



On CR Paneitz operators and CR pluriharmonic functions

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Abstract Let $(X, T^{1,0}X)$ be a compact orientable embeddable three dimensional strongly pseudoconvex CR manifold and let P be the associated CR Paneitz operator. In this paper, we show that (I) P is self-adjoint and P has L^2 closed range. Let N and Π be the associated partial inverse and the orthogonal projection onto $\text{Ker } P$ respectively, then N and Π enjoy some regularity properties. (II) Let \hat{P} and \hat{P}_0 be the space of L^2 CR pluriharmonic functions and the space of real part of L^2 global CR functions respectively. Let S be the associated Szegő projection and let τ, τ_0 be the orthogonal projections onto \hat{P} and \hat{P}_0 respectively. Then, $\Pi = S + \bar{S} + F_0$, $\tau = S + \bar{S} + F_1$, $\tau_0 = S + \bar{S} + F_2$, where F_0, F_1, F_2 are smoothing operators on X . In particular, Π, τ and τ_0 are Fourier integral operators with complex phases and $\hat{P}^\perp \cap \text{Ker } P$, $\hat{P}_0^\perp \cap \hat{P}$, $\hat{P}_0^\perp \cap \text{Ker } P$ are all finite dimensional subspaces of $C^\infty(X)$ (it is well-known that $\hat{P}_0 \subset \hat{P} \subset \text{Ker } P$). (III) $\text{Spec } P$ is a discrete subset of \mathbb{R} and for every $\lambda \in \text{Spec } P$, $\lambda \neq 0$, λ is an eigenvalue of P and the associated eigenspace $H_\lambda(P)$ is a finite dimensional subspace of $C^\infty(X)$.

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1 Introduction and statement of the main results

Let $(X, T^{1,0}X)$ be a compact orientable embeddable strongly pseudoconvex CR manifold of dimension three. Let P be the associated Paneitz operator and let \hat{P} be the space of L^2 CR pluriharmonic functions. The operator P and the space \hat{P} play important roles in CR embedding problems and CR conformal geometry (see [2–4,9]). The operator

$$P : \text{Dom } P \subset L^2(X) \rightarrow L^2(X)$$

is a real, symmetric, fourth order non-hypoelliptic partial differential operator and \hat{P} is an infinite dimensional subspace of $L^2(X)$. In CR embedding problems and CR conformal geometry, it is crucial to be able to answer the following fundamental analytic problems about P and \hat{P} (see [2–4,9]):

- (I) Is P self-adjoint? Does P has L^2 closed range? What is $\text{Spec } P$?
- (II) If we have $Pu = f$, where f is in some Sobolev space $H^s(X)$, $s \in \mathbb{Z}$, and $u \perp \text{Ker } P$. Can we have $u \in H^{s'}(X)$, for some $s' \in \mathbb{Z}$?
- (III) The kernel of P studied first by Hirachi [9] contains CR pluriharmonic functions(see also Lee [11]). The question asked by Hirachi is whether there is anything else in the kernel? In [4], they asked further that is the kernel of P a direct sum of a finite dimensional subspace with CR pluriharmonic functions?
- (IV) Let Π be the orthogonal projection onto $\text{Ker } P$ and let τ be the orthogonal projection onto \hat{P} . Let $\Pi(x, y)$ and $\tau(x, y)$ denote the distribution kernels of Π and τ respectively. The P' operator introduced in Case and Yang [2] plays a critical role in CR conformal geometry. To understand the operator P' , it is crucial to be able to know the exactly forms of $\Pi(x, y)$ and $\tau(x, y)$.

The purpose of this work is to answer the above questions. On the other hand, in several complex variables, the study of the associated Szegő projection S and τ are classical subjects. The operator S is well-understood; S is a Fourier integral operator with complex phase (see Boutet de Monvel-Sjöstrand [1,10]). But for τ , there are fewer results. In this paper, by using the Paneitz operator P , we could prove that τ is also a complex Fourier integral operator and $\tau = S + \bar{S} + F_1$, F_1 is a smoothing operator. It is quite interesting to see if the result hold in dimension ≥ 5 . We hope that the Paneitz operator P will be interesting for complex analysts and will be useful in several complex variables.

We now formulate the main results. We refer to Sect. 2 for some standard notations and terminology used here.

Let $(X, T^{1,0}X)$ be a compact orientable 3-dimensional strongly pseudoconvex CR manifold, where $T^{1,0}X$ is a CR structure of X . We assume throughout that it is CR embeddable in some \mathbb{C}^N , for some $N \in \mathbb{N}$. Fix a contact form $\theta \in C^\infty(X, T^*X)$ compactable with the CR structure $T^{1,0}X$. Then, $(X, T^{1,0}X, \theta)$ is a 3-dimensional

pseudohermitian manifold. Let $T \in C^\infty(X, TX)$ be the real non-vanishing global vector field given by

$$\begin{aligned} \langle d\theta, T \wedge u \rangle &= 0, \quad \forall u \in T^{1,0}X \oplus T^{0,1}X, \\ \langle \theta, T \rangle &= -1. \end{aligned}$$

Let $\langle \cdot | \cdot \rangle$ be the Hermitian inner product on $\mathbb{C}TX$ given by

$$\begin{aligned} \langle Z_1 | Z_2 \rangle &= -\frac{1}{2i} \langle d\theta, Z_1 \wedge \bar{Z}_2 \rangle, \quad Z_1, Z_2 \in T^{1,0}X, \\ T^{1,0}X \perp T^{0,1}X &:= \overline{T^{1,0}X}, \quad T \perp (T^{1,0}X \oplus T^{0,1}X), \quad \langle T | T \rangle = 1. \end{aligned}$$

The Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ induces a Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}T^*X$. Let $T^{*0,1}X$ be the bundle of $(0, 1)$ forms of X . Take $\theta \wedge d\theta$ be the volume form on X , we then get natural inner products on $C^\infty(X)$ and $\Omega^{0,1}(X) := C^\infty(X, T^{*0,1}X)$ induced by $\theta \wedge d\theta$ and $\langle \cdot | \cdot \rangle$. We shall use $(\cdot | \cdot)$ to denote these inner products and use $\|\cdot\|$ to denote the corresponding norms. Let $L^2(X)$ and $L^2_{(0,1)}(X)$ denote the completions of $C^\infty(X)$ and $\Omega^{0,1}(X)$ with respect to $(\cdot | \cdot)$ respectively. Let

$$\square_b := \bar{\partial}_b^* \bar{\partial}_b : C^\infty(X) \rightarrow C^\infty(X)$$

be the Kohn Laplacian (see [10]), where $\bar{\partial}_b : C^\infty(X) \rightarrow \Omega^{0,1}(X)$ is the tangential Cauchy–Riemann operator and $\bar{\partial}_b^* : \Omega^{0,1}(X) \rightarrow C^\infty(X)$ is the formal adjoint of $\bar{\partial}_b$ with respect to $(\cdot | \cdot)$. That is, $(\bar{\partial}_b f | g) = (f | \bar{\partial}_b^* g)$, for every $f \in C^\infty(X)$, $g \in \Omega^{0,1}(X)$.

Let \mathcal{P} be the set of all CR pluriharmonic functions on X . That is,

$$\begin{aligned} \mathcal{P} &= \{u \in C^\infty(X, \mathbb{R}); \forall x_0 \in X, \text{ there is a } f \in C^\infty(X) \\ &\quad \text{with } \bar{\partial}_b f = 0 \text{ near } x_0 \text{ and } \operatorname{Re} f = u \text{ near } x_0\}. \end{aligned} \tag{1.1}$$

The Paneitz operator

$$P : C^\infty(X) \rightarrow C^\infty(X)$$

can be characterized as follows (see section 4 in [2] and Lee [11]): P is a fourth order partial differential operator, real, symmetric, $\mathcal{P} \subset \operatorname{Ker} P$ and

$$\begin{aligned} Pf &= \square_b \bar{\square}_b f + L_1 \circ L_2 f + L_3 f, \quad \forall f \in C^\infty(X), \\ L_1, L_2, L_3 &\in C^\infty(X, T^{1,0}X \oplus T^{0,1}X). \end{aligned} \tag{1.2}$$

We extend P to L^2 space by

$$\begin{aligned} P : \operatorname{Dom} P &\subset L^2(X) \rightarrow L^2(X), \\ \operatorname{Dom} P &= \left\{ u \in L^2(X); Pu \in L^2(X) \right\}. \end{aligned} \tag{1.3}$$

Let $\hat{\mathcal{P}} \subset L^2(X)$ be the completion of \mathcal{P} with respect to $(\cdot | \cdot)$. Then,

$$\hat{\mathcal{P}} \subset \text{Ker } P.$$

Put

$$\mathcal{P}_0 = \{\text{Re } f \in C^\infty(X, \mathbb{R}); f \in C^\infty(X) \text{ is a global CR function on } X\}$$

and let $\hat{\mathcal{P}}_0 \subset L^2(X)$ be the completion of \mathcal{P}_0 with respect to $(\cdot | \cdot)$. It is clearly that $\hat{\mathcal{P}}_0 \subset \hat{\mathcal{P}} \subset \text{Ker } P$. Let

$$\begin{aligned} \tau &: L^2(X) \rightarrow \hat{\mathcal{P}}, \\ \tau_0 &: L^2(X) \rightarrow \hat{\mathcal{P}}_0, \end{aligned} \tag{1.4}$$

be the orthogonal projections.

We recall

Definition 1.1 Suppose Q is a closed densely defined self-adjoint operator

$$Q : \text{Dom } Q \subset H \rightarrow \text{Ran } Q \subset H,$$

where H is a Hilbert space. Suppose that Q has closed range. By the partial inverse of Q , we mean the bounded operator $M : H \rightarrow \text{Dom } Q$ such that

$$\begin{aligned} QM + \pi &= I \text{ on } H, \\ MQ + \pi &= I \text{ on } \text{Dom } Q, \end{aligned}$$

where $\pi : H \rightarrow \text{Ker } Q$ is the orthogonal projection.

Let $\Omega \subset X$ be an open set. For any continuous operator $A : C_0^\infty(\Omega) \rightarrow \mathcal{D}'(\Omega)$, throughout this paper, we write $A \equiv 0$ (on Ω) if A is a smoothing operator on Ω (see Sect. 2).

The main purpose of this work is to prove the following

Theorem 1.2 *With the notations and assumptions above,*

$$P : \text{Dom } P \subset L^2(X) \rightarrow L^2(X)$$

is self-adjoint and P has L^2 closed range. Let $N : L^2(X) \rightarrow \text{Dom } P$ be the partial inverse and let $\Pi : L^2(X) \rightarrow \text{Ker } P$ be the orthogonal projection. Then,

$$\begin{aligned} \Pi, \tau, \tau_0 &: H^s(X) \rightarrow H^s(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \\ N &: H^s(X) \rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \end{aligned} \tag{1.5}$$

$$\Pi \equiv \tau \text{ on } X, \quad \Pi \equiv \tau_0 \text{ on } X \tag{1.6}$$

and the kernel $\Pi(x, y) \in \mathcal{D}'(X \times X)$ of Π satisfies

$$\Pi(x, y) \equiv \int_0^\infty e^{i\varphi(x,y)t} a(x, y, t) dt + \int_0^\infty e^{-i\bar{\varphi}(x,y)t} \bar{a}(x, y, t) dt, \tag{1.7}$$

where

$$\begin{aligned} \varphi &\in C^\infty(X \times X), \quad \text{Im } \varphi(x, y) \geq 0, \quad d_x \varphi|_{x=y} = -\theta(x), \\ \varphi(x, y) &= -\bar{\varphi}(y, x), \\ \varphi(x, y) &= 0 \text{ if and only if } x = y, \end{aligned} \tag{1.8}$$

(see Theorem 1.9 and Theorem 1.11 for more properties of the phase φ), and

$$\begin{aligned} a(x, y, t) &\in S_{\text{cl}}^1(X \times X \times]0, \infty[), \\ a(x, y, t) &\sim \sum_{j=0}^\infty a_j(x, y) t^{1-j} \text{ in } S_{1,0}^1(X \times X \times]0, \infty[), \\ a_j(x, y) &\in C^\infty(X \times X), \quad j = 0, 1, \dots, \\ a_0(x, x) &= \frac{1}{2} \pi^{-n}, \quad \forall x \in X. \end{aligned} \tag{1.9}$$

(See Sect. 2 and Definition 2.1 for the precise meanings of the notation \equiv and the Hörmander symbol spaces $S_{\text{cl}}^1(X \times X \times]0, \infty[)$ and $S_{1,0}^1(X \times X \times]0, \infty[)$).

Remark 1.3 With the notations and assumptions used in Theorem 1.2, it is easy to see that Π is real, that is $\Pi = \bar{\Pi}$.

Remark 1.4 With the notations and assumptions used in Theorem 1.2, let $S : L^2(X) \rightarrow \text{Ker } \bar{\partial}_b$ be the Szegő projection. That is, S is the orthogonal projection onto $\text{Ker } \bar{\partial}_b = \{u \in L^2(X); \bar{\partial}_b u = 0\}$ with respect to $(\cdot | \cdot)$. In view of the proof of Theorem 1.2 (see Sect. 4), we see that $\Pi \equiv S + \bar{S}$ on X .

We have the classical formulas

$$\int_0^\infty e^{-tx} t^m dt = \begin{cases} m! x^{-m-1}, & \text{if } m \in \mathbb{Z}, \quad m \geq 0, \\ \frac{(-1)^m}{(-m-1)!} x^{-m-1} \left(\log x + c - \sum_1^{-m-1} \frac{1}{j} \right), & \text{if } m \in \mathbb{Z}, \quad m < 0. \end{cases} \tag{1.10}$$

Here $x \neq 0$, $\text{Re } x \geq 0$ and c is the Euler constant, i.e. $c = \lim_{m \rightarrow \infty} (\sum_1^m \frac{1}{j} - \log m)$. Note that

$$\int_0^\infty e^{i\varphi(x,y)t} \sum_{j=0}^\infty a_j(x, y) t^{1-j} dt = \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty e^{-t(-i(\varphi(x,y)+i\varepsilon))} \sum_{j=0}^\infty a_j(x, y) t^{1-j} dt. \tag{1.11}$$

We have the following corollary of Theorem 1.2

Corollary 1.5 *With the notations and assumptions used in Theorem 1.2, there exist $F_1, G_1, \in C^\infty(X \times X)$ such that*

$$\begin{aligned} \Pi(x, y) &= F_1(-i\varphi(x, y))^{-2} + G_1 \log(-i\varphi(x, y)) \\ &\quad + \overline{F}_1(i\overline{\varphi}(x, y))^{-2} + \overline{G}_1 \log(i\overline{\varphi}(x, y)). \end{aligned}$$

Moreover, we have

$$\begin{aligned} F_1 &= a_0(x, y) + a_1(x, y)(-i\varphi(x, y)) + f_1(x, y)(-i\varphi(x, y))^2, \\ G_1 &\equiv \sum_0^\infty \frac{(-1)^{k+1}}{k!} a_{2+k}(x, y)(-i\varphi(x, y))^k, \end{aligned} \tag{1.12}$$

where $a_j(x, y), j = 0, 1, \dots,$ are as in (1.9) and $f_1(x, y) \in C^\infty(X \times X)$.

Remark 1.6 It should be mentioned that Hirachi [8] derived the leading order asymptotics for the Szegő kernel. The key feature of Hirachi’s work is the identification of the coefficient for the logarithm term of the Szegő kernel. From Hirachi’s result, we can determine the logarithm terms in Corollary 1.5 and it is possible to derive the full asymptotics for $\Pi(x, y)$ by Hirachi’s method.

Put

$$\begin{aligned} \hat{\mathcal{P}}^\perp &:= \left\{ u \in L^2(X); (u | f) = 0, \forall f \in \hat{\mathcal{P}} \right\}, \\ \hat{\mathcal{P}}_0^\perp &:= \left\{ v \in L^2(X); (v | g) = 0, \forall g \in \hat{\mathcal{P}}_0 \right\}. \end{aligned}$$

From (1.6) and some standard argument in functional analysis (see Sect. 4), we deduce

Corollary 1.7 *With the notations and assumptions above, we have*

$$\hat{\mathcal{P}}^\perp \cap \text{Ker } P \subset C^\infty(X), \quad \hat{\mathcal{P}}_0^\perp \cap \text{Ker } P \subset C^\infty(X), \quad \hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}} \subset C^\infty(X)$$

and $\hat{\mathcal{P}}^\perp \cap \text{Ker } P, \hat{\mathcal{P}}_0^\perp \cap \text{Ker } P, \hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}}$ are all finite dimensional.

We have the orthogonal decompositions

$$\begin{aligned} \text{Ker } P &= \hat{\mathcal{P}}^\perp \oplus \left(\hat{\mathcal{P}}^\perp \cap \text{Ker } P \right), \\ \text{Ker } P &= \hat{\mathcal{P}}_0^\perp \oplus \left(\hat{\mathcal{P}}_0^\perp \cap \text{Ker } P \right), \\ \hat{\mathcal{P}} &= \hat{\mathcal{P}}_0 \oplus \left(\hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}} \right). \end{aligned} \tag{1.13}$$

From Corollary 1.7, we know that $\hat{\mathcal{P}}^\perp \cap \text{Ker } P, \hat{\mathcal{P}}_0^\perp \cap \text{Ker } P, \hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}}$ are all finite dimensional subsets of $C^\infty(X)$.

Since P is self-adjoint, $\text{Spec } P \subset \mathbb{R}$. In Sect. 5, we establish spectral theory for P .

Theorem 1.8 *With the notations and assumptions above, $\text{Spec } P$ is a discrete subset in \mathbb{R} and for every $\lambda \in \text{Spec } P, \lambda \neq 0, \lambda$ is an eigenvalue of P and the eigenspace*

$$H_\lambda(P) := \{u \in \text{Dom } P; Pu = \lambda u\}$$

is a finite dimensional subspace of $C^\infty(X)$.

1.1 The phase φ

In this section, we collect some properties of the phase function φ . We refer the reader to [10] for the proofs.

The following result describes the phase function φ in local coordinates.

Theorem 1.9 *With the assumptions and notations used in Theorem 1.2, for a given point $x_0 \in X$, let $\{Z_1\}$ be an orthonormal frame of $T^{1,0}X$ in a neighbourhood of x_0 , i.e. $\mathcal{L}_{x_0}(Z_1, \bar{Z}_1) = 1$. Take local coordinates $x = (x_1, x_2, x_3)$, $z = x_1 + ix_2$, defined on some neighbourhood of x_0 such that $\theta(x_0) = dx_3$, $x(x_0) = 0$, and for some $c \in \mathbb{C}$,*

$$Z_1 = \frac{\partial}{\partial z} - i\bar{z} \frac{\partial}{\partial x_3} - cx_3 \frac{\partial}{\partial x_3} + O(|x|^2),$$

where $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2})$. Set $y = (y_1, y_2, y_3)$, $w = y_1 + iy_2$. Then, for φ in Theorem 1.2, we have

$$\text{Im } \varphi(x, y) \geq c \sum_{j=1}^2 |x_j - y_j|^2, \quad c > 0, \tag{1.14}$$

in some neighbourhood of $(0, 0)$ and

$$\begin{aligned} \varphi(x, y) &= -x_3 + y_3 + i |z - w|^2 \\ &+ (i(\bar{z}w - z\bar{w}) + c(-zx_3 + wy_3) \\ &+ \bar{c}(-\bar{z}x_3 + \bar{w}y_3)) + (x_3 - y_3)f(x, y) + O(|(x, y)|^3), \end{aligned} \tag{1.15}$$

where f is smooth and satisfies $f(0, 0) = 0$, $f(x, y) = \bar{f}(y, x)$.

Definition 1.10 With the assumptions and notations used in Theorem 1.2, let $\varphi_1(x, y), \varphi_2(x, y) \in C^\infty(X \times X)$. We assume that $\varphi_1(x, y)$ and $\varphi_2(x, y)$ satisfy (1.8) and (1.14). We say that $\varphi_1(x, y)$ and $\varphi_2(x, y)$ are equivalent on X if for any $b_1(x, y, t) \in S_{\text{cl}}^1(X \times X \times]0, \infty[)$ we can find $b_2(x, y, t) \in S_{\text{cl}}^1(X \times X \times]0, \infty[)$ such that

$$\int_0^\infty e^{i\varphi_1(x,y)t} b_1(x, y, t) dt \equiv e^{i\varphi_2(x,y)t} b_2(x, y, t) dt \quad \text{on } X$$

and vice versa.

We characterize the phase φ

Theorem 1.11 *With the assumptions and notations used in Theorem 1.2, let $\varphi_1(x, y) \in C^\infty(X \times X)$. We assume that $\varphi_1(x, y)$ satisfies (1.8) and (1.14). $\varphi_1(x, y)$ and $\varphi(x, y)$ are equivalent on X in the sense of Definition 1.10 if and only if there is a function $h \in C^\infty(X \times X)$ such that $\varphi_1(x, y) - h(x, y)\varphi(x, y)$ vanishes to infinite order at $x = y$, for every $(x, x) \in X \times X$.*

2 Preliminaries

We shall use the following notations: \mathbb{R} is the set of real numbers, $\overline{\mathbb{R}}_+ := \{x \in \mathbb{R}; x \geq 0\}$, $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An element $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of \mathbb{N}_0^3 will be called a multiindex, the size of α is: $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. For $m \in \mathbb{N}$, we write $\alpha \in \{1, \dots, m\}^3$ if $\alpha_j \in \{1, \dots, m\}$, $j = 1, 2, 3$. We write $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$, $x = (x_1, x_2, x_3)$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$, $\partial_{x_j} = \frac{\partial}{\partial x_j}$, $\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha}$.

In this paper, we let TX and T^*X denote the tangent bundle of X and the cotangent bundle of X respectively. The complexified tangent bundle of X and the complexified cotangent bundle of X will be denoted by $\mathbb{C}TX$ and $\mathbb{C}T^*X$ respectively. We write $\langle \cdot, \cdot \rangle$ to denote the pointwise duality between TX and T^*X . We extend $\langle \cdot, \cdot \rangle$ bilinearly to $\mathbb{C}TX \times \mathbb{C}T^*X$. Let E be a C^∞ vector bundle over X . The fiber of E at $x \in X$ will be denoted by E_x . Let $Y \subset X$ be an open set. From now on, the spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $C^\infty(Y, E)$ and $\mathcal{D}'(Y, E)$ respectively. Let $\mathcal{E}'(Y, E)$ be the subspace of $\mathcal{D}'(Y, E)$ whose elements have compact support in Y . For $m \in \mathbb{R}$, we let $H^m(Y, E)$ denote the Sobolev space of order m of sections of E over Y . Put

$$H_{\text{loc}}^m(Y, E) = \{u \in \mathcal{D}'(Y, E); \varphi u \in H^m(Y, E), \forall \varphi \in C_0^\infty(Y)\},$$

$$H_{\text{comp}}^m(Y, E) = H_{\text{loc}}^m(Y, E) \cap \mathcal{E}'(Y, E).$$

Let $\Omega \subset X$ be an open set. If $A : C_0^\infty(\Omega) \rightarrow \mathcal{D}'(\Omega)$ is continuous, we write $K_A(x, y)$ or $A(x, y)$ to denote the distribution kernel of A . The following two statements are equivalent

- (a) A is continuous: $\mathcal{E}'(\Omega) \rightarrow C^\infty(\Omega)$,
- (b) $K_A \in C^\infty(\Omega \times \Omega)$.

If A satisfies (a) or (b), we say that A is smoothing. Let $B : C_0^\infty(\Omega) \rightarrow \mathcal{D}'(\Omega)$ be a continuous operator. We write $A \equiv B$ (on Ω) if $A - B$ is a smoothing operator. We say that A is properly supported if $\text{Supp } K_A \subset \Omega \times \Omega$ is proper. That is, the two projections: $t_x : (x, y) \in \text{Supp } K_A \rightarrow x \in \Omega$, $t_y : (x, y) \in \text{Supp } K_A \rightarrow y \in \Omega$ are proper (i.e. the inverse images of t_x and t_y of all compact subsets of Ω are compact). Let $H(x, y) \in \mathcal{D}'(\Omega \times \Omega)$. We write H to denote the unique continuous operator $C_0^\infty(\Omega) \rightarrow \mathcal{D}'(\Omega)$ with distribution kernel $H(x, y)$. In this work, we identify H with $H(x, y)$.

We recall Hörmander symbol spaces

Definition 2.1 Let $M \subset \mathbb{R}^N$ be an open set, $0 \leq \rho \leq 1, 0 \leq \delta \leq 1, m \in \mathbb{R}, N_1 \in \mathbb{N}$. $S_{\rho, \delta}^m(M \times \mathbb{R}^{N_1})$ is the space of all $a \in C^\infty(M \times \mathbb{R}^{N_1})$ such that for all compact $K \Subset M$

and all $\alpha \in \mathbb{N}_0^N, \beta \in \mathbb{N}_0^{N_1}$, there is a constant $C > 0$ such that

$$\left| \partial_x^\alpha \partial_\theta^\beta a(z, \theta) \right| \leq C(1 + |\theta|)^{m - \rho|\beta| + \delta|\alpha|}, \quad (x, \theta) \in K \times \mathbb{R}^{N_1}.$$

We say that $S_{\rho, \delta}^m$ is the space of symbols of order m type (ρ, δ) . Put

$$S^{-\infty}(M \times \mathbb{R}^{N_1}) := \bigcap_{m \in \mathbb{R}} S_{\rho, \delta}^m(M \times \mathbb{R}^{N_1}).$$

Let $a_j \in S_{\rho, \delta}^{m_j}(M \times \mathbb{R}^{N_1}), j = 0, 1, 2, \dots$ with $m_j \rightarrow -\infty, j \rightarrow \infty$. Then there exists $a \in S_{\rho, \delta}^{m_0}(M \times \mathbb{R}^{N_1})$ unique modulo $S^{-\infty}(M \times \mathbb{R}^{N_1})$, such that $a - \sum_{j=0}^{k-1} a_j \in S_{\rho, \delta}^{m_k}(M \times \mathbb{R}^{N_1})$ for $k = 0, 1, 2, \dots$

If a and a_j have the properties above, we write $a \sim \sum_{j=0}^\infty a_j$ in $S_{\rho, \delta}^{m_0}(M \times \mathbb{R}^{N_1})$.

Let $S_{cl}^m(M \times \mathbb{R}^{N_1})$ be the space of all symbols $a(x, \theta) \in S_{1,0}^m(M \times \mathbb{R}^{N_1})$ with

$$a(x, \theta) \sim \sum_{j=0}^\infty a_{m-j}(x, \theta) \text{ in } S_{1,0}^m(M \times \mathbb{R}^{N_1}),$$

with $a_k(x, \theta) \in C^\infty(M \times \mathbb{R}^{N_1})$ positively homogeneous of degree k in θ , that is, $a_k(x, \lambda\theta) = \lambda^k a_k(x, \theta), \lambda \geq 1, |\theta| \geq 1$.

By using partition of unity, we extend the definitions above to the cases when M is a smooth paracompact manifold and when we replace $M \times \mathbb{R}^{N_1}$ by T^*M .

Let $\Omega \subset X$ be an open set. Let $a(x, \xi) \in S_{\frac{1}{2}, \frac{1}{2}}^k(T^*\Omega)$. We can define

$$A(x, y) = \frac{1}{(2\pi)^3} \int e^{i\langle x-y, \xi \rangle} a(x, \xi) d\xi$$

as an oscillatory integral and we can show that

$$A : C_0^\infty(\Omega) \rightarrow C^\infty(\Omega)$$

is continuous and has unique continuous extension:

$$A : \mathcal{E}'(\Omega) \rightarrow \mathcal{D}'(\Omega).$$

Definition 2.2 Let $k \in \mathbb{R}$. A pseudodifferential operator of order k type $(\frac{1}{2}, \frac{1}{2})$ is a continuous linear map $A : C_0^\infty(\Omega) \rightarrow \mathcal{D}'(\Omega)$ such that the distribution kernel of A is

$$A(x, y) = \frac{1}{(2\pi)^3} \int e^{i\langle x-y, \xi \rangle} a(x, \xi) d\xi$$

with $a \in S_{\frac{1}{2}, \frac{1}{2}}^k(T^*\Omega)$. We call $a(x, \xi)$ the symbol of A . We shall write $L_{\frac{1}{2}, \frac{1}{2}}^k(\Omega)$ to denote the space of pseudodifferential operators of order k type $(\frac{1}{2}, \frac{1}{2})$.

We recall the following classical result of Calderon–Vaillancourt (see chapter XVIII of Hörmander [7]).

Proposition 2.3 *If $A \in L^k_{\frac{1}{2}, \frac{1}{2}}(\Omega)$. Then,*

$$A : H^s_{\text{comp}}(\Omega) \rightarrow H^{s-k}_{\text{loc}}(\Omega)$$

is continuous, for all $s \in \mathbb{R}$. Moreover, if A is properly supported, then

$$A : H^s_{\text{loc}}(\Omega) \rightarrow H^{s-k}_{\text{loc}}(\Omega)$$

is continuous, for all $s \in \mathbb{R}$.

3 Microlocal analysis for \square_b

We will reduce the analysis of the Paneitz operator to the analysis of Kohn Laplacian. We extend $\bar{\partial}_b$ to L^2 space by $\bar{\partial}_b : \text{Dom } \bar{\partial}_b \subset L^2(X) \rightarrow L^2_{(0,1)}(X)$, where $\text{Dom } \bar{\partial}_b := \{u \in L^2(X); \bar{\partial}_b u \in L^2_{(0,1)}(X)\}$. Let

$$\bar{\partial}_b^* : \text{Dom } \bar{\partial}_b^* \subset L^2_{(0,1)}(X) \rightarrow L^2(X)$$

be the L^2 adjoint of $\bar{\partial}_b$. The Gaffney extension of Kohn Laplacian is given by

$$\begin{aligned} \square_b &= \bar{\partial}_b^* \bar{\partial}_b : \text{Dom } \square_b \subset L^2(X) \rightarrow L^2(X), \\ \text{Dom } \square_b &:= \left\{ u \in L^2(X); u \in \text{Dom } \bar{\partial}_b, \bar{\partial}_b u \in \text{Dom } \bar{\partial}_b^* \right\}. \end{aligned} \tag{3.1}$$

It is well-known that \square_b is a positive self-adjoint operator. Moreover, the characteristic manifold of \square_b is given by

$$\Sigma = \{(x, \xi) \in T^*X; \xi = \lambda\theta(x), \lambda \neq 0\}. \tag{3.2}$$

Since X is embeddable, \square_b has L^2 closed range. Let $G : L^2(X) \rightarrow \text{Dom } \square_b$ be the partial inverse and let $S : L^2(X) \rightarrow \text{Ker } \square_b$ be the orthogonal projection (Szegő projection). Then,

$$\begin{aligned} \square_b G + S &= I \text{ on } L^2(X), \\ G \square_b + S &= I \text{ on } \text{Dom } \square_b. \end{aligned} \tag{3.3}$$

In [10], we proved that $G \in L^{-1}_{\frac{1}{2}, \frac{1}{2}}(X)$, $S \in L^0_{\frac{1}{2}, \frac{1}{2}}(X)$ and we got explicit formulas of the kernels $G(x, y)$ and $S(x, y)$.

We introduce some notations. Let $\Omega \subset X$ be an open set with real local coordinates $x = (x_1, x_2, x_3)$. Let $f, g \in C^\infty(\Omega)$. We write $f \asymp g$ if for every compact set $K \subset \Omega$ there is a constant $c_K > 0$ such that $f \leq c_K g$ and $g \leq c_K f$ on K . We need

Definition 3.1 $a(t, x, \eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*\Omega)$ is quasi-homogeneous of degree j if $a(t, x, \lambda\eta) = \lambda^j a(\lambda t, x, \eta)$ for all $\lambda > 0$.

We introduce some symbol classes

Definition 3.2 Let $\mu > 0$. We say that $a(t, x, \eta) \in \widetilde{S}_\mu^m(\overline{\mathbb{R}}_+ \times T^*\Omega)$ if $a(t, x, \eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*\Omega)$ and there is a $a(x, \eta) \in S_{1,0}^m(T^*\Omega)$ such that for all indices $\alpha, \beta \in \mathbb{N}_0^3, \gamma \in \mathbb{N}_0$, every compact set $K \Subset \Omega$, there exists a constant $c_{\alpha,\beta,\gamma} > 0$ independent of t such that for all $t \in \overline{\mathbb{R}}_+$,

$$\left| \partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta (a(t, x, \eta) - a(x, \eta)) \right| \leq c_{\alpha,\beta,\gamma} e^{-t\mu|\eta|} (1 + |\eta|)^{m+\gamma-|\beta|}, \quad x \in K, |\eta| \geq 1.$$

The following is well-known (see [10])

Theorem 3.3 With the assumptions and notations above, $G \in L^{\frac{-1}{2}, \frac{1}{2}}(X), S \in L^{\frac{0}{\frac{1}{2}, \frac{1}{2}}}(X), S(x, y) \equiv \int e^{i\varphi(x,y)t} a(x, y, t) dt$, where $\varphi(x, y) \in C^\infty(X \times X)$ is as in (1.8) and

$$\begin{aligned} a(x, y, t) &\in S_{cl}^1(X \times X \times]0, \infty[), \\ a(x, y, t) &\sim \sum_{j=0}^\infty a_j(x, y) t^{1-j} \text{ in } S_{1,0}^1(X \times X \times]0, \infty[), \\ a_j(x, y) &\in C^\infty(X \times X), \quad j = 0, 1, \dots, \\ a_0(x, x) &= \frac{1}{2} \pi^{-n}, \quad \forall x \in X, \end{aligned}$$

and on every open local coordinate patch $\Omega \subset X$ with real local coordinates $x = (x_1, x_2, x_3)$, we have

$$G(x, y) \equiv \int \int_0^\infty e^{i(\psi(t,x,\eta) - \langle y, \eta \rangle)} - t \left(i\psi'_t(t, x, \eta) a(t, x, \eta) + \frac{\partial a}{\partial t}(t, x, \eta) \right) dt d\eta, \tag{3.4}$$

where $a(t, x, \eta) \in \widetilde{S}_\mu^0(\overline{\mathbb{R}}_+ \times T^*\Omega), \psi(t, x, \eta) \in \widetilde{S}_\mu^1(\overline{\mathbb{R}}_+ \times T^*\Omega)$ for some $\mu > 0, \psi(t, x, \eta)$ is quasi-homogeneous of degree 1, $\psi'_t(t, x, \eta) = \frac{\partial \psi}{\partial t}(t, x, \eta), \psi(0, x, \eta) = \langle x, \eta \rangle, \text{Im } \psi \geq 0$ with equality precisely on $(\{0\} \times T^*\Omega \setminus \{0\}) \cup (\mathbb{R}_+ \times \Sigma)$,

$$\psi(t, x, \eta) = \langle x, \eta \rangle \text{ on } \Sigma, \quad d_{x,\eta}(\psi - \langle x, \eta \rangle) = 0 \text{ on } \Sigma,$$

and

$$\text{Im } \psi(t, x, \eta) \asymp \left(|\eta| \frac{t|\eta|}{1+t|\eta|} \right) \left(\text{dist} \left(\left(x, \frac{\eta}{|\eta|} \right), \Sigma \right) \right)^2, \quad t \geq 0, |\eta| \geq 1. \tag{3.5}$$

(See Theorem 3.4 below for the meaning of the integral (3.4).)

Proof We only sketch the proof. For all the details, we refer the reader to Part I in [10]. We use the heat equation method. We work with some real local coordinates $x = (x_1, x_2, x_3)$ defined on Ω . We consider the problem

$$\begin{cases} (\partial_t + \square_b)u(t, x) = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u(0, x) = v(x). \end{cases} \tag{3.6}$$

We look for an approximate solution of (3.6) of the form $u(t, x) = A(t)v(x)$,

$$A(t)v(x) = \frac{1}{(2\pi)^3} \int e^{i(\psi(t,x,\eta) - \langle y, \eta \rangle)} \alpha(t, x, \eta) v(y) dy d\eta \tag{3.7}$$

where formally

$$\alpha(t, x, \eta) \sim \sum_{j=0}^{\infty} \alpha_j(t, x, \eta),$$

with $\alpha_j(t, x, \eta)$ quasi-homogeneous of degree $-j$.

The full symbol of \square_b equals $\sum_{j=0}^2 p_j(x, \xi)$, where $p_j(x, \xi)$ is positively homogeneous of order $2 - j$ in the sense that

$$p_j(x, \lambda \eta) = \lambda^{2-j} p_j(x, \eta), \quad |\eta| \geq 1, \lambda \geq 1.$$

We apply $\partial_t + \square_b$ formally inside the integral in (3.7) and then introduce the asymptotic expansion of $\square_b(\alpha e^{i\psi})$. Set $(\partial_t + \square_b)(\alpha e^{i\psi}) \sim 0$ and regroup the terms according to the degree of quasi-homogeneity. The phase $\psi(t, x, \eta)$ should solve

$$\begin{cases} \frac{\partial \psi}{\partial t} - i p_0(x, \psi'_x) = O(|\text{Im } \psi|^N), \quad \forall N \geq 0, \\ \psi|_{t=0} = \langle x, \eta \rangle, \end{cases} \tag{3.8}$$

where $\psi'_x = (\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_3})$. Note that $p_0(x, \xi)$ is a polynomial with respect to ξ . This equation can be solved with $\text{Im } \psi(t, x, \eta) \geq 0$ and the phase $\psi(t, x, \eta)$ is quasi-homogeneous of degree 1. Moreover,

$$\begin{aligned} \psi(t, x, \eta) &= \langle x, \eta \rangle \text{ on } \Sigma, \quad d_{x,\eta}(\psi - \langle x, \eta \rangle) = 0 \text{ on } \Sigma, \\ \text{Im } \psi(t, x, \eta) &\asymp \left(|\eta| \frac{t |\eta|}{1 + t |\eta|} \right) \left(\text{dist} \left(\left(x, \frac{\eta}{|\eta|} \right), \Sigma \right) \right)^2, \quad |\eta| \geq 1. \end{aligned}$$

Furthermore, there exists $\psi(\infty, x, \eta) \in C^\infty(\Omega \times \mathbb{R}^3)$ with a uniquely determined Taylor expansion at each point of Σ such that for every compact set $K \subset \Omega \times \mathbb{R}^3$ there is a constant $c_K > 0$ such that

$$\text{Im } \psi(\infty, x, \eta) \geq c_K |\eta| \left(\text{dist} \left(\left(x, \frac{\eta}{|\eta|} \right), \Sigma \right) \right)^2, \quad |\eta| \geq 1.$$

If $\lambda \in C(T^*\Omega \setminus 0)$, $\lambda > 0$ is positively homogeneous of degree 1 and $\lambda|_\Sigma < \min \lambda_j$, $\lambda_j > 0$, where $\pm i\lambda_j$ are the non-vanishing eigenvalues of the fundamental matrix of \square_b , then the solution $\psi(t, x, \eta)$ of (3.8) can be chosen so that for every compact set $K \subset \Omega \times \mathbb{R}^3$ and all indices α, β, γ , there is a constant $c_{\alpha,\beta,\gamma,K}$ such that

$$\left| \partial_x^\alpha \partial_\eta^\beta \partial_t^\gamma (\psi(t, x, \eta) - \psi(\infty, x, \eta)) \right| \leq c_{\alpha,\beta,\gamma,K} e^{-\lambda(x,\eta)t} \quad \text{on } \overline{\mathbb{R}}_+ \times K.$$

We obtain the transport equations

$$\begin{cases} T(t, x, \eta, \partial_t, \partial_x)\alpha_0 = O(|\text{Im } \psi|^N), \quad \forall N, \\ T(t, x, \eta, \partial_t, \partial_x)\alpha_j + l_j(t, x, \eta, \alpha_0, \dots, \alpha_{j-1}) = O(|\text{Im } \psi|^N), \quad \forall N, \quad j \in \mathbb{N}. \end{cases} \tag{3.9}$$

It was proved in [10] that (3.9) can be solved. Moreover, there exist positively homogeneous functions of degree $-j$

$$\alpha_j(\infty, x, \eta) \in C^\infty(T^*\Omega), \quad j = 0, 1, 2, \dots,$$

such that $\alpha_j(t, x, \eta)$ converges exponentially fast to $\alpha_j(\infty, x, \eta)$, $t \rightarrow \infty$, for all $j \in \mathbb{N}_0$. Set

$$\tilde{G} = \frac{1}{(2\pi)^3} \int \int_0^\infty e^{i(\psi(t,x,\eta) - \langle y, \eta \rangle)} - t \left(i\psi'_t(t, x, \eta)\alpha(t, x, \eta) + \frac{\partial \alpha}{\partial t}(t, x, \eta) \right) dt d\eta$$

and

$$\tilde{S} = \frac{1}{(2\pi)^3} \int e^{i(\psi(\infty,x,\eta) - \langle y, \eta \rangle)} \alpha(\infty, x, \eta) d\eta,$$

where $\psi'_t(t, x, \eta) := \frac{\partial \psi}{\partial t}(t, x, \eta)$. We can show that \tilde{G} is a pseudodifferential operator of order -1 type $(\frac{1}{2}, \frac{1}{2})$, \tilde{S} is a pseudodifferential operator of order 0 type $(\frac{1}{2}, \frac{1}{2})$ satisfying

$$\tilde{S} + \square_b \tilde{G} \equiv I, \quad \square_b \tilde{S} \equiv 0.$$

Moreover, from global theory of complex Fourier integral operators, we can show that $\tilde{S} \equiv \int e^{i\varphi(x,y)t} a(x, y, t) dt$. Furthermore, by using some standard argument in functional analysis, we can show that $\tilde{G} \equiv G$, $\tilde{S} \equiv S$.

Until further notice, we work in an open local coordinate patch $\Omega \subset X$ with real local coordinates $x = (x_1, x_2, x_3)$. The following is well-known (see Chapter 5 in [10])

Theorem 3.4 *With the notations and assumptions used in Theorem 3.3, let $\chi \in C_0^\infty(\mathbb{R}^3)$ be equal to 1 near the origin. Put*

$$G_\varepsilon(x, y) = \int \int_0^\infty e^{i(\psi(t,x,\eta) - \langle y, \eta \rangle)} - t \left(i\psi'_t(t, x, \eta)a(t, x, \eta) + \frac{\partial a}{\partial t}(t, x, \eta) \right) \chi(\varepsilon\eta) dt d\eta,$$

where $\psi(t, x, \eta), a(t, x, \eta)$ are as in (3.4). For $u \in C^\infty_0(\Omega)$, we can show that

$$Gu := \lim_{\varepsilon \rightarrow 0} \int G_\varepsilon(x, y)u(y)dy \in C^\infty(\Omega)$$

and

$$G : C^\infty_0(\Omega) \rightarrow C^\infty(\Omega),$$

$$u \rightarrow \lim_{\varepsilon \rightarrow 0} \int G_\varepsilon(x, y)u(y)dy$$

is continuous.

Moreover, $G \in L^{-1}_{\frac{1}{2}, \frac{1}{2}}(\Omega)$ with symbol

$$\int_0^\infty e^{i(\psi(t,x,\eta) - \langle x, \eta \rangle)} - t \left(i\psi'_t(t, x, \eta)a(t, x, \eta) + \frac{\partial a}{\partial t}(t, x, \eta) \right) dt \in S^{-1}_{\frac{1}{2}, \frac{1}{2}}(T^*\Omega).$$

We need the following (see Lemma 5.13 in [10] for a proof)

Lemma 3.5 *With the notations and assumptions used in Theorem 3.3, for every compact set $K \subset \Omega$ and all $\alpha \in \mathbb{N}^3_0, \beta \in \mathbb{N}^3_0$, there exists a constant $c_{\alpha, \beta, K} > 0$ such that*

$$\left| \partial_x^\alpha \partial_\eta^\beta (e^{i(\psi(t,x,\eta) - \langle x, \eta \rangle)} t\psi'_t(t, x, \eta)) \right| \leq c_{\alpha, \beta, K} (1 + |\eta|)^{\frac{|\alpha| - |\beta|}{2}} e^{-t\mu|\eta|} e^{-\text{Im} \psi(t, x, \eta)} (1 + \text{Im} \psi(t, x, \eta))^{1 + \frac{|\alpha| + |\beta|}{2}},$$

(3.10)

where $x \in K, t \in \overline{\mathbb{R}}_+, |\eta| \geq 1$ and $\mu > 0$ is a constant independent of α, β and K .

In this work, we need

Theorem 3.6 *Let $L \in C^\infty(X, T^{1,0}X \oplus T^{0,1}X)$. Then, $L \circ G \in L^{-\frac{1}{2}}_{\frac{1}{2}, \frac{1}{2}}(X)$.*

Proof We work on an open local coordinate patch $\Omega \subset X$ with real local coordinates $x = (x_1, x_2, x_3)$. Let $l(x, \eta) \in C^\infty(T^*\Omega)$ be the symbol of L . Then, $l(x, \lambda\eta) = \lambda l(x, \eta), \lambda > 0$. It is well-known (see Chapter 5 in [10]) that

$$(LG)(x, y) \equiv \int e^{i\langle x - y, \eta \rangle} \alpha(x, \eta) d\eta,$$

where

$$\begin{aligned} \alpha(x, \eta) &= \alpha_0(x, \eta) + \alpha_1(x, \eta) \in S_{\frac{1}{2}, \frac{1}{2}}^0(T^*\Omega), \\ \alpha_0(x, \eta) &= \int e^{i(\psi(t,x,\eta) - \langle x, \eta \rangle)} (-1) l(x, \psi'_x(t, x, \eta)) t \psi'_t(t, x, \eta) a(t, x, \eta) dt, \\ \alpha_1(x, \eta) &\in S_{\frac{1}{2}, \frac{1}{2}}^{-1}(T^*\Omega). \end{aligned}$$

Here $a(t, x, \eta) \in \widetilde{S}_\mu^0(\overline{\mathbb{R}}_+ \times T^*\Omega)$, $\mu > 0$. We only need to prove that $\alpha_0(x, \eta) \in S_{\frac{1}{2}, \frac{1}{2}}^{-\frac{1}{2}}(T^*\Omega)$. Fix $\alpha, \beta \in \mathbb{N}_0^3$. From (3.10), (3.5) and notice that $l(x, \psi'_x(t, x, \eta)) = 0$ at Σ , we can check that

$$\begin{aligned} & \left| \partial_x^\alpha \partial_\eta^\beta \alpha_0(x, \eta) \right| \\ & \leq \sum_{|\alpha'| + |\alpha''| = |\alpha|, |\beta'| + |\beta''| = |\beta|} \int \left| \partial_x^{\alpha'} \partial_\eta^{\beta'} \left(e^{i(\psi(t,x,\eta) - \langle x, \eta \rangle)} t \psi'_t(t, x, \eta) \right) \right| \\ & \quad \times \left| \partial_x^{\alpha''} \partial_\eta^{\beta''} (l(x, \psi'_x(t, x, \eta)) a(t, x, \eta)) \right| dt \\ & \leq C_{\alpha, \beta} \sum_{|\alpha'| + |\alpha''| = |\alpha|, |\beta'| + |\beta''| = |\beta|} \int (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} e^{-t\mu|\eta|} e^{-\frac{1}{2} \text{Im } \psi(t,x,\eta)} \\ & \quad \times (1 + |\eta|)^{1 - |\beta''|} \left(\text{dist} \left(\left(x, \frac{\eta}{|\eta|} \right), \Sigma \right) \right)^{\max\{0, 1 - |\beta''|\}} dt \\ & \leq \widetilde{C}_{\alpha, \beta} \sum_{|\alpha'| + |\alpha''| = |\alpha|, |\beta'| + |\beta''| = |\beta|} \int (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} e^{-c \frac{t|\eta|^2}{1+t|\eta|}} \left(\text{dist} \left(\left(x, \frac{\eta}{|\eta|} \right), \Sigma \right) \right)^2 \\ & \quad \times e^{-t\mu|\eta|} (1 + |\eta|)^{1 - |\beta''|} \left(\text{dist} \left(\left(x, \frac{\eta}{|\eta|} \right), \Sigma \right) \right)^{\max\{0, 1 - |\beta''|\}} dt, \tag{3.11} \end{aligned}$$

where $c > 0, \mu > 0, C_{\alpha, \beta} > 0$ and $\widetilde{C}_{\alpha, \beta} > 0$ are constants.

When $|\beta''| = 0$, we have

$$\begin{aligned} & \int (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} e^{-c \frac{t|\eta|^2}{1+t|\eta|}} \left(\text{dist} \left(\left(x, \frac{\eta}{|\eta|} \right), \Sigma \right) \right)^2 \\ & \quad \times e^{-t\mu|\eta|} (1 + |\eta|)^{1 - |\beta''|} \left(\text{dist} \left(\left(x, \frac{\eta}{|\eta|} \right), \Sigma \right) \right) dt \\ & \leq \widetilde{c} \int (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} \frac{1}{\sqrt{t}} (1 + |\eta|)^{-|\beta''|} e^{-\frac{1}{2}t\mu|\eta|} dt \\ & \leq \widetilde{c}_1 \int_0^{\frac{1}{1+|\eta|}} (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} \frac{1}{\sqrt{t}} (1 + |\eta|)^{-|\beta''|} e^{-\frac{1}{2}t\mu|\eta|} dt \end{aligned}$$

$$\begin{aligned}
 & + \tilde{c}_2 \int_{\frac{1}{1+|\eta|}}^{\infty} (1 + |\eta|)^{\frac{|\alpha'|-|\beta'|}{2}} \frac{1}{\sqrt{t}} (1 + |\eta|)^{-|\beta''|} e^{-\frac{1}{2}t\mu|\eta|} dt \\
 & \leq \tilde{c}_3 (1 + |\eta|)^{-\frac{1}{2}-|\beta''|+\frac{|\alpha'|-|\beta'|}{2}}, \tag{3.12}
 \end{aligned}$$

where $|\eta| \geq 1, \tilde{c}_1 > 0, \tilde{c}_2 > 0$ and $\tilde{c}_3 > 0$ are constants.

When $|\beta''| \geq 1$, we have

$$\begin{aligned}
 & \int (1 + |\eta|)^{\frac{|\alpha'|-|\beta'|}{2}} e^{-c\frac{t|\eta|^2}{1+t|\eta|}(\text{dist}((x, \frac{\eta}{|\eta|}), \Sigma))^2} \\
 & \quad \times e^{-t\mu|\eta|} (1 + |\eta|)^{1-|\beta''|} dt \\
 & \leq \hat{c} \int (1 + |\eta|)^{\frac{|\alpha'|-|\beta'|}{2}} (1 + |\eta|)^{1-|\beta''|} e^{-t\mu|\eta|} dt \\
 & \leq \hat{c}_1 (1 + |\eta|)^{-|\beta''|+\frac{|\alpha'|-|\beta'|}{2}} \\
 & \leq \hat{c}_2 (1 + |\eta|)^{-\frac{1}{2}+\frac{|\alpha'|-|\beta'|}{2}-|\beta''|}, \tag{3.13}
 \end{aligned}$$

where $|\eta| \geq 1, \hat{c}_1 > 0, \hat{c}_2 > 0$ are constants.

From (3.11), (3.12) and (3.13), we conclude that $\alpha_0(x, \eta) \in S_{\frac{1}{2}, \frac{1}{2}}^{-\frac{1}{2}}(T^*\Omega)$. The theorem follows.

4 Microlocal Hodge decomposition theorems for P and the proof of Theorem 1.2

By using Theorems 3.3 and 3.6, we will establish microlocal Hodge decomposition theorems for P in this section. Let $G \in L_{\frac{1}{2}, \frac{1}{2}}^{-1}(X), S \in L_{\frac{1}{2}, \frac{1}{2}}^0(X)$ be as in Theorem 3.3. From (1.2) and (3.3), we have

$$\begin{aligned}
 P\bar{G}G & = (\square_b \bar{\square}_b + L_1 \circ L_2 + L_3)\bar{G}G \\
 & = \square_b(I - \bar{S})G + L_1 \circ L_2 \bar{G}G + L_3 \bar{G}G \\
 & = I - S - \square_b \bar{S}G + L_1 \circ L_2 \bar{G}G + L_3 \bar{G}G \\
 & = I - S - \bar{S}\square_b G + \bar{S}\square_b G - \square_b \bar{S}G + L_1 \circ L_2 \circ \bar{G}G + L_3 \bar{G}G \\
 & = I - S - \bar{S}(I - S) + [\bar{S}, \square_b]G + L_1 \circ L_2 \bar{G}G + L_3 \bar{G}G \\
 & = I - S - \bar{S} + \bar{S}S + [\bar{S}, \square_b]G + L_1 \circ L_2 \bar{G}G + L_3 \bar{G}G. \tag{4.1}
 \end{aligned}$$

We need

Lemma 4.1 *We have*

$$\begin{aligned}
 & [\bar{S}, \square_b]G + L_1 \circ L_2 \bar{G}G + L_3 \bar{G}G \\
 & : H^s(X) \rightarrow H^{s+\frac{1}{2}}(X) \text{ is continuous, for every } s \in \mathbb{Z}. \tag{4.2}
 \end{aligned}$$

Proof From Theorem 3.6, we see that $L_1 \circ L_2 \bar{G} \in L^{\frac{1}{2}, \frac{1}{2}}(X)$. Thus,

$$L_1 \circ L_2 \bar{G}G + L_3 \bar{G}G : H^s(X) \rightarrow H^{s+\frac{1}{2}}(X) \text{ is continuous, } \forall s \in \mathbb{Z}. \tag{4.3}$$

Since $\bar{\square}_b \bar{S} = \bar{S} \bar{\square}_b = 0$, we have

$$[\bar{S}, \square_b] = [\bar{S}, \square_b - \bar{\square}_b]. \tag{4.4}$$

Since the principal symbol of \square_b is real, $\square_b - \bar{\square}_b$ is a first order partial differential operator. From this observation and note that $\bar{S} \in L^0_{\frac{1}{2}, \frac{1}{2}}(X)$, it is not difficult to see that $[\bar{S}, \square_b - \bar{\square}_b] \in L^{\frac{1}{2}, \frac{1}{2}}(X)$. From this and (4.4), we conclude that

$$[\bar{S}, \square_b]G : H^s(X) \rightarrow H^{s+\frac{1}{2}}(X) \text{ is continuous, } \forall s \in \mathbb{Z}. \tag{4.5}$$

From (4.5) and (4.3), (4.2) follows.

We also need

Lemma 4.2 *We have $\bar{S}S \equiv 0$ on X , $S\bar{S} \equiv 0$ on X .*

Proof We first notice that $\bar{S} \circ S$ is smoothing away $x = y$. We have

$$\begin{aligned} \bar{S} \circ S(x, y) &\equiv \int_{\sigma>0, t>0} e^{-i\bar{\varphi}(x,w)\sigma + i\varphi(w,y)t} \bar{a}(x, w, \sigma) a(w, y, t) d\sigma dv_X(w) dt \\ &\equiv \int_{s>0, t>0} e^{it(-\bar{\varphi}(x,w)s + \varphi(w,y))} t \bar{a}(x, w, st) a(w, y, t) ds dv_X(w) dt, \end{aligned} \tag{4.6}$$

where $dv_X = \theta \wedge d\theta$ is the volume form. Take $\chi \in C^\infty_0(\mathbb{R}, [0, 1])$ with $\chi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, $\chi = 0$ on $] -\infty, -1] \cup [1, \infty[$. From (4.6), we have

$$\begin{aligned} \bar{S} \circ S(x, y) &\equiv I_\varepsilon + II_\varepsilon, \\ I_\varepsilon &= \int_{s>0, t>0} e^{it(-\bar{\varphi}(x,w)s + \varphi(w,y))} \chi\left(\frac{|x-w|^2}{\varepsilon}\right) t \bar{a}(x, w, st) a(w, y, t) ds dv_X(w) dt, \\ II_\varepsilon &= \int_{s>0, t>0} e^{it(-\bar{\varphi}(x,w)s + \varphi(w,y))} \left(1 - \chi\left(\frac{|x-w|^2}{\varepsilon}\right)\right) \\ &\quad \times t \bar{a}(x, w, st) a(w, y, t) ds dv_X(w) dt, \end{aligned} \tag{4.7}$$

where $\varepsilon > 0$ is a small constant. Since $\varphi(x, w) = 0$ if and only if $x = w$, we can integrate by parts with respect to s and conclude that II_ε is smoothing. Since $\bar{S} \circ S$ is smoothing away $x = y$, we may assume that $|x - y| < \varepsilon$. Since $d_w(-\bar{\varphi}(x, w)s + \varphi(w, y))|_{x=y=w} = -\omega_0(x)(s + 1) \neq 0$, if $\varepsilon > 0$ is small, we can integrate by parts

with respect to w and conclude that I_ε is smoothing. We get $\bar{S} \circ S \equiv 0$ on X . Similarly, we can repeat the procedure above and conclude that $S \circ \bar{S} \equiv 0$ on X . The lemma follows.

Put

$$R_0 = \bar{S}S + [\bar{S}, \square_b]G + L_1 \circ L_2 \bar{G}G + L_3 \bar{G}G. \tag{4.8}$$

From Lemmas 4.1 and 4.2, we see that

$$R_0 : H^s(X) \rightarrow H^{s+\frac{1}{2}}(X) \text{ is continuous, } \forall s \in \mathbb{Z}. \tag{4.9}$$

We can prove

Theorem 4.3 *With the assumptions and notations above, for every $m \in \mathbb{N}_0$, there are continuous operators*

$$R_m, A_m : C_0^\infty(X) \rightarrow \mathcal{D}'(X)$$

such that

$$\begin{aligned} PA_m + S + \bar{S} &= I + R_m, \\ A_m : H^s(X) &\rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \\ R_m : H^s(X) &\rightarrow H^{s+\frac{m+1}{2}}(X) \text{ is continuous, } \forall s \in \mathbb{Z}. \end{aligned} \tag{4.10}$$

Proof From (4.8) and (4.1), we have

$$P\bar{G}G + S + \bar{S} = I + R_0. \tag{4.11}$$

Since $(S + \bar{S})P = 0$, from (4.11), we have

$$(S + \bar{S})^2 = S + \bar{S} + (S + \bar{S})R_0. \tag{4.12}$$

From Lemma 4.2, we have

$$(S + \bar{S})^2 = S^2 + S\bar{S} + \bar{S}S + \bar{S}^2 \equiv S + \bar{S}. \tag{4.13}$$

From (4.12) and (4.13), we conclude that

$$(S + \bar{S})R_0 \equiv 0 \quad \text{on } X. \tag{4.14}$$

Fix $m \in \mathbb{N}_0$. From (4.11), we have

$$\begin{aligned} P\bar{G}G \left(I - R_0 + R_0^2 + \dots + (-R_0)^m \right) + (S + \bar{S}) \left(I - R_0 + R_0^2 + \dots + (-R_0)^m \right) \\ = (I + R_0) \left(I - R_0 + R_0^2 + \dots + (-R_0)^m \right) = I + R_0(-R_0)^m. \end{aligned} \tag{4.15}$$

From (4.14), we have

$$(S + \bar{S}) \left(I - R_0 + R_0^2 + \cdots + (-R_0)^m \right) = S + \bar{S} - F, \quad F \text{ is smoothing.} \quad (4.16)$$

Put $A_m = \bar{G}G(I - R_0 + R_0^2 + \cdots + (-R_0)^m)$, $R_m = R_0(-R_0)^m + F$. From (4.15), (4.16) and (4.9), we obtain

$$\begin{aligned} PA_m + S + \bar{S} &= I + R_m, \\ A_m : H^s(X) &\rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \\ R_m : H^s(X) &\rightarrow H^{s+\frac{m+1}{2}}(X) \text{ is continuous, } \forall s \in \mathbb{Z}. \end{aligned}$$

The theorem follows.

Lemma 4.4 *Let $u \in \text{Dom } P$. Then, $u = u^0 + (S + \bar{S})u$, for some $u^0 \in H^2(X) \cap \text{Dom } P$.*

Proof Fix $m \geq 3$, $m \in \mathbb{N}$. Let A_m, R_m be as in (4.10) and let A_m^* and R_m^* be the adjoints of A_m and R_m with respect to $(\cdot | \cdot)$ respectively. Then,

$$A_m^*P + S + \bar{S} = I + R_m^*. \quad (4.17)$$

Let $u \in \text{Dom } P$. Then, $Pu = v \in L^2(X)$. From (4.17), it is easy to see that

$$u = A_m^*v - R_m^*u + (S + \bar{S})u.$$

Since $A_m^*v - R_m^*u \in H^2(X)$ and $(S + \bar{S})u \in \text{Ker } P \subset \text{Dom } P$, the lemma follows.

Lemma 4.5 *Let $u \in \text{Dom } P$. Then,*

$$((S + \bar{S})u | Pg) = 0, \quad \forall g \in \text{Dom } P.$$

Proof Let $u, g \in \text{Dom } P$. Take $u_j, g_j \in C^\infty(X)$, $j = 1, 2, \dots$, $u_j \rightarrow u \in L^2(X)$ as $j \rightarrow \infty$ and $g_j \rightarrow g \in L^2(X)$ as $j \rightarrow \infty$. Then, $(S + \bar{S})u_j \rightarrow (S + \bar{S})u$ in $L^2(X)$ as $j \rightarrow \infty$ and $Pg_j \rightarrow Pg$ in $H^{-4}(X)$ as $j \rightarrow \infty$. Thus,

$$((S + \bar{S})u | Pg) = \lim_{j \rightarrow \infty} ((S + \bar{S})u_j | Pg) = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} ((S + \bar{S})u_j | Pg_k). \quad (4.18)$$

For any j, k , $((S + \bar{S})u_j | Pg_k) = (P(S + \bar{S})u_j | g_k) = 0$. From this observation and (4.18), the lemma follows.

Now, we can prove

Theorem 4.6 *The operator $P : \text{Dom } P \subset L^2(X) \rightarrow L^2(X)$ is self-adjoint.*

Proof Let $u, v \in \text{Dom } P$. From Lemma 4.4, we have

$$\begin{aligned} u &= u^0 + (S + \bar{S})u, \quad u^0 \in H^2(X) \cap \text{Dom } P, \\ v &= v^0 + (S + \bar{S})v, \quad v^0 \in H^2(X) \cap \text{Dom } P. \end{aligned}$$

From Lemma 4.5, we see that

$$(u | Pv) = (u^0 | Pv^0), \quad (Pu | v) = (Pu^0 | v^0). \tag{4.19}$$

Let $g_j, f_j \in C^\infty(X)$, $j = 1, 2, \dots$, $g_j \rightarrow u^0 \in H^2(X)$ as $j \rightarrow \infty$ and $f_j \rightarrow v^0 \in H^2(X)$ as $j \rightarrow \infty$. We have

$$\begin{aligned} (g_j | Pf_j) &= (g_j - u^0 | Pf_j) + (u^0 | Pf_j) \\ &= (g_j - u^0 | Pf_j) + (u^0 | Pv^0) + (u^0 | P(f_j - v^0)). \end{aligned} \tag{4.20}$$

Now,

$$\begin{aligned} |(g_j - u^0 | Pf_j)| &\leq C_0 \|g_j - u^0\|_2 \|Pf_j\|_{-2} \\ &\leq C_1 \|g_j - u^0\|_2 \|f_j\|_2 \rightarrow 0 \quad \text{as } j \rightarrow \infty \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} |(u^0 | P(f_j - v^0))| &\leq C_2 \|u^0\|_2 \|P(f_j - v^0)\|_{-2} \\ &\leq C_3 \|u^0\|_2 \|f_j - v^0\|_2 \rightarrow 0 \quad \text{as } j \rightarrow \infty, \end{aligned} \tag{4.22}$$

where $C_0 > 0, C_1 > 0, C_2 > 0, C_3 > 0$ are constants and $\|\cdot\|_s$ denotes the standard Sobolev norm of order s on X . From (4.20), (4.21) and (4.22), we obtain

$$(u^0 | Pv^0) = \lim_{j \rightarrow \infty} (g_j | Pf_j). \tag{4.23}$$

For each j , it is clearly that $(g_j | Pf_j) = (Pg_j | f_j)$. We can repeat the procedure above and conclude that

$$\lim_{j \rightarrow \infty} (Pg_j | f_j) = (Pu^0 | v^0).$$

From this observation, (4.23) and (4.19), we conclude that

$$(Pu | v) = (u | Pv), \quad \forall u, v \in \text{Dom } P. \tag{4.24}$$

Let $P^* : \text{Dom } P^* \subset L^2(X) \rightarrow L^2(X)$ be the Hilbert space adjoint of P . From (4.24), we deduce that $\text{Dom } P \subset \text{Dom } P^*$ and $Pu = P^*u, \forall u \in \text{Dom } P^*$.

Let $v \in \text{Dom } P^*$. By definition, there is a $f \in L^2(X)$ such that

$$(v | Pg) = (f | g), \quad \forall g \in \text{Dom } P.$$

Since $C^\infty(X) \subset \text{Dom } P, Pv = f$ in the sense of distribution. Since $f \in L^2(X), v \in \text{Dom } P$ and $Pv = P^*v = f$. The theorem follows.

We should noticed that Chiu [5] gave a detailed proof (and different proof) of Theorem 4.6.

Theorem 4.7 *The operator $P : \text{Dom } P \subset L^2(X) \rightarrow L^2(X)$ has closed range.*

Proof Fix $m \in \mathbb{N}_0$, let A_m, R_m be as in (4.10) and let A_m^* and R_m^* be the adjoints of A_m and R_m with respect to $(\cdot | \cdot)$ respectively. Then,

$$A_m^*P + S + \bar{S} = I + R_m^*. \tag{4.25}$$

Now, we claim that there is a constant $C > 0$ such that

$$\|Pu\| \geq C \|u\|, \quad \forall u \in \text{Dom } P \cap (\text{Ker } P)^\perp. \tag{4.26}$$

If the claim is not true, then we can find $u_j \in \text{Dom } P \cap (\text{Ker } P)^\perp$ with $\|u_j\| = 1, j = 1, 2, \dots$, such that

$$\|Pu_j\| \leq \frac{1}{j} \|u_j\|, \quad j = 1, 2, \dots \tag{4.27}$$

From (4.25), we have

$$u_j = A_m^*Pu_j + (S + \bar{S})u_j - R_mu_j, \quad j = 1, 2, \dots \tag{4.28}$$

Since $u_j \in (\text{Ker } P)^\perp, j = 1, 2, \dots$, and $P(S + \bar{S}) = 0$, we have

$$(S + \bar{S})u_j = 0, \quad j = 1, 2, \dots \tag{4.29}$$

From (4.28) and (4.29), we get

$$u_j = A_m^*Pu_j - R_mu_j, \quad j = 1, 2, \dots \tag{4.30}$$

From (4.30) and Rellich’s theorem, we can find subsequence $\{u_{j_s}\}_{s=1}^\infty, 1 \leq j_1 < j_2 < \dots, u_{j_s} \rightarrow u$ in $L^2(X)$. From (4.27), we see that $Pu = 0$. Hence, $u \in \text{Ker } P$. Since $u_j \in (\text{Ker } P)^\perp, j = 1, 2, \dots$, we get a contradiction. The claim (4.26) follows. From (4.26), the theorem follows.

In view of Theorems 4.6 and 4.7, we know that P is self-adjoint and P has closed range. Let $N : L^2(X) \rightarrow \text{Dom } P$ be the partial inverse and let $\Pi : L^2(X) \rightarrow \text{Ker } P$ be the orthogonal projection. We can prove

Theorem 4.8 *With the notations and assumptions above, we have*

$$\begin{aligned} \Pi &: H^s(X) \rightarrow H^s(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \\ N &: H^s(X) \rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \\ \Pi &\equiv S + \bar{S}. \end{aligned} \tag{4.31}$$

Proof Fix $m \in \mathbb{N}_0$. Let A_m, R_m be as in Theorem 4.3. Then,

$$PA_m + S + \bar{S} = I + R_m.$$

Thus,

$$\Pi + \Pi R_m = \Pi(PA_m + S + \bar{S}) = \Pi(S + \bar{S}) = S + \bar{S}. \tag{4.32}$$

From (4.10) and (4.32), we have

$$\Pi - (S + \bar{S}) : H^{-\frac{m+1}{2}}(X) \rightarrow L^2(X) \text{ is continuous.} \tag{4.33}$$

By taking adjoint in (4.33), we get

$$\Pi - (S + \bar{S}) : L^2(X) \rightarrow H^{\frac{m+1}{2}}(X) \text{ is continuous.} \tag{4.34}$$

From (4.33) and (4.34), we have

$$(\Pi - (S + \bar{S}))^2 : H^{-\frac{m+1}{2}}(X) \rightarrow H^{\frac{m+1}{2}}(X) \text{ is continuous.} \tag{4.35}$$

Now,

$$\begin{aligned} (\Pi - (S + \bar{S}))^2 &= \Pi - \Pi(S + \bar{S}) - (S + \bar{S})\Pi + (S + \bar{S})^2 \\ &= \Pi - (S + \bar{S}) - (S + \bar{S}) + S + \bar{S} + S\bar{S} + \bar{S}S \\ &\equiv \Pi - (S + \bar{S}) \text{ (here we used Lemma 4.2)}. \end{aligned} \tag{4.36}$$

From (4.35) and (4.36), we conclude that

$$\Pi - (S + \bar{S}) : H^{-\frac{m+1}{2}}(X) \rightarrow H^{\frac{m+1}{2}}(X) \text{ is continuous.}$$

Since m is arbitrary, we get

$$\Pi \equiv S + \bar{S}. \tag{4.37}$$

Now,

$$N(PA_m + S + \bar{S}) = N(I + R_m). \tag{4.38}$$

Note that $NP = I - \Pi$, $N\Pi = 0$. From this observation, we have

$$N(PA_m + S + \bar{S}) = (I - \Pi)A_m + NF, \tag{4.39}$$

where $F \equiv 0$ [here we used (4.37)]. From (4.39) and (4.38), we have

$$N - A_m = -\Pi A_m + NF - NR_m. \tag{4.40}$$

From (4.37) and (4.40), we have

$$\begin{aligned} N - A_m^* &= -A_m^* \Pi + F^* N - R_m^* N \\ &= -A_m^* \Pi + F^* (-\Pi A_m + NF - NR_m + A_m) \\ &\quad - R_m^* (-\Pi A_m + NF - NR_m + A_m) \\ &: H^s(X) \rightarrow H^{s+2}(X) \text{ is continuous, } \forall -\frac{m+1}{2} \leq s \leq \frac{m-3}{2}, \quad s \in \mathbb{Z}, \end{aligned} \tag{4.41}$$

where A_m^* , F^* , R_m^* are adjoints of A_m , F , R_m respectively. Note that

$$A_m^* : H^s(X) \rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}.$$

From this observation, (4.41) and note that m is arbitrary, we conclude that

$$N : H^s(X) \rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}.$$

The theorem follows.

Let τ and τ_0 be as in (1.4). Now, we can prove

Theorem 4.9 *We have $\tau \equiv \Pi$ on X , $\tau_0 \equiv \Pi$ on X .*

Proof Since $\hat{P} \subset \text{Ker } P$, we have $\Pi\tau = \tau$. From this observation and (4.31), we get

$$(S + \bar{S})\tau - \tau = F\tau, \tag{4.42}$$

where F is a smoothing operator. It is clearly that $(S + \bar{S})\tau = \tau(S + \bar{S}) = S + \bar{S}$. From this observation and (4.42), we get $S + \bar{S} - \tau = F\tau$ and hence $S + \bar{S} - \tau = \tau F^*$, where F^* is the adjoint of F . Thus,

$$(S + \bar{S} - \tau)(S + \bar{S} - \tau) = F\tau^2 F^* \equiv 0. \tag{4.43}$$

Now,

$$\begin{aligned} (S + \bar{S} - \tau)^2 &= (S + \bar{S})^2 - (S + \bar{S})\tau - \tau(S + \bar{S}) + \tau^2 \\ &= S + S\bar{S} + \bar{S}S + \bar{S} - S - \bar{S} - S - \bar{S} + \tau \\ &\equiv \tau - (S + \bar{S}) \text{ (here we used Lemma 4.2)}. \end{aligned} \tag{4.44}$$

From (4.44), (4.43) and (4.31), we get $\tau \equiv \Pi$.

Similarly, we can repeat the procedure above and conclude that $\tau_0 \equiv \Pi$. The theorem follows.

From Theorems 4.6, 4.7, 4.8, 4.9 and 3.3, we get Theorem 1.2.

Corollary 4.10 *We have*

$$\hat{\mathcal{P}}^\perp \cap \text{Ker } P \subset C^\infty(X), \quad \hat{\mathcal{P}}_0^\perp \cap \text{Ker } P \subset C^\infty(X), \quad \hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}} \subset C^\infty(X)$$

and $\hat{\mathcal{P}}^\perp \cap \text{Ker } P, \hat{\mathcal{P}}_0^\perp \cap \text{Ker } P, \hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}}$ are all finite dimensional.

Proof If $\hat{\mathcal{P}}^\perp \cap \text{Ker } P$ is infinite dimensional, then we can find

$$f_j \in \hat{\mathcal{P}}^\perp \cap \text{Ker } P, \quad j = 1, 2, \dots,$$

such that $(f_j | f_k) = \delta_{j,k}, j, k = 1, 2, \dots$, where $\delta_{j,k} = 1$ if $j = k, \delta_{j,k} = 0$ if $j \neq k$. Since $f_j \in \text{Ker } P, j = 1, 2, \dots, f_j = \Pi f_j, j = 1, 2, \dots$. From Theorem 4.9, we have

$$f_j = \tau f_j + F f_j, \quad j = 1, 2, 3, \dots, \tag{4.45}$$

where F is a smoothing operator. Since $f_j \in \hat{\mathcal{P}}^\perp, j = 1, 2, \dots, \tau f_j = 0, j = 1, 2, \dots$. From this observation and (4.45), we get

$$f_j = F f_j, \quad j = 1, 2, 3, \dots. \tag{4.46}$$

From (4.46) and Rellich’s theorem, we can find subsequence $\{f_{j_s}\}_{s=1}^\infty, 1 \leq j_1 < j_2 < \dots, f_{j_s} \rightarrow f$ in $L^2(X)$. Since $(f_j | f_k) = \delta_{j,k}, j, k = 1, 2, \dots$, we get a contradiction. Thus, $\hat{\mathcal{P}}^\perp \cap \text{Ker } P$ is finite dimensional. Let $\{f_1, f_2, \dots, f_d\}$ be an orthonormal frame of $\hat{\mathcal{P}}^\perp \cap \text{Ker } P, d < \infty$. As (4.46), we have $f_j = F f_j, j = 1, 2, \dots, d$. Thus, $f_j \in C^\infty(X), j = 1, 2, \dots, d$, and hence $\hat{\mathcal{P}}^\perp \cap \text{Ker } P \subset C^\infty(X)$.

We can repeat the procedure above and conclude that $\hat{\mathcal{P}}_0^\perp \cap \text{Ker } P \subset C^\infty(X), \hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}} \subset C^\infty(X), \hat{\mathcal{P}}_0^\perp \cap \text{Ker } P, \hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}}$ are all finite dimensional.

5 Spectral theory for P

In this section, we will prove Theorem 1.8. For any $\lambda > 0$, put

$$\Pi_{[-\lambda, \lambda]} := E([-\lambda, \lambda]),$$

where E denotes the spectral measure for P (see section 2 in Davies [6], for the precise meaning of spectral measure). We need

Theorem 5.1 Fix $\lambda > 0$. We have $P\Pi_{[-\lambda,\lambda]} \equiv 0$ on X .

Proof As before, let N be the partial inverse of P and let Π be the orthogonal projection onto $\text{Ker } P$. We have

$$NP + \Pi = I. \tag{5.1}$$

From (5.1), we have

$$NP^2\Pi_{[-\lambda,\lambda]} = P\Pi_{[-\lambda,\lambda]}. \tag{5.2}$$

From (1.5), (5.2) and notice that

$$P^2\Pi_{[-\lambda,\lambda]} : L^2(X) \rightarrow L^2(X) \text{ is continuous,}$$

we conclude that

$$P\Pi_{[-\lambda,\lambda]} : L^2(X) \rightarrow H^2(X) \text{ is continuous.} \tag{5.3}$$

Similarly, we can repeat the procedure above and deduce that

$$P^2\Pi_{[-\lambda,\lambda]} : L^2(X) \rightarrow H^2(X) \text{ is continuous.} \tag{5.4}$$

From (5.4), (5.2) and (1.5), we get

$$P\Pi_{[-\lambda,\lambda]} : L^2(X) \rightarrow H^4(X) \text{ is continuous.}$$

Continuing in this way, we conclude that

$$P\Pi_{[-\lambda,\lambda]} : L^2(X) \rightarrow H^m(X) \text{ is continuous, } \forall m \in \mathbb{N}_0. \tag{5.5}$$

Note that $P\Pi_{[-\lambda,\lambda]} = \Pi_{[-\lambda,\lambda]}P = (P\Pi_{[-\lambda,\lambda]})^*$, where $(P\Pi_{[-\lambda,\lambda]})^*$ is the adjoint of $P\Pi_{[-\lambda,\lambda]}$. By taking adjoint in (5.5), we get

$$\Pi_{[-\lambda,\lambda]}P = P\Pi_{[-\lambda,\lambda]} : H^{-m}(X) \rightarrow L^2(X) \text{ is continuous, } \forall m \in \mathbb{N}_0.$$

Hence,

$$(P\Pi_{[-\lambda,\lambda]})^2 = P^2\Pi_{[-\lambda,\lambda]} : H^{-m}(X) \rightarrow H^m(X) \text{ is continuous, } \forall m \in \mathbb{N}_0. \tag{5.6}$$

From (5.6), (5.2) and (1.5), the theorem follows.

We need

Theorem 5.2 For any $\lambda > 0$, $\Pi_{[-\lambda,\lambda]} \equiv \Pi$ on X .

Proof From (5.1) and Theorem 5.1, we get

$$\Pi\Pi_{[-\lambda,\lambda]} \equiv \Pi_{[-\lambda,\lambda]} \text{ on } X. \tag{5.7}$$

On the other hand, it is clearly that $\Pi\Pi_{[-\lambda,\lambda]} = \Pi$. From this observation and (5.7), the theorem follows.

Now, we can prove

Theorem 5.3 *Spec P is a discrete subset in \mathbb{R} and for every $\lambda \in \text{Spec P}$, $\lambda \neq 0$, λ is an eigenvalue of P and the eigenspace*

$$H_\lambda(\text{P}) := \{u \in \text{Dom P}; Pu = \lambda u\}$$

is a finite dimensional subspace of $C^\infty(X)$.

Proof Since P has L^2 closed range, there is a $\mu > 0$ such that $\text{Spec P} \subset]-\infty, -\mu] \cup [\mu, \infty[$. Fix $\lambda > \mu$. Put $\Pi_{[-\lambda, -\mu] \cup [\mu, \lambda]} := E([-\lambda, -\mu] \cup [\mu, \lambda])$. Note that

$$\Pi_{[-\lambda, -\mu] \cup [\mu, \lambda]} = \Pi_{[-\lambda, \lambda]} - \Pi_{[-\frac{\mu}{2}, \frac{\mu}{2}]}.$$

From this observation and Theorem 5.2, we see that

$$\Pi_{[-\lambda, -\mu] \cup [\mu, \lambda]} \equiv 0. \tag{5.8}$$

We claim that $\text{Spec P} \cap \{[-\lambda, -\mu] \cup [\mu, \lambda]\}$ is discrete. If not, we can find $f_j \in \text{Rang } E([-\lambda, -\mu] \cup [\mu, \lambda])$, $j = 1, 2, \dots$, with $(f_j | f_k) = \delta_{j,k}$, $j, k = 1, 2, \dots$. Note that

$$f_j = \Pi_{[-\lambda, -\mu] \cup [\mu, \lambda]} f_j, \quad j = 1, 2, \dots$$

From this observation, (5.8) and Rellich’s theorem, we can find subsequence $\{f_{j_s}\}_{s=1}^\infty$, $1 \leq j_1 < j_2 < \dots$, $f_{j_s} \rightarrow f$ in $L^2(X)$. Since $(f_j | f_k) = \delta_{j,k}$, $j, k = 1, 2, \dots$, we get a contradiction. Thus, $\text{Spec P} \cap \{[-\lambda, -\mu] \cup [\mu, \lambda]\}$ is discrete. Hence Spec P is a discrete subset in \mathbb{R} .

Let $r \in \text{Spec P}$, $r \neq 0$. Since Spec P is discrete, $P - r$ has L^2 closed range. If $P - r$ is injective, then $\text{Range}(P - r) = L^2(X)$ and

$$(P - r)^{-1} : L^2(X) \rightarrow L^2(X)$$

is continuous. We get a contradiction. Hence r is an eigenvalue of Spec P . Put

$$H_r(\text{P}) := \{u \in \text{Dom P}; Pu = ru\}.$$

We can repeat the procedure above and conclude that $\dim H_r(\text{P}) < \infty$. Take $0 < \mu_0 < \lambda_0$ so that $r \in \{[-\lambda_0, -\mu_0] \cup [\mu_0, \lambda_0]\}$. From Theorem 5.2, we see that

$$\Pi_{[-\lambda_0, -\mu_0] \cup [\mu_0, \lambda_0]} \equiv 0.$$

Since

$$H_r(\text{P}) = \{\Pi_{[-\lambda_0, -\mu_0] \cup [\mu_0, \lambda_0]} f; f \in H_r(\text{P})\},$$

$H_r(\text{P}) \subset C^\infty(X)$. The theorem follows.

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