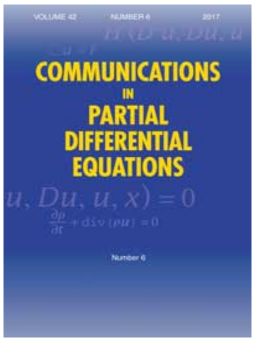

Communications in Partial Differential Equations

ISSN: 0360-5302 (Print) 1532-4133 (Online) Journal homepage: http://www.tandfonline.com/loi/lpde20

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To cite this article: Chin-Yu Hsiao & George Marinescu (2017) Berezin–Toeplitz quantization for lower energy forms, Communications in Partial Differential Equations, 42:6, 895-942, DOI: 10.1080/03605302.2017.1330340

To link to this article: https://doi.org/10.1080/03605302.2017.1330340

Accepted author version posted online: 05 Jun 2017.
Published online: 21 Jul 2017.

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COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
2017, VOL. 42, NO. 6, 895–942
https://doi.org/10.1080/03605302.2017.1330340

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Berezin–Toeplitz quantization for lower energy forms

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ABSTRACT

Let M be an arbitrary complex manifold and let L be a Hermitian holomorphic line bundle over M. We introduce the Berezin–Toeplitz quantization of the open set of M where the curvature on L is nondegenerate. In particular, we quantize any manifold admitting a positive line bundle. The quantum spaces are the spectral spaces corresponding to $\left[0, k^{-N}\right]$, where $N > 1$ is fixed, of the Kodaira Laplace operator acting on forms with values in tensor powers L^k. We establish the asymptotic expansion of associated Toeplitz operators and their composition in the semiclassical limit $k \to \infty$ and we define the corresponding star-product. If the Kodaira Laplace operator has a certain spectral gap this method yields quantization by means of harmonic forms. As applications, we obtain the Berezin–Toeplitz quantization for semi-positive and big line bundles.

ARTICLE HISTORY
Received 10 March 2015
Accepted 28 February 2017

KEYWORDS
Berezin–Toeplitz quantization; Kodaira Laplace operator; local holomorphic Morse inequalities; Melin–Sjöstrand stationary phase method; positive line bundle; spectral asymptotics

MATHEMATICS SUBJECT CLASSIFICATION
53D50; 32A25; 47B35; 81S10

1. Introduction and statement of the main results

The aim of the geometric quantization theory of Kostant and Souriau is to relate the classical observables (smooth functions) on a phase space (a symplectic manifold) to the quantum observables (bounded linear operators) on the quantum space (sections of a line bundle). Berezin–Toeplitz quantization is a particularly efficient version of the geometric quantization theory [2, 3, 14, 21, 22, 31]. Toeplitz operators and more generally Toeplitz structures were introduced in geometric quantization by Berezin [3] and Boutet de Monvel–Guillemin [6]. We refer to [22, 26, 30] for reviews of Berezin–Toeplitz quantization.

The setting of Berezin–Toeplitz quantization on Kähler manifolds is the following. Let (M, ω, J) be a Kähler manifold of $\dim_{\mathbb{C}} M = n$ with Kähler form ω and complex structure J. Let (L, h) be a holomorphic Hermitian line bundle on X, and let ∇^L be the holomorphic Hermitian connection on (L, h) with curvature $R^L = (\nabla^L)^2$. We assume that (L, h, ∇^L) is a prequantum line bundle, i.e.,

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{1.1}$$

Let $g^{TM} := \omega(\cdot, J\cdot)$ be the J-Riemannian metric on TM. The Riemannian volume form of g^{TM} is denoted by dv_M. On the space of smooth sections with compact support $\mathscr{C}_0^\infty(M, L^k)$ we introduce the L^2-scalar product associated to the metrics h and the Riemannian volume

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form dv_M by

$$\langle s_1, s_2\rangle = \int_M \langle s_1(x), s_2(x)\rangle_{h^k}\, dv_M(x)\,. \tag{1.2}$$

The completion of $\mathscr{C}_0^\infty(M,L^k)$ with respect to (1.2) is denoted as usual by $L^2(M,L^k)$. We denote by $H^0_{(2)}(M,L^k)$ the closed subspace of $L^2(M,L^k)$ consisting of holomorphic sections. The Bergman projection is the orthogonal projection $P_k : L^2(M,L^k) \to H^0_{(2)}(M,L^k)$. For a bounded function $f \in \mathscr{C}^\infty(M)$, set

$$T_{f,k} : L^2(M,L^k) \longrightarrow L^2(M,L^k)\,, \quad T_{f,k} = P_k f P_k\,, \tag{1.3}$$

where the action of f is the pointwise multiplication by f. The map which associates to $f \in \mathscr{C}^\infty(M)$ the family of bounded operators $\{T_{f,k}\}$ on $L^2(M,L^k)$ is called the *Berezin–Toeplitz quantization*. A *Toeplitz operator* is a sequence $\{T_k\}_{k\in\mathbb{N}}$ of bounded linear endomorphisms of $L^2(M,L^k)$ verifying $T_k = P_k\, T_k\, P_k$, such that there exist a sequence $g_\ell \in \mathscr{C}^\infty(M)$ such that for any $p \geqslant 0$, there exists $C_p > 0$ with $\|T_k - \sum_{\ell=0}^p T_{g_\ell,k}\, k^{-\ell}\|_{op} \leqslant C_p\ k^{-p-1}$ for any $k \in \mathbb{N}$, where $\|\cdot\|_{op}$ denotes the operator norm on the space of bounded operators.

Assume now that (M,ω,J) is a compact Kähler manifold. Then Bordemann et al. [5] and Schlichenmaier [29] (using the analysis of Toeplitz structures of Boutet de Monvel–Guillemin [6]), Charles [7] (inspired by semiclassical analysis of Boutet de Monvel–Guillemin [6]) and Ma–Marinescu [25] (using the expansion of the Bergman kernel [9, 24]) showed that the composition of two Toeplitz operators is a Toeplitz operator, in the sense that for any $f,g \in \mathscr{C}^\infty(M)$, the product $T_{f,k}\, T_{g,k}$ has an asymptotic expansion

$$T_{f,k}\, T_{g,k} = \sum_{p=0}^\infty T_{C_p(f,g),k}\, k^{-p} + \mathcal{O}(k^{-\infty}) \tag{1.4}$$

where C_p are bidifferential operators of order $\leqslant 2r$, satisfying $C_0(f,g) = fg$ and $C_1(f,g) - C_1(g,f) = \sqrt{-1}\,\{f,g\}$. Here $\{\,\cdot\,,\,\cdot\,\}$ is the Poisson bracket on $(M, 2\pi\omega)$. We deduce from (1.4),

$$[T_{f,k}\,, T_{g,k}] = \frac{\sqrt{-1}}{k} T_{\{f,g\},k} + \mathcal{O}(k^{-2})\,. \tag{1.5}$$

In [24, 25] Ma–Marinescu extended the Berezin–Toeplitz quantization to symplectic manifolds and orbifolds by using as quantum space the kernel of the Dirac operator acting on powers of the prequantum line bundle twisted with an arbitrary vector bundle with arbitrary metric on manifolds. Recently, Charles [8] introduced a semiclassical approach for symplectic manifolds inspired from the Boutet de Monvel–Guillemin theory [6].

In this paper, we extend the Berezin–Toeplitz quantization in several directions. Firstly, we consider an arbitrary Hermitian manifold (M,Θ,J) endowed with arbitrary Hermitian holomorphic line bundle (L,h) and we quantize the open set $M(0)$ where the curvature of (L,h) is positive. Since there are no holomorphic L^2 sections in general, we use as quantum spaces the spectral spaces of the Kodaira Laplacian $\Box_k^{(0)}$ on $L^2(M,L^k)$, corresponding to energy less than k^{-N}, $N > 1$ fixed, decaying to 0 polynomially in k, as $k \to \infty$. Secondly, we consider the same construction for the Kodaira Laplacian $\Box_k^{(q)}$ acting on $(0,q)$-forms. In this case, we quantize the open set $M(q)$ where the curvature of (L,h) is nondegenerate and has exactly q negative eigenvalues (and hence $n-q$ positive ones). Quantization using $(0,q)$-forms was introduced in [24, Section 8.2] for bundles with mixed curvature of signature

$(q, n-q)$ everywhere on a compact manifold. It was based on the asymptotic of Bergman kernel developed in Ma and Marinescu [23].

The idea underlying the construction used in this paper comes from the local holomorphic Morse inequalities [4, 11, 18, 24]. Roughly speaking, the harmonic $(0, q)$-forms with values in L^k tend to concentrate on $M(q)$ as $k \to \infty$. More precisely, the semiclassical limit of the kernel of the spectral projectors considered above was determined in [18, Theorem 1.1], see also [18, Theorems 1.6 –1.10] for important particular cases. This is the main technical ingredient used in this paper, which is in turn based on techniques of microlocal and semiclassical analysis [13, 28], especially the stationary phase method of Melin–Sjöstrand [28].

We now formulate the main results. We refer to Section 2 for some standard notations and terminology used here. We are working in the following general setting:

(A) (M, Θ, J) is a Hermitian manifold of complex dimension n, where Θ is a smooth positive $(1,1)$-form and J is the complex structure. Moreover, (L, h) is a holomorphic Hermitian line bundle over M, where h is the Hermitian fiber metric on L, and $q \in \{0, 1, \ldots, n\}$.
(B) $f, g \in \mathscr{C}^\infty(M)$ are smooth bounded functions.

Let $g^{TM}_\Theta(\cdot,\cdot) = \Theta(\cdot, J\cdot)$ be the Riemannian metric on TM induced by Θ and J and let $\langle\cdot,\cdot\rangle$ be the Hermitian metric on $\mathbb{C}TM := TM \otimes_\mathbb{R} \mathbb{C}$ induced by g^{TM}_Θ. The Riemannian volume form dv_M of (M, Θ) satisfies $dv_M = \Theta^n/n!$. For every $q = 0, 1, \ldots, n$, the Hermitian metric $\langle\cdot,\cdot\rangle$ on $TM \otimes_\mathbb{R} \mathbb{C}$ induces a Hermitian metric $\langle\cdot,\cdot\rangle$ on $\Lambda^{0,q}(T^*M)$ the bundle of $(0, q)$ forms of M.

We will denote by ϕ the local weights of the Hermitian metric h on L (see (2.1)). Let ∇^L be the holomorphic Hermitian connection on (L, h) with curvature $R^L = (\nabla^L)^2$. We will identify the curvature form R^L with the Hermitian matrix $\dot{R}^L \in \mathscr{C}^\infty(M, \mathrm{End}(T^{1,0}M))$ satisfying for every $U, V \in T^{1,0}_x M$, $x \in M$,

$$\langle R^L(x), U \wedge \overline{V} \rangle = \langle \dot{R}^L(x) U, V \rangle. \tag{1.6}$$

Let $\det \dot{R}^L(x) := \mu_1(x) \ldots \mu_n(x)$, where $\{\mu_j(x)\}_{j=1}^n$, are the eigenvalues of $\dot{R}^L$ with respect to $\langle\cdot,\cdot\rangle$. For $j \in \{0, 1, \ldots, n\}$, let

$$M(j) = \big\{x \in M;\ \dot{R}^L(x) \text{ is nondegenerate and has exactly } j \text{ negative eigenvalues}\big\}. \tag{1.7}$$

We denote by W the subbundle of rank j of $T^{1,0}M|_{M(j)}$ generated by the eigenvectors corresponding to negative eigenvalues of $\dot{R}^L$. Then $\det \overline{W}^* := \Lambda^j \overline{W}^* \subset \Lambda^{0,j}(T^*M)|_{M(j)}$ is a rank one sub-bundle. Here $\overline{W}^*$ is the dual bundle of the complex conjugate bundle of W and $\Lambda^j \overline{W}^*$ is the vector space of all finite sums of $v_1 \wedge \cdots \wedge v_j$, $v_1, \ldots, v_j \in \overline{W}^*$. We denote by $I_{\det \overline{W}^*} \in \mathrm{End}(\Lambda^{0,j}(T^*M))$ the orthogonal projection from $\Lambda^{0,j}(T^*M)$ onto $\det \overline{W}^*$.

For $k > 0$, let (L^k, h^k) be the kth tensor power of the line bundle (L, h). Let $(\cdot,\cdot)_k$ and $(\cdot,\cdot)$ denote the global L^2 inner products on $\Omega^{0,q}_0(M, L^k)$ and $\Omega^{0,q}_0(M)$ induced by $\langle\cdot,\cdot\rangle$ and h^k, respectively (see (2.2)). We denote by $L^2_{(0,q)}(M, L^k)$ and $L^2_{(0,q)}(M)$ the completions of $\Omega^{0,q}_0(M, L^k)$ and $\Omega^{0,q}_0(M)$ with respect to $(\cdot,\cdot)_k$ and $(\cdot,\cdot)$, respectively.

Let $\Box^{(q)}_k$ be the Kodaira Laplacian acting on $(0, q)$–forms with values in L^k, cf. (2.6). We denote by the same symbol $\Box^{(q)}_k$ the Gaffney extension of the Kodaira Laplacian, cf. (2.9). It is well-known that $\Box^{(q)}_k$ is self-adjoint and the spectrum of $\Box^{(q)}_k$ is contained in $\overline{\mathbb{R}}_+$ (see [24, Proposition 3.1.2]). For a Borel set $B \subset \mathbb{R}$ let $E(B)$ be the spectral projection of $\Box^{(q)}_k$ corresponding to the set B, where E is the spectral measure of $\Box^{(q)}_k$ (see Davies [10, Section 2]) and for $\lambda \in \mathbb{R}$ we set $E_\lambda = E\big((-\infty,\lambda]\big)$ and

$$\mathscr{E}^q_\lambda(M,L^k) = \operatorname{Range} E_\lambda \subset L^2_{(0,q)}(M,L^k)\,. \tag{1.8}$$

If $\lambda = 0$, then $\mathscr{E}^q_0(M,L^k) = \operatorname{Ker}\Box^{(q)}_k =: \mathscr{H}^q(M,L^k)$ is the space of global harmonic sections. The *spectral projection* of $\Box^{(q)}_k$ is the orthogonal projection

$$P^{(q)}_{k,\lambda} : L^2_{(0,q)}(M,L^k) \to \mathscr{E}^q_\lambda(M,L^k)\,. \tag{1.9}$$

Fix $f \in \mathscr{C}^\infty(M)$ be a bounded function. Let $\lambda \geq 0$. The *Berezin–Toeplitz quantization for* $\mathscr{E}^q_\lambda(M,L^k)$ is the operator

$$T^{(q),f}_{k,\lambda} := P^{(q)}_{k,\lambda} \circ f \circ P^{(q)}_{k,\lambda} : L^2_{(0,q)}(M,L^k) \to \mathscr{E}^q_\lambda(M,L^k). \tag{1.10}$$

Let $T^{(q),f}_{k,\lambda}(\cdot\,,\cdot)$ be the Schwartz kernel of $T^{(q),f}_{k,\lambda}$, see (2.13), (2.14). Since $\Box^{(q)}_k$ is elliptic, we have $T^{(q),f}_{k,\lambda}(\cdot\,,\cdot) \in \mathscr{C}^\infty\big(M \times M, (L^k \otimes \Lambda^{0,q}(T^*M)) \boxtimes (L^k \otimes \Lambda^{0,q}(T^*M))^*\big)$.

Let $A_k : L^2_{(0,q)}(M,L^k) \to L^2_{(0,q)}(M,L^k)$ be a k-dependent continuous operator with smooth kernel $A_k(x,y)$ and let $D_0, D_1 \Subset M$ be open trivializations with trivializing sections s and $\widehat{s}$, respectively. In this paper, we will identify A_k and $A_k(x,y)$ on $D_0 \times D_1$ with the localized operators $A_{k,s,\widehat{s}}$ and $A_{k,s,\widehat{s}}(x,y)$, respectively (see (2.3)).

The first main result of this work is the following.

Theorem 1.1. *Under the assumptions (A) and (B) let $j \in \{0,1,\dots,n\}$ and $D_0, D_1 \Subset M$ on which L is trivial. Suppose that one of the following conditions is fulfilled:*
(i) $D_0 \Subset M(j)$ *and* $j \neq q$,
(ii) $D_0 \Subset M(q)$ *and* $\overline{D}_0 \bigcap \overline{D}_1 = \emptyset$.
Then, for every $N > 1$, $m \in \mathbb{N}$, there exists $C_{N,m} > 0$ independent of k such that

$$\left| T^{(q),f}_{k,k^{-N}}(x,y) \right|_{\mathscr{C}^m(D_0\times D_1)} \leq C_{N,m} k^{2n-\frac{N}{2}+2m}. \tag{1.11}$$

If $D_0 \Subset M(q)$ there exists a symbol

$$b_f(x,y,k) \in S^n(1; D_0 \times D_0, \Lambda^{0,q}(T^*M) \boxtimes (\Lambda^{0,q}(T^*M))^*)$$

and a phase function $\Psi \in \mathscr{C}^\infty(D_0 \times D_0)$ such that for every $N > 1$, $m \in \mathbb{N}$, there exists $\widetilde{C}_{N,m} > 0$ independent of k such that

$$\left| T^{(q),f}_{k,k^{-N}}(x,y) - e^{ik\Psi(x,y)} b_f(x,y,k) \right|_{\mathscr{C}^m(D_0\times D_0)} \leq \widetilde{C}_{N,m} k^{2n-\frac{N}{2}+2m}, \tag{1.12}$$

where $b_f(x,y,k) \sim \sum_{j=0}^\infty b_{f,j}(x,y)k^{n-j}$ in the sense of Definition 2.1 and

$$b_{f,0}(x,x) = (2\pi)^{-n} f(x) \left|\det \dot{R}^L(x)\right| I_{\det \overline{W}^*}(x), \quad x \in D_0, \tag{1.13}$$

and

$$\begin{gathered} \Psi(x,y) \in \mathscr{C}^\infty(D_0 \times D_0), \quad \Psi(x,y) = -\overline{\Psi}(y,x)\,, \\ \exists\, c > 0 : \operatorname{Im}\Psi \geq c\left|x-y\right|^2\,, \quad \Psi(x,y) = 0 \Leftrightarrow x = y\,. \end{gathered} \tag{1.14}$$

We collect more properties for the phase Ψ in Theorem 3.3. The results says that, roughly speaking, the Toeplitz kernel $T^{(q),f}_{k,k^{-N}}(\cdot,\cdot)$ acting on $(0,q)$-forms, decays rapidly as $k\to\infty$ outside $M(q)$ and off-diagonal, and admits an asymptotic expansion on the set $M(q)$.

Let $\ell, m\in\mathbb{N}$ be fixed and choose $N\geq 2(n+\ell+2m+1)$. Then we deduce from (1.12) that

$$T^{(q),f}_{k,k^{-N}}(x,x)=\sum_{r=0}^{\ell} b_{f,r}(x,x)k^{n-r}+O(k^{n-\ell-1})\ \text{ in } \mathscr{C}^m(D_0), D_0\Subset M(q). \tag{1.15}$$

Note that if M is compact complex manifold endowed with a positive line bundle L (i.e., $M(0)=M$) we have by [27, Theorem 0.1] for any $\ell, m\in\mathbb{N}$,

$$T^{(0),f}_{k,0}(x,x)=\sum_{r=0}^{\ell} b_{f,r}(x,x)k^{n-r}+O(k^{n-\ell-1})\ \text{ in } \mathscr{C}^m(M). \tag{1.16}$$

Actually, in this case, due to the spectral gap of the Kodaira Laplacian [24, Theorem 1.5.5] we have $T^{(0)}_{f,k,k^{-N}}=T^{(0)}_{f,k,0}$ for k large enough, so (1.15) follows from (1.16). The expansion (1.15) bears resemblance to the expansion of the Toeplitz kernels for functions $f\in\mathscr{C}^p(M)$ (see [1, (3.19)]), for arbitrary $p\in\mathbb{N}$. In (1.15) the upper bound for the order of expansion ℓ is due to the size k^{-N} of the spectral parameter, while in case of symbols of class $\mathscr{C}^p(M)$ is due to the order of differentiability p.

It is interesting to note that Theorem 1.1 and the following results provide a generalization of various expansions for Toeplitz operators in the case of an arbitrary complex manifold endowed with a positive line bundle. In this case, we have simply $M=M(0)$. Of course, in such generality, the quantum spaces have to be spectral spaces $\mathscr{E}^q_{k^{-N}}(M,L^k)$.

The first three coefficients of the kernel expansions of Toeplitz operators and of their composition for the quantization of a compact Kähler manifold with positive line bundle were calculated by Ma–Marinescu [27] in the presence of a twisting vector bundle E and later by Hsiao [17] for $E=\mathbb{C}$. Both [17, 27] work with a general not necessarily Kähler base metric Θ which might not be polarized, that is, $\Theta\neq\frac{\sqrt{-1}}{2\pi}R^L$ in general. We will calculate the top coefficients $b_{f,1}(x,x)$ and $b_{f,2}(x,x)$ of the expansion (1.12) in Section 7. The coefficients $b_{f,0}(x,x)$ and $b_{f,1}(x,x)$ were given in [7] for $E=\mathbb{C}$ and $\Theta=\frac{\sqrt{-1}}{2\pi}R^L$. It is a remarkable manifestation of universality, that the coefficients for the quantization with holomorphic sections [17, 27] and for the quantization with spectral spaces used in this paper are given by the same formulas. We refer to [32] for an interpretation in graph-theoretic terms of the Toeplitz kernel expansion. The formulas from [27] play an essential role in the quantization of the Mabuchi energy [15] and Laplace operator [20]. On the set where the curvature of L is degenerate we have the following behavior.

Theorem 1.2. *Under the general assumptions (A) and (B), set*

$$M_{\deg}=\left\{x\in M;\ \dot{R}^L \text{ is degenerate at } x\in M\right\}.$$

Then for every $x_0\in M_{\deg}$, $\varepsilon>0$, $N>1$ and every $j\in\{0,1,\ldots,n\}$, there exist a neighborhood U of x_0 and $k_0>0$, such that for all $k\geq k_0$ we have

$$\left|T^{(j),f}_{k,k^{-N}}(x,x)\right|\leq\varepsilon k^n,\quad x\in U. \tag{1.17}$$

We consider next the composition of two Berezin–Toeplitz quantizations. We have first the following expansion of the kernels of Toeplitz operators.

Theorem 1.3. *Under the assumptions (A) and (B) let $j \in \{0, 1, \dots, n\}$ and $D_0, D_1 \Subset M$ on which L is trivial. Suppose that one of the following conditions is fulfilled:*
(i) $D_0 \Subset M(j)$ *and* $j \neq q$,
(ii) $D_0 \Subset M(q)$ *and* $\overline{D}_0 \bigcap \overline{D}_1 = \emptyset$.
Then, for every $N > 1$, $m \in \mathbb{N}$, there exists $C_{N,m} > 0$ independent of k such that

$$\Big|\big(T^{(q),f}_{k,k^{-N}} \circ T^{(q),g}_{k,k^{-N}}\big)(x,y)\Big|_{\mathscr{C}^m(D_0\times D_1)} \leq C_{N,m}k^{3n-\frac{N}{2}+2m}. \tag{1.18}$$

If $D_0 \Subset M(q)$ there exists a symbol

$$b_{f,g}(x,y,k) \in S^n(1; D_0 \times D_0, \Lambda^{0,q}(T^*M) \boxtimes (\Lambda^{0,q}(T^*M))^*)$$

such that for every $N > 1$, $m \in \mathbb{N}$, there exists $\widetilde{C}_{N,m} > 0$ independent of k such that

$$\Big|\big(T^{(q),f}_{k,k^{-N}} \circ T^{(q),g}_{k,k^{-N}}\big)(x,y) - e^{ik\Psi(x,y)}b_{f,g}(x,y,k)\Big|_{\mathscr{C}^m(D_0\times D_0)} \leq \widetilde{C}_{N,m}k^{3n-\frac{N}{2}+2m}, \tag{1.19}$$

where $b_{f,g}(x,y,k) \sim \sum_{j=0}^{\infty} b_{f,g,j}(x,y)k^{n-j}$ in the sense of Definition 2.1 and

$$b_{f,g,0}(x,x) = (2\pi)^{-n}f(x)g(x)\big|\det \dot{R}^L(x)\big|I_{\det \overline{W}^*}(x), \quad x \in D_0, \tag{1.20}$$

and $\Psi(x,y) \in \mathscr{C}^\infty(D_0 \times D_0)$ is as in Theorem 1.1.

It should be noticed that Theorem 1.3 holds for any Hermitian manifold M, not necessarily compact. Note that the estimates in Theorem 1.3 involve the power $k^{3n-\frac{N}{2}+2m}$ compared to $k^{2n-\frac{N}{2}+2m}$ in Theorem 1.1. We will explain why there are different exponents $3n$ and $2n$ in the proof of Theorem 6.1.

We will calculate the top coefficients $b_{f,1}(x,x)$, $b_{f,2}(x,x)$ and $b_{f,g,1}(x,x)$, $b_{f,g,2}(x,x)$ of the expansions (1.12) and (1.19) in Section 7 (see Theorems 7.1 and 7.4).

We come now to the asymptotic expansion of the composition of two Toeplitz operators in the operator norm. Let $A_k : L^2(M,L^k) \to L^2(M,L^k)$ be k-dependent continuous operator. We say that $A_k = \mathcal{O}(k^m + k^{m_1})$ as $k \to \infty$, locally in the L^2 operator norm if for any $\chi, \chi_1 \in \mathscr{C}^\infty_0(M)$, there exists $C > 0$ independent of k such that $\|\chi A_k \chi_1\|_{op} \leq C(k^m + k^{m_1})$, for k large, where $\|\cdot\|_{op}$ denotes the L^2 operator norm. We also denote by $\langle\cdot,|\cdot\rangle_\omega$ the Hermitian metric on $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ induced by $\omega := \frac{\sqrt{-1}}{2\pi}R^L$.

Theorem 1.4. *Under the assumptions (A) and (B) suppose moreover that $f, g \in \mathscr{C}^\infty(M)$ have compact support in $M(0)$. Then for every $N > 1$, there exist functions $C_p(f,g) \in \mathscr{C}^\infty_0(M(0))$, $p \in \mathbb{N}$, such that for any $\ell \in \mathbb{N}$ the product $T^{(0),f}_{k,k^{-N}}\, T^{(0),g}_{k,k^{-N}}$ has the asymptotic expansion*

$$T^{(0),f}_{k,k^{-N}} \circ T^{(0),g}_{k,k^{-N}} = \sum_{p=0}^{\ell} T^{(0),C_p(f,g)}_{k,k^{-N}}\, k^{-p} + \mathcal{O}(k^{-\ell-1} + k^{3n-\frac{N}{2}}), \quad k \to \infty, \tag{1.21}$$

locally in the L^2 operator norm. Moreover,

$$C_0(f,g) = fg\,, \quad C_1(f,g) = -\frac{1}{2\pi}\langle\, \partial f\,,\, \partial\overline{g}\,\rangle_\omega\,, \tag{1.22}$$

and therefore the commutator of two Toeplitz operators satisfies

$$\left[T^{(0),f}_{k,k^{-N}}, T^{(0),g}_{k,k^{-N}}\right] = \frac{\sqrt{-1}}{k} T^{(0),\{f,g\}}_{k,k^{-N}} + \mathcal{O}(k^{-2} + k^{3n-\frac{N}{2}}), \ \ k \to \infty, \tag{1.23}$$

where $\{f, g\}$ *is the Poisson bracket on* $(M(0), 2\pi\omega)$.

We will give formulas for the coefficients $C_j(f, g)$, $j = 0, 1, 3$, in Corollary 7.5. They have the same form as those in the expansion of the Toeplitz operators acting on spaces of holomorphic sections, see [17, (1.29)], [27, (0.20)]. Formula (1.23) represents the semiclassical correspondence principle between classical and quantum observables. Theorem 1.4 allows us to introduce a star-product on the set where a line positive is positive, see Remark 6.5.

As an application of Theorems 1.1 and 1.2, we obtain:

Theorem 1.5. *Assume (A) and (B) are fulfilled and let* $N > 2n$. *Then*

$$T^{(q),f}_{k,k^{-N}}(x,x) = k^n (2\pi)^{-n} \left|\det \dot{R}^L(x)\right| f(x) I_{\det \overline{W}^*}(x) + O(k^{n-1}),\ k \to \infty, \tag{1.24}$$

locally uniformly on $M(q)$*, for every* $D \Subset M$*, there exists* $C_D > 0$ *independent of* k *such that*

$$\left|T^{(q),f}_{k,k^{-N}}(x,x)\right| \le C_D k^n, \ \ \forall x \in D, \tag{1.25}$$

and if $\mathbb{1}_{M(q)}$ *denotes the characteristic function of* $M(q)$*, we have the pointwise convergence:*

$$\lim_{k\to\infty} k^{-n} T^{(q),f}_{k,k^{-N}}(x,x) = (2\pi)^{-n} f(x) \left|\det \dot{R}^L(x)\right| \mathbb{1}_{M(q)}(x) I_{\det \overline{W}^*}(x), \ \ x \in M. \tag{1.26}$$

Since $L^k_x \boxtimes (L^k_x)^* \cong \mathbb{C}$, we can identify $T^{(q),f}_{k,\lambda}(x,x)$ to an element of $\mathrm{End}(\Lambda^{0,q}_x(T^*M)$. Then

$$M \ni x \longmapsto T^{(q),f}_{k,\lambda}(x,x) \in \mathrm{End}(\Lambda^{0,q}_x(T^*M)) \tag{1.27}$$

is a smooth section of $\mathrm{End}(\Lambda^{0,q}(T^*M))$. Let $\mathrm{Tr}\, T^{(q),f}_{k,\lambda}(x,x)$ denote the trace of $T^{(q),f}_{k,\lambda}(x,x)$ with respect to $\langle \cdot, \cdot \rangle$. When M is compact, we define

$$\mathrm{Tr}\, T^{(q),f}_{k,\lambda} := \int_M \mathrm{Tr}\, T^{(q),f}_{k,\lambda}(x,x) dv_M(x). \tag{1.28}$$

For $\lambda = 0$, we set $T^{(q),f}_k := T^{(q),f}_{k,0}$, $T^{(q),f}_k(x,y) := T^{(q),f}_{k,0}(x,y)$, $\mathrm{Tr}\, T^{(q),f}_k(x,x) := \mathrm{Tr}\, T^{(q),f}_{k,0}(x,x)$, $\mathrm{Tr}\, T^{(q),f}_k := \mathrm{Tr}\, T^{(q),f}_{k,0}$.

From (1.24)–(1.26), we get Weyl's formula for Berezin–Toeplitz quantization.

Theorem 1.6. *Assume (A) and (B) are fulfilled and let* $N > 2n$. *If* M *is compact, then*

$$\mathrm{Tr}\, T^{(q),f}_{k,k^{-N}} = k^n (2\pi)^{-n} \int_{M(q)} f(x) \left|\det \dot{R}^L(x)\right| dv_M(x) + o(k^n),\ \ k \to \infty. \tag{1.29}$$

From Theorem 1.6 we deduce the following (see Section 8).

Theorem 1.7. *Under assumptions (A) and (B) suppose that* M *is compact and* $M(q-1) = \emptyset$, $M(q+1) = \emptyset$. *Then*

$$\lim_{k\to\infty} \left|k^{-n} T^{(q),f}_k(x,x) - (2\pi)^{-n} f(x) \left|\det \dot{R}^L(x)\right| \mathbb{1}_{M(q)}(x) I_{\det \overline{W}^*}(x)\right| = 0 \, in\, L^1_{(0,q)}(M). \tag{1.30}$$

In particular,

$$\operatorname{Tr} T_k^{(q),f} = k^n(2\pi)^{-n}\int_{M(q)} f(x)\left|\det \dot{R}^L(x)\right| dv_M(x) + o(k^n) \quad \text{as } k\to\infty. \tag{1.31}$$

Let's consider $q = 0$ and $f \equiv 1$ in (1.31). If $M(1) = \emptyset$, we obtain $\dim H^0(M, L^k) = k^n(2\pi)^{-n}\int_{M(0)}\left|\det \dot{R}^L(x)\right| dv_M(x) + o(k^n)$ as $k\to\infty$. Therefore, $\dim H^0(M, L^k) \sim k^n$ as $k\to\infty$, provided $M(0) \neq \emptyset$ and $M(1) = \emptyset$. This is a form of Demailly's criterion for a line bundle to be big, which answers the Grauert–Riemenschneider conjecture, see [11], [24, Theorem 2.2.27].

We wish now to link the quantization scheme, we proposed above by using spectral spaces $\mathscr{E}^q_{k^{-N}}(M, L^k)$ to the more traditional quantization using holomorphic sections (or, more generally, harmonic forms). For this purpose we need the notion of $O(k^{-N})$ small spectral gap property introduced in [18, Definition 1.5]:

Definition 1.8. Let $D \subset M$. We say that $\Box_k^{(q)}$ has $O(k^{-N})$ *small spectral gap on D* if there exist constants $C_D > 0$, $N \in \mathbb{N}$, $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$ and $u \in \Omega_0^{0,q}(D, L^k)$, we have

$$\left\|(I - P_{k,0}^{(q)})u\right\|_k \leq C_D\, k^N \left\|\Box_k^{(q)} u\right\|_k.$$

Let $D_0, D_1 \subset M$ be open sets and $A_k, C_k : \Omega_0^{0,q}(D_1) \to \Omega^{0,q}(D_0)$ be k-dependent continuous operators with smooth kernels $A_k(x,y), C_k(x,y) \in \mathscr{C}^\infty(D_0\times D_1, \Lambda^{0,q}(T^*M)\boxtimes(\Lambda^{0,q}(T^*M))^*)$. We write $A_k \equiv C_k \mod O(k^{-\infty})$ locally uniformly on $D_0\times D_1$ or $A_k(x,y)\equiv C_k(x,y) \mod O(k^{-\infty})$ locally uniformly on $D_0\times D_1$ if

$$\left|\partial_x^\alpha \partial_y^\beta (A_k(x,y) - C_k(x,y))\right| = O(k^{-N})$$

uniformly on every compact set in $D_0\times D_1$, for all $\alpha, \beta \in \mathbb{N}_0^{2n}$ and every $N > 1$.

The following result describes the asymptotics of the kernels of Toeplitz operators corresponding to *harmonic forms* in the case of small spectral gap.

Theorem 1.9. *Under the assumptions (A) and (B) let $j \in \{0, 1, \ldots, n\}$ and $D_0, D_1 \Subset M$ on which L is trivial. Suppose that one of the following conditions is fulfilled:*
(i) $D_0 \Subset M(j)$ *and* $j \neq q$,
(ii) $D_0 \Subset M(q)$, $\Box_k^{(q)}$ *has an* $O(k^{-N})$ *small spectral gap on* D_0 *and* $\overline{D}_0 \bigcap \overline{D}_1 = \emptyset$.
Then

$$T_k^{(q),f}(x,y) \equiv 0 \mod O(k^{-\infty}) \text{ locally uniformly on } D_0\times D_1,$$
$$(T_k^{(q),f}\circ T_k^{(q),g})(x,y) \equiv 0 \mod O(k^{-\infty}) \text{ locally uniformly on } D_0\times D_1.$$

Assume that $D_0 \Subset M(q)$ and $\Box_k^{(q)}$ has an $O(k^{-N})$ small spectral gap on D_0. Then,

$$T_k^{(q),f}(x,y) \equiv e^{ik\Psi(x,y)} b_f(x,y,k) \mod O(k^{-\infty}) \text{ locally uniformly on } D_0\times D_0,$$
$$(T_k^{(q),f}\circ T_k^{(q),g})(x,y) \equiv e^{ik\Psi(x,y)} b_{f,g}(x,y,k) \mod O(k^{-\infty}) \text{ locally uniformly on } D_0\times D_0,$$

where $b_f(x,y,k), b_{f,g}(x,y,k) \in S^n(1; D_0\times D_0, \Lambda^{0,q}(T^*M)\boxtimes(\Lambda^{0,q}(T^*M))^*)$ are as in (1.12) and (1.19), respectively, and $\Psi(x,y) \in \mathscr{C}^\infty(D_0\times D_0)$ is as in Theorem 1.1.

There are several geometric situations when there exists a spectral gap. For example, if L is a positive line bundle on a compact manifold M, or more generally, if L is uniformly positive on a complete manifold (M,Θ) with $\sqrt{-1}R^{K_M^*}$ and $\partial\Theta$ bounded below, then the Kodaira Laplacian $\Box_k^{(0)}$ has a "large" spectral gap on M, that is, there exists a constant $C>0$ such that for all k we have $\inf\{\lambda\in\operatorname{Spec}(\Box_k^{(0)});\,\lambda\neq 0\}\geq Ck$, (see [24, Theorem 1.5.5], [24, Theorem 6.1.1, (6.1.8)]). Therefore, we can recover from Theorem 1.9 results about quantization of noncompact manifolds, such as [24, Theorem 7.5.1], [25, Theorem 5.3], [26, Theorem 2.30].

In this paper, as applications of Theorem 1.9, we establish Berezin–Toeplitz quantization for semipositive and big line bundles. We assume now that (M,Θ) is compact and we set

$$\operatorname{Herm}(L)=\left\{\text{singular Hermitian metrics on } L\right\},$$

$$\mathcal{M}(L)=\Big\{h\in\operatorname{Herm}(L);\ h \text{ is smooth outside a proper analytic set}$$

$$\text{and the curvature current of } h \text{ is strictly positive}\Big\}.$$

Note that by Siu's criterion [24, Theorem 2.2.27], L is big under the hypotheses of Theorem 1.10 below. By [24, Lemma 2.3.6], $\mathcal{M}(L)\neq\emptyset$. Set

$$M':=\left\{p\in M;\ \exists\, h\in\mathcal{M}(L) \text{ with } h \text{ smooth near } p\right\}. \tag{1.32}$$

Theorem 1.10. *Let (M,Θ) be a compact Hermitian manifold. Let $(L,h)\to M$ be a Hermitian holomorphic line bundle with smooth Hermitian metric h having semipositive curvature and with $M(0)\neq\emptyset$. Let $f,g\in\mathscr{C}^\infty(M)$ and let $D_0\Subset M(0)\bigcap M'$ be an open set on which L is trivial. Then*

$$T_k^{(0),f}(x,y)\equiv e^{ik\Psi(x,y)}b_f(x,y,k)\quad \text{mod } O(k^{-\infty}) \text{ locally uniformly on } D_0\times D_0,$$

$$(T_k^{(0),f}\circ T_k^{(0),g})(x,y)\equiv e^{ik\Psi(x,y)}b_{f,g}(x,y,k)\quad \text{mod } O(k^{-\infty}) \text{ locally uniformly on } D_0\times D_0,$$

where $b_f(x,y,k), b_{f,g}(x,y,k)\in S^n(1;D_0\times D_0)$ are as in (1.12) and (1.19), respectively, and $\Psi(x,y)\in\mathscr{C}^\infty(D_0\times D_0)$ is as in Theorem 1.1.

Let us consider a singular Hermitian holomorphic line bundle $(L,h)\to M$ (see e.g., [24, Definition 2.3.1]). We assume that h is smooth outside a proper analytic set Σ and the curvature current of h is strictly positive. The metric h induces singular Hermitian metrics h^k on L^k. We denote by $\mathscr{I}(h^k)$ the Nadel multiplier ideal sheaf associated to h^k and by $H^0(M,L^k\otimes\mathscr{I}(h^k))\subset H^0(M,L^k)$ the space of global sections of the sheaf $\mathscr{O}(L^k)\otimes\mathscr{I}(h^k)$ (see (2.12)), where $H^0(M,L^k):=\left\{u\in\mathscr{C}^\infty(M,L^k);\ \overline{\partial}_k u=0\right\}$. We denote by $(\cdot\,,\cdot)_k$ the natural inner products on $\mathscr{C}^\infty(M,L^k\otimes\mathscr{I}(h^k))$ induced by h and the volume form dv_M on M (see (2.11) and see also (2.10) for the precise meaning of $\mathscr{C}^\infty(M,L^k\otimes\mathscr{I}(h^k))$). The (multiplier ideal) Bergman kernel of $H^0(M,L^k)\mathscr{I}(h^k))$ is the orthogonal projection

$$P_{k,\mathscr{I}}^{(0)}: L^2(M,L^k)\to H^0(M,L^k\otimes\mathscr{I}(h^k)). \tag{1.33}$$

Let $f\in\mathscr{C}^\infty(M)$. The multiplier ideal Berezin–Toeplitz operator is

$$T_{k,\mathscr{I}}^{(0),f}:=P_{k,\mathscr{I}}^{(0)}\circ f\circ P_{k,\mathscr{I}}^{(0)}: L^2(M,L^k)\to H^0(M,L^k\otimes\mathscr{I}(h^k)) \tag{1.34}$$

where we denote by f the multiplication operator on $L^2(M, L^k)$ by f. Let $T^{(0),f}_{k,\mathscr{I}}(x,y)$ be the distribution kernel of $T^{(0),f}_{k,\mathscr{I}}$. Note that $T^{(0),f}_{k,\mathscr{I}}(x,y) \in \mathscr{C}^\infty((M\setminus\Sigma) \times (M\setminus\Sigma), (L^k)^* \boxtimes L^k)$.

Theorem 1.11. *Let (L, h) be a singular Hermitian holomorphic line bundle over a compact Hermitian manifold (M, Θ). We assume that h is smooth outside a proper analytic set Σ and the curvature current of h is strictly positive. Let $f, g \in \mathscr{C}^\infty(M)$. Let $D_0 \subset M\setminus\Sigma$ be an open set on which L is trivial. Then*

$$T^{(0),f}_{k,\mathscr{I}}(x,y) \equiv e^{ik\Psi(x,y)} b_f(x,y,k) \mod O(k^{-\infty}) \text{ locally uniformly on } D_0 \times D_0,$$

$$(T^{(0),f}_{k,\mathscr{I}} \circ T^{(0),g}_{k,\mathscr{I}})(x,y) \equiv e^{ik\Psi(x,y)} b_{f,g}(x,y,k) \mod O(k^{-\infty}) \text{ locally uniformly on } D_0 \times D_0,$$

where $b_f(x,y,k), b_{f,g}(x,y,k) \in S^n(1; D_0 \times D_0)$ are as in (1.12) and (1.19), respectively, and $\Psi(x,y) \in \mathscr{C}^\infty(D_0 \times D_0)$ is as in Theorem 1.1.

The paper is organized as follows. In Section 2, we collect terminology, definitions and statements we use throughout. In Sections 3 and 4 prove the off-diagonal decay for the kernels $P^{(q)}_{k,k^{-N}}(\cdot,\cdot)$ and $T^{(q),f}_{k,k^{-N}}(\cdot,\cdot)$. In Section 5, we establish the full asymptotic of the Berezin–Toeplitz kernels $T^{(q),f}_{k,k^{-N}}(\cdot,\cdot)$ and prove Theorem 1.1. Section 6 is devoted to the expansion of the composition of two Toeplitz operators and contains the proof of Theorems 1.3, 1.4, and 1.9–1.11. In Section 7, we calculate the leading coefficients of the various expansions we established. Finally, in Section 8, we prove Theorems 1.2 and 1.7.

2. Preliminaries

Some standard notations. We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural numbers and by $\mathbb{R}$ the set of real numbers. We use the standard notations w^α, ∂^α_x for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, $w \in \mathbb{C}^m$, $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_m})$.

Let Ω be a $\mathscr{C}^\infty$ paracompact manifold equipped with a smooth density of integration. We let $T\Omega$ and $T^*\Omega$ denote the tangent bundle of Ω and the cotangent bundle of Ω, respectively. The complexified tangent bundle of Ω and the complexified cotangent bundle of Ω will be denoted by $\mathbb{C}T\Omega := T\Omega \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C}T^*\Omega := T^*\Omega \otimes_{\mathbb{R}} \mathbb{C}$, respectively. We write $\langle\cdot,\cdot\rangle$ to denote the pointwise duality between $T\Omega$ and $T^*\Omega$. We extend $\langle\cdot,\cdot\rangle$ bilinearly to $(T\Omega \otimes_{\mathbb{R}} \mathbb{C}) \times (T^*\Omega \otimes_{\mathbb{R}} \mathbb{C})$.

Let E be a $\mathscr{C}^\infty$ vector bundle over Ω. We write E^* to denote the dual bundle of E. The fiber of E at $x \in \Omega$ will be denoted by E_x. We denote by End (E) the vector bundle over Ω with fiber $\mathrm{End}(E)_x = \mathrm{End}(E_x)$ over $x \in \Omega$.

Let F be a vector bundle over another $\mathscr{C}^\infty$ paracompact manifold Ω'. We introduce the vector bundle $F \boxtimes E^* = \pi_1^*(F) \otimes \pi_2^*(E^*)$ over $\Omega' \times \Omega$, where π_1 and π_2 are the projections of $\Omega' \times \Omega$ on the first and second factor (see [24, p. 337]). The fiber of $F \boxtimes E^*$ over $(x,y) \in \Omega' \times \Omega$ consists of the linear maps from E_y to F_x.

Let $Y \subset \Omega$ be an open set and take any L^2 inner product on $\mathscr{C}^\infty_0(Y, E)$. By using this L^2 inner product, in this paper, we will consider a distribution section of E over Y is a continuous linear form on $\mathscr{C}^\infty_0(Y, E)$. From now on, the spaces distribution sections of E over Y will be denoted by $\mathscr{D}'(Y, E)$. Let $\mathscr{E}'(Y, E)$ be the subspace of $\mathscr{D}'(Y, E)$ whose elements have compact support in Y. For $m \in \mathbb{R}$, we let $H^m(Y,E)$ denote the Sobolev space of order m of sections of E over Y. Put

$$H^m_{\mathrm{loc}}(Y,E) = \left\{u \in \mathscr{D}'(Y,E);\ \varphi u \in H^m(Y,E),\ \varphi \in \mathscr{C}^\infty_0(Y)\right\},$$

$$H^m_{\mathrm{comp}}(Y,E) = H^m_{\mathrm{loc}}(Y,E) \cap \mathscr{E}'(Y,E).$$

Let M be a complex manifold of dimension n. We always assume that M is paracompact. Let $T^{1,0}M$ and $T^{0,1}M$ denote the holomorphic tangent bundle of M and the antiholomorphic tangent bundle of M, respectively. Let $\Lambda^{1,0}(T^*M)$ be the holomorphic cotangent bundle of M and let $\Lambda^{0,1}(T^*M)$ be the antiholomorphic cotangent bundle of M. For $p,q \in \mathbb{N}$, let $\Lambda^{p,q}(T^*M) = \Lambda^p(\Lambda^{1,0}(T^*M)) \otimes \Lambda^q(\Lambda^{0,1}(T^*M))$ be the bundle of (p,q) forms of M.

For an open set $D \subset M$ we let $\Omega^{p,q}(D)$ denote the space of smooth sections of $\Lambda^{p,q}(T^*M)$ over D and let $\Omega^{0,q}_0(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D. Similarly, if E is a vector bundle over D, then we let $\Omega^{p,q}(D,E)$ denote the space of smooth sections of $\Lambda^{p,q}(T^*M)\otimes E$ over D. Let $\Omega^{p,q}_0(D,E)$ be the subspace of $\Omega^{p,q}(D,E)$ whose elements have compact support in D.

For a multi-index $J = (j_1,\ldots,j_q) \in \{1,\ldots,n\}^q$ we set $|J| = q$. We say that J is strictly increasing if $1 \leqslant j_1 < j_2 < \cdots < j_q \leqslant n$. Let $\{e_1,\ldots,e_n\}$ be a local frame for $\Lambda^{0,1}(T^*M)$ on an open set $D \subset M$. For a multi-index $J = (j_1,\ldots,j_q) \in \{1,\ldots,n\}^q$, we put $e^J = e_{j_1} \wedge \cdots \wedge e_{j_q}$. Let E be a vector bundle over D and let $f \in \Omega^{0,q}(D,E)$. f has the local representation

$$f|_D = \sum_{|J|=q}{}' f_J(z) e^J,$$

where $\sum'$ means that the summation is performed only over strictly increasing multi-indices and $f_J \in \mathscr{C}^\infty(D,E)$.

Metric data. Let (M,Θ) be a complex manifold of dimension n, where Θ is a smooth positive $(1,1)$ form, which induces a Hermitian metric $\langle\cdot,\cdot\rangle$ on the holomorphic tangent bundle $T^{1,0}M$. In local holomorphic coordinates $z = (z_1,\ldots,z_n)$, if $\Theta = \sqrt{-1}\sum_{j,k=1}^n \Theta_{j,k} dz_j \wedge d\overline{z}_k$, then $\langle \frac{\partial}{\partial z_j} \mid \frac{\partial}{\partial z_k} \rangle = \Theta_{j,k}, j,k = 1,\ldots,n$. We extend the Hermitian metric $\langle\cdot,\cdot\rangle$ to $TM \otimes_{\mathbb{R}} \mathbb{C}$ in a natural way. The Hermitian metric $\langle\cdot,\cdot\rangle$ on $TM \otimes_{\mathbb{R}} \mathbb{C}$ induces a Hermitian metric on $\Lambda^{p,q}(T^*M)$ also denoted by $\langle\cdot,\cdot\rangle$.

Let (L,h) be a Hermitian holomorphic line bundle over M, where the Hermitian metric on L is denoted by h. Until further notice, we assume that h is smooth. Given a local holomorphic frame s of L on an open subset $D \subset M$ we define the associated local weight of h by

$$|s(x)|^2_h = e^{-2\phi(x)}, \quad \phi \in \mathscr{C}^\infty(D,\mathbb{R}). \tag{2.1}$$

Let $R^L = (\nabla^L)^2$ be the Chern curvature of L, where ∇^L is the Hermitian holomorphic connection. Then $R^L|_D = 2\partial\overline{\partial}\phi$.

Let L^k, $k > 0$, be the kth tensor power of the line bundle L. The Hermitian fiber metric on L induces a Hermitian fiber metric on L^k that we shall denote by h^k. If s is a local trivializing holomorphic section of L then s^k is a local trivializing holomorphic section of L^k. For $p,q \in \mathbb{N}$, the Hermitian metric $\langle\cdot,\cdot\rangle$ on $\Lambda^{p,q}(T^*M)$ and h^k induce a Hermitian metric on $\Lambda^{p,q}(T^*M)\otimes L^k$, denoted by $\langle\cdot,\cdot\rangle_{h^k}$. For $s \in \Omega^{p,q}(M,L^k)$, we denote the pointwise norm $|s(x)|^2_{h^k} := \langle s(x), s(x)\rangle_{h^k}$. We take $dv_M = dv_M(x)$ as the induced volume form on M.

The L^2–Hermitian inner products on the spaces $\Omega_0^{p,q}(M, L^k)$ and $\Omega_0^{p,q}(M)$ are given by

$$(s_1, s_2)_k = \int_M \langle s_1(x), s_2(x) \rangle_{h^k} \, dv_M(x) \,, \quad s_1, s_2 \in \Omega_0^{p,q}(M, L^k),$$

$$(f_1, f_2) = \int_M \langle f_1(x), f_2(x) \rangle \, dv_M(x) \,, \quad f_1, f_2 \in \Omega_0^{p,q}(M). \tag{2.2}$$

$$\|s\|_k^2 = (s, s)_k, \quad s \in \Omega_0^{p,q}(M, L^k), \qquad \|f\|^2 := (f, f), \quad f \in \Omega_0^{p,q}(M).$$

Let $A_k : L^2_{(0,q)}(M, L^k) \to L^2_{(0,q)}(M, L^k)$ be a k-dependent continuous operator with smooth kernel $A_k(x, y)$. Let s, $\widehat{s}$ be local trivializing holomorphic sections of L on $D_0 \Subset M$, $D_1 \Subset M$, respectively, $|s|_h^2 = e^{-2\phi}$, $|\widehat{s}|_h^2 = e^{-2\widehat{\phi}}$, where D_0, D_1 are open sets. The localized operator of A_k on $D_0 \times D_1$ is given by

$$A_{k,s,\widehat{s}} : \Omega_0^{0,q}(D_1) \to \Omega^{0,q}(D_0), \quad u \longmapsto s^{-k} e^{-k\phi} (A_k \widehat{s}^k e^{k\widehat{\phi}} u), \tag{2.3}$$

and let $A_{k,s,\widehat{s}}(x, y) \in \mathscr{C}^\infty(D_0 \times D_1, \Lambda^{0,q}(T^*M)) \boxtimes (\Lambda^{0,q}(T^*M))^*)$ be the distribution kernel of $A_{k,s,\widehat{s}}$. For $s = \widehat{s}$, $D_0 = D_1$, we set

$$A_{k,s} := A_{k,s,s} \,, \quad A_{k,s}(x, y) := A_{k,s,s}(x, y). \tag{2.4}$$

A self-adjoint extension of the Kodaira Laplacian. We denote by

$$\overline{\partial}_k : \Omega^{0,r}(M, L^k) \to \Omega^{0,r+1}(M, L^k) \,, \; \overline{\partial}_k^* : \Omega^{0,r+1}(M, L^k) \to \Omega^{0,r}(M, L^k) \tag{2.5}$$

the Cauchy–Riemann operator acting on sections of L^k and its formal adjoint with respect to $(\,\cdot\,|\,\cdot\,)_k$, respectively. Let

$$\Box_k^{(q)} := \overline{\partial}_k \overline{\partial}_k^* + \overline{\partial}_k^* \overline{\partial}_k : \Omega^{0,q}(M, L^k) \to \Omega^{0,q}(M, L^k) \tag{2.6}$$

be the Kodaira Laplacian acting on $(0, q)$–forms with values in L^k. We extend $\overline{\partial}_k$ to $L^2_{(0,r)}(M, L^k)$ by

$$\overline{\partial}_k : \mathrm{Dom}\, \overline{\partial}_k \subset L^2_{(0,r)}(M, L^k) \to L^2_{(0,r+1)}(M, L^k) \,, \tag{2.7}$$

where $\mathrm{Dom}\, \overline{\partial}_k := \{u \in L^2_{(0,r)}(M, L^k); \; \overline{\partial}_k u \in L^2_{(0,r+1)}(M, L^k)\}$, where $\overline{\partial}_k u$ is defined in the sense of distributions. We also write

$$\overline{\partial}_k^* : \mathrm{Dom}\, \overline{\partial}_k^* \subset L^2_{(0,r+1)}(M, L^k) \to L^2_{(0,r)}(M, L^k) \tag{2.8}$$

to denote the Hilbert space adjoint of $\overline{\partial}_k$ in the L^2 space with respect to $(\,\cdot\,,\,\cdot\,)_k$. Let $\Box_k^{(q)}$ denote the Gaffney extension of the Kodaira Laplacian given by

$$\mathrm{Dom}\, \Box_k^{(q)} = \left\{ s \in L^2_{(0,q)}(M, L^k); \; s \in \mathrm{Dom}\, \overline{\partial}_k \cap \mathrm{Dom}\, \overline{\partial}_k^*, \;\; \overline{\partial}_k s \in \mathrm{Dom}\, \overline{\partial}_k^*, \; \overline{\partial}_k^* s \in \mathrm{Dom}\, \overline{\partial}_k \right\}, \tag{2.9}$$

and $\Box_k^{(q)} s = \overline{\partial}_k \overline{\partial}_k^* s + \overline{\partial}_k^* \overline{\partial}_k s$ for $s \in \mathrm{Dom}\, \Box_k^{(q)}$. By a result of Gaffney [24, Proposition 3.1.2], $\Box_k^{(q)}$ is a positive self-adjoint operator. Note that if M is complete, the Kodaira Laplacian $\Box_k^{(q)}$ is essentially self-adjoint [24, Corollary 3.3.4] and the Gaffney extension coincides with the Friedrichs extension of $\Box_k^{(q)}$.

Consider a singular Hermitian metric h on a holomorphic line bundle L over M. If h_0 is a smooth Hermitian metric on L then $h = h_0 e^{-2\varphi}$ for some function $\varphi \in L^1_{loc}(M, \mathbb{R})$. The *Nadel* *multiplier ideal sheaf* of h is defined by $\mathscr{I}(h) = \mathscr{I}(\varphi)$; the definition does not depend on the choice of h_0. Recall that the *Nadel multiplier ideal sheaf* $\mathscr{I}(\varphi) \subset \mathscr{O}_M$ is the ideal subsheaf of germs of holomorphic functions $f \in \mathscr{O}_{M,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in local coordinates near x for all $x \in M$. Put

$$\mathscr{C}^\infty(M, L \otimes \mathscr{I}(h)) := \left\{ S \in \mathscr{C}^\infty(M, L);\ \int_M \left|S\right|_h^2 dv_M = \int_M \left|S\right|_{h_0}^2 e^{-2\varphi}\, dv_M < \infty \right\}, \tag{2.10}$$

where $|\cdot|_h$ and $|\cdot|_{h_0}$ denote the pointwise norms for sections induced by h and h_0, respectively. With the help of h and the volume form dv_M we define an L^2 inner product on $\mathscr{C}^\infty(M, L \otimes \mathscr{I}(h))$:

$$(S, S')_h = \int_M \langle S, S' \rangle_{h_0} e^{-2\varphi} dv_M\,, \quad S, S' \in \mathscr{C}^\infty(M, L \otimes \mathscr{I}(h))\,. \tag{2.11}$$

The singular Hermitian metric h induces a singular Hermitian metric $h^k = h_0^k e^{-2k\varphi}$ on L^k, $k > 0$. We denote by $(\,\cdot\,,\cdot\,)_k$ the natural inner products on $\mathscr{C}^\infty(M, L^k \otimes \mathscr{I}(h^k))$ defined as in (2.11) and by $L^2(M, L^k)$ the completion of $\mathscr{C}^\infty(M, L^k \otimes \mathscr{I}(h^k))$ with respect to $(\,\cdot\,,\cdot\,)_k$. The space of global sections in the sheaf $\mathscr{O}(L^k) \otimes \mathscr{I}(h^k)$ is given by

$$\begin{aligned} &H^0(M, L^k \otimes \mathscr{I}(h^k)) \\ &\quad = \left\{ s \in \mathscr{C}^\infty(M, L^k);\ \overline{\partial}_k s = 0,\ \int_M \left|s\right|_{h^k}^2 dv_M = \int_M \left|s\right|_{h_0^k}^2 e^{-2k\varphi}\, dv_M < \infty \right\}. \end{aligned} \tag{2.12}$$

Schwartz kernel theorem and semiclassical Hörmander symbol spaces. We recall here the Schwartz kernel theorem [16, Theorems 5.2.1, 5.2.6], [24, Thorem B.2.7]. Let Ω be a $\mathscr{C}^\infty$ paracompact manifold equipped with a smooth density of integration. Let E and F be smooth vector bundles over Ω. Any distribution ("kernel")

$$A(x, y) \in \mathscr{D}'(\Omega \times \Omega, F \boxtimes E^*), \tag{2.13}$$

defines a continuous operator

$$A : \mathscr{C}_0^\infty(\Omega, E) \to \mathscr{D}'(\Omega, F)\,, \quad \langle Au, v \rangle := \langle A(x, y), v(x) \otimes u(y) \rangle, \tag{2.14}$$

for any $u \in \mathscr{C}_0^\infty(\Omega, E)$, $v \in \mathscr{C}_0^\infty(\Omega, F)$. Conversely, any continuous linear operator $A : \mathscr{C}_0^\infty(\Omega, E) \to \mathscr{D}'(\Omega, F)$ is given by a distribution $A(x, y) \in \mathscr{D}'(\Omega \times \Omega, F \boxtimes E^*)$ as above, called the Schwartz distribution kernel of A. Moreover, the following two statements are equivalent

$$\begin{aligned} &\text{(a) } A \text{ is continuous: } \mathscr{E}'(\Omega, E) \to \mathscr{C}^\infty(\Omega, F), \\ &\text{(b) } A(x, y) \in \mathscr{C}^\infty(\Omega \times \Omega, F \boxtimes E^*). \end{aligned} \tag{2.15}$$

If A satisfies (a) or (b), we say that A is a *smoothing operator*. Furthermore, A is smoothing if and only if $A : H^s_{\rm comp}(\Omega, E) \to H^{s+N}_{\rm loc}(\Omega, F)$ is continuous, for all $N \geq 0$, $s \in \mathbb{R}$. Let $A, B : \mathscr{C}_0^\infty(\Omega, E) \to \mathscr{D}'(\Omega, F)$ be continuous operators. We write $A \equiv B$ or $A(x, y) \equiv B(x, y)$ if $A - B$ is a smoothing operator.

We say that A is properly supported if the restrictions to $\operatorname{Supp} A(\cdot, \cdot)$ of the projections π_1 and π_2 from $\Omega \times \Omega$ to the first and second factor are proper.

We say that A is smoothing away the diagonal if $\chi_1 A \chi_2$ is smoothing for all $\chi_1, \chi_2 \in \mathscr{C}_0^\infty(\Omega)$ with $\operatorname{Supp} \chi_1 \cap \operatorname{Supp} \chi_2 = \emptyset$.

We recall the definition of semiclassical Hörmander symbol spaces [13, Chapter 8]:

Definition 2.1. Let U be an open set in $\mathbb{R}^N$. Let

$$S(1) = S(1;U) := \Big\{a \in \mathscr{C}^\infty(U) \,|\, \forall \alpha \in \mathbb{N}_0^N : \sup_{x\in U} \big|\partial^\alpha a(x)\big| < \infty\Big\},$$

$$S^0(1;U) := \Big\{(a(\cdot,k))_{k\in\mathbb{N}} \in \mathscr{C}^\infty(U)^{\mathbb{N}} \,|\, \forall \alpha \in \mathbb{N}_0^N \;:\; \sup_{k\in\mathbb{N}} \sup_{x\in U} \big|\partial^\alpha a(x,k)\big| < \infty\Big\}.$$

For $m \in \mathbb{R}$ let

$$S^m(1;U) = \Big\{(a(\cdot,k))_{k\in\mathbb{N}} \in \mathscr{C}^\infty(U)^{\mathbb{N}} \,|\, (k^{-m}a(\cdot,k)) \in S^0(1;U)\Big\}.$$

Hence $(a(\cdot,k)) \in S^m(1;U)$ if for every $\alpha \in \mathbb{N}_0^N$, there exists $C_\alpha > 0$, such that $|\partial^\alpha a(\cdot,k)| \leq C_\alpha k^m$ on W. Consider a sequence $a_j \in S^{m_j}(1)$, $j \in \mathbb{N}$, where $m_j \searrow -\infty$, and let $a \in S^{m_0}(1)$. We say that

$$a(\cdot,k) \sim \sum_{j=0}^\infty a_j(\cdot,k), \;\text{ in } S^{m_0}(1),$$

if for every $\ell \in \mathbb{N}$ we have $a - \sum_{j=0}^{\ell} a_j \in S^{m_{\ell+1}}(1)$. For a given sequence a_j as above, we can always find such an asymptotic sum a, which is unique up to an element in $S^{-\infty}(1) = S^{-\infty}(1;U) := \cap_m S^m(1)$. We define $S^m(1;Y,E)$ in the natural way, where Y is a smooth paracompact manifold and E is a vector bundle over Y.

3. Spectral kernel estimates away the diagonal

The goal of this section is to prove the off-diagonal decay for the kernel $P^{(q)}_{k,k^{-N}}(\cdot,\cdot)$ of the spectral projection $P^{(q)}_{k,k^{-N}}$. For this purpose, we introduce a localization of the projection. Let $s, \widehat{s}$ be local trivializing holomorphic sections of L on $D_0 \Subset M$, $D_1 \Subset M$, respectively, $|s|_h^2 = e^{-2\phi}$, $|\widehat{s}|_h^2 = e^{-2\widehat{\phi}}$, where D_0, D_1 are open sets. We denote by $P^{(q)}_{k,k^{-N},s,\widehat{s}}$ the localization given by (2.3).

Let $\{e_1, e_2, \ldots, e_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ be orthonormal frames of $\Lambda^{0,1}(T^*M)$ on D_0 and D_1, respectively. Then,

$$\big\{e^J;\, |J| = q,\;\; J \text{ is strictly increasing}\big\},\;\; \big\{w^J;\, |J| = q,\;\; J \text{ is strictly increasing}\big\}$$

are orthonormal frames of $\Lambda^{0,q}(T^*M)$ on D_0 and D_1, respectively. We write

$$\begin{aligned} &P^{(q)}_{k,k^{-N},s,\widehat{s}}(x,y) = \sum_{|I|=|J|=q}{}' P^{(q),I,J}_{k,k^{-N},s,\widehat{s}}(x,y) e^I(x) \wedge (w^J(y))^\dagger, \\ &P^{(q),I,J}_{k,k^{-N},s,\widehat{s}}(x,y) \in \mathscr{C}^\infty(D_0 \times D_1), \forall\, |I| = |J| = q, I, J \text{ are strictly increasing}, \end{aligned} \tag{3.1}$$

in the sense that for every $u = \sum_{|J|=q}{}' u_J w^J \in \Omega_0^{0,q}(D_1)$, we have

$$(P^{(q)}_{k,k^{-N},s,\widehat{s}}u)(x) = \sum_{|I|=|J|=q}{}' e^I(x) \otimes \int P^{(q),I,J}_{k,k^{-N},s,\widehat{s}}(x,y) u_J(y) dv_M(y). \tag{3.2}$$

The goal of this section is to prove the following.

Theorem 3.1. *With the notations used above, we assume that* $D_0 \Subset M(j)$, $j \neq q$, $j \in \{0, 1, \ldots, n\}$ *or* $D_0 \Subset M(q)$ *and* $\overline{D}_0 \bigcap \overline{D}_1 = \emptyset$. *Then, for every* $N > 1$, $m \in \mathbb{N}$, *there exists* $C_{N,m} > 0$ *independent of k, such that for all strictly increasing I, J with* $|I| = |J| = q$,

$$\left|P^{(q),I,J}_{k,k^{-N},s,\widehat{s}}(x,y)\right|_{\mathscr{C}^m(D_0\times D_1)} \leq C_{N,m}k^{2n-\frac{N}{2}+2m}.$$

As preparation, we recall the next result, established in [18, Theorems 4.11 and 4.12]. The localization $P^{(q)}_{k,k^{-N},s}$ is defined as in (2.4).

Theorem 3.2. *With the notations used above, assume that* $D_0 \Subset M(q)$. *Then, for every* $N > 1$, $m \in \mathbb{N}$, *there exists* $C_{N,m} > 0$ *independent of k such that*

$$\left|P^{(q)}_{k,k^{-N},s}(x,y) - e^{ik\Psi(x,y)}b(x,y,k)\right|_{\mathscr{C}^m(D_0\times D_0)} \leq C_{N,m}k^{3n-N+2m},$$

where

$$b(x,y,k) \in S^n(1; D_0 \times D_0, \Lambda^{0,q}(T^*M) \boxtimes (\Lambda^{0,q}(T^*M))^*),$$

$$b(x,y,k) \sim \sum_{j=0}^{\infty} b_j(x,y)k^{n-j} \text{ in } S^n(1; D_0 \times D_0, \Lambda^{0,q}(T^*M) \boxtimes (\Lambda^{0,q}(T^*M))^*),$$

$$b_j(x,y) \in \mathscr{C}^\infty(D_0 \times D_0, (\Lambda^{0,q}(T^*M) \boxtimes (\Lambda^{0,q}(T^*M))^*),\ j = 0, 1, 2, \ldots,$$

$$b_0(x,x) = (2\pi)^{-n}\left|\det \dot{R}^L(x)\right| I_{\det \overline{W}^*}(x),\ \ \forall x \in D_0,$$

and $b(x,y,k)$ *is properly supported and* $\Psi(x,y) \in \mathscr{C}^\infty(D_0 \times D_0)$ *is as in Theorem 1.1.*

Assume that $D_0 \Subset M(j)$, $j \neq q$, $j \in \{0, 1, 2, \ldots, n\}$. *Then, for every* $N > 1$, $m \in \mathbb{N}$, *there exists* $\widetilde{C}_{N,m} > 0$ *independent of k such that*

$$\left|P^{(q)}_{k,k^{-N},s}(x,y)\right|_{\mathscr{C}^m(D_0\times D_0)} \leq \widetilde{C}_{N,m}k^{3n-N+2m}.$$

The following properties of the phase function Ψ follow also from [18, Theorem 3.8].

Theorem 3.3. *With the assumptions and notations used in Theorem 1.1, for a given point* $p \in D_0$, *let* $x = z = (z_1, \ldots, z_n)$ *be local holomorphic coordinates centered at p satisfying*

$$\Theta(p) = \sqrt{-1}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j\,,$$
$$\phi(z) = \sum_{j=1}^{n} \lambda_j \left|z_j\right|^2 + O(|z|^3)\,,\ z \text{ near } p\,,\ \{\lambda_j\}_{j=1}^n \subset \mathbb{R}\backslash\{0\}\,, \tag{3.3}$$

then we have near $(0,0)$,

$$\Psi(z,w) = i\sum_{j=1}^{n} \left|\lambda_j\right| \left|z_j - w_j\right|^2 + i\sum_{j=1}^{n} \lambda_j(\overline{z}_j w_j - z_j\overline{w}_j) + O(|(z,w)|^3). \tag{3.4}$$

Moreover, when $q = 0$, we have

$$\Psi(z,w) = i\big(\phi(z)+\phi(w)\big) - 2i \sum_{\alpha,\beta\in\mathbb{N},|\alpha|+|\beta|\le N} \frac{\partial^{|\alpha|+|\beta|}\phi}{\partial z^\alpha \partial \overline{z}^\beta}(0)\frac{z^\alpha}{\alpha!}\frac{\overline{w}^\beta}{\beta!} + O(|(z,w)|^{N+1}), \tag{3.5}$$

for every $N \in \mathbb{N}$.

Fix $N > 1$. Let $\{g_1(x), g_2(x), \dots, g_{d_k}(x)\}$ be an orthonormal frame for $\mathscr{E}_{k^{-N}}(M, L^k)$, where $d_k \in \mathbb{N}\cup\{\infty\}$. On D_0, D_1, we write

$$\begin{aligned} g_j(x) &= s^k(x)\widetilde{g}_j(x),\ \ \widetilde{g}_j(x) = \sideset{}{'}\sum_{|J|=q} \widetilde{g}_{j,J}(x)e^J(x) \text{ on } D_0,\ \ j=1,\dots,d_k, \\ g_j(x) &= \widehat{s}^k(x)\widehat{g}_j(x),\ \ \widehat{g}_j(x) = \sideset{}{'}\sum_{|J|=q} \widehat{g}_{j,J}(x)w^J(x) \text{ on } D_1,\ \ j=1,\dots,d_k. \end{aligned} \tag{3.6}$$

It is not difficult to check that for every strictly increasing I, J, with $|I| = |J| = q$, we have

$$\begin{aligned} P^{(q),I,J}_{k,k^{-N},s,\widehat{s}}(x,y) &= \sum_{j=1}^{d_k} \widetilde{g}_{j,I}(x)e^{-k\phi(x)}\overline{\widehat{g}_{j,J}}(y)e^{-k\widehat{\phi}(y)}, \\ P^{(q),I,J}_{k,k^{-N},s}(x,y) &= \sum_{j=1}^{d_k} \widetilde{g}_{j,I}(x)e^{-k\phi(x)}\overline{\widetilde{g}_{j,J}}(y)e^{-k\widehat{\phi}(y)}. \end{aligned} \tag{3.7}$$

Lemma 3.4. *Assume that $D_0 \Subset M(j)$, $j \neq q$. Then, for every $N > 1$, $m \in \mathbb{N}$, there exists $C_{N,m} > 0$ independent of k such that for every strictly increasing I, J, with $|I| = |J| = q$,*

$$\left| P^{(q),I,J}_{k,k^{-N},s,\widehat{s}}(x,y) \right|_{\mathscr{C}^m(D_0\times D_1)} \le C_{N,m}k^{2n-\frac{N}{2}+2m}.$$

Proof. Fix I, J are strictly increasing, $|I| = |J| = q$, and let $\alpha, \beta \in \mathbb{N}_0^{2n}$. By (3.7), we have

$$\left|\partial_x^\alpha \partial_y^\beta P^{(q),I,J}_{k,k^{-N},s,\widehat{s}}(x,y)\right| \le \sqrt{\sum_{j=1}^{d_k}\left|\partial_x^\alpha\big(\widetilde{g}_{j,I}(x)e^{-k\phi(x)}\big)\right|^2}\sqrt{\sum_{j=1}^{d_k}\left|\partial_y^\beta\big(\widehat{g}_{j,J}(y)e^{-k\widehat{\phi}(y)}\big)\right|^2}. \tag{3.8}$$

In view of Theorem 3.2, we see that

$$\sum_{j=1}^{d_k}\left|\partial_x^\alpha\big(\widetilde{g}_{j,I}(x)e^{-k\phi(x)}\big)\right|^2 \le C_\alpha k^{3n-N+4|\alpha|}, \text{ on } D_0, \tag{3.9}$$

where $C_\alpha > 0$ is a constant independent of k. Moreover, it is known (see [18, Theorem 4.3]) that

$$\sum_{j=1}^{d_k}\left|\partial_y^\beta\big(\widehat{g}_{j,J}(y)e^{-k\widehat{\phi}(y)}\big)\right|^2 \le C_\beta k^{n+4|\beta|} \text{ on } D_1, \tag{3.10}$$

where $C_\beta > 0$ is a constant independent of k. From (3.8) to (3.10), the lemma follows. □

Lemma 3.4 provides the proof of Theorem 3.1 in the case $D_0 \Subset M(j)$, $j \neq q$. Now, we assume that $D_0 \Subset M(q)$. Fix $p \in D_0$, I_0, J_0 strictly increasing with $|I_0| = |J_0| = q$, and $\alpha, \beta \in \mathbb{N}_0^{2n}$. Put

$$\begin{aligned}
&\partial_x^\alpha \partial_y^\alpha e^{ik\Psi(x,y)} b(x,y,k) = e^{ik\Psi(x,y)} a(x,y,k) = \sum_{|I|=|J|=q}{}' e^{ik\Psi(x,y)} a_{I,J}(x,y,k) e^I(x) \wedge (e^J(y))^\dagger,\\
&\partial_y^\alpha e^{ik\Psi(x,y)} b(x,y,k) = e^{ik\Psi(x,y)} d(x,y,k) = \sum_{|I|=|J|=q}{}' e^{ik\Psi(x,y)} d_{I,J}(x,y,k) e^I(x) \wedge (e^J(y))^\dagger,
\end{aligned} \tag{3.11}$$

where $\Psi(x,y)$ and $b(x,y,k)$ are as in Theorem 3.2 and $a_{I,J}(x,y,k), d_{I,J}(x,y,k) \in \mathscr{C}^\infty(D_0 \times D_0)$, for any I, J strictly increasing with $|I| = |J| = q$.

Lemma 3.5. *Assume that $a_{I_0,I_0}(p,p,k) \leq 0$. Then, there exists $C_{\alpha,\beta} > 0$ independent of k and the point p such that*

$$\left|\partial_x^\alpha \partial_y^\beta P^{(q),I_0,J_0}_{k,k^{-N},s,\hat{s}}(p,y)\right| \leq C_{\alpha,\beta} k^{2n-\frac{N}{2}+2|\alpha|+2|\beta|}, \quad \forall y \in D_1.$$

Proof. In view of Theorems 3.2 and (3.7), we see that

$$\begin{aligned}
\sum_{j=1}^{d_k} \left|\partial_x^\alpha (\widetilde{g}_{j,I_0} e^{-k\phi})(p))\right|^2 &= \partial_x^\alpha \partial_y^\alpha P^{(q),I_0,I_0}_{k,k^{-N},s}(p,p) \leq \left|\partial_x^\alpha \partial_y^\alpha P^{(q),I_0,I_0}_{k,k^{-N},s}(p,p) - a_{I_0,I_0}(p,p)\right|\\
&\leq C_\alpha k^{3n-N+4|\alpha|},
\end{aligned} \tag{3.12}$$

where $C_\alpha > 0$ is a constant independent of k and the point p. From (3.12), (3.8), and (3.10), the lemma follows. ☐

Now, we assume that $a_{I_0,I_0}(p,p) > 0$. Take $\chi \in \mathscr{C}_0^\infty(\mathbb{R},[0,1])$ so that $\chi = 1$ if $|x| \leq 1$, $\chi = 0$ if $|x| > 2$. Put

$$\begin{aligned}
&\widetilde{u}_k(x) = \frac{1}{\sqrt{a_{I_0,I_0}(p,p,k)}} e^{ik\Psi(x,p)} \chi\left(\frac{|x-p|^2}{\varepsilon}\right) \sum_{|I|=q}{}' d_{I,I_0}(x,p,k) e^I(x) \in \Omega_0^{0,q}(D_0),\\
&u_k(x) = s^k(x)\widetilde{u}_k(x) e^{k\phi(x)} \in \Omega_0^{0,q}(D_0, L^k),
\end{aligned} \tag{3.13}$$

where $\varepsilon > 0$ is a small constant and $d_{I,I_0}(x,y)$ is as in (3.11). We need

Lemma 3.6. *We have*

$$u_k(x) \equiv \frac{P^{(q)}_{k,k^{-N}} u_k(x)}{\|P^{(q)}_{k,k^{-N}} u_k\|_k} \quad \text{mod } O(k^{-\infty}) \text{ on } M,$$

that is, for every local trivializing holomorphic section s_1 of L on an open set $W \Subset M$, $|s_1|_h^2 = e^{-2\phi_1}$, we have

$$s_1^{-k} e^{-k\phi_1} u_k(x) \equiv s_1^{-k} e^{-k\phi} \frac{P^{(q)}_{k,k^{-N}} u_k(x)}{\|P^{(q)}_{k,k^{-N}} u_k\|_k} \quad \text{mod } O(k^{-\infty}) \text{ on } W.$$

Proof. It is known from [18, Theorems 3.11 and 3.12] that

$$\int e^{ik\Psi(x,y)}e^{-ik\overline{\Psi}(x,y)}\left|\chi\left(\frac{|x-y|^2}{\varepsilon}\right)\right|^2\sideset{}{'}\sum_{|I|=q}\left|d_{I,I_0}(x,y)\right|^2 dv_M(x)$$
$$\equiv a_{I_0,I_0}(y,y,k) \mod O(k^{-\infty}). \tag{3.14}$$

From (3.14), it is easy to see that

$$\|u_k\|_k \equiv 1 \mod O(k^{-\infty}). \tag{3.15}$$

Moreover, we have by [18, Theorem 3.11],

$$\left\|(\Box_k^{(q)})^j u_k\right\|_k \equiv 0 \mod O(k^{-\infty}),\ j=1,2,\ldots. \tag{3.16}$$

From (3.16), we have

$$\left\|u_k - P^{(q)}_{k,k^{-N}}u_k\right\|_k \le k^N\left\|\Box_k^{(q)}u_k\right\|_k \equiv 0 \mod O(k^{-\infty}). \tag{3.17}$$

From (3.16) and semiclassical Gårding inequalities (see [18, Lemma 4.1]), we obtain

$$u_k \equiv P^{(q)}_{k,k^{-N}}u_k \mod O(k^{-\infty}). \tag{3.18}$$

From (3.18), (3.15), the lemma follows. □

Lemma 3.7. *With the notations and assumptions above, assume that for k large,* dist $(p,y) \ge c\frac{\log k}{\sqrt{k}}$, $\forall y \in D_1$, *where $c > 0$ is a constant independent of k. Then, there exists $C_{\alpha,\beta} > 0$ independent of K and the point p such that*

$$\left|\partial_x^\alpha\partial_y^\beta P^{(q),I_0,J_0}_{k,k^{-N},s,\widehat{s}}(p,y)\right| \le C_{\alpha,\beta}k^{2n-\frac{N}{2}+2(|\alpha|+|\beta|)},\ \ \forall y\in D_1.$$

Proof. Let us choose

$$g_1 = \frac{P^{(q)}_{k,k^{-N}}u_k}{\|P^{(q)}_{k,k^{-N}}u_k\|_k} \tag{3.19}$$

in the orthonormal frame $\{g_1(x), g_2(x),\ldots,g_{d_k}(x)\}$ of $\mathscr{E}_{k^{-N}}(M,L^k)$ (see (3.6)). From (3.11), (3.13), and Lemma 3.6, it is not difficult to check that

$$\left|\partial_x^\alpha(\widetilde{g}_{1,I_0}e^{-k\phi})(p)\right|^2 \equiv a_{I_0,I_0}(p,p) \mod O(k^{-\infty}). \tag{3.20}$$

From (3.10), (3.11), and Theorem 3.2, we conclude that

$$\sum_{j=2}^{d_k}\left|\partial_x^\alpha(\widetilde{g}_{j,I_0}e^{-k\phi})(p)\right|^2 \le k^{3n-N+4|\alpha|}, \tag{3.21}$$

where $C > 0$ is a constant independent of k and the point p. From (3.7), (3.10), and (3.21), we have

$$\left|\partial_x^\alpha\partial_y^\beta P^{(q),I_0,J_0}_{k,k^{-N},s,\widehat{s}}(p,y)\right| \le \left|\partial_x^\alpha(\widetilde{g}_{1,I_0}e^{-k\phi})(p)\right|\left|\partial_y^\beta(\widehat{g}_{1,J_0}e^{-k\widehat{\phi}})(y)\right|$$
$$+\sqrt{\sum_{j=2}^{d_k}\left|\partial_x^\alpha(\widetilde{g}_{j,I_0}e^{-k\phi})(p)\right|^2}\sqrt{\sum_{j=2}^{d_k}\left|\partial_y^\beta(\widehat{g}_{j,J_0}e^{-k\widehat{\phi}})(y)\right|^2}$$

$$\leq \left|\partial_x^\alpha(\widetilde{g}_{1,I_0}e^{-k\phi})(p)\right| \left|\partial_y^\beta(\widehat{g}_{1,J_0}e^{-k\widehat{\phi}})(y)\right| + C_1 k^{2n-\frac{N}{2}+2(|\alpha|+|\beta|)}, \tag{3.22}$$

where $C_1 > 0$ is a constant independent of k and the point p. From Lemma 3.6 and noting that $\widetilde{u}_k(y) \equiv 0 \mod O(k^{-\infty})$ if dist $(p,y) \geq c\frac{\log k}{\sqrt{k}}$, where $c > 0$ is a constant independent of k, we conclude that

$$\left|\partial_y^\beta(\widehat{g}_{1,J_0}e^{-k\widehat{\phi}})(y)\right| \equiv 0 \mod O(k^{-\infty}),\ \forall y \in D_1.$$

From this observation and (3.22), the lemma follows. □

From Lemmas 3.4, 3.5, 3.7, Theorem 3.1 follows.

We can repeat the proof of Theorem 3.1 and conclude:

Theorem 3.8. *Let s and $\widehat{s}$ be local trivializing holomorphic sections of L on open sets $D_0 \Subset M$, $D_1 \Subset M$, respectively, $|s|_h^2 = e^{-2\phi}$, $|\widehat{s}|_h^2 = e^{-2\widehat{\phi}}$. Assume that $D_0 \Subset M(j)$, $j \neq q$. Then*

$$P^{(q)}_{k,s,\widehat{s}}(x,y) \equiv 0 \mod O(k^{-\infty}) \text{ locally uniformly on } D_0 \times D_1.$$

Assume now that $D_0 \Subset M(q)$ and $\Box^{(q)}_k$ has an $O(k^{-N})$ small spectral gap on D_0. Suppose that $\overline{D}_0 \bigcap \overline{D}_1 = \emptyset$. Then,

$$P^{(q)}_{k,s,\widehat{s}}(x,y) \equiv 0 \mod O(k^{-\infty}) \text{ locally uniformly on } D_0 \times D_1.$$

4. Berezin–Toeplitz kernel estimates away the diagonal

In this section, we prove the off-diagonal decay of the kernel $T^{(q),f}_{k,k^{-N}}(\cdot,\cdot)$ of the Berezin–Toeplitz quantization (cf. (1.10)), where $f \in \mathscr{C}^\infty(M)$ is as usual a bounded function and $N > 1$. This yields one half of Theorem 1.1, i.e., (1.11).

We consider as before the localization of $T^{(q),f}_{k,k^{-N}}(\cdot,\cdot)$ as follows. Let s, $\widehat{s}$ be local trivializing holomorphic sections of L on open sets $D_0 \Subset M$, $D_1 \Subset M$, respectively, $|s|_h^2 = e^{-2\phi}$, $|\widehat{s}|_h^2 = e^{-2\widehat{\phi}}$. Let $\{e_1, e_2, \ldots, e_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ be orthonormal frames of $\Lambda^{0,1}(T^*M)$ on D_0 and D_1, respectively. Then, $\{e^J;\ |J| = q,\ J \text{ strictly increasing}\}$, $\{w^J;\ |J| = q,\ J \text{ strictly increasing}\}$ are orthonormal frames of $\Lambda^{0,q}(T^*M)$ on D_0 and D_1, respectively. As in (3.1), we write

$$T^{(q),f}_{k,k^{-N},s,\widehat{s}} = \sum_{|I|=|J|=q}{}' T^{(q),f,I,J}_{k,k^{-N},s,\widehat{s}}(x,y)e^I(x) \wedge (w^J(y))^\dagger,\quad T^{(q),f,I,J}_{k,k^{-N},s,\widehat{s}} \in \mathscr{C}^\infty(D_0 \times D_1). \tag{4.1}$$

Let $\{g_j\}_{j=1}^{d_k}$ and $\{\delta_j\}_{j=1}^{d_k}$ be orthonormal bases of $\mathscr{E}^q_{k^{-N}}(M, L^k)$, where $d_k \in \mathbb{N} \cup \{\infty\}$. On D_0, D_1, we write

$$\begin{aligned} g_j(x) &= s^k(x)\widetilde{g}_j(x),\ \ \widetilde{g}_j(x) = \sum_{|J|=q}{}'\widetilde{g}_{j,J}(x)e^J(x) \text{ on } D_0,\ \ j = 1, \ldots, d_k, \\ \delta_j(x) &= \widehat{s}^k(x)\widehat{\delta}_j(x),\ \ \widehat{\delta}_j(x) = \sum_{|J|=q}{}'\widehat{\delta}_{j,J}(x)w^J(x) \text{ on } D_1,\ \ j = 1, \ldots, d_k. \end{aligned} \tag{4.2}$$

It is not difficult to check that for every strictly increasing I, J, $|I| = |J| = q$, we have

$$T^{(q),f,I,J}_{k,k^{-N},s,\widehat{s}}(x,y) = \sum_{j,\ell=1}^{d_k} \widetilde{g}_{j,I}(x)e^{-k\phi(x)}(f\delta_\ell \,|\, g_j)_k \overline{\widehat{\delta}_{\ell,J}(y)}e^{-k\widehat{\phi}(y)}. \tag{4.3}$$

Lemma 4.1. *Assume that $D_0 \Subset M(j)$, $j \neq q$. Then, for every $N > 1$ and $m \in \mathbb{N}$, there exists $C_{N,m} > 0$ independent of k such that for every I, J strictly increasing, $|I| = |J| = q$, we have*

$$\left| T^{(q),f,I,J}_{k,k^{-N},s,\widehat{s}}(x,y) \right|_{\mathscr{C}^m(D_0\times D_1)} \leq C_{N,m} k^{2n-\frac{N}{2}+2m}.$$

Proof. Fix $\alpha, \beta \in \mathbb{N}_0^{2n}$, I_0, J_0 strictly increasing with $|I_0| = |J_0| = q$, and $(x_0, y_0) \in D_0 \times D_1$. Take $\{g_1, g_2, \ldots, g_{d_k}\}$ and $\{\delta_1, \delta_2, \ldots, \delta_{d_k}\}$ so that

$$\begin{aligned} \left| \partial_x^\alpha (\widetilde{g}_{1,I_0} e^{-k\phi})(x_0) \right|^2 &= \sum_{j=1}^{d_k} \left| \partial_x^\alpha (\widetilde{g}_{j,I_0} e^{-k\phi})(x_0) \right|^2, \\ \left| \partial_y^\beta (\widehat{\delta}_{1,J_0} e^{-k\widehat{\phi}})(y_0) \right|^2 &= \sum_{j=1}^{d_k} \left| \partial_y^\beta (\widehat{\delta}_{j,J_0} e^{-k\widehat{\phi}})(y_0) \right|^2. \end{aligned} \tag{4.4}$$

This is always possible, see [18, Proposition 4.5]. From (4.3) and (4.4), we see that

$$(\partial_x^\alpha \partial_y^\beta T^{(q),f,I_0,J_0}_{k,k^{-N},s,\widehat{s}})(x_0,y_0) = \partial_x^\alpha (\widetilde{g}_{1,I_0} e^{-k\phi})(x_0)(f\delta_1 \,|\, g_1)_k \partial_y^\beta (\overline{\widehat{\delta}_{1,J_0}} e^{-k\widehat{\phi}})(y_0). \tag{4.5}$$

In view of Theorems 3.2 and (3.10), we see that

$$\begin{aligned} \left| \partial_x^\alpha (\widetilde{g}_{1,I_0} e^{-k\phi})(x_0) \right|^2 &\leq C_\alpha k^{3n-N+4|\alpha|}, \\ \left| \partial_y^\beta (\widehat{\delta}_{1,J_0} e^{-k\widehat{\phi}})(y_0) \right|^2 &\leq C_\beta k^{n+4|\beta|}, \end{aligned} \tag{4.6}$$

where $C_\alpha > 0$, $C_\beta > 0$ are constants independent of k and the points x_0 and y_0. From (4.5) and (4.6), the lemma follows. □

Now, we assume that $D_0 \Subset M(q)$. Fix $D_0 \Subset \widetilde{D}_0 \Subset M(q)$ and take $\tau(x) \in \mathscr{C}_0^\infty(\widetilde{D}_0)$, $\tau = 1$ on D_0.

Lemma 4.2. *With the assumptions and notations above, for every $N > 1$ and $m \in \mathbb{N}$, there exists $C_{N,m} > 0$ independent of k such that*

$$\left| T^{(q),(1-\tau)f,I,J}_{k,k^{-N},s,\widehat{s}}(x,y) \right|_{\mathscr{C}^m(D_0\times D_1)} \leq C_{N,m} k^{2n-\frac{N}{2}+2m},$$

for every strictly increasing multi-indices I, J, $|I| = |J| = q$.

Proof. Fix $\alpha, \beta \in \mathbb{N}_0^{2n}$, $|I_0| = |J_0| = q$, I_0, J_0 are strictly increasing and $(p, y_0) \in D_0 \times D_1$. Take $\{\delta_1, \delta_2, \ldots, \delta_{d_k}\}$ so that

$$\left| \partial_y^\beta (\widehat{\delta}_{1,J_0} e^{-k\widehat{\phi}})(y_0) \right|^2 = \sum_{j=1}^{d_k} \left| \partial_y^\beta (\widehat{\delta}_{j,J_0} e^{-k\widehat{\phi}})(y_0) \right|^2. \tag{4.7}$$

Assume that $a_{I_0,I_0}(p,p,k) \leq 0$, where $a_{I,J}(x,y,k)$ is as in (3.11). From (3.12) and (3.10), we have

$$\begin{aligned}
&\left|\partial_x^\alpha \partial_y^\beta T^{(q),(1-\tau)f,I_0,J_0}_{k,k^{-N},s,\widehat{s}}(p,y_0)\right| \\
&= \left|\sum_{j=1}^{d_k} \partial_x^\alpha(\widetilde{g}_{j,I_0}e^{-k\phi})(p)(\,(1-\tau)f\delta_1\,|\,g_j\,)_k\partial_y^\beta(\overline{\widehat{\delta}_{1,J_0}}e^{-k\widehat{\phi}})(y_0)\right| \\
&\leq C\sqrt{\sum_{j=1}^{d_k}\left|\partial_x^\alpha(\widetilde{g}_{j,I_0}e^{-k\phi})(p)\right|^2}\left|\partial_y^\beta(\overline{\widehat{\delta}_{1,J_0}}e^{-k\widehat{\phi}})(y_0)\right| \\
&\leq C_{\alpha,\beta}(k^{3n-N+4|\alpha|}k^{n+4|\beta|})^{\frac{1}{2}} = C_{\alpha,\beta}k^{2n-\frac{N}{2}+2|\alpha|+2|\beta|},
\end{aligned} \tag{4.8}$$

where $C_{\alpha,\beta} > 0$, $C > 0$ are constants independent of k and the points p, y_0.

Now, we assume that $a_{I_0,I_0}(p,p,k) > 0$. We define now u_k is as in (3.13) and g_1 as in (3.19). Since $g_1 \equiv u_k \mod O(k^{-\infty})$ and $u_k(x) \equiv 0 \mod O(k^{-\infty})$ if $\mathrm{dist}\,(x,p) \geq c\frac{\log k}{\sqrt{k}}$, where $c > 0$ is a constant independent of k, we conclude that

$$\left|\partial_x^\alpha(\widetilde{g}_{1,I_0}e^{-k\phi})(p)(\,(1-\tau)f\delta_1\,|\,g_1\,)_k\partial_y^\beta(\overline{\widehat{\delta}_{1,J_0}}e^{-k\widehat{\phi}})(y_0)\right| \equiv 0 \mod O(k^{-\infty}). \tag{4.9}$$

From (3.21) and (3.10), we have

$$\begin{aligned}
&\left|\sum_{j=2}^{d_k} \partial_x^\alpha(\widetilde{g}_{j,I_0}e^{-k\phi})(p)(\,(1-\tau)f\delta_1\,|\,g_j\,)_k\partial_y^\beta(\overline{\widehat{\delta}_{1,J}}e^{-k\widehat{\phi}})(y_0)\right| \\
&\leq C_1\sqrt{\sum_{j=2}^{d_k}\left|\partial_x^\alpha(\widetilde{g}_{j,I_0}e^{-k\phi})(p)\right|^2}\left|\partial_y^\beta(\overline{\widehat{\delta}_{1,J}}e^{-k\widehat{\phi}})(y_0)\right| \\
&\leq \widetilde{C}_{\alpha,\beta}(k^{3n-N+4|\alpha|}k^{n+4|\beta|})^{\frac{1}{2}} = \widetilde{C}_{\alpha,\beta}k^{2n-\frac{N}{2}+2|\alpha|+2|\beta|},
\end{aligned} \tag{4.10}$$

where $C_1 > 0$, $\widetilde{C}_{\alpha,\beta} > 0$ are constants independent of k and the points p, y_0. From (4.9) and (4.10), we obtain

$$\left|\partial_x^\alpha \partial_y^\beta T^{(q),(1-\tau)f,I_0,J_0}_{k,k^{-N},s,\widehat{s}}(p,y_0)\right| \leq \widehat{C}_{\alpha,\beta}k^{2n-\frac{N}{2}+2|\alpha|+2|\beta|}, \tag{4.11}$$

where $\widehat{C}_{\alpha,\beta} > 0$ is a constant independent of k and the points p, y_0.

From (4.8) and (4.11), the lemma follows. □

Lemma 4.3. *With the assumptions and notations above, assume that $\overline{D}_0 \bigcap \overline{D}_1 = \emptyset$. Then, for every $N > 1$ and $m \in \mathbb{N}$, there exists $C_{N,m} > 0$ independent of k such that*

$$\left|T^{(q),\tau f,I,J}_{k,k^{-N},s,\widehat{s}}(x,y)\right|_{\mathscr{C}^m(D_0\times D_1)} \leq C_{N,m}k^{2n-\frac{N}{2}+2m},$$

for every I, J strictly increasing, $|I| = |J| = q$.

Proof. Fix $\alpha, \beta \in \mathbb{N}_0^{2n}$, $|I_0| = |J_0| = q$, I_0, J_0 are strictly increasing and $(p, y_0) \in D_0 \times D_1$. Take $\{\delta_1, \delta_2, \cdots, \delta_{d_k}\}$ so that

$$\left|\partial_y^\beta(\widehat{\delta}_{1,J_0} e^{-k\phi})(y_0)\right|^2 = \sum_{j=1}^{d_k} \left|\partial_y^\beta(\widehat{\delta}_{j,J_0} e^{-k\widehat{\phi}})(y_0)\right|^2. \tag{4.12}$$

Assume that $a_{I_0,I_0}(p,p,k) \leq 0$, where $a_{I,J}(x,y,k)$ is as in (3.11). We can repeat the procedure in the proof of Lemma 4.2 and conclude that

$$\left|\partial_x^\alpha \partial_y^\beta T^{(q),\tau f, I_0, J_0}_{k,k^{-N},s,\widehat{s}}(p, y_0)\right| \leq C_{\alpha,\beta} k^{2n-\frac{N}{2}+2|\alpha|+2|\beta|}, \tag{4.13}$$

where $C_{\alpha,\beta} > 0$ is a constant independent of k and the points p and y_0. Now, we assume that $a_{I_0,I_0}(p,p,k) > 0$. Let g_1 be as in (3.19), where u_k is as in (3.13). From Lemma 3.6 and (3.13), we have

$$\begin{aligned}
&\widetilde{g}_1(x)e^{-k\phi(x)} \equiv \frac{1}{\sqrt{a_{I_0,I_0}(p,p,k)}} e^{ik\Psi(x,p)} \chi\left(\frac{|x-p|^2}{\varepsilon}\right) {\sum_{|I|=q}}' d_{I,I_0}(x,p,k) e^I(x) \mod O(k^{-\infty}),\\
&\partial_x^\alpha(\widetilde{g}_{1,I_0} e^{-k\phi})(p) \equiv \sqrt{a_{I_0,I_0}(p,p,k)} \mod O(k^{-\infty}).
\end{aligned} \tag{4.14}$$

From (4.14), it is straightforward to see that for every $N \in \mathbb{N}$, there exists $C_N > 0$ independent of k and the points p and y_0 such that

$$\begin{aligned}
&\left|\partial_x^\alpha(\widetilde{g}_{1,I_0} e^{-k\phi})(p)(\,\tau f \delta_1 \,|\, g_1\,)_k \partial_y^\beta(\overline{\widehat{\delta}_{1,J_0}} e^{-k\widehat{\phi}})(y_0)\right|\\
&\leq \int e^{-k\mathrm{Im}\,\Psi(x,p)} \chi\left(\frac{|x-p|^2}{\varepsilon}\right) {\sum_{|I|=q}}' \left|d_{I,I_0}(x,p,k)\right| |\tau(x)| \left|f(x)\right| dv_M(x)\\
&\quad \times {\sum_{|I|=q}}' \sup\left\{\left|\widehat{\delta}_{1,I}(x)e^{-k\widehat{\phi}(x)} \partial_y^\beta(\overline{\widehat{\delta}_{1,J_0}} e^{-k\widehat{\phi}})(y_0)\right|;\, x \in \mathrm{Supp}\,\chi\left(\frac{|x-p|^2}{\varepsilon}\right) \Subset \widetilde{D}_0\right\}\\
&\quad + C_N k^{-N}.
\end{aligned} \tag{4.15}$$

From (3.11), we can check that

$${\sum_{|I|=q}}' \left|d_{I,I_0}(x,p,k)\right| \leq C_\alpha k^{n+|\alpha|}, \ \ \forall x \in \mathrm{Supp}\,\chi\left(\frac{|x-p|^2}{\varepsilon}\right), \tag{4.16}$$

where $C_\alpha > 0$ is a constant independent of k and the point p. F rom (4.16) and (3.4), it is not-difficult to check that

$$\int e^{-k\mathrm{Im}\,\Psi(x,p)} \chi\left(\frac{|x-p|^2}{\varepsilon}\right) {\sum_{|I|=q}}' \left|d_{I,I_0}(x,p,k)\right| |\tau(x)| \left|f(x)\right| dv_M(x) \leq C_0 k^{|\alpha|}, \tag{4.17}$$

where $C_0 > 0$ is a constant independent of k and the point p. Moreover, from Theorem 3.1, we see that

$${\sum_{|I|=q}}' \sup\left\{\left|\widehat{\delta}_{1,I}(x)e^{-k\widehat{\phi}(x)} \partial_y^\beta(\overline{\widehat{\delta}_{1,J_0}} e^{-k\widehat{\phi}})(y_0)\right|;\, x \in \mathrm{Supp}\,\chi\left(\frac{|x-p|^2}{\varepsilon}\right)\right\} \leq C_\beta k^{2n-\frac{N}{2}+2|\beta|}, \tag{4.18}$$

where $C_\beta > 0$ is a constant independent of k and the points p, y_0. From (4.15), (4.17), and (4.18), we conclude that

$$\left|\partial_x^\alpha(\widetilde{g}_{1,I_0}e^{-k\phi})(p)(\,\tau f\delta_1\,|\,g_1\,)_k\partial_y^\beta(\widehat{\delta}_{1,J_0}e^{-k\widehat{\phi}})(y_0)\right| \leq C_{\alpha,\beta}k^{2n-\frac{N}{2}+2|\alpha|+2|\beta|}, \tag{4.19}$$

where $C_{\alpha,\beta} > 0$ is a constant independent of k and the points p, y_0.

From (3.21) and (3.10), we have

$$\begin{aligned}
&\left|\sum_{j=2}^{d_k}\partial_x^\alpha(\widetilde{g}_{j,I_0}e^{-k\phi})(p)(\,\tau f\delta_1\,|\,g_j\,)_k\partial_y^\beta(\overline{\widehat{\delta}_{1,J_0}}e^{-k\widehat{\phi}})(y_0)\right| \\
&\leq C_2\sqrt{\sum_{j=2}^{d_k}\left|\partial_x^\alpha(\widetilde{g}_{j,I_0}e^{-k\phi})(p)\right|^2}\left|\partial_y^\beta(\overline{\widehat{\delta}_{1,J_0}}e^{-k\widehat{\phi}})(y_0)\right| \\
&\leq \widehat{C}_{\alpha,\beta}(k^{3n-N+4|\alpha|}k^{n+4|\beta|})^{\frac{1}{2}} = \widehat{C}_{\alpha,\beta}k^{2n-\frac{N}{2}+2|\alpha|+2|\beta|},
\end{aligned} \tag{4.20}$$

where $C_2 > 0$, $\widehat{C}_{\alpha,\beta} > 0$ are constants independent of k and the point p. From (4.19) and (4.20), the lemma follows. □

From Lemmas 4.1–4.3 we deduce:

Theorem 4.4. *Let $s, \widehat{s}$ be local trivializing holomorphic sections of L on $D_0 \Subset M$ and $D_1 \Subset M$, respectively. Assume that $D_0 \Subset M(j)$, $j \neq q$ or $D_0 \Subset M(q)$ and $\overline{D}_0 \bigcap \overline{D}_1 = \emptyset$. Then, for every $m \in \mathbb{N}$, $N > 1$, there exists $C_{N,m} > 0$ independent of k such that*

$$\left|T^{(q),f}_{k,k^{-N},s,\widehat{s}}(x,y)\right|_{\mathscr{C}^m(D_0\times D_1)} \leq C_{N,m}k^{2n-\frac{N}{2}+2m}.$$

Theorem 4.4 implies immediately one half of Theorem 1.1, more precisely (1.11).

We can repeat the proof of Theorem 4.4 and deduce:

Theorem 4.5. *Let $s, \widehat{s}$ be local trivializing holomorphic sections of L on $D_0 \Subset M$ and $D_1 \Subset M$, respectively. Assume that $D_0 \Subset M(j)$, $j \neq q$. Then,*

$$T^{(q),f}_{k,s,\widehat{s}}(x,y) \equiv 0 \mod O(k^{-\infty}) \text{ locally uniformly on } D_0\times D_1.$$

Assume that $D_0 \Subset M(q)$ and $\Box^{(q)}_k$ has $O(k^{-N})$ small spectral gap on D_0. Suppose that $\overline{D}_0 \bigcap \overline{D}_1 = \emptyset$. Then,

$$T^{(q),f}_{k,s,\widehat{s}}(x,y) \equiv 0 \mod O(k^{-\infty}) \text{ locally uniformly on } D_0\times D_1.$$

Let's explain why in Theorem 4.5, we have "$\equiv 0 \mod O(k^{-\infty})$". Recall that Theorem 4.4 is based on Theorem 3.2 which says that if $D_0 \Subset M(q)$, then, for every $N > 1$, $m \in \mathbb{N}$, there exists $C_{N,m} > 0$ independent of k such that

$$\left|P^{(q)}_{k,k^{-N},s}(x,y) - e^{ik\Psi(x,y)}b(x,y,k)\right|_{\mathscr{C}^m(D_0\times D_0)} \leq C_{N,m}k^{3n-N+2m}, \tag{4.21}$$

and if $D_0 \Subset M(j), j \neq q, j \in \{0, 1, 2, \ldots, n\}$, then, for every $N > 1$, $m \in \mathbb{N}$, there exists $\widetilde{C}_{N,m} > 0$ independent of k such that

$$\left|P^{(q)}_{k,k^{-N},s}(x, y)\right|_{\mathscr{C}^m(D_0\times D_0)} \leq \widetilde{C}_{N,m} k^{3n-N+2m}. \tag{4.22}$$

The estimates $\lesssim k^{3n-N+2m}$ in (4.21) and (4.22) imply that we have the estimate $\lesssim k^{2n-\frac{N}{2}+2m}$ in Theorem 4.4. Now, we consider the Bergman kernel. As in Theorem 3.2, assume that $D_0 \Subset M(q)$ and $\Box^{(q)}_k$ has $O(k^{-N})$ small spectral gap on D_0, then

$$P^{(q)}_{k,s}(x, y) - e^{ik\Psi(x,y)} b(x, y, k) \equiv 0 \mod O(k^{-\infty}) \text{ on } D_0 \times D_0. \tag{4.23}$$

Moreover, if $D_0 \Subset M(j), j \neq q, j \in \{0, 1, 2, \ldots, n\}$, then

$$P^{(q)}_{k,s}(x, y) \equiv 0 \mod O(k^{-\infty}) \text{ on } D_0 \times D_0, \tag{4.24}$$

by Theorems 4.12 and 4.14 in [27]. From (4.23) and (4.24), we can repeat the proof of Theorem 4.4 and deduce that in Theorem 4.5, we actually have "$\equiv 0 \mod O(k^{-\infty})$".

5. Asymptotic expansion of Berezin–Toeplitz quantization

In this section, we will establish the full asymptotic expansion for the kernel of the Toeplitz kernel $T^{(q),f}_{k,k^{-N}}(\cdot, \cdot)$ corresponding to lower energy forms. This leads to the proof of Theorem 1.1.

Let s be a local trivializing holomorphic section of L on an open set $D \Subset M$, $|s|^2_h = e^{-2\phi}$. Fix $N > 1$. We assume that $D \Subset M(q)$. Put

$$S_k(x, y) := e^{ik\Psi(x,y)} b(x, y, k), \tag{5.1}$$

where $\Psi(x, y)$ and $b(x, y, k)$ are as in Theorem 3.2. Fix an open set $D_0 \Subset D$ and $\tau \in \mathscr{C}^\infty_0(D)$ with $\tau = 1$ on D_0. Put

$$R_k(x, y) = \int (P^{(q)}_{k,k^{-N},s}(x, z) - S_k(x, z))\tau(z)f(z)P^{(q)}_{k,k^{-N},s}(z, y)dv_M(z). \tag{5.2}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal frame of $\Lambda^{0,1}(T^*M)$ on D. Then,

$$\left\{e^J;\ |J| = q, J \text{ is strictly increasing}\right\}$$

is an orthonormal frame of $\Lambda^{0,q}(T^*M)$ on D. As in (3.1), we write

$$\begin{aligned}
R_k(x, y) &= \sum_{|I|=|J|=q}{}' R^{I,J}_k(x, y)e^I(x) \wedge (e^J(y))^\dagger,\\
S_k(x, y) &= \sum_{|I|=|J|=q}{}' S^{I,J}_k(x, y)e^I(x) \wedge (e^J(y))^\dagger = \sum_{|I|=|J|=q}{}' e^{ik\Psi(x,y)} b_{I,J}(x, y, k)e^I(x) \wedge (e^J(y))^\dagger,\\
P^{(q)}_{k,k^{-N},s}(x, y) &= \sum_{|I|=|J|=q}{}' P^{(q),I,J}_{k,k^{-N},s}(x, y)e^I(x) \wedge (e^J(y))^\dagger.
\end{aligned} \tag{5.3}$$

It is easy to see that for every $|I| = |J| = q$, I, J are strictly increasing, we have

$$R^{I,J}_k(x, y) = \sum_{|K|=q}{}' \int (P^{(q),I,K}_{k,k^{-N},s}(x, z) - S^{I,K}_k(x, z))\tau(z)f(z)P^{(q),K,J}_{k,k^{-N},s}(z, y)dv_M(z). \tag{5.4}$$

Take $\{g_1(x), g_2(x), \dots, g_{d_k}(x)\}$ and $\{\delta_1(x), \delta_2(x), \dots, \delta_{d_k}(x)\}$ be orthonormal frames for $\mathscr{E}_{k-N}(M, L^k)$, where $d_k \in \mathbb{N} \cup \{\infty\}$. On D, we write

$$\begin{aligned} g_j(x) &= s^k(x)\widetilde{g}_j(x), \ \ \widetilde{g}_j(x) = \sum_{|J|=q}{}^{'}\widetilde{g}_{j,J}(x)e^J(x) \text{ on } D, \ \ j = 1, \dots, d_k, \\ \delta_j(x) &= s^k(x)\widetilde{\delta}_j(x), \ \ \widetilde{\delta}_j(x) = \sum_{|J|=q}{}^{'}\widetilde{\delta}_{j,J}(x)e^J(x) \text{ on } D, \ \ j = 1, \dots, d_k. \end{aligned} \tag{5.5}$$

Lemma 5.1. *With the assumptions and notations above, for every $N > 1$ and $m \in \mathbb{N}$, there exists $C_{N,m} > 0$ independent of k such that*

$$\left|R_k(x,y)\right|_{\mathscr{C}^m(D\times D)} \leq C_{N,m}k^{2n-\frac{N}{2}+2m}.$$

Proof. Fix $(p, y_0) \in D \times D$, strictly increasing I_0, J_0, $|I_0| = |J_0| = q$, and $\alpha, \beta \in \mathbb{N}_0^{2n}$. Assume that $a_{I_0,I_0}(p,p,k) \leq 0$, where $a_{I,J}(x,y,k)$ is as in (3.11). In view of the proof of Lemma 3.5, we see that

$$\left|\partial_x^\alpha \partial_y^\alpha P^{(q),I_0,I_0}_{k,k^{-N},s}(p,p)\right| + \left|a_{I_0,I_0}(p,p)\right| \leq C_\alpha k^{3n-N+4|\alpha|}, \tag{5.6}$$

where $C_\alpha > 0$ is a constant independent of k and the point p. It is not difficult to see that for every $|I| = |J| = q$, I, J are strictly increasing, we have

$$\begin{aligned} R_k^{I,J}(x,y) = \sum_{|K|=q}{}^{'}\int \Big(&\sum_{j=1}^{d_k}\widetilde{g}_{j,I}(x)e^{-k\phi(x)}\overline{\widetilde{g}_{j,K}}(z)e^{-k\phi(z)} - S_k^{I,K}(x,z)\Big)\tau(z)f(z) \\ &\times \Big(\sum_{\ell=1}^{d_k}\widetilde{\delta}_{\ell,K}(z)e^{-k\phi(z)}\overline{\widetilde{\delta}_{\ell,J}}(y)e^{-k\phi(y)}\Big)dv_M(z). \end{aligned} \tag{5.7}$$

Take $\{g_1, g_2, \dots, g_{d_k}\}$ and $\{\delta_1, \delta_2, \dots, \delta_{d_k}\}$ so that

$$\begin{aligned} \sum_{j=1}^{d_k}\left|\partial_x^\alpha(\widetilde{g}_{j,I_0}e^{-k\phi})(p)\right|^2 &= \partial_x^\alpha\partial_y^\alpha P^{(q)}_{k,k^{-N},s}(p,p) = \left|\partial_x^\alpha(\widetilde{g}_{1,I_0}e^{-k\phi})(p)\right|^2, \\ \sum_{j=1}^{d_k}\left|\partial_x^\beta(\widetilde{\delta}_{j,J_0}e^{-k\phi})(y_0)\right|^2 &= \partial_x^\beta\partial_y^\beta P^{(q)}_{k,k^{-N},s}(y_0,y_0) = \left|\partial_x^\beta(\widetilde{\delta}_{1,J_0}e^{-k\phi})(y_0)\right|^2. \end{aligned} \tag{5.8}$$

From (5.8) and (5.7), we get

$$\begin{aligned} \partial_x^\alpha\partial_y^\beta R_k^{I_0,J_0}(p,y_0) = \sum_{|K|=q}{}^{'}\int \Big(&\partial_x^\alpha(\widetilde{g}_{1,I_0}e^{-k\phi})(p)\overline{\widetilde{g}_{1,K}}(z)e^{-k\phi(z)} - \partial_x^\alpha S_k^{I_0,K}(p,z)\Big)\tau(z)f(z) \\ &\times \widetilde{\delta}_{1,K}(z)e^{-k\phi(z)}\partial_y^\beta\overline{(\widetilde{\delta}_{1,J_0}e^{-k\phi})}(y_0)dv_M(z). \end{aligned} \tag{5.9}$$

From (5.6), (5.8), and (3.10), we have

$$\left|\sum_{|K|=q}{}' \int \left(\partial_x^\alpha (\widetilde{g}_{1,I_0} e^{-k\phi})(p)\overline{\widetilde{g}_{1,K}}(z)e^{-k\phi(z)}\tau(z)f(z)\widetilde{\delta}_{1,K}(z)\partial_y^\beta(\overline{\widetilde{\delta}_{1,J_0}}e^{-k\phi})(y_0)e^{-k\phi(z)}dv_M(z)\right|$$

$$\leq C\sqrt{\partial_x^\alpha \partial_y^\alpha P^{(q),I_0,I_0}_{k,k^{-N},s}(p,p)}\left|\partial_y^\beta(\overline{\widetilde{\delta}_{1,J_0}}e^{-k\phi})(y_0)\right|$$

$$\leq C_{\alpha,\beta} k^{\frac{3}{2}n-\frac{N}{2}+2|\alpha|} k^{\frac{n}{2}+2|\beta|} = C_{\alpha,\beta} k^{2n-\frac{N}{2}+2|\alpha|+2|\beta|}, \tag{5.10}$$

where $C > 0$, $C_{\alpha,\beta} > 0$ are constants independent of k and the points p and y_0.

It is known by [18, Theorem 3.11] that

$$\sum_{|K|=q}{}' \int \left|\partial_x^\alpha S_k^{I_0,K}(p,z)\right|^2 |\tau(z)|^2 e^{-2k\phi(z)} dv_M(z) \equiv a_{I_0,I_0}(p,p,k) \mod O(k^{-\infty}). \tag{5.11}$$

From (5.11) and (5.6), we obtain

$$\left|\sum_{|K|=q}{}' \int \partial_x^\alpha S_k^{I_0,K}(p,z)\tau(z)f(z)\widetilde{\delta}_{1,K}(z)e^{-k\phi(z)}\partial_y^\beta(\overline{\widetilde{\delta}_{1,J_0}}e^{-k\phi})(y_0)dv_M(z)\right|$$

$$\leq \widetilde{C}_{\alpha,\beta} k^{2n-\frac{N}{2}+2|\alpha|+2|\beta|}, \tag{5.12}$$

where $\widetilde{C}_{\alpha,\beta} > 0$ is a constant independent of k and the points p and y_0. From (5.12), (5.10), and (5.9), we deduce that

$$\left|\partial_x^\alpha \partial_y^\beta R_k^{I_0,J_0}(p,y_0)\right| \leq \widehat{C}_{\alpha,\beta} k^{2n-\frac{N}{2}+2|\alpha|+2|\beta|}, \tag{5.13}$$

where $\widehat{C}_{\alpha,\beta} > 0$ is a constant independent of k and the points p and y_0.

Now, we assume that $a_{I_0,I_0}(p,p,k) > 0$. Take $g_1 = (P^{(q)}_{k,k^{-N}} u_k)/|u_k|_{h^k}$, where u_k is as in (3.13). From Theorem 3.2 and Lemma 3.6, we can check that for every $N > 0$, there is $C_N > 0$ independent of k and the point p such that

$$\left|\sum_{|K|=q}{}' \left(\partial_x^\alpha (\widetilde{g}_{1,I_0} e^{-k\phi})(p)\overline{\widetilde{g}_{1,K}(z)}e^{-k\phi(z)} - \partial_x^\alpha S_k^{I_0,K}(p,z)\right)\right| \leq C_N k^{-N}, \ \forall z \in D \tag{5.14}$$

and

$$\sum_{j=2}^{d_k} \left|\partial_x^\alpha (\widetilde{g}_{j,I_0} e^{-k\phi})(p)\right|^2 \leq C_\alpha k^{3n-N+4|\alpha|}, \tag{5.15}$$

where $C_\alpha > 0$ is a constant independent of k and the point p. Take $\{\delta_1, \delta_2, \ldots, \delta_{d_k}\}$ so that

$$\left|\partial_x^\beta (\widetilde{\delta}_{1,J_0} e^{-k\phi})(y_0)\right|^2 = \partial_x^\beta \partial_y^\beta P^{(q),J_0,J_0}_{k,k^{-N},s}(y_0,y_0). \tag{5.16}$$

From (5.7), (5.14), (5.15), (5.16), and (3.10), we have

$$\left|\partial_x^\alpha \partial_y^\beta R_k^{I_0,J_0}(p,y_0)\right| \leq C\sqrt{\sum_{j=2}^{d_k}\left|\partial_x^\alpha(\widetilde{g}_{j,I_0}e^{-k\phi})(p)\right|^2}\left|\partial_y^\beta(\overline{\widetilde{\delta}_{1,J_0}}e^{-k\phi})(y_0)\right| + C_N k^{-N}$$
$$\leq C_{\alpha,\beta}k^{2n-\frac{N}{2}+2|\alpha|+2|\beta|} + C_N k^{-N}, \tag{5.17}$$

for every $N > 0$, where $C, C_N, C_{\alpha,\beta} > 0$ are independent of k and the points p and y_0. From (5.17) and (5.13), the lemma follows. □

Put

$$\widetilde{R}_k(x,y) = \int S_k(x,z)\tau(z)f(z)\Big(P^{(q)}_{k,k^{-N},s}(z,y) - S_k(z,y)\Big)dv_M(z). \tag{5.18}$$

We can repeat the proof of Lemma 5.1 and conclude:

Lemma 5.2. *With the assumptions and notations above, for every $N > 1$ and $m \in \mathbb{N}$, there exists $\widetilde{C}_{N,m} > 0$ independent of k such that*

$$\left|\widetilde{R}_k(x,y)\right|_{\mathscr{C}^m(D\times D)} \leq \widetilde{C}_{N,m}k^{2n-\frac{N}{2}+2m}.$$

Lemma 5.3. *We have*

$$\int S_k(x,z)\tau(z)f(z)S_k(z,y)dv_M(z) \equiv e^{ik\Psi(x,y)}b_f(x,y,k) \mod O(k^{-\infty})$$

locally uniformly on $D\times D$, where $b_f(x,y,k) \sim \sum_{j=0}^{\infty} b_{f,j}(x,y)k^{n-j}$ in $S^n(1;D\times D,\Lambda^{0,q}(T^*M))\boxtimes(\Lambda^{0,q}(T^*M))^*)$, $b_{f,0}(x,x) = (2\pi)^{-n}f(x)\left|\det\dot{R}^L(x)\right| I_{\det\overline{W}^*}(x)$, for any $x \in D_0$.

Proof. From the stationary phase formula of Melin–Sjöstrand [28], there is a complex phase function $\Psi_1(x,y) \in \mathscr{C}^\infty(D\times D)$ with $\Psi_1(x,x) = 0$, $\operatorname{Im}\Psi_1(x,y) \geq c\left|x-y\right|^2$ on $D\times D$, for some $c > 0$, such that for every bounded function $f \in \mathscr{C}^\infty(M)$, we have

$$\int S_k(x,z)\tau(z)f(z)S_k(z,y)dv_M(z) \equiv e^{ik\Psi_1(x,y)}\widetilde{b}_f(x,y,k) \mod O(k^{-\infty}) \tag{5.19}$$

locally uniformly on $D\times D$, where

$$\widetilde{b}_f(x,y,k) \sim \sum_{j=0}^{\infty}\widetilde{b}_{f,j}(x,y)k^{n-j} \text{ in } S^n(1;D\times D,\Lambda^{0,q}(T^*M))\boxtimes(\Lambda^{0,q}(T^*M))^*),$$

with $\widetilde{b}_{f,j} \in \mathscr{C}^\infty(D\times D,\Lambda^{0,q}(T^*M))\boxtimes(\Lambda^{0,q}(T^*M))^*)$, $j \in \mathbb{N}$. Moreover, for all $x \in D_0$ we have $b_{f,0}(x,x) = (2\pi)^{-n}f(x)\left|\det\dot{R}^L(x)\right| I_{\det\overline{W}^*}(x)$. Basically, here we used the fact that composition of complex Fourier integral operators is still a complex Fourier integral operator. Take $f = 1$. Fix $D' \Subset \{\tau = 1\}$. We claim that

$$\int S_k(x,z)\tau(z)S_k(z,y)dv_M(z) \equiv e^{ik\Psi(x,y)}b(x,y,k) \mod O(k^{-\infty}) \tag{5.20}$$

locally uniformly on $D' \times D'$, where

$$b(x, y, k) \sim \sum_{j=0}^{\infty} b_j(x, y) k^{n-j} \text{ in } S^n(1; D' \times D', \Lambda^{0,q}(T^*M) \boxtimes (\Lambda^{0,q}(T^*M))^*),$$

with $b_j \in \mathscr{C}^\infty(D' \times D', \Lambda^{0,q}(T^*M) \boxtimes (\Lambda^{0,q}(T^*M))^*)$, $j \in \mathbb{N}$. The S_k constructed in [18] is called approximated Szegö kernel. S_k satisfies

$$S_k \circ S_k \equiv S_k \mod O(k^{-\infty}) \text{ on } D \tag{5.21}$$

(see [18, Theorems 3.11 and 3.12]). Relation (5.21) says that

$$\int S_k(x, z) S_k(z, y) dv_M(z) \equiv S_k(x, y) = e^{ik\Psi(x,y)} b(x, y, k) \mod O(k^{-\infty}) \tag{5.22}$$

locally uniformly on $D \times D$. Since S_k is properly supported (see the discussion after (2.15) for the meaning of properly supported), the integral (5.22) is well-defined. Now,

$$\begin{aligned}\int S_k(x, z)\tau(z) S_k(z, y) dv_M(z) &= \int S_k(x, z) S_k(z, y) dv_M(z) \\ &\quad - \int S_k(x, z)(1 - \tau(z)) S_k(z, y) dv_M(z).\end{aligned} \tag{5.23}$$

Note that $S_k(x, y) = O(k^{-\infty})$ if $|x - y| \geq c$, for some $c > 0$. From this observation, we conclude that for $(x, y) \in D' \times D'$ (recall that $D' \Subset \{\tau = 1\}$)

$$\begin{aligned}&\int S_k(x, z)(1 - \tau(z)) S_k(z, y) dv_M(z) \\ &= \int_{z \notin \overline{D'}} S_k(x, z)(1 - \tau(z)) S_k(z, y) dv_M(z) \equiv 0 \mod O(k^{-\infty})\end{aligned} \tag{5.24}$$

locally uniformly on $D' \times D'$. From (5.22), (5.23), and (5.24), we get (5.20).

We claim that

$$\Psi(x, y) - \Psi_1(x, y) \text{ vanishes to infinite order on diag}\,(D' \times D'). \tag{5.25}$$

Note that $\Psi(x, x) = \Psi_1(x, x) = 0$. We assume that there are $\alpha_0, \beta_0 \in \mathbb{N}_0^{2n}$, $|\alpha_0| + |\beta_0| \geq 1$ and $(x_0, x_0) \in D' \times D'$, such that

$$\begin{aligned}&\partial_x^{\alpha_0} \partial_y^{\beta_0} \Big(\Psi_1(x, y) - \Psi(x, y)\Big)|_{(x_0,x_0)} \neq 0, \\ &\partial_x^{\alpha} \partial_y^{\beta} \Big(\Psi_1(x, y) - \Psi(x, y)\Big)|_{(x_0,x_0)} = 0, \ \ \forall \alpha, \beta \in \mathbb{N}_0^{2n}, |\alpha| + |\beta| < |\alpha_0| + |\beta_0|.\end{aligned} \tag{5.26}$$

From (5.19) and (5.20), we have

$$e^{ik(\Psi_1(x,y) - \Psi(x,y))} \widetilde{b}_1(x, y, k) - b(x, y, k) = e^{-ik\Psi(x,y)} F_k(x, y) \text{ on } D' \times D', \tag{5.27}$$

where $F_k \equiv 0 \mod O(k^{-\infty})$ locally uniformly on $D' \times D'$. From (5.26), it is easy to see that

$$\begin{aligned}&\lim_{k \to \infty} k^{-n-1} \partial_x^{\alpha_0} \partial_y^{\beta_0} \Big(e^{ik(\Psi_1(x,y) - \Psi(x,y))} \widetilde{b}_1(x, y, k) - b(x, y, k)\Big)|_{(x_0,x_0)} \\ &= i\partial_x^{\alpha_0} \partial_y^{\beta_0} \Big(\Psi(x, y) - \Psi_1(x, y)\Big)|_{(x_0,x_0)} \widetilde{b}_{1,0}(x_0, x_0) \neq 0.\end{aligned} \tag{5.28}$$

It is obviously that

$$\lim_{k\to\infty} k^{-n-1}\partial_x^{\alpha_0}\partial_y^{\beta_0}\left(e^{-ik\Psi(x,y)}F_k(x,y)\right)|_{(x_0,x_0)} = 0. \tag{5.29}$$

From (5.27) to (5.29), we get a contradiction. The claim follows. Since τ is arbitrary, Ψ and Ψ_1 are independent of τ, we conclude that $\Psi_1(x,y) - \Psi(x,y)$ vanishes to infinite order on $D \times D$. Thus, we can replace Ψ_1 by Ψ in (5.19). The lemma follows. □

From Lemmas 5.1–5.3 and Lemma 4.2, we obtain the following.

Theorem 5.4. *With the notations above, let s be local trivializing holomorphic section of L on $D_0 \Subset M$. Assume that $D_0 \Subset M(q)$. Then, for every $N > 1$, $m \in \mathbb{N}$, there exists $\widetilde{C}_{N,m} > 0$ independent of k such that*

$$\left|T^{(q),f}_{k,k^{-N},s}(x,y) - e^{ik\Psi(x,y)}b_f(x,y,k)\right|_{\mathscr{C}^m(D_0\times D_0)} \leq \widetilde{C}_{N,m}k^{2n-\frac{N}{2}+2m},$$

where

$$\begin{aligned}
&b_f(x,y,k) \in S^n(1; D_0 \times D_0, \Lambda^{0,q}(T^*M)) \boxtimes (\Lambda^{0,q}(T^*M))^*),\\
&b_f(x,y,k) \sim \sum_{j=0}^{\infty} b_{f,j}(x,y)k^{n-j} \text{ in } S^n(1; D_0 \times D_0, \Lambda^{0,q}(T^*M)) \boxtimes (\Lambda^{0,q}(T^*M))^*),\\
&b_{f,0}(x,x) = (2\pi)^{-n}f(x)\left|\det \dot{R}^L(x)\right|I_{\det \overline{W}^*}(x), \ \ \forall x \in D_0,
\end{aligned} \tag{5.30}$$

and Ψ is as in Theorem 3.2.

Proof of Theorem 1.1. From Theorems 4.4 and 5.4, Theorem 1.1 follows. □

6. Asymptotics of the composition of Toeplitz operators

In this section, we establish the expansion of the composition of two Toeplitz operators and prove Theorems 1.3, 1.4, 1.9, 1.10, and 1.11.

Let $f, g \in \mathscr{C}^\infty(M)$ be bounded. For $\lambda \geq 0$, put

$$T^{(q),f,g}_{k,\lambda} := T^{(q),f}_{k,\lambda} \circ T^{(q),g}_{k,\lambda} : L^2_{(0,q)}(M, L^k) \to \mathscr{E}^q_\lambda(M, L^k)$$

and set $T^{(q),f,g}_k := T^{(q),f,g}_{k,0}$.

Theorem 6.1. *Let $s, \widehat{s}$ be local trivializing holomorphic sections of L on $D_0 \Subset M$ and $D_1 \Subset M$, respectively, $|s|^2_h = e^{-2\phi}$, $|\widehat{s}|^2_h = e^{-2\widehat{\phi}}$, where D_0 and D_1 are open sets. Assume that $D_0 \Subset M(j)$, $j \neq q$ or $D_0 \Subset M(q)$ and $\overline{D}_0 \bigcap \overline{D}_1 = \emptyset$. Then, for every $m \in \mathbb{N}$, $N > 1$, there exists $C_{N,m} > 0$ independent of k such that*

$$\left|T^{(q),f,g}_{k,k^{-N},s,\widehat{s}}(x,y)\right|_{\mathscr{C}^m(D_0\times D_1)} \leq C_{N,m}k^{3n-\frac{N}{2}+2m}. \tag{6.1}$$

Proof. The proof is similar to the proof of Theorem 4.4, so we will insist here on the appearance of the exponent $3n$ in the power of k in (6.1), compared to $2n$ in the previous estimates. The argument holds for any complex manifold, *not necessarily compact*. For simplicity, we only consider $q = 0$. Let $s, \widehat{s}$ be local trivializing holomorphic sections of L on $\widetilde{D}_0 \Subset M$ and $\widetilde{D}_1 \Subset M$, respectively, $|s|^2_h = e^{-2\phi}$, $|\widehat{s}|^2_h = e^{-2\widehat{\phi}}$, where $\widetilde{D}_0$ and $\widetilde{D}_1$ are open sets. Fix $D_0 \Subset \widetilde{D}_0$, $D_1 \Subset \widetilde{D}_1$, D_0 and D_1 are open sets. Let $\tau \in \mathscr{C}^\infty_0(\widetilde{D}_0)$ and $\tau = 1$ on D_0. We will first show that how to estimate the kernel of $T^{(0),(1-\tau)f,g}_{k,k^{-N}}(x, y)$ on $D_0 \times D_1$. Take

$$\left\{\alpha_1(x), \alpha_2(x), \ldots, \alpha_{d_k}(x)\right\}, \quad \left\{\delta_1(x), \delta_2(x), \ldots, \delta_{d_k}(x)\right\}$$

be orthonormal frames for $\mathscr{E}^0_{k^{-N}}(M, L^k)$, where $d_k \in \mathbb{N} \cup \{\infty\}$. On $\widetilde{D}_0$, we write $\alpha_j(x) = s^k(x)\widetilde{\alpha}_j(x)$, $j = 1, \ldots, d_k$. On $\widetilde{D}_1$, we write $\delta_j(x) = \hat{s}^k(x)\hat{\delta}_j(x)$, $j = 1, \ldots, d_k$. For every $y \in \widetilde{D}_1$, put

$$T^{(0),g}_{k,k^{-N}}(x, y)(\hat{s}^{-k}e^{-k\widehat{\phi}})(y) := \sum_{j,\ell=1}^{d_k} \alpha_j(x)(g\delta_\ell \,|\, \alpha_j)_k \overline{\widehat{\delta}_\ell}(y)e^{-k\widehat{\phi}}(y). \tag{6.2}$$

Since $\sum_{j=1}^{d_k} \left|\alpha_j(x)\right|^2_{h^k}$ and $\sum_{j=1}^{d_k} \left|\delta_j(x)\right|^2_{h^k}$ converge locally uniformly in C^∞ topology, for fixed y, $T^{(0),g}_{k,k^{-N}}(\cdot, y)(\hat{s}^{-k}e^{-k\widehat{\phi}})(y)$ is a smooth section of L^k. It is easy to see that for every $(x, y) \in D_0 \times D_1$,

$$T^{(0),(1-\tau)f,g}_{k,k^{-N},s,\hat{s}}(x, y) = \sum_{j=1}^{d_k} e^{-k\phi(x)}\widetilde{\alpha}_j(x)\Big(((1-\tau)f)(\cdot)T^{(0),g}_{k,k^{-N}})(\cdot, y)(\hat{s}^{-k}e^{-k\hat{\phi}})(y) \,|\, \alpha_j(\cdot)\Big)_k. \tag{6.3}$$

When M is compact, d_k is finite and $d_k \approx k^n$ and it is easy to estimate (6.3). When M is *noncompact*, d_k could be infinite, so to estimate (6.3) we need a more detailed analysis. Now, we fix $p \in D_0$. From Theorem 3.2 and Lemma 3.6, we can find $v_k \in \mathscr{E}^0_{k^{-N}}(M, L^k)$ with $\|v_k\|_k = 1$,

$$\int_{M\setminus D_0} |v_k|^2_{h^k}\, dv_M = O(k^{-\infty}) \tag{6.4}$$

and

$$\left|P^{(0)}_{k,k^{-N},s}(p, p) - \left|v_k(p)\right|^2_{h^k}\right| \lesssim k^{3n-N}. \tag{6.5}$$

We take $\alpha_1 = v_k$ and obtain from (6.5) that

$$\sum_{j=2}^{d_k} e^{-2k\phi(p)} \left|\widetilde{\alpha}_j(p)\right|^2 \lesssim k^{3n-N}. \tag{6.6}$$

Now,

$$\begin{aligned}
&\left|e^{-k\phi(p)}\widetilde{\alpha}_1(p)\Big(((1-\tau)f)(\cdot)T^{(0),g}_{k,k^{-N}})(\cdot, y)(\hat{s}^{-k}e^{-k\hat{\phi}})(y) \,|\, \alpha_1(\cdot)\Big)_k\right| \\
&\quad \leq \left|e^{-k\phi(x)}\widetilde{\alpha}_1(p)\right| \left\|T^{(0),g}_{k,k^{-N}}(\cdot, y)(\hat{s}^{-k}e^{-k\widehat{\phi}})(y)\right\|_k \left\|(1-\tau)f\alpha_1\right\|_k.
\end{aligned} \tag{6.7}$$

We claim that

$$\left\|T^{(0),g}_{k,k^{-N}}(\cdot, y)(\hat{s}^{-k}e^{-k\widehat{\phi}})(y)\right\|^2_k \lesssim k^n \text{ locally uniformly on } y \in D_1. \tag{6.8}$$

Fix $y_0 \in D_1$. We take $\{\delta_1(x), \delta_2(x), \ldots, \delta_{d_k}(x)\}$ so that $\hat{\delta}_1(y_0) \neq 0, \hat{\delta}_j(y_0) = 0, j = 2, 3, \ldots, d_k$. Then,

$$T^{(0),g}_{k,k-N}(x, y_0)(\hat{s}^{-k} e^{-k\widehat{\phi}})(y_0) = \sum_{j=1}^{d_k} \alpha_j(x)(g\delta_1 \,|\, \alpha_j)_k \overline{\widehat{\delta_1}}(y_0) e^{-k\widehat{\phi}(y_0)}. \tag{6.9}$$

From (6.9), we can check that

$$\begin{aligned}\left\| T^{(0),g}_{k,k-N}(\cdot, y_0) \right\|_k^2 &= \sum_{j=1}^{d_k} \|\alpha_j\|_k^2 \left|(g\delta_1 \,|\, \alpha_j)_k\right|^2 \left|\overline{\widehat{\delta_1}}(y_0)\right|^2 e^{-2k\widehat{\phi}(y_0)} \\ &= \left\| P^{(0)}_{k,k-N}(g\delta_1) \right\|_k^2 e^{-2k\hat{\phi}(y_0)} \left|\overline{\widehat{\delta_1}}(y_0)\right|^2 .\end{aligned} \tag{6.10}$$

Since g is a bounded function, $\|P^{(0)}_{k,k-N}(g\delta_1)\|_k^2 \leq C$, for some constant $C > 0$ independent of k. Moreover, $e^{-2k\hat{\phi}(y_0)} \left|\overline{\widehat{\delta_1}}(y_0)\right|^2 \lesssim k^n$ locally uniformly on D_1. From this observation and (6.10), the estimate (6.8) follows. Relation (6.4) yields

$$\left\|(1-\tau)f\alpha_1\right\|_k = O(k^{-\infty}). \tag{6.11}$$

From (6.7), (6.8) , (6.11) and since $\left|e^{-k\phi(p)}\widetilde{\alpha}_1(p)\right| \lesssim k^{\frac{n}{2}}$, we conclude that

$$\left| e^{-k\phi(p)}\widetilde{\alpha}_1(p) \Big(((1-\tau)f)(\cdot) T^{(0),g}_{k,k-N})(\cdot, y)(\hat{s}^{-k} e^{-k\hat{\phi}})(y) \,|\, \alpha_1(\cdot) \Big)_k \right| = O(k^{-\infty}). \tag{6.12}$$

From (6.6) and (6.8), we have

$$\begin{aligned}&\left| \sum_{j=2}^{d_k} e^{-k\phi(p)}\widetilde{\alpha}_j(p) \Big(((1-\tau)f)(\cdot) T^{(0),g}_{k,k-N})(\cdot, y)(\hat{s}^{-k} e^{-k\hat{\phi}})(y) \,|\, \alpha_j(\cdot) \Big)_k \right| \\ &\leq \sqrt{\sum_{j=2}^{d_k} e^{-2k\phi(p)} \left|\widetilde{\alpha}_j(p)\right|^2} \sqrt{\sum_{j=2}^{d_k} \left| \Big(((1-\tau)f)(\cdot) T^{(0),g}_{k,k-N})(\cdot, y)(\hat{s}^{-k} e^{-k\hat{\phi}})(y) \,|\, \alpha_j(\cdot) \Big)_k \right|^2} \\ &\leq \sqrt{\sum_{j=2}^{d_k} e^{-2k\phi(p)} \left|\widetilde{\alpha}_j(p)\right|^2} \sqrt{\sum_{j=1}^{d_k} \left| \Big(((1-\tau)f)(\cdot) T^{(0),g}_{k,k-N})(\cdot, y)(\hat{s}^{-k} e^{-k\hat{\phi}})(y) \,|\, \alpha_j(\cdot) \Big)_k \right|^2} \\ &= \sqrt{\sum_{j=2}^{d_k} e^{-2k\phi(p)} \left|\widetilde{\alpha}_j(p)\right|^2} \left\| P^{(0)}_{k,k-N}(((1-\tau)f)(\cdot) T^{(0),g}_{k,k-N})(\cdot, y)(\hat{s}^{-k} e^{-k\hat{\phi}})(y)) \right\|_k \\ &\leq \sqrt{\sum_{j=2}^{d_k} e^{-2k\phi(p)} \left|\widetilde{\alpha}_j(p)\right|^2} \left\| ((1-\tau)f)(\cdot) T^{(0),g}_{k,k-N})(\cdot, y)(\hat{s}^{-k} e^{-k\hat{\phi}})(y) \right\|_k \\ &\lesssim k^{2n-N}.\end{aligned} \tag{6.13}$$

Note that here we still get the exponent $2n - N$. From (6.3), (6.12), and (6.13), we get

$$\left| T^{(0),(1-\tau)f,g}_{k,k-N,s,\hat{s}}(x, y) \right| \lesssim k^{2n-N} \text{ locally uniformly on } D_0 \times D_1. \tag{6.14}$$

Thus, to estimate $T^{(0),f,g}_{k,k^{-N},s,\hat{s}}(x,y)$, we only need to estimate $T^{(0),\tau f,g}_{k,k^{-N},s,\hat{s}}(x,y)$. Let $\hat{\tau}\in\mathscr{C}^\infty_0(\widetilde{D}_1)$ and $\hat{\tau}=1$ on D_1. We can repeat the procedure above and conclude that

$$\left|T^{(0),\tau f,(1-\hat{\tau})g}_{k,k^{-N},s,\hat{s}}(x,y)\right|\lesssim k^{2n-N}\ \text{locally uniformly on } D_0\times D_1. \tag{6.15}$$

Thus, to estimate $T^{(0),f,g}_{k,k^{-N},s,\hat{s}}(x,y)$, we only need to estimate $T^{(0),\tau f,\hat{\tau}g}_{k,k^{-N},s,\hat{s}}(x,y)$. We now explain how to estimate $T^{(0),\tau f,\hat{\tau}g}_{k,k^{-N},s,\hat{s}}(x,y)$. Take $\tau_1(x)\in\mathscr{C}^\infty_0(\widetilde{D}_0)$, $\tau_1=1$ on $\operatorname{Supp}\tau$. We have

$$\begin{aligned}
T^{(0),\tau f,\hat{\tau}g}_{k,k^{-N}}&=\widetilde{T}^{(0),\tau f,\hat{\tau}g}_{k,k^{-N}}+\hat{T}^{(0),\tau f,\hat{\tau}g}_{k,k^{-N}},\\
\widetilde{T}^{(0),\tau f,\hat{\tau}g}_{k,k^{-N}}&=P^{(0)}_{k,k^{-N}}\tau fP^{(0)}_{k,k^{-N}}\tau_1P^{(0)}_{k,k^{-N}}\hat{\tau}gP^{(0)}_{k,k^{-N}}=T^{(0),\tau f}_{k,k^{-N}}\tau_1T^{(0),\hat{\tau}g}_{k,k^{-N}},\\
\hat{T}^{(0),\tau f,\hat{\tau}g}_{k,k^{-N}}&=P^{(0)}_{k,k^{-N}}\tau fP^{(0)}_{k,k^{-N}}(1-\tau_1)P^{(0)}_{k,k^{-N}}\hat{\tau}gP^{(0)}_{k,k^{-N}}=T^{(0),\tau f}_{k,k^{-N}}(1-\tau_1)T^{(0),\hat{\tau}g}_{k,k^{-N}}.
\end{aligned} \tag{6.16}$$

The estimate of $\widetilde{T}^{(0),\tau f,\hat{\tau}g}_{k,k^{-N},}$ is as compact case since

$$\widetilde{T}^{(0),\tau f,\hat{\tau}g}_{k,k^{-N},s,\hat{s}}(x,y)=\int T^{(0),\tau f}_{k,k^{-N},s,s}(x,z)\tau_1(z)T^{(0),\hat{\tau}g}_{k,k^{-N},s,\hat{s}}(z,y)dv_M(z)$$

and the integral is over some compact set of M. We only need to show how to estimate $\hat{T}^{(0),\tau f,\hat{\tau}g}_{k,k^{-N},s,\hat{s}}(x,y)$. Note that

$$\hat{T}^{(0),\tau f,\hat{\tau}g}_{k,k^{-N}}=P^{(0)}_{k,k^{-N}}\tau fT^{(0),1-\tau_1}_{k,k^{-N}}\hat{\tau}gP^{(0)}_{k,k^{-N}}. \tag{6.17}$$

From (6.17), it is easy to see that

$$\begin{aligned}
&\hat{T}^{(0),\tau f,\hat{\tau}g}_{k,k^{-N},s,\hat{s}}(x,y)\\
&\quad=\sum_{j,\ell=1}^{d_k}\widetilde{\alpha}_j(x)e^{-k\phi(x)}\overline{\hat{\delta}_\ell(y)}e^{-k\hat{\phi}(y)}\\
&\qquad\times\int T^{(0),1-\tau_1}_{k,k^{-N},s,\hat{s}}(z,u)\hat{\tau}(u)g(u)\hat{\delta}_\ell(u)\tau(z)f(z)\overline{\widetilde{\alpha}_j(z)}e^{-k\hat{\phi}(u)-k\phi(z)}dv_M(u)dv_M(z).
\end{aligned} \tag{6.18}$$

Now, fix $x_0\in D_0$ and $y_0\in D_1$. We take $\{\alpha_1(x),\alpha_2(x),\ldots,\alpha_{d_k}(x)\}$ and $\{\delta_1(x),\delta_2(x),\ldots,\delta_{d_k}(x)\}$ so that $\widetilde{\alpha}_1(x_0)\neq0$, $\widetilde{\alpha}_j(x_0)=0$, $j=2,3,\ldots,d_k$, $\hat{\delta}_1(y_0)\neq0$, $\hat{\delta}_j(y_0)=0$, $j=2,3,\ldots,d_k$. Thus,

$$\begin{aligned}
&\hat{T}^{(0),\tau f,\hat{\tau}g}_{k,k^{-N},s,\hat{s}}(x_0,y_0)\\
&\quad=\widetilde{\alpha}_1(x_0)e^{-k\phi(x_0)}\overline{\hat{\delta}_1(y_0)}e^{-k\hat{\phi}(y_0)}\\
&\qquad\times\int T^{(0),1-\tau_1}_{k,k^{-N},s,\hat{s}}(z,u)\hat{\tau}(u)g(u)\hat{\delta}_1(u)\tau(z)f(z)\overline{\widetilde{\alpha}_1(z)}e^{-k\hat{\phi}(u)-k\phi(z)}dv_M(u)dv_M(z).
\end{aligned} \tag{6.19}$$

From Lemma 4.2, we see that $|T^{(0),1-\tau_1}_{k,k^{-N},s,\hat{s}}(z,u)| \lesssim k^{2n-N}$ locally uniformly on $\widetilde{D}_0 \times \widetilde{D}_1$. From this observation and since $|\widetilde{\alpha}_1(x_0)e^{-k\phi(x_0)}\overline{\widetilde{\delta}_1(y_0)e^{-k\hat{\phi}(y_0)}}| \lesssim k^n$, we deduce that

$$\left|\hat{T}^{(0),\tau f,\hat{\tau}g}_{k,k^{-N},s,\hat{s}}(x_0,y_0)\right| \lesssim k^{3n-N}.$$

Here we get the power $3n$. □

We have moreover:

Theorem 6.2. *With the notations above, let s be local trivializing holomorphic section of L on $D_0 \Subset M$. Assume that $D_0 \Subset M(q)$. Then, for every $N > 1$, $m \in \mathbb{N}$, there exists $\widetilde{C}_{N,m} > 0$ independent of k such that*

$$\left|T^{(q),f,g}_{k,k^{-N},s}(x,y) - e^{ik\Psi(x,y)}b_{f,g}(x,y,k)\right|_{\mathscr{C}^m(D_0\times D_0)} \le \widetilde{C}_{N,m}k^{3n-\frac{N}{2}+2m},$$

where

$$\begin{aligned}
&b_{f,g}(x,y,k) \in S^n(1;D_0\times D_0,\Lambda^{0,q}(T^*M))\boxtimes(\Lambda^{0,q}(T^*M))^*),\\
&b_{f,g}(x,y,k) \sim \sum_{j=0}^{\infty} b_{f,g,j}(x,y)k^{n-j} \text{ in } S^n(1;D_0\times D_0,\Lambda^{0,q}(T^*M))\boxtimes(\Lambda^{0,q}(T^*M))^*),\\
&b_{f,g,0}(x,x) = (2\pi)^{-n}f(x)g(x)\left|\det \dot{R}^L(x)\right|I_{\det \overline{W}^*}(x),\ \ \forall x \in D_0,
\end{aligned} \tag{6.20}$$

and Ψ is as in Theorem 3.2.

Proof. The proof of this theorem is similar to the proof of Theorem 5.4. We only give the outline of the proof and for simplicity we consider only $q=0$. Fix $D_0 \subset \widetilde{D}_0 \Subset M(q)$ and take $\tau(x) \in \mathscr{C}^\infty_0(\widetilde{D}_0)$, $\tau = 1$ on D_0. We may assume that the section s defined on $\widetilde{D}_0$. We can repeat the proof of Lemma 4.2 with minor changes and conclude that for every $N > 1$ and $m \in \mathbb{N}$, there is $C_{N,m} > 0$ independent of k such that

$$\begin{aligned}
&\left|T^{(0),(1-\tau)f,g}_{k,k^{-N},s}(x,y)\right|_{\mathscr{C}^m(D_0\times D_0)} \le C_{N,m}k^{3n-\frac{N}{2}+2m},\\
&\left|T^{(0),\tau f,(1-\tau)g}_{k,k^{-N},s}(x,y)\right|_{\mathscr{C}^m(D_0\times D_0)} \le C_{N,m}k^{3n-\frac{N}{2}+2m}.
\end{aligned} \tag{6.21}$$

From (6.21), we only need to consider $T^{(0),\tau f,\tau g}_{k,k^{-N},s}$. Take $\tau_1(x) \in \mathscr{C}^\infty_0(\widetilde{D}_0)$, $\tau_1 = 1$ on $\operatorname{Supp}\tau$. We have

$$\begin{aligned}
T^{(0),\tau f,\tau g}_{k,k^{-N}} &= \widetilde{T}^{(0),\tau f,\tau g}_{k,k^{-N}} + \widehat{T}^{(0),\tau f,\tau g}_{k,k^{-N}},\\
\widetilde{T}^{(0),\tau f,\tau g}_{k,k^{-N}} &= P^{(0)}_{k,k^{-N}}\tau f P^{(0)}_{k,k^{-N}}\tau_1 P^{(0)}_{k,k^{-N}}\tau g P^{(0)}_{k,k^{-N}} = T^{(0),\tau f}_{k,k^{-N}}\tau_1 T^{(0),\tau g}_{k,k^{-N}},\\
\widehat{T}^{(0),\tau f,\tau g}_{k,k^{-N}} &= P^{(0)}_{k,k^{-N}}\tau f P^{(0)}_{k,k^{-N}}(1-\tau_1)P^{(0)}_{k,k^{-N}}\tau g P^{(0)}_{k,k^{-N}} = T^{(0),\tau f}_{k,k^{-N}}(1-\tau_1)T^{(0),\tau g}_{k,k^{-N}}.
\end{aligned} \tag{6.22}$$

Let $\widetilde{T}^{(0),\tau f,\tau g}_{k,k^{-N},s}(x,y), \widehat{T}^{(0),\tau f,\tau g}_{k,k^{-N},s}(x,y) \in \mathscr{C}^\infty(\widetilde{D}_0\times\widetilde{D}_0,\Lambda^{0,q}(T^*M))\boxtimes(\Lambda^{0,q}(T^*M))^*)$ be the distribution kernels of $s^{-k}e^{-k\phi}\widetilde{T}^{(0),\tau f,\tau g}_{k,k^{-N}}s^k e^{k\phi}$ and $s^{-k}e^{-k\phi}\widehat{T}^{(0),\tau f,\tau g}_{k,k^{-N}}s^k e^{k\phi}$, respectively. We have

$$T^{(0),\tau f,\tau g}_{k,k^{-N},s}(x,y) = \widetilde{T}^{(0),\tau f,\tau g}_{k,k^{-N},s}(x,y) + \widehat{T}^{(0),\tau f,\tau g}_{k,k^{-N},s}(x,y). \tag{6.23}$$

We first consider $\widehat{T}^{(0),\tau f,\tau g}_{k,k^{-N},s}(x,y)$. Take

$$\{\alpha_1(x),\alpha_2(x),\ldots,\alpha_{d_k}(x)\},\quad \{\delta_1(x),\delta_2(x),\ldots,\delta_{d_k}(x)\}$$

be orthonormal frames for $\mathscr{E}^0_{k^{-N}}(M,L^k)$, where $d_k\in\mathbb{N}\cup\{\infty\}$. On $\widetilde{D}_0$, we write

$$\alpha_j(x)=s^k(x)\widetilde{\alpha}_j(x),\ \ \delta_j(x)=s^k(x)\widetilde{\delta}_j(x),\ \ j=1,\ldots,d_k.$$

It is straightforward to check that

$$\begin{aligned}\widehat{T}^{(0),\tau f,\tau g}_{k,k^{-N},s}(x,y)&=\sum_{j,s=1}^{d_k}\widetilde{\alpha}_j(x)e^{-k\phi(x)}\overline{\widetilde{\delta}_s(y)}e^{-k\phi(y)}\\&\quad\times\int T^{(0),1-\tau_1}_{k,k^{-N},s}(z,u)\tau(u)g(u)\widetilde{\delta}_s(u)\tau(z)f(z)\overline{\widetilde{\alpha}_j(z)}e^{-k\phi(u)-k\phi(z)}dv_M(u)dv_M(z).\end{aligned}\tag{6.24}$$

From Lemmas 4.2, (3.10), and (6.24), it is not difficult to see that for every $N>1$ and $m\in\mathbb{N}$, there exists $C_{N,m}>0$ independent of k such that

$$\left|\widehat{T}^{(0),\tau f,\tau g}_{k,k^{-N},s}(x,y)\right|_{\mathscr{C}^m(D_0\times D_0)}\leq C_{N,m}k^{3n-\frac{N}{2}+2m}.\tag{6.25}$$

We now consider $\widetilde{T}^{(0),\tau f,\tau g}_{k,k^{-N},s}(x,y)$. We have

$$\widetilde{T}^{(0),\tau f,\tau g}_{k,k^{-N},s}(x,y)=\int T^{(0),\tau f}_{k,k^{-N},s}(x,z)\tau_1(z)T^{(0),\tau g}_{k,k^{-N},s}(z,y)dv_M(z).\tag{6.26}$$

Put

$$S_{k,f}(x,y)=e^{ik\Psi(x,y)}b_f(x,y,k),\ \ S_{k,g}(x,y)=e^{ik\Psi(x,y)}b_g(x,y,k),$$

where $\Psi(x,y)$ is as in Theorem 3.2 and $b_f(x,y,k),b_g(x,y,k)\in S^n(1;\widetilde{D}_0\times\widetilde{D}_0)$ are as in Theorem 5.4. Put

$$A_k(x,y)=\int T^{(0),\tau f}_{k,k^{-N},s}(x,z)\tau_1(z)T^{(0),\tau g}_{k,k^{-N},s}(z,y)dv_M(z)-\int S_{k,f}(x,z)\tau_1(z)S_{k,g}(z,y)dv_M(z).$$

From Theorem 5.4, it is straightforward to see that for every $N>1$ and $m\in\mathbb{N}$, there exists $C_{N,m}>0$ independent of k such that

$$\left|A_k(x,y)\right|_{\mathscr{C}^m(D_0\times D_0)}\leq C_{N,m}k^{3n-\frac{N}{2}+2m}.\tag{6.27}$$

We claim that

$$\int S_{k,f}(x,z)\tau_1(z)S_{k,g}(z,y)dv_M(z)\equiv e^{ik\Psi(x,y)}b_{f,g}(x,y,k)\mod O(k^{-\infty})\tag{6.28}$$

locally uniformly on $\widetilde{D}_0\times\widetilde{D}_0$, where $b_{f,g}(x,y,k)\in S^n(1;\widetilde{D}_0\times\widetilde{D}_0)$,

$$b_{f,g}(x,y,k)\sim\sum_{j=0}^{\infty}b_{f,g,j}(x,y)k^{n-j}\text{ in }S^n(1;\widetilde{D}_0\times\widetilde{D}_0),$$

with $b_{f,g,j}\in\mathscr{C}^\infty(\widetilde{D}_0\times\widetilde{D}_0)$, and $b_{f,g,0}(x,x)=(2\pi)^{-n}\tau(x)^2\tau_1(x)f(x)g(x)\left|\det\dot{R}^L(x)\right|$, $x\in\widetilde{D}_0$.

We use now the theory of complex Fourier integral operator, in particular the fact that composition of complex Fourier integral operators is still a complex Fourier integral operator.

Indeed, the complex stationary phase formula of Melin–Sjöstrand [28] tells us that there is a complex phase $\Psi_1(x,y) \in \mathscr{C}^\infty(\widetilde{D}_0 \times \widetilde{D}_0)$ with $\operatorname{Im}\Psi_1(x,y) \approx \left|x-y\right|^2$, such that for any $A(x,y) = e^{ik\Psi(x,y)}a(x,y,k)$, $C(x,y) = e^{ik\Psi(x,y)}c(x,y,k)$, where $a(x,y,k), c(x,y,k) \in S^n(1;\widetilde{D}_0 \times \widetilde{D}_0)$, and every $\chi \in \mathscr{C}^\infty_0(\widetilde{D}_0)$, we have

$$\int A(x,z)\chi(z)B(z,y)dv_M(z) \equiv e^{ik\Psi_1(x,y)}h(x,y,k) \mod O(k^{-\infty}) \tag{6.29}$$

locally uniformly on $\widetilde{D}_0 \times \widetilde{D}_0$, where $h(x,y,k) \in S^n(1;\widetilde{D}_0 \times \widetilde{D}_0)$,

$$h(x,y,k) \sim \sum_{j=0}^{\infty} h_j(x,y)k^{n-j} \text{ in } S^n(1;\widetilde{D}_0 \times \widetilde{D}_0)$$

and $h_0(x,x) = (2\pi)^{-n}\chi(x)a_0(x,x)c_0(x,x), x \in \widetilde{D}_0$, where a_0 and c_0 denote the leading terms of $a(x,y,k)$ and $c(x,y,k)$, respectively. In the proof of Lemma 5.3, we proved that $\Psi(x,y) - \Psi_1(x,y)$ vanishes to infinite order on $x = y$ (see (5.25)). Thus, we can replace Ψ_1 in (6.29) by Ψ and we get (6.28).

From (6.28), (6.27), (6.25), (6.23), and (6.21), the theorem follows. □

Proof of Theorem 1.3. Theorems 6.1 and 6.2 yield immediately Theorem 1.3. □

Proof of Theorem 1.9. This follows by using the asymptotics of the Bergman kernel proved in [18, Theorem 1.6] in the case of an $O(k^{-N})$ small spectral gap and adapting the proofs of Theorems 1.1 and 1.3 to the current situation. □

Proof of Theorem 1.10. By [18, Theorem 8.2], we know that $\Box^{(0)}_k$ has an $O(k^{-N})$ small spectral gap on every $D \Subset M' \cap M(0)$. This observation and Theorem 1.9 yield Theorem 1.10. □

Proof of Theorem 1.11. $M\setminus\Sigma$ is a noncompact complex manifold. Let $\Box^{(0)}_k$ be the Gaffney extension of Kodaira Laplacian on $M\setminus\Sigma$ and let $P^{(0)}_{k,M\setminus\Sigma}$ be the associated Bergman projection. By a result of Skoda (see [18, Lemma 7.2]), we know that

$$P^{(0)}_{k,\mathscr{I}} = P^{(0)}_{k,M\setminus\Sigma} \text{ on } M\setminus\Sigma. \tag{6.30}$$

Moreover, we know that $\Box^{(0)}_k$ has $O(k^{-N})$ small spectral gap on every $D \Subset M\setminus\Sigma$ (see [18, Theorem 9.1]). This observation, (6.30) and Theorem 1.9 imply Theorem 1.11. □

In the following, we will prove Theorem 1.4. Fix $N > 1$. Let $f, g \in \mathscr{C}^\infty_0(D)$, $D \Subset M(0)$. For simplicity, we may assume that $L|_D$ is trivial and let s be a local trivializing holomorphic section of L on D, $|s|^2_h = e^{-2\phi}$. Take $\tau \in C^\infty_0(D)$ with $\tau = 1$ on $\operatorname{Supp} f \cup \operatorname{Supp} g$. Put

$$R_k = T^{(0),f}_{k,k^{-N}}T^{(0),g}_{k,k^{-N}} - \tau T^{(0),f}_{k,k^{-N}}T^{(0),g}_{k,k^{-N}}\tau. \tag{6.31}$$

We can repeat the proof of Lemma 4.2 with minor changes and obtain:

Lemma 6.3. *Let s_1, s_2 be local trivializing holomorphic sections of L on $D_1 \Subset M$ and $D_2 \Subset M$, respectively, where D_1 and D_2 are open sets. Then, for every $m \in \mathbb{N}$, there exists $C_m > 0$*

independent of k such that

$$\left|R_{k,s_1,s_2}(x,y)\right|_{\mathscr{C}^m(D_1\times D_2)} \leq C_m k^{3n-\frac{N}{2}+2m},$$

where $R_{k,s_1,s_2}(x,y)$ *denotes the distribution kernel of* $R_{k,s_1,s_2} := s_1^{-k}e^{-k\phi_1}R_k s_2^k e^{k\phi_2}$.

In particular, $T^{(0),f}_{k,k^{-N}}T^{(0),g}_{k,k^{-N}} - \tau T^{(0),f}_{k,k^{-N}}T^{(0),g}_{k,k^{-N}}\tau = \mathcal{O}(k^{3n-\frac{N}{2}})$ *locally in the* L^2 *operator norm.*

Let $b_{f,g}(x,y,k) \in S^n(1; D \times D)$ be as in Theorem 1.3. Then

$$b_{f,g}(x,y,k) \sim \sum_{j=0}^{\infty} b_{f,g,j}(x,y)k^{n-j} \text{ in } S^n(1; D \times D).$$

Since $f, g \in \mathscr{C}_0^\infty(D)$, we can take $b_{f,g}(x,y,k), b_{f,g,j}(x,y) \in \mathscr{C}_0^\infty(D \times D)$, $j \in \mathbb{N}$. Note that $b_{f,g}(x,y,k)$ and $b_{f,g,j}(x,y)$ have uniquely determined Taylor expansion at $x = y$. Consider

$$\begin{aligned} B_k : L^2(M, L^k) &\to L^2(M, L^k) \\ u &\mapsto s^k e^{k\phi}\tau \int e^{ik\Psi(x,y)} b_{f,g}(x,y,k)s^{-k}e^{-k\phi(y)}\tau(y)u(y)dv_M(y). \end{aligned}$$

In view of Theorem 1.3 and Lemma 6.3, we see that

$$B_k - T^{(0),f}_{k,k^{-N}}T^{(0),g}_{k,k^{-N}} = \mathcal{O}(k^{3n-\frac{N}{2}}) \text{ locally in the } L^2 \text{ operator norm.} \tag{6.32}$$

Lemma 6.4. *For any* $p \in \mathbb{N}$ *there exist* $C_p(f,g) \in \mathscr{C}_0^\infty(D)$ *such that*

$$b_{f,g}(x,y,k) \sim \sum_{p=0}^{\infty} b_{C_p(f,g)}(x,y,k)k^{-p} \text{ in } S^n(1; D \times D),$$

where $b_{C_p(f,g)}(x,y,k) \in S^n(1; D \times D)$ *for each* $p \in \mathbb{N}$.

Proof. Set

$$C_0(f,g) = fg \in \mathscr{C}_0^\infty(D). \tag{6.33}$$

From (1.20) and (1.13), we see that

$$b_{f,g,0}(x,x) = b_{C_0(f,g),0}(x,x), \quad \forall x \in D. \tag{6.34}$$

Note that $b_{f,g,0}(x,y)$ and $b_{C_0(f,g),0}(x,y)$ are holomorphic with respect to x and

$$b_{f,g,0}(x,y) = \overline{b}_{f,g,0}(y,x), \quad b_{C_0(f,g),0}(x,y) = \overline{b}_{C_0(f,g),0}(y,x).$$

From this observation and (6.34), it is easy to see that $b_{f,g,0}(x,y) - b_{C_0(f,g),0}(x,y)$ vanishes to infinite order on $x = y$. Thus, we can take $b_{C_0(f,g),0}(x,y)$ so that $b_{C_0(f,g),0}(x,y) = b_{f,g,0}(x,y)$ and hence $b_{f,g}(x,y,k) - b_{C_0(f,g)}(x,y,k) \in S^{n-1}(1, D \times D)$. Consider the expansion

$$b_{f,g}(x,y,k) - b_{C_0(f,g)}(x,y,k) \sim \sum_{j=0}^{\infty} a_j(x,y)k^{n-1-j} \text{ in } S^{n-1}(1; D \times D), \tag{6.35}$$

where $a_j(x,y) \in \mathscr{C}_0^\infty(D)$, $j \in \mathbb{N}$. Set

$$C_1(f,g)(x) = (2\pi)^n a_0(x,x)\left|\det \dot{R}^L(x)\right|^{-1} \in \mathscr{C}_0^\infty(D). \tag{6.36}$$

From (1.13), we have $b_{C_1(f,g),0}(x,x) = a_0(x,x)$ and as in the discussion above, we can take $b_{C_1(f,g),0}(x,y)$ so that $b_{C_1(f,g),0}(x,y) = a_0(x,y)$ and hence

$$b_{f,g}(x,y,k) - b_{C_0(f,g)}(x,y,k) - \frac{1}{k} b_{C_1(f,g)}(x,y,k) \in S^{n-2}(1, D \times D).$$

Continuing inductively, the lemma follows. □

Proof of Theorem 1.4. Let $a(x,y,k) \in S^{n-j_0}(1, D \times D)$, $j_0 \in \mathbb{N}$. Consider the operator

$$\begin{aligned} A_k : L^2(M, L^k) &\to L^2(M, L^k) \\ u &\mapsto s^k e^{k\phi} \tau \int e^{ik\Psi(x,y)} a(x,y,k) s^{-k} e^{-k\phi(y)} \tau(y) u(y) dv_M(y). \end{aligned}$$

By [18, Theorem 3.11], we have

$$A_k = \mathcal{O}(k^{-j_0}) \text{ locally in the } L^2 \text{ operator norm.} \tag{6.37}$$

For every $p \in \mathbb{N}$ put

$$\begin{aligned} B_{k,p} : L^2(M, L^k) &\to L^2(M, L^k) \\ u &\mapsto s^k e^{k\phi} \tau \int e^{ik\Psi(x,y)} b_{C_p(f,g)}(x,y,k) s^{-k} e^{-k\phi(y)} \tau(y) u(y) dv_M(y). \end{aligned}$$

As in (6.32), we can check that for $p = 0, 1, 2, \ldots$,

$$B_{k,p} - T^{(0),C_p(f,g)}_{k,k^{-N}} = \mathcal{O}(k^{3n-\frac{N}{2}}) \text{ locally in the } L^2 \text{ operator norm.} \tag{6.38}$$

Moreover, from (6.37) and Lemma 6.4, we have

$$B_k - \sum_{p=0}^{\ell} B_{k,p} k^{-p} = \mathcal{O}(k^{-\ell-1}) \text{ locally in the } L^2 \text{ operator norm}, \quad \ell = 0, 1, 2, \ldots. \tag{6.39}$$

From (6.32), (6.38), and (6.39), we conclude that

$$T^{(0),f}_{k,k^{-N}} T^{(0),g}_{k,k^{-N}} - \sum_{p=0}^{\ell} T^{(0),C_p(f,g)}_{k,k^{-N}} k^{-p} = \mathcal{O}(k^{-\ell-1} + k^{3n-\frac{N}{2}}), \quad \ell = 0, 1, 2, \ldots,$$

locally in the L^2 operator norm. Moreover, we have $C_0(f,g) = C_0(g,f) = fg$ by (6.33). We also have $C_1(f,g) = -\frac{1}{2\pi} \langle \partial f \,|\, \overline{\partial g} \rangle_\omega$ by (7.36), so as in [27, (0.23)] we obtain

$$C_1(f,g) - C_1(g,f) = \sqrt{-1}\{f,g\}, \tag{6.40}$$

where $\{f,g\}$ is the Poisson bracket of the functions f, g with respect to the symplectic form $2\pi\omega$ on $M(0)$ (see also [25, (4.89)], [24, (7.4.3)]). Therefore (1.23) follows. □

Recall that the Poisson bracket $\{\cdot\,,\,\cdot\}$ on $(M, 2\pi\omega)$ is defined as follows. For $f, g \in \mathscr{C}^\infty(M)$, let ξ_f be the Hamiltonian vector field generated by f, which is defined by $2\pi\omega(\xi_f, \cdot) = df$. Then

$$\{f,g\} := \xi_f(dg). \tag{6.41}$$

Remark 6.5. Berezin introduced in his ground-breaking work [3] a star-product by using Toeplitz operators. Formal star-products are known to exist on symplectic manifolds by

De Wilde and Lecomte [12] and Fedosov [14]. The Berezin–Toeplitz star-product gives a concrete geometric realization of such product. For compact Kähler manifold the Berezin–Toeplitz star product was introduced in Karabegov and Schlichenmaier [19] and Schlichenmaier [29]. For general compact symplectic manifolds this was realized in Ma and Marinescu [24, 25] by using Toeplitz operators obtained by projecting on the kernel of the Dirac operator. Due to Theorem 1.4, we can also define an associative star-product on the set $M(0)$ where a holomorphic line bundle $L \to M$ is positive, namely by setting for any $f, g \in \mathscr{C}^\infty_0(M(0))$,

$$f * g := \sum_{k=0}^{\infty} C_k(f,g)\hbar^k \in \mathscr{C}^\infty(X)[[\hbar]]. \tag{6.42}$$

7. Calculation of the leading coefficients

In this section, we will give formulas for the top coefficients of the expansion (1.13) in the case $q = 0$, cf. Theorem 7.1. We introduce the geometric objects used in Theorem 7.1 below. Consider the $(1,1)$-form on M,

$$\omega := \frac{\sqrt{-1}}{2\pi} R^L. \tag{7.1}$$

On $M(0)$ the $(1,1)$-form ω is positive and induces a Riemannian metric $g^{TM}_\omega(\cdot,\cdot) = \omega(\cdot, J\cdot)$. In local holomorphic coordinates $z = (z_1, \ldots, z_n)$, put

$$\begin{aligned} \omega &= \sqrt{-1}\sum_{j,k=1}^{n} \omega_{j,k} dz_j \wedge d\bar{z}_k, \\ \Theta &= \sqrt{-1}\sum_{j,k=1}^{n} \Theta_{j,k} dz_j \wedge d\bar{z}_k. \end{aligned} \tag{7.2}$$

We notice that $\Theta_{j,k} = \langle \frac{\partial}{\partial z_j} \mid \frac{\partial}{\partial z_k} \rangle$, $\omega_{j,k} = \langle \frac{\partial}{\partial z_j} \mid \frac{\partial}{\partial z_k} \rangle_\omega$, $j, k = 1, \ldots, n$. Put

$$h = \left(h_{j,k}\right)_{j,k=1}^{n}, \quad h_{j,k} = \omega_{k,j}, \quad j,k = 1,\ldots,n, \tag{7.3}$$

and $h^{-1} = \left(h^{j,k}\right)_{j,k=1}^{n}$, h^{-1} is the inverse matrix of h. The complex Laplacian with respect to ω is given by

$$\triangle_\omega = (-2)\sum_{j,k=1}^{n} h^{j,k} \frac{\partial^2}{\partial z_j \partial \bar{z}_k}. \tag{7.4}$$

We notice that $h^{j,k} = \langle dz_j \mid dz_k \rangle_\omega$, $j, k = 1, \ldots, n$. Put

$$\begin{aligned} V_\omega &:= \det\left(\omega_{j,k}\right)_{j,k=1}^{n}, \\ V_\Theta &:= \det\left(\Theta_{j,k}\right)_{j,k=1}^{n} \end{aligned} \tag{7.5}$$

and set

$$\begin{aligned} r &= \triangle_\omega \log V_\omega, \\ \hat{r} &= \triangle_\omega \log V_\Theta. \end{aligned} \tag{7.6}$$

r is called the scalar curvature with respect to ω. Let $R_\Theta^{\det}$ be the curvature of the canonical line bundle $K_M = \det \Lambda^{1,0}(T^*M)$ with respect to the real two form Θ. We recall that

$$R_\Theta^{\det} = -\overline{\partial}\partial \log V_\Theta. \tag{7.7}$$

Let ∇_ω^{TM} be the Levi–Civita connection on $(M(0), g_\omega^{TM})$, $R_\omega^{TM} = (\nabla_\omega^{TM})^2$ its curvature. Let h be as in (7.3). Put $\theta = h^{-1}\partial h = \left(\theta_{j,k}\right)_{j,k=1}^n$, $\theta_{j,k} \in \Lambda^{1,0}(T^*M), j,k = 1,\ldots,n$. θ is the Chern connection matrix with respect to ω. Then,

$$\begin{aligned}
&R_\omega^{TM} = \overline{\partial}\theta = \left(\overline{\partial}\theta_{j,k}\right)_{j,k=1}^n = \left(\mathcal{R}_{j,k}\right)_{j,k=1}^n \in \mathscr{C}^\infty(M, \Lambda^{1,1}(T^*M) \otimes \mathrm{End}\,(T^{1,0}M)),\\
&R_\omega^{TM}(\overline{U}, V) \in \mathrm{End}\,(T^{1,0}M),\ \ \forall U, V \in T^{1,0}M, \\
&R_\omega^{TM}(\overline{U}, V)\xi = \sum_{j,k=1}^n \langle\, \mathcal{R}_{j,k} \,|\, \overline{U} \wedge V \,\rangle\, \xi_k \frac{\partial}{\partial z_j},\ \ \xi = \sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j},\ \ U, V \in T^{1,0}M.
\end{aligned} \tag{7.8}$$

We denote by $\langle\,\cdot\,,\cdot\,\rangle_\omega$ the pointwise Hermitian metrics induced by g_ω^{TM} on $\Lambda^{p,q}(T^*M) \otimes \Lambda^{r,s}(T^*M)$, $p,q,r,s \in \{0,1,\ldots,n\}$, and by $|\cdot|_\omega$ the corresponding norms.

Set

$$\left|R_\omega^{TM}\right|_\omega^2 := \sum_{j,k,s,t=1}^n \left|\langle\, R_\omega^{TM}(\overline{e}_j, e_k)e_s \,|\, e_t \,\rangle_\omega\right|^2, \tag{7.9}$$

where $e_1,\ldots,e_n$ is an orthonormal frame for $T^{1,0}M$ with respect to $\langle\,\cdot\,,\cdot\,\rangle_\omega$. It is straightforward to see that the definition of $\left|R_\omega^{TM}\right|_\omega^2$ is independent of the choices of orthonormal frames. Thus, $\left|R_\omega^{TM}\right|_\omega^2$ is globally defined. The Ricci curvature with respect to ω is given by

$$\mathrm{Ric}_\omega := -\sum_{j=1}^n \langle\, R_\omega^{TM}(\cdot, e_j)\cdot \,|\, e_j \,\rangle_\omega, \tag{7.10}$$

where $e_1,\ldots,e_n$ is an orthonormal frame for $T^{1,0}M$ with respect to $\langle\,\cdot\,,\cdot\,\rangle_\omega$. That is,

$$\langle\, \mathrm{Ric}_\omega \,|\, U \wedge V \,\rangle = -\sum_{j=1}^n \langle\, R_\omega^{TX}(U, e_j)V \,|\, e_j \,\rangle_\omega,\ \ U, V \in TM \otimes_{\mathbb{R}} \mathbb{C}.$$

Ric_ω is a global $(1,1)$ form.

Let

$$D^{1,0} : \mathscr{C}^\infty(M, \Lambda^{1,0}(T^*M)) \to \mathscr{C}^\infty(M, \Lambda^{1,0}(T^*M) \otimes \Lambda^{1,0}(T^*M)) \tag{7.11}$$

be the $(1,0)$ component of the Chern connection on $\Lambda^{1,0}(T^*M)$ induced by $\langle\,\cdot\,,\cdot\,\rangle_\omega$. That is, in local coordinates $z = (z_1,\ldots,z_n)$, put

$$A = \left(a_{j,k}\right)_{j,k=1}^n,\ \ a_{j,k} = \langle\, dz_k \,|\, dz_j \,\rangle_\omega,\ \ j,k = 1,\ldots,n,$$

and set

$$\mathcal{A} = A^{-1}\partial A = \left(\alpha_{j,k}\right)_{j,k=1}^n,\ \ \alpha_{j,k} \in \Lambda^{1,0}(T^*M),\ \ j,k = 1,\ldots,n. \tag{7.12}$$

Then, for $u = \sum_{j=1}^n u_j dz_j \in \mathscr{C}^\infty(M, \Lambda^{1,0}(T^*M))$, we have

$$D^{1,0}u = \sum_{j=1}^n \partial u_j \otimes dz_j + \sum_{j,k=1}^n u_j \alpha_{k,j} \otimes dz_k \in \mathscr{C}^\infty(M, \Lambda^{1,0}(T^*M) \otimes \Lambda^{1,0}(T^*M)).$$

Theorem 7.1. *With the assumptions and notations used in Theorem 1.1, the coefficients $b_{f,1}(x,x)$ and $b_{f,2}(x,x)$ in the expansion (1.12) for $q=0$ have the following form: for every $x \in D_0$,*

$$b_{f,1}(x,x) = (2\pi)^{-n} f(x) \det \dot{R}^L(x)\Big(\frac{1}{4\pi}\widehat{r} - \frac{1}{8\pi} r\Big)(x) + (2\pi)^{-n} \det \dot{R}^L(x)\Big(-\frac{1}{4\pi}\triangle_\omega f\Big)(x), \tag{7.13}$$

$$\begin{aligned} b_{f,2}(x,x) = {} & (2\pi)^{-n} f(x) \det \dot{R}^L(x)\Big(\frac{1}{128\pi^2} r^2 - \frac{1}{32\pi^2} r\widehat{r} + \frac{1}{32\pi^2}(\widehat{r})^2 - \frac{1}{32\pi^2}\triangle_\omega \widehat{r} \\ & - \frac{1}{8\pi^2}\left|R^{\det}_\Theta\right|^2_\omega + \frac{1}{8\pi^2}\langle \operatorname{Ric}_\omega \,|\, R^{\det}_\Theta \rangle_\omega + \frac{1}{96\pi^2}\triangle_\omega r - \frac{1}{24\pi^2}\left|\operatorname{Ric}_\omega\right|^2_\omega \\ & + \frac{1}{96\pi^2}\left|R^{TX}_\omega\right|^2_\omega\Big)(x) + (2\pi)^{-n}\det \dot{R}^L(z)\Big(\frac{1}{16\pi^2}(\triangle_\omega f)\Big(-\widehat{r} + \frac{1}{2} r\Big) \\ & - \frac{1}{4\pi^2}\langle \overline{\partial}\partial f \,|\, R^{\det}_\Theta \rangle_\omega + \frac{1}{8\pi^2}\langle \overline{\partial}\partial f \,|\, \operatorname{Ric}_\omega \rangle_\omega + \frac{1}{32\pi^2}\triangle^2_\omega f\Big)(x). \end{aligned} \tag{7.14}$$

The formulas given in Theorem 7.1 simplify if we assume that $\omega = \Theta$. In this case, we have $V_\omega = V_\Theta$ and $r = \widehat{r}$. See also [26, Section 2.7], [27, Remark 0.5], concerning the calculation of the coefficients for an arbitrary underlying Hermitian metric Θ.

Let $q = 0$ and let

$$S_k(x,y) = e^{ik\Psi(x,y)} b(x,y,k)$$

be as in (5.1). Note that $b(x,y,k) \in S^n(1; D_0 \times D_0)$,

$$b(x,y,k) \sim \sum_{j=0}^{\infty} b_j(x,y) k^{n-j} \text{ in } S^n(1; D_0 \times D_0),$$

where $b_j(x,y) \in \mathscr{C}^\infty(D_0 \times D_0)$, $j \in \mathbb{N}$, and $b_0(x,x) = (2\pi)^{-n}\left|\det \dot{R}^L(x)\right|$. We have

$$\int S_k(x,z,k) f(z) S_k(z,y,k)\, dv_M(z) \equiv e^{ik\Psi(x,y)} b_f(x,y,k) \mod O(k^{-\infty}) \tag{7.15}$$

locally uniformly on $D_0 \times D_0$, where $b_f(x,y,k) \in S^n(1; D_0 \times D_0)$,

$$b_f(x,y,k) \sim \sum_{j=0}^{\infty} b_{f,j}(x,y) k^{n-j} \text{ in } S^n(1; D_0 \times D_0), \tag{7.16}$$

where $b_{f,j}(x,y) \in \mathscr{C}^\infty(D_0 \times D_0)$, $j \in \mathbb{N}$, and $b_{0,f}(x,x) = (2\pi)^{-n} f(x)\left|\det \dot{R}^L(x)\right|$. In this section, we will calculate $b_{1,f}(x,x)$ and $b_{2,f}(x,x)$, $x \in D_0$. Fix $p \in D_0$. In a small neighborhood of the point p there exist local coordinates $z = (z_1, \ldots, z_n) = x = (x_1, \ldots, x_{2n})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, and a local frame s of L, $|s|^2_h = e^{-2\phi}$ so that

$$\begin{aligned} & z(p) = 0, \\ & \phi(z) = \sum_{j=1}^{n} \lambda_j \left|z_j\right|^2 + \phi_1(z), \end{aligned}$$

$$\phi_1(z)=O(|z|)^4),\quad \frac{\partial^{|\alpha|+|\beta|}\phi_1}{\partial z^\alpha\partial\bar z^\beta}(0)=0\ \text{ for } \alpha,\beta\in\mathbb{N}^n, |\alpha|\leq 1 \text{ or } |\beta|\leq 1\,,$$
$$\Theta(z)=\sqrt{-1}\sum_{j=1}^n dz_j\wedge d\bar z_j+O(|z|).\tag{7.17}$$

Until further notice, we work with this local coordinates x and we identify p with the point $x=z=0$. It is well-known (see [18, Section 4.5]) that for every $N\in\mathbb{N}$, we have

$$\Psi(z,0)=i\phi(z)+O(|z|^N)\,,\ \ \Psi(0,z)=i\phi(z)+O(|z|^N).\tag{7.18}$$

We have

$$\int S_k(0,z,k)f(z)S_k(z,0,k)dv_M(z)$$
$$=\int_{D_0}e^{ik(\Psi(0,z)+\Psi(z,0))}b(0,z,k)b(z,0,k)f(z)V_\Theta(z)d\lambda(z)+r_k,\tag{7.19}$$

where $d\lambda(z)=2^n dx_1dx_2\cdots dx_{2n}$, $dv_M(z)=V_\Theta(z)d\lambda(z)$ and

$$\lim_{k\to\infty}\frac{r_k}{k^N}=0,\ \ \forall N\geq 0.$$

We notice that since $b(z,w,k)$ is properly supported, we have

$$b(0,z,k)\in\mathscr{C}_0^\infty(D_0),\ \ b(z,0,k)\in\mathscr{C}_0^\infty(D_0).\tag{7.20}$$

We recall the stationary phase formula of Hörmander (see [16, Theorem 7.7.5]).

Theorem 7.2. *Let $K\subset D$ be a compact set and N a positive integer. If $u\in\mathscr{C}_0^\infty(K)$, $F\in\mathscr{C}^\infty(D)$ and* $\operatorname{Im}F\geq 0$ *in D,* $\operatorname{Im}F(0)=0$, $F'(0)=0$, $\det F''(0)\neq 0$, $F'\neq 0$ *in $K\setminus\{0\}$ then*

$$\left|\int e^{ikF(z)}u(z)V_\Theta(z)d\lambda(z)-2^n e^{ikF(0)}\det\left(\frac{kF''(0)}{2\pi i}\right)^{-\frac12}\sum_{j<N}k^{-j}L_ju\right|$$
$$\leq Ck^{-N}\sum_{|\alpha|\leq 2N}\sup\left|\partial_x^\alpha u\right|,\ \ k>0,\tag{7.21}$$

where C is uniform when F runs in a relatively compact set of $\mathscr{C}^\infty(D)$, $\frac{|x|}{|F'(x)|}$ has a uniform bound and

$$L_ju=\sum_{\nu-\mu=j}\sum_{2\nu\geq 3\mu}i^{-j}2^{-\nu}\langle F''(0)^{-1}D,D\rangle^\nu\frac{(h^\mu V_\Theta u)(0)}{\nu!\mu!}.\tag{7.22}$$

Here

$$h(x)=F(x)-F(0)-\frac12\langle F''(0)x,x\rangle\tag{7.23}$$

and $D=\begin{pmatrix}-i\partial_{x_1}\\ \vdots\\ -i\partial_{x_{2n}}\end{pmatrix}$.

We now apply (7.21) to the integral in (7.19). Put

$$F(z) = \Psi(0,z) + \Psi(z,0).$$

From (7.17) and (7.18), we see that

$$F(z) = 2i\sum_{j=1}^{n}\lambda_j\left|z_j\right|^2 + 2i\phi_1(z) + O(|z|^N),\ \ \forall N\geq 0,$$
$$h(z) = 2i\phi_1(z) + O(|z|^N),\ \ \forall N\geq 0, \tag{7.24}$$

where h is given by (7.23). Moreover, we can check that

$$\det\left(\frac{kF''(0)}{2\pi i}\right)^{-\frac{1}{2}} = k^{-n}\pi^n 2^{-n}\lambda_1^{-1}\lambda_2^{-1}\cdots\lambda_n^{-1} = k^{-n}\pi^n\big(\det\dot{R}^L(0)\big)^{-1} \tag{7.25}$$

and

$$\langle F''(0)^{-1}D, D\rangle = i\triangle_0\,, \qquad \triangle_0 = \sum_{j=1}^{n}\frac{1}{\lambda_j}\frac{\partial^2}{\partial\bar{z}_j\partial z_j}\,. \tag{7.26}$$

From (7.24), (7.26) and using that $h = O(|z|^4)$, it is not difficult to see that

$$L_j(b(0,z,k)b(z,0,k)f) = \sum_{\nu-\mu=j}\sum_{2\nu\geq 4\mu}(-1)^{\mu}2^{-j}\frac{\triangle_0^{\nu}\big(\phi_1^{\mu}V_{\Theta}b(0,z,k)b(z,0,k)f\big)(0)}{\nu!\mu!}, \tag{7.27}$$

where L_j is given by (7.22). We notice that

$$b(0,z,k) \equiv \sum_{j=0}^{\infty}b_j(0,z)k^{n-j} \mod O(k^{-\infty}),\ \ b(z,0,k) \equiv \sum_{j=0}^{\infty}b_j(z,0)k^{n-j} \mod O(k^{-\infty}).$$

From this observation, (7.27) becomes:

$$L_j\big(b(0,z,k)b(z,0,k)f\big) = \sum_{\nu-\mu=j}\sum_{2\nu\geq 4\mu}\sum_{0\leq s+t\leq N}(-1)^{\mu}2^{-j}\frac{k^{2n-s-t}\triangle_0^{\nu}\big(\phi_1^{\mu}V_{\Theta}fb_s(0,z)b_t(z,0)\big)(0)}{\nu!\mu!} + O(k^{2n-N-1}), \tag{7.28}$$

for all $N\geq 0$. From (7.28), (7.25), (7.21), (7.19) and (7.15), we get

$$b_f(0,0,k) = (2\pi)^n(\det\dot{R}^L(0))^{-1} \times\sum_{j=0}^{N}k^{n-j}\Big(\sum_{0\leq m\leq j}\sum_{\nu-\mu=m}\sum_{2\nu\geq 4\mu}\sum_{s+t=j-m}(-1)^{\mu}2^{-m}\frac{\triangle_0^{\nu}\big(\phi_1^{\mu}V_{\Theta}fb_s(0,z)b_t(z,0)\big)(0)}{\nu!\mu!}\Big) + O(k^{n-N-1}),\ \ \forall N\geq 0. \tag{7.29}$$

Combining (7.29) with (7.16), we obtain

Theorem 7.3. *The coefficients $b_{f,j}$ of the expansion (7.16) of $b_f(x,y,k)$, are given by*

$$b_{f,j}(0,0) = (2\pi)^n(\det \dot{R}^L(0))^{-1} \times \sum_{0\le m\le j}\sum_{\nu-\mu=m}\sum_{2\nu\ge 4\mu}\sum_{s+t=j-m}(-1)^{\mu}2^{-m}\frac{\triangle_0^{\nu}\big(\phi_1^{\mu}V_{\Theta}fb_s(0,z)b_t(z,0)\big)(0)}{\nu!\mu!}, \tag{7.30}$$

for all $j=0,1,\ldots$. In particular,

$$b_{f,0}(0,0) = (2\pi)^n(\det \dot{R}^L(0))^{-1}f(0)b_0(0,0)^2, \tag{7.31}$$

$$b_{f,1}(0,0) = (2\pi)^n(\det \dot{R}^L(0))^{-1}\Big(2f(0)b_0(0,0)b_1(0,0) + \frac{1}{2}\triangle_0\big(V_{\Theta}fb_0(0,z)b_0(z,0)\big)(0) - \frac{1}{4}\triangle_0^2\big(\phi_1V_{\Theta}fb_0(0,z)b_0(z,0)\big)(0)\Big) \tag{7.32}$$

and

$$\begin{aligned}b_{f,2}(0,0) = (2\pi)^n(\det \dot{R}^L(0))^{-1}\Big(&2f(0)b_0(0,0)b_2(0,0) + f(0)b_1(0,0)^2\\ &+\frac{1}{2}\triangle_0\big(V_{\Theta}f(b_0(0,z)b_1(z,0)+b_1(0,z)b_0(z,0))\big)(0)\\ &-\frac{1}{4}\triangle_0^2\big(\phi_1V_{\Theta}f(b_0(0,z)b_1(z,0)+b_1(0,z)b_0(z,0))\big)(0)\\ &+\frac{1}{8}\triangle_0^2\big(V_{\Theta}fb_0(0,z)b_0(z,0)\big)(0) - \frac{1}{24}\triangle_0^3\big(\phi_1V_{\Theta}fb_0(0,z)b_0(z,0)\big)(0)\\ &+\frac{1}{192}\triangle_0^4\big(\phi_1^2V_{\Theta}fb_0(0,z)b_0(z,0)\big)(0)\Big).\end{aligned} \tag{7.33}$$

In [18, Section 4.5], we determined all the derivatives of $b_0(x,y)$, $b_1(x,y)$, $b_2(x,y)$ at $(0,0)$. From this observation and Theorem 7.3, we can repeat the procedure in [17, Section 4] and obtain Theorem 7.1. Since the calculation is the same, we omit the details.

Let $D^{1,0}: \mathscr{C}^{\infty}(M,\Lambda^{1,0}(T^*M)) \to \mathscr{C}^{\infty}(M,\Lambda^{1,0}(T^*M)\otimes\Lambda^{1,0}(T^*M))$ be the $(1,0)$ component of the Chern connection on $\Lambda^{1,0}(T^*M)$ induced by $\langle\cdot,\cdot\rangle_{\omega}$ (see the discussion after (7.11)). From Theorem 7.1 and the proof of Theorem 1.3, we can repeat the proof of [17, Theorem 1.5] and get the following (see also Ma–Marinescu [24] for another method).

Theorem 7.4. *With the notations as in Theorem 1.3, let $q=0$. Then, for $b_{f,g,1}$, $b_{f,g,2}$ in (1.19), we have*

$$b_{f,g,1}(x) = b_{fg,1}(x) + (2\pi)^{-n}\det\dot{R}^L(x)\Big(-\frac{1}{2\pi}\langle\,\partial f\,|\,\partial\overline{g}\,\rangle_{\omega}\Big)(x), \tag{7.34}$$

$$
\begin{aligned}
b_{f,g,2}(x) &= b_{fg,2}(x) \\
&\quad + (2\pi)^{-n} \det \dot{R}^L(x)\Big(-\frac{1}{4\pi^2}\langle\, \overline{\partial} g \wedge \partial f \,|\, \mathrm{Ric}_\omega \,\rangle_\omega + \frac{1}{4\pi^2}\langle\, \overline{\partial} g \wedge \partial f \,|\, R^{\det}_\Theta \,\rangle_\omega \\
&\quad + \frac{1}{8\pi^2}\langle\, \partial\triangle_\omega f \,|\, \partial\overline{g} \,\rangle_\omega + \frac{1}{8\pi^2}\langle\, \overline{\partial}\triangle_\omega g \,|\, \overline{\partial}\,\overline{f} \,\rangle_\omega - \frac{1}{8\pi^2}\langle\, D^{1,0}\partial f \,|\, D^{1,0}\partial\overline{g} \,\rangle_\omega \\
&\quad - \frac{1}{4\pi^2}\langle\, \overline{\partial}\partial f \,|\, \overline{\partial}\partial\overline{g} \,\rangle_\omega + \frac{1}{8\pi^2}\langle\, \partial f \,|\, \partial\overline{g} \,\rangle_\omega(-\widehat{r} + \frac{1}{2}r)\Big)(x). \qquad (7.35)
\end{aligned}
$$

Corollary 7.5. *The coefficients $C_1(f,g)$ and $C_2(f,g)$ of the expansion (1.21) of the composition $T^{(0),f}_{k,k^{-N}} \circ T^{(0),g}_{k,k^{-N}}$ of two Toeplitz operators are given by*

$$C_1(f,g) = -\frac{1}{2\pi}\langle\, \partial f \,|\, \partial\overline{g} \,\rangle_\omega\,, \qquad (7.36)$$

$$C_2(f,g) = \frac{1}{8\pi^2}\langle\, D^{1,0}\partial f \,|\, D^{1,0}\partial\overline{g} \,\rangle_\omega + \frac{1}{4\pi^2}\langle\, \overline{\partial} g \wedge \partial f \,|\, R^{det}_\Theta \,\rangle_\omega\,. \qquad (7.37)$$

Proof. Formula (7.36) follows from (7.34) and

$$b_{1,f,g} = b_{1,fg} + b_{0,C_1(f,g)} = b_{1,fg} + (2\pi)^{-n}(\det \dot{R}^L)C_1(f,g)\,,$$

see [17, (5.21)] or [27, (5.76)]. Formula (7.37) follows as in [17, Section 5.3]. □

8. Behavior on the degenerate set and the Weyl law

In this section, we will prove Theorems 1.2 and 1.7. We recall first the following.

Theorem 8.1 ([18, Theorem 1.3]). *Set*

$$M_{\deg} = \Big\{ x \in M;\ \dot{R}^L \text{ is degenerate at } x \in M \Big\}.$$

Then for every $x_0 \in M_{\deg}$, $\varepsilon > 0$, $N > 1$ and every $m \in \{0, 1, \dots, n\}$, there exist a neighborhood U of x_0 and $k_0 > 0$, such that for all $k \geq k_0$ we have

$$\Big| P^{(m)}_{k,k^{-N}}(x,x) \Big| \leq \varepsilon k^n, \quad x \in U. \qquad (8.1)$$

Proof of Theorem 1.2. Fix $x_0 \in M_{\deg}$, $\varepsilon > 0$ and $m \in \{0, 1, \dots, n\}$. Let U be a small neighborhood of x_0 as in Theorem 8.1. Let p be any point of U and let s be a local section of L defined in a small open set $D \Subset U$ of p, $|s|^2_h = e^{-2\phi}$. Fix $|I_0| = |J_0| = q$, I_0, J_0 are strictly increasing. Take $\big\{\alpha_1(x), \alpha_2(x), \dots, \alpha_{d_k}(x)\big\}$ and $\big\{\beta_1(x), \beta_2(x), \dots, \beta_{d_k}(x)\big\}$ be orthonormal frames for $\mathscr{E}_{k^{-N}}(M, L^k)$ so that

$$\Big|\widetilde{\alpha}_{1,I_0}(p)e^{-k\phi(p)}\Big|^2 = \sum_{j=1}^{d_k}\Big|\widetilde{\alpha}_{j,I_0}(p)e^{-k\phi(p)}\Big|^2,$$

$$\Big|\widetilde{\beta}_{1,J_0}(p)e^{-k\phi(p)}\Big|^2 = \sum_{j=1}^{d_k}\Big|\widetilde{\beta}_{j,J_0}(p)e^{-k\phi(p)}\Big|^2,$$

where $d_k \in \mathbb{N} \cup \{\infty\}$ and on D, we write

$$\alpha_j(x) = s^k(x)\widetilde{\alpha}_j(x),\ \ \widetilde{\alpha}_j(x) = \sum_{|J|=q}{}^{'} \widetilde{\alpha}_{j,J}(x)e^J(x) \text{ on } D,\ \ j = 1, \ldots, d_k,$$

$$\beta_j(x) = s^k(x)\widetilde{\beta}_j(x),\ \ \widetilde{\beta}_j(x) = \sum_{|J|=q}{}^{'} \widetilde{\beta}_{j,J}(x)e^J(x) \text{ on } D,\ \ j = 1, \ldots, d_k.$$

We have

$$\begin{aligned} T^{(q),f,I_0,J_0}_{k,k^{-N}}(p,p) &= \sum_{j,\ell=1}^{d_k} \widetilde{\alpha}_{j,I_0}(p)e^{-k\phi(p)}(f\beta_\ell \,|\, \alpha_j)_k \overline{\widetilde{\beta}_{\ell,J_0}(p)}e^{-k\phi(p)} \\ &= \widetilde{\alpha}_{1,I_0}(p)e^{-k\phi(p)}(f\beta_1 \,|\, \alpha_1)_k \overline{\widetilde{\beta}_{1,J_0}(p)}e^{-k\phi(p)}. \end{aligned} \tag{8.2}$$

From (8.2), it is not difficult to see that

$$\left| T^{(q),f,I_0,J_0}_{k,k^{-N}}(p,p) \right| \leq \sup\left\{ \left| f(x) \right| ;\, x \in M \right\} \left| P^{(q)}_{k,k^{-N}}(p,p) \right|. \tag{8.3}$$

From (8.3) and (8.1), the theorem follows. □

We now prove Theorem 1.7. We introduce some notations. For $\lambda \geq 0$, put

$$P^{(q)}_{k,0<\mu\leq\lambda} := E(]0,\lambda]),\ \ \mathscr{E}^q_{0<\mu\leq\lambda}(M,L^k) := \operatorname{Rang} E(]0,\lambda]).$$

Recall that E denotes the spectral measure of $\Box^{(q)}_k$. Let $P^{(q)}_{k,0<\mu\leq\lambda}(\cdot\,,\cdot)$ be the Schwartz kernel of $P^{(q)}_{k,0<\mu\leq\lambda}$. The trace of $P^{(q)}_{k,0<\mu\leq\lambda}(x,x)$ is given by

$$\operatorname{Tr} P^{(q)}_{k,0<\mu\leq\lambda}(x,x) := \sum_{j=1}^{d} \left\langle P^{(q)}_{k,0<\mu\leq\lambda}(x,x)\, e_{J_j}(x) \,|\, e_{J_j}(x) \right\rangle,$$

where $e_{J_1}, \ldots, e_{J_d}$ is a local orthonormal basis of $\Lambda^{0,q}(T^*M)$ with respect to $\langle\cdot\,,\cdot\rangle$. Now, we assume that M is compact. We need the following.

Lemma 8.2. *There exists $C > 0$ independent of k such that*

$$\left| T^{(q),f}_k(x,x) - T^{(q),f}_{k,k^{-N}}(x,x) \right| \leq Ck^{\frac{n}{2}} \sqrt{\operatorname{Tr} P^{(q)}_{k,0<\mu\leq k^{-N}}(x,x)},\ \ \forall x \in M. \tag{8.4}$$

Proof. Let p be any point of M and let s be a local trivializing holomorphic section of L defined in a small open set $D \Subset U$ of p, $|s|^2_h = e^{-2\phi}$. Fix $|I_0| = |J_0| = q$, I_0, J_0 are strictly increasing. Take $\left\{\alpha_1(x), \alpha_2(x), \ldots, \alpha_{m_k}(x)\right\}$, $\left\{\beta_1(x), \beta_2(x), \ldots, \beta_{m_k}(x)\right\}$ to be orthonormal frames for $\mathscr{E}^q_0(M,L^k)$ and

$$\left\{\alpha_{m_k+1}(x), \alpha_{m_k+2}(x), \ldots, \alpha_{d_k}(x)\right\},\ \ \left\{\beta_{m_k+1}(x), \beta_{m_k+2}(x), \ldots, \beta_{d_k}(x)\right\}$$

to be orthonormal frames for $\mathscr{E}^q_{0<\mu\leq k^{-N}}(M,L^k)$ so that

$$\left|\widetilde{\alpha}_{1,I_0}(p)e^{-k\phi(p)}\right|^2 = \sum_{j=1}^{m_k} \left|\widetilde{\alpha}_{j,I_0}(p)e^{-k\phi(p)}\right|^2,$$

$$\left|\widetilde{\alpha}_{m_k+1,I_0}(p)e^{-k\phi(p)}\right|^2 = \sum_{j=m_k+1}^{d_k} \left|\widetilde{\alpha}_{j,I_0}(p)e^{-k\phi(p)}\right|^2,$$

$$\left|\widetilde{\beta}_{1,J_0}(p)e^{-k\phi(p)}\right|^2=\sum_{j=1}^{m_k}\left|\widetilde{\beta}_{j,J_0}(p)e^{-k\phi(p)}\right|^2,$$

$$\left|\widetilde{\beta}_{m_k+1,J_0}(p)e^{-k\phi(p)}\right|^2=\sum_{j=m_k+1}^{d_k}\left|\widetilde{\beta}_{j,J_0}(p)e^{-k\phi(p)}\right|^2,$$

where $d_k\in\mathbb{N}\cup\{\infty\}$ and on D, we write

$$\alpha_j(x)=s^k(x)\widetilde{\alpha}_j(x),\ \ \widetilde{\alpha}_j(x)=\sum_{|J|=q}{}^{'}\widetilde{\alpha}_{j,J}(x)e^J(x)\text{ on }D,\ \ j=1,\dots,d_k,$$

$$\beta_j(x)=s^k(x)\widetilde{\beta}_j(x),\ \ \widetilde{\beta}_j(x)=\sum_{|J|=q}{}^{'}\widetilde{\beta}_{j,J}(x)e^J(x)\text{ on }D,\ \ j=1,\dots,d_k.$$

We have

$$\begin{aligned}&T_k^{(q),f,I_0,J_0}(p,p)\\&=\widetilde{\alpha}_{1,I_0}(p)e^{-k\phi(p)}(f\beta_1\,|\,\alpha_1)_k\overline{\widetilde{\beta}_{1,J_0}(p)}e^{-k\phi(p)}\\&=T_{k,k^{-N}}^{(q),f,I_0,J_0}(p,p)-\widetilde{\alpha}_{m_k+1,I_0}(p)e^{-k\phi(p)}(f\beta_1\,|\,\alpha_{m_k+1})_k\overline{\widetilde{\beta}_{1,J_0}(p)}e^{-k\phi(p)}\\&\quad-\widetilde{\alpha}_{m_k+1,I_0}(p)e^{-k\phi(p)}(f\beta_{m_k+1}\,|\,\alpha_{m_k+1})_k\overline{\widetilde{\beta}_{m_k+1,J_0}(p)}e^{-k\phi(p)}\\&\quad-\widetilde{\alpha}_{1,I_0}(p)e^{-k\phi(p)}(f\beta_{m_k+1}\,|\,\alpha_1)_k\overline{\widetilde{\beta}_{m_k+1,J_0}(p)}e^{-k\phi(p)}.\end{aligned}\tag{8.5}$$

From (8.5), it is easy to see that

$$\left|T_k^{(q),f,I_0,J_0}(p,p)-T_{k,k^{-N}}^{(q),f,I_0,J_0}(p,p)\right|\leq C_1k^{\frac{n}{2}}\sqrt{\operatorname{Tr}P^{(q)}_{k,0<\mu\leq k^{-N}}(p,p)},$$

where $C_1>0$ is a constant independent of k and the point p. The lemma follows. □

Proof of Theorem 1.7. Since $M(q-1)=\emptyset$ and $M(q+1)=\emptyset$, it is known [18, Corollary 1.4], that for every $N>1$,

$$\dim\mathscr{E}^{q-1}_{k^{-N}}(M,L^k)=o(k^n),\ \ \dim\mathscr{E}^{q+1}_{k^{-N}}(M,L^k)=o(k^n).\tag{8.6}$$

Moreover, it is easy to see that

$$\dim\mathscr{E}^{q}_{0<\mu\leq k^{-N}}(M,L^k)\leq\dim\mathscr{E}^{q-1}_{k^{-N}}(M,L^k)+\dim\mathscr{E}^{q+1}_{k^{-N}}(M,L^k).\tag{8.7}$$

From (8.6) and (8.7), we have

$$\begin{aligned}\int_M\sqrt{\operatorname{Tr}P^{(q)}_{k,0<\mu\leq k^{-N}}(x,x)}dv_M(x)&\leq C_0\sqrt{\int_M\operatorname{Tr}P^{(q)}_{k,0<\mu\leq k^{-N}}(x,x)dv_M(x)}\\&=C_0\sqrt{\dim\mathscr{E}^q_{0<\mu\leq k^{-N}}(M,L^k)}\\&=o(k^{\frac{n}{2}}),\end{aligned}\tag{8.8}$$

where $C_0 > 0$ is a constant independent of k. From (8.8) and (8.4), we conclude that

$$\lim_{k\to\infty} k^{-n}\left|T_k^{(q),f}(x,x) - T_{k,k^{-N}}^{(q),f}(x,x)\right| = 0 \text{ in } L^1_{(0,q)}(M). \tag{8.9}$$

In view of Theorem 1.5, we see that

$$\lim_{k\to\infty}\left|k^{-n}T_{k,k^{-N}}^{(q),f}(x,x) - (2\pi)^{-n}\left|\det \dot{R}^L(x)\right| f(x)\mathbb{1}_{M(q)}(x) I_{\det \overline{W}^*}(x)\right| = 0 \tag{8.10}$$

in $L^1_{(0,q)}(M)$. From (8.9) and (8.10), the theorem follows. □

Acknowledgment

We are grateful to the referee for several suggestions which led to the improvement of the presentation.

Funding

The first author was supported by Taiwan Ministry of Science of Technology project 104-2628-M-001-003-MY2, the Golden-Jade fellowship of Kenda Foundation, Academia Sinica Career Development Award, and partially funded through the Institutional Strategy of the University of Cologne within the German Excellence Initiative. Second author partially supported by DFG funded projects SFB/TR 12, MA 2469/2-2, SFB TRR 191.

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