2$ for cubes is due to M. Lacey and B. Wick [LaWi], and corrects that used in early versions of our paper [SaShUr5].

5.1. **The Monotonicity Lemma.** For $0\leq\alpha<n$ and $m\in\mathbb{R}_{+}$, we recall the m-weighted fractional Poisson integral

$$\mathrm{P}_{m}^{\alpha}\left(J,\mu\right)\equiv\int_{\mathbb{R}^{n}}\frac{\left|J\right|^{\frac{m}{n}}}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_{J}\right|\right)^{n+m-\alpha}}d\mu\left(y\right),$$

where $\mathrm{P}_{1}^{\alpha}\left(J,\mu\right)=\mathrm{P}^{\alpha}\left(J,\mu\right)$ is the standard Poisson integral. The next lemma holds for quasicubes and common point masses with the same proof as in [SaShUr7].

Lemma 6 (Monotonicity)**.** *Suppose that I and J are quasicubes in $\mathbb{R}^{n}$ such that $J\subset 2J\subset I$, and that μ is a signed measure on $\mathbb{R}^{n}$ supported outside I. Finally suppose that T^{α} is a standard α-fractional singular integral on $\mathbb{R}^{n}$ with $0<\alpha<n$. Then we have the estimate*

$$\left\|\triangle_{J}^{\omega}T^{\alpha}\mu\right\|_{L^{2}(\omega)}\lesssim\Phi^{\alpha}\left(J,\left|\mu\right|\right), \tag{23}$$

where for a positive measure ν,

$$\begin{aligned}\Phi^{\alpha}\left(J,\nu\right)^{2} &\equiv \left(\frac{\mathrm{P}^{\alpha}\left(J,\nu\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\|\triangle_{J}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}+\left(\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J,\nu\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\|\mathbf{x}-\mathbf{m}_{J}\right\|_{L^{2}(\mathbf{1}_{J}\omega)}^{2}\ ,\\ \mathbf{m}_{J} &\equiv \mathbb{E}_{J}^{\omega}\mathbf{x}=\frac{1}{\left|J\right|_{\omega}}\int_{J}\mathbf{x}d\omega.\end{aligned}$$

5.2. **The Energy Lemma.** Suppose now we are given a subset $\mathcal{H}$ of the dyadic quasigrid $\Omega\mathcal{D}^{\omega}$. Let $\mathsf{P}_{\mathcal{H}}^{\omega}=\sum_{J\in\mathcal{H}}\triangle_{J}^{\omega}$ be the corresponding ω-quasiHaar projection. We define $\mathcal{H}^{\ast}\equiv\bigcup_{J\in\mathcal{H}}\left\{J^{\prime}\in\Omega\mathcal{D}^{\omega}:J^{\prime}\subset J\right\}$.

The next lemma also holds for quasicubes and common point masses with the same proof as in [SaShUr7].

Lemma 7 (**Energy Lemma**)**.** *Let J be a quasicube in $\Omega\mathcal{D}^{\omega}$. Let Ψ_{J} be an $L^{2}\left(\omega\right)$ function supported in J and with ω-integral zero, and denote its quasiHaar support by $\mathcal{H}=\operatorname{supp}\widehat{\Psi_{J}}\equiv\left\{K\in\Omega\mathcal{D}^{\omega}:\widehat{\Psi_{J}}\left(K\right)\neq 0\right\}$. Let ν be a positive measure supported in $\mathbb{R}^{n}\setminus\gamma J$ with $\gamma\geq 2$, and for each $J^{\prime}\in\mathcal{H}$, let $\nu_{J^{\prime}}=\varphi_{J^{\prime}}\nu$ with $\left|\varphi_{J^{\prime}}\right|\leq 1$. Let T^{α} be a standard α-fractional singular integral operator with $0\leq\alpha<n$. Then with $\delta^{\prime}=\frac{\delta}{2}$ we have*

$$\begin{aligned}\left|\sum_{J^{\prime}\in\mathcal{H}}\left\langle T^{\alpha}\left(\nu_{J^{\prime}}\right),\triangle_{J^{\prime}}^{\omega}\Psi_{J}\right\rangle_{\omega}\right| &\lesssim \left\|\Psi_{J}\right\|_{L^{2}(\omega)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\nu\right)}{\left|J\right|^{\frac{1}{n}}}\right)\left\|\mathsf{P}_{\mathcal{H}}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}\\ &\quad+\left\|\Psi_{J}\right\|_{L^{2}(\omega)}\frac{1}{\gamma^{\delta^{\prime}}}\left(\frac{\mathrm{P}_{1+\delta^{\prime}}^{\alpha}\left(J,\nu\right)}{\left|J\right|^{\frac{1}{n}}}\right)\left\|\mathsf{P}_{\mathcal{H}^{\ast}}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}\\ &\lesssim \left\|\Psi_{J}\right\|_{L^{2}(\omega)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\nu\right)}{\left|J\right|^{\frac{1}{n}}}\right)\left\|\mathsf{P}_{\mathcal{H}^{\ast}}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)},\end{aligned}$$

and in particular the 'pivotal' bound

$$\left|\left\langle T^{\alpha}\left(\nu\right),\Psi_{J}\right\rangle_{\omega}\right|\leq C\left\|\Psi_{J}\right\|_{L^{2}(\omega)}\mathrm{P}^{\alpha}\left(J,\left|\nu\right|\right)\sqrt{\left|J\right|_{\omega}}\ .$$

Remark 4. *The first term on the right side of the energy inequality above is the 'big' Poisson integral P^{α} times the 'small' energy term $\left\|\mathsf{P}_{\mathcal{H}}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}$ that is additive in $\mathcal{H}$, while the second term on the right is the 'small' Poisson integral $\mathrm{P}_{1+\delta^{\prime}}^{\alpha}$ times the 'big' energy term $\left\|\mathsf{P}_{\mathcal{H}^{\ast}}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}$ that is no longer additive in $\mathcal{H}$. The first term presents no problems in subsequent analysis due solely to the additivity of the 'small' energy term. It is the second term that must be handled by special methods. For example, in the Intertwining Proposition below, the interaction of the singular integral occurs with a pair of quasicubes $J\subset I$ at* highly separated *levels, where the goodness of J can exploit the decay $\delta^{\prime}$ in the kernel of the 'small' Poisson integral $\mathrm{P}_{1+\delta^{\prime}}^{\alpha}$ relative to the 'big' Poisson integral P^{α}, and results in a bound directly by the quasienergy condition. On the* *other hand, in the local recursion of M. Lacey at the end of the paper, the separation of levels in the pairs $J \subset I$ can be as* little *as a fixed parameter $\boldsymbol{\rho}$, and here we must first separate the stopping form into two sublinear forms that involve the two estimates respectively. The form corresponding to the smaller Poisson integral $\mathrm{P}^{\alpha}_{1+\delta'}$ is again handled using goodness and the decay δ' in the kernel, while the form corresponding to the larger Poisson integral P^{α} requires the stopping time and recursion argument of M. Lacey.*

6. Preliminaries of NTV type

An important reduction of our theorem is delivered by the following two lemmas, the first of which is due to Nazarov, Treil and Volberg in the case of one dimension (see [NTV4] and [Vol]), and the second of which is a bilinear Carleson embedding. The proofs given there do not extend in standard ways to higher dimensions with common point masses, and we use the quasiweak boundedness property to handle the case of touching quasicubes, and an application of Schur's Lemma to handle the case of separated quasicubes. The first lemma below is Lemmas 8.1 and 8.7 in [LaWi] but with the larger constant $\mathcal{A}_2^{\alpha}$ there in place of the smaller constant A_2^{α} here. We emphasize that only the offset A_2^{α} condition is needed with testing and weak boundedness in these preliminary estimates.

Lemma 8. *Suppose T^{α} is a standard fractional singular integral with $0 \leq \alpha < n$, and that all of the quasicubes $I \in \Omega\mathcal{D}^{\sigma}, J \in \Omega\mathcal{D}^{\omega}$ below are good with goodness parameters ε and $\mathbf{r}$. Fix a positive integer $\boldsymbol{\rho} > \mathbf{r}$. For $f \in L^2(\sigma)$ and $g \in L^2(\omega)$ we have*

$$\sum_{\substack{(I,J)\in\Omega\mathcal{D}^{\sigma}\times\Omega\mathcal{D}^{\omega}\\ 2^{-\boldsymbol{\rho}}\ell(I)\leq\ell(J)\leq 2^{\boldsymbol{\rho}}\ell(I)}} \left|\left\langle T_{\sigma}^{\alpha}\left(\triangle_I^{\sigma} f\right), \triangle_J^{\omega} g\right\rangle_{\omega}\right| \lesssim \left(\mathfrak{T}_{\alpha} + \mathfrak{T}_{\alpha}^{*} + \mathcal{WBP}_{T^{\alpha}} + \sqrt{A_2^{\alpha}}\right) \left\|f\right\|_{L^2(\sigma)} \left\|g\right\|_{L^2(\omega)} \tag{24}$$

and

$$\sum_{\substack{(I,J)\in\Omega\mathcal{D}^{\sigma}\times\Omega\mathcal{D}^{\omega}\\ I\cap J=\emptyset \text{ and } \frac{\ell(J)}{\ell(I)}\notin\left[2^{-\boldsymbol{\rho}},2^{\boldsymbol{\rho}}\right]}} \left|\left\langle T_{\sigma}^{\alpha}\left(\triangle_I^{\sigma} f\right), \triangle_J^{\omega} g\right\rangle_{\omega}\right| \lesssim \sqrt{A_2^{\alpha}} \left\|f\right\|_{L^2(\sigma)} \left\|g\right\|_{L^2(\omega)}, \tag{25}$$

where the implied constants depend only on n, α and T^{α}.

Lemma 9. *Suppose T^{α} is a standard fractional singular integral with $0 \leq \alpha < n$, that all of the quasicubes $I \in \Omega\mathcal{D}^{\sigma}, J \in \Omega\mathcal{D}^{\omega}$ below are good, that $\boldsymbol{\rho} > \mathbf{r}$, that $f \in L^2(\sigma)$ and $g \in L^2(\omega)$, that $\mathcal{F} \subset \Omega\mathcal{D}^{\sigma}$ and $\mathcal{G} \subset \Omega\mathcal{D}^{\omega}$ are σ-Carleson and ω-Carleson collections respectively, i.e.,*

$$\sum_{F'\in\mathcal{F}:\ F'\subset F} \left|F'\right|_{\sigma} \lesssim \left|F\right|_{\sigma}, \qquad F\in\mathcal{F}, \text{ and } \sum_{G'\in\mathcal{G}:\ G'\subset G} \left|G'\right|_{\omega} \lesssim \left|G\right|_{\omega}, \qquad G\in\mathcal{G},$$

that there are numerical sequences $\{\alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ and $\{\beta_{\mathcal{G}}(G)\}_{G\in\mathcal{G}}$ such that

$$\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)^2 \left|F\right|_{\sigma} \leq \left\|f\right\|_{L^2(\sigma)}^2 \text{ and } \sum_{G\in\mathcal{G}} \beta_{\mathcal{G}}(G)^2 \left|G\right|_{\sigma} \leq \left\|g\right\|_{L^2(\sigma)}^2 , \tag{26}$$

and finally that for each pair of quasicubes $(I,J) \in \Omega\mathcal{D}^{\sigma} \times \Omega\mathcal{D}^{\omega}$, there are bounded functions $\beta_{I,J}$ and $\gamma_{I,J}$ supported in $I \setminus 2J$ and $J \setminus 2I$ respectively, satisfying

$$\left\|\beta_{I,J}\right\|_{\infty}, \left\|\gamma_{I,J}\right\|_{\infty} \leq 1.$$

Then

$$\sum_{\substack{(F,J)\in\mathcal{F}\times\Omega\mathcal{D}^{\omega}\\ F\cap J=\emptyset \text{ and } \ell(J)\leq 2^{-\boldsymbol{\rho}}\ell(F)}} \left|\left\langle T_{\sigma}^{\alpha}\left(\beta_{F,J}\mathbf{1}_F\alpha_{\mathcal{F}}(F)\right), \triangle_J^{\omega} g\right\rangle_{\omega}\right| + \sum_{\substack{(I,G)\in\Omega\mathcal{D}^{\sigma}\times\mathcal{G}\\ I\cap G=\emptyset \text{ and } \ell(I)\leq 2^{-\boldsymbol{\rho}}\ell(G)}} \left|\left\langle T_{\sigma}^{\alpha}\left(\triangle_I^{\sigma} f\right), \gamma_{I,G}\mathbf{1}_G\beta_{\mathcal{G}}(G)\right\rangle_{\omega}\right| \tag{27}$$
$$\lesssim \sqrt{A_2^{\alpha}} \left\|f\right\|_{L^2(\sigma)} \left\|g\right\|_{L^2(\omega)}.$$

See [SaShUr6] for complete details of the proofs when common point masses are permitted.

Remark 5. *If $\mathcal{F}$ and $\mathcal{G}$ are σ-Carleson and ω-Carleson collections respectively, and if $\alpha_{\mathcal{F}}(F) = \mathbb{E}_F^{\sigma}|f|$ and $\beta_{\mathcal{G}}(G) = \mathbb{E}_G^{\omega}|g|$, then the 'quasi' orthogonality condition (26) holds (here 'quasi' has a different meaning than quasi), and this special case of Lemma 9 serves as a basic example.*

Remark 6. *Lemmas 8 and 9 differ mainly in that an orthogonal collection of quasiHaar projections is replaced by a 'quasi' orthogonal collection of indicators* $\{\mathbf{1}_F \alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$. *More precisely, the main difference between (25) and (27) is that a quasiHaar projection* $\triangle_I^\sigma f$ *or* $\triangle_J^\omega g$ *has been replaced with a constant multiple of an indicator* $\mathbf{1}_F \alpha_{\mathcal{F}}(F)$ *or* $\mathbf{1}_G \beta_{\mathcal{G}}(G)$, *and in addition, a bounded function is permitted to multiply the indicator of the quasicube having larger sidelength.*

7. Corona Decompositions and splittings

We will use two different corona constructions, namely a Calderón-Zygmund decomposition and an energy decomposition of NTV type, to reduce matters to the stopping form, the main part of which is handled by Lacey's recursion argument. We will then iterate these coronas into a double corona. We first recall our basic setup. For convenience in notation we will sometimes suppress the dependence on α in our nonlinear forms, but will retain it in the operators, Poisson integrals and constants. We will assume that the good/bad quasicube machinery of Nazarov, Treil and Volberg [Vol] is in force here as in [SaShUr7]. Let $\Omega\mathcal{D}^\sigma = \Omega\mathcal{D}^\omega$ be an $(\mathbf{r},\varepsilon)$-good quasigrid on $\mathbb{R}^n$, and let $\{h_I^{\sigma,a}\}_{I\in\Omega\mathcal{D}^\sigma,\ a\in\Gamma_n}$ and $\left\{h_J^{\omega,b}\right\}_{J\in\Omega\mathcal{D}^\omega,\ b\in\Gamma_n}$ be corresponding quasiHaar bases as described above, so that

$$f=\sum_{I\in\Omega\mathcal{D}^\sigma}\triangle_I^\sigma f \text{ and } g=\sum_{J\in\Omega\mathcal{D}^\omega}\triangle_J^\omega g\ ,$$

where the quasiHaar projections $\triangle_I^\sigma f$ and $\triangle_J^\omega g$ vanish if the quasicubes I and J are not good. Recall that we must show the bilinear inequality (17), i.e. $|\mathcal{T}^\alpha(f,g)|\leq\mathfrak{N}_{T^\alpha}\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}$.

We now proceed for the remainder of this section to follow the development in [SaShUr7], pointing out just the highlights, and referring to [SaShUr7] for proofs, when no changes are required by the inclusion of quasicubes and common point masses.

7.1. The Calderón-Zygmund corona.

We now introduce a stopping tree $\mathcal{F}$ for the function $f\in L^2(\sigma)$. Let $\mathcal{F}$ be a collection of Calderón-Zygmund stopping quasicubes for f, and let $\Omega\mathcal{D}^\sigma=\bigcup_{F\in\mathcal{F}}\mathcal{C}_F$ be the associated corona decomposition of the dyadic quasigrid $\Omega\mathcal{D}^\sigma$. See below and also [SaShUr7] for the standard definitions of corona, etc.

For a quasicube $I\in\Omega\mathcal{D}^\sigma$ let $\pi_{\Omega\mathcal{D}^\sigma}I$ be the $\Omega\mathcal{D}^\sigma$-parent of I in the quasigrid $\Omega\mathcal{D}^\sigma$, and let $\pi_{\mathcal{F}}I$ be the smallest member of $\mathcal{F}$ that contains I. For $F,F'\in\mathcal{F}$, we say that F' is an $\mathcal{F}$-child of F if $\pi_{\mathcal{F}}(\pi_{\Omega\mathcal{D}^\sigma}F')=F$ (it could be that $F=\pi_{\Omega\mathcal{D}^\sigma}F'$), and we denote by $\mathfrak{C}_{\mathcal{F}}(F)$ the set of $\mathcal{F}$-children of F. For $F\in\mathcal{F}$, define the projection $\mathsf{P}_{\mathcal{C}_F}^\sigma$ onto the linear span of the quasiHaar functions $\{h_I^{\sigma,a}\}_{I\in\mathcal{C}_F,\ a\in\Gamma_n}$ by

$$\mathsf{P}_{\mathcal{C}_F}^\sigma f=\sum_{I\in\mathcal{C}_F}\triangle_I^\sigma f=\sum_{I\in\mathcal{C}_F,\ a\in\Gamma_n}\langle f,h_I^{\sigma,a}\rangle_\sigma\, h_I^{\sigma,a}.$$

The standard properties of these projections are

$$f=\sum_{F\in\mathcal{F}}\mathsf{P}_{\mathcal{C}_F}^\sigma f,\qquad\int\left(\mathsf{P}_{\mathcal{C}_F}^\sigma f\right)\sigma=0,\qquad\|f\|_{L^2(\sigma)}^2=\sum_{F\in\mathcal{F}}\left\|\mathsf{P}_{\mathcal{C}_F}^\sigma f\right\|_{L^2(\sigma)}^2 .$$

7.2. The energy corona.

We also impose a quasienergy corona decomposition as in [NTV4] and [LaSaUr2].

Definition 9. *Given a quasicube* S_0, *define* $\mathcal{S}(S_0)$ *to be the maximal subquasicubes* $I\subset S_0$ *such that*

$$\sum_{J\in\mathcal{M}_{\boldsymbol{\tau}-\mathrm{deep}}(I)}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{S_0\setminus\gamma J}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\geq C_{\mathrm{energy}}\left[\left(\mathcal{E}_\alpha^{\mathrm{strong}}\right)^2+A_2^\alpha+A_2^{\alpha,\mathrm{punct}}\right]\ |I|_\sigma\ ,\tag{28}$$

where $\mathcal{E}_\alpha^{\mathrm{strong}}$ *is the constant in the strong quasienergy condition defined in Definition 5, and* C_{energy} *is a sufficiently large positive constant depending only on* $\boldsymbol{\tau}\geq\mathbf{r},n$ *and* α. *Then define the* σ*-energy stopping quasicubes of* S_0 *to be the collection*

$$\mathcal{S}=\{S_0\}\cup\bigcup_{n=0}^\infty\mathcal{S}_n$$

where $\mathcal{S}_0=\mathcal{S}(S_0)$ *and* $\mathcal{S}_{n+1}=\bigcup_{S\in\mathcal{S}_n}\mathcal{S}(S)$ *for* $n\geq0$.

From the quasienergy condition in Definition 5 we obtain the σ-Carleson estimate

$$\sum_{S\in\mathcal{S}:\ S\subset I}|S|_{\sigma}\leq 2\,|I|_{\sigma}\,,\qquad I\in\Omega\mathcal{D}^{\sigma}. \tag{29}$$

Finally, we record the reason for introducing quasienergy stopping times. If

$$X_{\alpha}\left(\mathcal{C}_S\right)^2\equiv\sup_{I\in\mathcal{C}_S}\frac{1}{|I|_{\sigma}}\sum_{J\in\mathcal{M}_{\tau-\mathrm{deep}}(I)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{S\setminus\gamma J}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \tag{30}$$

is (the square of) the α-*stopping quasienergy* of the weight pair (σ,ω) with respect to the corona $\mathcal{C}_S$, then we have the *stopping quasienergy bounds*

$$X_{\alpha}\left(\mathcal{C}_S\right)\leq\sqrt{C_{\mathrm{energy}}}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{strong}}\right)^2+A_2^{\alpha}+A_2^{\alpha,\mathrm{punct}}},\qquad S\in\mathcal{S}, \tag{31}$$

where $A_2^{\alpha}+A_2^{\alpha,\mathrm{punct}}$ and the the strong quasienergy constant $\mathcal{E}_{\alpha}^{\mathrm{strong}}$ are controlled by assumption.

7.3. **General stopping data.** It is useful to extend our notion of corona decomposition to more general stopping data. Our general definition of stopping data will use a positive constant $C_0\geq 4$.

Definition 10. *Suppose we are given a positive constant $C_0\geq 4$, a subset $\mathcal{F}$ of the dyadic quasigrid $\Omega\mathcal{D}^{\sigma}$ (called the stopping times), and a corresponding sequence $\alpha_{\mathcal{F}}\equiv\{\alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ of nonnegative numbers $\alpha_{\mathcal{F}}(F)\geq 0$ (called the stopping data). Let $(\mathcal{F},\prec,\pi_{\mathcal{F}})$ be the tree structure on $\mathcal{F}$ inherited from $\Omega\mathcal{D}^{\sigma}$, and for each $F\in\mathcal{F}$ denote by $\mathcal{C}_F=\{I\in\Omega\mathcal{D}^{\sigma}:\pi_{\mathcal{F}}I=F\}$ the corona associated with F:*

$$\mathcal{C}_F=\left\{I\in\Omega\mathcal{D}^{\sigma}:I\subset F\text{ and }I\not\subset F'\text{ for any }F'\prec F\right\}.$$

We say the triple $(C_0,\mathcal{F},\alpha_{\mathcal{F}})$ constitutes stopping data *for a function $f\in L^1_{loc}(\sigma)$ if*

1. $\mathbb{E}_I^{\sigma}|f|\leq\alpha_{\mathcal{F}}(F)$ *for all* $I\in\mathcal{C}_F$ *and* $F\in\mathcal{F}$,
2. $\sum_{F'\preceq F}|F'|_{\sigma}\leq C_0\,|F|_{\sigma}$ *for all* $F\in\mathcal{F}$,
3. $\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)^2\,|F|_{\sigma}\leq C_0^2\,\|f\|_{L^2(\sigma)}^2$,
4. $\alpha_{\mathcal{F}}(F)\leq\alpha_{\mathcal{F}}(F')$ *whenever* $F',F\in\mathcal{F}$ *with* $F'\subset F$.

Definition 11. *If $(C_0,\mathcal{F},\alpha_{\mathcal{F}})$ constitutes (general)* stopping data *for a function $f\in L^1_{loc}(\sigma)$, we refer to the orthogonal decomposition*

$$f=\sum_{F\in\mathcal{F}}\mathsf{P}_{\mathcal{C}_F}^{\sigma}f;\qquad\mathsf{P}_{\mathcal{C}_F}^{\sigma}f\equiv\sum_{I\in\mathcal{C}_F}\triangle_I^{\sigma}f,$$

as the (general) corona decomposition *of f associated with the stopping times $\mathcal{F}$.*

Property (1) says that $\alpha_{\mathcal{F}}(F)$ bounds the quasiaverages of f in the corona $\mathcal{C}_F$, and property (2) says that the quasicubes at the tops of the coronas satisfy a Carleson condition relative to the weight σ. Note that a standard 'maximal quasicube' argument extends the Carleson condition in property (2) to the inequality

$$\sum_{F'\in\mathcal{F}:\ F'\subset A}|F'|_{\sigma}\leq C_0\,|A|_{\sigma}\ \text{ for all open sets }A\subset\mathbb{R}^n. \tag{32}$$

Property (3) is the 'quasi' orthogonality condition that says the sequence of functions $\{\alpha_{\mathcal{F}}(F)\,\mathbf{1}_F\}_{F\in\mathcal{F}}$ is in the vector-valued space $L^2\left(\ell^2;\sigma\right)$, and property (4) says that the control on stopping data is nondecreasing on the stopping tree $\mathcal{F}$. We emphasize that we are *not* assuming in this definition the stronger property that there is $C>1$ such that $\alpha_{\mathcal{F}}(F')>C\alpha_{\mathcal{F}}(F)$ whenever $F',F\in\mathcal{F}$ with $F'\subsetneqq F$. Instead, the properties (2) and (3) substitute for this lack. Of course the stronger property *does* hold for the familiar *Calderón-Zygmund* stopping data determined by the following requirements for $C>1$,

$$\mathbb{E}_{F'}^{\sigma}|f|>C\mathbb{E}_F^{\sigma}|f|\ \text{ whenever }F',F\in\mathcal{F}\text{ with }F'\subsetneqq F,\qquad\mathbb{E}_I^{\sigma}|f|\leq C\mathbb{E}_F^{\sigma}|f|\ \text{ for }I\in\mathcal{C}_F,$$

which are themselves sufficiently strong to automatically force properties (2) and (3) with $\alpha_{\mathcal{F}}(F)=\mathbb{E}_F^{\sigma}|f|$.

We have the following useful consequence of (2) and (3) that says the sequence $\{\alpha_{\mathcal{F}}(F)\,\mathbf{1}_F\}_{F\in\mathcal{F}}$ has a *'quasi' orthogonal* property relative to f with a constant C_0' depending only on C_0:

$$\left\|\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)\,\mathbf{1}_F\right\|_{L^2(\sigma)}^2\leq C_0'\,\|f\|_{L^2(\sigma)}^2\,. \tag{33}$$

We will use a construction that permits *iteration* of general corona decompositions.

Lemma 10. *Suppose that* $(C_0,\mathcal{F},\alpha_{\mathcal{F}})$ *constitutes* stopping data *for a function* $f\in L^1_{loc}(\sigma)$*, and that for each* $F\in\mathcal{F}$*,* $\left(C_0,\mathcal{K}(F),\alpha_{\mathcal{K}(F)}\right)$ *constitutes* stopping data *for the corona projection* $\mathsf{P}^{\sigma}_{\mathcal{C}_F}f$*, where in addition* $F\in\mathcal{K}(F)$*. There is a positive constant* C_1*, depending only on* C_0*, such that if*

$$\begin{aligned} \mathcal{K}^*(F) &\equiv \left\{K\in\mathcal{K}(F)\cap\mathcal{C}_F:\alpha_{\mathcal{K}(F)}(K)\geq\alpha_{\mathcal{F}}(F)\right\} \\ \mathcal{K} &\equiv \bigcup_{F\in\mathcal{F}}\mathcal{K}^*(F)\cup\{F\}, \\ \alpha_{\mathcal{K}}(K) &\equiv \begin{array}{ccc} \alpha_{\mathcal{K}(F)}(K) & \text{for} & K\in\mathcal{K}^*(F)\setminus\{F\} \\ \max\left\{\alpha_{\mathcal{F}}(F),\alpha_{\mathcal{K}(F)}(F)\right\} & \text{for} & K=F \end{array}, \qquad \text{for } F\in\mathcal{F}, \end{aligned}$$

the triple $(C_1,\mathcal{K},\alpha_{\mathcal{K}})$ *constitutes* stopping data *for* f*. We refer to the collection of quasicubes* $\mathcal{K}$ *as the* iterated *stopping times, and to the orthogonal decomposition* $f=\sum_{K\in\mathcal{K}}P_{\mathcal{C}^{\mathcal{K}}_K}f$ *as the* iterated *corona decomposition of* f*, where*

$$\mathcal{C}^{\mathcal{K}}_K\equiv\{I\in\Omega\mathcal{D}:I\subset K\text{ and }I\not\subset K'\text{ for }K'\prec_{\mathcal{K}}K\}.$$

Note that in our definition of $(C_1,\mathcal{K},\alpha_{\mathcal{K}})$ we have 'discarded' from $\mathcal{K}(F)$ all of those $K\in\mathcal{K}(F)$ that are not in the corona $\mathcal{C}_F$, and also all of those $K\in\mathcal{K}(F)$ for which $\alpha_{\mathcal{K}(F)}(K)$ is strictly less than $\alpha_{\mathcal{F}}(F)$. Then the union over F of what remains is our new collection of stopping times. We then define stopping data $\alpha_{\mathcal{K}}(K)$ according to whether or not $K\in\mathcal{F}$: if $K\notin\mathcal{F}$ but $K\in\mathcal{C}_F$ then $\alpha_{\mathcal{K}}(K)$ equals $\alpha_{\mathcal{K}(F)}(K)$, while if $K\in\mathcal{F}$, then $\alpha_{\mathcal{K}}(K)$ is the larger of $\alpha_{\mathcal{K}(F)}(F)$ and $\alpha_{\mathcal{F}}(K)$. See [SaShUr7] for a proof.

7.4. **Doubly iterated coronas and the NTV quasicube size splitting.** Let

$$\mathcal{NTV}_\alpha\equiv\sqrt{\mathcal{A}^{\alpha}_2+\mathcal{A}^{\alpha,*}_2+A^{\alpha,\mathrm{punct}}_2+A^{\alpha,*,\mathrm{punct}}_2}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}^*_{T^\alpha}\ .$$

Here is a brief schematic diagram of the decompositions, with bounds in $\square$:

$$\boxed{\begin{array}{ccccccc}
\langle T^\alpha_\sigma f,g\rangle_\omega & & & & & & \\
\downarrow & & & & & & \\
\mathsf{B}_{\Subset_{\boldsymbol{\rho}}}(f,g) & + & \mathsf{B}_{\boldsymbol{\rho}\Supset}(f,g) & + & \mathsf{B}_{\cap}(f,g) & + & \mathsf{B}_{\diagup}(f,g) \\
\downarrow & & \boxed{\text{dual}} & & \boxed{\mathcal{NTV}_\alpha} & & \boxed{\mathcal{NTV}_\alpha} \\
\downarrow & & & & & & \\
\mathsf{T}_{\text{diagonal}}(f,g) & + & \mathsf{T}_{\text{far below}}(f,g) & + & \mathsf{T}_{\text{far above}}(f,g) & + & \mathsf{T}_{\text{disjoint}}(f,g) \\
\downarrow & & \downarrow & & \boxed{\emptyset} & & \boxed{\emptyset} \\
\downarrow & & \downarrow & & & & \\
\mathsf{B}^{A}_{\Subset_{\boldsymbol{\rho}}}(f,g) & & \mathsf{T}^1_{\text{far below}}(f,g) & + & \mathsf{T}^2_{\text{far below}}(f,g) & & \\
\downarrow & & \boxed{\mathcal{NTV}_\alpha+\mathcal{E}^{\text{strong}}_\alpha} & & \boxed{\mathcal{NTV}_\alpha} & & \\
\downarrow & & & & & & \\
\mathsf{B}^{A}_{stop}(f,g) & + & \mathsf{B}^{A}_{paraproduct}(f,g) & + & \mathsf{B}^{A}_{neighbour}(f,g) & & \\
\boxed{\mathcal{E}^{\text{strong}}_\alpha+\sqrt{A^\alpha_2}} & & \boxed{\mathfrak{T}_{T^\alpha}} & & \boxed{\sqrt{A^\alpha_2}} & &
\end{array}}$$

We begin with the NTV *quasicube size splitting* of the inner product $\langle T^\alpha_\sigma f,g\rangle_\omega$ - and later apply the iterated corona construction to the Calderón–Zygmund corona and the energy corona in order to bound the below form $\mathsf{B}_{\Subset_{\boldsymbol{\rho}}}(f,g)$ - that splits the pairs of quasicubes (I,J) in a simultaneous quasiHaar decomposition of f and g into four groups, namely those pairs that:

(1) are below the size diagonal and $\boldsymbol{\rho}$-deeply embedded,
(2) are above the size diagonal and $\boldsymbol{\rho}$-deeply embedded,
(3) are disjoint, and
(4) are of $\boldsymbol{\rho}$-comparable size.

More precisely we have

$$\begin{aligned}\langle T_\sigma^\alpha f,g\rangle_\omega &= \sum_{I\in\Omega\mathcal{D}^\sigma,\ J\in\Omega\mathcal{D}^\omega} \langle T_\sigma^\alpha(\triangle_I^\sigma f),(\triangle_I^\omega g)\rangle_\omega \\ &= \sum_{\substack{I\in\Omega\mathcal{D}^\sigma,\ J\in\Omega\mathcal{D}^\omega\\ J\Subset_\rho I}} \langle T_\sigma^\alpha(\triangle_I^\sigma f),(\triangle_J^\omega g)\rangle_\omega + \sum_{\substack{I\in\Omega\mathcal{D}^\sigma,\ J\in\Omega\mathcal{D}^\omega\\ J_\rho\Supset I}} \langle T_\sigma^\alpha(\triangle_I^\sigma f),(\triangle_J^\omega g)\rangle_\omega \\ &\quad+ \sum_{\substack{I\in\Omega\mathcal{D}^\sigma,\ J\in\Omega\mathcal{D}^\omega\\ J\cap I=\emptyset}} \langle T_\sigma^\alpha(\triangle_I^\sigma f),(\triangle_J^\omega g)\rangle_\omega + \sum_{\substack{I\in\Omega\mathcal{D}^\sigma,\ J\in\Omega\mathcal{D}^\omega\\ 2^{-\rho}\le\ell(J)/\ell(I)\le 2^\rho}} \langle T_\sigma^\alpha(\triangle_I^\sigma f),(\triangle_J^\omega g)\rangle_\omega \\ &= \mathsf{B}_{\Subset_\rho}(f,g)+\mathsf{B}_{_\rho\Supset}(f,g)+\mathsf{B}_\cap(f,g)+\mathsf{B}_{/}(f,g)\,.\end{aligned}$$

Lemma 8 in the section on NTV preliminaries show that the *disjoint* and *comparable* forms $\mathsf{B}_\cap(f,g)$ and $\mathsf{B}_{/}(f,g)$ are both bounded by the $\mathcal{A}_2^\alpha + A_2^{\alpha,\mathrm{punct}}$, quasitesting and quasiweak boundedness property constants. The *below* and *above* forms are clearly symmetric, so we need only consider the form $\mathsf{B}_{\Subset_\rho}(f,g)$, to which we turn for the remainder of the proof. For this we need functional energy.

Definition 12. *Let $\mathfrak{F}_\alpha$ be the smallest constant in the 'functional quasienergy' inequality below, holding for all $h\in L^2(\sigma)$ and all σ-Carleson collections $\mathcal{F}$ with Carleson norm $C_\mathcal{F}$ bounded by a fixed constant C:*

$$\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)} \left(\frac{\mathrm{P}^\alpha(J,h\sigma)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^\omega_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}};J}\mathbf{x}\right\|^2_{L^2(\omega)} \le \mathfrak{F}_\alpha\|h\|_{L^2(\sigma)}\,. \tag{34}$$

Several ingredients now come into play in order to reduce control of the below form $\mathsf{B}_{\Subset_\rho}(f,g)$ to the functional energy constant $\mathcal{F}_\alpha$ and the stopping form $\mathsf{B}^A_{stop}(f,g)$;

1. (1) starting with the doubly iterated corona of Calderón-Zygmund and energy in Lemma 10 in order to obtain the decomposition into $\mathsf{T}_{\mathrm{diagonal}}$, $\mathsf{T}_{\text{far below}}$, $\mathsf{T}_{\text{far above}}$ and $\mathsf{T}_{\mathrm{disjoint}}$,
2. (2) continuing with an adaptation of the Intertwining Proposition from [SaShUr7] to include quasicubes and common point masses so as to bound the forms $\mathsf{T}^1_{\text{far below}}$ and $\mathsf{T}^2_{\text{far below}}(f,g)$ using the functional energy constant $\mathcal{F}_\alpha$,
3. (3) and followed by the NTV decomposition into paraproduct, neighbour and stopping forms.

The adaptation of the Intertwining Proposition to include quasicubes and common point masses is easy because the measures ω and σ only 'see each other' in the proof through the energy Muckenhoupt conditions $A_2^{\alpha,\mathrm{energy}}$ and $A_2^{\alpha,*,\mathrm{energy}}$, and the straightforward details can be found in [SaShUr6]. Thus we now turn to the difficult task of controlling the functional energy constant $\mathcal{F}_\alpha$ by the Muckenhoupt and energy side conditions.

8. Control of functional energy by energy modulo $\mathcal{A}_2^\alpha$ and $A_2^{\alpha,\mathrm{punct}}$

Now we arrive at one of our main propositions in the proof of our theorem. We show that the functional quasienergy constants $\mathfrak{F}_\alpha$ as in (34) are controlled by $\mathcal{A}_2^\alpha$, $A_2^{\alpha,\mathrm{punct}}$ and both the *strong* quasienergy constant $\mathcal{E}_\alpha^{\mathrm{strong}}$ defined in Definition 5. The proof of this fact is further complicated when common point masses are permitted, accounting for the inclusion of the punctured Muckenhoupt condition $A_2^{\alpha,\mathrm{punct}}$. But apart from this difference, the proof here is essentially the same as that in [SaShUr7], where common point masses were prohibited. As a consequence we will refer to [SaShUr7] in many of the places where the arguments are unchanged. A complete and detailed proof can of course be found in [SaShUr6].

Proposition 1. *We have*

$$\mathfrak{F}_\alpha \lesssim \mathcal{E}_\alpha^{\mathrm{strong}} + \sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,*}} + \sqrt{A_2^{\alpha,\mathrm{punct}}} \text{ and } \mathfrak{F}_\alpha^* \lesssim \mathcal{E}_\alpha^{\mathrm{strong},*} + \sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,*}} + \sqrt{A_2^{\alpha,*,\mathrm{punct}}}\,.$$

To prove this proposition, we fix $\mathcal{F}$ as in (34), and set

$$\mu \equiv \sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)} \left\|\mathsf{P}^\omega_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)}\cdot\delta_{(c_J,\ell(J))} \text{ and } d\overline{\mu}(x,t)\equiv\frac{1}{t^2}d\mu(x,t)\ , \tag{35}$$

where $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)$ consists of the maximal $\mathbf{r}$-deeply embedded subquasicubes of F, and where $\delta_{(c_J,\ell(J))}$ denotes the Dirac unit mass at the point $(c_J,\ell(J))$ in the upper half-space $\mathbb{R}^{n+1}_+$. Here J is a dyadic quasicube with center c_J and side length $\ell(J)$. For convenience in notation, we denote for any dyadic quasicube J the localized projection $\mathsf{P}^{\omega}_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}};J}$ given by

$$\mathsf{P}^{\omega}_{F,J} \equiv \mathsf{P}^{\omega}_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}};J} = \sum_{J'\subset J:\ J'\in\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}} \triangle^{\omega}_{J'}.$$

We emphasize that the quasicubes $J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)$ are not necessarily good, but that the subquasicubes $J' \subset J$ arising in the projection $\mathsf{P}^{\omega}_{F,J}$ are good. We can replace $\mathbf{x}$ by $\mathbf{x}-\mathbf{c}$ inside the projection for any choice of $\mathbf{c}$ we wish; the projection is unchanged. More generally, δ_q denotes a Dirac unit mass at a point q in the upper half-space $\mathbb{R}^{n+1}_+$.

We prove the two-weight inequality

$$\left\|\mathbb{P}^{\alpha}(f\sigma)\right\|_{L^2(\mathbb{R}^{n+1}_+,\overline{\mu})} \lesssim \left(\mathcal{E}_{\alpha}^{\mathrm{strong}} + \sqrt{\mathcal{A}_2^{\alpha}} + \sqrt{\mathcal{A}_2^{\alpha,*}} + \sqrt{A_2^{\alpha,\mathrm{punct}}}\right)\|f\|_{L^2(\sigma)}, \tag{36}$$

for all nonnegative f in $L^2(\sigma)$, noting that $\mathcal{F}$ and f are *not* related here. Above, $\mathbb{P}^{\alpha}(\cdot)$ denotes the α-fractional Poisson extension to the upper half-space $\mathbb{R}^{n+1}_+$,

$$\mathbb{P}^{\alpha}\nu(x,t) \equiv \int_{\mathbb{R}^n} \frac{t}{\left(t^2+|x-y|^2\right)^{\frac{n+1-\alpha}{2}}}d\nu(y),$$

so that in particular

$$\|\mathbb{P}^{\alpha}(f\sigma)\|^2_{L^2(\mathbb{R}^{n+1}_+,\overline{\mu})} = \sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(F)} \mathbb{P}^{\alpha}(f\sigma)(c(J),\ell(J))^2 \left\|\mathsf{P}^{\omega}_{F,J}\frac{x}{|J|^{\frac{1}{n}}}\right\|^2_{L^2(\omega)},$$

and so (36) proves the first line in Proposition 1 upon inspecting (34). Note also that we can equivalently write $\|\mathbb{P}^{\alpha}(f\sigma)\|_{L^2(\mathbb{R}^{n+1}_+,\overline{\mu})} = \left\|\widetilde{\mathbb{P}}^{\alpha}(f\sigma)\right\|_{L^2(\mathbb{R}^{n+1}_+,\mu)}$ where $\widetilde{\mathbb{P}}^{\alpha}\nu(x,t) \equiv \frac{1}{t}\mathbb{P}^{\alpha}\nu(x,t)$ is the renormalized Poisson operator. Here we have simply shifted the factor $\frac{1}{t^2}$ in $\overline{\mu}$ to $\left|\widetilde{\mathbb{P}}^{\alpha}(f\sigma)\right|^2$ instead, and we will do this shifting often throughout the proof when it is convenient to do so.

The characterization of the two-weight inequality for fractional and Poisson integrals in [Saw] was stated in terms of the collection $\mathcal{P}^n$ of cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. It is a routine matter to pullback the Poisson inequality under a globally biLipschitz map $\Omega:\mathbb{R}^n\to\mathbb{R}^n$, then apply the theorem in [Saw] (as a black box), and then to pushforward the conclusions of the theorems so as to extend these characterizations of fractional and Poisson integral inequalities to the setting of quasicubes $Q\in\Omega\mathcal{P}^n$ and quasitents $Q\times[0,\ell(Q)]\subset\mathbb{R}^{n+1}_+$ with $Q\in\Omega\mathcal{P}^n$. Using this extended theorem for the two-weight Poisson inequality, we see that inequality (36) requires checking these two inequalities for dyadic quasicubes $I\in\Omega\mathcal{D}$ and quasiboxes $\widehat{I}=I\times[0,\ell(I))$ in the upper half-space $\mathbb{R}^{n+1}_+$:

$$\int_{\mathbb{R}^{n+1}_+}\mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)(x,t)^2\,d\overline{\mu}(x,t) \equiv \|\mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)\|^2_{L^2(\widehat{I},\overline{\mu})} \lesssim \left(\left(\mathcal{E}_{\alpha}^{\mathrm{strong}}\right)^2 + \mathcal{A}_2^{\alpha} + \mathcal{A}_2^{\alpha,*} + A_2^{\alpha,\mathrm{punct}}\right)\sigma(I), \tag{37}$$

$$\int_{\mathbb{R}^n}[\mathbb{Q}^{\alpha}(t\mathbf{1}_{\widehat{I}}\overline{\mu})]^2d\sigma(x) \lesssim \left(\left(\mathcal{E}_{\alpha}^{\mathrm{strong}}\right)^2 + \mathcal{A}_2^{\alpha} + \mathcal{A}_2^{\alpha,*} + A_2^{\alpha,\mathrm{punct}}\right)\int_{\widehat{I}}t^2d\overline{\mu}(x,t), \tag{38}$$

for all *dyadic* quasicubes $I\in\Omega\mathcal{D}$, and where the dual Poisson operator $\mathbb{Q}^{\alpha}$ is given by

$$\mathbb{Q}^{\alpha}(t\mathbf{1}_{\widehat{I}}\overline{\mu})(x) = \int_{\widehat{I}}\frac{t^2}{(t^2+|x-y|^2)^{\frac{n+1-\alpha}{2}}}d\overline{\mu}(y,t).$$

It is important to note that we can choose for $\Omega\mathcal{D}$ any fixed dyadic quasigrid, the compensating point being that the integrations on the left sides of (37) and (38) are taken over the entire spaces $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^n$ respectively.

Remark 7. *There is a gap in the proof of the Poisson inequality at the top of page 542 in* [Saw]*. However, this gap can be fixed as in* [SaWh, p. 861]*.*

8.1. **Poisson testing.** We now turn to proving the Poisson testing conditions (37) and (38). The same testing conditions have been considered in [SaShUr5] but in the setting of no common point masses, and the proofs there carry over to the situation here, but careful attention must now be paid to the possibility of common point masses. In [Hyt2] Hytönen circumvented this difficulty by introducing a Poisson operator 'with holes', which was then analyzed using shifted dyadic grids, but part of his argument was heavily dependent on the dimension being $n=1$, and the extension of this argument to higher dimensions is feasible (see earlier versions of this paper on the *arXiv*), but technically very involved. We circumvent the difficulty of permitting common point masses here instead by using the energy Muckenhoupt constants $A_2^{\alpha,\mathrm{energy}}$ and $A_2^{\alpha,*,\mathrm{energy}}$, which require control by the punctured Muckenhoupt constants $A_2^{\alpha,\mathrm{punct}}$ and $A_2^{\alpha,*,\mathrm{punct}}$. The following elementary Poisson inequalities (see e.g. [Vol]) will be used extensively.

Lemma 11. *Suppose that J,K,I are quasicubes in $\mathbb{R}^n$, and that μ is a positive measure supported in $\mathbb{R}^n\setminus I$. If $J\subset K\subset 2K\subset I$, then*

$$\frac{\mathrm{P}^\alpha\left(J,\mu\right)}{\left|J\right|^{\frac{1}{n}}}\lesssim\frac{\mathrm{P}^\alpha\left(K,\mu\right)}{\left|K\right|^{\frac{1}{n}}}\lesssim\frac{\mathrm{P}^\alpha\left(J,\mu\right)}{\left|J\right|^{\frac{1}{n}}},$$

while if $2J\subset K\subset I$, then

$$\frac{\mathrm{P}^\alpha\left(K,\mu\right)}{\left|K\right|^{\frac{1}{n}}}\lesssim\frac{\mathrm{P}^\alpha\left(J,\mu\right)}{\left|J\right|^{\frac{1}{n}}}.$$

Now we record the bounded overlap of the projections $\mathsf{P}_{F,J}^\omega$.

Lemma 12. *Suppose $\mathsf{P}_{F,J}^\omega$ is as above and fix any $I_0\in\Omega\mathcal{D}$, so that I_0, F and J all lie in a common quasigrid. If $J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(F\right)$ for some $F\in\mathcal{F}$ with $F\supsetneqq I_0\supset J$ and $\mathsf{P}_{F,J}^\omega\neq0$, then*

$$F=\pi_{\mathcal{F}}^{(\ell)}I_0\text{ for some }0\leq\ell\leq\boldsymbol{\tau}.$$

As a consequence we have the bounded overlap,

$$\#\left\{F\in\mathcal{F}:J\subset I_0\subsetneqq F\text{ for some }J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(F\right)\text{ with }\mathsf{P}_{F,J}^\omega\neq0\right\}\leq\boldsymbol{\tau}.$$

Finally we record the only places in the proof where the *refined* quasienergy conditions are used. This lemma will be used in bounding both of the local Poisson testing conditions. Recall that $\mathcal{A}\Omega\mathcal{D}$ consists of all alternate $\Omega\mathcal{D}$-dyadic quasicubes where K is alternate dyadic if it is a union of 2^n $\Omega\mathcal{D}$-dyadic quasicubes K' with $\ell\left(K'\right)=\frac{1}{2}\ell\left(K\right)$. See [SaShUr7] for a proof when common point masses are prohibited, and the presence of common point masses here requires no change.

Remark 8. *The following lemma is another of the key results on the way to the proof of our theorem, and is an analogue of the corresponding lemma from* [SaShUr5]*, but with the right hand side involving only the plugged energy constants and the energy Muckenhoupt constants.*

Lemma 13. *Let $\Omega\mathcal{D},\mathcal{F}\subset\Omega\mathcal{D}$ be quasigrids and $\left\{\mathsf{P}_{F,J}^\omega\right\}_{\substack{F\in\mathcal{F}\\J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}}$ be as above with J,F in the dyadic quasigrid $\Omega\mathcal{D}$. For any alternate quasicube $I\in\mathcal{A}\Omega\mathcal{D}$ define*

$$B\left(I\right)\equiv\sum_{F\in\mathcal{F}:\ F\supsetneqq I'\text{ for some }I'\in\mathfrak{C}(I)}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_I\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_{F,J}^\omega\mathbf{x}\right\|_{L^2(\omega)}^2\ .\tag{39}$$

Then

$$B\left(I\right)\lesssim\boldsymbol{\tau}\left(\left(\mathcal{E}_\alpha^{\mathrm{strong}}\right)^2+A_2^{\alpha,\mathrm{energy}}\right)\left|I\right|_\sigma\ .\tag{40}$$

8.2. **The forward Poisson testing inequality.** Fix $I\in\Omega\mathcal{D}$. We split the integration on the left side of (37) into a local and global piece:

$$\int_{\mathbb{R}_+^{n+1}}\mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)^2d\overline{\mu}=\int_{\widehat{I}}\mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)^2d\overline{\mu}+\int_{\mathbb{R}_+^{n+1}\setminus\widehat{I}}\mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)^2d\overline{\mu}\equiv\mathbf{Local}\left(I\right)+\mathbf{Global}\left(I\right),$$

where more explicitly,

$$\mathbf{Local}\left(I\right)\equiv\int_{\widehat{I}}\left[\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(x,t\right)\right]^{2}d\overline{\mu}\left(x,t\right);\qquad\overline{\mu}\equiv\frac{1}{t^{2}}\mu,\tag{41}$$

$$\text{i.e. }\overline{\mu}\equiv\sum_{J\in\Omega\mathcal{D}}\frac{1}{\ell\left(J\right)^{2}}\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\left(\mathbf{r},\varepsilon\right)-\mathrm{deep}}\left(F\right)}\left\Vert\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\Vert_{L^{2}\left(\omega\right)}^{2}\cdot\delta_{\left(c_{J},\ell\left(J\right)\right)}.$$

Here is a brief schematic diagram of the decompositions, with bounds in $\boxed{}$, used in this subsection:

$$\boxed{\begin{array}{ccc}\mathbf{Local}\left(I\right) & & \\ \downarrow & & \\ \mathbf{Local}^{\mathrm{plug}}\left(I\right) & + & \mathbf{Local}^{\mathrm{hole}}\left(I\right)\\ & & \boxed{\left(\mathcal{E}_{\alpha}^{\mathrm{strong}}\right)^{2}}\\ \downarrow & & \\ \downarrow & & \\ A & + & B\\ \boxed{\left(\mathcal{E}_{\alpha}^{\mathrm{strong}}\right)^{2}} & & \boxed{\left(\mathcal{E}_{\alpha}^{\mathrm{strong}}\right)^{2}+A_{2}^{\alpha,\mathrm{energy}}}\end{array}}$$

and

$$\boxed{\begin{array}{ccccccc}\mathbf{Global}\left(I\right) & & & & & & \\ \downarrow & & & & & & \\ A & + & B & + & C & + & D\\ \boxed{A_{2}^{\alpha}} & & \boxed{A_{2}^{\alpha}+A_{2}^{\alpha,\mathrm{energy}}} & & \boxed{\mathcal{A}_{2}^{\alpha,*}} & & \boxed{\mathcal{A}_{2}^{\alpha,*}+A_{2}^{\alpha,\mathrm{energy}}+A_{2}^{\alpha,\mathrm{punct}}}\end{array}}.$$

An important consequence of the fact that I and J lie in the same quasigrid $\Omega\mathcal{D}=\Omega\mathcal{D}^{\omega}$, is that

$$\left(c\left(J\right),\ell\left(J\right)\right)\in\widehat{I}\textbf{ if and only if }J\subset I.\tag{42}$$

We thus have

$$\begin{aligned}\mathbf{Local}\left(I\right)&=\int_{\widehat{I}}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(x,t\right)^{2}d\overline{\mu}\left(x,t\right)\\ &=\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}\left(F\right):\ J\subset I}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(c_{J},\left|J\right|^{\frac{1}{n}}\right)^{2}\left\Vert\mathsf{P}_{F,J}^{\omega}\frac{\mathbf{x}}{\left|J\right|^{\frac{1}{n}}}\right\Vert_{L^{2}\left(\omega\right)}^{2}\\ &\approx\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}\left(F\right):\ J\subset I}\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I}\sigma\right)^{2}\left\Vert\mathsf{P}_{F,J}^{\omega}\frac{\mathbf{x}}{\left|J\right|^{\frac{1}{n}}}\right\Vert_{L^{2}\left(\omega\right)}^{2}\\ &\lesssim\mathbf{Local}^{\mathrm{plug}}\left(I\right)+\mathbf{Local}^{\mathrm{hole}}\left(I\right),\end{aligned}$$

where the 'plugged' local sum $\mathbf{Local}^{\mathrm{plug}}\left(I\right)$ is given by

$$\begin{aligned}\mathbf{Local}^{\mathrm{plug}}\left(I\right)&\equiv\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}\left(F\right):\ J\subset I}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{F\cap I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\Vert\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\Vert_{L^{2}\left(\omega\right)}^{2}\\ &=\left\{\sum_{F\in\mathcal{F}:\ F\subset I}+\sum_{F\in\mathcal{F}:\ F\supsetneqq I}\right\}\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}\left(F\right):\ J\subset I}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{F\cap I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\Vert\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\Vert_{L^{2}\left(\omega\right)}^{2}\\ &=A+B.\end{aligned}$$

Then a *trivial* application of the deep quasienergy condition (where 'trivial' means that the outer decomposition is just a single quasicube) gives

$$\begin{aligned}A&\leq\sum_{F\in\mathcal{F}:\ F\subset I}\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}\left(F\right)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{F}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\Vert\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\Vert_{L^{2}\left(\omega\right)}^{2}\\ &\leq\sum_{F\in\mathcal{F}:\ F\subset I}\left(\mathcal{E}_{\alpha}^{\mathrm{strong}}\right)^{2}\left|F\right|_{\sigma}\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{strong}}\right)^{2}\left|I\right|_{\sigma}\ ,\end{aligned}$$

since $\left\|\mathsf{P}_{F,J}^{\omega}x\right\|_{L^2(\omega)}^2 \leq \left\|\mathsf{P}_J^{\mathrm{good},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2$, where we recall that the quasienergy constant $\mathcal{E}_\alpha^{\mathrm{strong}}$ is defined in Definition 5. We also used that the stopping quasicubes $\mathcal{F}$ satisfy a σ-Carleson measure estimate,

$$\sum_{F\in\mathcal{F}:\ F\subset F_0} |F|_\sigma \lesssim |F_0|_\sigma .$$

Lemma 13 applies to the remaining term B to obtain the bound

$$B \lesssim \boldsymbol{\tau}\left(\left(\mathcal{E}_\alpha^{\mathrm{strong}}\right)^2 + A_2^{\alpha,\mathrm{energy}}\right)|I|_\sigma \ .$$

It remains then to show the inequality with 'holes', where the support of σ is restricted to the complement of the quasicube F. Thus for $J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)$ we may use $I\setminus F$ in the argument of the Poisson integral. We consider

$$\mathbf{Local}^{\mathrm{hole}}(I) = \sum_{F\in\mathcal{F}}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus F}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \ .$$

Lemma 14. *We have*

$$\mathbf{Local}^{\mathrm{hole}}(I) \lesssim \left(\mathcal{E}_\alpha^{\mathrm{strong}}\right)^2 |I|_\sigma \ . \tag{43}$$

Details are left to the reader,or see [SaShUr7] or [SaShUr6] for a proof. This completes the proof of

(44)

$$\begin{aligned}\mathbf{Local}(L) &\approx \sum_{F\in\mathcal{F}}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset L}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{L}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\lesssim \left(\left(\mathcal{E}_\alpha^{\mathrm{strong}}\right)^2 + A_2^{\alpha,\mathrm{energy}}\right)|L|_\sigma, \quad L\in\Omega\mathcal{D}.\end{aligned}$$

8.2.1. *The alternate local estimate.* For future use, we prove a strengthening of the local estimate $\mathbf{Local}(L)$ to *alternate* quasicubes $M\in\mathcal{A}\Omega\mathcal{D}$.

Lemma 15. *With notation as above and $M\in\mathcal{A}\Omega\mathcal{D}$ an alternate quasicube, we have*

(45)

$$\begin{aligned}\mathbf{Local}(M) &\equiv \sum_{F\in\mathcal{F}}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset M}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{M}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\lesssim \left(\left(\mathcal{E}_\alpha^{\mathrm{strong}}\right)^2 + A_2^{\alpha,\mathrm{energy}}\right)|M|_\sigma, \quad M\in\mathcal{A}\Omega\mathcal{D}.\end{aligned}$$

Again details are left to the reader, or see [SaShUr7] or [SaShUr6] for a proof.

8.2.2. *The global estimate.* Now we turn to proving the following estimate for the global part of the first testing condition (37):

$$\mathbf{Global}(I) = \int_{\mathbb{R}_+^{n+1}\setminus\widehat{I}} \mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)^2 d\overline{\mu} \lesssim \mathcal{A}_2^{\alpha,*}|I|_\sigma .$$

We begin by decomposing the integral on the right into four pieces. As a particular consequence of Lemma 12, we note that given J, there are at most a fixed number $\boldsymbol{\tau}$ of $F\in\mathcal{F}$ such that $J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(F)$. We have:

$$\begin{aligned}\int_{\mathbb{R}_+^{n+1}\setminus\widehat{I}} \mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)^2 d\mu &\leq \sum_{J:\ (c_J,\ell(J))\in\mathbb{R}_+^{n+1}\setminus\widehat{I}} \mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)\left(c_J,\ell(J)\right)^2 \sum_{\substack{F\in\mathcal{F}\\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}}\left\|\mathsf{P}_{F,J}^{\omega}\frac{\mathbf{x}}{|J|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2 \\ &= \left\{\sum_{\substack{J\cap 3I=\emptyset\\ \ell(J)\leq\ell(I)}} + \sum_{J\subset 3I\setminus I} + \sum_{\substack{J\cap I=\emptyset\\ \ell(J)>\ell(I)}} + \sum_{J\supsetneqq I}\right\}\mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)\left(c_J,\ell(J)\right)^2 \sum_{\substack{F\in\mathcal{F}:\\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}}\left\|\mathsf{P}_{F,J}^{\omega}\frac{\mathbf{x}}{|J|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2 \\ &= A+B+C+D.\end{aligned}$$

Terms A, B and C are handled almost the same as in [SaShUr7], and we leave them for the reader. As always complete details are in [SaShUr6].

Finally, we turn to term D which is significantly different due to the presence of common point masses, more precisely a new *'preparation to puncture'* argument arises which is explained in detail below. The quasicubes J occurring here are included in the set of ancestors $A_k \equiv \pi_{\Omega\mathcal{D}}^{(k)} I$ of I, $1 \leq k < \infty$.

$$\begin{aligned}
D &= \sum_{k=1}^{\infty} \mathbb{P}^{\alpha}\left(\mathbf{1}_I \sigma\right)\left(c\left(A_k\right), \left|A_k\right|^{\frac{1}{n}}\right)^2 \sum_{\substack{F \in \mathcal{F}: \\ A_k \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \left\| \mathsf{P}_{F,A_k}^{\omega} \frac{\mathbf{x}}{\left|A_k\right|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\
&= \sum_{k=1}^{\infty} \mathbb{P}^{\alpha}\left(\mathbf{1}_I \sigma\right)\left(c\left(A_k\right), \left|A_k\right|^{\frac{1}{n}}\right)^2 \sum_{\substack{F \in \mathcal{F}: \\ A_k \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \sum_{J' \in \mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}:\ J' \subset A_k \setminus I} \left\| \triangle_{J'}^{\omega} \frac{\mathbf{x}}{\left|A_k\right|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\
&\quad + \sum_{k=1}^{\infty} \mathbb{P}^{\alpha}\left(\mathbf{1}_I \sigma\right)\left(c\left(A_k\right), \left|A_k\right|^{\frac{1}{n}}\right)^2 \sum_{\substack{F \in \mathcal{F}: \\ A_k \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \sum_{J' \in \mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}:\ J' \subset I} \left\| \triangle_{J'}^{\omega} \frac{\mathbf{x}}{\left|A_k\right|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\
&\quad + \sum_{k=1}^{\infty} \mathbb{P}^{\alpha}\left(\mathbf{1}_I \sigma\right)\left(c\left(A_k\right), \left|A_k\right|^{\frac{1}{n}}\right)^2 \sum_{\substack{F \in \mathcal{F}: \\ A_k \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \sum_{J' \in \mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}:\ I \subsetneqq J' \subset A_k} \left\| \triangle_{J'}^{\omega} \frac{\mathbf{x}}{\left|A_k\right|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\
&\equiv D_{\mathrm{disjoint}} + D_{\mathrm{descendent}} + D_{\mathrm{ancestor}} .
\end{aligned}$$

We thus have from Lemma 12 again,

$$\begin{aligned}
D_{\mathrm{disjoint}} &= \sum_{k=1}^{\infty} \mathbb{P}^{\alpha}\left(\mathbf{1}_I \sigma\right)\left(c\left(A_k\right), \left|A_k\right|^{\frac{1}{n}}\right)^2 \\
&\qquad \times \sum_{\substack{F \in \mathcal{F}: \\ A_k \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \sum_{J' \in \mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}:\ J' \subset A_k \setminus I} \left\| \triangle_{J'}^{\omega} \frac{\mathbf{x}}{\left|A_k\right|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\
&\lesssim \sum_{k=1}^{\infty} \left(\frac{\left|I\right|_{\sigma} \left|A_k\right|^{\frac{1}{n}}}{\left|A_k\right|^{1+\frac{1-\alpha}{n}}} \right)^2 \boldsymbol{\tau}\ \left|A_k \setminus I\right|_{\omega} = \boldsymbol{\tau} \left\{ \frac{\left|I\right|_{\sigma}}{\left|I\right|^{1-\frac{\alpha}{n}}} \sum_{k=1}^{\infty} \frac{\left|I\right|^{1-\frac{\alpha}{n}}}{\left|A_k\right|^{2\left(1-\frac{\alpha}{n}\right)}} \left|A_k \setminus I\right|_{\omega} \right\} \left|I\right|_{\sigma} \\
&\lesssim \boldsymbol{\tau} \left\{ \frac{\left|I\right|_{\sigma}}{\left|I\right|^{1-\frac{\alpha}{n}}} \mathcal{P}^{\alpha}\left(I, \mathbf{1}_{I^c}\omega\right) \right\} \left|I\right|_{\sigma} \lesssim \boldsymbol{\tau} \mathcal{A}_2^{\alpha,*} \left|I\right|_{\sigma} ,
\end{aligned}$$

since

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\left|I\right|^{1-\frac{\alpha}{n}}}{\left|A_k\right|^{2\left(1-\frac{\alpha}{n}\right)}} \left|A_k \setminus I\right|_{\omega} &= \int \sum_{k=1}^{\infty} \frac{\left|I\right|^{1-\frac{\alpha}{n}}}{\left|A_k\right|^{2\left(1-\frac{\alpha}{n}\right)}} \mathbf{1}_{A_k \setminus I}(x)\, d\omega(x) \\
&= \int \sum_{k=1}^{\infty} \frac{1}{2^{2\left(1-\frac{\alpha}{n}\right)k}} \frac{\left|I\right|^{1-\frac{\alpha}{n}}}{\left|I\right|^{2\left(1-\frac{\alpha}{n}\right)}} \mathbf{1}_{A_k \setminus I}(x)\, d\omega(x) \\
&\lesssim \int_{I^c} \left(\frac{\left|I\right|^{\frac{1}{n}}}{\left[\left|I\right|^{\frac{1}{n}} + \mathrm{quasidist}(x, I)\right]^2} \right)^{n-\alpha} d\omega(x) = \mathcal{P}^{\alpha}\left(I, \mathbf{1}_{I^c}\omega\right) .
\end{aligned}$$

The next term $D_{\text{descendent}}$ satisfies

$$\begin{aligned} D_{\text{descendent}} &\lesssim \sum_{k=1}^{\infty}\left(\frac{|I|_\sigma\,|A_k|^{\frac{1}{n}}}{|A_k|^{1+\frac{1-\alpha}{n}}}\right)^2 \boldsymbol{\tau}\left\|\mathsf{P}_I^{\text{good},\omega}\frac{\mathbf{x}}{2^k|I|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2 \\ &= \boldsymbol{\tau}\sum_{k=1}^{\infty}2^{-2k(n-\alpha+1)}\left(\frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}}\right)^2\left\|\mathsf{P}_I^{\text{good},\omega}\frac{\mathbf{x}}{|I|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2 \\ &\lesssim \boldsymbol{\tau}\left\{\frac{|I|_\sigma\left\|\mathsf{P}_I^{\text{good},\omega}\frac{\mathbf{x}}{|I|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2}{|I|^{2\left(1-\frac{\alpha}{n}\right)}}\right\}|I|_\sigma \lesssim \boldsymbol{\tau}A_2^{\alpha,\text{energy}}\,|I|_\sigma\ . \end{aligned}$$

Finally for D_{ancestor} we note that each J' is of the form $J'=A_\ell\equiv\pi_{\Omega\mathcal{D}}^{(\ell)}I$ for some $\ell\geq 1$, and that there are at most $C\boldsymbol{\tau}$ pairs (F,A_k) with $k\geq\ell$ such that $A_k\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F)$ and $J'=A_\ell\in\mathcal{C}_F^{\text{good},\boldsymbol{\tau}-\text{shift}}$. Now we write

$$\begin{aligned} D_{\text{ancestor}} &= \sum_{k=1}^{\infty}\mathbb{P}^\alpha(\mathbf{1}_I\sigma)\left(c(A_k),|A_k|^{\frac{1}{n}}\right)^2 \sum_{\substack{F\in\mathcal{F}:\\ A_k\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F)}}\ \sum_{J'\in\mathcal{C}_F^{\text{good},\boldsymbol{\tau}-\text{shift}}:\ I\subsetneqq J'\subset A_k}\left\|\triangle_{J'}^\omega\frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2 \\ &\lesssim \boldsymbol{\tau}\sum_{k=1}^{\infty}\left(\frac{|I|_\sigma\,|A_k|^{\frac{1}{n}}}{|A_k|^{1+\frac{1-\alpha}{n}}}\right)^2\sum_{\ell=1}^{k}\left\|\triangle_{A_\ell}^\omega\frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2 \\ &\leq \boldsymbol{\tau}\sum_{k=1}^{\infty}\left(\frac{|I|_\sigma\,|A_k|^{\frac{1}{n}}}{|A_k|^{1+\frac{1-\alpha}{n}}}\right)^2\left\|\mathsf{P}_{A_k}^{\text{good},\omega}\frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2 . \end{aligned}$$

It is at this point that we must invoke a new *'prepare to puncture'* argument. Now define $\widetilde{\omega}=\omega-\omega(\{p\})\,\delta_p$ where p is an atomic point in I for which

$$\omega(\{p\})=\sup_{q\in\mathfrak{P}_{(\sigma,\omega)}:\ q\in I}\omega(\{q\}).$$

(If ω has no atomic point in common with σ in I set $\widetilde{\omega}=\omega$.) Then we have $|I|_{\widetilde{\omega}}=\omega\left(I,\mathfrak{P}_{(\sigma,\omega)}\right)$ and

$$\frac{|I|_{\widetilde{\omega}}}{|I|^{\left(1-\frac{\alpha}{n}\right)}}\frac{|I|_\sigma}{|I|^{\left(1-\frac{\alpha}{n}\right)}}=\frac{\omega\left(I,\mathfrak{P}_{(\sigma,\omega)}\right)}{|I|^{\left(1-\frac{\alpha}{n}\right)}}\frac{|I|_\sigma}{|I|^{\left(1-\frac{\alpha}{n}\right)}}\leq A_2^{\alpha,\text{punct}}.$$

A key observation, already noted in the proof of Lemma 2 above, is that

$$\|\triangle_K^\omega\mathbf{x}\|_{L^2(\omega)}^2=\left\{\begin{array}{ccc}\|\triangle_K^\omega(\mathbf{x}-\mathbf{p})\|_{L^2(\omega)}^2 & \text{if} & p\in K\\ \|\triangle_K^\omega\mathbf{x}\|_{L^2(\widetilde{\omega})}^2 & \text{if} & p\notin K\end{array}\right.\leq\ell(K)^2\,|K|_{\widetilde{\omega}}\,,\qquad\text{for all }K\in\Omega\mathcal{D}\,, \tag{46}$$

and so, as in the proof of Lemma 2,

$$\left\|\mathsf{P}_{A_k}^{\text{good},\omega}\frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2\leq 3\,|A_k|_{\widetilde{\omega}}\ .$$

Then we continue with

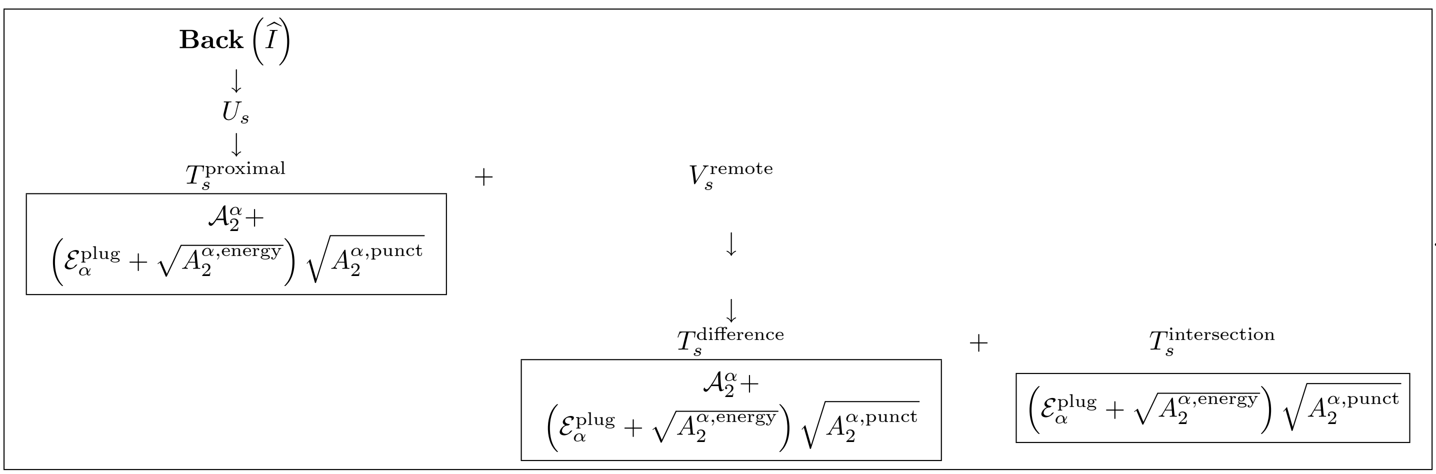
$$
\begin{aligned}
& \boldsymbol{\tau} \sum_{k=1}^{\infty} \left(\frac{|I|_\sigma \, |A_k|^{\frac{1}{n}}}{|A_k|^{1+\frac{1-\alpha}{n}}} \right)^2 \left\| \mathsf{P}_{A_k}^{\mathrm{good},\omega} \frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\
\lesssim \; & \boldsymbol{\tau} \sum_{k=1}^{\infty} \left(\frac{|I|_\sigma \, |A_k|^{\frac{1}{n}}}{|A_k|^{1+\frac{1-\alpha}{n}}} \right)^2 |A_k|_{\widetilde{\omega}} \\
= \; & \boldsymbol{\tau} \sum_{k=1}^{\infty} \left(\frac{|I|_\sigma}{|A_k|^{1-\frac{\alpha}{n}}} \right)^2 |A_k \setminus I|_\omega + \boldsymbol{\tau} \sum_{k=1}^{\infty} \left(\frac{|I|_\sigma}{2^{k(n-\alpha)} \, |I|^{1-\frac{\alpha}{n}}} \right)^2 |I|_{\widetilde{\omega}} \\
\lesssim \; & \boldsymbol{\tau} \left(\mathcal{A}_2^{\alpha,*} + A_2^{\alpha,\mathrm{punct}} \right) |I|_\sigma \,,
\end{aligned}
$$

where the inequality $\sum_{k=1}^{\infty} \left(\frac{|I|_\sigma}{|A_k|^{1-\frac{\alpha}{n}}} \right)^2 |A_k \setminus I|_\omega \lesssim \mathcal{A}_2^{\alpha,*} |I|_\sigma$ is already proved above in the estimate for D_{disjoint}.

8.3. **The backward Poisson testing inequality.** Fix $I \in \Omega\mathcal{D}$. It suffices to prove

$$
\mathbf{Back}\left(\widehat{I}\right) \equiv \int_{\mathbb{R}^n} \left[\mathbb{Q}^\alpha \left(t \mathbf{1}_{\widehat{I}} \overline{\mu} \right)(y) \right]^2 d\sigma(y) \lesssim \left\{ \mathcal{A}_2^\alpha + \left(\mathcal{E}_\alpha^{\mathrm{plug}} + \sqrt{A_2^{\alpha,\mathrm{energy}}} \right) \sqrt{A_2^{\alpha,\mathrm{punct}}} \right\} \int_{\widehat{I}} t^2 d\overline{\mu}(x,t). \tag{47}
$$

Note that in dimension $n = 1$, Hytönen obtained in [Hyt2] the simpler bound A_2^α for the term analogous to (47). Here is a brief schematic diagram of the decompositions, with bounds in $\square$, used in this subsection:



Using (42) we see that the integral on the right hand side of (47) is

$$
\int_{\widehat{I}} t^2 d\overline{\mu} = \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J \subset I} \left\| \mathsf{P}_{F,J}^\omega \mathbf{x} \right\|_{L^2(\omega)}^2 . \tag{48}
$$

where $\mathsf{P}_{F,J}^\omega$ was defined earlier.

We now compute using (42) again that

$$
\begin{aligned}
\mathbb{Q}^\alpha \left(t \mathbf{1}_{\widehat{I}} \overline{\mu} \right)(y) &= \int_{\widehat{I}} \frac{t^2}{\left(t^2 + |x-y|^2 \right)^{\frac{n+1-\alpha}{2}}} d\overline{\mu}(x,t) \\
&\approx \sum_{F \in \mathcal{F}} \sum_{\substack{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F) \\ J \subset I}} \frac{\left\| \mathsf{P}_{F,J}^\omega \mathbf{x} \right\|_{L^2(\omega)}^2}{\left(|J|^{\frac{1}{n}} + |y - c_J| \right)^{n+1-\alpha}},
\end{aligned} \tag{49}
$$

and then expand the square and integrate to obtain that the term $\mathbf{Back}\left(\widehat{I}\right)$ is

$$\sum_{\substack{F\in\mathcal{F}\\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)\\ J\subset I}}\ \sum_{\substack{F'\in\mathcal{F}:\\ J'\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(F'\right)\\ J'\subset I}}\int_{\mathbb{R}^n}\frac{\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{n+1-\alpha}}\frac{\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\right)^{n+1-\alpha}}d\sigma\left(y\right).$$

By symmetry we may assume that $\ell\left(J'\right)\leq\ell\left(J\right)$. We fix an integer s, and consider those quasicubes J and J' with $\ell\left(J'\right)=2^{-s}\ell\left(J\right)$. For fixed s we will control the expression

$$\begin{aligned} U_s &\equiv \sum_{F,F'\in\mathcal{F}}\ \sum_{\substack{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F),\ J'\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(F'\right)\\ J,J'\subset I,\ \ell\left(J'\right)=2^{-s}\ell(J)}} \\ &\quad\times\int_{\mathbb{R}^n}\frac{\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{n+1-\alpha}}\frac{\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\right)^{n+1-\alpha}}d\sigma\left(y\right), \end{aligned}$$

by proving that

$$U_s\lesssim 2^{-\delta s}\left\{\mathcal{A}_2^{\alpha}+\left(\mathcal{E}_{\alpha}^{\mathrm{strong}}+\sqrt{A_2^{\alpha,\mathrm{energy}}}\right)\sqrt{A_2^{\alpha,\mathrm{punct}}}\right\}\int_{\widehat{I}}t^2d\overline{\mu},\qquad\text{where }\delta=\frac{1}{2n}. \tag{50}$$

With this accomplished, we can sum in $s\geq 0$ to control the term $\mathbf{Back}\left(\widehat{I}\right)$. The remaining details of the proof are very similar to the corresponding arguments in [SaShUr7], with the only exception being the repeated use of the *'prepare to puncture'* argument above whenever the measures σ and ω can 'see each other' in an estimate. We refer the reader to [SaShUr6] for complete details[9].

9. The stopping form

This section is virtually unchanged from the corresponding section in [SaShUr7], so we content ourselves with a brief recollection. In the one-dimensional setting of the Hilbert transform, Hytönen [Hyt2] observed that "...the innovative verification of the local estimate by Lacey [Lac] is already set up in such a way that it is ready for us to borrow as a black box." The same observation carried over in spirit regarding the adaptation of Lacey's recursion and stopping time to proving the local estimate in [SaShUr7]. However, that adaptation involved the splitting of the stopping form into two sublinear forms, the first handled by methods in [LaSaUr2], and the second by the methods in [Lac]. The arguments are little changed when including common point masses, and we leave them for the reader (or see [SaShUr6] for the proofs written out in detail).

10. Energy dispersed measures

In this final section we prove that the energy side conditions in our main theorem hold if both measures are appropriately energy dispersed. We begin with the definitions of energy dispersed and reversal of energy.

10.1. **Energy dispersed measures and reversal of energy.** Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$. Recall that for $0\leq k\leq n$, we denote by $\mathcal{L}_k^n$ the collection of all k-dimensional planes in $\mathbb{R}^n$, and for a quasicube J, we define the *k-dimensional second moment* $\mathsf{M}_k^n\left(J,\mu\right)$ of μ on J by

$$\mathsf{M}_k^n\left(J,\mu\right)^2\equiv\inf_{L\in\mathcal{L}_k^n}\int_J\mathrm{dist}\left(x,L\right)^2d\mu\left(x\right).$$

Finally we defined μ to be *k-energy dispersed* if there is $c>0$ such that

$$\mathsf{M}_k^n\left(J,\mu\right)\geq c\mathsf{M}_0^n\left(J,\mu\right),\qquad\text{for all quasicubes }J\text{ in }\mathbb{R}^n.$$

[9] In [SaShUr5] and [SaShUr7] the bound for term B in the global estimate was mistakenly claimed without proof to be simply $\mathcal{A}_2^{\alpha}$ instead of the correct bound $\mathcal{A}_2^{\alpha}+\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}+\sqrt{A_2^{\alpha,\mathrm{energy}}}\right)\sqrt{A_2^{\alpha,\mathrm{punct}}}$ given in [SaShUr6].

In order to introduce a useful reformulation of the k-dimensional second moment, we will use the observation that minimizing k-planes L pass through the center of mass. More precisely, for any k-plane $L \in \mathcal{L}_k^n$ such that $\int_A \operatorname{dist}(x, L)^2 d\mu(x)$ is minimized, where A is a set of positive μ-measure, we claim that

$$\mathbb{E}_A^\mu x \in L\ .$$

Indeed, if we rotate coordinates so that $L = \left\{ \left(x^1, ..., x^k, a^{k+1}, ..., a^n\right) : \left(x^1, ..., x^k\right) \in \mathbb{R}^k \right\}$, then

$$\begin{aligned} \int_A \operatorname{dist}(x, L)^2 d\mu(x) &= \int_A \sum_{j=k+1}^n \left(x^j - a^j\right)^2 d\mu(x) \\ &= \sum_{j=k+1}^n \left[\int_A \left(x^j\right)^2 d\mu(x) - 2a^j \int_A x^j d\mu(x) + \left(a^j\right)^2 \int_A d\mu(x) \right] \\ &= \sum_{j=k+1}^n \left[\int_A \left(x^j\right)^2 d\mu(x) + \left(\int_A d\mu(x)\right) \left\{ \left(a^j\right)^2 - 2\frac{\int_A x^j d\mu(x)}{\int_A d\mu(x)} a^j \right\} \right] \end{aligned}$$

is minimized over $a^{k+1}, ..., a^n$ when

$$a^j = \frac{\int_A x^j d\mu(x)}{\int_A d\mu(x)} = \left(\mathbb{E}_A^\mu x\right)^j, \qquad k+1 \le j \le n.$$

This shows that the point $\mathbb{E}_A^\mu x$ belongs to the k-plane L.

Now we can obtain our reformulation of the k-dimensional second moment. Let $\mathcal{S}_k^n$ denote the collection of k-dimenional subspaces in $\mathbb{R}^n$. If $\mathcal{P}_S$ denotes orthogonal projection onto the subspace $S \in \mathcal{S}_{n-k}^n$ where $S = L_0^\perp$ and $L_0 \in \mathcal{S}_k^n$ is the subspace parallel to L, then we have the variance identity,

$$\begin{aligned} \mathsf{M}_k^n(J, \mu)^2 &= \inf_{L \in \mathcal{L}_k^n} \int_J \operatorname{dist}(x, L)^2 d\mu(x) = \inf_{S \in \mathcal{S}_{n-k}^n} \int_J \left|\mathcal{P}_S x - \mathcal{P}_S\left(\mathbb{E}_J^\mu x\right)\right|^2 d\mu(x) \\ &= \frac{1}{2} \inf_{S \in \mathcal{S}_{n-k}^n} \frac{1}{|J|_\mu} \int_J \int_J \left|\mathcal{P}_S x - \mathcal{P}_S y\right|^2 d\mu(x)\, d\mu(y) \\ &= \frac{1}{2} \inf_{L_0 \in \mathcal{S}_k^n} \frac{1}{|J|_\mu} \int_J \int_J \operatorname{dist}(x, L_0 + y)^2 d\mu(x)\, d\mu(y), \end{aligned} \tag{51}$$

since $\mathcal{P}_S\left(\mathbb{E}_J^\mu x\right) = \mathbb{E}_J^\mu\left(\mathcal{P}_S x\right)$. Here we have used in the first line the fact that the minimizing k-planes L pass through the center of mass $\mathbb{E}_J^\mu x$ of x in J.

Note that if μ is supported on a k-dimensional plane L in $\mathbb{R}^n$, then $\mathsf{M}_k^n(J, \mu)$ vanishes for all quasicubes J. On the other hand, $\mathsf{M}_0^n(J, \mu)$ is positive for any quasicube J on which the restriction of μ is *not* a point mass, and we conclude that measures μ supported on a k-plane. and whose restriction to J is not a point mass, are *not* k-energy dispersed. Thus $\mathsf{M}_k^n(J, \mu)$ measures the extent to which a certain 'energy' of μ is not localized to a k-plane. In this final section we will prove the necessity of the energy conditions for boundedness of the vector Riesz transform $\mathbf{R}^{\alpha,n}$ when the locally finite Borel measures σ and ω on $\mathbb{R}^n$ are k-energy dispersed with

$$\left\{ \begin{array}{lll} n-k < \alpha < n,\ \alpha \neq n-1 & \text{if} & 1 \le k \le n-2 \\ 0 \le \alpha < n,\ \alpha \neq 1, n-1 & \text{if} & k = n-1 \end{array} \right. . \tag{52}$$

Now we recall the definition of strong energy reversal from [SaShUr2]. We say that a vector $\mathbf{T}^\alpha = \{T_\ell^\alpha\}_{\ell=1}^2$ of α-fractional transforms in the plane has *strong* reversal of ω-energy on a cube J if there is a positive constant C_0 such that for all $2 \le \gamma \le 2^{\mathbf{r}(1-\varepsilon)}$ and for all positive measures μ supported outside γJ, we have the inequality

$$\mathbb{E}_J^\omega\left[\left(\mathbf{x} - \mathbb{E}_J^\omega \mathbf{x}\right)^2\right] \left(\frac{\mathrm{P}^\alpha(J, \mu)}{|J|^{\frac{1}{n}}}\right)^2 = \mathsf{E}(J, \omega)^2\, \mathrm{P}^\alpha(J, \mu)^2 \le C_0\ \mathbb{E}_J^\omega \left|\mathbf{T}^\alpha \mu - \mathbb{E}_J^{d\omega} \mathbf{T}^\alpha \mu\right|^2, \tag{53}$$

Now note that if ω is k-energy dispersed, then we have

$$\mathsf{E}(J, \omega)^2 = \frac{1}{|J|_\omega |J|^{\frac{2}{n}}} \mathsf{M}_0^n(J, \omega)^2 \lesssim \frac{1}{|J|_\omega |J|^{\frac{2}{n}}} \mathsf{M}_k^n(J, \omega)^2 \equiv \mathsf{E}_k(J, \omega)^2,$$

and where we have defined on the right hand side the analogous notion of energy $\mathsf{E}_k(J,\omega)$ in terms of $\mathsf{M}_k(J,\omega)$, and which is smaller than $\mathsf{E}(J,\omega)$. We now state the main result of this first subsection.

Lemma 16. *Let $0 \leq \alpha < n$. Suppose that ω is k-energy dispersed and that k and α satisfy (52). Then the α-fractional Riesz transform $\mathbf{R}^{\alpha,n} = \{R_\ell^{n,\alpha}\}_{\ell=1}^n$ has strong reversal (53) of ω-energy on* all *cubes J provided γ is chosen large enough depending only on n and α.*

In [SaShUr4] we showed that energy reversal can fail spectacularly for measures in general, but left open the possibility of reversing at least one direction in the energy for $\mathbf{R}^{\alpha,n}$ when $\alpha \neq 1$ in the plane $n=2$, and we will show in the next subsection that this is indeed possible, with even more directions included in higher dimensions.

10.2. **Fractional Riesz transforms and semi-harmonicity.** Now we fix $1 \leq \ell \leq n$ and write $x = (x', x'')$ with $x' = (x_1, ..., x_\ell) \in \mathbb{R}^\ell$ and $x'' = (x_{\ell+1}, ..., x_n) \in \mathbb{R}^{n-\ell}$ (when $\ell = n$ we have $x = x'$). Then we compute for β real that

$$\begin{aligned}
\triangle_{x'} |x|^\beta &= \triangle_{x'} \left(|x'|^2 + |x''|^2\right)^{\frac{\beta}{2}} = \nabla_{x'} \cdot \nabla_{x'} \left(|x'|^2 + |x''|^2\right)^{\frac{\beta}{2}} \\
&= \nabla_{x'} \cdot \left\{\frac{\beta}{2}\left(|x'|^2 + |x''|^2\right)^{\frac{\beta}{2}-1} 2x'\right\} = \beta \nabla_{x'} \cdot \left\{x' \left(|x'|^2 + |x''|^2\right)^{\frac{\beta}{2}-1}\right\} \\
&= \beta\left\{(\nabla_{x'} \cdot x')\left(|x'|^2 + |x''|^2\right)^{\frac{\beta-2}{2}} + x' \cdot \nabla_{x'}\left(|x'|^2 + |x''|^2\right)^{\frac{\beta-2}{2}}\right\} \\
&= \beta\left\{\ell\left(|x'|^2 + |x''|^2\right)^{\frac{\beta-2}{2}} + x' \cdot \frac{\beta-2}{2}\left(|x'|^2 + |x''|^2\right)^{\frac{\beta-2}{2}-1} 2x'\right\} \\
&= \beta\left\{\ell\left(|x'|^2 + |x''|^2\right)^{\frac{\beta-2}{2}} + (\beta-2)|x'|^2\left(|x'|^2 + |x''|^2\right)^{\frac{\beta-4}{2}}\right\} \\
&= \beta\left\{\ell\left(|x'|^2 + |x''|^2\right)\left(|x'|^2 + |x''|^2\right)^{\frac{\beta-4}{2}} + (\beta-2)|x'|^2\left(|x'|^2 + |x''|^2\right)^{\frac{\beta-4}{2}}\right\} \\
&= \beta\left\{(\ell+\beta-2)|x'|^2 + \ell|x''|^2\right\}\left(|x'|^2 + |x''|^2\right)^{\frac{\beta-4}{2}}.
\end{aligned}$$

The case of interest for us is when $\beta = \alpha - n + 1$, since then

$$\triangle_{x'} |x|^\beta = \nabla_{x'} \cdot \nabla_{x'} |x|^{\alpha-n+1} = \nabla_{x'} \cdot \nabla |x|^{\alpha-n+1} = c_{\alpha,n} \nabla_{x'} \cdot \mathbf{K}^{\alpha,n}(x), \tag{54}$$

where $\mathbf{K}^{\alpha,n}$ is the vector convolution kernel of the α-fractional Riesz transform $\mathbf{R}^{\alpha,n}$. Now if $\ell = 1$ in this case, then the factor

$$F_{\ell,\beta}(x) \equiv (\ell+\beta-2)|x'|^2 + \ell|x''|^2$$

is $(\beta-1)|x'|^2 + |x''|^2$, and thus in dimension $n \geq 2$, the factor $F_{1,\beta}(x)$ will be of one sign for all x if and only if $\alpha - n + 1 = \beta > 1$, i.e. $\alpha > n$, which is of no use since the Riesz transform $\mathbf{R}^{\alpha,n}$ is defined only for $0 \leq \alpha < n$.

Thus we must assume $\ell \geq 2$ and $\beta = \alpha - n + 1$ when $n \geq 2$. Under these assumptions, we then note that $F_{\ell,\beta}(x)$ will be of one sign for all x if $\ell + \beta - 2 > 0$, i.e. $\alpha > n + 1 - \ell$, in which case we conclude that

$$\begin{aligned}
\left|\triangle_{x'} |x|^{\alpha-n+1}\right| &= |\alpha-n+1|\left\{(\ell+\alpha-n-1)|x'|^2 + \ell|x''|^2\right\}\left(|x'|^2 + |x''|^2\right)^{\frac{\alpha-n-3}{2}} \\
&\approx \left(|x'|^2 + |x''|^2\right)^{\frac{\alpha-n-1}{2}} = |x|^{\alpha-n-1}, \qquad \text{for } \alpha \neq n-1.
\end{aligned} \tag{55}$$

When $\ell = n$, this shows that $\left|\triangle_x |x|^{\alpha-n+1}\right| \approx |x|^{\alpha-n-1}$ for $\alpha > 1$ with $\alpha \neq n-1$. But in the case $\ell = n$ we can obtain more. Indeed, since x'' is no longer present, we have for $0 \leq \alpha < 1$ that

$$\triangle_x |x|^{\alpha-n+1} \approx |x|^{\alpha-n-1}.$$

(This includes dimension $n = 1$ but only for $0 < \alpha < 1$).

We summarize these results as follows. For dimension $n \geq 2$ and $x = (x', x'')$ with $x' \in \mathbb{R}^\ell$ and $x'' \in \mathbb{R}^{n-\ell}$, we have

$$\left| \triangle_{x'} |x|^{\alpha-n+1} \right| \approx |x|^{\alpha-n-1},$$

provided

$$\begin{array}{ll} \textbf{either } 2 \leq \ell \leq n-1 \text{ and } n+1-\ell < \alpha < n \text{ with } \alpha \neq n-1, \\ \textbf{or } \ell = n \text{ and } 0 \leq \alpha < n \text{ with } \alpha \neq 1, n-1. \end{array} \tag{56}$$

Thus the two cases not included are $\alpha = 1$ and $\alpha = n-1$. The case $\alpha = 1$ is not included since $|x|^{\alpha-n+1} = |x|^{2-n}$ is the fundamental solution of the Laplacian for $n > 2$ and constant for $n = 2$. The case $\alpha = n-1$ is not included since $|x|^{\alpha-n+1} = 1$ is constant.

So we now suppose that α and ℓ are as in (56), and we consider ℓ-planes L intersecting the cube J. Recall that the trace of a matrix is invariant under rotations. Thus for each such ℓ-plane L, and for $z \in J \cap L$, we have from (54) and (55), and with $\mathbf{I}^{\alpha+1,n}\mu(z) \equiv \int_{\mathbb{R}^n} |z-y|^{\alpha+1-n} d\mu(y)$ denoting the convolution of $|x|^{\alpha+1-n}$ with μ, that

$$|\nabla_L \mathbf{R}^{\alpha,n}\mu(z)| \gtrsim |\text{trace}\, \nabla_L \mathbf{R}^{\alpha,n}\mu(z)| = \left|\triangle_L \mathbf{I}^{\alpha+1,n}\mu(z)\right| \approx \int |y-z|^{\alpha-n-1} d\mu(y) \approx \frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}}, \tag{57}$$

where ∇_L denotes the gradient in the ℓ-plane L, i.e. $\nabla_L = \mathcal{P}_S \nabla$ where S is the subspace parallel to L and $\mathcal{P}_S$ is orthogonal projection onto S, and where we assume that the positive measure μ is supported outside the expanded cube γJ.

We now claim that for every $z \in J \cap L$, the full matrix gradient $\nabla \mathbf{R}^{\alpha,n}\mu(z)$ is 'missing' at most $\ell - 1$ 'large' directions, i.e. has at least $n - \ell + 1$ eigenvalues each of size at least $c\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}}$. Indeed, to see this, suppose instead that the matrix $\nabla \mathbf{R}^{\alpha,n}\mu(z)$ has at most $n - \ell$ eigenvalues of size at least $c\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}}$. Then there is an ℓ-dimensional subspace S such that

$$|\nabla_S \mathbf{R}^{\alpha,n}\mu(z)| = |(\mathcal{P}_S \nabla)\mathbf{R}^{\alpha,n}\mu(z)| = |\mathcal{P}_S(\nabla \mathbf{R}^{\alpha,n}\mu(z))| \leq c\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}},$$

which contradicts (57) if c is chosen small enough. This proves our claim, and moreover, it satisfies the quantitative quadratic estimate

$$|\xi \cdot \nabla \mathbf{R}^{\alpha,n}\mu(z)\,\xi| \geq c\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}} |\xi|^2,$$

for all vectors ξ in some $(n-\ell+1)$-dimensional subspace

$$\mathsf{S}_z^{n-\ell+1} \equiv \text{Span}\left\{\mathbf{v}_z^1, ..., \mathbf{v}_z^{n-\ell+1}\right\} \in \mathcal{S}_{n-\ell+1}^n,$$

with $\mathbf{v}_z^j \in \mathbb{S}^{n-1}$ for $1 \leq j \leq n-\ell+1$.

It is convenient at this point to let

$$k = \ell - 1,$$

so that $1 \leq k \leq n-1$ and the assumptions (56) become

$$\begin{array}{ll} \textbf{either } 1 \leq k \leq n-2 \text{ and } n-k < \alpha < n \text{ with } \alpha \neq n-1, \\ \textbf{or } k = n-1 \text{ and } 0 \leq \alpha < n \text{ with } \alpha \neq 1, n-1, \end{array} \tag{58}$$

and our conclusion becomes

$$|\xi \cdot \nabla \mathbf{R}^{\alpha,n}\mu(z)\,\xi| \geq c\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}} |\xi|^2, \qquad \xi \in \mathsf{S}_z^{n-k},\ z \in J. \tag{59}$$

10.2.1. *Proof of strong reversal of energy.* We are now in a position to prove the strong reversal of energy for Riesz transforms in Lemma 16.

Proof. (of Lemma 16) Recall that $\mathsf{E}_k(J,\omega)^2=\inf_{L\in\mathcal{L}_k^n}\frac{1}{|J|_\omega}\int_J\left(\frac{\operatorname{dist}(x,L)}{|J|^{\frac{1}{n}}}\right)^2d\omega(x)$ and

$$\frac{1}{|J|_\omega}\int_J\left(\frac{\operatorname{dist}(x,L)}{|J|^{\frac{1}{n}}}\right)^2d\omega(x)=\frac{1}{2}\frac{1}{|J|_\omega}\int_J\frac{1}{|J|_\omega}\int_J\left(\frac{\operatorname{dist}(x,z+L_0)}{|J|^{\frac{1}{n}}}\right)^2d\omega(x)\,d\omega(z)\,, \tag{60}$$

where we recall that $L_0\in\mathcal{S}_k^n$ is parallel to L. The real matrix

$$M(x)\equiv\nabla\mathbf{R}^{\alpha,n}\mu(x)\,,\qquad x\in J, \tag{61}$$

is a scalar multiple of the Hessian of $|x|^{\alpha+1}$, hence is symmetric, and so we can rotate coordinates to diagonalize the matrix,

$$M(x)=\begin{bmatrix}\lambda_1(x)&0&\cdots&0\\0&\lambda_2(x)&&\vdots\\\vdots&&\ddots&0\\0&\cdots&0&\lambda_n(x)\end{bmatrix},$$

where $|\lambda_1(x)|\leq|\lambda_2(x)|\leq...\leq|\lambda_n(x)|$. We now fix $x=c_J$ to be the center of J in the matrix $M(c_J)$ and fix the eigenvalues corresponding to $M(c_J)$:

$$|\lambda_1|\leq|\lambda_2|\leq...\leq|\lambda_n|\,,\qquad\lambda_j\equiv\lambda_j(c_J)\,,$$

and define also the subspaces S^{n-i} to be $\mathsf{S}^{n-i}_{c_J}$ for $1\leq i\leq k$. Note that we then have $\mathsf{S}^{n-i}=\operatorname{Span}\{\mathbf{e}_{i+1},...,\mathbf{e}_n\}$. Let L_z^i be the i-plane

$$L_z^i\equiv z+\left(\mathsf{S}^{n-i}\right)^\perp=\left\{\left(u^1,...,u^i,z^{i+1},...,z^n\right):\left(u^1,...,u^i\right)\in\mathbb{R}^i\right\}. \tag{62}$$

By (59) we have

$$|\lambda_{k+1}|\geq c\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}}.$$

For convenience define $|\lambda_0|\equiv0$ and then define $0\leq m\leq k$ be the unique integer such that

$$|\lambda_m|<c\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}}\leq|\lambda_{m+1}|\,. \tag{63}$$

Now consider the largest $0\leq\ell\leq m$ that satisfies

$$|\lambda_\ell|\leq\gamma^{-\frac{1}{2n}}|\lambda_{\ell+1}|\,. \tag{64}$$

Note that this use of ℓ is quite different than that used in (56).

So suppose first that ℓ satisfies $1\leq\ell\leq m$ and is the largest index satisfying (64). Then if $\ell<m$ we have $|\lambda_i|>\gamma^{-\frac{1}{2n}}|\lambda_{i+1}|$ for $\ell+1\leq i\leq m$, and so both

$$\begin{aligned}|\lambda_{\ell+1}|&>\gamma^{-\frac{1}{2n}}|\lambda_{\ell+2}|>...>\gamma^{-\frac{m-\ell}{2n}}|\lambda_{m+1}|\geq\gamma^{-\frac{m-\ell}{2n}}c\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}},\\|\lambda_1|&\leq...\leq|\lambda_\ell|\leq\gamma^{-\frac{1}{2n}}|\lambda_{\ell+1}|\,.\end{aligned} \tag{65}$$

Both inequalities in the display above also hold for $\ell=m$ by (63) and (64). Roughly speaking, in this case where $1\leq\ell\leq m$, the gradient of $\mathbf{R}^{\alpha,n}\mu$ has modulus at least $|\lambda_{\ell+1}|$ in the directions of $\mathbf{e}_{\ell+1},...,\mathbf{e}_n$, while the gradient of $\mathbf{R}^{\alpha,n}\mu$ has modulus at most $\gamma^{-\frac{1}{2n}}|\lambda_{\ell+1}|$ in the directions of $\mathbf{e}_1,...,\mathbf{e}_\ell$.

Recall that $\mathsf{S}^{n-\ell}=\mathsf{S}^{n-\ell}_{c_J}$ is the subspace on which the symmetric matrix $M(c_J)=\nabla(\mathbf{R}^{\alpha,n}\mu)(c_J)$ has energy $\xi^{\mathrm{tr}}M(c_J)\,\xi$ bounded below by $|\lambda_{\ell+1}|$. Now we proceed to show that

$$|\lambda_{\ell+1}|^2|J|^{\frac{2}{n}}\,\mathsf{E}(J,\omega)^2\lesssim\frac{1}{|J|_\omega^2}\int_J\int_J|\mathbf{R}^{\alpha,n}\mu(x)-\mathbf{R}^{\alpha,n}\mu(z)|^2\,d\omega(x)\,d\omega(z)\,. \tag{66}$$

We will use our hypothesis that ω is k-energy dispersed to obtain

$$\mathsf{E}(J,\omega)\leq\mathsf{E}_k(J,\omega)\leq\mathsf{E}_m(J,\omega)\leq\mathsf{E}_\ell(J,\omega)$$

since $\ell \leq m \leq k$. To prove (66), we take $L_z \equiv L_z^\ell$ as in (62) and begin with

$$\begin{aligned} \operatorname{dist}(x, L_z)^2 &= \operatorname{dist}\left(x, z + \left(\mathsf{S}^{n-\ell}\right)^\perp\right)^2 \\ &= (x_{\ell+1} - z_{\ell+1})^2 + \ldots + (x_n - z_n)^2 = |x'' - z''|^2 , \end{aligned} \tag{67}$$

where $x = (x', x'')$ with $x' \in \mathbb{R}^\ell$ and $x'' \in \mathbb{R}^{n-\ell}$, and $L_z = \left\{(u', z'') : u' \in \mathbb{R}^\ell\right\}$. Now for $x, z \in J$ we take $\xi \equiv \left(0, \frac{x''-z''}{|x''-z''|}\right) \in \mathsf{S}^{n-\ell}$ (where $\frac{0}{0} = 0$). We use the estimate

$$|J|^{\frac{1}{n}} \left\|\nabla^2 \mathbf{R}^{\alpha,n}\mu\right\|_{L^\infty(J)} \lesssim |J|^{\frac{1}{n}} \int_{\mathbb{R}^n \setminus \gamma J} \frac{d\mu(y)}{|y - c_J|^{n-\alpha+2}} \lesssim \frac{1}{\gamma} \int_{\mathbb{R}^n \setminus \gamma J} \frac{d\mu(y)}{|y - c_J|^{n-\alpha+1}} \approx \frac{1}{\gamma} \frac{\mathrm{P}^\alpha(J, \mu)}{|J|^{\frac{1}{n}}}, \tag{68}$$

to obtain

$$\begin{aligned} &\frac{1}{|J|_\omega^2} \int_J \int_J \left(\left\|\nabla^2 \mathbf{R}^{\alpha,n}\mu\right\|_{L^\infty(J)} |x - z|\, |J|^{\frac{1}{n}}\right)^2 d\omega(x)\, d\omega(z) \\ \lesssim\ & \frac{1}{\gamma^2} \mathrm{P}^\alpha(J, \mu)^2 \frac{1}{|J|_\omega^2} \int_J \int_J \left(\frac{|x - z|}{|J|^{\frac{1}{n}}}\right)^2 d\omega(x)\, d\omega(z) = \frac{1}{\gamma^2} \mathrm{P}^\alpha(J, \mu)^2\, \mathsf{E}(J, \omega)^2 . \end{aligned} \tag{69}$$

We then start with a decomposition into big B and small S pieces,

$$\begin{aligned} & \frac{1}{|J|_\omega^2} \int_J \int_J \left|\mathbf{R}^{\alpha,n}\mu(x) - \mathbf{R}^{\alpha,n}\mu(z)\right|^2 d\omega(x)\, d\omega(z) \\ \gtrsim\ & \frac{1}{|J|_\omega^2} \int_J \int_J \left|\mathbf{R}^{\alpha,n}\mu(z', x'') - \mathbf{R}^{\alpha,n}\mu(z', z'')\right|^2 d\omega(x)\, d\omega(z) \\ & - \frac{1}{|J|_\omega^2} \int_J \int_J \left|\mathbf{R}^{\alpha,n}\mu(x', x'') - \mathbf{R}^{\alpha,n}\mu(z', x'')\right|^2 d\omega(x)\, d\omega(z) \\ \equiv\ & B - S. \end{aligned}$$

For $w \in J$ we have

$$\begin{aligned} \left|\nabla \mathbf{R}^{\alpha,n}\mu(w) - M(c_J)\right| &= \left|\nabla \mathbf{R}^{\alpha,n}\mu(w) - \nabla \mathbf{R}^{\alpha,n}\mu(c_J)\right| \\ &\lesssim |w - c_J| \left\|\nabla^2 \mathbf{R}^{\alpha,n}\mu\right\|_{L^\infty(J)} \lesssim \frac{1}{\gamma} \frac{\mathrm{P}^\alpha(J, \mu)}{|J|^{\frac{1}{n}}}, \end{aligned} \tag{70}$$

from (68), and this inequality will allow us to replace x or z by c_J at appropriate places in the estimates below, introducing a harmless error. We now use the second inequality in (65) with the diagonal form of $M(c_J) = \nabla \mathbf{R}^{\alpha,n}\mu(c_J)$, along with the error estimates (69) and (70), to control S by

$$\begin{aligned} S \leq\ & \frac{1}{|J|_\omega^2} \int_J \int_J \left|(x' - z') \cdot \nabla' \mathbf{R}^{\alpha,n}\mu(x)\right|^2 d\omega(x)\, d\omega(z) \\ & + \frac{1}{|J|_\omega^2} \int_J \int_J \left\{\left\|\nabla^2 \mathbf{R}^{\alpha,n}\mu\right\|_{L^\infty(J)} |x' - z'|^2\right\}^2 d\omega(x)\, d\omega(z) \\ \lesssim\ & \frac{1}{|J|_\omega^2} \int_J \int_J \left|(x' - z') \cdot \nabla' \mathbf{R}^{\alpha,n}\mu(c_J)\right|^2 d\omega(x)\, d\omega(z) \\ & + \frac{1}{|J|_\omega^2} \int_J \int_J \left\{\left\|\nabla^2 \mathbf{R}^{\alpha,n}\mu\right\|_{L^\infty(J)} |x' - z'|\, |J|^{\frac{1}{n}}\right\}^2 d\omega(x)\, d\omega(z) , \end{aligned}$$

and then continuing with

$$\begin{aligned} S \lesssim\ & \frac{1}{|J|_\omega^2} \int_J \int_J \left\{|x' - z'|\ \ |\lambda_\ell|\right\}^2 d\omega(x)\, d\omega(z) + \frac{1}{\gamma^2} \mathrm{P}^\alpha(J, \mu)^2\, \mathsf{E}(J, \omega)^2 \\ \lesssim\ & \frac{1}{\gamma} |\lambda_{\ell+1}|^2 \frac{1}{|J|_\omega^2} \int_J \int_J |x - z|^2 d\omega(x)\, d\omega(z) + \frac{1}{\gamma^2} \mathrm{P}^\alpha(J, \mu)^2\, \mathsf{E}(J, \omega)^2 \\ =\ & \frac{1}{\gamma} |J|^{\frac{2}{n}} |\lambda_{\ell+1}|^2\, \mathsf{E}(J, \omega)^2 + \frac{1}{\gamma^2} \mathrm{P}^\alpha(J, \mu)^2\, \mathsf{E}(J, \omega)^2 , \end{aligned}$$

which is small enough to be absorbed later on in the proof. To bound term B from below we use (70) in

$$\begin{aligned}\mathbf{R}^{\alpha,n}\mu(z',x'')-\mathbf{R}^{\alpha,n}\mu(z',z'') &= (x''-z'')\cdot\nabla''\mathbf{R}^{\alpha,n}\mu(z)+O\left(\left\|\nabla^2\mathbf{R}^{\alpha,n}\mu\right\|_{L^\infty(J)}|x-z|^2\right)\\ &= (x''-z'')\cdot\nabla''\mathbf{R}^{\alpha,n}\mu(c_J)+O\left(\left\|\nabla^2\mathbf{R}^{\alpha,n}\mu\right\|_{L^\infty(J)}|x-z|\,|J|^{\frac{1}{n}}\right),\end{aligned}$$

and then (59) with the choice $\xi\equiv\left(0,\frac{x''-z''}{|x''-z''|}\right)\in\mathsf{S}^{n-\ell}$, to obtain

$$\begin{aligned}|x''-z''|\,|\lambda_{\ell+1}| &\le |x''-z''|\left|\left(\xi\cdot\nabla''\right)\mathbf{R}^{\alpha,n}\mu(c_J)\cdot\xi\right|\\ &= \left|(x''-z'')\cdot\nabla''\mathbf{R}^{\alpha,n}\mu(c_J)\cdot\xi\right|\\ &\le \left|(x''-z'')\cdot\nabla''\mathbf{R}^{\alpha,n}\mu(c_J)\right|\\ &\le \left|\mathbf{R}^{\alpha,n}\mu(z',x'')-\mathbf{R}^{\alpha,n}\mu(z',z'')\right|+O\left(\left\|\nabla^2\mathbf{R}^{\alpha,n}\mu\right\|_{L^\infty(J)}|x-z|\,|J|^{\frac{1}{n}}\right).\end{aligned}$$

Then using (69) and (70) we continue with

$$\begin{aligned}&\frac{1}{|J|_\omega^2}\int_J\int_J\left|\mathbf{R}^{\alpha,n}\mu(z',x'')-\mathbf{R}^{\alpha,n}\mu(z',z'')\right|^2d\omega(x)\,d\omega(z)\\ \gtrsim\ &|\lambda_{\ell+1}|^2\frac{1}{|J|_\omega^2}\int_J\int_J|x''-z''|^2\,d\omega(x)\,d\omega(z)-\frac{1}{\gamma^2}\mathrm{P}^\alpha(J,\mu)^2\,\mathsf{E}(J,\omega)^2,\end{aligned}$$

and then

$$\begin{aligned}(71)\qquad &|\lambda_{\ell+1}|^2|J|^{\frac{2}{n}}\mathsf{E}(J,\omega)^2\le C|\lambda_{\ell+1}|^2|J|^{\frac{2}{n}}\mathsf{E}_\ell(J,\omega)^2\\ =\ &|\lambda_{\ell+1}|^2\frac{1}{|J|_\omega^2}\int_J\int_J\operatorname{dist}(x,L_z)^2\,d\omega(x)\,d\omega(z)=|\lambda_{\ell+1}|^2\frac{1}{|J|_\omega^2}\int_J\int_J|x''-z''|^2\,d\omega(x)\,d\omega(z)\\ \lesssim\ &\frac{1}{|J|_\omega^2}\int_J\int_J\left|\mathbf{R}^{\alpha,n}\mu(z',x'')-\mathbf{R}^{\alpha,n}\mu(z',z'')\right|^2d\omega(x)\,d\omega(z)+\frac{1}{\gamma^2}\mathrm{P}^\alpha(J,\mu)^2\,\mathsf{E}(J,\omega)^2\\ \lesssim\ &\frac{1}{|J|_\omega^2}\int_J\int_J\left|\mathbf{R}^{\alpha,n}\mu(x)-\mathbf{R}^{\alpha,n}\mu(z)\right|^2d\omega(x)\,d\omega(z)+S+\frac{1}{\gamma^2}\mathrm{P}^\alpha(J,\mu)^2\,\mathsf{E}(J,\omega)^2\\ \lesssim\ &\frac{1}{|J|_\omega^2}\int_J\int_J\left|\mathbf{R}^{\alpha,n}\mu(x)-\mathbf{R}^{\alpha,n}\mu(z)\right|^2d\omega(x)\,d\omega(z)+\frac{1}{\gamma}|\lambda_{\ell+1}|^2|J|^{\frac{2}{n}}\mathsf{E}(J,\omega)^2,\end{aligned}$$

since $\frac{1}{\gamma^2}\mathrm{P}^\alpha(J,\mu)^2\,\mathsf{E}(J,\omega)^2\le\frac{1}{\gamma}|J|^{\frac{2}{n}}|\lambda_{\ell+1}|^2\,\mathsf{E}(J,\omega)^2$ for γ large enough depending only on n and α. Finally then, for γ large enough depending only on n and α we can absorb the last term on the right hand side of (71) into the left hand side to obtain (66):

$$|\lambda_{\ell+1}|^2|J|^{\frac{2}{n}}\mathsf{E}(J,\omega)^2\lesssim\frac{1}{|J|_\omega^2}\int_J\int_J\left|\mathbf{R}^{\alpha,n}\mu(x)-\mathbf{R}^{\alpha,n}\mu(z)\right|^2d\omega(x)\,d\omega(z).$$

But since $\gamma^{-\frac{m-\ell}{2n}}c\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}}\le|\lambda_{\ell+1}|$ by (65), we have obtained

$$\begin{aligned}\mathrm{P}^\alpha(J,\mu)^2\,\mathsf{E}(J,\omega)^2 &\le \frac{1}{c^2}\gamma|\lambda_{\ell+1}|^2|J|^{\frac{2}{n}}\mathsf{E}(J,\omega)^2\\ &\lesssim \frac{1}{|J|_\omega^2}\int_J\int_J\left|\mathbf{R}^{\alpha,n}\mu(x)-\mathbf{R}^{\alpha,n}\mu(z)\right|^2d\omega(x)\,d\omega(z),\end{aligned}$$

which is the strong reverse energy inequality for J since

$$\frac{1}{2|J|_\omega^2}\int_J\int_J\left|\mathbf{R}^{\alpha,n}\mu(x)-\mathbf{R}^{\alpha,n}\mu(z)\right|^2d\omega(x)\,d\omega(z)=\mathbb{E}_J^\omega\left|\mathbf{R}^{\alpha,n}\mu-\mathbb{E}_J^{d\omega}\mathbf{R}^{\alpha,n}\mu\right|^2.$$

This completes the proof of strong reversal of energy under the assumption that $1\le\ell\le m$.

If instead $\ell=0$, then $|\lambda_i|>\gamma^{-\frac{1}{2n}}|\lambda_{i+1}|$ for all $1\le i\le m$, and so the smallest eigenvalue satisfies

$$|\lambda_1|>\gamma^{-\frac{1}{2n}}|\lambda_2|>\gamma^{-\frac{2}{2n}}|\lambda_3|>\ldots>\gamma^{-\frac{k}{2n}}|\lambda_{m+1}|>\gamma^{-\frac{1}{2}}c\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}}.$$

In this case the arguments above show that

$$\begin{aligned}\left(\gamma^{-\frac{1}{2}}c\frac{\mathrm{P}^{\alpha}\left(J,\mu\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\mathsf{E}\left(J,\omega\right)^{2} &\lesssim \frac{1}{\left|J\right|_{\omega}^{2}}\int_{J}\int_{J}\left|\mathbf{R}^{\alpha,n}\mu\left(x\right)-\mathbf{R}^{\alpha,n}\mu\left(z\right)\right|^{2}d\omega\left(x\right)d\omega\left(z\right)\\ &\quad+\frac{1}{\gamma^{2}}\mathrm{P}^{\alpha}\left(J,\mu\right)^{2}\mathsf{E}\left(J,\omega\right)^{2},\end{aligned}$$

which again yields the strong reverse energy inequality for J since the second term on the right hand side can then be absorbed into the left hand side for γ sufficiently large depending only on n and α. □

10.3. **Necessity of the energy conditions.** Now we demonstrate in a standard way the necessity of the energy conditions for the vector Riesz transform $\mathbf{R}^{\alpha,n}$ when the measures σ and ω are appropriately energy dispersed. Indeed, we can then establish the inequality

$$\mathcal{E}_{\alpha}^{\mathrm{strong}}\lesssim\sqrt{\mathcal{A}_{2}^{\alpha}}+\mathfrak{T}_{\mathbf{R}^{\alpha,n}}.$$

So assume that (58) holds. We use Lemma 16 to obtain that the α-fractional Riesz transform $\mathbf{R}^{\alpha,n}$ has strong reversal of ω-energy on *all* quasicubes J. Then we use the next lemma to obtain the energy condition $\mathcal{E}_{\alpha}^{\mathrm{strong}}\lesssim\mathfrak{T}_{\mathbf{R}^{n,\alpha}}+\sqrt{A_{2}^{\alpha}}$.

Lemma 17. *Let $0\leq\alpha<n$ and suppose that $\mathbf{R}^{\alpha,n}$ has strong reversal of ω-energy on* all *quasicubes J. Then we have the energy condition inequality,*

$$\mathcal{E}_{\alpha}^{\mathrm{strong}}\lesssim\mathfrak{T}_{\mathbf{T}^{n,\alpha}}+\sqrt{A_{2}^{\alpha,\mathrm{punct}}}\ .$$

Proof. Fix $\gamma\geq2$ large enough depending only on n and α, and fix goodness parameters $\mathbf{r}$ and ε so that $\gamma\leq2^{\mathbf{r}(1-\varepsilon)}$. Then Lemma 16 holds. From the strong reversal of ω-energy with $d\mu\equiv\mathbf{1}_{I_{r}\setminus\gamma J}d\sigma$, we have

$$\begin{aligned}&\mathsf{E}\left(J,\omega\right)^{2}\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I_{r}\setminus\gamma J}d\sigma\right)^{2}\\ \leq\ & C\ \mathbb{E}_{J}^{\omega}\left|\mathbf{T}^{\alpha}\left(\mathbf{1}_{I_{r}\setminus\gamma J}d\sigma\right)-\mathbb{E}_{J}^{d\omega}\mathbf{T}^{\alpha}\left(\mathbf{1}_{I_{r}\setminus\gamma J}d\sigma\right)\right|^{2}\\ \lesssim\ & \mathbb{E}_{J}^{\omega}\left|\mathbf{T}^{\alpha}\left(\mathbf{1}_{I_{r}\setminus\gamma J}d\sigma\right)\right|^{2}\lesssim\mathbb{E}_{J}^{\omega}\left|\mathbf{T}^{\alpha}\left(\mathbf{1}_{I_{r}}d\sigma\right)\right|^{2}+\mathbb{E}_{J}^{\omega}\left|\mathbf{T}^{\alpha}\left(\mathbf{1}_{\gamma J}d\sigma\right)\right|^{2},\end{aligned}$$

and so

$$\begin{aligned}\sum_{J\in M_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I_{r})}\left|J\right|_{\omega}\mathsf{E}\left(J,\omega\right)^{2}\mathrm{P}^{\alpha}\left(J,\mu\right)^{2} &\lesssim \sum_{J}\int_{J}\left|\mathbf{T}^{\alpha}\left(\mathbf{1}_{I_{r}}d\sigma\right)(x)\right|^{2}d\omega\left(x\right)+\sum_{J}\int_{J}\left|\mathbf{T}^{\alpha}\left(\mathbf{1}_{\gamma J}d\sigma\right)(x)\right|^{2}d\omega\left(x\right)\\ &\lesssim \int_{I_{r}}\left|\mathbf{T}^{\alpha}\left(\mathbf{1}_{I_{r}}d\sigma\right)(x)\right|^{2}d\omega\left(x\right)+\sum_{J}\int_{\gamma J}\left|\mathbf{T}^{\alpha}\left(\mathbf{1}_{\gamma J}d\sigma\right)(x)\right|^{2}d\omega\left(x\right)\\ &\lesssim \mathfrak{T}_{\mathbf{T}^{n,\alpha}}\left|I_{r}\right|_{\sigma}+\sum_{J}\mathfrak{T}_{\mathbf{T}^{n,\alpha}}\left|\gamma J\right|_{\sigma}\lesssim\mathfrak{T}_{\mathbf{T}^{n,\alpha}}\left|I_{r}\right|_{\sigma}\end{aligned}$$

since $\gamma J\subset I_{r}$ for $\gamma\leq2^{\mathbf{r}(1-\varepsilon)}$, and since the quasicubes γJ have bounded overlap (see [SaShUr6, Lemma 2 in v3]). We also have

$$\sum_{J\in M_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I_{r})}\left|J\right|_{\omega}\mathsf{E}\left(J,\omega\right)^{2}\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\gamma J}d\sigma\right)^{2}\lesssim\sum_{J\in M_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I_{r})}A_{2}^{\alpha,\mathrm{energy}}\left|\gamma J\right|_{\sigma}\lesssim A_{2}^{\alpha,\mathrm{energy}}\left|I_{r}\right|_{\sigma}$$

by the bounded overlap of the quasicubes γJ in I_{r} once more. We can now easily complete the proof of $\mathcal{E}_{\alpha}^{\mathrm{strong}}\lesssim\mathfrak{T}_{\mathbf{T}^{n,\alpha}}+\sqrt{A_{2}^{\alpha,\mathrm{punct}}}$. □

References

[And] T. C. Anderson, *A new sufficent two-weighted bump assumption for L^p boundedness of Calderón-Zygmund operators,* Proc. Amer. Math. Soc. **143**, No. 8, (2015), 3573-3586.

[CoFe] R. R. Coifman and C. L. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals,* Studia Math. **51** (1974), 241-250.

[DaJo] Guy David, and Jean-Lin Journé, *A boundedness criterion for generalized Calderón-Zygmund operators,* Ann. of Math. (2) **120** (1984), 371–397, MR763911 (85k:42041).

[HuMuWh] R. Hunt, B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for the conjugate function and the Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227-251.

[Hyt] Tuomas Hytönen, *On Petermichl's dyadic shift and the Hilbert transform*, C. R. Math. Acad. Sci. Paris **346** (2008), MR2464252.

[Hyt2] Tuomas Hytönen, *The two weight inequality for the Hilbert transform with general measures,* *arXiv:1312.0843v2*.

[HyLaPe] Tuomas Hytönen, Michael T. Lacey, and C. Pérez, *Sharp weighted bounds for the q-variation of singular integrals*, Bull. Lon. Math. Soc. **45** (2013), 529-540.

[KaLiPeWa] A. Kairema, J. Li, M. C. Pereyra and L. Ward, *Haar bases on quasi-metric spaces, and dyadic structure theorems for function spaces on product spaces of homogeneous type,* J. Funct. Anal. **271**, Issue 7, (2016), 1793-1843.

[Lac] Michael T. Lacey, *Two weight inequality for the Hilbert transform: A real variable characterization, II*, Duke Math. J. Volume **163**, Number 15 (2014), 2821-2840.

[Lac2] Michael T. Lacey, *The two weight inequality for the Hilbert transform: a primer*, to appear in this volume in honor of Cora Sadosky, arXiv:1304.5004v1.

[Lac3] Michael T. Lacey, *On the separated bumps conjecture for Calderón-Zygmund operators,* , arXiv:1310.3507v4.

[LaSaUr1] Michael T. Lacey, Eric T. Sawyer, and Ignacio Uriarte-Tuero, *A characterization of two weight norm inequalities for maximal singular integrals with one doubling measure,* Analysis & PDE, Vol. **5** (2012), No. 1, 1-60.

[LaSaUr2] Michael T. Lacey, Eric T. Sawyer, and Ignacio Uriarte-Tuero, *A Two Weight Inequality for the Hilbert transform assuming an energy hypothesis,* Journal of Functional Analysis, Volume **263** (2012), Issue 2, 305-363.

[LaSaShUr] Michael T. Lacey, Eric T. Sawyer, Chun-Yen Shen and Ignacio Uriarte-Tuero, *The Two weight inequality for Hilbert transform, coronas, and energy conditions,* arXiv: (2011).

[LaSaShUr2] Michael T. Lacey, Eric T. Sawyer, Chun-Yen Shen and Ignacio Uriarte-Tuero, *Two Weight Inequality for the Hilbert Transform: A Real Variable Characterization,* arXiv:1201.4319 (2012).

[LaSaShUr3] Michael T. Lacey, Eric T. Sawyer, Chun-Yen Shen and Ignacio Uriarte-Tuero, *Two weight inequality for the Hilbert transform: A real variable characterization I*, Duke Math. J, Volume **163**, Number 15 (2014), 2795-2820.

[LaSaShUrWi] Michael T. Lacey, Eric T. Sawyer, Chun-Yen Shen, Ignacio Uriarte-Tuero and Brett D. Wick, *Two weight inequalities for the Cauchy transform from* $\mathbb{R}$ to $\mathbb{C}_+$, *arXiv:1310.4820v4*.

[LaWi1] Michael T. Lacey and Brett D. Wick, *Two weight inequalities for the Cauchy transform from* $\mathbb{R}$ to $\mathbb{C}_+$, *arXiv:1310.4820v1*.

[LaWi] Michael T. Lacey and Brett D. Wick, *Two weight inequalities for Riesz transforms: uniformly full dimension weights*, *arXiv:1312.6163v1,v2,v3*.

[NTV1] F. Nazarov, S. Treil and A. Volberg, *The Bellman function and two weight inequalities for Haar multipliers*, J. Amer. Math. Soc. **12** (1999), 909-928, MR{1685781 (2000k:42009)}.

[NTV2] F. Nazarov, S. Treil and A. Volberg, *The Tb-theorem on non-homogeneous spaces,* Acta Math. **190** (2003), no. 2, MR 1998349 (2005d:30053).

[NTV4] F. Nazarov, S. Treil and A. Volberg, *Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures*, preprint (2004) arxiv:1003.1596.

[Per] C. Pérez, *Two weighted inequalities for potential and fractional type maximal operators*, Indiana Univ. Math. J. **43** (1994), no. 2, 663–683.

[Saw1] E. Sawyer, *A characterization of a two weight norm inequality for maximal operators*, Studia Math. **75** (1982), 1-11, MR{676801 (84i:42032)}.

[Saw] E. Sawyer, *A characterization of two weight norm inequalities for fractional and Poisson integrals*, Trans. A.M.S. **308** (1988), 533-545, MR{930072 (89d:26009)}.

[SaShUr2] Eric T. Sawyer, Chun-Yen Shen and Ignacio Uriarte-Tuero, A *two weight theorem for α-fractional singular integrals with an energy side condition*, arXiv:1302.5093v8.

[SaShUr3] Eric T. Sawyer, Chun-Yen Shen and Ignacio Uriarte-Tuero, *A geometric condition, necessity of energy, and two weight boundedness of fractional Riesz transforms*, arXiv:1310.4484v1.

[SaShUr4] Eric T. Sawyer, Chun-Yen Shen and Ignacio Uriarte-Tuero, *A note on failure of energy reversal for classical fractional singular integrals*, IMRN, Volume **2015**, Issue 19, 9888-9920.

[SaShUr5] Eric T. Sawyer, Chun-Yen Shen and Ignacio Uriarte-Tuero, A *two weight theorem for α-fractional singular integrals with an energy side condition and quasicube testing*, arXiv:1302.5093v10.

[SaShUr6] Eric T. Sawyer, Chun-Yen Shen and Ignacio Uriarte-Tuero, A *two weight theorem for α-fractional singular integrals with an energy side condition, quasicube testing and common point masses*, arXiv:1505.07816v2,v3.

[SaShUr7] Eric T. Sawyer, Chun-Yen Shen and Ignacio Uriarte-Tuero, A *two weight theorem for α-fractional singular integrals with an energy side condition*, Revista Mat. Iberoam. **32** (2016), no. 1, 79-174.

[SaShUr8] Eric T. Sawyer, Chun-Yen Shen and Ignacio Uriarte-Tuero, The *two weight $T1$ theorem for fractional Riesz transforms when one measure is supported on a curve*, arXiv:1505.07822v4.

[SaShUr9] Eric T. Sawyer, Chun-Yen Shen and Ignacio Uriarte-Tuero, *Failure of necessity of the energy condition*, arXiv:16072.06071v2.

[SaWh] E. Sawyer and R. L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, *Amer. J. Math.* **114** (1992), 813-874.

[Ste] E. M. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, N. J., 1993.

[Vol] A. Volberg, *Calderón-Zygmund capacities and operators on nonhomogeneous spaces,* CBMS Regional Conference Series in Mathematics (2003), MR{2019058 (2005c:42015)}.

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