SZEGÖ KERNEL ASYMPTOTICS FOR HIGH POWER OF CR LINE BUNDLES AND KODAIRA EMBEDDING THEOREMS ON CR MANIFOLDS

CHIN-YU HSIAO

Abstract. Let X be an abstract not necessarily compact orientable CR manifold of dimension $2n-1$, $n \geqslant 2$, and let L^k be the k-th tensor power of a CR complex line bundle L over X. Given $q \in \{0, 1, \ldots, n-1\}$, let $\Box^{(q)}_{b,k}$ be the Gaffney extension of Kohn Laplacian for $(0,q)$ forms with values in L^k. For $\lambda \geq 0$, let $\Pi^{(q)}_{k,\leq\lambda} := E((-\infty, \lambda])$, where E denotes the spectral measure of $\Box^{(q)}_{b,k}$. In this work, we prove that $\Pi^{(q)}_{k,\leq k^{-N_0}} F^*_k$, $F_k \Pi^{(q)}_{k,\leq k^{-N_0}} F^*_k$, $N_0 \geq 1$, admit asymptotic expansions with respect to k on the non-degenerate part of the characteristic manifold of $\Box^{(q)}_{b,k}$, where F_k is some kind of microlocal cut-off function. Moreover, we show that $F_k \Pi^{(q)}_{k,\leq 0} F^*_k$ admits a full asymptotic expansion with respect to k if $\Box^{(q)}_{b,k}$ has small spectral gap property with respect to F_k and $\Pi^{(q)}_{k,\leq 0}$ is k-negligible away the diagonal with respect to F_k. By using these asymptotics, we establish almost Kodaira embedding theorems on CR manifolds and Kodaira embedding theorems on CR manifolds with transversal CR S^1 action.

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1. Introduction and statement of the main results

The problem of local and global embedding CR manifolds is prominent in areas such as complex analysis, partial differential equations and differential geometry. Consider X a compact CR manifold of dimension $2n-1$, $n\geq 2$. When X is strongly pseudoconvex and dimension of X is greater than five, a classical theorem of L. Boutet de Monvel [5] asserts that X can be globally CR embedded into $\mathbb{C}^N$, for some $N\in\mathbb{N}$. When the Levi form of X has mixed signature, then the space of global CR functions is finite dimensional (could be even trivial) and moreover many interesting examples live in the projective space (e. g. the quadric $\{[z]\in\mathbb{CP}^{N-1};\,|z_1|^2+\ldots+|z_q|^2-|z_{q+1}|^2-\ldots-|z_N|^2=0\}$). It is thus natural to consider a setting analogue to the Kodaira embedding theorem and ask if X can be embedded into the projective space by means of CR sections of a CR line bundle of positive curvature. For this purpose it is important to study the asymptotic behaviour of the associated Szegö kernel and study if there are a lot of CR sections in high powers of the line bundle. This was initiated in [15], [17](see also Marinescu [20]).

Other developments recently concerned the Bergman kernel for a high power of a holomorphic line bundle. The Bergman kernel is the smooth kernel of the orthogonal projection onto the space of L^2-integrable holomorphic sections. The study of the asymptotic behaviour of the Bergman kernel is an active research subject in complex geometry and is closely related to topics like the structure of algebraic manifolds, the existence of canonical Kähler metrics , Berezin-Toeplitz quantization and equidistribution of zeros of holomorphic sections(see [18]). It is quite interesting to consider CR analogue of the Bergman kernel asymptotic expansion and to study the influence of the asymptotics in CR geometry as in the complex case. This direction could become a research area in CR geometry.

The purpose of this work is to completely study the asymptotic behaviour of the Szegö kernel associated to a hypoelliptic operator $\Box^{(q)}_{b,k}$ with respect to a high power of a CR line bundle. The difficulty of this problem comes from the presence of positive eigenvalues of the curvature of the line bundle and negative eigenvalues of the Levi form of X and hence the semi-classical characteristic manifold of $\Box^{(q)}_{b,k}$ is always degenerate at some point. This difficulty is also closely related to the fact that in the global L^2-estimates for the $\overline{\partial}_b$-operator of Kohn-Hörmander there is a curvature term from the line bundle as well from the Levi form and, in general, it is very difficult to control the sign of the total curvature contribution. In this work, we introduce some kind of microlocal cut-off function F_k and we prove that $\Pi^{(q)}_{k,\leq k^{-N_0}}F^*_k$, $F_k\Pi^{(q)}_{k,\leq k^{-N_0}}F^*_k$, $N_0\geq 1$, admit asymptotic expansions on the non-degenerate part of the characteristic manifold of $\Box^{(q)}_{b,k}$, where for $\lambda\geq 0$, $\Pi^{(q)}_{k,\leq\lambda}:=E((-\infty,\lambda])$, E is the spectral measure of $\Box^{(q)}_{b,k}$. Moreover, we show that $F_k\Pi^{(q)}_{k,\leq 0}F^*_k$ admits a full asymptotic expansion if $\Box^{(q)}_{b,k}$ has small spectral gap property with respect to F_k and $\Pi^{(q)}_{k,\leq 0}$ is k-negligible away the diagonal with respect to F_k. By using these asymptotics, we establish almost Kodaira embedding theorems on CR manifolds and Kodaira embedding theorems on CR manifolds with transversal CR S^1 action. From the analytic view point, this work can be seen as a completely semi-classical study of some kind of hypoelliptic operators.

We now formulate the main results. We refer to section 3 for some standard notations and terminology used here.

Let $(X, T^{1,0}X)$ be a paracompact orientable not necessarily compact CR manifold of dimension $2n-1$, $n \geqslant 2$, with a Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X$ is orthogonal to $T^{0,1}X$ and $\langle u\,|\,v\rangle$ is real if u, v are real tangent vectors. For every $q = 0, 1, \ldots, n-1$, the Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$ induces a Hermitian metric $\langle\cdot|\cdot\rangle$ on $T^{*0,q}X$ the bundle of $(0,q)$ forms of X. Let (L, h^L) be a CR Hermitian line bundle over X, where the Hermitian fiber metric on L is denoted by h^L. We will denote by ϕ the local weights of the Hermitian metric(see (3.5)). For $k > 0$, let (L^k, h^{L^k}) be the k-th tensor power of the line bundle (L, h^L). We denote by $dv_X = dv_X(x)$ the volume form on X induced by the fixed Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$. Then we get natural global L^2 inner products $(\cdot|\cdot)_{h^{L^k}}$, $(\cdot|\cdot)$ on $\Omega^{0,q}_0(X, L^k)$ and $\Omega^{0,q}_0(X)$ respectively. We denote by $L^2_{(0,q)}(X, L^k)$ and $L^2_{(0,q)}(X)$ the completions of $\Omega^{0,q}_0(X, L^k)$ and $\Omega^{0,q}_0(X)$ with respect to $(\cdot|\cdot)_{h^{L^k}}$ and $(\cdot|\cdot)$ respectively. Let

$$\Box^{(q)}_{b,k} : \mathrm{Dom}\,\Box^{(q)}_{b,k} \subset L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$$

be the Gaffney extension of the Kohn Laplacian(see (3.9)). By a result of Gaffney, for every $q = 0, 1, \ldots, n-1$, $\Box^{(q)}_{b,k}$ is a positive self-adjoint operator (see Proposition 3.1.2 in Ma-Marinescu [18]). That is, $\Box^{(q)}_{b,k}$ is self-adjoint and the spectrum of $\Box^{(q)}_{b,k}$ is contained in $\overline{\mathbb{R}}_+$, $q = 0, 1, \ldots, n-1$. For a Borel set $B \subset \mathbb{R}$ we denote by $E(B)$ the spectral projection of $\Box^{(q)}_{b,k}$ corresponding to the set B, where E is the spectral measure of $\Box^{(q)}_{b,k}$ (see section 2 in Davies [7], for the precise meanings of spectral projection and spectral measure). For $\lambda \geq 0$, we set

$$\begin{aligned} &H^q_{b,\leq\lambda}(X, L^k) := \mathrm{Ran}\,E\big((-\infty, \lambda]\big) \subset L^2_{(0,q)}(X, L^k)\,, \\ &H^q_{b,>\lambda}(X, L^k) := \mathrm{Ran}\,E\big((\lambda, \infty)\big) \subset L^2_{(0,q)}(X, L^k). \end{aligned} \tag{1.1}$$

For $\lambda = 0$, we denote

$$H^q_b(X, L^k) := H^q_{b,\leq 0}(X, L^k) = \mathrm{Ker}\,\Box^{(q)}_{b,k}. \tag{1.2}$$

For $\lambda \geq 0$, let

$$\begin{aligned} &\Pi^{(q)}_{k,\leq\lambda} : L^2_{(0,q)}(X, L^k) \to H^q_{b,\leq\lambda}(X, L^k), \\ &\Pi^{(q)}_{k,>\lambda} : L^2_{(0,q)}(X, L^k) \to H^q_{b,>\lambda}(X, L^k), \end{aligned} \tag{1.3}$$

be the orthogonal projections with respect to $(\cdot|\cdot)_{h^{L^k}}$ and let $\Pi^{(q)}_{k,\leq\lambda}(x,y) \in \mathscr{D}'(X\times X, (T^{*0,q}_yX\otimes L^k_y)\boxtimes(T^{*0,q}_xX \otimes L^k_x))$ and $\Pi^{(q)}_{k,>\lambda}(x,y) \in \mathscr{D}'(X\times X, (T^{*0,q}_yX\otimes L^k_y)\boxtimes(T^{*0,q}_xX\otimes L^k_x))$ denote the distribution kernels of $\Pi^{(q)}_{k,\leq\lambda}$ and $\Pi^{(q)}_{k,>\lambda}$ respectively. For $\lambda = 0$, we denote $\Pi^{(q)}_k := \Pi^{(q)}_{k,\leq 0}$, $\Pi^{(q)}_k(x,y) := \Pi^{(q)}_{k,\leq 0}(x,y)$.

Let s be a local trivialization of L on an open set $D \Subset X$, $|s|^2_{h^L} = e^{-2\phi}$. We assume that $Y(q)$ holds at each point of D(see Definition 3.2, for the precise meaning of condition $Y(q)$). By the L^2 estimates of Kohn (see Folland-Kohn [8] and Chen-Shaw [6]), we see that for every $\lambda \geq 0$,

$$\Pi^{(q)}_{k,\leq\lambda}(x,y) \in C^\infty(D\times D, (T^{*0,q}_yX\otimes L^k_y)\boxtimes(T^{*0,q}_xX\otimes L^k_x)). \tag{1.4}$$

Let Σ be the semi-classical characteristic manifold of $\Box^{(q)}_{b,k}$ on D(see (4.11)). We have

$$\Sigma = \big\{(x,\xi) \in T^*D;\, \xi = \lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x), \lambda \in \mathbb{R}\big\}$$

(see Proposition 4.2), where $\omega_0 \in C^\infty(X, T^*X)$ is the uniquely determined global 1-form(see the discussion before (3.1)). For $x \in D$, let M^ϕ_x be the Hermitian quadratic form on $T^{1,0}_xX$ given by Definition 3.4 and let $\mathcal{L}_x$ be the Levi form at x with respect to ω_0(see Definition 3.1). Let σ denote the canonical two form on T^*D. We can show that σ is non-degenerate at $\rho = (p, \lambda_0\omega_0(p) - 2\mathrm{Im}\,\overline{\partial}_b\phi(p)) \in \Sigma$, $p \in D$, if and only if the Hermitian quadratic form $M^\phi_p - 2\lambda_0\mathcal{L}_p$ is non-degenerate(see Theorem 4.5). From this, it is easy to see that if M^ϕ_p has q negative and $n-1-q$ positive eigenvalues, then σ is degenerate at $(p, \lambda\omega_0(p) - 2\mathrm{Im}\,\overline{\partial}_b\phi(p)) \in \Sigma$, for some $\lambda \in \mathbb{R}$. Fix $(n_-, n_+) \in \mathbb{N}^2_0$, $n_- + n_+ = n-1$. Put

$$\begin{aligned} \Sigma' = &\{(x, \lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)) \in T^*D\textstyle\bigcap\Sigma; \\ &M^\phi_x - 2\lambda\mathcal{L}_x \text{ is non-degenerate of constant signature } (n_-, n_+)\}. \end{aligned} \tag{1.5}$$

Let s be a local trivializing section of L on an open subset $D \Subset X$ and $|s|^2_{h^L} = e^{-2\phi}$. Let $B_k : \Omega^{0,q}_0(D) \to \mathscr{D}'(D, T^{*0,q}X)$ be a k-dependent continuous operator. We write $B_k(x,y)$ to denote the distribution kernel of B_k. B_k is called k-negligible (on D) if B_k is smoothing and the kernel $B_k(x,y)$ of B_k satisfies $\left|\partial^\alpha_x\partial^\beta_y B_k(x,y)\right| = O(k^{-N})$ locally uniformly on every compact set in $D \times D$, for all multi-indices α, β and all $N \in \mathbb{N}$. Let $C_k : \Omega^{0,q}_0(D) \to \mathscr{D}'(D, T^{*0,q}X)$ be another k-dependent continuous operator. We write $B_k \equiv C_k \mod O(k^{-\infty})$ (on D) or $B_k(x,y) \equiv C_k(x,y) \mod O(k^{-\infty})$ (on D) if $B_k - C_k$ is k-negligible on D. Let $A_k : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$ be a continuous operator. We define the localized operator (with respect to the trivializing section s) of A_k by

$$\begin{aligned}\hat{A}_{k,s} : L^2_{(0,q)}(D)\bigcap\mathscr{E}'(D, T^{*0,q}X) &\to L^2_{(0,q)}(D),\\ u &\to e^{-k\phi}s^{-k}A_k(s^ke^{k\phi}u).\end{aligned} \tag{1.6}$$

We write $A_k \equiv 0 \mod O(k^{-\infty})$ on D if $\hat{A}_{k,s} \equiv 0 \mod O(k^{-\infty})$ on D. For $\lambda \geq 0$, we write $\hat{\Pi}^{(q)}_{k,\leq\lambda,s}$ to denote the localized operator of $\Pi^{(q)}_{k,\leq\lambda}$. We denote $\hat{\Pi}^{(q)}_{k,s} := \hat{\Pi}^{(q)}_{k,\leq 0,s}$.

1.1. **Main results: Szegö kernel asymptotics for lower energy forms and almost Kodaira embedding Theorems on CR manifolds.** One of the main results of this work is the following

Theorem 1.1. *With the notations and assumptions used above, let s be a local trivializing section of L on an open subset $D \Subset X$ and $|s|^2_{h^L} = e^{-2\phi}$. Fix $q \in \{0, 1, \ldots, n-1\}$ and suppose that $Y(q)$ holds on D. We assume that there exist a $\lambda_0 \in \mathbb{R}$ and $x_0 \in D$ such that $M^\phi_{x_0} - 2\lambda_0\mathcal{L}_{x_0}$ is non-degenerate of constant signature (n_-, n_+). We fix $D_0 \Subset D$, D_0 open, and let V be any bounded open set of T^*D with $\overline{V} \subset T^*D$, $\overline{V}\bigcap\Sigma \subset \Sigma'$, where Σ' is given by (1.5). Let*

$$\hat{\mathcal{I}}_k \equiv \frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta \mod O(k^{-\infty}) \text{ at } T^*D_0\bigcap\Sigma \tag{1.7}$$

be a properly supported classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$ with $\alpha(x,\eta,k) = 0$ if $|\eta| > M$, for some large $M > 0$ and $\operatorname{Supp}\alpha(x,\eta,k)\bigcap T^*D_0 \Subset V$. *Let $\hat{\mathcal{I}}^*_k$ be the adjoint of $\hat{\mathcal{I}}_k$ with respect to $(\cdot\,|\,\cdot)$. Then for every $N_0 \geq 1$ and every $D' \Subset D_0$, $\alpha, \beta \in \mathbb{N}^{2n-1}_0$, there is a constant $C_{D',\alpha,\beta,N_0} > 0$ independent of k, such that*

$$\begin{aligned}&\left|\partial^\alpha_x\partial^\beta_y\big((\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}^*_k)(x,y) - \int e^{ik\varphi(x,y,s)}a(x,y,s,k)ds\big)\right|\\ &\leq C_{D',\alpha,\beta,N_0}k^{3n+2|\beta|+|\alpha|-N_0-2} \quad \text{on } D'\times D',\\ &\left|\partial^\alpha_x\partial^\beta_y\big((\hat{\mathcal{I}}_k\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}^*_k)(x,y) - (\int e^{ik\varphi(x,y,s)}g(x,y,s,k)ds\big)\right|\\ &\leq C_{D',\alpha,\beta,N_0}k^{3n+2|\beta|+|\alpha|-N_0-2} \quad \text{on } D'\times D',\end{aligned} \tag{1.8}$$

where $a(x,y,s,k) = g(x,y,s,k) = 0$ if $q \neq n_-$, $a(x,y,s,k)$, $g(x,y,s,k) \in S^n_{\mathrm{loc}}\big(1;\Omega, T^{*0,q}X \boxtimes T^{*0,q}X\big)\bigcap C^\infty_0\big(\Omega, T^{*0,q}X\boxtimes T^{*0,q}X\big)$ are as in (7.71) and (7.72) if $q = n_-$ and

$$\begin{aligned}&\varphi(x,y,s) \in C^\infty(\Omega), \quad \operatorname{Im}\varphi(x,y,s) \geq 0, \quad \forall (x,y,s) \in \Omega,\\ &d_x\varphi(x,y,s)|_{x=y} = -2\operatorname{Im}\overline{\partial}_b\phi(x) + s\omega_0(x), \quad d_y\varphi(x,y,s)|_{x=y} = 2\operatorname{Im}\overline{\partial}_b\phi(x) - s\omega_0(x),\\ &\operatorname{Im}\varphi(x,y,s) + \left|\frac{\partial\varphi}{\partial s}(x,y,s)\right| \geq c\,|x-y|^2,\ c > 0 \text{ is a constant}, \forall (x,y,s) \in \Omega,\\ &\varphi(x,y,s) = 0 \text{ and } \frac{\partial\varphi}{\partial s}(x,y,s) = 0 \text{ if and only if } x = y.\end{aligned} \tag{1.9}$$

Here

$$\begin{aligned}\Omega :=&\{(x,y,s) \in D\times D\times\mathbb{R};\ (x, -2\operatorname{Im}\overline{\partial}_b\phi(x) + s\omega_0(x)) \in V\bigcap\Sigma,\\ &(y, -2\operatorname{Im}\overline{\partial}_b\phi(y) + s\omega_0(y)) \in V\bigcap\Sigma,\ |x-y| < \varepsilon, \text{ for some } \varepsilon > 0\}.\end{aligned} \tag{1.10}$$

We refer the reader to Definition 6.2, Definition 6.3 and Definition 6.1 for the definition of classical semi-classical pseudodifferential operators and the precise meanings of (1.7) and the Hörmander symbol spaces $S^0_{\mathrm{loc,cl}}$ and S^0_{loc}.

For more properties for the phase φ, see Theorem 2.1, Theorem 2.2 and Theorem 2.4.

Basically, Theorem 1.1 says that $(\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}^*_k)(x,y)$, $(\hat{\mathcal{I}}_k\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}^*_k)(x,y)$ are asymptotically close to the complex Fourier integral operators $\int e^{ik\varphi(x,y,s)}a(x,y,s,k)ds$, $\int e^{ik\varphi(x,y,s)}g(x,y,s,k)ds$ if N_0 large. We will show in section 9 that under certain conditions, $(\hat{\mathcal{I}}_k\hat{\Pi}^{(q)}_{k,s}\hat{\mathcal{I}}^*_k)(x,y)$ is a complex Fourier integral operator $\int e^{ik\varphi(x,y,s)}g(x,y,s,k)ds$(see Theorem 1.5).

Since $\operatorname{Im}\varphi(x,y,s)+\left|\frac{\partial\varphi}{\partial s}(x,y,s)\right|\geq c\left|x-y\right|^2$, $c>0$ is a constant, we can integrate by parts with respect to s and conclude that the integral $\int e^{ik\varphi(x,y,s)}b(x,y,s,k)ds$, $b(x,y,s,k)\in S^n_{\rm loc}\left(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\right)\bigcap C^\infty_0\left(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\right)$, is k-negligible away $x=y$. Thus, we can take $\varepsilon>0$ in (1.10) to be any small positive constant.

Definition 1.2. We say that L is positive if for every $x\in X$ there is a $\eta\in\mathbb{R}$ such that the Hermitian quadratic form $M^\phi_x-2\eta\mathcal{L}_x$ is positive definite.

In view of Proposition 3.5, we see that Definition 1.2 is well-defined.

Definition 1.3. Let $(X,T^{1,0}X)$ be a compact orientable CR manifold of dimension $2n-1$, $n\geqslant 2$. We say that X can be almost CR embedded into projective space if for every $m\in\mathbb{N}_0$ and $\varepsilon>0$ there is an embedding $\Phi_{\varepsilon,m}:X\to\mathbb{CP}^N$, for some $N\in\mathbb{N}$, $N\geq 2$, such that

$$\left\|d\Phi_{\varepsilon,m}(T^{1,0}X)-\mathbb{C}T\Phi_{\varepsilon,m}(X)\bigcap T^{1,0}\mathbb{CP}^N\right\|_{C^m(\mathbb{CP}^N,\mathbb{C}T\mathbb{CP}^N)}<\varepsilon.$$

We recall that a smooth map $\Phi:X\to\mathbb{CP}^N$, $N\in\mathbb{N}$, $N\geq 2$, is a CR embedding if Φ is an embedding and $d\Phi(T^{1,0}X)=\mathbb{C}T\Phi(X)\bigcap T^{1,0}\mathbb{CP}^N$.

By using Theorem 1.1, we establish in section 8 the almost Kodaira embedding theorems on CR manifolds (see Theorem 8.11)

Theorem 1.4. *Let $(X,T^{1,0}X)$ be a compact orientable CR manifold of dimension $2n-1$, $n\geqslant 2$. If X admits a positive CR line bundle L over X, then X can be almost CR embedded into projective space.*

It should be mentioned that Theorem 1.4 is in the spirit of the almost symplectic and almost isometric embedding of compact symplectic manifolds by Borthwick-Uribe [4], Ma-Marinescu [19] and Shiffman-Zelditch [23]. Especially, in Ma-Marinescu [19] the spectral spaces of the Bochner Laplacian are used to obtain the embedding.

1.2. **Main results: Szegö kernel asymptotics.** In view of Theorem 1.1, we see that if $\Box^{(q)}_{b,k}$ has spectral gap $\geq k^{-M}$, for some $M>0$, the operator $(\hat{\mathcal{I}}_k\hat{\Pi}^{(q)}_{k,s}\hat{\mathcal{I}}^*_k)(x,y)$ admits a full asymptotic expansion. But in general, it is very difficult to see that if $\Box^{(q)}_{b,k}$ has spectral gap. We then impose some mild semi-classical local conditions and we show that a certain conjugation of the Szegö projection by some kind of pseudodifferential operator is a Fourier integral operator under these semi-classical local conditions. More precisely, we have

Theorem 1.5. *With the notations and assumptions used above, let s be a local trivializing section of L on an open subset $D\Subset X$ and $\left|s\right|^2_{h^L}=e^{-2\phi}$. Fix $q\in\{0,1,\ldots,n-1\}$ and suppose that $Y(q)$ holds on D. We assume that there exist a $\lambda_0\in\mathbb{R}$ and $x_0\in D$ such that $M^\phi_{x_0}-2\lambda_0\mathcal{L}_{x_0}$ is non-degenerate of constant signature (n_-,n_+). Let $F_k:L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k)$ be a continuous operator and let $F^*_k:L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k)$ be the Hilbert space adjoint of F_k with respect to $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$. Let $\hat{F}_{k,s}$ and $\hat{F}^*_{k,s}$ be the localized operators of F_k and F^*_k respectively. We fix $D_0\Subset D$, D_0 open and let V be any bounded open set of T^*D with $\overline{V}\subset T^*D$, $\overline{V}\bigcap\Sigma\subset\Sigma'$, where Σ' is as in* (1.5). *Assume that*

$$\hat{F}_{k,s}-A_k=O(k^{-\infty}):H^s_{\rm comp}(D,T^{*0,q}X)\to H^s_{\rm loc}(D,T^{*0,q}X),\quad\forall s\in\mathbb{N}_0,$$

where $A_k\equiv\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta\mod O(k^{-\infty})$ at $T^*D_0\bigcap\Sigma$ is a classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$ with $\alpha(x,\eta,k)=0$ if $|\eta|>M$, for some large $M>0$ and $\operatorname{Supp}\alpha(x,\eta,k)\bigcap T^*D_0\Subset V$. Put $P_k:=F_k\Pi^{(q)}_kF^*_k$

and let $\hat{P}_{k,s}$ be the localized operator of P_k. Assume that $\Box^{(q)}_{b,k}$ has $O(k^{-n_0})$ small spectral gap on D with respect to F_k and $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to F_k on D. If $q \neq n_-$, then

$$\hat{P}_{k,s}(x,y) \equiv 0 \mod O(k^{-\infty}) \text{ on } D_0.$$

If $q = n_-$, then

$$\hat{P}_{k,s}(x,y) \equiv \int e^{ik\varphi(x,y,s)} g(x,y,s,k) ds \mod O(k^{-\infty}) \text{ on } D_0,$$

where $\varphi(x,y,s) \in C^\infty(\Omega)$ is as in Theorem 1.1 and

$$g(x,y,s,k) \in S^n_{\mathrm{loc}}\left(1;\Omega, T^{*0,q}X \boxtimes T^{*0,q}X\right) \bigcap C^\infty_0\left(\Omega, T^{*0,q}X \boxtimes T^{*0,q}X\right)$$

is as in (7.71) and (7.72). Here Ω is given by (1.10).

We refer the reader to Definition 9.1 and Definition 9.2 for the precise meanings of "$O(k^{-n_0})$ small spectral gap on D with respect to F_k" and "k-negligible away the diagonal with respect to F_k on D".

Remark 1.6. With the assumptions and notations used in Theorem 1.5, since we only assume some local properties of F_k on D, we don't know what is F_k outside D. In order to get an asymptotic expansion for the Szegö kernel, we need to assume that $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to F_k on D. If X is compact and F_k is a global classical semi-classical pseudodifferential operator on X(see Definition 9.3), we can show that(see Proposition 9.4) $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to F_k on every local trivialization $D \Subset X$. Furthermore, if X is non-compact and F_k is properly supported on $D \Subset X$, then $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to F_k on D(see also Proposition 9.4).

Remark 1.7. With the assumptions and notations used in Theorem 1.5, let

$$\left\{f_1 \in H^0_b(X,L^k), \ldots, f_{d_k} \in H^0_b(X,L^k)\right\}$$

be an orthonormal frame of the space $H^0_b(X,L^k)$, where $d_k \in \mathbb{N}_0 \bigcup \{\infty\}$. It is easy to see that

$$\hat{P}_{k,s}(x,x) = \sum_{j=1}^{d_k} \left|(F_k f_j)(x)\right|^2_{h^{L^k}}, \quad \forall x \in D.$$

Theorem 1.5 implies that if $\Box^{(q)}_{b,k}$ has $O(k^{-n_0})$ small spectral gap on D with respect to F_k and $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to F_k on D, then

$$\textstyle\sum_{j=1}^{d_k} \left|(F_k f_j)(x)\right|^2_{h^{L^k}} \equiv 0 \mod O(k^{-\infty}) \text{ on } D_0 \text{ when } q \neq n_-$$

and

$$\textstyle\sum_{j=1}^{d_k} \left|(F_k f_j)(x)\right|^2_{h^{L^k}} \equiv \sum_{j=1}^{\infty} k^{n-j} b_j(x) \mod O(k^{-\infty}) \text{ on } D_0 \text{ when } q = n_-,$$

where $b_j(x) \in C^\infty_0(D)$, $j = 0,1,\ldots$, and for every $x \in D_0$, $b_0(x) = \int g_0(x,x,s)ds$, $g_0(x,y,s)$ is given by (7.72).

After proving Theorem 1.4 and Theorem 1.5, we asked the following two questions: When we can get "true" Kodaira embedding Theorems on CR manifolds? Can we find some non-trivial examples for Theorem 1.5? In order to answer these questions, let's study carefully some CR submanifolds of projective space. We consider $\mathbb{CP}^{N-1}$, $N \geq 4$. Let $[z] = [z_1,\ldots,z_N]$ be the homogeneous coordinates of $\mathbb{CP}^{N-1}$. Put

$$X := \left\{[z_1,\ldots,z_N] \in \mathbb{CP}^{N-1};\, \lambda_1\left|z_1\right|^2 + \cdots + \lambda_m\left|z_m\right|^2 + \lambda_{m+1}\left|z_{m+1}\right|^2 + \cdots + \lambda_N\left|z_N\right|^2 = 0\right\},$$

where $m \in \mathbb{N}$ and $\lambda_j \in \mathbb{R}$, $j = 1,\ldots,N$. Then, X is a compact CR manifold of dimension $2(N-1)-1$ with CR structure $T^{1,0}X := T^{1,0}\mathbb{CP}^{N-1} \bigcap \mathbb{C}TX$. Now, we assume that $\lambda_1 < 0, \lambda_2 < 0, \ldots, \lambda_m < 0$, $\lambda_{m+1} > 0, \lambda_{m+2} > 0, \ldots, \lambda_N > 0$, where $m \geq 2$, $N - m \geq 2$. Then, it is easy to see that the Levi form has at least one negative and one positive eigenvalues at each point of X. Thus, $Y(0)$ holds at each point of X. X admits a S^1 action:

$$\begin{gathered} S^1 \times X \to X, \\ e^{i\theta} \circ [z_1,\ldots,z_m,z_{m+1},\ldots,z_N] \to [e^{i\theta}z_1,\ldots,e^{i\theta}z_m,z_{m+1},\ldots,z_N], \quad \theta \in [-\pi,\pi[. \end{gathered} \tag{1.11}$$

Since $(z_1, \ldots, z_m) \neq 0$ on X, this S^1 action is well-defined. Let $T \in C^\infty(X, TX)$ be the real vector field given by

$$Tu = \frac{\partial}{\partial \theta}(u(e^{i\theta}x))|_{\theta=0}, \quad u \in C^\infty(X). \tag{1.12}$$

It is easy to see that $[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X)$ and $T(x) \oplus T^{1,0}_x X \oplus T^{0,1}_x X = \mathbb{C}T_xX$(we say that the S^1 action is CR and transversal, see Definition 10.1).

Let $E \to \mathbb{CP}^{N-1}$ be the canonical line bundle with respect to the standard Fubini-Study metric. Consider $L := E|_X$. Then, it is easy to see that(see section 10.1) L is a T-rigid positive CR line bundle over $(X, T^{1,0}X)$(see Definition 1.10, for the meaning of "positive T-rigid CR line bundle"). Thus, we ask the following question

Question 1.8. Let $(X, T^{1,0}X)$ be a compact CR manifold with a transversal CR S^1 action and let T be the global vector field induced by the S^1 action. If there is a T-rigid positive CR line bundle over X, then can X be CR embedded into $\mathbb{CP}^N$, for some $N \in \mathbb{N}$?

In section 10, we study "CR manifolds with transversal CR S^1 actions" and fortunately, we found that if $Y(q)$ holds and M^ϕ_x is non-degenerate of constant signature, then we can always find a continuous operator $F_k : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$ such that the assumptions in Theorem 1.5 hold and by using Theorem 1.5, we solve Question 1.8 completely.

1.3. **Main results: Sezgö kernel asymptotics and Kodairan embedding Theorems on CR manifolds with transversal CR S^1 actions.** Let $(X, T^{1,0}X)$ be a CR manifold. We assume that X admits a transversal CR S^1 action: $S^1 \times X \to X$ (see Definition 10.1). We write $e^{i\theta}$, $0 \leq \theta < 2\pi$, to denote the S^1 action and we let T be the global vector field induced by the S^1 action(see (10.1)). Note that we don't assume that the S^1 action is globally free.

We show in Theorem 10.5 that there is a T-rigid Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ (see Definition 10.2 and Definition 10.3) such that $T^{1,0}X \perp T^{0,1}X$, $T \perp (T^{1,0}X \oplus T^{0,1}X)$, $\langle T | T \rangle = 1$ and $\langle u | v \rangle$ is real if u, v are real tangent vectors. Until further notice, we fix a T-rigid Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X \perp T^{0,1}X$, $T \perp (T^{1,0}X \oplus T^{0,1}X)$, $\langle T | T \rangle = 1$ and $\langle u | v \rangle$ is real if u, v are real tangent vectors.

Let L be a T-rigid CR line bundle over $(X, T^{1,0}X)$ with a T-rigid Hermitian fiber metric h^L on L(see Definition 10.7 and Definition 10.8). For $k > 0$, as before, we shall consider (L^k, h^{L^k}) and we will use the same notations as before. Since L is T-rigid, Tu is well-defined, for every $u \in \Omega^{0,q}(X, L^k)$. For $m \in \mathbb{Z}$, put

$$A^{0,q}_m(X, L^k) := \left\{u \in \Omega^{0,q}(X, L^k);\, Tu = imu\right\} \tag{1.13}$$

and let $\mathcal{A}^{0,q}_m(X, L^k) \subset L^2_{(0,q)}(X, L^k)$ be the completion of $A^{0,q}_m(X, L^k)$ with respect to $(\cdot | \cdot)_{h^{L^k}}$. For $m \in \mathbb{Z}$, let

$$Q^{(q)}_{m,k} : L^2_{(0,q)}(X, L^k) \to \mathcal{A}^{0,q}_m(X, L^k) \tag{1.14}$$

be the orthogonal projection with respect to $(\cdot | \cdot)_{h^{L^k}}$. Fix $\delta > 0$. Take $\tau_\delta(x) \in C^\infty_0(]-\delta, \delta[)$, $0 \leq \tau_\delta \leq 1$ and $\tau_\delta = 1$ on $[-\frac{\delta}{2}, \frac{\delta}{2}]$. Let $F^{(q)}_{\delta,k} : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$ be the continuous map given by

$$\begin{split} F^{(q)}_{\delta,k} : L^2_{(0,q)}(X, L^k) &\to L^2_{(0,q)}(X, L^k), \\ u &\to \sum_{m \in \mathbb{Z}} \tau_\delta(\frac{m}{k})(Q^{(q)}_{m,k}u). \end{split} \tag{1.15}$$

One of the main results of this work is the following

Theorem 1.9. *Let $(X, T^{1,0}X)$ be a compact CR manifold with a transversal CR S^1 action and let T be the global vector field induced by the S^1 action. Let L be a T-rigid CR line bundle over X with a T-rigid Hermitian fiber metric h^L. We assume that $Y(q)$ holds at each point of X and M^ϕ_x is non-degenerate of constant signature (n_-, n_+), for every $x \in X$. Let s be a local trivializing section of L on an open set $D \subset X$, $|s|^2_{h^L} = e^{-2\phi}$. Fix $D_0 \Subset D$. Let $F^{(q)}_{\delta,k} : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$ be* *the continuous operator given by* (1.15) *and let* $F^{(q),*}_{\delta,k}: L^2_{(0,q)}(X,L^k)\rightarrow L^2_{(0,q)}(X,L^k)$ *be the adjoint of* $F^{(q)}_{\delta,k}$ *with respect to* $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$. *Put* $P_k:=F^{(q)}_{\delta,k}\Pi^{(q)}_kF^{(q),*}_{\delta,k}$ *and let* $\hat{P}_{k,s}$ *be the localized operator of* P_k. *Assume that* $\delta>0$ *is small. If* $q\neq n_-$, *then* $\hat{P}_{k,s}\equiv 0 \mod O(k^{-\infty})$ *on* D_0. *If* $q=n_-$, *then*

$$\hat{P}_{k,s}(x,y)\equiv\int e^{ik\varphi(x,y,s)}g(x,y,s,k)ds \mod O(k^{-\infty}) \text{ on } D_0,$$

where $\varphi(x,y,s)\in C^\infty(\Omega)$ *is as in Theorem 1.1 and*

$$\begin{aligned}
&g(x,y,s,k)\in S^n_{\mathrm{loc}}\big(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big)\bigcap C^\infty_0\big(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\\
&g(x,y,s,k)\sim\sum^\infty_{j=0}g_j(x,y,s)k^{n-j} \text{ in } S^n_{\mathrm{loc}}\big(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\\
&g_j(x,y,s)\in C^\infty_0\big(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\quad j=0,1,2,\ldots,
\end{aligned}$$

and for every $(x,x,s)\in\Omega$, $x\in D_0$, $g_0(x,x,s)=(2\pi)^{-n}\left|\det\big(M^\phi_x-2s\mathcal{L}_x\big)\right|\left|\tau_\delta(s)\right|^2\pi_{(x,s,n_-)}$. *Here* Ω *is given by* (1.10), $\pi_{(x,s,n_-)}$ *is as in Theorem 7.12 and* τ_δ *is as in the discussion after* (1.14).

Definition 1.10. Let $(X,T^{1,0}X)$ be a CR manifold with a transversal CR S^1 action and let T be the global vector field induced by the S^1 action. We say that L is a T-rigid positive CR line bundle over X if L is a T-rigid CR line bundle over X and there is a T-rigid Hermitian fiber metric h^L on L such that M^ϕ_x is positive, for every $x\in X$, where ϕ is the local weights of h^L.

By using Theorem 1.9, we can repeat the proof of Theorem 1.4 and establish Kodaira embedding theorems on CR manifolds with transversal CR S^1 actions

Theorem 1.11. *Let* $(X,T^{1,0}X)$ *be a compact CR manifold with a transversal CR* S^1 *action and let* T *be the global vector field induced by the* S^1 *action. If there is a* T*-rigid positive CR line bundle over* X, *then* X *can be CR embedded into* $\mathbb{CP}^N$, *for some* $N\in\mathbb{N}$.

In section 11, we also establish Szegö kernel asymptotis on some non-compact CR manifolds (see Theorem 11.5).

The layout of this paper is as follows. In section 2, we collect some properties for the phase $\varphi(x,y,s)$. In section 3, we collect some notations, definitions and statements we use throughout. In section 4, we write down $\Box^{(q)}_{b,k}$ in a local trivialization and we get the formula for the characteristic manifold for $\Box^{(q)}_{b,k}$ and we prove that the canonical two form σ is non-degenerate at $\rho=(p,\lambda_0\omega_0(p)-2\mathrm{Im}\,\overline\partial_b\phi(p))\in\Sigma$ if and only if the Hermitian quadratic form $M^\phi_p-2\lambda_0\mathcal{L}_p$ is non-degenerate(see Theorem 4.5). In section 5, by using the identity $k=e^{-ikx_{2n}}\big(-i\frac{\partial}{\partial x_{2n}}(e^{ikx_{2n}})\big)$, we introduce the local operator $\Box^{(q)}_s$ defined on some open set of $\mathbb{R}^{2n}$ and by using the heat equation method, we establish microlocal Hodge decomposition theorems for $\Box^{(q)}_s$. Moreover, we study the phase φ carefully and we prove Theorem 2.1 and Theorem 2.2. In section 6, we reduce the semi-classical analysis of $\Box^{(q)}_{b,k}$ to the microlocal analysis of $\Box^{(q)}_s$ and we establish semi-classical Hodge decomposition theorems for $\Box^{(q)}_{b,k}$ in non-degenerate part of Σ and we calculate the leading terms of the asymptotic expansions in (7.71). We notice that it is possible to consider semi-classical heat equation for $\Box^{(q)}_{b,k}$ and establish semi-classical Hodge decomposition theorems for $\Box^{(q)}_{b,k}$ directly. In a further publication, we will study Kohn Laplacian on a CR manifold with codimension≥ 2 and the local operator $\Box^{(q)}_s$ is similar to some kind of Kohn Laplacian on a CR manifold with codimension 2. Hence, we decide to reduce the semi-classical analysis of $\Box^{(q)}_{b,k}$ to the microlocal analysis of $\Box^{(q)}_s$. In section 7, by using the scaling technique introduced in [15] and the semi-classical Hodge decomposition theorems in section 6, we prove Theorem 1.1. In section 8, by using Theorem 1.1, we establish almost Kodaira embedding theorems on CR manifolds and therefore we prove Theorem 1.4. It should be mentioned that the method in section 8 works well in complex case(the complex case is much more simpler than CR case) and we can give a pure analytic proof of classical Kodaira embedding theorem. In section 9, we prove Theorem 1.5. In section 10, we introduce and study CR manifolds with transversal CR S^1 action. We show that there is a T-rigid Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathbb{C}TX$(see Theorem 10.5) and we prove Theorem 1.9. Moreover, by using Theorem 1.9, we can repeat the proof of Theorem 1.4

and establish Kodaira embedding theorems on CR manifolds with transversal CR S^1 action. For simplicity, we only give the outline of the proof of Theorem 1.11. In section 11, by using Hörmander's L^2 estimates, we establish the small spectral gap property for $\Box^{(0)}_{b,k}$ with respect to $F^{(0)}_{\delta,k}$ and we prove Theorem 11.5. Finally, in section 12, by using global theory of complex Fourier integral operators of Melin-Sjöstrand [21], we prove Theorem 2.4.

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2. More properties of the phase $\varphi(x,y,s)$

In this section, we collect some properties of the phase $\varphi(x,y,s)$ for the convenience of the reader.

We can estimate $\operatorname{Im}\varphi(x,y,s)$ in some local coordinates

Theorem 2.1. *With the assumptions and notations used in Theorem 1.1, fix $p\in D$. We take local coordinates $x=(x_1,\ldots,x_{2n-1})$ defined in a small neighbourhood of p so that $\omega_0(p)=dx_{2n-1}$, $T^{1,0}_pX\oplus T^{0,1}_pX=\Big\{\sum^{2n-2}_{j=1}a_j\frac{\partial}{\partial x_j};\, a_j\in\mathbb{R}, j=1,\ldots,2n-2\Big\}$. If D is small enough, then there is a constant $c>0$ such that*

$$\begin{aligned}&\operatorname{Im}\varphi(x,y,s)\geq c\left|x'-y'\right|^2,\quad \forall (x,y,s)\in\Omega,\\ &\operatorname{Im}\varphi(x,y,s)+\left|\frac{\partial\varphi}{\partial s}(x,y,s)\right|\geq c\big(\left|x_{2n-1}-y_{2n-1}\right|+\left|x'-y'\right|^2\big),\quad \forall (x,y,s)\in\Omega,\end{aligned}\tag{2.1}$$

where $x'=(x_1,\ldots,x_{2n-2})$, $y'=(y_1,\ldots,y_{2n-2})$, $\left|x'-y'\right|^2=\sum^{2n-2}_{j=1}\left|x_j-y_j\right|^2$.

In section 5.4, we determine the tangential Hessian of $\varphi(x,y,s)$

Theorem 2.2. *With the assumptions and notations used in Theorem 1.1, fix $(p,p,s_0)\in\Omega$, and let $\overline{Z}_{1,s_0},\ldots,\overline{Z}_{n-1,s_0}$ be an orthonormal frame of $T^{1,0}_xX$ varying smoothly with x in a neighbourhood of p, for which the Hermitian quadratic form $M^\phi_x-2s_0\mathcal{L}_x$ is diagonalized at p. That is,*

$$M^\phi_p\big(\overline{Z}_{j,s_0}(p),Z_{t,s_0}(p)\big)-2s_0\mathcal{L}_p\big(\overline{Z}_{j,s_0}(p),Z_{t,s_0}(p)\big)=\lambda_j(s_0)\delta_{j,t},\quad j,t=1,\ldots,n-1.$$

Assume that $\lambda_j(s_0)<0$, $j=1,\ldots,n_-$, $\lambda_j(s_0)>0$, $j=n_-+1,\ldots,n-1$. Let $x=(x_1,\ldots,x_{2n-1})$, $z_j=x_{2j-1}+ix_{2j}$, $j=1,\ldots,n-1$, be local coordinates of X defined in some small neighbourhood of p such that

$$\begin{aligned}&x(p)=0,\quad \omega_0(p)=dx_{2n-1},\quad T(p)=-\frac{\partial}{\partial x_{2n-1}}(p),\\ &\langle\,\frac{\partial}{\partial x_j}(p)\,|\,\frac{\partial}{\partial x_t}(p)\,\rangle=2\delta_{j,t},\quad j,t=1,\ldots,2n-2,\\ &\overline{Z}_{j,s_0}(p)=\frac{\partial}{\partial z_j}+i\sum^{n-1}_{t=1}\tau_{j,t}\overline{z}_t\frac{\partial}{\partial x_{2n-1}}+c_jx_{2n-1}\frac{\partial}{\partial x_{2n-1}}+O(|x|^2),\quad j=1,\ldots,n-1,\\ &\phi(x)=\beta x_{2n-1}+\sum^{n-1}_{j=1}\big(\alpha_jz_j+\overline{\alpha}_j\overline{z}_j\big)+\frac{1}{2}\sum^{n-1}_{l,t=1}\mu_{t,l}z_t\overline{z}_l+\sum^{n-1}_{l,t=1}\big(a_{l,t}z_lz_t+\overline{a}_{l,t}\overline{z}_l\overline{z}_t\big)\\ &\quad+\sum^{n-1}_{j=1}\big(d_jz_jx_{2n-1}+\overline{d}_j\overline{z}_jx_{2n-1}\big)+O(|x_{2n-1}|^2)+O(|x|^3),\end{aligned}\tag{2.2}$$

where $\beta\in\mathbb{R}$, $\tau_{j,t},c_j,\alpha_j,\mu_{j,t},a_{j,t},d_j\in\mathbb{C}$, $\mu_{j,t}=\overline{\mu}_{t,j}$, $j,t=1,\ldots,n-1$. (This is always possible, see [2, p. 157–160]). We also write $y=(y_1,\ldots,y_{2n-1})$, $w_j=y_{2j-1}+iy_{2j}$, $j=1,\ldots,n-1$. Then, in *some small neighbourhood of* (p,p),

(2.3)

$$\begin{aligned}
&\varphi(x,y,s_0)\\
&=-i\sum_{j=1}^{n-1}\alpha_j(z_j-w_j)+i\sum_{j=1}^{n-1}\overline{\alpha}_j(\overline{z}_j-\overline{w}_j)+s_0(x_{2n-1}-y_{2n-1})-\frac{i}{2}\sum_{j,l=1}^{n-1}(a_{l,j}+a_{j,l})(z_jz_l-w_jw_l)\\
&\quad+\frac{i}{2}\sum_{j,l=1}^{n-1}(\overline{a}_{l,j}+\overline{a}_{j,l})(\overline{z}_j\overline{z}_l-\overline{w}_j\overline{w}_l)+\frac{1}{2}\sum_{j,l=1}^{n-1}\Big(is_0(\overline{\tau}_{l,j}-\tau_{j,l})+(\overline{\tau}_{l,j}+\tau_{j,l})\beta\Big)(z_j\overline{z}_l-w_j\overline{w}_l)\\
&\quad+\sum_{j=1}^{n-1}(-ic_j\beta-s_0c_j-id_j)(z_jx_{2n-1}-w_jy_{2n-1})+\sum_{j=1}^{n-1}(i\overline{c}_j\beta-s_0\overline{c}_j+i\overline{d}_j)(\overline{z}_jx_{2n-1}-\overline{w}_jy_{2n-1})\\
&\quad-\frac{i}{2}\sum_{j=1}^{n-1}\lambda_j(s_0)(z_j\overline{w}_j-\overline{z}_jw_j)+\frac{i}{2}\sum_{j=1}^{n-1}\left|\lambda_j(s_0)\right|\left|z_j-w_j\right|^2+(x_{2n-1}-y_{2n-1})f(x,y,s_0)+O(\left|(x,y)\right|^3),\\
&\qquad f\in C^\infty,\quad f(0,0)=0.
\end{aligned}$$

Definition 2.3. With the assumptions and notations used in Theorem 1.1, let $\varphi_1(x,y,s),\varphi_2(x,y,s)\in C^\infty(\Omega)$. We assume that $\varphi_1(x,y,s)$ and $\varphi_2(x,y,s)$ satisfy (1.9) and (2.1). We say that $\varphi_1(x,y,s)$ and $\varphi_2(x,y,s)$ are equivalent on Ω if for any

$$b_1(x,y,s,k)\in S^m_{\rm loc}\left(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\right)\bigcap C^\infty_0\left(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\right),\quad m\in\mathbb{Z},$$

we can find

$$b_2(x,y,s,k)\in S^m_{\rm loc}\left(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\right)\bigcap C^\infty_0\left(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\right)$$

such that

$$\int e^{ik\varphi_1(x,y,s)}b_1(x,y,s,k)ds\equiv\int e^{ik\varphi_1(x,y,s)}b_2(x,y,s,k)ds\mod O(k^{-\infty})\ \text{ on } D$$

and vise versa.

Let $\varphi_1(x,y,s)\in C^\infty(\Omega)$. We assume that $\varphi_1(x,y,s)$ satisfies (1.9) and (2.1). Fix $p\in D$. We take local coordinates $x=(x_1,\ldots,x_{2n-1})$ defined in a small neighbourhood of p so that $\omega_0(p)=dx_{2n-1}$, $T^{1,0}_pX\oplus T^{0,1}_pX=\left\{\sum_{j=1}^{2n-2}a_j\frac{\partial}{\partial x_j};\,a_j\in\mathbb{R},j=1,\ldots,2n-2\right\}$. Put

$$-2{\rm Im}\,\overline{\partial}_b\phi(x)=\sum_{j=1}^{2n-1}\alpha_j(x)dx_j,\quad\omega_0(x)=\sum_{j=1}^{2n-1}\beta_j(x)dx_j.$$

We assume that D is a small open neighbourhood of p. In Lemma 5.26, we will show that if D is small enough then we can find $\hat\varphi_1(x,y,s)\in C^\infty(\Omega)$ such that $\hat\varphi_1(x,y,s)$ satisfies (1.9), (2.1), $\frac{\partial\hat\varphi_1}{\partial y_{2n-1}}(x,y,s)-(\alpha_{2n-1}(y)+s\beta_{2n-1}(y))$ vanishes to infinite order at $x=y$ and $\hat\varphi_1(x,y,s)$ and $\varphi_1(x,y,s)$ are equivalent on Ω in the sense of Definition 2.3. We characterize the phase φ(see section 5.3 and section 12)

Theorem 2.4. *With the assumptions and notations used in Theorem 1.1, put* $\widetilde\varphi(x,y,s):=-\overline{\varphi}(y,x,s)$*. Then,* $\widetilde\varphi(x,y,s)$ *and* $\varphi(x,y,s)$ *are equivalent on* Ω *in the sense of Definition 2.3. Moreover, let* $\varphi_1(x,y,s)\in C^\infty(\Omega)$*. We assume that* $\varphi_1(x,y,s)$ *satisfies* (1.9) *and* (2.1)*. If* D *is small enough, then* φ *and* φ_1 *are equivalent on* Ω *in the sense of Definition 2.3 if and only if there are functions*

$f \in C^\infty(\Omega)$, $g_j \in C^\infty(\Omega)$, $j = 0, 1, \ldots, 2n-1$, $p_j \in C^\infty(\Omega)$, $j = 1, \ldots, 2n-1$, such that

$$\begin{aligned}
&\frac{\partial \hat{\varphi}}{\partial s}(x,y,s) - f(x,y,s)\frac{\partial \hat{\varphi}_1}{\partial s}(x,y,s),\\
&\hat{\varphi}(x,y,s) - \hat{\varphi}_1(x,y,s) = g_0(x,y,s)\frac{\partial \hat{\varphi}}{\partial s}(x,y,s),\\
&\frac{\partial \hat{\varphi}}{\partial x_j}(x,y,s) - \frac{\partial \hat{\varphi}_1}{\partial x_j}(x,y,s) = g_j(x,y,s)\frac{\partial \hat{\varphi}}{\partial s}(x,y,s), \quad j = 1, 2, \ldots, 2n-1,\\
&\frac{\partial \hat{\varphi}}{\partial y_j}(x,y,s) - \frac{\partial \hat{\varphi}_1}{\partial y_j}(x,y,s) = p_j(x,y,s)\frac{\partial \hat{\varphi}}{\partial s}(x,y,s), \quad j = 1, 2, \ldots, 2n-1,
\end{aligned}$$

vanish to infinite order on $x = y$, for every $(x,y,s) \in \Omega$, where $\hat{\varphi}(x,y,s) \in C^\infty(\Omega)$, $\hat{\varphi}_1(x,y,s) \in C^\infty(\Omega)$ are as in the discussion after Definition 2.3.

3. Preliminaries

3.1. **Some standard notations.** We shall use the following notations: $\mathbb{R}$ is the set of real numbers, $\overline{\mathbb{R}}_+ := \{x \in \mathbb{R}; x \geq 0\}$, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. An element $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathbb{N}_0^n$ will be called a multiindex, the size of α is: $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and the length of α is $l(\alpha) = n$. For $m \in \mathbb{N}$, we write $\alpha \in \{1, \ldots, m\}^n$ if $\alpha_j \in \{1, \ldots, m\}$, $j = 1, \ldots, n$. We say that α is strictly increasing if $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. We write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $x = (x_1, \ldots, x_n)$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\partial_{x_j} = \frac{\partial}{\partial x_j}$, $\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha}$, $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, $D_x = \frac{1}{i}\partial_x$, $D_{x_j} = \frac{1}{i}\partial_{x_j}$. Let $z = (z_1, \ldots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, be coordinates of $\mathbb{C}^n$. We write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $\overline{z}^\alpha = \overline{z}_1^{\alpha_1} \cdots \overline{z}_n^{\alpha_n}$, $\frac{\partial^{|\alpha|}}{\partial z^\alpha} = \partial_z^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n}$, $\partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}})$, $j = 1, \ldots, n$. $\frac{\partial^{|\alpha|}}{\partial \overline{z}^\alpha} = \partial_{\overline{z}}^\alpha = \partial_{\overline{z}_1}^{\alpha_1} \cdots \partial_{\overline{z}_n}^{\alpha_n}$, $\partial_{\overline{z}_j} = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}})$, $j = 1, \ldots, n$.

Let M be a C^∞ paracompact manifold. We let TM or $T(M)$ and T^*M or $T^*(M)$ denote the tangent bundle of M and the cotangent bundle of M respectively. The complexified tangent bundle of M and the complexified cotangent bundle of M will be denoted by $\mathbb{C}TM$ and $\mathbb{C}T^*M$ respectively. We write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between TM and T^*M. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}TM \times \mathbb{C}T^*M$. Let E be a C^∞ vector bundle over M. The fiber of E at $x \in M$ will be denoted by E_x. Let F be another vector bundle over M. We write $E \boxtimes F$ to denote the vector bundle over $M \times M$ with fiber over $(x,y) \in M \times M$ consisting of the linear maps from E_x to F_y. Let $Y \subset M$ be an open set. From now on, the spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $C^\infty(Y,E)$ and $\mathscr{D}'(Y,E)$ respectively. Let $\mathscr{E}'(Y,E)$ be the subspace of $\mathscr{D}'(Y,E)$ whose elements have compact support in Y. For $m \in \mathbb{R}$, we let $H^m(Y,E)$ denote the Sobolev space of order m of sections of E over Y. Put

$$\begin{gathered}
H^m_{\rm loc}(Y,E) = \left\{u \in \mathscr{D}'(Y,E);\, \varphi u \in H^m(Y,E),\, \forall \varphi \in C_0^\infty(Y)\right\},\\
H^m_{\rm comp}(Y,E) = H^m_{\rm loc}(Y,E) \cap \mathscr{E}'(Y,E)\,.
\end{gathered}$$

Let E and F be C^∞ vector bundles over a paracompact C^∞ manifold M equipped with a smooth density of integration. If $A : C_0^\infty(M,E) \to \mathscr{D}'(M,F)$ is continuous, we write $K_A(x,y)$ or $A(x,y)$ to denote the distribution kernel of A. The following two statements are equivalent

(a) A is continuous: $\mathscr{E}'(M,E) \to C^\infty(M,F)$,
(b) $K_A \in C^\infty(M \times M, E_y \boxtimes F_x)$.

If A satisfies (a) or (b), we say that A is smoothing. Let $B : C_0^\infty(M,E) \to \mathscr{D}'(M,F)$ be a continuous operator. We write $A \equiv B$ if $A - B$ is a smoothing operator. We say that A is properly supported if $\mathrm{Supp}\,K_A \subset M \times M$ is proper. That is, the two projections: $t_x : (x,y) \in \mathrm{Supp}\,K_A \to x \in M$, $t_y : (x,y) \in \mathrm{Supp}\,K_A \to y \in M$ are proper (i.e. the inverse images of t_x and t_y of all compact subsets of M are compact).

Let $H(x,y) \in \mathscr{D}'(M \times M, E_y \boxtimes F_x)$. We write H to denote the unique continuous operator $C_0^\infty(M,E) \to \mathscr{D}'(M,F)$ with distribution kernel $H(x,y)$. In this work, we identify H with $H(x,y)$.

3.2. **Set up and Terminology.** Let $(X, T^{1,0}X)$ be a paracompact orientable not necessarily compact CR manifold of dimension $2n-1$, $n \geqslant 2$, where $T^{1,0}X$ is a CR structure of X. That is, $T^{1,0}X$ is a complex $n-1$ dimensional subbundle of the complexified tangent bundle $\mathbb{C}TX$, satisfying $T^{1,0}X \bigcap T^{0,1}X = \{0\}$, where $T^{0,1}X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = C^\infty(X, T^{1,0}X)$.

Fix a smooth Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ so that $T^{1,0}X$ is orthogonal to $T^{0,1}X := \overline{T^{1,0}X}$ and $\langle\, u \,|\, v \,\rangle$ is real if u, v are real tangent vectors. Then locally there is a real non-vanishing vector field T of length one which is pointwise orthogonal to $T^{1,0}X \oplus T^{0,1}X$. T is unique up to the choice of sign. For $u \in \mathbb{C}TX$, we write $|u|^2 := \langle\, u \,|\, u \,\rangle$. Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. They can be identified with subbundles of the complexified cotangent bundle $\mathbb{C}T^*X$. Define the vector bundle of $(0,q)$ forms by $T^{*0,q}X := \Lambda^q T^{*0,1}X$. The Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0,q)$ forms $T^{*0,q}X$, $q = 0, 1, \ldots, n-1$. We shall also denote all these induced metrics by $\langle \cdot \,|\, \cdot \rangle$. For $v \in T^{*0,q}X$, we write $|v|^2 := \langle\, v \,|\, v \,\rangle$, and for any $p = 0, 1, 2, \ldots, n-1$, let $v^{\wedge,*} : T^{*0,q+p}X \to T^{*0,p}X$ be the adjoint of $v^\wedge : T^{*0,p}X \to T^{*0,p+q}X$ with respect to $\langle \cdot \,|\, \cdot \rangle$. That is, $\langle\, v \wedge u \,|\, g \,\rangle = \langle\, u \,|\, v^{\wedge,*} g \,\rangle$, $\forall u \in T^{*0,p}X$, $g \in T^{*0,p+q}X$. Let $D \subset X$ be an open set. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q}X$ over D and let $\Omega^{0,q}_0(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D. Similarly, if E is a vector bundle over D, then we let $\Omega^{0,q}(D, E)$ denote the space of smooth sections of $T^{*0,q}X \otimes E$ over D and let $\Omega^{0,q}_0(D, E)$ be the subspace of $\Omega^{0,q}(D, E)$ whose elements have compact support in D.

Locally we can choose an orthonormal frame $\omega_1, \ldots, \omega_{n-1}$ of the bundle $T^{*1,0}X$. Then $\overline{\omega}_1, \ldots, \overline{\omega}_{n-1}$ is an orthonormal frame of the bundle $T^{*0,1}X$. The real $(2n-2)$ form $\omega = i^{n-1}\omega_1 \wedge \overline{\omega}_1 \wedge \cdots \wedge \omega_{n-1} \wedge \overline{\omega}_{n-1}$ is independent of the choice of the orthonormal frame. Thus ω is globally defined. Locally there is a real 1-form ω_0 of length one which is orthogonal to $T^{*1,0}X \oplus T^{*0,1}X$. The form ω_0 is unique up to the choice of sign. Since X is orientable, there is a nowhere vanishing $(2n-1)$ form Q on X. Thus, ω_0 can be specified uniquely by requiring that $\omega \wedge \omega_0 = fQ$, where f is a positive function. Therefore ω_0, so chosen, is globally defined. We call ω_0 the uniquely determined global real 1-form. We choose a vector field T so that

$$|T| = 1\,, \quad \langle\, T \,,\, \omega_0 \,\rangle = -1\,. \tag{3.1}$$

Therefore T is uniquely determined. We call T the uniquely determined global real vector field. We have the pointwise orthogonal decompositions:

$$\begin{aligned} \mathbb{C}T^*X &= T^{*1,0}X \oplus T^{*0,1}X \oplus \{\lambda\omega_0;\ \lambda \in \mathbb{C}\}\,, \\ \mathbb{C}TX &= T^{1,0}X \oplus T^{0,1}X \oplus \{\lambda T;\ \lambda \in \mathbb{C}\}\,. \end{aligned} \tag{3.2}$$

We recall

Definition 3.1. For $p \in X$, the Levi form $\mathcal{L}_p$ is the Hermitian quadratic form on $T^{1,0}_pX$ defined as follows. For any $U,\ V \in T^{1,0}_pX$, pick $\mathcal{U}, \mathcal{V} \in C^\infty(X, T^{1,0}X)$ such that $\mathcal{U}(p) = U$, $\mathcal{V}(p) = V$. Set

$$\mathcal{L}_p(U, \overline{V}) = \frac{1}{2i}\big\langle \big[\,\mathcal{U}\ ,\overline{\mathcal{V}}\,\big](p)\ ,\omega_0(p)\big\rangle\,, \tag{3.3}$$

where $\big[\,\mathcal{U}\ ,\overline{\mathcal{V}}\,\big] = \mathcal{U}\ \overline{\mathcal{V}} - \overline{\mathcal{V}}\ \mathcal{U}$ denotes the commutator of $\mathcal{U}$ and $\overline{\mathcal{V}}$. Note that $\mathcal{L}_p$ does not depend of the choices of $\mathcal{U}$ and $\mathcal{V}$.

Locally there is an orthonormal basis $\{\mathcal{U}_1, \ldots, \mathcal{U}_{n-1}\}$ of $T^{1,0}X$ with respect to $\langle \cdot \,|\, \cdot \rangle$ such that $\mathcal{L}_p$ is diagonal in this basis, $\mathcal{L}_p(\mathcal{U}_j, \overline{\mathcal{U}}_l) = \delta_{j,l}\lambda_j(p)$, where $\delta_{j,l} = 1$ if $j = l$, $\delta_{j,l} = 0$ if $j \neq l$. The entries $\{\lambda_1(p), \ldots, \lambda_{n-1}(p)\}$ are called the eigenvalues of the Levi form at $p \in X$ with respect to $\langle \cdot \,|\, \cdot \rangle$.

Definition 3.2. Given $q \in \{0, \ldots, n-1\}$, the Levi form is said to satisfy condition $Y(q)$ at $p \in X$, if $\mathcal{L}_p$ has at least either $\max\,(q+1, n-q)$ eigenvalues of the same sign or $\min\,(q+1, n-q)$ pairs of eigenvalues with opposite signs. Note that the sign of the eigenvalues does not depend on the choice of the metric $\langle \cdot \,|\, \cdot \rangle$.

Let

$$\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X) \tag{3.4}$$

be the tangential Cauchy-Riemann operator. We say that a function $u \in C^\infty(X)$ is Cauchy-Riemann (CR for short) if $\overline{\partial}_b u = 0$.

Definition 3.3. Let L be a complex line bundle over X. We say that L is a Cauchy-Riemann (CR) complex line bundle over X if its transition functions are CR.

From now on, we let (L, h^L) be a CR Hermitian line bundle over X, where the Hermitian fiber metric on L is denoted by h^L. We will denote by ϕ the local weights of the Hermitian metric. More precisely, if s is a local trivializing section of L on an open subset $D \subset X$, then the local weight of h^L with respect to s is the function $\phi \in C^\infty(D, \mathbb{R})$ for which

$$|s(x)|^2_{h^L} = e^{-2\phi(x)}, \quad x \in D. \tag{3.5}$$

Let L^k, $k > 0$, be the k-th tensor power of the line bundle L. The Hermitian fiber metric on L induces a Hermitian fiber metric on L^k that we shall denote by h^{L^k}. If s is a local trivializing section of L then s^k is a local trivializing section of L^k. The Hermitian metrics $\langle \cdot \,|\, \cdot \rangle$ on $T^{*0,q}X$ and h^{L^k} induce Hermitian metrics on $T^{*0,q}X \otimes L^k$, $q = 0, 1, \ldots, n-1$. We shall denote these induced metrics by $\langle \cdot \,|\, \cdot \rangle_{h^{L^k}}$. For $f \in \Omega^{0,q}(X, L^k)$, we denote the pointwise norm $|f(x)|^2_{h^{L^k}} := \langle f(x) \,|\, f(x) \rangle_{h^{L^k}}$. We write $\overline{\partial}_{b,k}$ to denote the tangential Cauchy-Riemann operator acting on forms with values in L^k, defined locally by:

$$\overline{\partial}_{b,k} : \Omega^{0,q}(X, L^k) \to \Omega^{0,q+1}(X, L^k), \quad \overline{\partial}_{b,k}(s^k u) := s^k \overline{\partial}_b u, \tag{3.6}$$

where s is a local trivialization of L on an open subset $D \subset X$ and $u \in \Omega^{0,q}(D)$. We denote by $dv_X = dv_X(x)$ the volume form on X induced by the fixed Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$. Then we get natural global L^2 inner products $(\cdot \,|\, \cdot)_{h^{L^k}}$, $(\cdot \,|\, \cdot)$ on $\Omega^{0,q}_0(X, L^k)$ and $\Omega^{0,q}_0(X)$ respectively. We denote by $L^2_{(0,q)}(X, L^k)$ and $L^2_{(0,q)}(X)$ the completions of $\Omega^{0,q}_0(X, L^k)$ and $\Omega^{0,q}_0(X)$ with respect to $(\cdot \,|\, \cdot)_{h^{L^k}}$ and $(\cdot \,|\, \cdot)$ respectively. We extend $(\cdot \,|\, \cdot)_{h^{L^k}}$ and $(\cdot \,|\, \cdot)$ to $L^2_{(0,q)}(X, L^k)$ and $L^2_{(0,q)}(X)$ in the standard way respectively. For $f \in \Omega^{0,q}(X, L^k)$, we denote $\|f\|^2_{h^{L^k}} := (\, f \,|\, f\,)_{h^{L^k}}$. Similarly, for $f \in \Omega^{0,q}(X)$, we denote $\|f\|^2 := (\, f \,|\, f\,)$. We extend $\overline{\partial}_{b,k}$ to $L^2_{(0,r)}(X, L^k)$, $r = 0, 1, \ldots, n-1$, by

$$\overline{\partial}_{b,k} : \operatorname{Dom} \overline{\partial}_{b,k} \subset L^2_{(0,r)}(X, L^k) \to L^2_{(0,r+1)}(X, L^k), \tag{3.7}$$

where $\operatorname{Dom} \overline{\partial}_{b,k} := \{u \in L^2_{(0,r)}(X, L^k); \overline{\partial}_{b,k} u \in L^2_{(0,r+1)}(X, L^k)\}$, where for any $u \in L^2_{(0,r)}(X, L^k)$, $\overline{\partial}_{b,k} u$ is defined in the sense of distribution. We also write

$$\overline{\partial}^*_{b,k} : \operatorname{Dom} \overline{\partial}^*_{b,k} \subset L^2_{(0,r+1)}(X, L^k) \to L^2_{(0,r)}(X, L^k) \tag{3.8}$$

to denote the Hilbert space adjoint of $\overline{\partial}_{b,k}$ in the L^2 space with respect to $(\cdot \,|\, \cdot)_{h^{L^k}}$. Let $\Box^{(q)}_{b,k}$ denote the (Gaffney extension) of the Kohn Laplacian given by

(3.9)
$$\operatorname{Dom} \Box^{(q)}_{b,k} = \{s \in L^2_{(0,q)}(X, L^k);\ s \in \operatorname{Dom} \overline{\partial}_{b,k} \cap \operatorname{Dom} \overline{\partial}^*_{b,k},\ \overline{\partial}_{b,k} s \in \operatorname{Dom} \overline{\partial}^*_{b,k},\ \overline{\partial}^*_{b,k} s \in \operatorname{Dom} \overline{\partial}_{b,k}\},$$

and $\Box^{(q)}_{b,k} s = \overline{\partial}_{b,k} \overline{\partial}^*_{b,k} s + \overline{\partial}^*_{b,k} \overline{\partial}_{b,k} s$ for $s \in \operatorname{Dom} \Box^{(q)}_{b,k}$.

We need

Definition 3.4. Let s be a local trivializing section of L on an open subset $D \subset X$ and ϕ the corresponding local weight as in (3.5). For $p \in D$, we define the Hermitian quadratic form M^ϕ_p on $T^{1,0}_p X$ by

$$M^\phi_p(U, \overline{V}) = \Big\langle U \wedge \overline{V}, d\big(\overline{\partial}_b \phi - \partial_b \phi\big)(p) \Big\rangle, \quad U, V \in T^{1,0}_p X, \tag{3.10}$$

where d is the usual exterior derivative and $\overline{\partial_b \phi} = \overline{\partial}_b \overline{\phi}$.

The definition of M^ϕ_p depends on the choice of local trivializations. The following is well-known (see Proposition 4.2 of [15])

Proposition 3.5. *Let $\widetilde{D}$ be another local trivialization with $D \cap \widetilde{D} \neq \emptyset$. Let $\widetilde{s}$ be a local trivializing section of L on $\widetilde{D}$, $|\widetilde{s}(x)|^2_{h^L} = e^{-2\widetilde{\phi}(x)}$, $\widetilde{\phi} \in C^\infty(\widetilde{D}, \mathbb{R})$, and $\widetilde{s} = gs$ on $D \cap \widetilde{D}$, for some non-zero CR function g. For $p \in D \bigcap \widetilde{D}$, we have*

$$M_p^{\phi} = M_p^{\widetilde{\phi}} + i\Big(\frac{Tg}{g} - \frac{T\,\overline{g}}{\overline{g}}\Big)(p)\mathcal{L}_p. \tag{3.11}$$

4. Semi-classical $\Box^{(q)}_{b,k}$ and the characteristic manifold for $\Box^{(q)}_{b,k}$

As before, let $s(x)$ be a local trivializing of L on some open subset $D \subset X$. We have the unitary identification

$$\begin{cases} C_0^\infty(D, T^{*0,q}X) \longleftrightarrow C_0^\infty(D, L^k \otimes T^{*0,q}X) \\ u \longleftrightarrow \widetilde{u} = (e^{\phi}s)^k u, \quad u = e^{-k\phi}s^{-k}\widetilde{u} \\ \overline{\partial}_{s,k} \longleftrightarrow \overline{\partial}_{b,k}, \quad \overline{\partial}_{s,k}u = e^{-k\phi}s^{-k}\overline{\partial}_{b,k}((e^{\phi}s)^k u), \\ \overline{\partial}^*_{s,k} \longleftrightarrow \overline{\partial}^*_{b,k}, \quad \overline{\partial}^*_{s,k}u = e^{-k\phi}s^{-k}\overline{\partial}^*_{b,k}((e^{\phi}s)^k u), \\ \Box^{(q)}_{s,k} \longleftrightarrow \Box^{(q)}_{b,k}, \quad \Box^{(q)}_{s,k}u = e^{-k\phi}s^{-k}\Box^{(q)}_{b,k}((e^{\phi}s)^k u). \end{cases} \tag{4.1}$$

We can check that

$$\overline{\partial}_{s,k} = \overline{\partial}_b + k(\overline{\partial}_b\phi)^{\wedge}, \tag{4.2}$$

$$\overline{\partial}^*_{s,k} = \overline{\partial}^*_b + k(\overline{\partial}_b\phi)^{\wedge,*} \tag{4.3}$$

and

$$\Box^{(q)}_{s,k} = \overline{\partial}_{s,k}\overline{\partial}^*_{s,k} + \overline{\partial}^*_{s,k}\overline{\partial}_{s,k}. \tag{4.4}$$

Here $\overline{\partial}^*_b : C^\infty(X, T^{*0,q+1}X) \to C^\infty(X, T^{*0,q}X)$ is the formal adjoint of $\overline{\partial}_b$ with respect to $(\,\cdot\,|\,\cdot\,)$.

For $p \in X$, we can choose an orthonormal frame $e_1(x), \ldots, e_{n-1}(x)$ for $T^{*0,1}_x X$ varying smoothly with x in a neighbourhood of p. Let $Z_j(x)$, $j = 1, \ldots, n-1$, denote the basis of $T^{0,1}_x X$, which is dual to $e_j(x)$, $j = 1, \ldots, n-1$. Let Z^*_j be the formal adjoint of Z_j with respect to $(\,\cdot\,|\,\cdot\,)$, $j = 1, \ldots, n-1$. That is, $(Z_j f \mid h) = (f \mid Z^*_j h)$, $f, h \in C^\infty(X)$. On scalar functions, we have

$$\overline{\partial}_{s,k} = \sum_{j=1}^{n-1} e^{\wedge}_j \circ (Z_j + kZ_j(\phi)).$$

If $f(x)e_{j_1} \wedge \cdots \wedge e_{j_q}$ is a typical term in a general $(0,q)$ form, we get

$$\begin{aligned} &\overline{\partial}_{s,k}(f(x)e_{j_1} \wedge \cdots \wedge e_{j_q}) \\ &= \sum_{j=1}^{n-1} \big(Z_j(f) + kZ_j(\phi)\big)e^{\wedge}_j e_{j_1} \wedge \cdots \wedge e_{j_q} \\ &+ \sum_{t=1}^{q} (-1)^{t-1} f(z) e_{j_1} \wedge \cdots \wedge (\overline{\partial}_b e_{j_t}) \wedge \cdots \wedge e_{j_q} \\ &=\Big(\sum_{j=1}^{n-1} \big(Z_j(f) + kZ_j(\phi)\big)e^{\wedge}_j\Big) e_{j_1} \wedge \cdots \wedge e_{j_q} \\ &+\Big(\sum_{j=1}^{n-1} (\overline{\partial}_b e_j)^{\wedge} e^{\wedge,*}_j\Big)(f(z)e_{j_1} \wedge \cdots \wedge e_{j_q}). \end{aligned}$$

So for the given orthonormal frame we have the identification

$$\overline{\partial}_{s,k} = \sum_{j=1}^{n-1} \Big(e^{\wedge}_j \circ (Z_j + kZ_j(\phi)) + (\overline{\partial}_b e_j)^{\wedge} e^{\wedge,*}_j\Big) \tag{4.5}$$

and correspondingly

$$\overline{\partial}^*_{s,k}=\sum_{j=1}^{n-1}\Big(e^{\wedge,*}_j\circ(Z^*_j+k\overline{Z}_j(\phi))+e^{\wedge}_j(\overline{\partial}_be_j)^{\wedge,*}\Big). \tag{4.6}$$

We have

$$\begin{aligned}\Box^{(q)}_{s,k}&=\overline{\partial}_{s,k}\overline{\partial}^*_{s,k}+\overline{\partial}^*_{s,k}\overline{\partial}_{s,k}\\&=\sum_{j,t=1}^{n-1}\Big[\big(e^{\wedge}_j\circ(Z_j+kZ_j(\phi))+(\overline{\partial}_be_j)^{\wedge}e^{\wedge,*}_j\big)\big(e^{\wedge,*}_t\circ(Z^*_t+k\overline{Z}_t(\phi))+e^{\wedge}_t(\overline{\partial}_be_t)^{\wedge,*}\big)\\&\quad+\big(e^{\wedge,*}_t\circ(Z^*_t+k\overline{Z}_t(\phi))+e^{\wedge}_t(\overline{\partial}_be_t)^{\wedge,*}\big)\big(e^{\wedge}_j\circ(Z_j+kZ_j(\phi))+(\overline{\partial}_be_j)^{\wedge}e^{\wedge,*}_j\big)\Big]\\&=\sum_{j,t=1}^{n-1}\Big[\big(e^{\wedge}_j\circ(Z_j+kZ_j(\phi))\big)\big(e^{\wedge,*}_t\circ(Z^*_t+k\overline{Z}_t(\phi))\big)+\big(e^{\wedge,*}_t\circ(Z^*_t+k\overline{Z}_t(\phi))\big)\\&\quad\big(e^{\wedge}_j\circ(Z_j+kZ_j(\phi))\big)\Big]+\varepsilon(Z+kZ(\phi))+\varepsilon(Z^*+k\overline{Z}(\phi))+f\\&=\sum_{j,t=1}^{n-1}\Big[e^{\wedge}_je^{\wedge,*}_t\circ(Z_j+kZ_j(\phi))(Z^*_t+k\overline{Z}_t(\phi))+e^{\wedge,*}_te^{\wedge}_j\circ(Z^*_t+k\overline{Z}_t(\phi))\\&\quad(Z_j+kZ_j(\phi))\Big]+\varepsilon(Z+kZ(\phi))+\varepsilon(Z^*+k\overline{Z}(\phi))+f\\&=\sum_{j,t=1}^{n-1}(e^{\wedge}_je^{\wedge,*}_t+e^{\wedge,*}_te^{\wedge}_j)\circ(Z^*_t+k\overline{Z}_t(\phi))(Z_j+kZ_j(\phi))\\&\quad+\sum_{j,t=1}^{n-1}e^{\wedge}_je^{\wedge,*}_t\circ[Z_j+kZ_j(\phi),Z^*_t+k\overline{Z}_t(\phi)]\\&\quad+\varepsilon(Z+kZ(\phi))+\varepsilon(Z^*+k\overline{Z}(\phi))+f,\end{aligned} \tag{4.7}$$

where $\varepsilon(Z+kZ(\phi))$ denotes remainder terms of the form $\sum a_j(Z_j+kZ_j\phi)$ with a_j smooth, matrix-valued and independent of k, for all j, and similarly for $\varepsilon(Z^*+k\overline{Z}(\phi))$ and f is a smooth function independent of k.

Note that

$$e^{\wedge}_je^{\wedge,*}_t+e^{\wedge,*}_te^{\wedge}_j=\delta_{j,t}. \tag{4.8}$$

Combining (4.7) with (4.8), we get the following

Proposition 4.1. *With the notations used before, using the identification* (4.1), *we can identify the Kohn Laplacian* $\Box^{(q)}_{b,k}$ *with*

$$\begin{aligned}\Box^{(q)}_{s,k}&=\overline{\partial}_{s,k}\overline{\partial}^*_{s,k}+\overline{\partial}^*_{s,k}\overline{\partial}_{s,k}\\&=\sum_{j=1}^{n-1}(Z^*_j+k\overline{Z}_j(\phi))(Z_j+kZ_j(\phi))\\&\quad+\sum_{j,t=1}^{n-1}e^{\wedge}_je^{\wedge,*}_t\circ[Z_j+kZ_j(\phi),Z^*_t+k\overline{Z}_t(\phi)]\\&\quad+\varepsilon(Z+kZ(\phi))+\varepsilon(Z^*+k\overline{Z}(\phi))+f,\end{aligned} \tag{4.9}$$

where $\varepsilon(Z+kZ(\phi))$ *denotes remainder terms of the form* $\sum a_j(Z_j+kZ_j(\phi))$ *with* a_j *smooth, matrix-valued and independent of* k, *for all* j, *and similarly for* $\varepsilon(Z^*+k\overline{Z}(\phi))$ *and* f *is a smooth function independent of* k.

We work with some real local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on D. Let $\xi = (\xi_1, \ldots, \xi_{2n-1})$ denote the dual variables of x. Then (x, ξ) are local coordinates of the cotangent bundle T^*D. Let $q_j(x, \xi)$ be the semi-classical principal symbol of $Z_j + kZ_j(\phi)$, $j = 1, \ldots, n-1$. Then the semi-classical principal symbol of $\Box^{(q)}_{s,k}$ is given by

$$p_0 = \sum_{j=1}^{n-1} \overline{q}_j q_j. \tag{4.10}$$

The characteristic manifold Σ of $\Box^{(q)}_{s,k}$ is

$$\begin{aligned}\Sigma &= \{(x,\xi) \in T^*D;\, p_0(x,\xi) = 0\} \\ &= \big\{(x,\xi) \in T^*D;\, q_1(x,\xi) = \cdots = q_{n-1}(x,\xi) = \overline{q}_1(x,\xi) = \cdots = \overline{q}_{n-1}(x,\xi) = 0\big\}.\end{aligned} \tag{4.11}$$

From (4.11), we see that p_0 vanishes to second order at Σ. We have the following

Proposition 4.2. *We have*

$$\Sigma = \big\{(x,\xi) \in T^*D;\, \xi = \lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x), \lambda \in \mathbb{R}\big\}. \tag{4.12}$$

Proof. For $p \in D$, we may choose local coordinates $x = (x_1, \ldots, x_{2n-1})$, such that $x(p) = 0$, $\omega_0(p) = \sqrt{2}dx_{2n-1}$, $(\frac{\partial}{\partial x_j}(p) \mid \frac{\partial}{\partial x_k}(p)) = 2\delta_{j,k}$, $j, k = 1, \ldots, 2n-1$ and $Z_j = \frac{1}{2}(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}})$ at p, $j = 1, \ldots, n-1$, where Z_j, $j = 1, \ldots, n-1$, are as in (4.9). Then the principal symbol of Z_j is $\frac{1}{2}i(\xi_{2j-1} + i\xi_{2j})$ at p, $j = 1, \ldots, n-1$. Thus

$$q_j(p,\xi) = \frac{1}{2}i(\xi_{2j-1} + i\xi_{2j}) + \frac{1}{2}\big(\frac{\partial\phi}{\partial x_{2j-1}}(p) + i\frac{\partial\phi}{\partial x_{2j}}(p)\big), \tag{4.13}$$

$j = 1, \ldots, n-1$. From (4.13), we see that $(p, \xi) \in \Sigma$ if and only if

$$\xi_{2j-1} = -\frac{\partial\phi}{\partial x_{2j}}(p), \quad \xi_{2j} = \frac{\partial\phi}{\partial x_{2j-1}}(p), \quad j = 1, \ldots, n-1. \tag{4.14}$$

Note that

$$(\overline{\partial}_b\phi)(p) = \sum_{j=1}^{n-1}\Big(\frac{1}{2}\big(\frac{\partial\phi}{\partial x_{2j-1}}(p) + i\frac{\partial\phi}{\partial x_{2j}}(p)\big)(dx_{2j-1} - idx_{2j})\Big).$$

Hence

$$\mathrm{Im}\,(\overline{\partial}_b\phi)(p) = \sum_{j=1}^{n-1}\Big(\frac{1}{2}\big(-\frac{\partial\phi}{\partial x_{2j-1}}(p)dx_{2j} + \frac{\partial\phi}{\partial x_{2j}}(p)dx_{2j-1}\big)\Big). \tag{4.15}$$

From (4.14) and (4.15), the proposition follows. $\square$

Let $\sigma = d\xi \wedge dx$ denote the canonical two form on T^*D. We are therefore interested in whether σ is non-degenerate at $\rho \in \Sigma$. We recall that σ is non-degenerate at $\rho \in \Sigma$ if $\sigma(u, v) = 0$ for all $v \in \mathbb{C}T_\rho\Sigma$, where $u \in \mathbb{C}T_\rho\Sigma$, then $u = 0$.

From now on, for any $f \in C^\infty(T^*D, \mathbb{C})$, we write H_f to denote the Hamilton field of f. That is, in local symplectic coordinates (x, ξ),

$$H_f = \sum_{j=1}^{2n-1}\Big(\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j}\Big).$$

For $f, g \in C^\infty(T^*D, \mathbb{C})$, $\{f, g\}$ denotes the Poisson bracket of f and g. We recall that $\{f, g\} = \sum_{s=1}^{2n-1}(\frac{\partial f}{\partial \xi_s}\frac{\partial g}{\partial x_s} - \frac{\partial f}{\partial x_s}\frac{\partial g}{\partial \xi_s})$.

First, we need

Lemma 4.3. *For $\rho = (p, \lambda_0\omega_0(p) - 2\mathrm{Im}\,\overline{\partial}_b\phi(p)) \in \Sigma$, we have*

$$\sigma(H_{q_j}, H_{q_t})|_\rho = 0, \quad j, t = 1, \ldots, n-1, \tag{4.16}$$

$$\sigma(H_{\overline{q}_j}, H_{\overline{q}_t})|_\rho = 0, \quad j, t = 1, \ldots, n-1, \tag{4.17}$$

and

$$\begin{aligned}\sigma(H_{\overline{q}_j}, H_{q_t})|_\rho = 2i\lambda_0\mathcal{L}_p(\overline{Z}_j, Z_t) + i < [\overline{Z}_j, Z_t](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p) > \\ - i(\overline{Z}_j Z_t + Z_t\overline{Z}_j)\phi(p), \quad j, t = 1, \ldots, n-1,\end{aligned} \tag{4.18}$$

where Z_j, $j = 1, \ldots, n-1$, are as in (4.9) *and q_j is the semi-classical principal symbol of $Z_j + kZ_j(\phi)$, $j = 1, \ldots, n-1$.*

Proof. We write $\rho = (p, \xi_0)$. It is straightforward to see that

$$\sigma(H_{q_j}, H_{q_t})|_\rho = \{q_j, q_t\}(\rho) = - < [Z_j, Z_t](p), \xi_0 > + i[Z_j, Z_t]\phi(p). \tag{4.19}$$

We have

$$\begin{aligned}&< [Z_j, Z_t](p), \xi_0 > = < [Z_j, Z_t](p), \lambda_0\omega_0(p) - 2\mathrm{Im}\,\overline{\partial}_b\phi(p) > \\ &= \lambda_0 < [Z_j, Z_t](p), \omega_0(p) > + i < [Z_j, Z_t](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p) > .\end{aligned} \tag{4.20}$$

Since $[Z_j, Z_t](p) \in T^{0,1}_p X$, we have

$$< [Z_j, Z_t](p), \omega_0(p) > = 0 \tag{4.21}$$

and

$$< [Z_j, Z_t](p), \partial_b\phi(p) > = 0. \tag{4.22}$$

Thus,

$$< [Z_j, Z_t](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p) > = < [Z_j, Z_t](p), \overline{\partial}_b\phi(p) > = [Z_j, Z_t]\phi(p). \tag{4.23}$$

From (4.20), (4.21) and (4.23), we get

$$< [Z_j, Z_t](p), \xi_0 > = i[Z_j, Z_t]\phi(p).$$

Combining this with (4.19), we get (4.16). The proof of (4.17) is the same.

As (4.19), it is straightforward to see that

$$\sigma(H_{\overline{q}_j}, H_{q_t})|_\rho = \left\{\overline{q}_j, q_t\right\}(\rho) = < [\overline{Z}_j, Z_t](p), \xi_0 > - i(\overline{Z}_j Z_t + Z_t\overline{Z}_j)\phi(p), \tag{4.24}$$

where $j, t = 1, \ldots, n-1$. We have

$$\begin{aligned}&< [\overline{Z}_j, Z_t](p), \xi_0 > = < [\overline{Z}_j, Z_t](p), \lambda_0\omega_0(p) - 2\mathrm{Im}\,\overline{\partial}_b\phi(p) > \\ &= \lambda_0 < [\overline{Z}_j, Z_t](p), \omega_0(p) > + i < [\overline{Z}_j, Z_t](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p) > \\ &= 2i\lambda_0\mathcal{L}_p(\overline{Z}_j, Z_t) + i < [\overline{Z}_j, Z_t](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p) > .\end{aligned} \tag{4.25}$$

Combining (4.25) with (4.24), (4.18) follows. □

The following is well-known (see Lemma 4.1 in [15])

Lemma 4.4. *For any $U, V \in T^{1,0}_p X$, pick $\mathcal{U}, \mathcal{V} \in C^\infty(D, T^{1,0}X)$ that satisfy $\mathcal{U}(p) = U$, $\mathcal{V}(p) = V$. Then,*

$$M^\phi_p(U, \overline{V}) = -\big\langle \big[\mathcal{U}, \overline{\mathcal{V}}\,\big](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p)\big\rangle + \big(\mathcal{U}\overline{\mathcal{V}} + \overline{\mathcal{V}}\mathcal{U}\big)\phi(p). \tag{4.26}$$

Now, we can prove

Theorem 4.5. *σ is non-degenerate at $\rho = (p, \lambda_0\omega_0(p) - 2\mathrm{Im}\,\overline{\partial}_b\phi(p)) \in \Sigma$ if and only if the Hermitian quadratic form $M^\phi_p - 2\lambda_0\mathcal{L}_p$ is non-degenerate.*

Proof. Note that

$$\Sigma = \left\{(x, \xi) \in T^*D;\ q_j(x, \xi) = \overline{q}_j(x, \xi) = 0, \quad j = 1, \ldots, n-1\right\}.$$

Let $\mathbb{C}T_\rho\Sigma$ and $\mathbb{C}T_\rho(T^*D)$ be the complexifications of $T_\rho\Sigma$ and $T_\rho(T^*D)$ respectively. Let $T_\rho\Sigma^\perp$ be the orthogonal to $\mathbb{C}T_\rho\Sigma$ in $\mathbb{C}T_\rho(T^*D)$ with respect to the canonical two form σ. We notice that $\dim_{\mathbb{C}} T_\rho\Sigma^\perp = 2n-2$. It is easy to check that

$$\sigma(v, H_{q_j})|_\rho = < dq_j(\rho), v >, \quad \sigma(v, H_{\overline{q}_j})|_\rho = < d\overline{q}_j(\rho), v >,$$

$j = 1, \ldots, n-1$, $v \in \mathbb{C}T_\rho(T^*D)$. Thus, if $v \in \mathbb{C}T_\rho\Sigma$, we get $\sigma(H_{q_j}, v)|_\rho = 0$, $\sigma(H_{\overline{q}_j}, v)|_\rho = 0$, $j = 1, \ldots, n-1$. We conclude that $H_{q_1}, \ldots, H_{q_{n-1}}, H_{\overline{q}_1}, \ldots, H_{\overline{q}_{n-1}}$ is a basis for $T_\rho\Sigma^\perp$.

Now, we assume that $M^\phi_p - 2\lambda_0\mathcal{L}_p$ is non-degenerate. Let $\nu \in \mathbb{C}T_\rho\Sigma\bigcap T_\rho\Sigma^\perp$. We write $\nu = \sum_{j=1}^{n-1}(\alpha_j H_{q_j}(\rho) + \beta_j H_{\overline{q}_j}(\rho))$. Since $\nu \in \mathbb{C}T_\rho\Sigma$, we have

$$\sigma(\nu, H_{q_t})|_\rho = \sigma(\nu, H_{\overline{q}_t})|_\rho = 0,$$

$t = 1, \ldots, n-1$. In view of (4.16), (4.17), (4.18) and (4.26), we see that

$$\begin{aligned}\sigma(\nu, H_{q_t})|_\rho &= \sum_{j=1}^{n-1}\beta_j\Big(2i\lambda_0\mathcal{L}_p(\overline{Z}_j, Z_t) - iM^\phi_p(\overline{Z}_j, Z_t)\Big)\\ &= 2i\lambda_0\mathcal{L}_p(Y, Z_t) - iM^\phi_p(Y, Z_t) = 0,\end{aligned} \tag{4.27}$$

for all $t = 1, \ldots, n-1$, where $Y = \sum_{j=1}^{n-1}\beta_j\overline{Z}_j(p) \in T^{1,0}_pX$. Since $-M^\phi_p + 2\lambda_0\mathcal{L}_p$ is non-degenerate, we get $Y = 0$. Thus, $\beta_j = 0$, $j = 1, \ldots, n-1$. Similarly, we can repeat the process above to show that $\alpha_j = 0$, $j = 1, \ldots, n-1$. We conclude that $\mathbb{C}T_\rho\Sigma\bigcap T_\rho\Sigma^\perp = 0$. Hence σ is non-degenerate at ρ.

Conversely, we assume that σ is non-degenerate at ρ. If for some $Y \in T^{1,0}_pX$, we have $M^\phi_p(Y, \overline{Z}) - 2\lambda_0\mathcal{L}_p(Y, \overline{Z}) = 0$ for all $Z \in T^{1,0}_pX$. We write $Y = \sum_{j=1}^{n-1}\beta_j\overline{Z}_j(p)$. As before, we can show that $\sigma(\sum_{j=1}^{n-1}\beta_j H_{\overline{q}_j}, H_{q_t})|_\rho = 0$ and $\sigma(\sum_{j=1}^{n-1}\beta_j H_{\overline{q}_j}, H_{\overline{q}_t})|_\rho = 0$ for all $t = 1, \ldots, n-1$. Thus, $\sum_{j=1}^{n-1}\beta_j H_{\overline{q}_j} \in (T_\rho\Sigma^\perp)^\perp = \mathbb{C}T_\rho\Sigma$. Hence, $\sum_{j=1}^{n-1}\beta_j H_{\overline{q}_j} \in \mathbb{C}T_\rho\Sigma\bigcap T_\rho\Sigma^\perp$. Since σ is non-degenerate at ρ, we get $\beta_j = 0$, $j = 1, \ldots, n-1$. Thus, $M^\phi_p - 2\lambda_0\mathcal{L}_p$ is non-degenerate. The theorem follows. $\square$

5. The heat equation for the local operatot $\Box^{(q)}_s$

In this section, we will introduce the local operator $\Box^{(q)}_s$. The goal of this section is to find a microlocal partial inverse and an approximate kernel for $\Box^{(q)}_s$ in some non-degenerate part of the characteristic manifold of $\Box^{(q)}_s$. In the next section, we will reduce the semi-classical analysis of the Kohn Laplacian $\Box^{(q)}_{b,k}$ to the microlocal analysis of the local operator $\Box^{(q)}_s$.

5.1. $\Box^{(q)}_s$ and the eikonal equation for $\Box^{(q)}_s$.

We first introduce some notations. Let Ω be an open set in $\mathbb{R}^N$ and let f, g be positive continuous functions on Ω. We write $f \asymp g$ if for every compact set $K \subset \Omega$ there is a constant $c_K > 0$ such that $f \leq c_K g$ and $g \leq c_K f$ on K.

Let s be a local trivializing section of L on an open subset $D \Subset X$ and $|s|^2_{h^L} = e^{-2\phi}$. In this section, we work with some real local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on D. We write $\xi = (\xi_1, \ldots, \xi_{2n-1})$ or $\eta = (\eta_1, \ldots, \eta_{2n-1})$ to denote the dual coordinates of x. We consider the domain $\hat{D} := D \times \mathbb{R}$. We write $\hat{x} := (x, x_{2n}) = (x_1, x_2, \ldots, x_{2n-1}, x_{2n})$ to denote the coordinates of $D \times \mathbb{R}$, where x_{2n} is the coordinate of $\mathbb{R}$. We write $\hat{\xi} := (\xi, \xi_{2n})$ or $\hat{\eta} := (\eta, \eta_{2n})$ to denote the dual coordinates of $\hat{x}$, where ξ_{2n} and η_{2n} denote the dual coordinate of x_{2n}. We shall use the following notations: $< x, \eta > := \sum_{j=1}^{2n-1}x_j\eta_j$, $< x, \xi > := \sum_{j=1}^{2n-1}x_j\xi_j$, $< \hat{x}, \hat{\eta} > := \sum_{j=1}^{2n}x_j\eta_j$, $< \hat{x}, \hat{\xi} > := \sum_{j=1}^{2n}x_j\xi_j$.

Let $T^{*0,q}\hat{D}$ be the bundle with fiber

$$T^{*0,q}_{\hat{x}}\hat{D} := \big\{u \in T^{*0,q}_xD, \hat{x} = (x, x_{2n})\big\}$$

at $\hat{x} \in \hat{D}$. From now on, for every point $\hat{x} = (x, x_{2n}) \in \hat{D}$, we identify $T^{*0,q}_{\hat{x}}\hat{D}$ with $T^{*0,q}_xX$. Let $\langle\,\cdot\,|\,\cdot\,\rangle$ be the Hermitian metric on $\mathbb{C}T^*\hat{D}$ given by $\langle\,\hat{\xi}\,|\,\hat{\eta}\,\rangle = \langle\,\xi\,|\,\eta\,\rangle + \xi_{2n}\overline{\eta_{2n}}$, $(\hat{x}, \hat{\xi}), (\hat{x}, \hat{\eta}) \in \mathbb{C}T^*\hat{D}$. Let $\Omega^{0,q}(\hat{D})$ denote the space of smooth sections of $T^{*0,q}\hat{D}$ over $\hat{D}$ and put $\Omega^{0,q}_0(\hat{D}) := \Omega^{0,q}(\hat{D})\bigcap\mathscr{E}'(\hat{D}, T^{*0,q}\hat{D})$. Using the identification

$$ku(x) = e^{-ikx_{2n}}(-i\frac{\partial}{\partial x_{2n}}(e^{ikx_{2n}}u(x)), \quad \forall u \in \Omega^{0,q}(D),$$

we consider the following operators

$$\begin{aligned}&\overline{\partial}_s : \Omega^{0,r}(\hat{D}) \to \Omega^{0,r+1}(\hat{D}), \ \ \overline{\partial}_{s,k}u = e^{-ikx_{2n+1}}\overline{\partial}_s(ue^{ikx_{2n}}), \quad \forall u \in \Omega^{0,r}(D),\\ &\overline{\partial}^*_s : \Omega^{0,r+1}(\hat{D}) \to \Omega^{0,r}(\hat{D}), \ \ \overline{\partial}^*_{s,k}u = e^{-ikx_{2n+1}}\overline{\partial}^*_s(ue^{ikx_{2n}}), \quad \forall u \in \Omega^{0,r+1}(D),\end{aligned} \tag{5.1}$$

where $r = 0, 1, \ldots, n-1$ and $\overline{\partial}_{s,k}$, $\overline{\partial}^*_{s,k}$ are given by (4.1). From (4.5) and (4.6), it is easy to see that

$$\begin{aligned}\overline{\partial}_s &= \sum_{j=1}^{n-1}\Big(e_j^\wedge \circ (Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}}) + (\overline{\partial}_b e_j)^\wedge e_j^{\wedge,*}\Big),\\ \overline{\partial}^*_s &= \sum_{j=1}^{n-1}\Big(e_j^{\wedge,*} \circ (Z_j^* - i\overline{Z}_j(\phi)\frac{\partial}{\partial x_{2n}}) + e_j^\wedge(\overline{\partial}_b e_j)^{\wedge,*}\Big),\end{aligned} \tag{5.2}$$

where $Z_1, \ldots, Z_{n-1}$, $Z_1^*, \ldots, Z_{n-1}^*$ and $e_1, \ldots, e_{n-1}$ are as in Proposition 4.1. Put

$$\Box_s^{(q)} := \overline{\partial}_s\overline{\partial}^*_s + \overline{\partial}^*_s\overline{\partial}_s : \Omega^{0,q}(\hat{D}) \to \Omega^{0,q}(\hat{D}). \tag{5.3}$$

From (5.1), we have

$$\Box_{s,k}^{(q)} u = e^{-ikx_{2n+1}}\Box_s^{(q)}(ue^{ikx_{2n}}), \quad \forall u \in \Omega^{0,q}(D), \tag{5.4}$$

where $\Box_{s,k}^{(q)}$ is given by (4.4).

Let $u(x) \in \Omega_0^{0,q}(\hat{D})$. Note that

$$k\int e^{-ikx_{2n}}u(x)dx_{2n} = \int i\frac{\partial}{\partial x_{2n}}(e^{-ikx_{2n}})u(x)dx_{2n} = \int e^{-ikx_{2n}}\big(-i\frac{\partial u}{\partial x_{2n}}(x)\big)dx_{2n}.$$

From this observation and explicit formulas of $\overline{\partial}_{s,k}$, $\overline{\partial}^*_{s,k}$, $\overline{\partial}_s$ and $\overline{\partial}^*_s$ (see (4.5), (4.6) and (5.2)), we conclude that

$$\Box_{s,k}^{(q)}\big(\int e^{-ikx_{2n}}u(x)dx_{2n}\big) = \int e^{-ikx_{2n}}(\Box_s^{(q)}u)(x)dx_{2n}, \tag{5.5}$$

for all $u(x) \in \Omega_0^{0,q}(\hat{D})$.

From (5.2) and (5.3), we can repeat the proof of Proposition 4.1 and conclude that

Proposition 5.1. *With the notations used before, we have*

$$\begin{aligned}\Box_s^{(q)} &= \overline{\partial}_s\overline{\partial}^*_s + \overline{\partial}^*_s\overline{\partial}_s\\ &= \sum_{j=1}^{n-1}(Z_j^* - i\overline{Z}_j(\phi)\frac{\partial}{\partial x_{2n}})(Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}})\\ &\quad + \sum_{j,t=1}^{n-1} e_j^\wedge e_t^{\wedge,*} \circ [Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}}, Z_t^* - i\overline{Z}_t(\phi)\frac{\partial}{\partial x_{2n}}]\\ &\quad + \varepsilon(Z - iZ(\phi)\frac{\partial}{\partial x_{2n}}) + \varepsilon(Z^* - i\overline{Z}(\phi)\frac{\partial}{\partial x_{2n}}) + \textit{zero order terms},\end{aligned} \tag{5.6}$$

where $\varepsilon(Z - iZ(\phi)\frac{\partial}{\partial x_{2n}})$ denotes remainder terms of the form $\sum a_j(Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}})$ with a_j smooth, matrix-valued, for all j, and similarly for $\varepsilon(Z^* - i\overline{Z}(\phi)\frac{\partial}{\partial x_{2n}})$.

In this section, we will study the heat equation $\partial_t + \Box_s^{(q)}$. Until further notice, we fix $q \in \{0, 1, \ldots, n-1\}$. First, we consider the problem

$$\begin{cases}(\partial_t + \Box_s^{(q)})u(t,\hat{x}) = 0 & \text{in } \mathbb{R}_+ \times \hat{D},\\ u(0,\hat{x}) = v(\hat{x}).\end{cases} \tag{5.7}$$

We need

Definition 5.2. Let $0 \le q_1 \le n-1$, $q_1 \in \mathbb{N}_0$. We say that $a(t,\hat{x},\hat{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*\hat{D}, T^{*0,q_1}\hat{D} \boxtimes T^{*0,q}\hat{D})$ is quasi-homogeneous of degree j if $a(t,\hat{x},\lambda\hat{\eta}) = \lambda^j a(\lambda t,\hat{x},\hat{\eta})$ for all $\lambda > 0$.

Definition 5.3. Let $0 \le q_1 \le n-1$, $q_1 \in \mathbb{N}_0$. We say that $b(\hat{x},\hat{\eta}) \in C^\infty(T^*\hat{D}, T^{*0,q_1}\hat{D} \boxtimes T^{*0,q}\hat{D})$ is positively homogeneous of degree j if $b(\hat{x},\lambda\hat{\eta}) = \lambda^j b(\hat{x},\hat{\eta})$ for all $\lambda > 0$.

Let $0 \leq q_1 \leq n-1$, $q_1 \in \mathbb{N}_0$. We look for an approximate solution of (5.7) of the form $u(t,\hat{x}) = A(t)v(\hat{x})$,

$$A(t)v(\hat{x}) = \frac{1}{(2\pi)^{2n}} \iint e^{i(\psi(t,\hat{x},\hat{\eta}) - <\hat{y},\hat{\eta}>)} a(t,\hat{x},\hat{\eta}) v(\hat{y}) d\hat{y} d\hat{\eta} \tag{5.8}$$

where formally $a(t,\hat{x},\hat{\eta}) \sim \sum_{j=0}^{\infty} a_j(t,\hat{x},\hat{\eta})$, $a_j(t,\hat{x},\hat{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*\hat{D}, T^{*0,q_1}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $a_j(t,\hat{x},\hat{\eta})$ is a quasi-homogeneous function of degree $-j$. The phase $\psi(t,\hat{x},\hat{\eta})$ should solve the eikonal equation

$$\begin{cases} \dfrac{\partial \psi}{\partial t} - i\hat{p}_0(\hat{x},\psi'_{\hat{x}}) = O(|\mathrm{Im}\,\psi|^N), \quad \forall N \geq 0, \\ \psi|_{t=0} = <\hat{x},\hat{\eta}> \end{cases} \tag{5.9}$$

with $\mathrm{Im}\,\psi \geq 0$, where $\hat{p}_0$ denotes the principal symbol of $\Box_s^{(q)}$. From (5.6), we have

$$\hat{p}_0 = \sum_{j=1}^{n-1} \overline{\hat{q}}_j \hat{q}_j, \tag{5.10}$$

where $\hat{q}_j$ is the principal sympol of $Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}}$, $j = 1,\ldots,n-1$. The characteristic manifold $\hat{\Sigma}$ of $\Box_s^{(q)}$ is given by

$$\hat{\Sigma} = \left\{(\hat{x},\hat{\xi}) \in T^*\hat{D};\ \hat{q}_1(\hat{x},\hat{\xi}) = \cdots = \hat{q}_{n-1}(\hat{x},\hat{\xi}) = \overline{\hat{q}}_1(\hat{x},\hat{\xi}) = \cdots = \overline{\hat{q}}_{n-1}(\hat{x},\hat{\xi}) = 0\right\}. \tag{5.11}$$

From (5.11), we see that $\hat{p}_0$ vanishes to second order at $\hat{\Sigma}$. Let $\hat{\sigma}$ denote the canonical two form on $T^*\hat{D}$. We can repeat the proofs of Proposition 4.2 and Theorem 4.5 with minor changes and conclude that

Theorem 5.4. *We have*

$$\hat{\Sigma} = \left\{(\hat{x},\hat{\xi}) \in T^*\hat{D};\ \hat{\xi} = (\lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n}, \xi_{2n}), \lambda \in \mathbb{R}\right\}. \tag{5.12}$$

Moreover, $\hat{\sigma}$ is non-degenerate at $\hat{\rho} = ((p,x_{2n}),(\lambda_0\omega_0(p) - 2\mathrm{Im}\,\overline{\partial}_b\phi(p),\xi_{2n})) \in \hat{\Sigma}$ if and only if the Hermitian quadratic form $\xi_{2n}M_p^\phi - 2\lambda_0\mathcal{L}_p$ is non-degenerate.

Until further notice, we assume that

$$\begin{gathered}\text{there exist } x_0 \in D \text{ and } \lambda_0 \in \mathbb{R} \text{ such that } M_{x_0}^\phi - 2\lambda_0\mathcal{L}_{x_0} \text{ is non-degenerate} \\ \text{of constant signature } (n_-,n_+) \text{ at each point of } D.\end{gathered} \tag{5.13}$$

Let V be a bounded open set of T^*D with $\overline{V} \subset T^*D$ and

$$\overline{V} \bigcap \Sigma \subset \Sigma', \tag{5.14}$$

where Σ' is given by (1.5). Put

$$U = \left\{(\hat{x},\hat{\xi}) \in T^*\hat{D};\ \hat{\xi} = (\xi_{2n}\xi,\xi_{2n}), (x,\xi) \in V, \xi_{2n} > 0\right\}. \tag{5.15}$$

U is a conic open set of $T^*\hat{D}$ and

$$\begin{aligned} U \bigcap \hat{\Sigma} \subset &\{(\hat{x},(\lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n},\xi_{2n}));\ \xi_{2n}M_x^\phi - 2\lambda\mathcal{L}_x \text{ is non-degenerate} \\ &\text{of constant signature } (n_-,n_+)\}.\end{aligned} \tag{5.16}$$

Since $\overline{V} \bigcap \Sigma \Subset \Sigma'$, it is not difficult to see that there is a constant $\mu > 0$ such that

$$\inf\{|\lambda|;\ \lambda\text{: eigenvalue of } \xi_{2n}M_x^\phi - 2\lambda\mathcal{L}_x,\ (\hat{x},\hat{\xi}) \in U \bigcap \hat{\Sigma}\} \geq \mu\xi_{2n}. \tag{5.17}$$

Until further notice, we work in U. Since $\hat{\sigma}$ is non-degenerate at each point of $U \bigcap \hat{\Sigma}$, (5.9) can be solved with $\mathrm{Im}\,\psi \geq 0$ on U. More precisely, we have the following

Theorem 5.5. *There exists $\psi(t,\hat{x},\hat{\eta})\in C^\infty(\overline{\mathbb{R}}_+\times U)$ such that $\psi(t,\hat{x},\hat{\eta})$ is quasi-homogeneous of degree 1 and $\mathrm{Im}\,\psi\geq 0$ and such that (5.9) holds where the error term is uniform on every set of the form $[0,T]\times K$ with $T>0$ and $K\subset U$ compact. Furthermore, ψ is unique up to a term which is $O(|\mathrm{Im}\,\psi|^N)$ locally uniformly for every N and*

$$\begin{aligned}&\psi(t,\hat{x},\hat{\eta})=<\hat{x},\hat{\eta}> \text{ on } \hat{\Sigma}\bigcap U,\\ &d_{\hat{x},\hat{\eta}}(\psi-<\hat{x},\hat{\eta}>)=0 \text{ on } \hat{\Sigma}\bigcap U.\end{aligned}\tag{5.18}$$

Moreover, we have

$$\mathrm{Im}\,\psi(t,\hat{x},\hat{\eta})\asymp\Big(|\hat{\eta}|\,\frac{t\,|\hat{\eta}|}{1+t\,|\eta|}\Big)\Big(\mathrm{dist}\,\big((\hat{x},\frac{\hat{\eta}}{|\hat{\eta}|}),\hat{\Sigma}\big)\Big)^2,\quad t\geq 0,\quad (\hat{x},\hat{\eta})\in U.\tag{5.19}$$

Theorem 5.6. *There exists a function $\psi(\infty,\hat{x},\hat{\eta})\in C^\infty(U)$ with a uniquely determined Taylor expansion at each point of $\hat{\Sigma}\bigcap U$ such that $\psi(\infty,\hat{x},\hat{\eta})$ is positively homogeneous of degree 1 and for every compact set $K\subset U$ there is a $c_K>0$ such that $\mathrm{Im}\,\psi(\infty,\hat{x},\hat{\eta})\geq c_K\,|\hat{\eta}|\Big(\mathrm{dist}\,\big((\hat{x},\frac{\hat{\eta}}{|\hat{\eta}|}),\hat{\Sigma}\big)\Big)^2$, $d_{\hat{x},\hat{\eta}}(\psi(\infty,\hat{x},\hat{\eta})-<\hat{x},\hat{\eta}>)=0$ on $U\bigcap\hat{\Sigma}$. If $\lambda\in C(U)$, $\lambda>0$ and $\lambda(\hat{x},\hat{\xi})<\min\lambda_j(\hat{x},\hat{\xi})$, for all $(\hat{x},\hat{\xi})=(\hat{x},(\lambda\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n},\xi_{2n}))\in\hat{\Sigma}\bigcap U$, where $\lambda_j(\hat{x},\hat{\xi})$ are the eigenvalues of the Hermitian quadratic form $\xi_{2n}M^\phi_x-2\lambda\mathcal{L}_x$, then the solution $\psi(t,\hat{x},\hat{\eta})$ of (5.9) can be chosen so that for every compact set $K\subset U$ and all indices α, β, γ, there is a constant $c_{\alpha,\beta,\gamma,K}>0$ such that*

$$\left|\partial^\alpha_{\hat{x}}\partial^\beta_{\hat{\eta}}\partial^\gamma_t(\psi(t,\hat{x},\hat{\eta})-\psi(\infty,\hat{x},\hat{\eta}))\right|\leq c_{\alpha,\beta,\gamma,K}e^{-\lambda(\hat{x},\hat{\eta})t}\ \text{ on } \overline{\mathbb{R}}_+\times K.\tag{5.20}$$

For the proofs of Theorem 5.5 and Theorem 5.6, we refer the reader to Menikoff-Sjöstrand [22] and [14]. Put

$$\mathrm{diag}'\Big((U\bigcap\hat{\Sigma})\times(U\bigcap\hat{\Sigma})\Big):=\Big\{(\hat{x},\hat{x},\hat{\xi},-\hat{\xi});\,(\hat{x},\hat{\xi})\in U\bigcap\hat{\Sigma}\Big\}.\tag{5.21}$$

We also need the following which is also well-known (see Proposition 3.5 of part I in [14])

Theorem 5.7. *The two phases $\psi(\infty,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>$, $-\overline{\psi}(\infty,\hat{y},\hat{\eta})+<\hat{x},\hat{\eta}>$ are equivalent for classical symbols at every point of $\mathrm{diag}'\Big((U\bigcap\hat{\Sigma})\times(U\bigcap\hat{\Sigma})\Big)$ in the sense of Melin-Sjöstrand [21].*

From (5.6), we can check that the principal symbol $\hat{p}_0$ has the following property:

$$\hat{p}_0((x,x_{2n}+\alpha),\hat{\eta})=\hat{p}_0(\hat{x},\hat{\eta}),\quad\forall\alpha\in\mathbb{R}.\tag{5.22}$$

Fix $\alpha\in\mathbb{R}$, we consider

$$\widetilde{\psi}(t,\hat{x},\hat{\eta}):=\psi(t,(x,x_{2n}+\alpha),\hat{\eta})-\alpha\eta_{2n}.$$

From (5.22), it is not difficult to see that $\widetilde{\psi}(t,\hat{x},\hat{\eta})$ solves (5.9). From Theorem 5.5, we see that

$$\widetilde{\psi}(t,\hat{x},\hat{\eta})-\psi(t,\hat{x},\hat{\eta})=\big(\psi(t,(x,x_{2n}+\alpha),\eta)-(x_{2n}+\alpha)\eta_{2n}\big)-\big(\psi(t,\hat{x},\hat{\eta})-x_{2n}\eta_{2n}\big)$$

vanishes to infinite order at $\hat{\Sigma}\bigcap U$, for all $\alpha\in\mathbb{R}$. This means that the Taylor expansions of $\psi(t,\hat{x},\hat{\eta})-x_{2n}\eta_{2n}$ at $\hat{\Sigma}\bigcap U$ do not depend on x_{2n}. Thus, $\frac{\partial\psi}{\partial x_{2n}}(t,\hat{x},\hat{\eta})-\hat{\eta}_{2n}$ vanishes to infinite order at $\hat{\Sigma}\bigcap U$. We conclude that $\psi(t,\hat{x},\hat{\eta})-\big(\psi(t,(x,0),\hat{\eta})+x_{2n}\eta_{2n}\big)$ vanishes to infinite order at $\hat{\Sigma}\bigcap U$. Since we only need to consider Taylor expansions at $\hat{\Sigma}\bigcap U$, from now on, we assume that

$$\psi(t,\hat{x},\hat{\eta})=\psi(t,(x,0),\hat{\eta})+x_{2n}\eta_{2n}.\tag{5.23}$$

Thus,

$$\psi(\infty,\hat{x},\hat{\eta})=\psi(\infty,(x,0),\hat{\eta})+x_{2n}\eta_{2n}.\tag{5.24}$$

5.2. **The transport equations for $\Box_s^{(q)}$.** We let the full symbol of $\Box_s^{(q)}$ be:

$$\text{full symbol of } \Box_s^{(q)} = \sum_{j=0}^{2} \hat{p}_j(\hat{x}, \hat{\xi}),$$

where $\hat{p}_j(\hat{x}, \hat{\xi})$ is positively homogeneous of order $2-j$. We apply $\partial_t + \Box_s^{(q)}$ formally under the integral in (5.8) and then introduce the asymptotic expansion of $\Box_s^{(q)}(ae^{i\psi})$ (see page 148 of [21]). Setting $(\partial_t + \Box_s^{(q)})(ae^{i\psi}) \sim 0$ and regrouping the terms according to the degree of quasi-homogeneity. We obtain the transport equations

$$\begin{cases} T(t, \hat{x}, \hat{\eta}, \partial_t, \partial_{\hat{x}})a_0 = O(\lvert \operatorname{Im} \psi \rvert^N), \ \forall N, \\ T(t, \hat{x}, \hat{\eta}, \partial_t, \partial_{\hat{x}})a_j + R_j(t, \hat{x}, \hat{\eta}, a_0, \ldots, a_{j-1}) = O(\lvert \operatorname{Im} \psi \rvert^N), \ \forall N. \end{cases} \tag{5.25}$$

Here

$$T(t, \hat{x}, \hat{\eta}, \partial_t, \partial_{\hat{x}}) = \partial_t - i \sum_{j=1}^{2n} \frac{\partial \hat{p}_0}{\partial \xi_j}(\hat{x}, \psi'_{\hat{x}}) \frac{\partial}{\partial x_j} + q(t, \hat{x}, \hat{\eta})$$

where

$$q(t, \hat{x}, \hat{\eta}) = \hat{p}_1(\hat{x}, \psi'_{\hat{x}}) + \frac{1}{2i} \sum_{j,t=1}^{2n} \frac{\partial^2 \hat{p}_0(\hat{x}, \psi'_{\hat{x}})}{\partial \xi_j \partial \xi_t} \frac{\partial^2 \psi(t, \hat{x}, \hat{\eta})}{\partial x_j \partial x_t}$$

and R_j is a linear differential operator acting on $a_0, a_1, \ldots, a_{j-1}$. We note that $q(t, \hat{x}, \hat{\eta}) \to q(\infty, \hat{x}, \hat{\eta})$ exponentially fast in the sense of (5.20) and the same is true for the coefficients of R_j, for all j.

We pause and introduce some notations. The subprincipal symbol of $\Box_s^{(q)}$ at $(\hat{x}, \hat{\xi}) \in \hat{\Sigma}$ is given by

$$\hat{p}_0^s(\hat{x}, \hat{\xi}) = \hat{p}_1(\hat{x}, \hat{\xi}) + \frac{i}{2} \sum_{j=1}^{2n} \frac{\partial^2 \hat{p}_0(\hat{x}, \hat{\xi})}{\partial \hat{x}_j \partial \hat{\xi}_j} \in T_{\hat{x}}^{*0,q} \hat{D} \boxtimes T_{\hat{x}}^{*0,q} \hat{D}. \tag{5.26}$$

Since $\hat{\Sigma}$ is doubly characteristic, it is well-known that the subprincipal symbol of $\Box_s^{(q)}$ is invariantly defined on $\hat{\Sigma}$ (see page 83 in Hörmander [12]). The fundamental matrix of $\hat{p}_0$ at $\hat{\rho} \in \hat{\Sigma}$ is the linear map $F(\hat{\rho})$ on $T_{\hat{\rho}}(T^* \hat{D})$ defined by

$$\hat{\sigma}(t, F(\rho)s) = < t, \hat{p}_0''(\hat{\rho})s >, \quad t, s \in T_{\hat{\rho}}(T^* \hat{D}), \tag{5.27}$$

where $\hat{p}_0''(\hat{\rho}) = \begin{pmatrix} \frac{\partial^2 \hat{p}_0}{\partial \hat{x} \partial \hat{x}}(\hat{\rho}) & \frac{\partial^2 \hat{p}_0}{\partial \hat{\xi} \partial \hat{x}}(\hat{\rho}) \\ \frac{\partial^2 \hat{p}_0}{\partial \hat{x} \partial \hat{\xi}}(\hat{\rho}) & \frac{\partial^2 \hat{p}_0}{\partial \hat{\xi} \partial \hat{\xi}}(\hat{\rho}) \end{pmatrix}$. For $\hat{\rho} \in \hat{\Sigma}$, let $\widetilde{\operatorname{tr}} F(\hat{\rho}) := \sum \lvert \mu_j \rvert$, where $\pm i\mu_j$ are non-vanishing eigenvalues of $F(\hat{\rho})$. For $\hat{\rho} = (\hat{x}, \hat{\xi}) \in \hat{\Sigma} \bigcap U$, put

$$\inf(\hat{p}_0^s(\hat{\rho}) + \frac{1}{2} \widetilde{\operatorname{tr}} F(\hat{\rho})) = \inf \left\{ \lambda;\ \lambda\text{: eigenvalue of } \hat{p}_0^s(\hat{\rho}) + \tfrac{1}{2} \widetilde{\operatorname{tr}} F(\hat{\rho}) \right\} \tag{5.28}$$

and set

$$N(\hat{p}_0^s(\hat{\rho}) + \frac{1}{2} \widetilde{\operatorname{tr}} F(\hat{\rho})) = \left\{ u \in T_{\hat{x}}^{*0,q} \hat{D};\ (\hat{p}_0^s(\hat{\rho}) + \frac{1}{2} \widetilde{\operatorname{tr}} F(\hat{\rho}))u = 0 \right\}. \tag{5.29}$$

We return to our situation. We can repeat the proof of Proposition 4.3 in part I of [14] with minor changes and obtain the following

Theorem 5.8. *Let $0 \leq q_1 \leq n-1$, $q_1 \in \mathbb{N}_0$. Let $c_j(\hat{x}, \hat{\eta}) \in C^\infty(U, T^{*0,q_1} \hat{D} \boxtimes T^{*0,q} \hat{D})$, $j = 0, 1, \ldots$, be positively homogeneous functions of degree $m - j$, $m \in \mathbb{Z}$. Then, we can find solutions $a_j(t, \hat{x}, \hat{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times T^* \hat{D}, T^{*0,q_1} \hat{D} \boxtimes T^{*0,q} \hat{D})$, $j = 0, 1, \ldots$, of the system (5.25) with $a_j(0, \hat{x}, \hat{\eta}) = c_j(\hat{x}, \hat{\eta})$, $j = 0, 1, \ldots$, where $a_j(t, \hat{x}, \hat{\eta})$ is a quasi-homogeneous function of degree $m - j$ such that $a_j(t, \hat{x}, \hat{\eta})$ has unique Taylor expansions on $\hat{\Sigma}$, for all j. Furthermore, let $\lambda(\hat{x}, \hat{\eta}) \in C(U)$ and $\lambda(\hat{x}, \hat{\eta}) < \inf(\hat{p}_0^s(\hat{x}, \hat{\eta}) + \frac{1}{2} \widetilde{\operatorname{tr}} F(\hat{x}, \hat{\eta}))$, for all $(\hat{x}, \hat{\eta}) \in \hat{\Sigma} \bigcap U$. Then for all indices α, β, γ, j and every compact set $K \Subset \hat{\Sigma} \bigcap U$ there exists a constant $c > 0$ such that*

$$\left\lvert \partial_t^\gamma \partial_{\hat{x}}^\alpha \partial_{\hat{\eta}}^\beta a_j(t, \hat{x}, \hat{\eta}) \right\rvert \leq c e^{-t\lambda(\hat{x}, \hat{\eta})} \text{ on } \overline{\mathbb{R}}_+ \times K. \tag{5.30}$$

Moreover, for every $\hat\rho_0=(\hat x_0,\hat\eta_0)\in\hat\Sigma\bigcap U$,

$$\begin{aligned}&\lim_{t\to\infty}a_0(t,\hat x_0,\hat\eta_0)\ \textit{exists},\\ &\big(\lim_{t\to\infty}a_0(t,\hat x_0,\hat\eta_0)\big)u\in N(\hat p^s_0(\hat\rho)+\frac12\widetilde{\operatorname{tr}}\,F(\hat\rho)),\quad\forall u\in T^{*0,q}_{\hat x}\hat D.\end{aligned}\tag{5.31}$$

We are therefore interested in whether $\inf(\hat p^s_0(\hat\rho)+\frac12\widetilde{\operatorname{tr}}\,F(\hat\rho))>0$, $\hat\rho\in U\bigcap\hat\Sigma$. We have the following

Theorem 5.9. *If* $q=n_-$, *then for all* $(\hat x,\hat\xi)\in\hat\Sigma\bigcap U$, *we have*

$$\inf(\hat p^s_0(\hat x,\hat\xi)+\frac12\widetilde{\operatorname{tr}}\,F(\hat x,\hat\xi))=0.\tag{5.32}$$

If $q\neq n_-$, *then there is a constant* $\mu_0>0$ *such that for all* $(\hat x,\hat\xi)\in\hat\Sigma\bigcap U$, *we have*

$$\inf(\hat p^s_0(\hat x,\hat\xi)+\frac12\widetilde{\operatorname{tr}}\,F(\hat x,\hat\xi))>\mu_0\xi_{2n}.\tag{5.33}$$

Proof. First, we compute the subprincipal symbol $\hat p^s_0(\hat\rho)$, $\hat\rho\in\hat\Sigma$. For an operator of the form $(Z_j^*-i\overline Z_j(\phi)\frac{\partial}{\partial x_{2n}})(Z_j-iZ_j(\phi)\frac{\partial}{\partial x_{2n}})$ this subprincipal symbol is given by $-\frac{1}{2i}\{\hat q_j,\overline{\hat q}_j\}$ and the contribution from the double sum in (5.6) to the subprincipal symbol of $\Box^{(q)}_s$ is $\frac1i\sum_{j,t=1}^{n-1}e^\wedge_je^{\wedge,*}_t\circ\{\hat q_j,\overline{\hat q}_t\}$, where $\hat q_j$ is the principal symbol of $Z_j-iZ_j(\phi)\frac{\partial}{\partial x_{2n}}$ and $\{\hat q_j,\overline{\hat q}_t\}$ denotes the Poisson bracket of $\hat q_j$ and $\overline{\hat q}_t$. We recall that $\{\hat q_j,\overline{\hat q}_t\}=\sum_{s=1}^{2n}(\frac{\partial\hat q_j}{\partial\xi_s}\frac{\partial\overline{\hat q}_t}{\partial x_s}-\frac{\partial\hat q_j}{\partial x_s}\frac{\partial\overline{\hat q}_t}{\partial\xi_s})$. We get the subprincipal symbol of $\Box^{(q)}_s$ on $\hat\Sigma$, $\hat p^s_0=\sum_{j=1}^{n-1}-\frac{1}{2i}\{\hat q_j,\overline{\hat q}_j\}+\sum_{j,t=1}^{n-1}e^\wedge_je^{\wedge,*}_t\frac1i\{\hat q_j,\overline{\hat q}_t\}$. For $\hat\rho=(\hat x,(\lambda\omega_0(x)-2\mathrm{Im}\,\overline\partial_b\phi(x)\xi_{2n},\xi_{2n}))\in\hat\Sigma$, from the proof of Lemma 4.3, we see that

$$\{\hat q_j,\overline{\hat q_t}\}(\hat\rho)=iM^\phi_x(\overline Z_j,Z_t)\xi_{2n}-2i\lambda\mathcal L_x(\overline Z_j,Z_t).\tag{5.34}$$

Thus,

$$\begin{aligned}\hat p^s_0(\hat\rho)=&\sum_{j=1}^{n-1}-\frac12\Big(M^\phi_x(\overline Z_j,Z_j)\xi_{2n}-2\lambda\mathcal L_x(\overline Z_j,Z_j)\Big)\\ &+\sum_{j,t=1}^{n-1}e^\wedge_je^{\wedge,*}_t\Big(M^\phi_x(\overline Z_j,Z_t)\xi_{2n}-2\lambda\mathcal L_x(\overline Z_j,Z_t)\Big),\end{aligned}\tag{5.35}$$

for all $\hat\rho=(\hat x,\hat\xi)=(\hat x,(\lambda\omega_0(x)-2\mathrm{Im}\,\overline\partial_b\phi(x)\xi_{2n},\xi_{2n}))\in\hat\Sigma$.

Now, we compute the fundamental matrix F of $\hat p_0$ at $\hat\rho\in\hat\Sigma$. From now on, for any $f\in C^\infty(T^*\hat D)$, we write H_f to denote the Hamilton field of f. We can choose the basis $H_{\hat q_1},\ldots,H_{\hat q_{n-1}},H_{\overline{\hat q}_1},\ldots,H_{\overline{\hat q}_{n-1}}$ for $T_\rho\hat\Sigma^\perp$, where $T_\rho\hat\Sigma^\perp$ is the orthogonal to $\mathbb CT_\rho\hat\Sigma$ in $\mathbb CT_\rho(T^*\hat D)$ with respect to canonical two form $\hat\sigma$. Since $\hat p_0=\sum_{j=1}^{n-1}\overline{\hat q}_j\hat q_j$, we have $H_{\hat p_0}=\sum_{j=1}^{n-1}\Big(\overline{\hat q}_jH_{\hat q_j}+\hat q_jH_{\overline{\hat q}_j}\Big)$. We compute the linearization of $H_{\hat p_0}$ at $\hat\rho$

$$H_{\hat p_0}\Big(\hat\rho+\sum(t_kH_{\hat q_k}+s_kH_{\overline{\hat q}_k})\Big)=O(|t,s|^2)+\sum_{j,k}t_k\{\hat q_k,\overline{\hat q}_j\}H_{\hat q_j}+\sum_{j,k}s_k\{\overline{\hat q}_k,\hat q_j\}H_{\overline{\hat q}_j}.$$

So $F(\hat\rho)$ is expressed in the basis $H_{\hat q_1},\ldots,H_{\hat q_{n-1}},H_{\overline{\hat q}_1},\ldots,H_{\overline{\hat q}_{n-1}}$ by

$$F(\hat\rho)=\begin{pmatrix}\{\hat q_t,\overline{\hat q}_j\}(\hat\rho)&0\\0&\{\overline{\hat q}_t,\hat q_j\}(\hat\rho)\end{pmatrix}.\tag{5.36}$$

Again, from (5.34), we see that the non-vanishing eigenvalues of $F(\hat\rho)$ are

$$\pm i\lambda_1(x,\lambda,\xi_{2n}),\ldots,\pm i\lambda_{n-1}(x,\lambda,\xi_{2n}),\tag{5.37}$$

where $\hat\rho=(\hat x,(\lambda\omega_0(x)-2\mathrm{Im}\,\overline\partial_b\phi(x)\xi_{2n},\xi_{2n}))$ and $\lambda_j(x,\lambda,\xi_{2n})$, $j=1,\ldots,n-1$, are the eigenvalues of the Hermitian quadratic form $M^\phi_x\xi_{2n}-2\lambda\mathcal L_x$.

To compute further, fix a point $\hat{\rho}_0=((p,x_{2n}),(\lambda\omega_0(p)-2\mathrm{Im}\,\overline{\partial}_b\phi(p)\xi_{2n},\xi_{2n}))\in U\bigcap\hat{\Sigma}$ and we may assume that the Hermitian quadratic form $\xi_{2n}M^\phi_p-2\lambda\mathcal{L}_p$ is diagonalized with respect to $\overline{Z}_j(p)$, $j=1,\ldots,n-1$. Thus,

$$\sum_{j,t=1}^{n-1}e^\wedge_j e^{\wedge,*}_t\Big(M^\phi_p(\overline{Z}_j,Z_t)\xi_{2n}-2\lambda\mathcal{L}_p(\overline{Z}_j,Z_t)\Big)=\sum_{j=1}^{n-1}e^\wedge_j e^{\wedge,*}_j\Big(M^\phi_p(\overline{Z}_j,Z_j)\xi_{2n}-2\lambda\mathcal{L}_p(\overline{Z}_j,Z_j)\Big).$$

From this, (5.35) and (5.37), we see that on $\hat{\Sigma}\bigcap U$ and on the space of $(0,q)$ forms, $\hat{p}^s_0(\hat{\rho})+\frac{1}{2}\widetilde{\mathrm{tr}}\,F(\hat{\rho})$, $\hat{\rho}=(\hat{x},(\lambda\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n},\xi_{2n}))\in\hat{\Sigma}\bigcap U$ has the eigenvalues

$$\begin{aligned}&\frac{1}{2}\sum_{j=1}^{n-1}|\lambda_j(x,\lambda,\xi_{2n})|-\frac{1}{2}\sum_{j\notin J}\lambda_j(x,\lambda,\xi_{2n})+\frac{1}{2}\sum_{j\in J}\lambda_j(x,\lambda,\xi_{2n}),\ |J|=q,\\&J=(j_1,j_2,\ldots,j_q),\ 1\le j_1<j_2<\cdots<j_q\le n-1,\end{aligned}\tag{5.38}$$

where $\lambda_j(x,\lambda,\xi_{2n})$, $j=1,\ldots,n-1$, are the eigenvalues of the Hermitian quadratic form $M^\phi_x\xi_{2n}-2\lambda\mathcal{L}_x$.

Note that $\xi_{2n}M^\phi_x-2\lambda\mathcal{L}_x$ is non-degenerate of constant signature (n_-,n_+), for every $(\hat{x},(\lambda\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n},\xi_{2n}))\in\hat{\Sigma}\bigcap U$ and there is a constant $\mu>0$ such that $|\lambda_j(x,\lambda,\xi_{2n})|>\mu\xi_{2n}$, $j=1,\ldots,n-1$, for all $(\hat{x},(\lambda\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n},\xi_{2n}))\in\hat{\Sigma}\bigcap U$ (see (5.17)). Combining this observation with (5.38), it is straightforward to see that (5.32) and (5.33) hold. ☐

Remark 5.10. With the notations and assumptions above, let $q=n_-$ and let

$$\rho_0=((x_0,(s_0\omega_0(x_0)-2\mathrm{Im}\,\overline{\partial}_b\phi(x_0))\in V\bigcap\Sigma.$$

Let $\overline{Z}_{1,s_0},\ldots,\overline{Z}_{n-1,s_0}$ be an orthonormal frame of $T^{1,0}_xX$ varying smoothly with x in a neighbourhood of p, for which the Hermitian quadratic form $M^\phi_x-2s_0\mathcal{L}_x$ is diagonalized at x_0. That is,

$$M^\phi_p\big(\overline{Z}_{j,s_0}(x_0),Z_{t,s_0}(x_0)\big)-2s_0\mathcal{L}_{x_0}\big(\overline{Z}_{j,s_0}(x_0),Z_{t,s_0}(x_0)\big)=\lambda_j(s_0)\delta_{j,t},\quad j,t=1,\ldots,n-1.$$

Assume that $\lambda_j(s_0)<0$, $j=1,\ldots,n_-$. Let $e_{1,s_0},\ldots,e_{n-1,s_0}$ denote the basis of $T^{*0,1}X$, which is dual to $Z_{1,s_0},\ldots,Z_{n-1,s_0}$. Put

$$\mathcal{N}(x_0,s_0,n_-):=\left\{ce_{1,s_0}(x_0)\wedge\cdots\wedge e_{n_-,s_0}(x_0)\in T^{*0,q}_{x_0}X;\ c\in\mathbb{C}\right\}.\tag{5.39}$$

From the proof of Theorem 5.9, it is not difficult to see that for every $\hat{\rho}=((x,x_{2n}),(\xi_{2n}\xi,\xi_{2n}))\in U\bigcap\hat{\Sigma}$, $\xi=s_0\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x)$, we have

$$N(\hat{p}^s_0(\hat{\rho})+\frac{1}{2}\widetilde{\mathrm{tr}}\,F(\hat{\rho}))=\mathcal{N}(x,s_0,n_-).\tag{5.40}$$

Put

$$\pi(U)=\left\{\hat{x}\in\hat{D};\ (\hat{x},\hat{\xi})\in U,\text{for some }\hat{\xi}\in\mathbb{R}^{2n}\right\}.\tag{5.41}$$

From Theroem 5.8 and Theorem 5.9, we get the following

Theorem 5.11. *Let $0\le q_1\le n-1$, $q_1\in\mathbb{N}_0$. Let $c_j(\hat{x},\hat{\eta})\in C^\infty(U,T^{*0,q_1}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $j=0,1,\ldots,$ be positively homogeneous functions of degree $m-j$, $m\in\mathbb{Z}$. Then, we can find solutions $a_j(t,\hat{x},\hat{\eta})\in C^\infty(\overline{\mathbb{R}}_+\times T^*\hat{D},T^{*0,q_1}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $j=0,1,\ldots,$ of the system (5.25) with $a_j(0,\hat{x},\hat{\eta})=c_j(\hat{x},\hat{\eta})$, $j=0,1,\ldots,$ where $a_j(t,\hat{x},\hat{\eta})$ is a quasi-homogeneous function of degree $m-j$, such that $a_0(t,\hat{x},\hat{\eta})$ satisfies (5.31) and for all $\alpha,\beta\in\mathbb{N}^{2n}_0$, $\gamma,j\in\mathbb{N}_0$, every $\varepsilon_0>0$ and compact set $K\Subset\pi(U)$, there is a constant $c>0$ such that*

$$\left|\partial^\gamma_t\partial^\alpha_{\hat{x}}\partial^\beta_{\hat{\eta}}a_j(t,\hat{x},\hat{\eta})\right|\le ce^{\varepsilon_0t|\eta_{2n}|}(1+|\hat{\eta}|)^{m-j-|\beta|+\gamma}\quad\text{on }\overline{\mathbb{R}}_+\times\big(K\times\mathbb{R}^{2n}\big)\bigcap(U\bigcap\hat{\Sigma})\big).\tag{5.42}$$

Furthermore, if $q\neq n_-$, then for all $\alpha,\beta\in\mathbb{N}^{2n}_0$, $\gamma,j\in\mathbb{N}_0$, and every compact set $K\Subset\pi(U)$, there is a constant $c>0$ such that

$$\left|\partial^\gamma_t\partial^\alpha_{\hat{x}}\partial^\beta_{\hat{\eta}}a_j(t,\hat{x},\hat{\eta})\right|\le ce^{-\mu_0t|\eta_{2n}|}(1+|\hat{\eta}|)^{m-j-|\beta|+\gamma}\quad\text{on }\overline{\mathbb{R}}_+\times\big(K\times\mathbb{R}^{2n}\big)\bigcap(U\bigcap\hat{\Sigma})),\tag{5.43}$$

where $\mu_0>0$ is a constant as in (5.33).

We introduce some symbol classes

Definition 5.12. Let $\mu \geq 0$ be a non-negative constant. For $0 \leq q_1, q_2 \leq n-1$, $q_1, q_2 \in \mathbb{N}_0$ and $m \in \mathbb{R}$, we say that $a \in \hat{S}^m_\mu(\overline{\mathbb{R}}_+ \times U, T^{*0,q_1}\hat{D} \boxtimes T^{*0,q_2}\hat{D})$ if $a \in C^\infty(\overline{\mathbb{R}}_+ \times U, T^{*0,q_1}\hat{D} \boxtimes T^{*0,q_2}\hat{D})$ and for all indices $\alpha, \beta \in \mathbb{N}_0^{2n}$, $\gamma \in \mathbb{N}_0$, every compact set $K \Subset \pi(U)$ and every $\varepsilon > 0$, there exists a constant $c > 0$ such that

$$\left|\partial_t^\gamma \partial_{\hat{x}}^\alpha \partial_{\hat{\eta}}^\beta a(t, \hat{x}, \hat{\eta})\right| \leq c e^{t(-\mu|\eta_{2n}| + \varepsilon|\eta_{2n}|)}(1 + |\eta|)^{m+\gamma-|\beta|},\ \hat{x} \in K, (\hat{x}, \hat{\eta}) \in U.$$

Remark 5.13. It is easy to see that we have the following properties:

(a) If $a \in \hat{S}^m_{\mu_1}$, $b \in \hat{S}^l_{\mu_2}$ then $ab \in \hat{S}^{m+l}_{\mu_1+\mu_2}$, $a + b \in \hat{S}^{\max(m,l)}_{\min(\mu_1,\mu_2)}$.

(b) If $a \in \hat{S}^m_\mu$ then $\partial_t^\gamma \partial_{\hat{x}}^\alpha \partial_{\hat{\eta}}^\beta a \in \hat{S}^{m-|\beta|+\gamma}_\mu$.

(c) If $a_j \in \hat{S}^{m_j}_\mu$, $j = 0, 1, 2, \ldots$ and $m_j \searrow -\infty$ as $j \to \infty$, then there exists $a \in \hat{S}^{m_0}_\mu$ such that $a - \sum_0^{v-1} a_j \in \hat{S}^{m_v}_\mu$, for all $v = 1, 2, \ldots$. Moreover, if $\hat{S}^{-\infty}_\mu$ denotes $\bigcap_{m\in\mathbb{R}} \hat{S}^m_\mu$ then a is unique modulo $\hat{S}^{-\infty}_\mu$.

If a and a_j have the properties of (c), we write $a \sim \sum_{j=0}^\infty a_j$ in $\hat{S}^{m_0}_\mu$.

From Theorem 5.11 and the standard Borel construction, we get the following

Theorem 5.14. *Let $0 \leq q_1 \leq n-1$, $q_1 \in \mathbb{N}_0$. Let $c_j(\hat{x}, \hat{\eta}) \in C^\infty(U, T^{*0,q_1}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $j = 0, 1, \ldots$, be positively homogeneous functions of degree $m - j$, $m \in \mathbb{Z}$. We can find solutions $a_j(t, \hat{x}, \hat{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, T^{*0,q_1}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $j = 0, 1, \ldots$ of the system* (5.25) *with $a_j(0, \hat{x}, \hat{\eta}) = c_j(\hat{x}, \hat{\eta})$, $j = 0, 1, \ldots$, where $a_j(t, \hat{x}, \hat{\eta})$ is a quasi-homogeneous function of degree $m - j$, such that $a_0(t, \hat{x}, \hat{\eta})$ satisfies* (5.31) *and $a_j \in \hat{S}^{m-j}_\mu(\overline{\mathbb{R}}_+ \times U, T^{*0,q_1}\hat{D}, \boxtimes T^{*0,q}\hat{D})$, $j = 0, 1, \ldots$, for some μ with $\mu > 0$ if $q \neq n_-$ and $\mu = 0$ if $q = n_-$.*

For $0 \leq q_1 \leq n-1$, $q_1 \in \mathbb{N}_0$, let $a_j(t, \hat{x}, \hat{\eta}) \in \hat{S}^{m-j}_\mu(\overline{\mathbb{R}}_+ \times U, T^{*0,q_1}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $j = 0, 1, \ldots$, be quasi-homogeneous functions of degree $m - j$, $m \in \mathbb{Z}$. Assume that $a_j(t, \hat{x}, \hat{\eta})$, $j = 0, 1, \ldots$, are the solutions of the system (5.25). Let $a(t, \hat{x}, \hat{\eta}) \sim \sum_{j=0}^\infty a_j(t, \hat{x}, \hat{\eta})$ in $\hat{S}^m_\mu(\overline{\mathbb{R}}_+ \times U, T^{*0,q_1}\hat{D} \boxtimes T^{*0,q}\hat{D})$. Put

$$(\partial_t + \Box_s^{(q)})(e^{i\psi(t,\hat{x},\hat{\eta})}a(t, \hat{x}, \hat{\eta})) = e^{i\psi(t,\hat{x},\hat{\eta})}b(t, \hat{x}, \hat{\eta}),$$

where

$$b(t, \hat{x}, \hat{\eta}) \sim \sum_{j=0}^\infty b_j(t, \hat{x}, \hat{\eta}) \text{ in } \hat{S}^{m+2}_\mu(\overline{\mathbb{R}}_+ \times U, T^{*0,q_1}\hat{D} \boxtimes T^{*0,q}\hat{D}),$$

$b_j \in \hat{S}^{m+2-j}_\mu(\overline{\mathbb{R}}_+ \times U, T^{*0,q_1}\hat{D}, T^{*0,q}\hat{D})$, b_j is a quasi-homogeneous function of degree $m + 2 - j$, $j = 0, 1, \ldots$.

Since $a_j(t, \hat{x}, \hat{\eta})$, $j = 0, 1, \ldots$, solve the transport equations (5.25), we have that for all $N \in \mathbb{N}$, every compact set $K \Subset \pi(U)$, $\varepsilon > 0$, and all indices $\alpha, \beta \in \mathbb{N}_0^{2n}$, there exists $c > 0$ such that

$$\left|\partial_{\hat{x}}^\alpha \partial_{\hat{\eta}}^\beta b\right| \leq c e^{\varepsilon t|\eta_{2n}|}\big(|\hat{\eta}|^{-N} + |\hat{\eta}|^{m+2-N}\,(\mathrm{Im}\,\psi(t, \hat{x}, \hat{\eta}))^N\big) \ \text{ on } \overline{\mathbb{R}}_+ \times \big(K \times \mathbb{R}^{2n}\big) \bigcap (U \bigcap \hat{\Sigma}). \tag{5.44}$$

Conversely, if $(\partial_t + \Box_s^{(q)})(e^{i\psi(t,\hat{x},\hat{\eta})}a(t, \hat{x}, \hat{\eta})) = e^{i\psi(t,\hat{x},\hat{\eta})}b(t, \hat{x}, \hat{\eta})$ and b satisfies the same kind of estimates as (5.44), then $a_j(t, \hat{x}, \hat{\eta})$, $j = 0, 1, \ldots$, solve the system (5.25) to infinite order at $\hat{\Sigma} \bigcap U$. From this observation and the particular structure of the problem, we will next show

Theorem 5.15. *Let $q = n_-$. Let $c_j(\hat{x}, \hat{\eta}) \in C^\infty(U, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $j = 0, 1, \ldots$, be positively homogeneous functions of degree $m - j$, $m \in \mathbb{Z}$. We can find solutions $a_j(t, \hat{x}, \hat{\eta}) \in \hat{S}^{m-j}_0(\overline{\mathbb{R}}_+ \times U, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $j = 0, 1, \ldots$ of the system* (5.25)*, where $a_j(t, \hat{x}, \hat{\eta})$ is a quasi-homogeneous function of degree $m - j$, for each j, with $a_j(0, \hat{x}, \hat{\eta}) = c_j(\hat{x}, \hat{\eta})$, $j = 0, 1, \ldots$,*

$$a_j(t, \hat{x}, \hat{\eta}) - a_j(\infty, \hat{x}, \hat{\eta}) \in \hat{S}^{m-j}_\mu(\overline{\mathbb{R}}_+ \times U, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D}), \quad j = 0, 1, 2, \ldots,$$

and for every $(\hat{x}, \hat{\eta}) = ((x, x_{2n}), (\eta_{2n}\eta, \eta_{2n})) \in U \bigcap \hat{\Sigma}$, $\eta = s_0\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)$, we have

$$a_0(\infty, \hat{x}, \hat{\eta})u \in \mathcal{N}(x, s_0, n_-), \quad \forall u \in T^{*0,q}_{\hat{x}}\hat{D}, \tag{5.45}$$

where $\mu > 0$ is a constant and $a_j(\infty, \hat{x}, \hat{\eta}) \in C^\infty(U, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $j = 0, 1, \ldots$, $a_j(\infty, \hat{x}, \hat{\eta})$ is a positively homogeneous function of degree $m - j$, for each j.

Proof. Let $\widetilde{a}_j(t,\hat{x},\hat{\eta})\in\hat{S}^{m-j}_0(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $j=0,1,\ldots$, be any solutions of the system (5.25) with $\widetilde{a}_j(0,\hat{x},\hat{\eta})=c_j(\hat{x},\hat{\eta})$, $j=0,1,\ldots$, where $\widetilde{a}_j(t,\hat{x},\hat{\eta})$ is a quasi-homogeneous function of degree $m-j$, $j=0,1,\ldots$. Set $\widetilde{a}(t,\hat{x},\hat{\eta})\sim\sum^\infty_{j=0}\widetilde{a}_j(t,\hat{x},\hat{\eta})$ in $\hat{S}^m_0(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$ and put

$$(\partial_t+\Box^{(q)}_s)(e^{i\psi(t,\hat{x},\hat{\eta})}\widetilde{a}(t,\hat{x},\hat{\eta}))=e^{i\psi(t,\hat{x},\hat{\eta})}\widetilde{b}(t,\hat{x},\hat{\eta}),$$

where

$$\widetilde{b}(t,\hat{x},\hat{\eta})\sim\textstyle\sum^\infty_{j=0}\widetilde{b}_j(t,\hat{x},\hat{\eta})\text{ in }\hat{S}^{m+2}_\mu(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D}),$$

$\widetilde{b}_j\in\hat{S}^{m+2-j}_\mu(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $\widetilde{b}_j$ is a quasi-homogeneous function of degree $m+2-j$, $j=0,1,\ldots$. Since $\widetilde{a}_j(t,\hat{x},\hat{\eta})$, $j=0,1,\ldots$, solve the transport equations (5.25), $\widetilde{b}(t,\hat{x},\hat{\eta})$ satisfies (5.44). Note that we have the interwing properties

$$\overline{\partial}_s\Box^{(q)}_s=\Box^{(q+1)}_s\overline{\partial}_s,\quad\overline{\partial}^*_s\Box^{(q)}_s=\Box^{(q-1)}_s\overline{\partial}^*_s.\tag{5.46}$$

Now,

$$\overline{\partial}_s(e^{i\psi}\widetilde{a})=e^{i\psi}c,\quad\overline{\partial}^*_s(e^{i\psi}\widetilde{a})=e^{i\psi}d,$$

$c\sim\sum^\infty_{j=0}c_j(t,\hat{x},\hat{\eta})$ in $\hat{S}^{m+1}_0(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q+1}\hat{D})$, $d\sim\sum^\infty_{j=0}d_j(t,\hat{x},\hat{\eta})$ in $\hat{S}^{m+1}_0(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q-1}\hat{D})$, where $c_j(t,\hat{x},\hat{\eta})$ and $d_j(t,\hat{x},\hat{\eta})$ are quasi-homogeneous functions of degree $m+1-j$, $j=0,1,\ldots$. From (5.46), we have $(\partial_t+\Box^{(q+1)}_s)(e^{i\psi}c)=e^{i\psi}e$, $(\partial_t+\Box^{(q-1)}_s)(e^{i\psi}d)=e^{i\psi}f$, where e and f satisfy (5.44). Since e and f satisfy (5.44), we deduce that c_j, $j=0,1,\ldots$, solve the system (5.25) and d_j, $j=0,1,\ldots$, solve the system (5.25) too. From Theorem 5.8 and Theorem 5.11, we see that $c_j(t,\hat{x},\hat{\eta})$, $d_j(t,\hat{x},\hat{\eta})$, $j=0,1,\ldots$, satisfy the same kind of estimates as (5.43). Now $\Box^{(q)}_s=\overline{\partial}_s{}^*\overline{\partial}_s+\overline{\partial}^*_s\overline{\partial}_s$, so $\Box^{(q)}_s(e^{i\psi}\widetilde{a})=e^{i\psi}g$, where g satisfies the same kind of estimates as (5.43). From this we see that $\partial_t(e^{i\psi}\widetilde{a})=e^{i\psi}h$, where h has the same properties as g. Since $h=i(\partial_t\psi)\widetilde{a}+\partial_t\widetilde{a}$ and $\partial_t\psi$ satisfy the same kind of estimates as (5.43), $\partial_t\widetilde{a}$ satisfies the same kind of estimates as (5.43). From the standard Borel construction, we can find $a_j(t,\hat{x},\hat{\eta})\in C^\infty(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $j=0,1,\ldots$, such that $a_j(t,\hat{x},\hat{\eta})-\widetilde{a}_j(t,\hat{x},\hat{\eta})$ vanishes to infinite order at each point of $\hat{\Sigma}\bigcap U$, $a_j(t,\hat{x},\hat{\eta})$ is a quasi-homogeneous function of degree $m-j$ and there is a $\mu>0$ such that $\partial_t a_j(t,\hat{x},\hat{\eta})\in\hat{S}^{m-j}_\mu(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $j=0,1,\ldots$. We conclude that we can find $a_j(\infty,\hat{x},\hat{\eta})\in C^\infty(U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$, where $a_j(\infty,\hat{x},\hat{\eta})$ is a positively homogeneous function of degree $m-j$, $j=0,1,\ldots$, such that $a_j(t,\hat{x},\hat{\eta})-a_j(\infty,\hat{x},\hat{\eta})\in\hat{S}^{m-j}_\mu(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $\mu>0$, $j=0,1,2,\ldots$.

Finally, from Theorem 5.8, (5.31) and (5.40), we obtain (5.45). The theorem follows. □

5.3. **Microlocal Hodge decomposition theorems for $\Box^{(q)}_s$ in U.** We use the same notations and assumptions as before. Fix $D_0\Subset D$, where D_0 is an open set of D. As before, we put $\hat{D}_0:=D_0\times\mathbb{R}$. We need the following which is essentially well-known (see Chapter 5 in part I of [14])

Proposition 5.16. *Let $\mu>0$ and let $b(t,\hat{x},\hat{\eta})\in\hat{S}^m_\mu(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $m\in\mathbb{R}$. We assume that $b(t,\hat{x},\hat{\eta})=0$ when $|\hat{\eta}|\leq1$ and for every $t\in\overline{\mathbb{R}}_+$, $\operatorname{Supp}b(t,\hat{x},\hat{\eta})\bigcap T^*\hat{D}_0\subset\overline{W}$, where $W\subset U$ is a conic open set with $\overline{W}\subset U$. Take $\tau(\hat{x},\hat{\eta})\in C^\infty(T^*\hat{D})$, $\tau=1$ on $\overline{W}$, $\tau=0$ outside U and τ is positively homogeneous of degree 0. Let $\chi\in C^\infty_0(\mathbb{R}^{2n})$ be equal to 1 near the origin. Put*

$$B_\epsilon(\hat{x},\hat{y})=\int(\int^\infty_0e^{i(\psi(t,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>)}b(t,\hat{x},\hat{\eta})dt)\chi(\epsilon\eta)\tau(\hat{x},\hat{\eta})d\hat{\eta}.$$

For $u\in\Omega^{0,q}_0(\hat{D})$, we have

$$\lim_{\epsilon\to0}\Big(\int B_\epsilon(\hat{x},\hat{y})u(\hat{y})d\hat{y}\Big)\in\Omega^{0,q}(\hat{D})$$

and the operator

$$\begin{aligned}B:\Omega^{0,q}_0(\hat{D})&\to\Omega^{0,q}(\hat{D})\\ u&\to\lim_{\epsilon\to0}\Big(\int B_\epsilon(\hat{x},\hat{y})u(\hat{y})dy\Big)\end{aligned}\tag{5.47}$$

is continuous and B has a unique continuous extension:

$$B:\mathscr{E}'(\hat{D},T^{*0,q}\hat{D})\to\mathscr{D}'(\hat{D},T^{*0,q}\hat{D})$$

and $B(x,y)\in C^\infty\big(\hat{D}\times\hat{D}\setminus\operatorname{diag}(\hat{D}\times\hat{D}),T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D}\big)$*, where* $B(\hat{x},\hat{y})$ *denotes the distribution kernel of* B*.*

Let $b(t,\hat{x},\hat{\eta})\in\hat{S}^m_\mu(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $\mu>0$. $m\in\mathbb{R}$. We assume that $b(t,\hat{x},\hat{\eta})=0$ when $|\hat{\eta}|\leq 1$ and for every $t\in\overline{\mathbb{R}}_+$, $\operatorname{Supp}b(t,\hat{x},\hat{\eta})\bigcap T^*\hat{D}_0\subset\overline{W}$, where $W\subset U$ is a conic open set with $\overline{W}\subset U$. Let

$$B:\Omega^{0,q}_0(\hat{D})\to\Omega^{0,q}(\hat{D}),\ \ \mathscr{E}'(\hat{D},T^{*0,q}\hat{D})\to\mathscr{D}'(\hat{D},T^{*0,q}\hat{D})$$

be the continuous operator given by (5.47). We formally write

$$B=B(\hat{x},\hat{y})=\int\Big(\int_0^\infty e^{i(\psi(t,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>)}b(t,\hat{x},\hat{\eta})dt\Big)\tau(\hat{x},\hat{\eta})d\hat{\eta}.$$

From now on, we identify B with $B(\hat{x},\hat{y})$.

Remark 5.17. Let $a(t,\hat{x},\hat{\eta})\in\hat{S}^m_0(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $m\in\mathbb{R}$. We assume that $a(t,\hat{x},\hat{\eta})=0$ if $|\eta|\leq 1$ and for every $t\in\overline{\mathbb{R}}_+$, $\operatorname{Supp}a(t,\hat{x},\hat{\eta})\bigcap T^*\hat{D}_0\subset\overline{W}$, where $W\subset U$ is a conic open set with $\overline{W}\subset U$ and $a(t,\hat{x},\hat{\eta})-a(\infty,\hat{x},\hat{\eta})\in\hat{S}^m_\mu(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $\mu>0$, where $a(\infty,\hat{x},\hat{\eta})\in C^\infty(U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$ and $\operatorname{Supp}a(\infty,\hat{x},\hat{\eta})\bigcap T^*\hat{D}_0\subset\overline{W}$. Then we can also define

$$A(\hat{x},\hat{y})=\int\Big(\int_0^\infty\big(e^{i(\psi(t,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>)}a(t,\hat{x},\eta)-e^{i(\psi(\infty,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>)}a(\infty,\hat{x},\hat{\eta})\big)dt\Big)\tau(\hat{x},\hat{\eta})d\hat{\eta}\tag{5.48}$$

as an oscillatory integral by the following formula:

$$A(\hat{x},\hat{y})=\int\Big(\int_0^\infty e^{i(\psi(t,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>)}(-t)\big(i\psi'_t(t,\hat{x},\hat{\eta})a(t,\hat{x},\hat{\eta})+a'_t(t,\hat{x},\hat{\eta})\big)dt\Big)\tau(\hat{x},\hat{\eta})d\hat{\eta}.$$

We notice that $(-t)\big(i\psi'_t(t,\hat{x},\hat{\eta})a(t,\hat{x},\hat{\eta})+a'_t(t,\hat{x},\hat{\eta})\big)\in\hat{S}^{m+1}_\mu$, $\mu>0$.

The oscillatory integral $A(\hat{x},\hat{y})$ defines a continuous operator

$$A:\Omega^{0,q}_0(\hat{D})\to\Omega^{0,q}(\hat{D}),\ \ \mathscr{E}'(\hat{D},T^{*0,q}\hat{D})\to\mathscr{D}'(\hat{D},T^{*0,q}\hat{D}).$$

We formally write

$$\begin{aligned}A&=A(\hat{x},\hat{y})\\&=\int\Big(\int_0^\infty\big(e^{i(\psi(t,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>)}a(t,\hat{x},\eta)-e^{i(\psi(\infty,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>)}a(\infty,\hat{x},\hat{\eta})\big)dt\Big)\tau(\hat{x},\hat{\eta})d\hat{\eta}.\end{aligned}$$

Let $m\in\mathbb{R}$, $0\leq\rho,\delta\leq 1$. Let Γ be a conic open set of $T^*\hat{D}$. Let $S^m_{\rho,\delta}(\Gamma,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$ denote the Hörmander symbol space on Γ with values in $T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D}$ of order m type (ρ,δ)(see Definition 1.1 of Grigis-Sjöstrand [9]) and let $S^m_{\rm cl}(\Gamma,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$ denote the space of classical symbols on Γ with values in $T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D}$ of order m (see page 35 of Grigis-Sjöstrand [9]). Let $B\subset D$ be an open set. Let $L^m_{\frac{1}{2},\frac{1}{2}}(B,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$ and $L^m_{\rm cl}(B,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$ denote the space of pseudodifferential operators on B of order m type $(\frac{1}{2},\frac{1}{2})$ from sections of $T^{*0,q}\hat{D}$ to sections of $T^{*0,q}\hat{D}$ and the space of classical pseudodifferential operators on B of order m from sections of $T^{*0,q}\hat{D}$ to sections of $T^{*0,q}\hat{D}$. The classical result of Calderon and Vaillancourt tells us that for any $A\in L^m_{\frac{1}{2},\frac{1}{2}}(B,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$,

$$A:H^s_{\rm comp}(B,T^{*0,q}\hat{D})\to H^{s-m}_{\rm loc}(B,T^{*0,q}\hat{D})\text{ is continuous, for every }s\in\mathbb{R}.\tag{5.49}$$

(See Hörmander [12], for a proof).

We can repeat the proofs of Lemma 5.14, Lemma 5.16 in [14] and obtain the following

Proposition 5.18. *Let* $a(t,\hat{x},\hat{\eta})\in\hat{S}^m_0(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$*,* $m\in\mathbb{R}$*. We assume* $a(t,\hat{x},\hat{\eta})=0$ *if* $|\eta|\leq 1$ *and for every* $t\in\overline{\mathbb{R}}_+$*,* $\operatorname{Supp}a(t,\hat{x},\hat{\eta})\bigcap T^*\hat{D}_0\subset\overline{W}$*, where* $W\subset U$ *is a conic open set with* $\overline{W}\subset U$ *and* $a(t,\hat{x},\hat{\eta})-a(\infty,\hat{x},\hat{\eta})\in\hat{S}^m_\mu(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$*,* $\mu>0$*, where* $a(\infty,\hat{x},\hat{\eta})\in$

$C^\infty(U, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$ *and* $\operatorname{Supp} a(\infty, \hat{x}, \hat{\eta}) \bigcap T^*\hat{D}_0 \subset \overline{W}$. *Take* $\tau(\hat{x}, \hat{\eta}) \in C^\infty(T^*\hat{D})$, $\tau = 1$ *on* $\overline{W}$, $\tau = 0$ *outside* U *and* τ *is positively homogeneous of degree* 0. *Let*

$$\begin{aligned} &A(\hat{x}, \hat{y}) \\ &= \frac{1}{(2\pi)^{2n}} \int \Big(\int_0^\infty \big(e^{i(\psi(t,\hat{x},\hat{\eta}) - <\hat{y},\hat{\eta}>)} a(t, \hat{x}, \hat{\eta}) - e^{i(\psi(\infty,\hat{x},\hat{\eta}) - <\hat{y},\hat{\eta}>)} a(\infty, \hat{x}, \hat{\eta})\big) dt \Big) \tau(\hat{x}, \hat{\eta}) d\hat{\eta} \end{aligned}$$

be the oscillatory integral as in (5.48). *Then* $A \in L^{m-1}_{\frac{1}{2},\frac{1}{2}}(\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$ *with symbol*

$$q(\hat{x}, \hat{\eta}) = \int_0^\infty \Big(e^{i(\psi(t,\hat{x},\hat{\eta}) - <\hat{x},\hat{\eta}>)} a(t, \hat{x}, \hat{\eta}) - e^{i(\psi(\infty,\hat{x},\hat{\eta}) - <\hat{x},\hat{\eta}>)} a(\infty, \hat{x}, \hat{\eta}) \Big) dt \tau(\hat{x}, \hat{\eta})$$

in $S^{m-1}_{\frac{1}{2},\frac{1}{2}}(T^*\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$.

We can repeat the proof Proposition 5.18 in [14] and conclude that

Proposition 5.19. *Let* $a(\infty, \hat{x}, \hat{\eta}) \in C^\infty(T^*\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $\operatorname{Supp} a(\infty, \hat{x}, \hat{\eta}) \bigcap T^*\hat{D}_0 \subset \overline{W}$, *be a classical symbol of order* m, *where* $W \subset U$ *is a conic open set with* $\overline{W} \subset U$. *Take* $\tau(\hat{x}, \hat{\eta}) \in C^\infty(T^*\hat{D})$, $\tau = 1$ *on* $\overline{W}$, $\tau = 0$ *outside* U *and* τ *is positively homogeneous of degree* 0. *Then*

$$a(\hat{x}, \hat{\eta}) = e^{i(\psi(\infty,x,\eta) - <x,\eta>)} a(\infty, \hat{x}, \hat{\eta}) \tau(\hat{x}, \hat{\eta}) \in S^m_{\frac{1}{2},\frac{1}{2}}(T^*\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D}).$$

We assume that $q \neq n_-$. Let $\widetilde{I} = (2\pi)^{-2n} \int e^{i<\hat{x}-\hat{y},\hat{\eta}>} c(\hat{x}, \hat{\eta}) d\hat{\eta}$ be a classical pseudodifferential operator on $\hat{D}$ of order 0 from sections of $T^{*0,q}\hat{D}$ to sections of $T^{*0,q}\hat{D}$ with $c(\hat{x}, \hat{\eta}) \in S^0_{\mathrm{cl}}(T^*\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $\operatorname{Supp} c(\hat{x}, \hat{\eta}) \bigcap T^*\hat{D}_0 \subset W$, where $W \subset U$ is a conic open set with $\overline{W} \subset U$. We have

$$c(\hat{x}, \hat{\eta}) \sim \sum_{j=0}^\infty c_j(\hat{x}, \hat{\eta})$$

in the Hörmander symbol space $S^0_{1,0}(T^*\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $c_j(\hat{x}, \hat{\eta}) \in C^\infty(U, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $\operatorname{Supp} c_j(\hat{x}, \hat{\eta}) \subset \overline{W}$, $j = 0, 1, \ldots$, are positively homogeneous functions of degree $-j$. Let

$$a_j(t, \hat{x}, \hat{\eta}) \in \hat{S}^{-j}_{\mu_0}(\overline{\mathbb{R}}_+ \times U, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D}), \quad j = 0, 1, \ldots,$$

where $\mu_0 > 0$ is as in Theorem 5.14, with $a_j(0, \hat{x}, \hat{\eta}) = c_j(\hat{x}, \hat{\eta})$, $j = 0, 1, \ldots$. Let $a(t, \hat{x}, \hat{\eta}) \sim \sum_{j=0}^\infty a_j(t, \hat{x}, \hat{\eta})$ in $\hat{S}^0_{\mu_0}(\overline{\mathbb{R}}_+ \times U, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$. Choose $\chi \in C_0^\infty(\mathbb{R}^{2n})$ so that $\chi(\hat{\eta}) = 1$ when $|\hat{\eta}| < 1$ and $\chi(\hat{\eta}) = 0$ when $|\hat{\eta}| > 2$. Take $\tau(\hat{x}, \hat{\eta}) \in C^\infty(T^*\hat{D})$, $\tau = 1$ on $\overline{W}$, $\tau = 0$ outside U and τ is positively homogeneous of degree 0. Set

$$A(\hat{x}, \hat{y}) = \frac{1}{(2\pi)^{2n}} \int \big(\int_0^\infty e^{i(\psi(t,\hat{x},\hat{\eta}) - <\hat{y},\hat{\eta}>)} a(t, \hat{x}, \hat{\eta})(1 - \chi(\hat{\eta})) \tau(\hat{x}, \hat{\eta}) dt \big) d\hat{\eta}. \tag{5.50}$$

We can repeat the proof of Proposition 6.3 in [14] with minor changes and conclude that

Theorem 5.20. *Assume that* $q \neq n_-$. *Let* $\widetilde{I} = (2\pi)^{-2n} \int e^{i<\hat{x}-\hat{y},\hat{\eta}>} c(\hat{x}, \hat{\eta}) d\hat{\eta}$ *be a classical pseudodifferential operator on* $\hat{D}$ *of order* 0 *from sections of* $T^{*0,q}\hat{D}$ *to sections of* $T^{*0,q}\hat{D}$ *with* $c(\hat{x}, \hat{\eta}) \in S^0_{\mathrm{cl}}(\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $\operatorname{Supp} c(\hat{x}, \hat{\eta}) \bigcap T^*\hat{D}_0 \subset \overline{W}$, *where* $W \subset U$ *is a conic open set with* $\overline{W} \subset U$. *Let* $A = A(x, y) \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$ *be as in* (5.50). *Then, on* $\hat{D}_0$,

$$\Box^{(q)}_s \circ A \equiv \widetilde{I}.$$

We assume that $q = n_-$. Let $\widetilde{I}$ be the classical pseudodifferential operator as in Theorem 5.20. Let

$$a_j(t, \hat{x}, \hat{\eta}) \in \hat{S}^{-j}_0(\overline{\mathbb{R}}_+ \times U, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D}), \quad j = 0, 1, \ldots,$$

and $a_j(\infty, \hat{x}, \hat{\eta}) \in C^\infty(U, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $j = 0, 1, \ldots$, be as in Theorem 5.15. We recall that for some $\mu > 0$,

$$a_j(t, \hat{x}, \hat{\eta}) - a_j(\infty, \hat{x}, \hat{\eta}) \in \hat{S}^{-j}_\mu(\overline{\mathbb{R}}_+ \times U, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D}), \quad j = 0, 1, \ldots,$$

and for every $(\hat{x}, \hat{\eta}) = ((x, x_{2n}), (\eta_{2n}\eta, \eta_{2n})) \in U \bigcap \hat{\Sigma}$, $\eta = s_0\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)$, we have

$$a_0(\infty, \hat{x}, \hat{\eta}) u \in \mathcal{N}(x, s_0, n_-), \quad \forall u \in T^{*0,q}_{\hat{x}}\hat{D}. \tag{5.51}$$

Let

$$a(\infty,\hat{x},\hat{\eta})\sim\sum_{j=0}^{\infty}a_j(\infty,\hat{x},\hat{\eta}) \tag{5.52}$$

in $S^0_{1,0}(U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$. Let

$$a(t,\hat{x},\hat{\eta})\sim\sum_{j=0}^{\infty}a_j(t,\hat{x},\hat{\eta}) \tag{5.53}$$

in $\hat{S}^0_0(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$. We take $a(t,\hat{x},\hat{\eta})$ so that for every compact set $K\subset\pi(U)$ and all indices $\alpha,\beta\in\mathbb{N}^{2n}_0$, $\gamma,l\in\mathbb{N}_0$, there exists $c>0$ independent of t, such that

$$\left|\partial^\gamma_t\partial^\alpha_{\hat{x}}\partial^\beta_{\hat{\eta}}\Big(a(t,\hat{x},\hat{\eta})-\sum_{j=0}^{l}a_j(t,\hat{x},\hat{\eta})\Big)\right|\leq c(1+|\hat{\eta}|)^{-l-1+\gamma-|\beta|}, \tag{5.54}$$

where $t\in\overline{\mathbb{R}}_+$, $\hat{x}\in K$, $(\hat{x},\hat{\eta})\in U$, $|\hat{\eta}|\geq 1$, and

$$a(t,\hat{x},\hat{\eta})-a(\infty,\hat{x},\hat{\eta})\in\hat{S}^0_\mu(\overline{\mathbb{R}}_+\times U,T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})\ \text{ with }\mu>0.$$

Choose $\chi\in C^\infty_0(\mathbb{R}^{2n})$ so that $\chi(\hat{\eta})=1$ when $|\hat{\eta}|<1$ and $\chi(\hat{\eta})=0$ when $|\hat{\eta}|>2$. Take $\tau(\hat{x},\hat{\eta})\in C^\infty(T^*\hat{D})$, $\tau=1$ on $\overline{W}$, $\tau=0$ outside U and τ is positively homogeneous of degree 0. Set

$$\begin{aligned}G(\hat{x},\hat{y})=\frac{1}{(2\pi)^{2n}}\int\Big(\int_0^\infty&\big(e^{i(\psi(t,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>)}a(t,\hat{x},\hat{\eta})\\&-e^{i(\psi(\infty,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>)}a(\infty,\hat{x},\hat{\eta})\big)(1-\chi(\hat{\eta}))\tau(\hat{x},\hat{\eta})dt\Big)d\hat{\eta}.\end{aligned} \tag{5.55}$$

Put

$$S(\hat{x},\hat{y})=\frac{1}{(2\pi)^{2n}}\int e^{i(\psi(\infty,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>)}a(\infty,\hat{x},\hat{\eta})(1-\chi(\hat{\eta}))\tau(\hat{x},\hat{\eta})d\hat{\eta}. \tag{5.56}$$

We can repeat the proof of Proposition 6.5 in [14] with minor changes and obtain

Theorem 5.21. *We assume that* $q=n_-$. *Let* $\widetilde{I}=(2\pi)^{-2n}\int e^{i<\hat{x}-\hat{y},\hat{\eta}>}c(\hat{x},\hat{\eta})d\hat{\eta}$ *be a classical pseudodifferential operator on* $\hat{D}$ *of order* 0 *from sections of* $T^{*0,q}\hat{D}$ *to sections of* $T^{*0,q}\hat{D}$ *with* $c(\hat{x},\hat{\eta})\in S^0_{\rm cl}(T^*\hat{D},T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$, $\operatorname{Supp}c(\hat{x},\hat{\eta})\bigcap T^*\hat{D}_0\subset\overline{W}$, *where* $W\subset U$ *is a conic open set with* $\overline{W}\subset U$. *Let* $G=G(\hat{x},\hat{y})\in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\hat{D},T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$ *be as in* (5.55) *and let* $S=S(\hat{x},\hat{y})\in L^0_{\frac{1}{2},\frac{1}{2}}(\hat{D},T^{*0,q}\hat{D}\boxtimes T^{*0,q}\hat{D})$ *be as in* (5.56). *Then,*

$$S+\Box^{(q)}_s\circ G\equiv\widetilde{I}\ \text{ on }\hat{D}_0,\quad\Box^{(q)}_s\circ S\equiv 0\ \text{ on }\hat{D}.$$

Now, we study the distribution kernel $S(\hat{x},\hat{y})$ of S. Fix $p\in D$ and assume that D is a small open neighbourhood of p. We take local coordinates $x=(x_1,\ldots,x_{2n-1})$ so that $x(p)=0$, $\omega_0(p)=(0,0,\ldots,1)\in\mathbb{R}^{2n-1}$ and

$$-2\mathrm{Im}\,\overline{\partial}_b\phi(p)=(\alpha_1,\ldots,\alpha_{2n-2},0):=(\alpha,0)\in\mathbb{R}^{2n-1}.$$

Thus, $((p,x_{2n}),\hat{\xi})\in\hat{\Sigma}$ if and only if $\hat{\xi}=(\xi_{2n}\alpha_1,\xi_{2n}\alpha_2,\ldots,\xi_{2n}\alpha_{2n-2},\xi_{2n}\lambda,\xi_{2n})$, $\lambda\in\mathbb{R}$. We need

Lemma 5.22. *We have*

$$\det\left(\frac{\partial^2\psi}{\partial\eta_j\partial\eta_t}(\infty,(p,x_{2n}),(\alpha_1,\ldots,\alpha_{2n-2},\lambda,1))\right)^{2n-2}_{j,t=1}\neq 0,$$

for every $((p,x_{2n}),(\alpha_1,\ldots,\alpha_{2n-2},\lambda,1))\in U$.

Proof. We first claim that

$$\Big(\tfrac{\partial^2\mathrm{Im}\,\psi}{\partial\eta_j\partial\eta_t}(\infty,(p,x_{2n}),(\alpha_1,\ldots,\alpha_{2n-2},\lambda,1))\Big)^{2n-2}_{j,t=1}\ \text{is positive definite,} \tag{5.57}$$

for every $((p,x_{2n}),(\alpha_1,\ldots,\alpha_{2n-2},\lambda,1))\in U$. We consider Taylor expansion of

$$\operatorname{Im}\psi(\infty,(p,x_{2n}),(\eta_1,\ldots,\eta_{2n-2},\lambda,1))$$

at $((p,x_{2n}),(\alpha_1,\ldots,\alpha_{2n-2},\lambda,1))$:

$$\begin{aligned}
&\operatorname{Im}\psi(\infty,(p,x_{2n}),(\eta_1,\ldots,\eta_{2n-2},\lambda,1))\\
&=\frac{1}{2}\sum_{j,t=1}^{2n-2}\frac{\partial^2\operatorname{Im}\psi}{\partial\eta_j\partial\eta_t}(\infty,(p,x_{2n}),(\alpha_1,\ldots,\alpha_{2n-2},\lambda,1))(\eta_j-\alpha_j)(\eta_t-\alpha_t)\\
&\quad+O(\sum_{j=1}^{2n-2}|\eta_j-\alpha_j|^3).
\end{aligned}\tag{5.58}$$

Here we use the fact that $(\operatorname{Im}d_\eta\psi(\infty,(p,x_{2n}),(\alpha_1,\ldots,\alpha_{2n-2},\lambda,1))=0$ (see (5.18)). From Theorem 5.6, it is straightforward to see that

$$\operatorname{Im}\psi(\infty,(p,x_{2n}),(\eta_1,\ldots,\eta_{2n-2},\lambda,1))\asymp|\eta-\alpha|^2\,,$$

for $(\eta_1,\ldots,\eta_{2n-2})$ is in some small neighbourhood of $(\alpha_1,\ldots,\alpha_{2n-2})$. From this and (5.58), we conclude that

$$\left(\frac{\partial^2\operatorname{Im}\psi}{\partial\eta_j\partial\eta_t}(\infty,(p,x_{2n}),(\alpha_1,\ldots,\alpha_{2n-2},\lambda,1))\right)_{j,t=1}^{2n-2}$$

is positive definite. The claim (5.57) follows.

Put $A=\left(\frac{\partial^2\operatorname{Im}\psi}{\partial\eta_j\partial\eta_t}(\infty,(p,x_{2n}),(\alpha_1,\ldots,\alpha_{2n-2},\lambda,1))\right)_{j,t=1}^{2n-2}=\operatorname{Re}A+i\operatorname{Im}A$. Let $u\in\mathbb{C}^{2n-2}$. If $Au=0$, then $<(\operatorname{Re}A+i\operatorname{Im}A)u,\overline{u}>=<(\operatorname{Re}A)u,\overline{u}>+i<(\operatorname{Im}A)u,\overline{u}>=0$. Thus, $<(\operatorname{Re}A)u,\overline{u}>=<(\operatorname{Im}A)u,\overline{u}>=0$. Since $\operatorname{Im}A$ is positive definite, we conclude that $u=0$. The lemma follows. $\square$

Put

$$-2\operatorname{Im}\overline{\partial}_b\phi(x)=(\alpha_1(x),\ldots,\alpha_{2n-1}(x)),\quad \omega_0(x)=(\beta_1(x),\ldots,\beta_{2n-1}(x)),\quad x\in D.\tag{5.59}$$

From Lemma 5.22, we may take V and D small enough so that

$$\det\left(\frac{\partial^2\psi}{\partial\eta_j\partial\eta_k}(\infty,\hat{x},\hat{\eta})\right)_{j,k=1}^{2n-2}\neq0,\quad\forall(\hat{x},\hat{\eta})\in U$$

and

$$\beta_{2n-1}(x)\geq\frac{1}{2},\quad\forall x\in D.\tag{5.60}$$

Set

$$\widetilde{a}(\hat{x},\hat{\eta}):=a(\infty,\hat{x},\hat{\eta})(1-\chi(\hat{\eta}))\tau(\hat{x},\hat{\eta}).$$

Since $\tau(\hat{x},\hat{\eta})=0$ outside U, $\widetilde{a}(\hat{x},\hat{\eta})=0$ if $\eta_{2n}\leq0$. We have

$$\begin{aligned}
S(\hat{x},\hat{y})&=\frac{1}{(2\pi)^{2n}}\int e^{i(\psi(\infty,\hat{x},\hat{\eta})-<\hat{y},\hat{\eta}>)}\widetilde{a}(\hat{x},\hat{\eta})d\hat{\eta}\\
&=\frac{1}{(2\pi)^{2n}}\int_{t>0}e^{it(\psi(\infty,\hat{x},(w,1))-<\hat{y},(w,1)>)}t^{2n-1}\widetilde{a}(\hat{x},(tw,t))dwdt,
\end{aligned}\tag{5.61}$$

where $\eta=(\eta_1,\ldots,\eta_{2n-1})=tw$, $\eta_{2n}=t$, $w=(w_1,\ldots,w_{2n-1})\in\mathbb{R}^{2n-1}$. Let $w_{2n-1}=\alpha_{2n-1}(x)+s\beta_{2n-1}(x)$ and put $w'=(w_1,\ldots,w_{2n-2})$ in (5.61), we get

$$\begin{aligned}
&S(\hat{x},\hat{y})\\
&=\frac{1}{(2\pi)^{2n}}\int_{t>0}e^{it(\psi(\infty,\hat{x},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1))-<\hat{y},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1)>)}\times\\
&\qquad t^{2n-1}\beta_{2n-1}(x)\widetilde{a}(\hat{x},(tw',t\alpha_{2n-1}(x)+ts\beta_{2n-1}(x),t))dw'dsdt.
\end{aligned}\tag{5.62}$$

Note that $\beta_{2n-1}(x) \geq \frac{1}{2}$, for every $x \in D$. The stationary phase method of Melin and Sjöstrand (see page 148 of [21]) then permits us to carry out the w' integration in (5.62), to get

$$S(\hat{x}, \hat{y}) \equiv \int e^{it\Phi(\hat{x}, \hat{y}, s)} b(\hat{x}, \hat{y}, s, t) ds dt \tag{5.63}$$

with

$$b(\hat{x}, \hat{y}, s, t) \sim \sum_{j=0}^{\infty} b_j(\hat{x}, \hat{y}, s) t^{n-j} \tag{5.64}$$

in $S^n_{1,0}(\hat{\Omega}\times]0,\infty[, T^{*0,q}_{\hat{y}}\hat{D} \boxtimes T^{*0,q}_{\hat{x}}\hat{D})$, $\operatorname{Supp} b(\hat{x}, \hat{y}, s, t) \subset \hat{\Omega} \times \mathbb{R}_+$, where

$$\begin{aligned} \hat{\Omega} :=& \{(\hat{x}, \hat{y}, s) \in \hat{D} \times \hat{D} \times \mathbb{R};\ (\hat{x}, (-2\operatorname{Im} \overline{\partial}_b \phi(x) + s\omega_0(x), 1)) \in U \bigcap \hat{\Sigma}, \\ & (\hat{y}, (-2\operatorname{Im} \overline{\partial}_b \phi(y) + s\omega_0(y), 1)) \in U \bigcap \hat{\Sigma},\ |\hat{x} - \hat{y}| < \varepsilon, \text{ for some } \varepsilon > 0\} \end{aligned} \tag{5.65}$$

and

$$\operatorname{Supp} b_j(\hat{x}, \hat{y}, s) \subset \hat{\Omega}, \quad b_j(\hat{x}, \hat{y}, s) \in C^\infty(\hat{\Omega}, T^{*0,q}_{\hat{y}}\hat{D} \boxtimes T^{*0,q}_{\hat{x}}\hat{D}), \quad j = 0, 1, \ldots,$$

and $\Phi(\hat{x}, \hat{y}, s) \in C^\infty(\hat{\Omega})$ is the corresponding critical value. Since V is bounded, there is a constant $M > 0$ so that $|s| < M$, for every $(\hat{x}, \hat{y}, s) \in \hat{\Omega}$. Since S is a pseudodifferential operator, $S(\hat{x}, \hat{y})$ is smoothing away the diagonal $\hat{x} = \hat{y}$. We can take $\varepsilon > 0$ in (5.65) to be any small positive constant. That is, we may assume that $\Phi(\hat{x}, \hat{y}, s)$ and $b_j(\hat{x}, \hat{y}, s)$, $j = 0, 1, \ldots$, are supported in some small neighbourhood of $\hat{x} = \hat{y}$.

From (5.24), it is straightforward to see that we can take $\Phi(\hat{x}, \hat{y}, s)$ so that

$$\Phi(\hat{x}, \hat{y}, s) = x_{2n} - y_{2n} + \varphi(x, y, s), \quad \varphi(x, y, s) \in C^\infty(\Omega), \tag{5.66}$$

where

$$\begin{aligned} \Omega :=& \{(x, y, s) \in D \times D \times \mathbb{R};\ (x, -2\operatorname{Im} \overline{\partial}_b \phi(x) + s\omega_0(x)) \in V \bigcap \Sigma, \\ & (y, -2\operatorname{Im} \overline{\partial}_b \phi(y) + s\omega_0(y)) \in V \bigcap \Sigma,\ |x - y| < \varepsilon, \text{ for some } \varepsilon > 0\}. \end{aligned} \tag{5.67}$$

Since

$$d_{w'}(\psi(\infty, \hat{x}, (w', \alpha_{2n-1}(x) + s\beta_{2n-1}(x), 1)) - < \hat{y}, (w', \alpha_{2n-1}(x) + s\beta_{2n-1}(x), 1)) = 0$$

at $\hat{x} = \hat{y}$, $w' = (\alpha_1(x) + s\beta_1(x), \ldots, \alpha_{2n-2}(x) + s\beta_{2n-2}(x))$, it follows that when $\hat{x} = \hat{y}$, the corresponding critical point is $w' = (\alpha_1(x) + s\beta_1(x), \ldots, \alpha_{2n-2}(x) + s\beta_{2n-2}(x))$ and consequently for every $(\hat{x}, \hat{x}, s) \in \hat{\Omega}$ and every $(x, x, s) \in \Omega$,

$$\varphi(x, x, s) = 0, \tag{5.68}$$

$$\varphi'_x(x, x, s) = (\alpha_1(x) + s\beta_1(x), \ldots, \alpha_{2n-1}(x) + s\beta_{2n-1}(x)) = -2\operatorname{Im} \overline{\partial}_b \phi(x) + s\omega_0(x), \tag{5.69}$$

$$\varphi'_y(x, x, s) = 2\operatorname{Im} \overline{\partial}_b \phi(x) - s\omega_0(x). \tag{5.70}$$

Moreover, from the process above and (5.51), it is easy to see that

$$b_0(\hat{x}, \hat{x}, s) : T^{*0,q}_{\hat{x}}\hat{D} \to \mathcal{N}(x, s, n_-), \quad \forall (\hat{x}, \hat{x}, s) \in \hat{\Omega}, \tag{5.71}$$

where $b_0(\hat{x}, \hat{y}, s)$ is as in (5.64).

The following is essentially well-known (see page 147 of [21] or Proposition B.14 of paper I in [13]).

Proposition 5.23. *With the notations used above, if D and V are small enough, then there is a constant $c > 0$ such that*

$$\begin{aligned} \operatorname{Im} \varphi(x, y, s) \geq c \inf_{w' \in \Lambda} \Big(& \operatorname{Im} \psi(\infty, \hat{x}, (w', \alpha_{2n-1}(x) + s\beta_{2n-1}(x), 1)) \\ & + |d_{w'}\big(\psi(\infty, \hat{x}, (w', \alpha_{2n-1}(x) + s\beta_{2n-1}(x), 1)) - < \hat{y}, (w', \alpha_{2n-1}(x) + s\beta_{2n-1}(x), 1) >)|^2 \Big), \end{aligned} \tag{5.72}$$

for all $(x, y, s) \in \Omega$, where Λ is some open set of the origin in $\mathbb{R}^{2n-2}$.

From now on, we take D and V small enough so that (5.72) holds. We need

Theorem 5.24. *With the notations used above, there is a constant $c > 0$ such that*

$$\operatorname{Im}\varphi(x,y,s) \geq c\,|x'-y'|^2\,, \quad \forall (x,y,s) \in \Omega, \tag{5.73}$$

where $x' = (x_1, \ldots, x_{2n-2})$, $y' = (y_1, \ldots, y_{2n-2})$.

Moreover, if $\varepsilon > 0$ is small enough (ε is as in (5.65)) then there is a constant $c_1 > 0$ such that

$$\operatorname{Im}\varphi(x,y,s) + \left|\frac{\partial\varphi}{\partial s}(x,y,s)\right| \geq c_1\big(\,|x_{2n-1}-y_{2n-1}| + |x'-y'|^2\big), \quad \forall (x,y,s) \in \Omega, \tag{5.74}$$

and

$$\Phi(\hat{x},\hat{y},s) = 0 \text{ and } \tfrac{\partial\Phi}{\partial s}(\hat{x},\hat{y},s) = \tfrac{\partial\varphi}{\partial s}(x,y,s) = 0 \text{ if and only if } x = y. \tag{5.75}$$

Proof. From

$$\begin{aligned}
&\psi(\infty,\hat{x},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1)) - <\hat{y},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1)> \\
&=<\hat{x}-\hat{y},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1)> \\
&\quad + O(|w'-(\alpha_1(x)+s\beta_1(x),\ldots,\alpha_{2n-2}(x)+s\beta_{2n-2}(x))|^2)
\end{aligned}$$

we can check that

$$\begin{aligned}
&d_{w'}\big(\psi(\infty,\hat{x},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1)) - <\hat{y},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1)>\big) \\
&=<x'-y',dw'> + O(|w'-(\alpha_1(x)+s\beta_1(x),\ldots,\alpha_{2n-2}(x)+s\beta_{2n-2}(x))|).
\end{aligned}$$

Thus, there are constants c_1, $c_2 > 0$ such that

$$\begin{aligned}
&\big|d_{w'}\big(\psi(\infty,\hat{x},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1)) - <\hat{y},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1)>\big)\big|^2 \\
&\geq c_1\,|x'-y'|^2 - c_2\,|w'-(\alpha_1(x)+s\beta_1(x),\ldots,\alpha_{2n-2}(x)+s\beta_{2n-2}(x))|^2\,.
\end{aligned}$$

If $\frac{c_1}{2}\,|(x'-y')|^2 \geq c_2\,|w'-(\alpha_1(x)+s\beta_1(x),\ldots,\alpha_{2n-2}(x)+s\beta_{2n-2}(x))|^2$, then

$$\begin{aligned}
&\big|d_{w'}\big(\psi(\infty,\hat{x},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1)) - <\hat{y},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1)>\big)\big|^2 \\
&\geq \frac{c_1}{2}\,|(x'-y')|^2\,.
\end{aligned} \tag{5.76}$$

Now, we assume that

$$|(x'-y')|^2 \leq \frac{2c_2}{c_1}\,|w'-(\alpha_1(x)+s\beta_1(x),\ldots,\alpha_{2n-2}(x)+s\beta_{2n-2}(x))|^2\,.$$

From Theorem 5.6, we have

$$\begin{aligned}
&\operatorname{Im}\psi(\infty,\hat{x},(w',\alpha_{2n-1}(x)+s\beta_{2n-1}(x),1)) \\
&\geq c_3\,|w'-(\alpha_1(x)+s\beta_1(x),\ldots,\alpha_{2n-2}(x)+s\beta_{2n-2}(x))|^2 \geq \frac{c_1c_3}{2c_2}\,|(x'-y')|^2\,,
\end{aligned} \tag{5.77}$$

where c_3 is a positive constant. From (5.76), (5.77) and Proposition 5.23, (5.73) follows.

Now, we prove (5.74). In view of (5.69) and (5.70), we see that $\varphi(x,y,s) =< -2\operatorname{Im}\overline{\partial}_b\phi(x) + s\omega_0(x), x-y> + O(|x-y|^2)$. Thus,

$$\frac{\partial\varphi}{\partial s}(x,y,s) =<\omega_0(x),x-y> + O(|x-y|^2) = \sum_{j=1}^{2n-1}\beta_j(x)(x_j-y_j) + O(|x-y|^2). \tag{5.78}$$

Since $\beta_{2n-1}(x) \geq \frac{1}{2}$ for every $x \in D$, we conclude that if $\varepsilon > 0$ is small then there are constants $c_2 > 0$, $c_3 > 0$, such that

$$\left|\frac{\partial\varphi}{\partial s}(x,y,s)\right| \geq c_2\,|x_{2n-1}-y_{2n-1}| - c_3\,|x'-y'|^2\,, \quad \forall (x,y,s) \in \Omega.$$

Combining this with (5.73), we obtain (5.74).

Finally, from (5.73) and (5.74), it is easy to see that (5.75) holds and the theorem follows. $\square$

From now on, we take $\varepsilon > 0$ small enough so that (5.74) and (5.75) hold.

Remark 5.25. The phase $t\Phi(\hat{x},\hat{y},s)$ is not positively homogeneous with respect to (s,t). Since $t>0$ on $\hat{\Omega}$, we put $\Phi_0(\hat{x},\hat{y},s,t):=t\Phi(\hat{x},\hat{y},\frac{s}{t})$, $\Phi_0(\hat{x},\hat{y},s,t)\in C^\infty(\hat{\Omega}_0)$, where

$$\begin{aligned}\hat{\Omega}_0=\{&(\hat{x},\hat{y},s,t)\in\hat{D}\times\hat{D}\times\mathbb{R}\times\mathbb{R}_+;\ (\hat{x},(-2\mathrm{Im}\,\overline{\partial}_b\phi(x)t+s\omega_0(x),t))\in U\bigcap\hat{\Sigma},\\&(\hat{y},(-2\mathrm{Im}\,\overline{\partial}_b\phi(y)t+s\omega_0(y),t))\in U\bigcap\hat{\Sigma},\ \ |\hat{x}-\hat{y}|<\varepsilon\}.\end{aligned}$$

It is easy to see that Φ_0 is a complex valued phase function in the sense of Melin-Sjöstrand(see Definition 3.5 in [21]). By using the identification $t\Phi\leftrightarrow\Phi_0$, the framework of complex Fourier integral operators in [21] works well in this non-homogeneous case.

We pause and introduce some notations. Let $g(x,y,s)\in C^\infty(\Omega)$ be a complex valued smooth function. Assume that $g(x,y,s)=0$ if and only if $x=y$. For $p\in D$, put

(5.79)
$$\begin{aligned}&T_{(p,p,s)}H_g\\&=\left\{(a_1,\ldots,a_{2n-1},b_1,\ldots,b_{2n-1})\in\mathbb{C}^{2n-1}\times\mathbb{C}^{2n-1};\ \sum_{j=1}^{2n-1}\big(a_j\frac{\partial^2 g}{\partial x_j\partial s}(p,p,s)+b_j\frac{\partial^2 g}{\partial y_j\partial s}(p,p,s)\big)=0\right\}.\end{aligned}$$

The tangential Hessian of $g(x,y,s)$ at $(p,p,s)\in\Omega$ is the bilinear map $\mathrm{Hess}\,(g,T_{(p,p,s)}H_g):T_{(p,p,s)}H_g\times T_{(p,p,s)}H_g\to\mathbb{C}$ given by

$$\begin{aligned}&\mathrm{Hess}\,(g,T_{(p,p,s)}H_g):T_{(p,p,s)}H_g\times T_{(p,p,s)}H_g\to\mathbb{C},\\&\qquad\qquad(u,v)\to<g''(p,p)u,v>,\quad u,v\in T_{(p,p,s)}H_g,\end{aligned}\tag{5.80}$$

where $g''=\begin{bmatrix}g''_{xx}&g''_{xy}\\g''_{yx}&g''_{yy}\end{bmatrix}$. More precisely, if we put $u=(a_1,\ldots,a_{2n-1},b_1,\ldots,b_{2n-1})\in\mathbb{C}^{2n-1}\times\mathbb{C}^{2n-1}$, $v=(c_1,\ldots,c_{2n-1},d_1,\ldots,d_{2n-1})\in\mathbb{C}^{2n-1}\times\mathbb{C}^{2n-1}$, then

$$\begin{aligned}&<g''(p,p)u,v>\\&=\sum_{s,t=1}^{2n-1}\Big(c_sa_t\frac{\partial^2 g}{\partial x_s\partial x_t}(p,p,s)+d_sa_t\frac{\partial^2 g}{\partial y_s\partial x_t}(p,p,s)+c_sb_t\frac{\partial^2 g}{\partial x_s\partial y_t}(p,p,s)+d_sb_t\frac{\partial^2 g}{\partial y_s\partial y_t}(p,p,s).\end{aligned}$$

In view of (5.69) and (5.70), it is easy to see that $T_{(p,p,s)}H_\varphi$ is spanned by

$$(u,v),\ \ \big(T(p),T(p)\big),\ \ u,v\in T^{1,0}_pX\oplus T^{0,1}_pX.\tag{5.81}$$

Let $\mathcal{U}$ be an open set in $\mathbb{R}^N$. We let $\mathcal{U}^{\mathbb{C}}$ be an almost comlexification of $\mathcal{U}$. That is, $\mathcal{U}^{\mathbb{C}}$ is an open set in $\mathbb{C}^N$ with $\mathcal{U}^{\mathbb{C}}\bigcap\mathbb{R}^N=\mathcal{U}$. For any smooth function $f\in C^\infty(\mathcal{U})$, we write $\widetilde{f}\in C^\infty(\mathcal{U}^{\mathbb{C}})$ to denote an almost analytic extension of f. (See Chapter 1 of Melin-Sjöstrand [21], for the precise meaning of "almost analytic extension "). We need

Lemma 5.26. *Let $\upsilon(x,y,s)\in C^\infty(\Omega)$. We assume that $\upsilon(x,y,s)$ satisfies* (5.68)*,* (5.69)*,* (5.70)*,* (5.73) *and* (5.74)*. If D is small enough then for every $(x_0,x_0,s_0)\in\Omega$, we can find a function $\hat{\upsilon}(x,y,s)\in C^\infty(\Lambda)$, where $\Lambda\subset\Omega$ is a small neighbourhood of (x_0,x_0,s_0), such that $\hat{\upsilon}(x,y,s)$ satisfies* (5.68)*,* (5.69)*,* (5.70)*,* (5.73) *and* (5.74) *and $\frac{\partial\hat{\upsilon}}{\partial y_{2n-1}}(x,y,s)-(\alpha_{2n-1}(y)+s\beta_{2n-1}(y))$ vanishes to infinite order at $x=y$, $\mathrm{Hess}\,(\upsilon,T_{(x,x,s)}H_\upsilon)=\mathrm{Hess}\,(\hat{\upsilon},T_{(x,x,s)}H_{\hat{\upsilon}})$, $\forall(x,x,s)\in\Lambda$, and $t\Upsilon(\hat{x},\hat{y},s):=t(x_{2n}-y_{2n}+\upsilon(x,y,s))$ and $t\hat{\Upsilon}(\hat{x},\hat{y},s):=t(x_{2n}-y_{2n}+\hat{\upsilon}(x,y,s))$ are equivalent for classical symbols at every point of*

$$\mathrm{diag}'\Big((U\bigcap\hat{\Sigma})\times(U\bigcap\hat{\Sigma})\Big)\bigcap\Big\{(\hat{x},\hat{x},td_{\hat{x}}\Upsilon(\hat{x},\hat{x},s),-td_{\hat{x}}\Upsilon(\hat{x},\hat{x},s))\in T^*\hat{D};\ (x,x,s)\in\Lambda,t>0\Big\}$$

in the sense of Melin-Sjöstrand [21]*.(Remind that $\mathrm{diag}'\Big((U\bigcap\hat{\Sigma})\times(U\bigcap\hat{\Sigma})\Big)$ is given by* (5.21)*.)*

Proof. We first claim that we can find $g(x,y,s)\in C^\infty(\Lambda)$ with $g(x,x,s)=s$, where $\Lambda\subset\Omega$ is a small neighbourhood of (x_0,x_0,s_0), such that if we put

$$\upsilon_1(x,y,s):=\widetilde{\upsilon}(x,y,g(x,y,s))$$

then $\frac{\partial \upsilon_1}{\partial y_{2n-1}}(x,y,s)-(\alpha_{2n-1}(y)+s\beta_{2n-1}(y))$ vanishes to infinite order at $x=y$. We formally set $g(y,y,s)=s$ and $\frac{\partial \upsilon_1}{\partial y_{2n-1}}(x,y,s)=\alpha_{2n-1}(y)+s\beta_{2n-1}(y)\mod O(|x-y|^\infty)$. Then,

$$\begin{aligned}&\frac{\partial\widetilde{\upsilon}}{\partial y_{2n-1}}(x,y,g(x,y,s))+\frac{\partial\widetilde{\upsilon}}{\partial s}(x,y,g(x,y,s))\frac{\partial g}{\partial y_{2n-1}}(x,y,s)\\&=\alpha_{2n-1}(y)+s\beta_{2n-1}(y)\mod O(|x-y|^\infty).\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{\partial^2\widetilde{\upsilon}}{\partial y_{2n-1}^2}(x,y,g(x,y,s))+2\frac{\partial^2\widetilde{\upsilon}}{\partial s\partial y_{2n-1}}(x,y,g(x,y,s))\frac{\partial g}{\partial y_{2n-1}}(x,y,s)\\&+\frac{\partial^2\widetilde{\upsilon}}{\partial s^2}(x,y,g(x,y,s))(\frac{\partial g}{\partial y_{2n-1}}(x,y,s))^2+\frac{\partial\widetilde{\upsilon}}{\partial s}(x,y,g(x,y,s))\frac{\partial^2 g}{\partial y_{2n-1}^2}(x,y,s)\\&=\frac{\partial\alpha_{2n-1}}{\partial y_{2n-1}}(y)+s\frac{\partial\beta_{2n-1}}{\partial y_{2n-1}}(y)\mod O(|x-y|^\infty).\end{aligned}\tag{5.82}$$

Note that $\frac{\partial^2\widetilde{\upsilon}}{\partial s\partial y_{2n-1}}(y,y,g(y,y,s))=\frac{\partial^2\widetilde{\upsilon}}{\partial s\partial y_{2n-1}}(y,y,s)\neq 0$(see (5.78)) and

$$\frac{\partial\widetilde{\upsilon}}{\partial s}(y,y,g(y,y,s))=\frac{\partial\widetilde{\upsilon}}{\partial s^2}(y,y,g(y,y,s))=0.$$

From this observation and (5.82), we can determine $\frac{\partial g}{\partial y_{2n-1}}(x,y,s)|_{x=y}$. Continuing in this way, we can determine $\frac{\partial^{|\alpha|}g}{\partial y^\alpha}(x,y,s)|_{x=y}$, for every multiindex $\alpha=(\alpha_1,\ldots,\alpha_{2n-1})\in\mathbb{N}_0^{2n-1}$. By using Borel construction, the claim follows.

Since $\alpha_{2n-1}(y)+s\beta_{2n-1}(y)$ is real, we have

$$\frac{\partial^{|\alpha|+1}\mathrm{Im}\,\upsilon_1}{\partial y^\alpha\partial y_{2n-1}}(x,y,s)|_{x=y}=0,\tag{5.83}$$

for every multiindex $\alpha=(\alpha_1,\ldots,\alpha_{2n-1})\in\mathbb{N}_0^{2n-1}$. Moreover, from (5.73), it is straightforward to see that if D is small enough then,

$$\left(\tfrac{\partial^2\mathrm{Im}\,\upsilon_1}{\partial y_j\partial y_t}(x,y,s)|_{x=y}\right)_{j,t=1}^{2n-2}\text{ is positive definite at each point of }(x,x,s)\in\Lambda.\tag{5.84}$$

From (5.83) and (5.84), we deduce that for every $N>0$, there is a $C_N>0$, such that

$$\mathrm{Im}\,\upsilon_1(x,y,s)+\frac{1}{C_N}\left|x-y\right|^N\geq C_N\left|x'-y'\right|^2,\quad\forall(x,y,s)\in\Lambda.\tag{5.85}$$

From (5.83), (5.85) and the standard Borel construction, we can find $\hat{\upsilon}(x,y,s)\in C^\infty(\Lambda)$ such that $\hat{\upsilon}(x,y,s)-\upsilon_1(x,y,s)$ vanishes to infinite order at $x=y$ and $\mathrm{Im}\,\hat{\upsilon}(x,y,s)\geq C_0\left|x'-y'\right|^2$, $\forall(x,y,s)\in\Lambda$, where $C_0>0$ is a constant. Since $\hat{\upsilon}(x,y,s)-\upsilon_1(x,y,s)$ vanishes to infinite order at $x=y$, $\upsilon(x,y,s)$ satisfies (5.68), (5.69), (5.70), (5.74) and $\frac{\partial\hat{\upsilon}}{\partial y_{2n-1}}(x,y,s)-(\alpha_{2n-1}(y)+s\beta_{2n-1}(y))$ vanishes to infinite order at $x=y$.

Now, we prove that $t\Upsilon(\hat{x},\hat{y},s):=t(x_{2n}-y_{2n}+\upsilon(x,y,s))$ and $t\hat{\Upsilon}(\hat{x},\hat{y},s):=t(x_{2n}-y_{2n}+\hat{\upsilon}(x,y,s))$ are equivalent for classical symbols at every point of

$$\mathrm{diag}'\Big((U\bigcap\hat{\Sigma})\times(U\bigcap\hat{\Sigma})\Big)\bigcap\Big\{(\hat{x},\hat{x},td_{\hat{x}}\Upsilon(\hat{x},\hat{x},s),-td_{\hat{x}}\Upsilon(\hat{x},\hat{x},s))\in T^*\hat{D};\,(x,x,s)\in\Lambda,t>0\Big\}.$$

Fix $(\hat{x}_0,\hat{x}_0,s_0)\in\hat{\Omega}$, $(x,x,s)\in\Lambda$, $t_0>0$ and set

$$\begin{aligned}&\hat{x}_0=(x_0,x_{n,0}),\quad x_0\in\mathbb{R}^{2n-1},\quad(x_0,x_0,s_0)\in\Omega,\\&(\hat{x}_0,\hat{\xi}_0)=(\hat{x}_0,t_0\frac{\partial\hat{\Upsilon}}{\partial\hat{x}}(\hat{x}_0,\hat{x}_0,s_0))=(\hat{x}_0,t_0\frac{\partial\Upsilon}{\partial\hat{x}}(\hat{x}_0,\hat{x}_0,s_0))\in\Big(U\bigcap\hat{\Sigma}\Big)\bigcap T^*\hat{D}.\end{aligned}$$

Let $\hat{W}$ be a small neighbourhood of $(\hat{x}_0,\hat{x}_0,s_0)$ and let I_0 be a neighbourhood of t_0 in $\mathbb{R}_+$. Put

$$
\begin{aligned}
\Lambda_{\widetilde{t\hat{\Upsilon}}} :=&\{(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{t}\frac{\partial\widetilde{\hat{\Upsilon}}}{\partial\widetilde{x}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}),\widetilde{t}\frac{\partial\widetilde{\hat{\Upsilon}}}{\partial\widetilde{y}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}))\in\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n};\\
&\widetilde{\hat{\Upsilon}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=0,\frac{\partial\widetilde{\hat{\Upsilon}}}{\partial\widetilde{s}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=0,(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})\in\hat{W}^{\mathbb{C}},\widetilde{t}\in I_0^{\mathbb{C}}\},\\
\Lambda_{\widetilde{t\Upsilon}} :=&\{(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{t}\frac{\partial\widetilde{\Upsilon}}{\partial\widetilde{x}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}),\widetilde{t}\frac{\partial\widetilde{\Upsilon}}{\partial\widetilde{y}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}))\in\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n};\\
&\widetilde{\Upsilon}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=0,\frac{\partial\widetilde{\Upsilon}}{\partial\widetilde{s}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=0,(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})\in\hat{W}^{\mathbb{C}},\widetilde{t}\in I_0^{\mathbb{C}}\}.
\end{aligned}
\tag{5.86}
$$

From global theory of complex Fourier integral operators of Melin-Söstrand [21], we know that $t\hat{\Upsilon}(\hat{x},\hat{y},s)$ and $t\Upsilon(\hat{x},\hat{y},s)$ are equivalent for classical symbols at $(\hat{x}_0,\hat{x}_0,\hat{\xi}_0,-\hat{\xi}_0)\in\operatorname{diag}'\Big((U\bigcap\hat{\Sigma})\times(U\bigcap\hat{\Sigma})\Big)$ in the sense of Melin-Sjöstrand [21] if and only if $\Lambda_{\widetilde{t\hat{\Upsilon}}}$ and $\Lambda_{\widetilde{t\Upsilon}}$ are equivalent in the sense that there is a neighbourhood Q of $(\hat{x}_0,\hat{x}_0,\hat{\xi}_0,-\hat{\xi}_0)$ in $\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n}$, such that for every $N>0$, we have

$$
\begin{aligned}
&\operatorname{dist}(z,\Lambda_{\widetilde{t\hat{\Upsilon}}})\leq C_N\left|\operatorname{Im}z\right|^N,\quad\forall z\in Q\bigcap\Lambda_{\widetilde{t\Upsilon}},\\
&\operatorname{dist}(z_1,\Lambda_{\widetilde{t\Upsilon}})\leq C_N\left|\operatorname{Im}z_1\right|^N,\quad\forall z_1\in Q\bigcap\Lambda_{\widetilde{t\hat{\Upsilon}}},
\end{aligned}
\tag{5.87}
$$

where $C_N>0$ is independent of z and z_1. Put $t\Upsilon_1(\hat{x},\hat{y},s)=t(x_{2n}-y_{2n}+\upsilon_1(x,y,s))$. We take almost analytic extensions of $t\hat{\Upsilon}$, $t\Upsilon$ and $t\Upsilon_1$ such that

$$
\begin{aligned}
&\widetilde{t\hat{\Upsilon}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=\widetilde{t}\widetilde{\hat{\Upsilon}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=\widetilde{t}(\widetilde{x}_{2n}-\widetilde{y}_{2n})+\widetilde{t}\widetilde{\hat{\upsilon}}(\widetilde{x},\widetilde{y},\widetilde{s}),\\
&\widetilde{t\Upsilon}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=\widetilde{t}\,\widetilde{\Upsilon}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=\widetilde{t}(\widetilde{x}_{2n}-\widetilde{y}_{2n})+\widetilde{t}\widetilde{\upsilon}(\widetilde{x},\widetilde{y},\widetilde{s}),\\
&\widetilde{t\Upsilon_1}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=\widetilde{t}\,\widetilde{\Upsilon_1}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=\widetilde{t}(\widetilde{x}_{2n}-\widetilde{y}_{2n})+\widetilde{t}\widetilde{\upsilon}_1(\widetilde{x},\widetilde{y},\widetilde{s}),
\end{aligned}
\tag{5.88}
$$

and near $(\hat{x}_0,\hat{x}_0,\hat{\xi}_0,-\hat{\xi}_0)$, we have

$$
\Lambda_{\widetilde{t\Upsilon}}=\Lambda_{\widetilde{t\Upsilon_1}},
\tag{5.89}
$$

where $\Lambda_{\widetilde{t\Upsilon_1}}$ is defined as in (5.86), $(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})\in\hat{W}^{\mathbb{C}}$, $\widetilde{t}\in I_0^{\mathbb{C}}$. Thus, we only need to prove that $\Lambda_{\widetilde{t\hat{\Upsilon}}}$ and $\Lambda_{\widetilde{t\Upsilon_1}}$ are equivalent in the sense of (5.87).

Since $\hat{\upsilon}(x,y,s)-\upsilon_1(x,y,s)$ vanishes to infinite order at $x=y$, it is straightforward to see that (see section 12) there is a neighbourhood Q of $(\hat{x}_0,\hat{x}_0,\hat{\xi}_0,-\hat{\xi}_0)$ in $\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n}$, such that for every $N>0$ and every $z=(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{t}\frac{\partial\widetilde{\Upsilon}}{\partial\widetilde{x}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}),\widetilde{t}\frac{\partial\widetilde{\Upsilon}}{\partial\widetilde{y}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}))\in Q\bigcap\Lambda_{\widetilde{t\Upsilon}}$, we have

$$
\operatorname{dist}(z,\Lambda_{\widetilde{t\hat{\Upsilon}}})\leq C_N\Big(\left|\operatorname{Im}(\widetilde{x}',\widetilde{y},\widetilde{s})\right|^N+\left|\operatorname{Re}\widetilde{x}'-\operatorname{Re}\widetilde{y}\right|^N\Big),
\tag{5.90}
$$

where $C_N>0$ is independent of $z\in Q$. Moreover, we can repeat the process in section 12 and conclude that if Q is small enough then there is a constant $C_1>0$ independent of $z\in Q$ such that

$$
\left|\operatorname{Im}(\widetilde{x}',\widetilde{y},\widetilde{s})\right|+\left|\operatorname{Re}\widetilde{x}'-\operatorname{Re}\widetilde{y}\right|\leq C_1\left|\operatorname{Im}z\right|,
\tag{5.91}
$$

for every $z=(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{t}\frac{\partial\widetilde{\Upsilon}}{\partial\widetilde{x}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}),\widetilde{t}\frac{\partial\widetilde{\Upsilon}}{\partial\widetilde{y}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}))\in Q\bigcap\Lambda_{\widetilde{t\Upsilon}}$. From (5.90) and (5.91), we get the first formula in (5.87). Similarly, we can repeat the process above and conclude the second formula in (5.87). Moreover, from the construction above, it is easy to see that

$$
\operatorname{Hess}(\upsilon,T_{(x,x,s)}H_\upsilon)=\operatorname{Hess}(\hat{\upsilon},T_{(x,x,s)}H_{\hat{\upsilon}}),\quad\forall(x,x,s)\in\Lambda.
$$

The lemma follows. □

Definition 5.27. Let $\Phi_1(\hat{x},\hat{y},s)=x_{2n}-y_{2n}+\varphi_1(x,y,s)\in C^\infty(\hat{\Omega})$, $\Phi_2(\hat{x},\hat{y},s)=x_{2n}-y_{2n}+\varphi_2(x,y,s)\in C^\infty(\hat{\Omega})$, $,\varphi_1(x,y,s),\varphi_2(x,y,s)\in C^\infty(\Omega)$. We assume that φ_1 and φ_2 satisfy (5.68), (5.69), (5.70), (5.73) and (5.74). Let $(x_0,x_0,s_0)\in\Omega$. From Lemma 5.26, we know that in some small neighbourhood $\Lambda\subset\Omega$ of (x_0,x_0,s_0), there are $\hat{\varphi}_1(x,y,s),\hat{\varphi}_2(x,y,s)\in C^\infty(\Lambda)$ such that φ_1

and φ_2 satisfy (5.68), (5.69), (5.70), (5.73), (5.74) and $\frac{\partial\hat\varphi_1}{\partial y_{2n-1}}(x,y,s)-(\alpha_{2n-1}(y)+s\beta_{2n-1}(y))$ and $\frac{\partial\hat\varphi_2}{\partial y_{2n-1}}(x,y,s)-(\alpha_{2n-1}(y)+s\beta_{2n-1}(y))$ vanish to infinite order at $x=y$, $\mathrm{Hess}\,(\varphi_1,T_{(x,x,s)}H_{\varphi_1})=\mathrm{Hess}\,(\hat\varphi_1,T_{(x,x,s)}H_{\hat\varphi_1})$, $\forall(x,x,s)\in\Lambda$, $\mathrm{Hess}\,(\varphi_2,T_{(x,x,s)}H_{\varphi_2})=\mathrm{Hess}\,(\hat\varphi_2,T_{(x,x,s)}H_{\hat\varphi_2})$, $\forall(x,x,s)\in\Lambda$ and $t\hat\Phi_1(\hat x,\hat y,s):=t(x_{2n}-y_{2n}+\hat\varphi_1(x,y,s))$ and $t\Phi_1(\hat x,\hat y,s)$ are equivalent for classical symbols at every point of

$$\mathrm{diag}'\Big((U\bigcap\hat\Sigma)\times(U\bigcap\hat\Sigma)\Big)\bigcap\Big\{(\hat x,\hat x,td_{\hat x}\Phi_1(\hat x,\hat x,s),-td_{\hat x}\Phi_1(\hat x,\hat x,s))\in T^*\hat D;\ (x,x,s)\in\Lambda,t>0\Big\}$$

in the sense of Melin-Sjöstrand [21], $t\hat\Phi_2(\hat x,\hat y,s):=t(x_{2n}-y_{2n}+\hat\varphi_2(x,y,s))$ and $t\Phi_2(\hat x,\hat y,s)$ are equivalent for classical symbols at every point of

$$\mathrm{diag}'\Big((U\bigcap\hat\Sigma)\times(U\bigcap\hat\Sigma)\Big)\bigcap\Big\{(\hat x,\hat x,td_{\hat x}\Phi_2(\hat x,\hat x,s),-td_{\hat x}\Phi_2(\hat x,\hat x,s))\in T^*\hat D;\ (x,x,s)\in\Lambda,t>0\Big\}$$

in the sense of Melin-Sjöstrand [21]. We say that $\varphi_1(x,y,s)$ and $\varphi_2(x,y,s)$ are equivalent at (x_0,x_0,s) if there are functions $f\in C^\infty(\Lambda')$, $g_j\in C^\infty(\Lambda')$, $j=0,1,\ldots,2n-1$, $p_j\in C^\infty(\Lambda')$, $j=1,\ldots,2n-1$, such that

$$\begin{aligned}
&\frac{\partial\hat\varphi_1}{\partial s}(x,y,s)-f(x,y,s)\frac{\partial\hat\varphi_2}{\partial s}(x,y,s),\\
&\hat\varphi_1(x,y,s)-\hat\varphi_2(x,y,s)=g_0(x,y,s)\frac{\partial\hat\varphi_1}{\partial s}(x,y,s),\\
&\frac{\partial\hat\varphi_1}{\partial x_j}(x,y,s)-\frac{\partial\hat\varphi_2}{\partial x_j}(x,y,s)=g_j(x,y,s)\frac{\partial\hat\varphi_1}{\partial s}(x,y,s),\quad j=1,2,\ldots,2n-1,\\
&\frac{\partial\hat\varphi_1}{\partial y_j}(x,y,s)-\frac{\partial\hat\varphi_2}{\partial y_j}(x,y,s)=p_j(x,y,s)\frac{\partial\hat\varphi_1}{\partial s}(x,y,s),\quad j=1,2,\ldots,2n-1,
\end{aligned}$$

vanish to infinite order on $x=y$, for every $(x,y,s)\in\Lambda'$, where $\Lambda'\subset\Lambda$ is a small neighbourhood of (x_0,x_0,s_0).

We have

Theorem 5.28. *Let* $\Phi_1(\hat x,\hat y,s)=x_{2n}-y_{2n}+\varphi_1(x,y,s)\in C^\infty(\hat\Omega)$, $\varphi_1(x,y,s)\in C^\infty(\Omega)$. *Assume that* φ_1 *satisfis* (5.68), (5.69), (5.70), (5.73) *and* (5.74). *Then* $t\Phi(\hat x,\hat y,s)$ *and* $t\Phi_1(\hat x,\hat y,s)$ *are equivalent for classical symbols at every point of*

$$\mathrm{diag}'\Big((U\bigcap\hat\Sigma)\times(U\bigcap\hat\Sigma)\Big)\bigcap\Big\{(\hat x,\hat x,\hat\xi,-\hat\xi);\ (\hat x,\hat\xi)\in T^*\hat D\Big\}.$$

(remind that $\mathrm{diag}'\Big((U\bigcap\hat\Sigma)\times(U\bigcap\hat\Sigma)\Big)$ *is given by* (5.21)*) in the sense of Melin-Sjöstrand* [21] *if and only if* $\varphi(x,y,s)$ *and* $\varphi_1(x,y,s)$ *are equivalent at each point of* Ω *in the sense of Definition 5.27.*

The proof is straightforward and follows from global theory of complex Fourier integral operators of Melin-Sjöstrand [21]. We put the proof in section 12.

We notice that $\psi(\infty,\hat x,\hat\eta)-<\hat y,\hat\eta>$ and $t\Phi(\hat x,\hat y,s)$ are equivalent for classical symbols at every point of $\mathrm{diag}'\Big((U\bigcap\hat\Sigma)\times(U\bigcap\hat\Sigma)\Big)\bigcap\Big\{(\hat x,\hat x,\hat\xi,-\hat\xi);\ (\hat x,\hat\xi)\in T^*\hat D\Big\}$ in the sense of Melin-Sjöstrand [21]. Consider

$$-\overline\Phi(\hat y,\hat x,s)=x_{2n}-y_{2n}-\overline\varphi(y,x,s).$$

From Theorem 5.7, we see that $t\Phi(\hat x,\hat y,s)$ and $-t\overline\Phi(\hat y,\hat x,s)$ are equivalent for classical symbols at every point of $\mathrm{diag}'\Big((U\bigcap\hat\Sigma)\times(U\bigcap\hat\Sigma)\Big)$ in the sense of Melin-Sjöstrand [21]. Note that $-\overline\varphi(y,x,s)$ satisfies (5.68), (5.69), (5.70), (5.73) and (5.74). From Theorem 5.28, we see that $\varphi(x,y,s)$ and $-\overline\varphi(y,x,s)$ are equivalent at each point of Ω in the sense of Definition 5.27.

Summing up, we obtain the main result of this section

Theorem 5.29. *With the notations and assumptions above. Let* $S=S(\hat x,\hat y)\in L^0_{\frac12,\frac12}(\hat D,T^{*0,q}\hat D\boxtimes T^{*0,q}\hat D)$ *be as in Theorem 5.21. Then, on* $\hat D$, *we have*

$$S(\hat x,\hat y)\equiv\int e^{it\Phi(\hat x,\hat y,s)}b(\hat x,\hat y,s,t)dsdt \tag{5.92}$$

with

$$b(\hat{x},\hat{y},s,t)\sim\sum_{j=0}^{\infty}b_j(\hat{x},\hat{y},s)t^{n-j} \tag{5.93}$$

in $S^n_{1,0}(\hat{\Omega}\times]0,\infty[,T^{*0,q}_{\hat{y}}\hat{D}\boxtimes T^{*0,q}_{\hat{x}}\hat{D})$*,* $\mathrm{Supp}\,b(\hat{x},\hat{y},s,t)\subset\hat{\Omega}\times\mathbb{R}_+$*,*

$$b_0(\hat{x},\hat{x},s):T^{*0,q}_{\hat{x}}\hat{D}\to\mathcal{N}(x,s,n_-),\quad\forall(\hat{x},\hat{x},s)\in\hat{\Omega}, \tag{5.94}$$

where $\mathcal{N}(x,s,n_-)$ *is given by* (5.39)*,*

$$\begin{split}\hat{\Omega}:=&\{(\hat{x},\hat{y},s)\in\hat{D}\times\hat{D}\times\mathbb{R};\;(\hat{x},(-2\mathrm{Im}\,\overline{\partial}_b\phi(x)+s\omega_0(x),1))\in U\bigcap\hat{\Sigma},\\&(\hat{y},(-2\mathrm{Im}\,\overline{\partial}_b\phi(y)+s\omega_0(y),1))\in U\bigcap\hat{\Sigma},\;|\hat{x}-\hat{y}|<\varepsilon,\text{ for some }\varepsilon>0\},\end{split} \tag{5.95}$$

$$\mathrm{Supp}\,b_j(\hat{x},\hat{y},s)\subset\hat{\Omega},\quad b_j(\hat{x},\hat{y},s)\in C^\infty(\hat{\Omega},T^{*0,q}_{\hat{y}}\hat{D}\boxtimes T^{*0,q}_{\hat{x}}\hat{D}),\quad j=0,1,\ldots,$$

$$\begin{aligned}&\Phi(\hat{x},\hat{y},s)=x_{2n}-y_{2n}+\varphi(x,y,s),\\&\varphi(x,y,s)\in C^\infty(\Omega),\quad\Omega=\left\{(x,y,s)\in D\times D\times\mathbb{R};\;(\hat{x},\hat{y},s)\in\hat{\Omega}\right\},\end{aligned}$$

and $\varphi(x,y,s)$ *satisfies* (5.68)*,* (5.69)*,* (5.70)*,* (5.73) *and* (5.74)*. Furthermore,* $\varphi(x,y,s)$ *and* $-\overline{\varphi}(y,x,s)$ *are equivalent at each point of* Ω *in the sense of Definition 5.27.*

Moreover, the phase $t\Phi(\hat{x},\hat{y},s)$ *can be characterized as follows: Let* $\Phi_1(\hat{x},\hat{y},s)=x_{2n}-y_{2n}+\varphi_1(x,y,s)\in C^\infty(\hat{\Omega})$*,* $\varphi_1(x,y,s)\in C^\infty(\Omega)$*. We assume that* φ_1 *satisfies* (5.68)*,* (5.69)*,* (5.70)*,* (5.73) *and* (5.74)*. Then* $t\Phi(\hat{x},\hat{y},s)$ *and* $t\Phi_1(\hat{x},\hat{y},s)$ *are equivalent for classical symbols at every point of*

$$\mathrm{diag}'\Big((U\bigcap\hat{\Sigma})\times(U\bigcap\hat{\Sigma})\Big)\bigcap\Big\{(\hat{x},\hat{x},\hat{\xi},-\hat{\xi});\;(\hat{x},\hat{\xi})\in T^*\hat{D}\Big\}$$

in the sense of Melin-Sjöstrand [21] *if and only if* $\varphi(x,y,s)$ *and* $\varphi_1(x,y,s)$ *are equivalent at each point of* Ω *in the sense of Definition 5.27.*

5.4. **The tangential Hessian of** $\varphi(x,y,s)$**.** In this section, we will calculate the tangential Hessian of $\varphi(x,y,s)$ and we will use the same notations as before. Let $x=(x_1,\ldots,x_{2n-1})$ be local coordinates on D. The following is straightforward. We omit the proof.

Proposition 5.30. *Let* $\varphi_1(x,y,s),\varphi_2(x,y,s)\in C^\infty(\hat{\Omega})$*. We assume that* φ_1 *and* φ_2 *satisfy* (5.68)*,* (5.69)*,* (5.70)*,* (5.73) *and* (5.74)*. Then,* $T_{(x,x,s)}H_{\varphi_1}=T_{(x,x,s)}H_{\varphi_2}$*, for every* $(x,x,s)\in\Omega$*. Assume further that* $\varphi_1(x,y,s)$ *and* $\varphi_2(x,y,s)$ *are equivalent at each point of* Ω *in the sense of Definition 5.27. Then,* $\mathrm{Hess}\,(\varphi_1,T_{(x,x,s)}H_{\varphi_1})=\mathrm{Hess}\,(\varphi_2,T_{(x,x,s)}H_{\varphi_2})$*,* $\forall(x,x,s)\in\Omega$*.*

In particular, if we put $\varphi_1(x,y,s)=-\overline{\varphi}(y,x,s)$ *then* $\mathrm{Hess}\,(\varphi,T_{(x,x,s)}H_\varphi)=\mathrm{Hess}\,(\varphi_1,T_{(x,x,s)}H_{\varphi_1})$*,* $\forall(x,x,s)\in\Omega$*.*

From Proposition 5.30, we know that the tangential Hessian of φ at $(x,x,s)\in\Omega$ is uniquely determined in the equivalence class of φ in the sense of Definition 5.27. In the rest of this section, we will determine the tangential Hessian of $\varphi(x,y,s)$ at $(x,x,s)\in\Omega$. Until further notice, we fix $(p,p,s_0)\in\Omega$. Recall that (see (5.14) and (5.67)) $M^\phi_p-2s_0\mathcal{L}_p$ is non-degenerate of constant signature (n_-,n_+). We can repeat the proof of Lemma 8.1 in [14] with minor change and conclude that

Proposition 5.31. *Let* $\overline{Z}_{1,s_0},\ldots,\overline{Z}_{n-1,s_0}$ *be an orthonormal frame of* $T^{1,0}_xX$ *varying smoothly with* x *in a neighbourhood of* p*, for which the Hermitian quadratic form* $M^\phi_x-2s_0\mathcal{L}_x$ *is diagonalized at* p*. That is,*

$$M^\phi_p\big(\overline{Z}_{j,s_0}(p),Z_{t,s_0}(p)\big)-2s_0\mathcal{L}_p\big(\overline{Z}_{j,s_0}(p),Z_{t,s_0}(p)\big)=\lambda_j(s_0)\delta_{j,t},\quad j,t=1,\ldots,n-1. \tag{5.96}$$

Assume that $\lambda_j(s_0)<0$*,* $j=1,\ldots,n_-$*,* $\lambda_j(s_0)>0$*,* $j=n_-+1,\ldots,n-1$*. Let* $x=(x_1,\ldots,x_{2n-1})$ *be local coordinates of* X *defined in some small neighbourhood of* p *such that* $x(p)=0$*. Let* $h_j(x,\xi)$ *be* *the principal symbol of Z_{j,s_0}, $j=1,\ldots,n-1$. Then in some open neighbourhood $W\subset\Omega$ of (p,p,s_0), there exist $g_j(x,y,s)\in C^\infty(W)$, $j=1,\ldots,n-1$, such that*

(5.97)
$$\begin{aligned}&\overline{h}_j(x,\varphi'_x(x,y,s_0))+(\overline{Z}_{j,s_0}\phi)(x)=g_j(x,y,s_0)\frac{\partial\varphi}{\partial s}(x,y,s_0)+O(|(x,y)|^2),\quad j=1,\ldots,n_-,\\&h_j(x,\varphi'_x(x,y,s_0))+(Z_{j,s_0}\phi)(x)=g_j(x,y,s_0)\frac{\partial\varphi}{\partial s}(x,y,s_0)+O(|(x,y)|^2),\quad j=n_-+1,\ldots,n-1.\end{aligned}$$

Let $\overline{Z}_{1,s_0},\ldots,\overline{Z}_{n-1,s_0}$ be as in Proposition 5.31. We take local coordinates $x=(x_1,\ldots,x_{2n-1})$, $z_j=x_{2j-1}+ix_{2j}$, $j=1,\ldots,n-1$, defined in some small neighbourhood of p so that (2.2) hold. From (2.2), (5.69) and (5.70) it is not difficult to see that

$$\begin{aligned}&\frac{\partial^2\varphi}{\partial s\partial x_j}(p,p,s_0)=\frac{\partial^2\varphi}{\partial s\partial y_j}(p,p,s_0)=0,\quad j=1,\ldots,2n-2,\\&\frac{\partial^2\varphi}{\partial s\partial x_{2n-1}}(p,p,s_0)=1,\quad\frac{\partial^2\varphi}{\partial s\partial y_{2n-1}}(p,p,s_0)=-1.\end{aligned}\tag{5.98}$$

From (5.98), it is easy to see that to determine the tangential Hessian of $\varphi(x,y,s)$ at (p,p,s_0) is equivalent to determine

(5.99)
$$\begin{aligned}&\frac{\partial^2\varphi}{\partial x_j\partial x_l}(p,p,s_0),\quad\frac{\partial^2\varphi}{\partial x_j\partial y_l}(p,p,s_0),\quad\frac{\partial^2\varphi}{\partial y_j\partial y_l}(p,p,s_0),\quad j,l=1,\ldots,2n-2,\\&(\frac{\partial^2\varphi}{\partial x_j\partial x_{2n-1}}+\frac{\partial^2\varphi}{\partial x_j\partial y_{2n-1}})(p,p,s_0),\quad(\frac{\partial^2\varphi}{\partial y_j\partial x_{2n-1}}+\frac{\partial^2\varphi}{\partial y_j\partial y_{2n-1}})(p,p,s_0),\quad j=1,\ldots,2n-2,\\&(\frac{\partial^2\varphi}{\partial x^2_{2n-1}}+2\frac{\partial^2\varphi}{\partial x_{2n-1}\partial y_{2n-1}}+\frac{\partial^2\varphi}{\partial y^2_{2n-1}})(p,p,s_0).\end{aligned}$$

From (2.2), (3.3) and (4.26), it is straightforward to check that

$$\begin{aligned}&M^\phi_p\big(\overline{Z}_{j,s_0}(p),Z_{l,s_0}(p)\big)=(i\tau_{j,l}-i\overline{\tau}_{l,j})\beta+\mu_{j,l},\quad j,l=1,\ldots,n-1,\\&\mathcal{L}_p\big(\overline{Z}_{j,s_0}(p),Z_{l,s_0}(p)\big)=-\frac{1}{2}(\tau_{j,l}+\overline{\tau}_{l,j}),\quad j,l=1,\ldots,n-1.\end{aligned}\tag{5.100}$$

Since $M^\phi_p-2s_0\mathcal{L}_p$ is diagonal in the basis $\big\{\overline{Z}_{1,s_0},\ldots,\overline{Z}_{n-1,s_0}\big\}$, we have

$$(i\tau_{j,l}-i\overline{\tau}_{l,j})\beta+\mu_{j,l}+s_0(\tau_{j,l}+\overline{\tau}_{l,j})=\lambda_j(s_0)\delta_{j,l},\quad j,l=1,\ldots,n-1.\tag{5.101}$$

We write $y=(y_1,\ldots,y_{2n-1})$, $w_j=y_{2j-1}+iy_{2j}$, $j=1,\ldots,n-1$,

$$\frac{\partial}{\partial w_j}=\frac{1}{2}(\frac{\partial}{\partial y_{2j-1}}-i\frac{\partial}{\partial y_{2j}}),\quad\frac{\partial}{\partial\overline{w}_j}=\frac{1}{2}(\frac{\partial}{\partial y_{2j-1}}+i\frac{\partial}{\partial y_{2j}}),\quad j=1,\ldots,n-1.$$

From (5.97) and (2.2), we can check that

$$\begin{aligned}&-i\frac{\partial\varphi}{\partial z_j}(x,y,s_0)+\sum_{t=1}^{n-1}\tau_{j,t}\overline{z}_t\frac{\partial\varphi}{\partial x_{2n-1}}(x,y,s_0)-ic_jx_{2n-1}\frac{\partial\varphi}{\partial x_{2n-1}}(x,y,s)+(\overline{Z}_{j,s_0}\phi)(x)\\&=g_j(x,y,s_0)\frac{\partial\varphi}{\partial s}(x,y,s_0)+O(|(x,y)|^2),\quad j=1,\ldots,n_-,\\&i\frac{\partial\varphi}{\partial\overline{z}_j}(x,y,s_0)+\sum_{t=1}^{n-1}\overline{\tau}_{j,t}z_t\frac{\partial\varphi}{\partial x_{2n-1}}(x,y,s_0)+i\overline{c}_jx_{2n-1}\frac{\partial\varphi}{\partial x_{2n-1}}(x,y,s_0)+(Z_{j,s_0}\phi)(x)\\&=g_j(x,y,s_0)\frac{\partial\varphi}{\partial s}(x,y,s_0)+O(|(x,y)|^2),\quad j=n_-+1,\ldots,n-1.\end{aligned}\tag{5.102}$$

From (2.2), (5.98), (5.102) and notice that $\frac{\partial\varphi}{\partial x_{2n-1}}(p,p,s_0)=s_0$, it is straightforward to see that

$$
\begin{aligned}
&\frac{\partial^2\varphi}{\partial z_j\partial z_l}(p,p,s_0)=-i(a_{l,j}+a_{j,l}),\quad 1\leq j\leq n_-,\quad 1\leq l\leq n-1,\\
&\frac{\partial^2\varphi}{\partial z_j\partial w_l}(p,p,s_0)=\frac{\partial^2\varphi}{\partial z_j\partial\overline{w}_l}(p,p,s_0)=0,\quad 1\leq j\leq n_-,\quad 1\leq l\leq n-1,\\
&\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_l}(p,p,s_0)=-is_0\tau_{j,l}+\tau_{j,l}\beta-\frac{i}{2}\mu_{j,l},\quad 1\leq j\leq n_-,\quad 1\leq l\leq n-1,\\
&\Big(\frac{\partial^2\varphi}{\partial z_j\partial x_{2n-1}}+\frac{\partial^2\varphi}{\partial z_j\partial y_{2n-1}}\Big)(p,p,s_0)=-ic_j\beta-s_0c_j-id_j,\quad 1\leq j\leq n_-,
\end{aligned}
\tag{5.103}
$$

and

$$
\begin{aligned}
&\frac{\partial^2\varphi}{\partial\overline{z}_j\partial\overline{z}_l}(p,p,s_0)=i(\overline{a}_{l,j}+\overline{a}_{j,l}),\quad n_-+1\leq j\leq n-1,\quad 1\leq l\leq n-1,\\
&\frac{\partial^2\varphi}{\partial\overline{z}_j\partial w_l}(p,p,s_0)=\frac{\partial^2\varphi}{\partial\overline{z}_j\partial\overline{w}_l}(p,p,s_0)=0,\quad n_-+1\leq j\leq n-1,\quad 1\leq l\leq n-1,\\
&\frac{\partial^2\varphi}{\partial\overline{z}_j\partial z_l}(p,p,s_0)=is_0\overline{\tau}_{j,l}+\overline{\tau}_{j,l}\beta+\frac{i}{2}\overline{\mu}_{j,l},\quad n_-+1\leq j\leq n-1,\quad 1\leq l\leq n-1,\\
&\Big(\frac{\partial^2\varphi}{\partial\overline{z}_j\partial x_{2n-1}}+\frac{\partial^2\varphi}{\partial\overline{z}_j\partial y_{2n-1}}\Big)(p,p,s_0)=i\overline{c}_j\beta-s_0\overline{c}_j+i\overline{d}_j,\quad n_-+1\leq j\leq n-1.
\end{aligned}
\tag{5.104}
$$

Put $\varphi_1(x,y,s)=-\overline{\varphi}(y,x,s)$. In view of Proposition 5.31, we know the Hess$\,(\varphi,T_{(p,p,s_0)}H_\varphi)=$ Hess$\,(\varphi_1,T_{(p,p,s_0)}H_{\varphi_1})$. From this observation, (5.103) and (5.104), we can check that

$$
\begin{aligned}
&\frac{\partial^2\varphi}{\partial\overline{w}_j\partial\overline{w}_l}(p,p,s_0)=-i(\overline{a}_{l,j}+\overline{a}_{j,l}),\quad 1\leq j\leq n_-,\quad 1\leq l\leq n-1,\\
&\frac{\partial^2\varphi}{\partial\overline{w}_j\partial\overline{z}_l}(p,p,s_0)=\frac{\partial^2\varphi}{\partial\overline{w}_j\partial z_l}(p,p,s_0)=0,\quad 1\leq j\leq n_-,\quad 1\leq l\leq n-1,\\
&\frac{\partial^2\varphi}{\partial\overline{w}_j\partial w_l}(p,p,s_0)=-is_0\overline{\tau}_{j,l}-\overline{\tau}_{j,l}\beta-\frac{i}{2}\overline{\mu}_{j,l},\quad 1\leq j\leq n_-,\quad 1\leq l\leq n-1,\\
&\Big(\frac{\partial^2\varphi}{\partial\overline{w}_j\partial x_{2n-1}}+\frac{\partial^2\varphi}{\partial\overline{w}_j\partial y_{2n-1}}\Big)(p,p,s_0)=-i\overline{c}_j\beta+s_0\overline{c}_j-i\overline{d}_j,\quad 1\leq j\leq n_-,\\
&\frac{\partial^2\varphi}{\partial w_j\partial w_l}(p,p,s_0)=i(a_{l,j}+a_{j,l}),\quad n_-+1\leq j\leq n-1,\quad 1\leq l\leq n-1,\\
&\frac{\partial^2\varphi}{\partial w_j\partial\overline{z}_l}(p,p,s_0)=\frac{\partial^2\varphi}{\partial w_j\partial z_l}(p,p,s_0)=0,\quad n_-+1\leq j\leq n-1,\quad 1\leq l\leq n-1,\\
&\frac{\partial^2\varphi}{\partial w_j\partial\overline{w}_l}(p,p,s_0)=is_0\tau_{j,l}-\tau_{j,l}\beta+\frac{i}{2}\mu_{j,l},\quad n_-+1\leq j\leq n-1,\quad 1\leq l\leq n-1,\\
&\Big(\frac{\partial^2\varphi}{\partial w_j\partial x_{2n-1}}+\frac{\partial^2\varphi}{\partial w_j\partial y_{2n-1}}\Big)(p,p,s_0)=ic_j\beta+s_0c_j+id_j,\quad n_-+1\leq j\leq n-1.
\end{aligned}
\tag{5.105}
$$

Fix $n_-+1\leq j,l\leq n-1$. We determine $\frac{\partial^2\varphi}{\partial z_j\partial z_l}(p,p,s_0)$. From the fact $\varphi(z,z,s)=0$, we can check that

$$
\frac{\partial^2\varphi}{\partial z_j\partial z_l}(p,p,s_0)+\frac{\partial^2\varphi}{\partial z_j\partial w_l}(p,p,s_0)+\frac{\partial^2\varphi}{\partial w_j\partial z_l}(p,p,s_0)+\frac{\partial^2\varphi}{\partial w_j\partial w_l}(p,p,s_0)=0.
\tag{5.106}
$$

From (5.106) and (5.105), we conclude that

$$
\frac{\partial^2\varphi}{\partial z_j\partial z_l}(p,p,s_0)=-i(a_{l,j}+a_{j,l}),\quad n_-+1\leq j,l\leq n-1.
\tag{5.107}
$$

We can repeat the procedure above several times and deduce (we omit the computations)

$$
\begin{aligned}
&\frac{\partial^2\varphi}{\partial\overline{z}_j\partial\overline{z}_l}(p,p,s_0)=i(\overline{a}_{l,j}+\overline{a}_{j,l}),\quad 1\le j,l\le n_-,\\
&\frac{\partial^2\varphi}{\partial\overline{z}_j\partial z_l}(p,p,s_0)=is_0\overline{\tau}_{j,l}+\overline{\tau}_{j,l}\beta+\frac{i}{2}\overline{\mu}_{j,l},\quad 1\le j\le n_-,\quad n_-+1\le l\le n-1,\\
&\frac{\partial^2\varphi}{\partial\overline{z}_j\partial\overline{w}_l}(p,p,s_0)=0,\quad 1\le j\le n_-,\quad n_-+1\le l\le n-1,\\
&\frac{\partial^2\varphi}{\partial z_j\partial w_l}(p,p,s_0)=0,\quad n_-+1\le j\le n-1,\quad 1\le l\le n_-,\\
&\frac{\partial^2\varphi}{\partial\overline{z}_j\partial w_l}(p,p,s_0)=is_0(\tau_{l,j}+\overline{\tau}_{j,l})+(\overline{\tau}_{j,l}-\tau_{l,j})\beta+i\mu_{l,j},\quad 1\le j,l\le n_-,\\
&\frac{\partial^2\varphi}{\partial z_j\partial\overline{w}_l}(p,p,s_0)=-is_0(\overline{\tau}_{l,j}+\tau_{j,l})+(\tau_{j,l}-\overline{\tau}_{l,j})\beta-i\mu_{j,l},\quad n_-+1\le j,l\le n-1,\\
&(\frac{\partial^2\varphi}{\partial z_j\partial x_{2n-1}}+\frac{\partial^2\varphi}{\partial z_j\partial y_{2n-1}})(p,p,s_0)=-ic_j\beta-s_0c_j-id_j,\quad n_-+1\le j\le n-1,\\
&(\frac{\partial^2\varphi}{\partial\overline{z}_j\partial x_{2n-1}}+\frac{\partial^2\varphi}{\partial\overline{z}_j\partial y_{2n-1}})(p,p,s_0)=i\overline{c}_j\beta-s_0\overline{c}_j+i\overline{d}_j,\quad 1\le j\le n_-.
\end{aligned}
\tag{5.108}
$$

Again, from the fact that $\mathrm{Hess}\,(\varphi,T_{(p,p,s_0)}H_\varphi)=\mathrm{Hess}\,(\varphi_1,T_{(p,p,s_0)}H_{\varphi_1})$, (5.107) and (5.108), we can check that

$$
\begin{aligned}
&\frac{\partial^2\varphi}{\partial\overline{w}_j\partial\overline{w}_l}(p,p,s_0)=-i(\overline{a}_{l,j}+\overline{a}_{j,l}),\quad n_-+1\le j,l\le n-1,\\
&\frac{\partial^2\varphi}{\partial w_j\partial w_l}(p,p,s_0)=i(a_{l,j}+a_{j,l}),\quad 1\le j,l\le n_-,\\
&\frac{\partial^2\varphi}{\partial w_j\partial\overline{w}_l}(p,p,s_0)=is_0\tau_{j,l}-\tau_{j,l}\beta+\frac{i}{2}\mu_{j,l},\quad 1\le j\le n_-,\quad n_-+1\le l\le n-1,\\
&(\frac{\partial^2\varphi}{\partial\overline{w}_j\partial x_{2n-1}}+\frac{\partial^2\varphi}{\partial\overline{w}_j\partial y_{2n-1}})(p,p,s_0)=-i\overline{c}_j\beta+s_0\overline{c}_j-i\overline{d}_j,\quad n_-+1\le j\le n-1,\\
&(\frac{\partial^2\varphi}{\partial w_j\partial x_{2n-1}}+\frac{\partial^2\varphi}{\partial w_j\partial y_{2n-1}})(p,p,s_0)=ic_j\beta+s_0c_j+id_j,\quad 1\le j\le n_-.
\end{aligned}
\tag{5.109}
$$

Moreover, from $\varphi(x,x,s)=0$, we conclude that

$$
(\frac{\partial^2\varphi}{\partial x_{2n-1}^2}+2\frac{\partial^2\varphi}{\partial x_{2n-1}\partial y_{2n-1}}+\frac{\partial^2\varphi}{\partial y_{2n-1}^2})(p,p,s_0)=0.
\tag{5.110}
$$

From (5.103), (5.104), (5.105), (5.107), (5.108), (5.109), (5.110) and (5.99), we completely determine the tangential Hessian of $\varphi(x,y,s)$ at (p,p,s_0). Summing up, we obtain Theorem 2.2.

6. Semi-classical Hodge decomposition theorems for $\Box^{(q)}_{s,k}$ in some non-degenerate part of Σ

In this section we apply the results about the Microlocal decomposition for $\Box^{(q)}_s$ previously in order to describe the semi-classical behaviour of $\Box^{(q)}_{s,k}$ in some non-degenerate part of Σ. We pause and introduce some notations and definitions. We first recall briefly the definition of semi-classical pseudodifferential operators. We need

Definition 6.1. Let W be an open set in $\mathbb{R}^N$. Let $S(1;W)=S(1)$ be the set of $a\in C^\infty(W)$ such that for every $\alpha\in\mathbb{N}_0^N$, there exists $C_\alpha>0$, such that $|\partial_x^\alpha a(x)|\le C_\alpha$ on W. If $a=a(x,k)$ depends on $k\in]1,\infty[$, we say that $a(x,k)\in S_{\rm loc}(1;W)=S_{\rm loc}(1)$ if $\chi(x)a(x,k)$ uniformly bounded in $S(1)$ when k varies in $]1,\infty[$, for any $\chi\in C_0^\infty(W)$. For $m\in\mathbb{R}$, we put $S^m_{\rm loc}(1;W)=S^m_{\rm loc}(1)=k^mS_{\rm loc}(1)$.

If $a_j \in S^{m_j}_{\mathrm{loc}}(1)$, $m_j \searrow -\infty$, we say that $a \sim \sum_{j=0}^{\infty} a_j$ in $S^{m_0}_{\mathrm{loc}}(1)$ if $a - \sum_{j=0}^{N_0} a_j \in S^{m_{N_0+1}}_{\mathrm{loc}}(1)$ for every N_0. For a given sequence a_j as above, we can always find such an asymptotic sum a and a is unique up to an element in $S^{-\infty}_{\mathrm{loc}}(1) = S^{-\infty}_{\mathrm{loc}}(1; W) := \bigcap_m S^m_{\mathrm{loc}}(1)$. We say that $a(x,k) \in S^{m_0}_{\mathrm{loc}}(1)$ is a classical symbol on W of order m_0 if

$$a(x,k) \sim \sum_{j=0}^{\infty} k^{m_0-j} a_j(x) \text{ in } S^{m_0}_{\mathrm{loc}}(1), \quad a_j(x) \in S_{\mathrm{loc}}(1),\ j = 0,1\ldots. \tag{6.1}$$

The set of all classical symbols on W of order m_0 is denoted by $S^{m_0}_{\mathrm{loc},\mathrm{cl}}(1) = S^{m_0}_{\mathrm{loc},\mathrm{cl}}(1; W)$.

Let E be a vector bundle over a smooth paracompact manifold Y. We extend the definitions above to the space of smooth sections of E over Y in the natural way and we write $S^m_{\mathrm{loc}}(1; Y, E)$ and $S^m_{\mathrm{loc},\mathrm{cl}}(1; Y, E)$ to denote the corresponding spaces.

Let W be an open set in $\mathbb{R}^N$ and let E and F be complex vector bundles over W with Hermitian metrics. For any k-dependent continuous function

$$F_k : H^s_{\mathrm{comp}}(W, E) \to H^{s'}_{\mathrm{loc}}(W, F), \quad s, s' \in \mathbb{R},$$

we write

$$F_k = O(k^{n_0}) : H^s_{\mathrm{comp}}(W, E) \to H^{s'}_{\mathrm{loc}}(W, F), \quad n_0 \in \mathbb{Z},$$

if for any $\chi_0, \chi_1 \in C^\infty_0(W)$, there is a positive constant $c > 0$ independent of k, such that

$$\|(\chi_0 F_k \chi_1)u\|_{s'} \le c k^{n_0} \|u\|_s\,, \quad \forall u \in H^s_{\mathrm{loc}}(W, E), \tag{6.2}$$

where $\|u\|_s$ is the usual Sobolev norm of order s.

A k-dependent continuous operator $A_k : C^\infty_0(W, E) \to \mathscr{D}'(W, F)$ is called k-negligible (on W) if A_k is smoothing and the kernel $A_k(x,y)$ of A_k satisfies $\left|\partial^\alpha_x \partial^\beta_y A_k(x,y)\right| = O(k^{-N})$ locally uniformly on every compact set in $W \times W$, for all multi-indices α, β and all $N \in \mathbb{N}$. A_k is k-negligible if and only if

$$A_k = O(k^{-N'}) : H^s_{\mathrm{comp}}(W, E) \to H^{s+N}_{\mathrm{loc}}(W, F)\,,$$

for all $N, N' \ge 0$ and $s \in \mathbb{Z}$. Let $C_k : C^\infty_0(W, E) \to \mathscr{D}'(W, F)$ be another k-dependent continuous operator. We write $A_k \equiv C_k \mod O(k^{-\infty})$ (on W) or $A_k(x,y) \equiv C_k(x,y) \mod O(k^{-\infty})$ (on W) if $A_k - C_k$ is k-negligible on W.

Definition 6.2. Let W be an open set in $\mathbb{R}^N$ and let E and F be complex vector bundles over W. A classical semi-classical pseudodifferential operator on W of order m from sections of E to sections of F is a k-dependent continuous operator $A_k : C^\infty_0(W, E) \to C^\infty(W, F)$ such that the distribution kernel $A_k(x,y)$ is given by the oscillatory integral

$$\begin{aligned} &A_k(x,y) \equiv \frac{k^N}{(2\pi)^N} \int e^{ik<x-y,\eta>} a(x,y,\eta,k) d\eta \mod O(k^{-\infty}),\\ &a(x,y,\eta,k) \in S^m_{\mathrm{loc},\mathrm{cl}}(1; W \times W \times \mathbb{R}^N, E \boxtimes F). \end{aligned}$$

We shall identify A_k with $A_k(x,y)$ and it is clearly that A_k has a unique continuous extension $\mathscr{E}'(W, E) \to \mathscr{D}'(W, F)$.

Definition 6.3. Let

$$\hat{\mathcal{I}}_k = \frac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik<x-y,\eta>} p(x,y,\eta,k) d\eta$$

be a classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$ with $p(x,y,\eta,k) \in S^0_{\mathrm{loc},\mathrm{cl}}(1; D \times D \times \mathbb{R}^{2n-1}, T^{*0,q}X \boxtimes T^{*0,q}X)$. Let Λ be an open set of T^*D. We write

$$\hat{\mathcal{I}}_k \equiv \tfrac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik<x-y,\eta>} q(x,y,\eta,k) d\eta \mod O(k^{-\infty}) \text{ at } \Lambda \bigcap \Sigma,$$

where $q(x,y,\eta,k) \in S^0_{\mathrm{loc},\mathrm{cl}}(1; D \times D \times \mathbb{R}^{2n-1}, T^{*0,q}X \boxtimes T^{*0,q}X)$, if

$$\hat{\mathcal{I}}_k \equiv \frac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik<x-y,\eta>} q(x,y,\eta,k) d\eta + \frac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik<x-y,\eta>} \beta(x,y,\eta,k) d\eta \mod O(k^{-\infty}),$$

where $\beta(x,y,\eta,k) \in S^0_{\mathrm{loc}}(1; D \times D \times \mathbb{R}^{2n-1}, T^{*0,q}X \boxtimes T^{*0,q}X)$ and there is a small neighbourhood Γ of $\Lambda \bigcap \Sigma$ such that $\beta(x,y,\eta,k) = 0$ if $(x,\eta) \in \Gamma$.

We return to our situation. Let s be a local trivializing section of L on an open subset $D \subset X$ and $|s|^2_{h^L} = e^{-2\phi}$. From now on, we assume that there exist a $\lambda_0 \in \mathbb{R}$ and $x_0 \in D$ such that $M^\phi_{x_0} - 2\lambda_0 \mathcal{L}_{x_0}$ is non-degenerate of constant signature (n_-, n_+). We fix $D_0 \Subset D$, D_0 open. We work with some real local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on D. We write $\xi = (\xi_1, \ldots, \xi_{2n-1})$ or $\eta = (\eta_1, \ldots, \eta_{2n-1})$ to denote the dual coordinates of x. We will use the same notations as in section 5. Note that we write $\hat{x} = (x_1, \ldots, x_{2n-1}, x_{2n})$ to denote the local coordinates of $\hat{D}$ and we write $\hat{\xi} = (\xi_1, \ldots, \xi_{2n-1}, \xi_{2n})$ or $\hat{\eta} = (\eta_1, \ldots, \eta_{2n-1}, \eta_{2n})$ to denote the dual coordinates of $\hat{x}$.

Let $\chi(x_{2n}), \chi_1(x_{2n}) \in C^\infty_0(\mathbb{R})$, $\chi, \chi_1 \geq 0$. We assume that $\chi_1 = 1$ on $\operatorname{Supp}\chi$. We take χ so that $\int \chi(x_{2n}) dx_{2n} = 1$. Put

$$\chi_k(x_{2n}) = e^{ikx_{2n}} \chi(x_{2n}). \tag{6.3}$$

Let V and U be as in (5.14) and (5.15) respectively. The following is straightforward and follows from the usual stationary phase formula and therefore we omit the proof.

Proposition 6.4. *With the notations before, let $q \in \{0, 1, \ldots, n-1\}$. Let*

$$\widetilde{\mathcal{I}}_k = \frac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik<x-y,\eta>} \alpha(x, \eta, k) d\eta$$

be a classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$ with $\alpha(x, \eta, k) \in S^0_{\mathrm{loc,cl}}(1; T^*D, T^{*0,q}X \boxtimes T^{*0,q}X)$, $\alpha(x, \eta, k) = 0$ if $|\eta| > M$, for some large $M > 0$, and $\operatorname{Supp}\alpha(x, \eta, k) \bigcap T^*D_0 \Subset V$. Then, there is a classical pseudodifferential operator $\widetilde{I} = (2\pi)^{-2n} \int e^{i<\hat{x}-\hat{y},\hat{\eta}>} c(\hat{x}, \hat{\eta}) d\hat{\eta}$ on $\hat{D}$ of order 0 from sections of $T^{*0,q}\hat{D}$ to sections of $T^{*0,q}\hat{D}$ with $c(\hat{x}, \hat{\eta}) \in S^0_{\mathrm{cl}}(T^*\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$, $\operatorname{Supp} c(\hat{x}, \hat{\eta}) \bigcap T^*\hat{D}_0 \subset \overline{W}$, where $W \subset U$ is a conic open set with $\overline{W} \subset U$, such that

$$\widetilde{\mathcal{I}}_k \equiv \widetilde{I}_k \mod O(k^{-\infty}) \quad \text{on } D,$$

where $\widetilde{I}_k$ is the continuous operator $C^\infty_0(D, T^{*0,q}X) \to C^\infty(D, T^{*0,q}X)$ given by

$$\begin{aligned} \widetilde{I}_k : C^\infty_0(D, T^{*0,q}X) &\to C^\infty(D, T^{*0,q}X), \\ u &\to \int e^{-ikx_{2n}} \chi_1(x_{2n}) \widetilde{I}(\chi_k u)(\hat{x}) dx_{2n}. \end{aligned}$$

Now, we assume that $q = n_-$. Let $\widetilde{\mathcal{I}}_k = \frac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik<x-y,\eta>} \alpha(x, \eta, k) d\eta$ be a classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$ with $\alpha(x, \eta, k) \in S^0_{\mathrm{loc,cl}}(1; T^*D, T^{*0,q}X \boxtimes T^{*0,q}X)$, $\alpha(x, \eta, k) = 0$ if $|\eta| > M$, for some large $M > 0$, and $\operatorname{Supp}\alpha(x, \eta, k) \bigcap T^*D_0 \Subset V$. Let $\widetilde{I}$ be as in Proposition 6.4 and let $S \in L^0_{\frac{1}{2},\frac{1}{2}}(\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$ and $G \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$ be as in Theorem 5.21. Then, we have

$$S + \Box^{(q)}_s \circ G \equiv \widetilde{I} \text{ on } \hat{D}_0, \quad \Box^{(q)}_s \circ S \equiv 0 \text{ on } \hat{D}. \tag{6.4}$$

Now, we assume that S and G are properly supported. Define

$$\begin{aligned} \mathcal{S}_k : H^s_{\mathrm{loc}}(D, T^{*0,q}X) &\to H^s_{\mathrm{loc}}(D, T^{*0,q}X), \quad \forall s \in \mathbb{N}_0, \\ u &\to \int e^{-ikx_{2n}} \chi_1(x_{2n}) S(\chi_k u)(\hat{x}) dx_{2n}. \end{aligned} \tag{6.5}$$

Let $u \in H^s_{\mathrm{loc}}(D, T^{*0,q}X)$, $s \in \mathbb{N}_0$. We have $\chi_k u \in H^s_{\mathrm{loc}}(\hat{D}, T^{*0,q}\hat{D})$. Since $S \in L^0_{\frac{1}{2},\frac{1}{2}}(\hat{D}, T^{*0,q}\hat{D} \boxtimes T^{*0,q}\hat{D})$, we see that $S(\chi_k u) \in H^s_{\mathrm{loc}}(\hat{D}, T^{*0,q}\hat{D})$. From this, it is not difficult to see that

$$\int e^{-ikx_{2n}} \chi_1(x_{2n}) S(\chi_k u)(\hat{x}) dx_{2n} \in H^s_{\mathrm{loc}}(D, T^{*0,q}X).$$

Thus, $\mathcal{S}_k$ is well-defined. Since S is properly supported, $\mathcal{S}_k$ is properly supported, too. Moreover, from (6.5) and the fact that $S : H^s_{\mathrm{comp}}(\hat{D}, T^{*0,q}\hat{D}) \to H^s_{\mathrm{comp}}(\hat{D}, T^{*0,q}\hat{D})$ is continuous, for every $s \in \mathbb{R}$, it is straightforward to check that

$$\mathcal{S}_k = O(k^s) : H^s_{\mathrm{comp}}(D, T^{*0,q}X) \to H^s_{\mathrm{comp}}(D, T^{*0,q}X), \tag{6.6}$$

for all $s\in\mathbb{N}_0$.

Let $\mathcal{S}_k^*:\mathscr{D}'(D,T^{*0,q}X)\to\mathscr{D}'(D,T^{*0,q}X)$ be the formal adjoint of $\mathcal{S}_k$ with respect to $(\,\cdot\,|\,\cdot\,)$. Then $\mathcal{S}_k^*$ is also properly supported. It is not difficult to see that

$$(\mathcal{S}_k^*v)(x)=\int\overline{\chi_k(x_{2n})}S^*(ve^{ix_{2n}k}\chi_1)(\hat{x})dx_{2n}\in\Omega_0^{0,q}(D),$$

for all $v\in\Omega_0^{0,q}(D)$. From this observation, we can check that

$$\mathcal{S}_k^*=O(k^s):H^s_{\rm comp}\,(D,T^{*0,q}X)\to H^s_{\rm comp}\,(D,T^{*0,q}X)\,,\quad\forall s\in\mathbb{N}_0. \tag{6.7}$$

From (5.5), we have

$$\begin{aligned}\Box^{(q)}_{s,k}\circ\Big(\int e^{-ikx_{2n}}\chi_1(x_{2n})S(\chi_ku)(\hat{x})dx_{2n}\Big)&=\int e^{-ikx_{2n}}\big(\Box^{(q)}_s(\chi_1S)\big)(\chi_ku)(\hat{x})dx_{2n}\\&=\int e^{-ikx_{2n}}\big(\Box^{(q)}_s(\chi_1S\widetilde{\chi})\big)(\chi_ku)(\hat{x})dx_{2n},\end{aligned} \tag{6.8}$$

where $\widetilde{\chi}\in C_0^\infty(\mathbb{R})$, $\widetilde{\chi}=1$ on ${\rm Supp}\,\chi$ and $\chi_1=1$ on ${\rm Supp}\,\widetilde{\chi}$ and $u\in\Omega_0^{0,q}(D_0)$. Note that $\Box^{(q)}_s(\chi_1S\widetilde{\chi})=\Box^{(q)}_s(S\widetilde{\chi})-\Box^{(q)}_s((1-\chi_1)S\widetilde{\chi})$. From Theorem 5.21, we know that $\Box^{(q)}_sS$ is smoothing and the kernel of S is smoothing away the diagonal. Thus, $(1-\chi_1)S\widetilde{\chi}$ is smoothing. It follows that $\Box^{(q)}_s((1-\chi_1)S\widetilde{\chi})$ is smoothing. We conclude that $\Box^{(q)}_s(\chi_1S\widetilde{\chi})$ is smoothing. Let $K(\hat{x},\hat{y})\in C^\infty$ be the distribution kernel of $\Box^{(q)}_s(\chi_1S\widetilde{\chi})$. From (6.8) and recall the form χ_k (see (6.3)), we see that the distribution kernel of $\Box^{(q)}_{s,k}\mathcal{S}_k$ is given by

$$(\Box^{(q)}_{s,k}\mathcal{S}_k)(x,y)=\int e^{-i(x_{2n}-y_{2n})k}K(\hat{x},\hat{y})\chi(y_{2n})dx_{2n}dy_{2n}. \tag{6.9}$$

For $N\in\mathbb{N}$, we have

$$\begin{aligned}\Big|k^N(\Box^{(q)}_{s,k}\mathcal{S}_k)(x,y)\Big|&=\Big|\int\big((-i\frac{\partial}{\partial y_{2n}})^Ne^{-i(x_{2n}-y_{2n})k})\big)K(\hat{x},\hat{y})\chi(y_{2n})dy_{2n}dx_{2n}\Big|\\&=\Big|\int e^{-i(x_{2n}-y_{2n})k}(i\frac{\partial}{\partial y_{2n}})^N\big(K(\hat{x},\hat{y})\chi(y_{2n})\big)dy_{2n}dx_{2n}\Big|.\end{aligned} \tag{6.10}$$

Thus, $(\Box^{(q)}_{s,k}\mathcal{S}_k)(x,y)=O(k^{-N})$, locally uniformly for all $N\in\mathbb{N}$, and similarly for the derivatives. We deduce that

$$\Box^{(q)}_{s,k}\mathcal{S}_k\equiv0\mod O(k^{-\infty})\ \text{ on } D. \tag{6.11}$$

Thus,

$$\mathcal{S}_k^*\Box^{(q)}_{s,k}\equiv0\mod O(k^{-\infty})\ \text{ on } D. \tag{6.12}$$

Define

$$\begin{aligned}\mathcal{G}_k:H^s_{\rm loc}\,(D,T^{*0,q}X)&\to H^{s+1}_{\rm loc}(D,T^{*0,q}X),\quad\forall s\in\mathbb{N}_0,\\u&\to\int e^{-ikx_{2n}}\chi_1G(\chi_ku)(\hat{x})dx_{2n}.\end{aligned} \tag{6.13}$$

As above, we can show that $\mathcal{G}_k$ is well-defined. Since G is properly supported, $\mathcal{G}_k$ is properly supported, too. Moreover, from (6.13) and the fact that $G:H^s_{\rm comp}\,(\hat{D},T^{*0,q}\hat{D})\to H^{s+1}_{\rm comp}\,(\hat{D},T^{*0,q}\hat{D})$ is continuous, for every $s\in\mathbb{R}$, it is straightforward to check that

$$\mathcal{G}_k=O(k^s):H^s_{\rm comp}\,(D,T^{*0,q}X)\to H^{s+1}_{\rm comp}\,(D,T^{*0,q}X), \tag{6.14}$$

for all $s\in\mathbb{N}_0$.

Let $\mathcal{G}_k^*:\mathscr{D}'(D,T^{*0,q}X)\to\mathscr{D}'(D,T^{*0,q}X)$ be the formal adjoint of $\mathcal{G}_k$ with respect to $(\,\cdot\,|\,\cdot\,)$. We can check that

$$(\mathcal{G}_k^*v)(x)=\int\overline{\chi_k(x_{2n})}G^*(ve^{ix_{2n}k}\chi_1)(\hat{x})dx_{2n}\in\Omega_0^{0,q}(D),$$

for all $v\in\Omega_0^{0,q}(D)$. Thus, $\mathcal{G}_k^*:\Omega_0^{0,q}(D)\to\Omega_0^{0,q}(D)$. Moreover, as before, we can show that

$$\mathcal{G}_k^*=O(k^s):H^s_{\rm comp}\,(D,T^{*0,q}X)\to H^{s+1}_{\rm comp}\,(D,T^{*0,q}X)\,,\quad\forall s\in\mathbb{N}_0. \tag{6.15}$$

Let $u\in\Omega^{0,q}_0(D_0)$. From (5.5), we have

$$\Box^{(q)}_{s,k}(\mathcal{G}_ku)=\Box^{(q)}_{s,k}\circ\Big(\int e^{-ikx_{2n}}\chi_1G(\chi_ku)dx_{2n}\Big)=\int e^{-ikx_{2n}}\big(\Box^{(q)}_s\chi_1G\widetilde\chi)(\chi_ku)(\hat x)dx_{2n},$$

where $\widetilde\chi$ is as in (6.8). Note that $\Box^{(q)}_s(\chi_1G\widetilde\chi)=\Box^{(q)}_s(G\widetilde\chi)-\Box^{(q)}_s((1-\chi_1)G\widetilde\chi)$. From (6.4) and Theorem 5.21, we know that $\Box^{(q)}_sG+S\equiv\widetilde I$ and the kernel of G is smoothing away the diagonal. Thus, $(1-\chi_1)G\widetilde\chi$ is smoothing. It follows that $\Box^{(q)}_s((1-\chi_1)G\widetilde\chi)$ is smoothing. We conclude that $\Box^{(q)}_s(\chi_1G\widetilde\chi)\equiv(\widetilde I-S)\widetilde\chi$. From this, we get

$$\begin{aligned}\Box^{(q)}_{s,k}(\mathcal{G}_ku)&=\int e^{-ikx_{2n}}\big((\widetilde I-S)(\chi_ku)\big)(\hat x)dx_{2n}+\int e^{-ikx_{2n}}F(\chi_ku)(\hat x)dx_{2n}\\&=\int e^{-ikx_{2n}}\chi_1\big((\widetilde I-S)(\chi_ku)\big)(\hat x)dx_{2n}+\int e^{-ikx_{2n}}(1-\chi_1)\big((\widetilde I-S)(\chi_ku)\big)(\hat x)dx_{2n}\\&\quad+\int e^{-ikx_{2n}}F(\chi_ku)(\hat x)dx_{2n}\\&=(\widetilde I_k-\mathcal{S}_k)u+\int e^{-ikx_{2n}}(1-\chi_1)\big((\widetilde I-S)(\chi_ku)\big)(\hat x)dx_{2n}\\&\quad+\int e^{-ikx_{2n}}F(\chi_ku)(\hat x)dx_{2n},\end{aligned}\tag{6.16}$$

where F is a smoothing operator. We can repeat the procedure as in (6.8) and conclude that the operator

$$u\to\int e^{-ikx_{2n}}F(\chi_ku)(\hat x)dx_{2n},\quad u\in\Omega^{0,q}_0(D_0),$$

is k-negligible. Similarly, since $(1-\chi_1)(\widetilde I-S)\chi$ is smoothing, the operator

$$u\to\int e^{-ikx_{2n}}(1-\chi_1)\big((\widetilde I-S)(\chi_ku)\big)(\hat x)dx_{2n},\quad u\in\Omega^{0,q}_0(D_0),$$

is also k-negligible. From this observation and note that $\widetilde{\mathcal{I}}_k\equiv\widetilde I_k\mod O(k^{-\infty})$, we obtain

$$\Box^{(q)}_{s,k}\mathcal{G}_k+\mathcal{S}_k\equiv\widetilde{\mathcal{I}}_k\mod O(k^{-\infty})\ \text{ on } D_0\tag{6.17}$$

and hence

$$\mathcal{G}^*_k\Box^{(q)}_{s,k}+\mathcal{S}^*_k\equiv\widetilde{\mathcal{I}}^*_k\mod O(k^{-\infty})\ \text{ on } D_0,\tag{6.18}$$

where $\widetilde{\mathcal{I}}^*_k$ is the formal adjoint of $\widetilde{\mathcal{I}}_k$ with respect to $(\,\cdot\,|\,\cdot\,)$.

We now consider more general situations. We recall Definition 6.3. Let

$$\hat{\mathcal{I}}_k=\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}p(x,y,\eta,k)d\eta$$

be a classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$ with $p(x,y,\eta,k)\in S^0_{\mathrm{loc},\mathrm{cl}}(1;D\times D\times\mathbb{R}^{2n-1},T^{*0,q}X\boxtimes T^{*0,q}X)$. We assume that $\hat{\mathcal{I}}_k\equiv\widetilde{\mathcal{I}}_k\mod O(k^{-\infty})$ at $T^*D_0\bigcap\Sigma$, where

$$\widetilde{\mathcal{I}}_k=\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta$$

with $\alpha(x,\eta,k)\in S^0_{\mathrm{loc},\mathrm{cl}}(1;T^*D,T^{*0,q}X\boxtimes T^{*0,q}X)$, $\alpha(x,\eta,k)=0$ if $|\eta|>M$, for some large $M>0$ and $\mathrm{Supp}\,\alpha(x,\eta,k)\bigcap T^*D_0\Subset V$. We write

$$\begin{aligned}&\hat{\mathcal{I}}_k\equiv\widetilde{\mathcal{I}}_k+\widetilde{\mathcal{I}}^1_k\mod O(k^{-\infty}),\\&\widetilde{\mathcal{I}}^1_k=\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\beta(x,y,\eta,k)d\eta,\end{aligned}$$

where $\beta(x,y,\eta,k)\in S^0_{\mathrm{loc},\mathrm{cl}}(1;D\times D\times\mathbb{R}^{2n-1},T^{*0,q}X\boxtimes T^{*0,q}X)$ and there is a small neighbourhood Γ of $T^*D_0\bigcap\Sigma$ such that $\beta(x,y,\eta,k)=0$ if $(x,\eta)\in\Gamma$. Let $\mathcal{G}_k$ and $\mathcal{S}_k$ be as in (6.13) and (6.6) respectively. Then, (6.17) and (6.18) hold. Since $\beta(x,y,\eta,k)=0$ if (x,η) is in some small neighbourhood of $T^*D_0\bigcap\Sigma$, it is clear that there is a properly supported continuous operator $\mathcal{G}^1_k=O(k^s):H^s_{\rm comp}\,(D,T^{*0,q}X)\rightarrow H^{s+1}_{\rm comp}\,(D,T^{*0,q}X)$, $\forall s\in\mathbb N_0$, such that $\Box^{(q)}_{s,k}\mathcal{G}^1_k\equiv\widetilde{\mathcal{I}}^1_k\mod O(k^{-\infty})$ on D_0. Put

$$\mathcal{N}_k:=\mathcal{G}_k+\mathcal{G}^1_k=O(k^s):H^s_{\rm comp}\,(D,T^{*0,q}X)\rightarrow H^{s+1}_{\rm comp}\,(D,T^{*0,q}X),\ \ \forall s\in\mathbb N_0. \tag{6.19}$$

Then, we have

$$\begin{aligned}&\mathcal{N}_k=O(k^s):H^s_{\rm comp}\,(D,T^{*0,q}X)\rightarrow H^{s+1}_{\rm comp}\,(D,T^{*0,q}X),\ \ \forall s\in\mathbb N_0,\\&\mathcal{N}^*_k=O(k^s):H^s_{\rm comp}\,(D,T^{*0,q}X)\rightarrow H^{s+1}_{\rm comp}\,(D,T^{*0,q}X),\ \ \forall s\in\mathbb N_0,\end{aligned} \tag{6.20}$$

and

$$\begin{aligned}&\Box^{(q)}_{s,k}\mathcal{N}_k+\mathcal{S}_k\equiv\hat{\mathcal{I}}_k\mod O(k^{-\infty})\ \text{ on } D_0,\\&\mathcal{N}^*_k\Box^{(q)}_{s,k}+\mathcal{S}^*_k\equiv\hat{\mathcal{I}}^*_k\mod O(k^{-\infty})\ \text{ on } D_0,\end{aligned} \tag{6.21}$$

where $\mathcal{N}^*_k$ and $\hat{\mathcal{I}}^*_k$ are the formal adjoints of $\hat{\mathcal{I}}_k$ and $\mathcal{N}_k$ with respect to $(\,\cdot\,|\,\cdot\,)$ respectively.

From (6.6), (6.7), (6.11), (6.12), (6.20) and (6.21), we get

Theorem 6.5. *Let s be a local trivializing section of L on an open subset $D\subset X$ and $|s|^2_{h^L}=e^{-2\phi}$. We assume that there exist a $\lambda_0\in\mathbb R$ and $x_0\in D$ such that $M^\phi_{x_0}-2\lambda_0\mathcal{L}_{x_0}$ is non-degenerate of constant signature (n_-,n_+). Let $q=n_-$. We fix $D_0\Subset D$, D_0 open. Let V be as in (5.14). Let*

$$\hat{\mathcal{I}}_k=\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}p(x,y,\eta,k)d\eta$$

be a classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$ with $p(x,y,\eta,k)\in S^0_{{\rm loc\,,cl}}\,(1;D\times D\times\mathbb R^{2n-1},T^{*0,q}X\boxtimes T^{*0,q}X)$. We assume that

$$\begin{aligned}&\hat{\mathcal{I}}_k\equiv\widetilde{\mathcal{I}}_k\mod O(k^{-\infty})\ \text{ at } T^*D_0\textstyle\bigcap\Sigma,\\&\widetilde{\mathcal{I}}_k=\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta,\\&\alpha(x,\eta,k)\sim\textstyle\sum_{j=0}\alpha_j(x,\eta)k^{-j}\ \text{ in } S^0_{\rm loc}\,(1;T^*D,T^{*0,q}X\boxtimes T^{*0,q}X),\\&\alpha_j(x,\eta)\in C^\infty(T^*D,T^{*0,q}D\boxtimes T^{*0,q}D),\ \ j=0,1,\ldots,\end{aligned} \tag{6.22}$$

where $\alpha(x,\eta,k)\in S^0_{{\rm loc\,,cl}}\,(1;T^*D,T^{*0,q}X\boxtimes T^{*0,q}X)$ with $\alpha(x,\eta,k)=0$ if $|\eta|>M$, for some large $M>0$ and ${\rm Supp}\,\alpha(x,\eta,k)\bigcap T^*D_0\Subset V$. Let $\mathcal{S}_k$, $\mathcal{G}_k$ and $\mathcal{N}_k$ be as in (6.5), (6.13) and (6.19) respectively. Then,

$$\begin{aligned}&\mathcal{S}^*_k,\mathcal{S}_k=O(k^s):H^s_{\rm comp}\,(D,T^{*0,q}X)\rightarrow H^s_{\rm comp}\,(D,T^{*0,q}X),\ \ \forall s\in\mathbb N_0,\\&\mathcal{G}^*_k,\mathcal{G}_k,\mathcal{N}^*_k,\mathcal{N}_k=O(k^s):H^s_{\rm comp}\,(D,T^{*0,q}X)\rightarrow H^{s+1}_{\rm comp}\,(D,T^{*0,q}X),\ \ \forall s\in\mathbb N_0,\end{aligned} \tag{6.23}$$

and we have

$$\Box^{(q)}_{s,k}\mathcal{S}_k\equiv0\mod O(k^{-\infty})\ \text{ on } D,\ \ \mathcal{S}^*_k\Box^{(q)}_{s,k}\equiv0\mod O(k^{-\infty})\ \text{ on } D, \tag{6.24}$$

$$\mathcal{S}_k+\Box^{(q)}_{s,k}\mathcal{G}_k\equiv\widetilde{\mathcal{I}}_k\mod O(k^{-\infty})\ \text{ on } D_0, \tag{6.25}$$

$$\mathcal{G}^*_k\Box^{(q)}_{s,k}+\mathcal{S}^*_k\equiv\widetilde{\mathcal{I}}^*_k\mod O(k^{-\infty})\ \text{ on } D_0, \tag{6.26}$$

$$\mathcal{S}_k+\Box^{(q)}_{s,k}\mathcal{N}_k\equiv\hat{\mathcal{I}}_k\mod O(k^{-\infty})\ \text{ on } D_0, \tag{6.27}$$

$$\mathcal{N}^*_k\Box^{(q)}_{s,k}+\mathcal{S}^*_k\equiv\hat{\mathcal{I}}^*_k\mod O(k^{-\infty})\ \text{ on } D_0, \tag{6.28}$$

where $\mathcal{S}^*_k$, $\mathcal{G}^*_k$, $\mathcal{N}^*_k$, $\widetilde{\mathcal{I}}^*_k$ and $\hat{\mathcal{I}}^*_k$ are the formal adjoints of $\mathcal{S}_k$, $\mathcal{G}_k$, $\mathcal{N}_k$, $\widetilde{\mathcal{I}}_k$ and $\hat{\mathcal{I}}_k$ with respect to $(\,\cdot\,|\,\cdot\,)$ respectively and $\Box^{(q)}_{s,k}$ is given by (4.4).

We notice that $\mathcal{S}_k$, $\mathcal{S}^*_k$, $\mathcal{G}_k$, $\mathcal{G}^*_k$, $\mathcal{N}_k$, $\mathcal{N}^*_k$, are all properly supported on D. We need

Theorem 6.6. *With the notations and assumptions above, let $\mathcal{S}_k$ be as in Theorem 6.5. Then, $\mathcal{S}_k$ is a smoothing operator and the kernel of $\mathcal{S}_k$ satisfies*

$$\mathcal{S}_k(x,y) \equiv \int e^{ik\varphi(x,y,s)}a(x,y,s,k)ds \mod O(k^{-\infty}) \quad \text{on } D \tag{6.29}$$

with

$$\begin{aligned}
&a(x,y,s,k) \in S^n_{\mathrm{loc}}\big(1;\Omega, T^{*0,q}X \boxtimes T^{*0,q}X\big) \bigcap C^\infty_0\big(\Omega, T^{*0,q}X \boxtimes T^{*0,q}X\big),\\
&a(x,y,s,k) \sim \sum_{j=0}^{\infty} a_j(x,y,s)k^{n-j} \text{ in } S^n_{\mathrm{loc}}\big(1;\Omega, T^{*0,q}X \boxtimes T^{*0,q}X\big),\\
&a_j(x,y,s) \in C^\infty_0\big(\Omega, T^{*0,q}X \boxtimes T^{*0,q}X\big), \quad j = 0,1,2,\ldots,\\
&a_0(x,x,s) : T^{*0,q}_x X \to \mathcal{N}(x,s,n_-), \quad \forall (x,x,s) \in \Omega,
\end{aligned} \tag{6.30}$$

and $\varphi(x,y,s)$ is as in Theorem 5.29 and (2.3)*, where $\mathcal{N}(x,s,n_-)$ is as in* (5.39)*,*

$$\begin{aligned}
\Omega := &\{(x,y,s) \in D \times D \times \mathbb{R};\ (x, -2\mathrm{Im}\,\overline{\partial}_b\phi(x) + s\omega_0(x)) \in V \textstyle\bigcap \Sigma,\\
&(y, -2\mathrm{Im}\,\overline{\partial}_b\phi(y) + s\omega_0(y)) \in V \textstyle\bigcap \Sigma,\ |x-y| < \varepsilon, \text{ for some } \varepsilon > 0\}.
\end{aligned}$$

Proof. Theorem 6.6 essentially follows from the stationary phase formula of Melin-Sjöstrand [21]. From the definition (6.5) of $\mathcal{S}_k$ and Theorem 5.29, we see that the distribution kernel of $\mathcal{S}_k$ is given by

(6.31)
$$\begin{aligned}
\mathcal{S}_k(x,y) &\equiv \int_{t\geq 0} e^{it\Phi(\hat{x},\hat{y},s)-ikx_{2n}+iky_{2n}} b(\hat{x},\hat{y},s,t)\chi_1(x_{2n})\chi(y_{2n})dx_{2n}dtdy_{2n}ds \mod O(k^{-\infty})\\
&\equiv \int_{t\geq 0} e^{it\varphi(x,y,s)+i(x_{2n}-y_{2n})(t-k)} b(\hat{x},\hat{y},s,t)\chi_1(x_{2n})\chi(y_{2n})dx_{2n}dtdy_{2n}ds \mod O(k^{-\infty})\\
&\equiv \int_{\sigma\geq 0} e^{ik\big(\varphi(x,y,s)\sigma+(x_{2n}-y_{2n})(\sigma-1)\big)} kb(\hat{x},\hat{y},s,k\sigma)\chi_1(x_{2n})\chi(y_{2n})dx_{2n}d\sigma dy_{2n}ds \mod O(k^{-\infty}),
\end{aligned}$$

where the integrals above are defined as oscillatory integrals and $t = k\sigma$. Let $\gamma(\sigma) \in C^\infty_0(\mathbb{R}_+)$ with $\gamma(\sigma) = 1$ in some small neighbourhood of 1. We introduce the cut-off functions $\gamma(\sigma)$ and $1-\gamma(\sigma)$ in the integral (6.31):

$$I_0(x,y) := \int_{\sigma\geq 0} e^{ik\big(\varphi(x,y,s)\sigma+(x_{2n}-y_{2n})(\sigma-1)\big)}\gamma(\sigma)kb(\hat{x},\hat{y},s,k\sigma)\chi_1(x_{2n})\chi(y_{2n})dx_{2n}d\sigma dy_{2n}ds, \tag{6.32}$$

(6.33)
$$I_1(x,y) := \int_{\sigma\geq 0} e^{ik\big(\varphi(x,y,s)\sigma+(x_{2n}-y_{2n})(\sigma-1)\big)}(1-\gamma(\sigma))kb(\hat{x},\hat{y},s,k\sigma)\chi_1(x_{2n})\chi(y_{2n})dx_{2n}d\sigma dy_{2n}ds\,,$$

so that $\mathcal{S}_k(x,y) \equiv I_0(x,y) + I_1(x,y) \mod O(k^{-\infty})$. First, we study $I_1(x,y)$. Note that when $\sigma \neq 1$, $d_{y_{2n}}\big(\varphi(x,y,s)\sigma + (x_{2n}-y_{2n})(\sigma-1)\big) = 1-\sigma \neq 0$. Thus, we can integrate by parts and get that I_1 is smoothing and

$$I_1(x,y) \equiv 0 \mod O(k^{-\infty}). \tag{6.34}$$

Next, we study the kernel $I_0(x,y)$. From (5.74), we may assume that $b(\hat{x},\hat{y},s,k\sigma)$ is supported in some small neighbourhood of $\hat{x} = \hat{y}$. We want to apply the stationary phase method of Melin and Sjöstrand (see page 148 of[21]) to carry out the $dx_{2n}d\sigma$ integration in (6.32). Put

$$\Psi(\hat{x},\hat{y},\sigma) := \varphi(x,y,s)\sigma + (x_{2n}-y_{2n})(\sigma-1).$$

We first notice that $d_\sigma\Psi(\hat{x},\hat{y},\sigma)|_{\hat{x}=\hat{y}} = 0$ and $d_{x_{2n}}\Psi(\hat{x},\hat{y},\sigma)|_{\sigma=1} = 0$. Thus, $x = y$ and $\sigma = 1$ are real critical points. Moreover, we can check that the Hessian of $\Psi(\hat{x},\hat{y},\sigma)$ at $\hat{x} = \hat{y}$, $\sigma = 1$, is given by

$$\begin{pmatrix} \Psi''_{\sigma\sigma}(\hat{x},\hat{x},1) & \Psi''_{x_{2n}\sigma}(\hat{x},\hat{x},1) \\ \Psi''_{\sigma x_{2n}}(\hat{x},\hat{x},1) & \Psi''_{x_{2n}x_{2n}}(\hat{x},\hat{x},1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, $\Psi(\hat{x},\hat{y},\sigma)$ is a non-degenerate complex valued phase function in the sense of Melin-Sjöstrand [21]. Let

$$\widetilde{\Psi}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{\sigma}) := \widetilde{\varphi}(\widetilde{x},\widetilde{y},s)\widetilde{\sigma} + (\widetilde{x}_{2n} - \widetilde{y}_{2n})(\widetilde{\sigma} - 1)$$

be an almost analytic extension of $\Psi(\hat{x},\hat{y},\hat{\sigma})$, where $\widetilde{\varphi}(\widetilde{x},\widetilde{y},s)$ is an almost analytic extension of $\varphi(x,y,s)$. Here we fix s. We can check that given y_{2n} and (x,y), $\widetilde{x}_{2n} = y_{2n} - \varphi(x,y,s)$, $\widetilde{\sigma} = 1$ are the solutions of

$$\frac{\partial\widetilde{\Psi}}{\partial\widetilde{\sigma}} = 0\,,\ \frac{\partial\widetilde{\Psi}}{\partial\widetilde{x}_{2n}} = 0.$$

From this and by the stationary phase formula of Melin-Sjöstrand [21], we get

$$I_0(x,y) \equiv \int e^{ik\varphi(x,y,s)}a(x,y,s,k)ds \mod O(k^{-\infty}), \tag{6.35}$$

where $a(x,y,s,k) \in S^n_{\rm loc}\big(1;\Omega,T^{*0,q}X \boxtimes T^{*0,q}X\big) \bigcap C^\infty_0\big(\Omega,T^{*0,q}X \boxtimes T^{*0,q}X\big)$,

$$a(x,y,s,k) \sim \textstyle\sum^\infty_{j=0} a_j(x,y,s)k^{n-j} \text{ in } S^n_{\rm loc}\big(1;\Omega,T^{*0,q}X \boxtimes T^{*0,q}X\big),$$

$a_j(x,y,s) \in C^\infty_0\big(\Omega,T^{*0,q}X \boxtimes T^{*0,q}X\big)$, $j = 0,1,2,\ldots$, and

$$a_0(x,y,s) = 2\pi\int \widetilde{b}_0((x,y_{2n} - \varphi(x,y,s)),\hat{y},s)\chi(y_{2n})\widetilde{\chi}_1(y_{2n} - \varphi(x,y,s))dy_{2n}, \tag{6.36}$$

where $\widetilde{\chi}_1$ and $\widetilde{b}_0$ are almost analytic extensions of b_0 and χ_1 respectively, b_0 is as in Theorem 5.29. From (6.36) and notice that $\chi_1 = 1$ on $\operatorname{Supp}\chi$, $\varphi(x,x,s) = 0$, we deduce that

$$a_0(x,x,s) = 2\pi\int \widetilde{b}_0((x,y_{2n},y,s)\chi(y_{2n})dy_{2n}. \tag{6.37}$$

From (5.94) and (6.37), we conclude that

$$a_0(x,x,s) : T^{*0,q}_x X \to \mathcal{N}(x,s,n_-),\ \ \forall (x,x,s) \in \Omega. \tag{6.38}$$

From (6.34), (6.35) and (6.38), the theorem follows. ☐

We need

Theorem 6.7. *With the notations and assumptions before, we have*

$$\begin{aligned} &\mathcal{S}^*_k\mathcal{S}_k \equiv \widetilde{\mathcal{I}}^*_k\mathcal{S}_k \mod O(k^{-\infty}) \ \ \text{on } D_0,\\ &\mathcal{S}^*_k\mathcal{S}_k \equiv \hat{\mathcal{I}}^*_k\mathcal{S}_k \mod O(k^{-\infty}) \ \ \text{on } D_0 \end{aligned} \tag{6.39}$$

and $\mathcal{S}^*_k\mathcal{S}_k$ is a smoothing operator and the kernel of $\mathcal{S}^*_k\mathcal{S}_k$ satisfies

$$(\mathcal{S}^*_k\mathcal{S}_k)(x,y) \equiv \int e^{ik\varphi(x,y,s)}g(x,y,s,k)ds \mod O(k^{-\infty}) \ \ \text{on } D_0 \tag{6.40}$$

with

$$\begin{aligned} &g(x,y,s,k) \in S^n_{\rm loc}\big(1;\Omega,T^{*0,q}X \boxtimes T^{*0,q}X\big) \bigcap C^\infty_0\big(\Omega,T^{*0,q}X \boxtimes T^{*0,q}X\big),\\ &g(x,y,s,k) \sim \sum^\infty_{j=0} g_j(x,y,s)k^{n-j} \ \text{in } S^n_{\rm loc}\big(1;\Omega,T^{*0,q}X \boxtimes T^{*0,q}X\big),\\ &g_j(x,y,s) \in C^\infty_0\big(\Omega,T^{*0,q}X \boxtimes T^{*0,q}X\big),\ \ j = 0,1,2,\ldots,\\ &g_0(x,x,s) = \alpha^*_0(x,d_x\varphi(x,x,s))a_0(x,x,s),\ \ \forall (x,x,s) \in \Omega, \end{aligned} \tag{6.41}$$

where $\widetilde{\mathcal{I}}^*_k$ is as in (6.22), $\alpha^*_0(x,\eta) : T^{*0,q}_x X \to T^{*0,q}_x X$ is the adjoint of $\alpha_0(x,\eta)$ with respect to the Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $T^{*0,q}_x X$, $\alpha_0(x,\eta)$ is as in (6.22).

Proof. From (6.26) and (6.28), we have $\widetilde{\mathcal{I}}^*_k\mathcal{S}_k \equiv (\mathcal{G}^*_k\Box^{(q)}_{s,k} + \mathcal{S}^*_k)\mathcal{S}_k \mod O(k^{-\infty})$ on D_0 and $\hat{\mathcal{I}}^*_k\mathcal{S}_k \equiv (\mathcal{N}^*_k\Box^{(q)}_{s,k} + \mathcal{S}^*_k)\mathcal{S}_k \mod O(k^{-\infty})$ on D_0. Since $\Box^{(q)}_{s,k}\mathcal{S}_k \equiv 0 \mod O(k^{-\infty})$ on D, (6.39) follows.

From Theorem 6.6, (6.39) and the stationary phase formula of Melin-Sjöstrand [21], we get (6.40) and (6.41). The theorem follows. ☐

In the rest of this section, we will compute the leading terms $a_0(x,x,s)$ and $g_0(x,x,s)$ in the asymptotic expansions (6.30) and (6.41) respectively.

As before, let $x=(x_1,\ldots,x_{2n-1})$ be local coordinates on D_0. We also write $y=(y_1,\ldots,y_{2n-1})$ and $u=(u_1,\ldots,u_{2n-1})$. On D_0, we put $dv_X(x)=m(x)dx_1dx_2\ldots dx_{2n-1}=m(x)dx$. From (6.29), we can check that

$$\mathcal{S}^*_k(x,u)=\int_{\sigma\geq 0}e^{-ik\overline{\varphi}(u,x,\sigma)}a^*(u,x,\sigma,k)d\sigma,$$

where $a^*(u,x,\sigma,k):T^{*0,q}_uX\to T^{*0,q}_xX$ is the adjoint of $a(u,x,\sigma,k):T^{*0,q}_xX\to T^{*0,q}_uX$ with respect to $\langle\cdot\,|\,\cdot\rangle$. Thus,

(6.42)
$$(\mathcal{S}^*_k\circ\mathcal{S}_k)(x,y)\equiv\int_{\sigma\geq 0,s\geq 0}e^{ik\bigl(-\overline{\varphi}(u,x,\sigma)+\varphi(u,y,s)\bigr)}a^*(u,x,\sigma,k)a(u,y,s,k)m(u)dud\sigma ds\mod O(k^{-\infty}).$$

Note that $\mathcal{S}^*_k(x,u)$ and $\mathcal{S}_k(u,y)$ are k-negligible outside $x=u$ and $u=y$ respectively and $d_u\bigl(-\overline{\varphi}(u,x,\sigma)+\varphi(u,y,s)\bigr)\neq 0$ if $\sigma\neq s$ and (x,y) is in some small neighbourhood of $x=y$. From this observation, we conclude that for every $\varepsilon>0$, we have

(6.43)
$$\begin{split}&(\mathcal{S}^*_k\circ\mathcal{S}_k)(x,y)\\&\equiv\int_{\sigma\geq 0,s\geq 0}e^{ik\bigl(-\overline{\varphi}(u,x,\sigma)+\varphi(u,y,s)\bigr)}a^*(u,x,\sigma,k)a(u,y,s,k)\mu(\frac{\sigma-s}{\varepsilon})\mu(\frac{\left|y-u\right|^2}{\varepsilon})\mu(\frac{\left|x-u\right|^2}{\varepsilon})m(u)dud\sigma ds\\&\quad\mod O(k^{-\infty}),\end{split}$$

where $\mu\in C^\infty_0(]-1,1[)$, $\mu=1$ on $[\frac{1}{2},\frac{1}{2}]$. We want to apply the stationary phase method of Melin and Sjöstrand (see page 148 of[21]) to carry out the $dud\sigma$ integration in (6.43). Put

$$\Xi(u,x,y,\sigma,s):=-\overline{\varphi}(u,x,\sigma)+\varphi(u,y,s). \tag{6.44}$$

From (5.68), (5.69), (5.70) and (5.73), it is easy to see that

$$\operatorname{Im}\Xi(u,x,y,\sigma,s)\geq 0,\quad d_u\Xi(u,x,y,\sigma,s)|_{u=x=y,\sigma=s}=0,\quad d_\sigma\Xi(u,x,y,\sigma,s)|_{u=x=y,\sigma=s}=0.$$

Thus, $x=y=u$, $\sigma=s$, x is real, are real critical points. Now, we will compute the Hessian of Ξ at $x=y=u$, $\sigma=s$. We write $H_\Xi(x,s)$ to denote the Hessian of Ξ at $x=y=u$, $\sigma=s$. $H_\Xi(x,s)$ has the following form:

$$H_\Xi(x,s)=\left[\begin{array}{cccc}\frac{\partial^2\Xi}{\partial\sigma\partial\sigma}|_{u=x=y,\sigma=s}&\frac{\partial^2\Xi}{\partial\sigma\partial u_1}|_{u=x=y,\sigma=s}&\cdots&\frac{\partial^2\Xi}{\partial\sigma\partial u_{2n-1}}|_{u=x=y,\sigma=s}\\\frac{\partial^2\Xi}{\partial u_1\partial\sigma}|_{u=x=y,\sigma=s}&\frac{\partial^2\Xi}{\partial u_1\partial u_1}|_{u=x=y,\sigma=s}&\cdots&\frac{\partial^2\Xi}{\partial u_1\partial u_{2n-1}}|_{u=x=y,\sigma=s}\\\vdots&\vdots&\vdots&\vdots\\\frac{\partial^2\Xi}{\partial u_{2n-1}\partial\sigma}|_{u=x=y,\sigma=s}&\frac{\partial^2\Xi}{\partial u_{2n-1}\partial u_1}|_{u=x=y,\sigma=s}&\cdots&\frac{\partial^2\Xi}{\partial u_{2n-1}\partial u_{2n-1}}|_{u=x=y,\sigma=s}\end{array}\right]. \tag{6.45}$$

We fix $(p,p,s_0)\in\Omega$, $p\in D_0$. Take local coordinates $x=(x_1,\ldots,x_{2n-1})$ so that (2.2) hold. It is easy to see that

$$m(p)=2^{n-1}. \tag{6.46}$$

From (2.3), it is straightforward to check that

$$\begin{split}&\frac{\partial^2\Xi}{\partial\sigma\partial\sigma}|_{u=x=y=p,\sigma=s_0}=\frac{\partial^2\Xi}{\partial u_1\partial\sigma}|_{u=x=y,\sigma=s}=\ldots=\frac{\partial^2\Xi}{\partial u_{2n-2}\partial\sigma}|_{u=x=y,\sigma=s}=0,\\&\frac{\partial^2\Xi}{\partial u_{2n-1}\partial\sigma}|_{u=x=y,\sigma=s}=-1,\\&\frac{\partial^2\Xi}{\partial u_{2j-1}\partial u_{2j-1}}|_{u=x=y=p,\sigma=s_0}=\frac{\partial^2\Xi}{\partial u_{2j}\partial u_{2j}}|_{u=x=y=p,\sigma=s_0}=2i\left|\lambda_j(s_0)\right|,\quad j=1,\ldots,n-1,\\&\frac{\partial^2\Xi}{\partial u_j\partial u_k}|_{u=x=y=p,\sigma=s_0}=0\ \text{ if } j\neq k \text{ and } j,k=1,\ldots,2n-2,\end{split} \tag{6.47}$$

where $\lambda_1(s_0),\ldots,\lambda_{n-1}(s_0)$ are as in Theorem 2.2. From (6.47), we see that

$$H_\Xi(p,s_0)=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1\\ 0 & 2i\left|\lambda_1(s_0)\right| & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & 2i\left|\lambda_1(s_0)\right| & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 2i\left|\lambda_{n-1}(s_0)\right| & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 2i\left|\lambda_{n-1}(s_0)\right| & 0\\ -1 & * & * & \cdots & * & * & * \end{bmatrix}. \tag{6.48}$$

Thus,

$$\det\Big(\frac{H_\Xi(p,s_0)}{2\pi i}\Big)=\frac{1}{4}\pi^{-2n}\left|\lambda_1(s_0)\right|^2\left|\lambda_2(s_0)\right|^2\cdots\left|\lambda_{n-1}(s_0)\right|^2. \tag{6.49}$$

Since $(p,p,s_0)\in\Omega$ is arbitrary, we conclude that $\det\big(\frac{H_\Xi(p,s_0)}{2\pi i}\big)\neq 0$, for every $(x,x,s)\in\Omega$. Hence, we can apply the stationary phase method of Melin and Sjöstrand (see page 148 of[21]) to carry out the $dud\sigma$ integration in (6.43) and obtain

$$(\mathcal{S}_k^*\mathcal{S}_k)(x,y)\equiv\int e^{ik\varphi_1(x,y,s)}h(x,y,s,k)ds \mod O(k^{-\infty}) \tag{6.50}$$

with

$$\begin{aligned} &h(x,y,s,k)\in S^n_{\rm loc}\big(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big)\bigcap C_0^\infty\big(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\\ &h(x,y,s,k)\sim\sum_{j=0}^{\infty}h_j(x,y,s)k^{n-j}\text{ in }S^n_{\rm loc}\big(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\\ &h_j(x,y,s)\in C_0^\infty\big(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\quad j=0,1,2,\ldots,\\ &h_0(x,x,s)=\Big(\det\big(\frac{H_\Xi(x,s)}{2\pi i}\big)\Big)^{-\frac12}a_0^*(x,x,s)a_0(x,x,s)m(x),\quad\forall (x,x,s)\in\Omega, \end{aligned} \tag{6.51}$$

and

$$\begin{aligned} &\varphi_1(x,x,s)=0,\quad d_x\varphi_1(x,x,s)=d_x\varphi(x,x,s),\quad d_y\varphi_1(x,x,s)=d_y\varphi(x,x,s),\ \forall (x,x,s)\in\Omega,\\ &{\rm Im}\,\varphi_1(x,y,s)\geq 0,\quad\forall (x,y,s)\in\Omega, \end{aligned} \tag{6.52}$$

where $a_0(x,y,s)$ is as in (6.30) and $a_0^*(x,x,s):T^{*0,q}_xX\to T^{*0,q}_xX$ is the adjoint of $a_0(x,x,s):T^{*0,q}_xX\to T^{*0,q}_xX$ with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$. We need

Lemma 6.8. *With the notations above, for every $(x,x,s)\in\Omega$, we have*

$$h_0(x,x,s)=g_0(x,x,s), \tag{6.53}$$

where $g_0(x,y,s)$ is as in (6.41).

Proof. Fix (x_0,x_0,s_0). Suppose that ${\rm Re}\,h_0(x_0,x_0,s_0)\neq{\rm Re}\,g_0(x_0,x_0,s_0)$. We may assume that ${\rm Re}\,h_0(x_0,x_0,s_0)<{\rm Re}\,g_0(x_0,x_0,s_0)$. Take $\epsilon_0>0$ be a small constant so that ${\rm Re}\,h_0(x_0,x_0,s)<{\rm Re}\,g_0(x_0,x_0,s)$, for every $|s-s_0|<\epsilon_0$. Let Σ' and V be as in (1.5) and (5.14) respectively. For every $\epsilon>0$, put

$$\Sigma'_{s_0,\epsilon}:=\big\{(x,s\omega_0(x)-2{\rm Im}\,\overline\partial_b\phi(x))\in\Sigma';\,|s-s_0|<\epsilon\big\}.$$

Let $r(x,\eta)\in C_0^\infty(V)$ with $r(x,\eta)\geq 0$, $r(x,\eta)=1$ on $\Sigma'_{s_0,\frac{\epsilon_0}{2}}$ and ${\rm Supp}\,r(x,\eta)\bigcap\Sigma\subset\Sigma'_{s_0,\epsilon_0}$. We remind that Σ is given by (4.11). Consider the classical semi-classical pseudodifferential operator:

$$R=\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}r(x,\eta)d\eta.$$

From (6.50), (6.51), (6.52), (5.69) and the stationary phase method of Melin and Sjöstrand (see page 148 of[21]), we have

$$
\begin{aligned}
&(R\mathcal{S}_k^*\mathcal{S}_k)(x,x) \sim \sum_{j=0}^{\infty} k^{-j} \int \vartheta_j(x,x,s)ds \text{ in } S^n_{\rm loc}\big(1;\Omega, T^{*0,q}X\boxtimes T^{*0,q}X\big),\\
&\vartheta_j(x,y,s)\in C^\infty_0\big(\Omega, T^{*0,q}X\boxtimes T^{*0,q}X\big),\quad j=0,1,2,\ldots,\\
&\vartheta_0(x,x,s)=r(x,s\omega_0(x)-2{\rm Im}\,\overline\partial_b\phi(x))h_0(x,x,s),\quad \forall (x,x,s)\in\Omega.
\end{aligned}
\tag{6.54}
$$

Similarly, from Theorem 6.7, we conclude that

$$
\begin{aligned}
&(R\mathcal{S}_k^*\mathcal{S}_k)(x,x) \sim \sum_{j=0}^{\infty} k^{-j} \int \zeta_j(x,x,s)ds \text{ in } S^n_{\rm loc}\big(1;\Omega, T^{*0,q}X\boxtimes T^{*0,q}X\big),\\
&\zeta_j(x,y,s)\in C^\infty_0\big(\Omega, T^{*0,q}X\boxtimes T^{*0,q}X\big),\quad j=0,1,2,\ldots,\\
&\zeta_0(x,x,s)=r(x,s\omega_0(x)-2{\rm Im}\,\overline\partial_b\phi(x))g_0(x,x,s),\quad \forall (x,x,s)\in\Omega.
\end{aligned}
\tag{6.55}
$$

From (6.54) and (6.55), we deduce that

$$
\int r(x_0,s\omega_0(x_0)-2{\rm Im}\,\overline\partial_b\phi(x_0)){\rm Re}\,h_0(x_0,x_0,s)ds=\int r(x_0,s\omega_0(x_0)-2{\rm Im}\,\overline\partial_b\phi(x_0)){\rm Re}\,g_0(x_0,x_0,s)ds.
\tag{6.56}
$$

Since ${\rm Supp}\,r(x,\eta)\bigcap\Sigma\subset\Sigma'_{s_0,\epsilon_0}$, we have

$$
\begin{aligned}
&\int r(x_0,s\omega_0(x_0)-2{\rm Im}\,\overline\partial_b\phi(x_0)){\rm Re}\,h_0(x_0,x_0,s)ds\\
&=\int_{|s-s_0|<\epsilon_0} r(x_0,s\omega_0(x_0)-2{\rm Im}\,\overline\partial_b\phi(x_0)){\rm Re}\,h_0(x_0,x_0,s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\int r(x_0,s\omega_0(x_0)-2{\rm Im}\,\overline\partial_b\phi(x_0)){\rm Re}\,g_0(x_0,x_0,s)ds\\
&=\int_{|s-s_0|<\epsilon_0} r(x_0,s\omega_0(x_0)-2{\rm Im}\,\overline\partial_b\phi(x_0)){\rm Re}\,g_0(x_0,x_0,s)ds.
\end{aligned}
$$

From this observation and (6.56), we deduce that

$$
\int_{|s-s_0|<\epsilon_0} r(x_0,s\omega_0(x_0)-2{\rm Im}\,\overline\partial_b\phi(x_0))\big({\rm Re}\,h_0(x_0,x_0,s)-{\rm Re}\,g_0(x_0,x_0,s)\big)ds=0.
$$

Since ${\rm Re}\,h_0(x_0,x_0,s)<{\rm Re}\,g_0(x_0,x_0,s)$, for every $|s-s_0|<\epsilon_0$, and $r(x_0,s\omega_0(x_0)-2{\rm Im}\,\overline\partial_b\phi(x_0))\geq 0$, $r(x_0,s\omega_0(x_0)-2{\rm Im}\,\overline\partial_b\phi(x_0))$ is not identically to zero function, we get a contradiction. Thus, ${\rm Re}\,h_0(x_0,x_0,s_0)={\rm Re}\,g_0(x_0,x_0,s_0)$.

We can repeat the procedure above and conclude that ${\rm Im}\,h_0(x_0,x_0,s_0)={\rm Im}\,g_0(x_0,x_0,s_0)$. Since (x_0,x_0,s_0) is arbitrary, the lemma follows. □

As before, we fix $(p,p,s_0)\in\Omega$ and take local coordinates $x=(x_1,\ldots,x_{2n-1})$ so that (2.2) hold. From (6.46), (6.49), (6.51) Lemma 6.8 and (6.41), we see that

$$
\begin{aligned}
g_0(p,p,s_0)&=(2\pi)^n\left|\lambda_1(s_0)\right|^{-1}\left|\lambda_2(s_0)\right|^{-1}\cdots\left|\lambda_{n-1}(s_0)\right|^{-1}a_0^*(p,p,s_0)a_0(p,p,s_0)\\
&=\alpha_0^*(p,s_0\omega_0(p)-2{\rm Im}\,\overline\partial_b\phi(p))a_0(p,p,s_0),
\end{aligned}
\tag{6.57}
$$

where $\alpha_0(x,\eta)$ is as in (6.22) and $\alpha^*_0(x,\eta):T^{*0,q}_xX\rightarrow T^{*0,q}_xX$ is the adjoint of $\alpha_0(x,\eta)$ with respect to the Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $T^{*0,q}_xX$. Let $\pi_{(p,s_0,n_-)}:T^{*0,q}_pX\rightarrow\mathcal{N}(p,s_0,n_-)$ be the orthogonal projection with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$. In view of (6.30), we know that $a_0(p,p,s_0):T^{*0,q}_pX\rightarrow\mathcal{N}(p,s_0,n_-)$. From (5.39), we see that $\dim\mathcal{N}(p,s_0,n_-)=1$. From this observation and (6.57), it is straightforward to see that

$$\begin{aligned}a_0(p,p,s_0) =&(2\pi)^{-n}\left|\lambda_1(s_0)\right|\left|\lambda_2(s_0)\right|\cdots\left|\lambda_{n-1}(s_0)\right|\pi_{(p,s_0,n_0)}\alpha(p,s_0\omega_0(p)-2\mathrm{Im}\,\overline{\partial}_b\phi(p)),\\ g_0(p,p,s_0) =&(2\pi)^{-n}\left|\lambda_1(s_0)\right|\left|\lambda_2(s_0)\right|\cdots\left|\lambda_{n-1}(s_0)\right|\times\\ &\qquad\alpha_0^*(p,s_0\omega_0(p)-2\mathrm{Im}\,\overline{\partial}_b\phi(p))\pi_{(p,s_0,n_-)}\alpha(p,s_0\omega_0(p)-2\mathrm{Im}\,\overline{\partial}_b\phi(p)).\end{aligned}$$

Summing up, we obtain

Theorem 6.9. *With the same same notations and assumptions as in Theorem 6.5, let $a_0(x,y,s)\in C^\infty_0(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X)$ and $g_0(x,y,s)\in C^\infty_0(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X)$ be as in (6.30) and (6.41) respectively. Fix $(p,p,s_0)\in\Omega$, $p\in D_0$, and let $\pi_{(p,s_0,n_-)}:T^{*0,q}_pX\to\mathcal{N}(p,s_0,n_-)$ be the orthogonal projection with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$, where $\mathcal{N}(p,s_0,n_-)$ is given by (5.39). Then,*

(6.58)

$$\begin{aligned}&a_0(p,p,s_0)=(2\pi)^{-n}\left|\det\big(M^\phi_p-2s_0\mathcal{L}_p\big)\right|\pi_{(p,s_0,n_-)}\alpha_0(p,s_0\omega_0(p)-2\mathrm{Im}\,\overline{\partial}_b\phi(p)),\\ &g_0(p,p,s_0)\\ &=(2\pi)^{-n}\left|\det\big(M^\phi_p-2s_0\mathcal{L}_p\big)\right|\alpha_0^*(p,s_0\omega_0(p)-2\mathrm{Im}\,\overline{\partial}_b\phi(p))\pi_{(p,s_0,n_-)}\alpha_0(p,s_0\omega_0(p)-2\mathrm{Im}\,\overline{\partial}_b\phi(p)),\end{aligned}$$

where $\alpha_0(x,\eta)$ is as in (6.22), $\alpha^*_0(x,\eta):T^{*0,q}_xX\to T^{*0,q}_xX$ is the adjoint of $\alpha_0(x,\eta)$ with respect to the Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $T^{*0,q}_xX$ and $\left|\det\big(M^\phi_p-2s_0\mathcal{L}_p\big)\right|=\left|\lambda_1(s_0)\right|\left|\lambda_2(s_0)\right|\cdots\left|\lambda_{n-1}(s_0)\right|$. Here $\lambda_1(s_0),\ldots,\lambda_{n-1}(s_0)$ are eigenvalues of the Hermitian quadratic form $M^\phi_p-2s_0\mathcal{L}_p$ with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$.

Using Theorem 5.20 and repeating the proof of Theorem 6.5 we conclude that

Theorem 6.10. *Let s be a local trivializing section of L on an open subset $D\subset X$ and $\left|s\right|^2_{h^L}=e^{-2\phi}$. We assume that there exist a $\lambda_0\in\mathbb{R}$ and $x_0\in D$ such that $M^\phi_{x_0}-2\lambda_0\mathcal{L}_{x_0}$ is non-degenerate of constant signature (n_-,n_+). Let $q\neq n_-$. We fix $D_0\Subset D$, D_0 open. Let V be as in (5.14). Let*

$$\hat{\mathcal{I}}_k\equiv\tfrac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta\mod O(k^{-\infty})\ \text{at}\ T^*D_0\bigcap\Sigma$$

be a classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$, where $\alpha(x,\eta,k)\in S^0_{\mathrm{loc\,,cl}}(1;T^*D,T^{*0,q}X\boxtimes T^{*0,q}X)$ with $\alpha(x,\eta,k)=0$ if $\left|\eta\right|>M$, for some large $M>0$ and $\mathrm{Supp}\,\alpha(x,\eta,k)\bigcap T^*D_0\Subset V$. Then, there exists a properly supported continuous operator

$$\mathcal{N}_k=O(k^s):H^s_{\mathrm{comp}}(D,T^{*0,q}X)\to H^{s+1}_{\mathrm{comp}}(D,T^{*0,q}X),\quad\forall s\in\mathbb{N}_0,$$

such that

$$\Box^{(q)}_{s,k}\mathcal{N}_k\equiv\hat{\mathcal{I}}_k\mod O(k^{-\infty})$$

on D_0, where $\Box^{(q)}_{s,k}$ is given by (4.4).

7. Szegö kernel asymptotics for lower energy forms

Let $\lambda\geq0$. We recall that (see (1.1)) $H^q_{b,\leq\lambda}(X,L^k)$ denote the spectral space of $\Box^{(q)}_{b,k}$ corresponding to energy less that λ and $\Pi^{(q)}_{k,\leq\lambda}:L^2_{(0,q)}(X,L^k)\to H^q_{b,\leq\lambda}(X,L^k)$ denote the orthogonal projection with respect to $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$. Fix $N_0\geq1$. In this section, we will study semi-classical asymptotic expansion of $\Pi^{(q)}_{k,\leq k^{-N_0}}$.

7.1. Asymptotic upper bounds.

Fix $N_0\geq1$. In this section we will give pointwise upper bounds for u and $\partial^\alpha u$, where $u\in H^q_{b,\leq k^{-N_0}}(X,L^k)$.

Let s be a local trivializing section of L on an open subset $D\subset X$ and $\left|s\right|^2_{h^L}=e^{-2\phi}$. Fix $p\in D$, let $U_1(y),\ldots,U_{n-1}(y)$ be an orthonormal frame of $T^{1,0}_yX$ varying smoothly with y in a neighbourhood of p, for which the Levi form is diagonal at p. We take local coordinates $x = (x_1, \ldots, x_{2n-2}, x_{2n-1}) = (z, x_{2n-1})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n-1$, defined on a small neighbourhood of p such that

$$
\begin{aligned}
& x(p) = 0, \quad \omega_0(p) = dx_{2n-1}, \quad T(p) = -\frac{\partial}{\partial x_{2n-1}}(p), \\
& \langle \frac{\partial}{\partial x_j}(p) \,|\, \frac{\partial}{\partial x_t}(p) \rangle = 2\delta_{j,t}, \quad j,t = 1, \ldots, 2n-2, \\
& U_j = \frac{\partial}{\partial z_j} - i\tau_j \overline{z}_j \frac{\partial}{\partial x_{2n-1}} - c_j x_{2n-1} \frac{\partial}{\partial x_{2n-1}} + O(|x|^2), \quad j = 1, \ldots, n-1, \\
& \phi = \sum_{j=1}^{n-1} (\alpha_j z_j + \overline{\alpha}_j \overline{z}_j) + \beta x_{2n-1} + \sum_{j,t=1}^{n-1} (a_{j,t} z_j z_t + \overline{a}_{j,t} \overline{z}_j \overline{z}_t) + \sum_{j,t=1}^{n-1} \mu_{j,t} \overline{z}_j z_t \\
& \quad + O(|z|\,|x_{2n-1}|) + O(|x_{2n-1}|^2) + O(|x|^3),
\end{aligned} \tag{7.1}
$$

where $\tau_j, \beta \in \mathbb{R}$, $j = 1, \ldots, n-1$, $\mu_{j,t}, c_j, \alpha_j, a_{j,t} \in \mathbb{C}$, $\mu_{j,t} = \overline{\mu}_{t,j}$, $j,t = 1, \ldots, n-1$ (This is always possible, see [2, p. 157–160]). Note that $\tau_1, \ldots, \tau_{n-1}$ are eigenvalues of $\mathcal{L}_p$ with respect to $\langle \cdot | \cdot \rangle$. We assume that this local coordinates are defined on D and until further notice, we work with this local coordinates and we identify D with some open set in $\mathbb{R}^{2n-1}$. Put

$$
R(x) = R(z, x_{2n-1}) = \sum_{j=1}^{n-1} \alpha_j z_j + \sum_{j,t=1}^{n-1} a_{j,t} z_j z_t, \tag{7.2}
$$

$$
\phi_0 = \phi - R(x) - \overline{R(x)} = \beta x_{2n-1} + \sum_{j,t=1}^{n-1} \mu_{j,t} \overline{z}_j z_t + O(|z|\,|x_{2n-1}|) + O(|x_{2n-1}|^2) + O(|x|^3). \tag{7.3}
$$

Let $(\cdot | \cdot)_{k\phi}$ and $(\cdot | \cdot)_{k\phi_0}$ be the inner products on the space $\Omega_0^{0,q}(D)$ defined as follows:

$$
(f \mid g)_{k\phi} = \int_D \langle f \,|\, g \rangle e^{-2k\phi} dv_X \,, \quad (f \mid g)_{k\phi_0} = \int_D \langle f \,|\, g \rangle e^{-2k\phi_0} dv_X \,,
$$

where $f, g \in \Omega_0^{0,q}(D)$. We denote by $L^2_{(0,q)}(D, k\phi)$ and $L^2_{(0,q)}(D, k\phi_0)$ the completions of $\Omega_0^{0,q}(D)$ with respect to $(\cdot | \cdot)_{k\phi}$ and $(\cdot | \cdot)_{k\phi_0}$, respectively. We have the unitary identification

$$
\begin{cases}
L^2_{(0,q)}(D, k\phi_0) \leftrightarrow L^2_{(0,q)}(D, k\phi) \\
\qquad\qquad u \to e^{2kR} u, \\
\quad u = e^{-2kR} v \leftarrow v.
\end{cases} \tag{7.4}
$$

Let $\overline{\partial}_b^{*,k\phi} : \Omega^{0,q+1}(D) \to \Omega^{0,q}(D)$ be the formal adjoint of $\overline{\partial}_b$ with respect to $(\cdot | \cdot)_{k\phi}$. Put

$$
\Box^{(q)}_{b,k\phi} = \overline{\partial}_b \overline{\partial}_b^{*,k\phi} + \overline{\partial}_b^{*,k\phi} \overline{\partial}_b : \Omega^{0,q}(D) \to \Omega^{0,q}(D) \,.
$$

Let $u \in \Omega^{0,q}(D, L^k)$. Then there exists $\hat{u} \in \Omega^{0,q}(D)$ such that $u = s^k \hat{u}$ and we have $\Box^{(q)}_{b,k} u = s^k \Box^{(q)}_{b,k\phi} \hat{u}$. In this section, we identify u with $\hat{u}$ and $\Box^{(q)}_{b,k}$ with $\Box^{(q)}_{b,k\phi}$. Note that $|u(0)|^2 = |\hat{u}(0)|^2 e^{-2k\phi(0)} = |\hat{u}(0)|^2$. Let $\widetilde{\overline{\partial}}_b : \Omega^{0,q}(D) \to \Omega^{0,q+1}(D)$ be the first order partial differential operator given by

$$
\begin{aligned}
& \widetilde{\overline{\partial}}_b : \Omega^{0,q}(D) \to \Omega^{0,q+1}(D), \\
& \overline{\partial}_b (e^{2kR} u) = e^{2kR} \widetilde{\overline{\partial}}_b u, \quad \forall u \in \Omega^{0,q}(D).
\end{aligned} \tag{7.5}
$$

Let $\widetilde{\overline{\partial}}_b^* : \Omega^{0,q+1}(D) \to \Omega^{0,q}(D)$ be the formal adjoint of $\widetilde{\overline{\partial}}_b$ with respect to $(\cdot | \cdot)_{k\phi_0}$. It is easy to see that

$$
\overline{\partial}_b^{*,k\phi} (e^{2kR} u) = e^{2kR} \widetilde{\overline{\partial}}_b^* u, \quad \forall u \in \Omega^{0,q+1}(D). \tag{7.6}
$$

Put

$$
\widetilde{\Box}^{(q)}_{b,k\phi} = \widetilde{\overline{\partial}}_b \widetilde{\overline{\partial}}_b^* + \widetilde{\overline{\partial}}_b^* \widetilde{\overline{\partial}}_b : \Omega^{0,q}(D) \to \Omega^{0,q}(D). \tag{7.7}
$$

From (7.5) and (7.6), we have

$$\Box^{(q)}_{b,k\phi}(e^{2kR}u) = e^{2kR}\widetilde{\Box}^{(q)}_{b,k\phi}u, \quad \forall u \in \Omega^{0,q}(D). \tag{7.8}$$

Until further notice, we fix $q \in \{0, 1, \ldots, n-1\}$ and we assume that $Y(q)$ holds at each point of D. For $r > 0$, let $D_r = \big\{x = (x_1, \ldots, x_{2n-1}) \in \mathbb{R}^{2n-1};\ |x_j| < r, \quad j = 1, \ldots, 2n-1\big\}$. Let F_k be the scaling map: $F_k(z, x_{2n-1}) = (\frac{z}{\sqrt{k}}, \frac{x_{2n-1}}{k})$. From now on, we assume that k is large enough so that $F_k(D_{\log k}) \subset D$. Let $(e_j(x))_{j=1,\ldots,n-1}$ denote the basis of $T^{*0,1}_x X$, dual to $(\overline{U}_j(x))_{j=1,\ldots,n-1}$. Let $J = (j_1, \ldots, j_q) \in \{1, \ldots, n-1\}^q$ be a multiindex. Define

$$e_J = e_{j_1} \wedge \cdots \wedge e_{j_q}, \quad \text{if } 1 \leq j_1, j_2, \ldots, j_q \leq n-1.$$

Then $\{e_J;\ J \in \{1, \ldots, n-1\}^q,\ J \text{ strictly increasing}\}$ is an orthonormal frame for $T^{*0,q}X$ over D. We define the scaled bundle $F_k^* T^{*0,q}X$ on $D_{\log k}$ to be the bundle whose fiber at $x \in D_{\log k}$ is

$$F_k^* T^{*0,q}_x X := \Big\{ {\sum_{J\in\{1,\ldots,n-1\}^q}}' a_J e_J(\tfrac{z}{\sqrt{k}}, \tfrac{x_{2n-1}}{k});\ a_J \in \mathbb{C}, \forall J \in \{1, \ldots, n-1\}^q \Big\},$$

where $\sum'$ means that the summation is performed only over strictly increasing multiindices. We take the Hermitian metric $\langle \cdot | \cdot \rangle_{F_k^*}$ on $F_k^* T^{*0,q}X$ so that at each point $x \in D_{\log k}$,

$$\Big\{ e_J\big(\tfrac{z}{\sqrt{k}}\ , \tfrac{x_{2n-1}}{k}\big)\ ;\ J \in \{1, \ldots, n-1\}^q,\ J \text{ strictly increasing} \Big\}$$

is an orthonormal basis for $F_k^* T^{*0,q}_x X$. For $r > 0$, let $F_k^*\Omega^{0,q}(D_r)$ denote the space of smooth sections of $F_k^* T^{*0,q}X$ over D_r. Let $F_k^*\Omega^{0,q}_0(D_r)$ be the subspace of $F_k^*\Omega^{0,q}(D_r)$ whose elements have compact support in D_r. Given $f \in \Omega^{0,q}(F_k(D_{\log k}))$ we write $f = {\sum_{|J|=q}}' f_J e_J$. We define the scaled form $F_k^* f \in F_k^*\Omega^{0,q}(D_{\log k})$ by:

$$F_k^* f = {\sum_{J\in\{1,\ldots,n-1\}^q}}' f_J\Big(\frac{z}{\sqrt{k}}\ , \frac{x_{2n-1}}{k}\Big) e_J\Big(\frac{z}{\sqrt{k}}\ , \frac{x_{2n-1}}{k}\Big).$$

It is well-known (see section 2.2 in [15]) that there is a scaled Laplacian $\widetilde{\Box}^{(q),(k)}_{b,k\phi} : F_k^*\Omega^{0,q}(D_{\log k}) \to F_k^*\Omega^{0,q}(D_{\log k})$ such that

$$\widetilde{\Box}^{(q),(k)}_{b,k\phi}(F_k^* u) = \frac{1}{k} F_k^*(\widetilde{\Box}^{(q)}_{b,k\phi} u), \quad \forall u \in \Omega^{0,q}(F_k(D_{\log k})), \tag{7.9}$$

and all the derivatives of the coefficients of the operator $\widetilde{\Box}^{(q),(k)}_{k\phi,(k)}$ are uniformly bounded in k on $D_{\log k}$. Let $D_r \subset D_{\log k}$ and let $W^s_{kF_k^*\phi_0}(D_r, F_k^* T^{*0,q}X)$, $s \in \mathbb{N}_0$, denote the Sobolev space of order s of sections of $F_k^* T^{*0,q}X$ over D_r with respect to the weight $e^{-2kF_k^*\phi_0}$. The Sobolev norm on this space is given by

$$\|u\|^2_{kF_k^*\phi_0,s,D_r} = {\sum_{\alpha\in\mathbb{N}_0^{2n-1},|\alpha|\leq s, J\in\{1,\ldots,n-1\}^q}}' \int_{D_r} |\partial_x^\alpha u_J|^2\, e^{-2kF_k^*\phi_0}(F_k^* m)(x)dx, \tag{7.10}$$

where $u = \sum'_{J\in\{1,\ldots,n-1\}^q} u_J e_J\big(\frac{z}{\sqrt{k}}, \frac{x_{2n-1}}{k}\big) \in W^s_{kF_k^*\phi_0}(D_r, F_k^* T^{*0,q}X)$ and $m(x)dx$ is the volume form. If $s = 0$, we write $\|\cdot\|_{kF_k^*\phi_0,D_r}$ to denote $\|\cdot\|_{kF_k^*\phi_0,0,D_r}$.

The following is well-known(see section 2.2 in [15])

Proposition 7.1. *Let $r > 0$ with $D_{2r} \subset D_{\log k}$ and let $s \in \mathbb{N}_0$. Then, there is a constant $C_{r,s} > 0$ independent of k and the point p, such that for every $u \in \Omega^{0,q}(D_{\log k})$,*

$$\|u\|^2_{kF_k^*\phi_0,s,D_r} \leqslant C_{r,s}\Big(\|u\|^2_{kF_k^*\phi_0,D_{2r}} + \sum_{j=1}^{s} \big\|(\widetilde{\Box}^{(q),(k)}_{b,k\phi})^j u\big\|^2_{kF_k^*\phi_0,D_{2r}} \Big). \tag{7.11}$$

Moreover, there exist a semi-norm P on $C^\infty(D_{2r})$ and a strictly positive continuous function $F : \mathbb{R} \to \mathbb{R}_+$, where P and F only depend on r and s independent of the point p and k, such that if we *put*

A

$= \{$*all the coefficients of* $\widetilde{\Box}^{(q),(k)}_{b,k\phi}$, $\widetilde{\overline{\partial}}_b$, $\widetilde{\overline{\partial}}^*_b$, $[\overline{U}_j, U_t]$, $\overline{U}_j$, U_t, $j,t = 1,\ldots,n-1$, *and* $kF^*_k\phi_0$, $F^*_k m\}$ *and* $B = \{$*all the eigenvalues of* $\mathcal{L}_p\}$, *then* $C_{r,s}$ *can be bounded by* $\sum_{f\in A} F(P(f)) + \sum_{\lambda\in B} F(\lambda)$.

We need

Lemma 7.2. *For k large and for every $\alpha \in \mathbb{N}_0^{2n-1}$, there is a constant $C_\alpha > 0$ independent of k and the point p, such that for all $u \in \Omega^{0,q}(D_{\log k})$ with $\|u\|_{kF^*_k\phi_0,D_{\log k}} \leq 1$ and $\left\|(\widetilde{\Box}^{(q),(k)}_{b,k\phi})^m u\right\|_{kF^*_k\phi_0,D_{\log k}} \leq k^{-m}$, $\forall m \in \mathbb{N}_0$, we have*

$$|(\partial^\alpha_x u)(0)| \leq C_\alpha. \tag{7.12}$$

Proof. Let $u \in \Omega^{0,q}(D_{\log k})$, $\|u\|_{kF^*_k\phi_0,D_{\log k}} \leq 1$, $\left\|(\widetilde{\Box}^{(q),(k)}_{b,k\phi})^m u\right\|_{kF^*_k\phi_0,D_{\log k}} \leq k^{-m}$, $\forall m \in \mathbb{N}_0$. By using Fourier transform, it is easy to see that (cf. Lemma 2.6 in [15])

$$|(\partial^\alpha_x u)(0)| \leq C\,\|u\|_{kF^*_k\phi_0,n+|\alpha|,D_r}\,, \tag{7.13}$$

for some $r > 0$, where $C > 0$ only depends on the dimension and the size of α. From (7.11), we see that

$$\begin{aligned}\|u\|^2_{kF^*_k\phi_0,n+|\alpha|,D_r} &\leq C_{r,n+|\alpha|}\Big(\|u\|^2_{kF^*_k\phi_0,D_{2r}} + \sum_{j=1}^{n+|\alpha|}\left\|(\widetilde{\Box}^{(q),(k)}_{b,k\phi})^j u\right\|_{kF^*_k\phi_0,D_{2r}}\Big)\\ &\leq C_{r,n+|\alpha|}\Big(1+\sum_{j=1}^{\infty}k^{-j}\Big) \leq \widetilde{C}_\alpha,\end{aligned} \tag{7.14}$$

where $\widetilde{C}_\alpha > 0$ is independent of k and the point p. Combining (7.14) with (7.13), (7.12) follows. $\square$

Now, we can prove

Theorem 7.3. *For every $\alpha \in \mathbb{N}_0^{2n-1}$, $D' \Subset D$, there is a constant $C_{\alpha,D'} > 0$ independent of k, such that for every $u \in H^q_{b,\leq k^{-N_0}}(X, L^k)$, $u|_D = s^k\widetilde{u}$, $\widetilde{u} \in \Omega^{0,q}(D)$, we have*

$$\left|(\partial^\alpha_x(\widetilde{u}e^{-k\phi}))(x)\right| \leq C_{\alpha,D'}k^{\frac{n}{2}+|\alpha|}\,\|u\|\,,\quad \forall x \in D'. \tag{7.15}$$

Remark 7.4. Let s_1 be another local frame of L on D, $|s_1|^2 = e^{-2\phi_1}$. We have $s_1 = gs$ for some CR function $g \in C^\infty(D)$, $g \neq 0$ on D. Let $u \in \Omega^{0,q}(D, L^k)$. On D, we write $u = s^k\widetilde{u} = s_1^k\widetilde{v}$. Then, we can check that

$$\widetilde{v}e^{-k\phi_1} = \widetilde{u}(\overline{g}^{\,1/2}g^{-1/2})^k e^{-k\phi}. \tag{7.16}$$

From (7.16), it is easy to see that if $\widetilde{u}$ satisfies (7.15), then $\widetilde{v}$ also satisfies (7.15). Thus, the conclusion of Theorem 7.3 makes sense.

Proof of Theorem 7.3. We may assume that $0 \in D'$. Let $u \in H^q_{b,\leq k^{-N_0}}(X, L^k)$, $u|_D = s^k\widetilde{u}$, $\widetilde{u} \in \Omega^{0,q}(D)$. We may assume that $F_k(D_{\log k}) \subset D$ and consider $\widetilde{u}|_{F_k(D_{\log k})}$. Set

$$\beta_k := k^{-\frac{n}{2}}F^*_k(e^{-2kR}\widetilde{u}) \in F^*_k\Omega^{0,q}(D_{\log k}).$$

We recall that R is given by (7.2). (See also (7.4).) We can check that

$$\|\beta_k\|_{kF^*_k\phi_0,D_{\log k}} \leq \|u\|_{h^{L^k}}\,. \tag{7.17}$$

Since $u \in H^q_{b,\leq k^{-N_0}}(X, L^k)$, we have $\left\|(\Box^{(q)}_{b,k})^m u\right\|_{h^{L^k}} \leq k^{-mN_0}\|u\|_{h^{L^k}}$ for all $m \in \mathbb{N}$. From this observation, (7.9) and (7.8), we have

$$\begin{aligned}\left\|(\widetilde{\Box}^{(q),(k)}_{b,k\phi})^m\beta_k\right\|_{kF^*_k\phi_0,D_{\log k}} &= \frac{1}{k^{m+\frac{n}{2}}}\left\|F^*_k\big((\widetilde{\Box}^{(q)}_{b,k\phi})^m e^{-2kR}\widetilde{u}\big)\right\|_{kF^*_k\phi_0,D_{\log k}}\\ &\leq \frac{1}{k^m}\left\|(\Box^{(q)}_{b,k})^m u\right\|_{h^{L^k}} \leq k^{-mN_0-m}\,\|u\|_{h^{L^k}}\,.\end{aligned} \tag{7.18}$$

From (7.17), (7.18) and Lemma 7.2, it is straightforward to see that for every $\alpha \in \mathbb{N}_0^{2n-1}$,

$$\left|k^{-\frac{n}{2}-\frac{|\alpha|}{2}}(\partial_x^\alpha \widetilde{u})(0)\right| \leq \hat{C}_\alpha k^{\frac{|\alpha|}{2}} \sum_{\gamma\in\mathbb{N}_0^{2n-1},|\gamma|\leq|\alpha|} |(\partial_x^\gamma \beta_k)(0)| \leq \widetilde{C}_\alpha \left\|u\right\|_{h^{L^k}},$$

where $\hat{C}_\alpha > 0$, $\widetilde{C}_\alpha > 0$ are constants independent of k and the point p. Thus, for every $\alpha \in \mathbb{N}_0^{2n}$, there is a constant $C_\alpha > 0$ independent of k and the point p, such that

$$\left|(\partial_x^\alpha(\widetilde{u}e^{-k\phi}))(0)\right| \leq C_\alpha k^{\frac{n}{2}+|\alpha|} \left\|u\right\|_{h^{L^k}}.$$

Let x_0 be another point of D'. We can repeat the procedure above and conclude that for every $\alpha \in \mathbb{N}_0^{2n-1}$, there is a $C_\alpha(x_0) > 0$ independent of k and the point x_0, such that

$$\left|(\partial_x^\alpha(\widetilde{u}e^{-k\phi}))(x_0)\right| \leq C_\alpha(x_0) k^{\frac{n}{2}+|\alpha|} \left\|u\right\|_{h^{L^k}}.$$

The theorem follows. □

We pause and introduce some notations. We identify $\mathbb{R}^{2n-1}$ with the Heisenberg group $H_n := \mathbb{C}^{n-1}\times\mathbb{R}$. Put

$$\psi_0(z,\theta) = \beta x_{2n-1} + \sum_{j,t=1}^{n-1} \mu_{j,t}\overline{z}_j z_t \in C^\infty(H_n,\mathbb{R}), \tag{7.19}$$

where β and $\mu_{j,t}$, $j,t = 1,\ldots,n-1$, are as in (7.1). Let $(\,\cdot\,|\,\cdot\,)_{\psi_0}$ be the inner product on $C_0^\infty(H_n)$ defined as follows:

$$(\,f\,|\,g\,)_{\psi_0} = \int_{H_n} f\overline{g}e^{-2\psi_0}d\lambda(x)\,, \quad f,g\in C_0^\infty(H_n),$$

where $d\lambda(x) = 2^{n-1}dx_1\cdots dx_{2n-1}$. For $f \in C^\infty(H_n)$, we write $\left\|f\right\|_{\psi_0}^2 := (\,f\,|\,f\,)_{\psi_0}$. Let $u(x) \in C^\infty(H_n)$. Fix $\alpha \in \mathbb{N}_0^{2n-1}$. Assume that $\left\|\partial_x^\alpha u\right\|_{\psi_0} < \infty$. Put

$$v_\alpha(x) = (\partial_x^\alpha u)(x)e^{-\beta x_{2n-1}}. \tag{7.20}$$

Set

$$\Phi_0 = \sum_{j,t=1}^{n-1} \mu_{j,\,t}\overline{z}_j z_t. \tag{7.21}$$

We have

$$\int_{H_n} |v_\alpha(x)|^2 e^{-2\Phi_0(z)} d\lambda(x) < \infty.$$

Let us denote by $L^2(H_n,\Phi_0)$ the completion of $C_0^\infty(H_n)$ with respect to the norm $\|\cdot\|_{\Phi_0}$, where

$$\|u\|_{\Phi_0}^2 = \int_{H_n} |u|^2 e^{-2\Phi_0} d\lambda(x)\,, \quad u\in C_0^\infty(H_n)\,.$$

Choose $\chi(x_{2n-1}) \in C_0^\infty(\mathbb{R})$ so that $\chi(x_{2n-1}) = 1$ when $|x_{2n-1}| < 1$ and $\chi(x_{2n-1}) = 0$ when $|x_{2n-1}| > 2$ and set $\chi_j(x_{2n-1}) = \chi(x_{2n-1}/j)$, $j\in\mathbb{N}$. Let

$$\hat{v}_{\alpha,j}(z,\eta) = \int_{\mathbb{R}} v(x)\chi_j(x_{2n-1})e^{-ix_{2n-1}\eta}dx_{2n-1} \in C^\infty(H_n), \quad j = 1,2,\ldots. \tag{7.22}$$

From Parseval's formula, we have

$$\begin{aligned}
&\int_{H_n} |\hat{v}_{\alpha,j}(z,\eta) - \hat{v}_{\alpha,t}(z,\eta)|^2 e^{-2\Phi_0(z)} d\eta dv(z)\\
&= 2\pi \int_{H_n} |v(x)|^2 |\chi_j(x_{2n-1}) - \chi_t(x_{2n-1})|^2 e^{-2\Phi_0(z)} d\eta dv(z) \to 0,\ j,t\to\infty,
\end{aligned}$$

where $dv(z)=2^{n-1}dx_1\cdots dx_{2n-2}$. Thus, there is $\hat{v}_\alpha(z,\eta)\in L^2(H_n,\Phi_0)$ such that $\hat{v}_{\alpha,j}(z,\eta)\to\hat{v}_\alpha(z,\eta)$ in $L^2(H_n,\Phi_0)$. We call $\hat{v}(z,\eta)$ the Fourier transform of $v_\alpha(x)$ with respect to x_{2n-1}. Formally,

$$\begin{aligned}\hat{v}_\alpha(z,\eta)&=\int_{\mathbb{R}} e^{-ix_{2n-1}\eta}v_\alpha(x)dx_{2n-1}\\&=\int_{\mathbb{R}} e^{-ix_{2n-1}\eta}(\partial_x^\alpha u)(x)e^{-\beta x_{2n-1}}dx_{2n-1}.\end{aligned}\tag{7.23}$$

Put

$$\mathbb{R}_p:=\left\{\eta\in\mathbb{R};\ M_p^\phi-2\eta\mathcal{L}_p \text{ is positive definite}\right\}.\tag{7.24}$$

We can check that

$$\mathbb{R}_p:=\left\{\eta\in\mathbb{R};\text{ the matrix }(\mu_{j,t})_{j,t=1}^{n-1}-\eta\left(\tau_j\delta_{j,t}\right)_{j,t=1}^{n-1}\text{ is positive definite}\right\}.$$

The following theorem is essentially well-known (see section 2 and section 3 in [15])

Theorem 7.5. *With the notations used before, fix $N_0\geq 1$. Let $\{k_j\}_{j=1}^\infty$ be a sequence with $0<k_1<k_2<k_3<\cdots$, $\lim_{j\to\infty}k_j=\infty$. Let $f_{k_j}\in H^0_{b,\leq k^{-N_0}}(X,L^k)$ with $\left\|f_{k_j}\right\|_{h^{L^{k_j}}}=1$, $j=1,2,\ldots$. On D, put $f_{k_j}=s^k\widetilde{f}_{k_j}$, $\widetilde{f}_{k_j}\in C^\infty(D)$, $j=1,2,\ldots$. Let $u_{k_j}=k_j^{-\frac{n}{2}}F^*_{k_j}(e^{-2k_jR}\widetilde{f}_{k_j})\in C^\infty(F_{k_j}(D_{\log k_j}))$. Identify u_{k_j} with a function on H_n by extending it with zero, for each j. Then there is a subsequence $\left\{u_{k_{j_s}}\right\}$ of $\left\{u_{k_j}\right\}$ such that $u_{k_{j_s}}$ converges uniformly with all its derivatives on any compact subset of H_n to a smooth function $u\in C^\infty(H_n)$ with $\|\partial_x^\alpha u\|_{\psi_0}<\infty$, for every $\alpha\in\mathbb{N}_0^{2n-1}$. Moreover, fix $\alpha\in\mathbb{N}_0^{2n-1}$ and let $\hat{v}_\alpha(z,\eta)\in L^2(H_n,\Phi_0)$ be as in (7.23). Then, for almost everywhere $\eta\in\mathbb{R}$,*

$$|\hat{v}_\alpha(z,\eta)|\leq f_\alpha(\eta)g_\alpha(z,\eta)1_{\mathbb{R}_p}(\eta),\quad\forall z\in C^\infty(\mathbb{C}^n),\tag{7.25}$$

where $f_\alpha(\eta)$ is a positive measurable function with $\int_{\mathbb{R}}|f_\alpha(\eta)|\,d\eta<C<\infty$, $C>0$ is a constant independent of the sequence $\left\{f_{k_j}\right\}_{j=1}^\infty$ and the point p, $g_\alpha(z,\eta)\in C^\infty(H_n,\overline{\mathbb{R}}_+)$, $1_{\mathbb{R}_p}(\eta)=1$ if $\eta\in\mathbb{R}_p$, $1_{\mathbb{R}_p}(\eta)=0$ if $\eta\notin\mathbb{R}_p$ and $|g_\alpha(0,\eta)|\,1_{\mathbb{R}_p}(\eta)\leq C_1$, $C_1>0$ is a constant independent of the sequence $\left\{f_{k_j}\right\}_{j=1}^\infty$ and the point p. Thus, fix $z\in\mathbb{C}^{n-1}$, $\int|\hat{v}_\alpha(z,\eta)|\,d\eta<\infty$. Furthermore, we have

$$(\partial_x^\alpha u)(x)e^{-\beta x_{2n-1}}=\frac{1}{2\pi}\int e^{ix_{2n-1}\eta}\hat{v}_\alpha(z,\eta)d\eta=\frac{1}{2\pi}\int e^{ix_{2n-1}\eta}\hat{v}_\alpha(z,\eta)1_{\mathbb{R}_p}(\eta)d\eta,\quad\forall x\in H_n.\tag{7.26}$$

Theorem 7.5 will only be used in section 8.

7.2. **Kernel of the spectral function.** We first introduce some notations. Let $(e_1,\ldots,e_{n-1})$ be a smooth local orthonormal frame of $T_x^{*0,1}X$ over an open set $D\subset X$. Then $(e^J:=e_{j_1}\wedge\cdots\wedge e_{j_q})_{1\leqslant j_1<j_2<\cdots<j_q\leq n-1}$ is an orthonormal frame of $T_x^{*0,q}X$ over D. For $f\in\Omega^{0,q}(D)$, we may write $f=\sideset{}{'}\sum_{J\in\{1,\ldots,n-1\}^q}f_Je^J$, with $f_J=\langle\, f\,|\,e^J\,\rangle\in C^\infty(D)$. We call f_J the component of f along e^J. Let $A:\Omega_0^{0,q}(D)\to\Omega^{0,q}(D)$ be a continuous operator with smooth kernel. We write

$$A(x,y)=\sideset{}{'}\sum_{I,J\in\{1,\ldots,n-1\}^q}e^I(x)A_{I,J}(x,y)e^J(y),\tag{7.27}$$

where $A_{I,J}\in C^\infty(D\times D)$ for all strictly increasing $I,J\in\{1,\ldots,n-1\}^q$, in the sense that

$$(Au)(x)=\sideset{}{'}\sum_{I,J\in\{1,\ldots,n-1\}^q}e^I(x)\int_D A_{I,J}(x,y)u_J(y)dv_X(y),\tag{7.28}$$

for all $u=\sideset{}{'}\sum_{J\in\{1,\ldots,n-1\}^q}u_Je^J\in\Omega_0^{0,q}(D)$. Let A^* be the formal adjoint of A with respect to $(\cdot\,|\,\cdot)$. We can check that

$$\begin{aligned}&A^*(x,y)=\sideset{}{'}\sum_{I,J\in\{1,\ldots,n-1\}^q}e^I(x)A^*_{I,J}(x,y)e^J(y),\\&A^*_{I,J}(x,y)=\overline{A_{J,I}(y,x)}\text{ for all strictly increasing }I,J\in\{1,\ldots,n-1\}^q.\end{aligned}\tag{7.29}$$

Let

$$B:\Omega^{0,q}(D)\to\Omega^{0,q}(D),\quad \Omega^{0,q}_0(D)\to\Omega^{0,q}_0(D),$$
$$B(x,y)=\sum\nolimits'_{I,J\in\{1,\ldots,n-1\}^q}e^I(x)B_{I,J}(x,y)e^J(y),$$

where $B_{I,J}(x,y)\in C^\infty(D\times D)$ for all strictly increasing $I,J\in\{1,\ldots,n-1\}^q$, be a properly supported smoothing operator. We write

$$(B\circ A)(x,y)=\sum\nolimits'_{I,J\in\{1,\ldots,n-1\}^q}e^I(x)(B\circ A)_{I,J}(x,y)e^J(y)$$

in the sense of (7.28). It is not difficult to see that

$$(B\circ A)_{I,J}(x,y)=\sum\nolimits'_{K\in\{1,\ldots,n-1\}^q}\int_D B_{I,K}(x,u)A_{K,J}(u,y)dv_X(u), \tag{7.30}$$

for all strictly increasing $I,J\in\{1,\ldots,n-1\}^q$.

Now, we return to our situation. Fix $q\in\{1,2,\ldots,n-1\}$. As before, let s be a local trivializing section of L on an open subset $D\subset X$ and $|s|^2_{h^L}=e^{-2\phi}$. Until further notice, we assume that $Y(q)$ holds at each point of D. Since $Y(q)$ holds at each point of D, by Kohn's L^2 estimates (see [6]), we have

$$\Pi^{(q)}_{k,\leq\lambda}(x,y)\in C^\infty\big(D\times D,(T^{*0,q}_yX\otimes L^k_y)\boxtimes(T^{*0,q}_xX\otimes L^k_x)\big),$$

for every $\lambda\geq0$. Fix $\lambda\geq0$. On $D\times D$, we write

$$\Pi^{(q)}_{k,\leq\lambda}(x,y)=s(x)^k\Pi^{(q)}_{k,\leq\lambda,s}(x,y)s^*(y)^k,$$

where $\Pi^{(q)}_{k,k^{-N_0},s}(x,y)$ is smooth on $D\times D$, so that for $x\in D$, $u\in\Omega^{0,q}_0(D,L^k)$,

$$\begin{aligned}(\Pi^{(q)}_{k,\leq\lambda}u)(x)&=s(x)^k\int_X\Pi^{(q)}_{k,\leq\lambda,s}(x,y)\langle\,u(y)\,,\,s^*(y)^k\,\rangle dv_X(y)\\&=s(x)^k\int_X\Pi^{(q)}_{k,\leq\lambda,s}(x,y)\widetilde{u}(y)dv_X(y),\quad u=s^k\widetilde{u},\quad\widetilde{u}\in\Omega^{0,q}_0(D).\end{aligned} \tag{7.31}$$

For $x=y$, we can check that $\Pi^{(q)}_{k,\leq\lambda,s}(x,x)\in C^\infty(D,T^{*0,q}X\boxtimes T^{*0,q}X)$ is independent of the choice of local frame s.

As (1.6), we define the localized spectral projection (with respect to the trivializing section s) by

$$\begin{aligned}\hat{\Pi}^{(q)}_{k,\leq\lambda,s}:L^2_{(0,q)}(D)\bigcap\mathscr{E}'(D,T^{*0,q}X)&\to\Omega^{0,q}(D),\\ u&\to e^{-k\phi}s^{-k}\Pi^{(q)}_{k,\leq\lambda}(s^ke^{k\phi}u).\end{aligned} \tag{7.32}$$

That is, if $\Pi^{(q)}_{k,\leq\lambda}(s^ke^{k\phi}u)=s^kv$ on D, then $\hat{\Pi}^{(q)}_{k,\leq\lambda,s}u=e^{-k\phi}v$. We notice that

$$\hat{\Pi}^{(q)}_{k,\leq\lambda,s}(x,y)=e^{-k\phi(x)}\Pi^{(q)}_{k,\leq\lambda,s}(x,y)e^{k\phi(y)}, \tag{7.33}$$

where $\hat{\Pi}^{(q)}_{k,\leq\lambda,s}(x,y)$ is the kernel of $\hat{\Pi}^{(q)}_{k,\leq\lambda,s}$ with respect to $(\,\cdot\,|\,\cdot\,)$ and $\Pi^{(q)}_{k,\leq\lambda,s}(x,y)$ is as in (7.31). When $\lambda=0$, we call $\hat{\Pi}^{(q)}_{k,\leq0,s}$ the localized Szegö projection and we set

$$\hat{\Pi}^{(q)}_{k,s}:=\hat{\Pi}^{(q)}_{k,\leq0,s}. \tag{7.34}$$

We write

$$\hat{\Pi}^{(q)}_{k,\leq\lambda,s}(x,y)=\sum\nolimits'_{I,J\in\{1,\ldots,n-1\}^q}e^I(x)\hat{\Pi}^{(q)}_{k,\leq\lambda,s,I,J}(x,y)e^J(y) \tag{7.35}$$

in the sense of (7.28), where $\hat{\Pi}^{(q)}_{k,\leq\lambda,s,I,J}\in C^\infty(D\times D)$, for all strictly increasing $I,J\in\{1,\ldots,n-1\}^q$. Since $\Pi^{(q)}_{k,\leq\lambda}$ is self-adjoint, we have

$$\hat{\Pi}^{(q)}_{k,\leq\lambda,s,I,J}(x,y)=\overline{\hat{\Pi}^{(q)}_{k,\leq\lambda,s,J,I}(y,x)}, \tag{7.36}$$

for all strictly increasing $I, J \in \{1, \ldots, n-1\}^q$.

Now, we fix $N_0 \geq 1$. Let $\{f_j\}_{j=1}^{d_k} \subset L^2_{(0,q)}(X, L^k)$ be an orthonormal frame for $H^q_{b,\leq k^{-N_0}}(X, L^k)$, $d_k \in \mathbb{N}_0 \bigcup \{\infty\}$. Note that $f_j|_D \in \Omega^{0,q}(D)$, $j = 1, \ldots, d_k$. For each j, we write

$$f_j|_D = \sum_{J\in\{1,\ldots,n-1\}^q}{}^{'} f_{j,J}(x)e^J(x), \quad f_{j,J} \in C^\infty(D, L^k) \text{ for all strictly increasing } J \in \{1, \ldots, n-1\}^q.$$

For $j = 1, \ldots, d_k$ and strictly increasing $J \in \{1, \ldots, n-1\}^q$ we define $\widetilde{f}_{j,J} \in C^\infty(D)$ and $\widetilde{f}_j \in \Omega^{0,q}(D)$ by

$$f_{j,J} = s^k \widetilde{f}_{j,J}, \quad \widetilde{f}_j = \sum_{J\in\{1,\ldots,n-1\}^q}{}^{'} \widetilde{f}_{j,J}(x)e^J(x). \tag{7.37}$$

Then, $f_j|_D = s^k \widetilde{f}_j$, $j = 1, \ldots, d_k$, and it is not difficult to see that

$$\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s,I,J}(x,y) = \sum_{j=1}^{d_k} \widetilde{f}_{j,I}(x)\overline{\widetilde{f}_{j,J}(y)}e^{-k(\phi(x)+\phi(y))}, \tag{7.38}$$

for all strictly increasing $I, J \in \{1, \ldots, n-1\}^q$. Since $\hat{\Pi}^{(q)}_{k,\leq\lambda,s,I,J}$ is smooth for every strictly increasing $I, J \in \{1, \ldots, n-1\}^q$, we conclude that for all $\alpha \in \mathbb{N}_0^{2n-1}$,

$$\textstyle\sum_{j=1}^{d_k} \left|(\partial_x^\alpha(\widetilde{f}_j e^{-k\phi}))(x)\right|^2 \text{ converges at each point of } x \in D. \tag{7.39}$$

Similarly, if $F : \mathscr{E}'(D, T^{*0,q}X) \to \mathscr{E}'(D, T^{*0,q}X)$ is a properly supported continuous operator such that for all $s \in \mathbb{N}_0$,

$$F : H^s_{\rm comp}(D, T^{*0,q}X) \to H^{s+s_0}_{\rm comp}(D, T^{*0,q}X)$$

is continuous, for some $s_0 \in \mathbb{R}$. Then, we can check that

$$\textstyle\sum_{j=1}^{d_k} \left|(F(\widetilde{f}_j e^{-k\phi}))(x)\right|^2 \text{ converges at each point of } x \in D. \tag{7.40}$$

Proposition 7.6. *With the notations used above, for every $\alpha \in \mathbb{N}_0^{2n-1}$, $D' \Subset D$, there is a constant $C_{\alpha,D'} > 0$ independent of k, such that*

$$\sum_{j=1}^{d_k} \left|(\partial_x^\alpha(\widetilde{f}_j e^{-k\phi}))(x)\right|^2 \leq C_{\alpha,D'} k^{n+2|\alpha|}, \quad \forall x \in D'. \tag{7.41}$$

Proof. Fix $\alpha \in \mathbb{N}_0^{2n-1}$ and $p \in D'$. We may assume that

$$\sum_{j=1}^{d_k} \left|(\partial_x^\alpha(\widetilde{f}_j e^{-k\phi}))(p)\right|^2 \neq 0.$$

Set

$$u(x) = \frac{1}{\sqrt{\sum_{j=1}^{d_k} \left|(\partial_x^\alpha(\widetilde{f}_j e^{-k\phi}))(p)\right|^2}} \sum_{j=1}^{d_k} f_j(x)\overline{(\partial_x^\alpha(\widetilde{f}_j e^{-k\phi}))(p)}.$$

Since $\sum_{j=1}^{d_k} \left|(\partial_x^\alpha(\widetilde{f}_j e^{-k\phi}))(p)\right|^2$ converges, it is easy to check that

$$u \in H^q_{b,\leq k^{-N_0}}(X, L^k), \quad \|u\|_{h^{L^k}} = 1.$$

On D, we write $u = s^k\widetilde{u}$, $\widetilde{u} \in \Omega^{0,q}(D)$. We can check that

$$\widetilde{u} = \frac{1}{\sqrt{\sum_{j=1}^{d_k} \left|(\partial_x^\alpha(\widetilde{f}_j e^{-k\phi}))(p)\right|^2}} \sum_{j=1}^{d_k} \widetilde{f}_j(x)\overline{(\partial_x^\alpha(\widetilde{f}_j e^{-k\phi}))(p)}. \tag{7.42}$$

In view of Theorem 7.3, we see that $\left|(\partial_x^\alpha(\widetilde{u}e^{-k\phi}))(p)\right| \leq C_\alpha k^{\frac{n}{2}+|\alpha|}$, with $C_\alpha > 0$ independent of k and of the point p. From (7.42), it is straightforward to see that

$$\left|(\partial_x^\alpha(\widetilde{u}e^{-k\phi}))(p)\right| = \sqrt{\sum_{j=1}^{d_k}\left|(\partial_x^\alpha(\widetilde{f}_j e^{-k\phi}))(p)\right|^2} \leq C_\alpha k^{\frac{n}{2}+|\alpha|}.$$

The proposition follows. □

Now, we assume that there exist a $\lambda_0 \in \mathbb{R}$ and $x_0 \in D$ such that $M^\phi_{x_0} - 2\lambda_0\mathcal{L}_{x_0}$ is non-degenerate of constant signature (n_-, n_+). Let $q = n_-$. We fix $D_0 \Subset D$, D_0 open. Let V be as in (5.14). Let

$$\hat{\mathcal{I}}_k \equiv \tfrac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta \mod O(k^{-\infty}) \text{ at } T^*D_0 \bigcap \Sigma$$

be a properly supported classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$, where $\alpha(x,\eta,k) \in S^0_{\mathrm{loc}\,,\mathrm{cl}}(1; T^*D, T^{*0,q}X \boxtimes T^{*0,q}X)$ with $\alpha(x,\eta,k) = 0$ if $|\eta| > M$, for some large $M > 0$ and $\operatorname{Supp}\alpha(x,\eta,k) \bigcap T^*D_0 \Subset V$. Let $\mathcal{S}_k$, $\mathcal{N}_k$ be as in Theorem 6.5. Let $\Box^{(q)}_{s,k}$ be as in (4.4). Then,

$$\begin{aligned}
&\Box^{(q)}_{s,k}\mathcal{N}_k + \mathcal{S}_k = \hat{\mathcal{I}}_k + H_k \ \text{ on } \mathscr{D}'(D_0, T^{*0,q}X),\\
&\mathcal{N}^*_k\Box^{(q)}_{s,k} + \mathcal{S}^*_k = \hat{\mathcal{I}}^*_k + H^*_k \ \text{ on } \mathscr{D}'(D_0, T^{*0,q}X),
\end{aligned}$$

where $H_k \equiv 0 \mod O(k^{-\infty})$ on D_0, $\mathcal{N}^*_k$, $\mathcal{S}^*_k$, $\hat{\mathcal{I}}^*_k$ and H^*_k are formal adjoints of $\mathcal{N}_k$, $\mathcal{S}_k$, $\hat{\mathcal{I}}_k$ and H_k with respect to $(\,\cdot\,|\,\cdot\,)$ respectively. Now,

$$\hat{\mathcal{I}}^*_k\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s} = (\mathcal{N}^*_k\Box^{(q)}_{s,k} - H^*_k + \mathcal{S}^*_k)\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s} = R + \mathcal{S}^*_k\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s} \ \text{ on } \mathscr{E}'(D_0, T^{*0,q}X), \tag{7.43}$$

where we denote

$$R = (\mathcal{N}^*_k\Box^{(q)}_{s,k} - H^*_k)\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s}.$$

We write

$$R(x,y) = {\sum_{I,J\in\{1,\ldots,n-1\}^q}}' e^I(x)R_{I,J}(x,y)e^J(y)$$

in the sense of (7.28), where $R_{I,J} \in C^\infty(D_0 \times D_0)$ for all strictly increasing $I, J \in \{1,\ldots,n-1\}^q$. From (7.38), it is straightforward to see that

$$\begin{aligned}
&R_{I,J}(x,y) = \sum_{j=1}^{d_k}\widetilde{g}_{j,I}(x)\overline{\widetilde{f}_{j,J}(y)}e^{-k\phi(y)},\\
&\widetilde{g}_j = (\mathcal{N}^*_k\Box^{(q)}_{s,k} - H^*_k)(\widetilde{f}_j e^{-k\phi})(x), \ \ \widetilde{g}_j(x) = {\sum_{I\in\{1,\ldots,n-1\}^q}}'\widetilde{g}_{j,I}(x)e^I(x), \ \ j = 1,\ldots,d_k,
\end{aligned} \tag{7.44}$$

for all strictly increasing $I, J \in \{1,\ldots,n-1\}^q$. From (7.40), we see that for all $\alpha \in \mathbb{N}_0^{2n-1}$,

$$\textstyle\sum_{j=1}^{d_k}\left|(\partial_x^\alpha\widetilde{g}_j)(x)\right|^2 \text{ converges at each point of } x \in D_0.$$

To estimate $R_{I,J}(x,y)$, we first need

Lemma 7.7. *With the notations and assumptions used above, for every $D' \Subset D_0$, $\alpha \in \mathbb{N}_0^{2n-1}$, there is a constant $C_{\alpha,D'} > 0$ independent of k, such that for all $u \in H^q_{b,\leq k^{-N_0}}(X, L^k)$, $\|u\|_{h^{L^k}} = 1$, $u|_{D_0} = s^k\widetilde{u}$, $\widetilde{u} \in \Omega^{0,q}(D_0)$, if we set $\widetilde{v}(x) = (\mathcal{N}^*_k\Box^{(q)}_{s,k} - H^*_k)(\widetilde{u}e^{-k\phi})$, then*

$$\left|(\partial_x^\alpha\widetilde{v})(x)\right| \leq C_{\alpha,D'}k^{\frac{5n}{2}+2|\alpha|-N_0-2}, \ \ \forall x \in D'.$$

Proof. Let $u \in H^q_{b,\leq k^{-N_0}}(X, L^k)$, $\|u\|_{h^{L^k}} = 1$, $u|_{D_0} = s^k\widetilde{u}$, $\widetilde{u} \in \Omega^{0,q}(D)$. Set $\widetilde{v}(x) = \mathcal{N}^*_k\Box^{(q)}_{s,k}(\widetilde{u}e^{-k\phi})$. We recall that (see (6.23))

$$\mathcal{N}^*_k = O(k^s) : H^s_{\mathrm{comp}}(D_0, T^{*0,q}X) \to H^{s+1}_{\mathrm{loc}}(D_0, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0. \tag{7.45}$$

Let $D' \Subset D'' \Subset D_0$. By using Fourier transforms, we see that for all $x \in D'$, we have

$$\left|(\partial_x^\alpha\widetilde{v})(x)\right| \leq C_\alpha\left\|\widetilde{v}\right\|_{n+|\alpha|,D''},$$

where C_α only depends on the dimension and the length of α. Here $\|.\|_{s,D''}$ denotes the usual Sobolev norm of order s on D''. From this observation and (7.45), we see that for every $N>0$,

$$|(\partial_x^\alpha \widetilde{v})(x)| \leq C_\alpha \|\widetilde{v}\|_{n+|\alpha|,D''} \leq C'_\alpha k^{n-1+|\alpha|} \left\|\Box_{s,k}^{(q)}(\widetilde{u}e^{-k\phi})\right\|_{n-1+|\alpha|,D''} + C_N k^{-N}, \tag{7.46}$$

where $C'_\alpha>0$, $C_N>0$ are independent of k and $\widetilde{u}$. Let $\Box_{b,k}^{(q)}u=f$, $f|_{D_0}=s^k\widetilde{f}$, $\widetilde{f}\in\Omega^{0,q}(D_0)$. We can check that $f\in H^q_{b,\leq k^{-N_0}}(X,L^k)$ and $\|f\|_{h^{L^k}}\leq k^{-N_0}$. From (4.1), we see that

$$\Box_{s,k}^{(q)}(e^{-k\phi}\widetilde{u})=e^{-k\phi}\widetilde{f}. \tag{7.47}$$

In view of Theorem 7.3, we know that for all $\beta\in\mathbb{N}_0^{2n-1}$,

$$\left|\partial_x^\beta(\Box_{s,k}^{(q)}(e^{-k\phi}\widetilde{u}))\right| = \left|\partial_x^\beta(e^{-k\phi}\widetilde{f})\right| \leq C_\beta k^{\frac{n}{2}+|\beta|}\|f\|_{h^{L^k}} \leq C_\beta k^{\frac{n}{2}+|\beta|-N_0} \text{ on } D'',$$

where $C_\beta>0$ is independent of k. Thus,

$$\left\|\Box_{s,k}^{(q)}(e^{-k\phi}\widetilde{u})\right\|_{n-1+|\alpha|,D''} \leq \widetilde{C}_\alpha k^{\frac{3n}{2}+|\alpha|-N_0-1}, \tag{7.48}$$

where $\widetilde{C}_\alpha>0$ is independent of k. Combining (7.48) with (7.46), the lemma follows. $\Box$

Lemma 7.8. *Let $\widetilde{g}_j(x)\in\Omega^{0,q}(D_0)$, $j=1,\ldots,d_k$, be as in (7.44). For every $D'\Subset D_0$, $\alpha\in\mathbb{N}_0^{2n-1}$, there is a constant $C_\alpha>0$ independent of k, such that for all $x\in D'$*

$$\sum_{j=1}^{d_k}|(\partial_x^\alpha\widetilde{g}_j)(x)|^2\leq C_\alpha k^{5n+4|\alpha|-2N_0-4}.$$

Proof. Fix $\alpha\in\mathbb{N}_0^{2n-1}$ and $p\in D'$. We may assume that $\sum_{j=1}^{d_k}|(\partial_x^\alpha\widetilde{g}_j)(p)|^2\neq0$. Set

$$h(x)=\frac{1}{\sqrt{\sum_{j=1}^{d_k}|(\partial_x^\alpha\widetilde{g}_j)(p)|^2}}\sum_{j=1}^{d_k}f_j(x)\overline{(\partial_x^\alpha\widetilde{g}_j)(p)}.$$

Since $\sum_{j=1}^{d_k}|(\partial_x^\alpha\widetilde{g}_j)(p)|^2$ converges, we can check that $h\in H^q_{b,\leq k^{-N_0}}(X,L^k)$, $\|h\|_{h^{L^k}}=1$. On D_0, we write $h=s^k\widetilde{h}$. We can check that

$$\mathcal{N}_k^*\Box_{s,k}^{(q)}(\widetilde{h}e^{-k\phi})=\frac{1}{\sqrt{\sum_{j=1}^{d_k}|(\partial_x^\alpha\widetilde{g}_j)(p)|^2}}\sum_{j=1}^{d_k}\widetilde{g}_j(x)\overline{(\partial_x^\alpha\widetilde{g}_j)(p)}.$$

In view of Lemma 7.7, we see that

$$\left|\partial_x^\alpha(\mathcal{N}_k^*\Box_{s,k}^{(q)}(\widetilde{h}e^{-k\phi}))(p)\right|=\sqrt{\sum_{j=1}^{d_k}|(\partial_x^\alpha\widetilde{g}_j)(p)|^2}\leq C_\alpha k^{\frac{5n}{2}+2|\alpha|-N_0-2},$$

where $C_\alpha>0$ is independent of k and the point p. The lemma follows. $\Box$

Now, we can prove

Proposition 7.9. *With the notations and assumptions used above, for every $D'\Subset D_0$, $\alpha,\beta\in\mathbb{N}_0^{2n-1}$, there is a constant $C_{\alpha,\beta}>0$ independent of k, such that*

$$\left|(\partial_x^\alpha\partial_y^\beta R_{I,J})(x,y)\right|\leq C_{\alpha,\beta}k^{3n+2|\alpha|+|\beta|-N_0-2},\quad \forall(x,y)\in D'\times D', \tag{7.49}$$

for all strictly increasing $I,J\in\{1,\ldots,n-1\}^q$, where $R_{I,J}(x,y)$ is as in (7.44).

Proof. Fix $p\in D'$ and $J\in\{1,\ldots,n-1\}^q$ strictly increasing. Let $\alpha,\beta\in\mathbb{N}_0^{2n-1}$. We may assume that $\sum_{j=1}^{d_k}\left|(\partial_y^\beta(\widetilde{f}_{j,J}e^{-k\phi}))(p)\right|^2\neq0$. Put

$$u(x)=\frac{1}{\sqrt{\sum_{j=1}^{d_k}\left|(\partial_y^\beta(\widetilde{f}_{j,J}e^{-k\phi}))(p)\right|^2}}\sum_{j=1}^{d_k}f_j(x)\overline{(\partial_y^\beta(\widetilde{f}_{j,J}e^{-k\phi}))(p)}. \tag{7.50}$$

Then, $u \in H^q_{b,\leq k^{-N_0}}(X, L^k)$, $\|u\|_{h^{L^k}} = 1$. On D_0, we write $u = s^k\widetilde{u}$, $\widetilde{u} = \sideset{}{'}\sum_{I\in\{1,\ldots,n-1\}^q}\widetilde{u}_Ie^I$. Put $\widetilde{v} = \mathcal{N}^*_k\Box^{(q)}_s(\widetilde{u}e^{-k\phi}) = \sideset{}{'}\sum_{I\in\{1,\ldots,n-1\}^q}\widetilde{v}_Ie^I \in \Omega^{0,q}(D)$. It is not difficult to check that

$$\widetilde{v} = \frac{1}{\sqrt{\sum_{j=1}^{d_k}\left|(\partial^\beta_y(\widetilde{f}_{j,J}e^{-k\phi}))(p)\right|^2}}\sum_{j=1}^{d_k}\widetilde{g}_j\overline{(\partial^\beta_y(\widetilde{f}_{j,J}e^{-k\phi}))(p)},$$

where $\{\widetilde{g}_j\}_{j=1}^{d_k}$ are as in (7.44). In view of Lemma 7.7, there exists $C_\alpha > 0$ independent of k and the point p such that $|(\partial^\alpha_x\widetilde{v})(x)| \leq C_\alpha k^{\frac{5n}{2}+2|\alpha|-N_0-2}$, for all $x \in D'$. In particular,

$$\begin{aligned}|(\partial^\alpha_x\widetilde{v}_I)(x)| &= \frac{1}{\sqrt{\sum_{j=1}^{d_k}\left|(\partial^\beta_y(\widetilde{f}_{j,J}e^{-k\phi}))(p)\right|^2}}\left|\sum_{j=1}^{d_k}(\partial^\alpha_x\widetilde{g}_{j,I})(x)\overline{(\partial^\beta_y(\widetilde{f}_{j,J}e^{-k\phi}))(p)}\right| \\ &\leq C_\alpha k^{\frac{5n}{2}+2|\alpha|-N_0-2}, \quad \forall x \in D',\end{aligned} \tag{7.51}$$

for all strictly increasing $I \in \{1,\ldots,n-1\}^q$. In view of Proposition 7.6, we see that

$$\sum_{j=1}^{d_k}\left|(\partial^\beta_y(\widetilde{f}_je^{-k\phi}))(p)\right|^2 \leq C_\beta k^{n+2|\beta|},$$

where $C_\beta > 0$ is independent of k and the point p. From this and (7.51), we conclude the existence of a constant $C_{\alpha,\beta} > 0$ independent of k and the point p with

$$\begin{aligned}\left|(\partial^\alpha_x\partial^\beta_yR_{I,J})(x,p)\right| &= \sqrt{\sum_{j=1}^{d_k}\left|(\partial^\beta_y(\widetilde{f}_{j,J}e^{-k\phi}))(p)\right|^2}\,|(\partial^\alpha_x\widetilde{v}_I)(x)| \\ &\leq C_{\alpha,\beta}k^{3n+2|\alpha|+|\beta|-N_0-2},\end{aligned}$$

for all $x \in D'$, all strictly increasing $I, J \in \{1,\ldots,n-1\}^q$. The proposition follows. $\square$

Let R^* be the formal adjoint R with respect to $(\cdot\,|\,\cdot)$. From (7.43), we have

$$\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}_k = R^* + \hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s}\mathcal{S}_k. \tag{7.52}$$

We also write

$$R^*(x,y) = \sideset{}{'}\sum_{I,J\in\{1,\ldots,n-1\}^q}e^I(x)R^*_{I,J}(x,y)e^J(y) \tag{7.53}$$

in the sense of (7.28), where $R^*_{I,J}(x,y) \in C^\infty(D_0\times D_0)$, for all strictly increasing $I, J \in \{1,\ldots,n-1\}^q$. Note that $R^*_{I,J}(x,y) = \overline{R_{J,I}(y,x)}$, for all strictly increasing $I, J \in \{1,\ldots,n-1\}^q$. Combining this observation with (7.49), we conclude that for every $D' \Subset D_0$, $\alpha, \beta \in \mathbb{N}^{2n-1}_0$, there is a constant $C_{\alpha,\beta} > 0$ independent of k, such that

$$\left|(\partial^\alpha_x\partial^\beta_yR^*_{I,J})(x,y)\right| \leq C_{\alpha,\beta}k^{3n+2|\beta|+|\alpha|-N_0-2}, \quad \forall (x,y) \in D'\times D', \tag{7.54}$$

for all strictly increasing $I, J \in \{1,\ldots,n-1\}^q$.

We consider $\hat{\mathcal{I}}^*_kR^*$. Note that $\hat{\mathcal{I}}^*_kR^*$ is a smoothing function on D and we write

$$(\hat{\mathcal{I}}^*_kR^*)(x,y) = \sideset{}{'}\sum_{I,J\in\{1,\ldots,n-1\}^q}e^I(x)(\hat{\mathcal{I}}^*_kR^*)_{I,J}(x,y)e^J(y)$$

in the sense of (7.28), where $(\hat{\mathcal{I}}^*_kR^*)_{I,J} \in C^\infty(D\times D)$ for all strictly increasing $I, J \in \{1,\ldots,n-1\}^q$. It is easy to see that

$$\hat{\mathcal{I}}^*_k = O(k^0) : H^s_{\mathrm{comp}}(D, T^{*0,q}X) \to H^s_{\mathrm{comp}}(D, T^{*0,q}X), \tag{7.55}$$

for every $s\in\mathbb{N}_0$. From (7.55), we can repeat the proof of Proposition 7.9 with minor change and conclude that

Proposition 7.10. *With the notations and assumptions used above, for every $D'\Subset D_0$, $\alpha,\beta\in\mathbb{N}_0^{2n-1}$, there is a constant $C_{\alpha,\beta}>0$ independent of k, such that*

$$\Bigl|\partial_x^\alpha\partial_y^\beta(\hat{\mathcal{I}}_k^*R^*)_{I,J}(x,y)\Bigr|\leq C_{\alpha,\beta}k^{3n+2|\alpha|+|\beta|-N_0-2},\ \ \forall(x,y)\in D'\times D',\tag{7.56}$$

for all strictly increasing $I,J\in\{1,\ldots,n-1\}^q$.

7.3. **Szegö kernel asymptotics for lower energy forms.** Let $\lambda\geq 0$ and let $H^q_{b,>\lambda}(X,L^k)$ and $\Pi^{(q)}_{b,>\lambda}$ be as in (1.1) and (1.3) respectively. It is well-known (see section 2 in [7]) that for all $\lambda>0$,

$$L^2_{(0,q)}(X,L^k)=H^q_{b,\leq\lambda}(X,L^k)\oplus H^q_{b,>\lambda}(X,L^k)\tag{7.57}$$

and

$$\|u\|^2_{h^{L^k}}\leq\frac{1}{\lambda}(\,\Box^{(q)}_{b,k}u\,|\,u\,)_{h^{L^k}},\ \ \forall u\in H^q_{b,>\lambda}(X,L^k)\bigcap{\rm Dom}\,\Box^{(q)}_{b,k}.\tag{7.58}$$

Let s be a local trivializing section of L on an open subset $D\subset X$ and $|s|^2_{h^L}=e^{-2\phi}$. Consider the localization

$$\begin{split}\hat\Pi^{(q)}_{k,>\lambda,s}:L^2_{(0,q)}(D)\cap\mathscr{E}'(D,T^{*0,q}X)\rightarrow L^2_{(0,q)}(D),\\ u\rightarrow e^{-k\phi}s^{-k}\Pi^{(q)}_{k,>\lambda}(s^ke^{k\phi}u).\end{split}\tag{7.59}$$

From (7.57), we have the decomposition

$$u=\hat\Pi^{(q)}_{k,\leq\lambda,s}u+\hat\Pi^{(q)}_{k,>\lambda,s}u,\ \ u\in\Omega^{0,q}_0(D).\tag{7.60}$$

We work with the same notations and assumptions as in section 7.2. Let $u\in H^{s_1}_{\rm comp}(D,T^{*0,q}X)$, $s_1\leq 0$, $s_1\in\mathbb{Z}$. From (7.60), we have

$$\mathcal{S}_ku=\hat\Pi^{(q)}_{k,\leq k^{-N_0},s}\mathcal{S}_ku+\hat\Pi^{(q)}_{k,>k^{-N_0},s}\mathcal{S}_ku.\tag{7.61}$$

From (7.59) and (7.58), we can check that

$$\begin{split}\Bigl\|\hat\Pi^{(q)}_{k,>k^{-N_0},s}\mathcal{S}_ku\Bigr\|_D&\leq\Bigl\|\Pi^{(q)}_{k,>k^{-N_0}}(s^ke^{k\phi}(\mathcal{S}_ku))\Bigr\|_{h^{L^k}}\leq k^{N_0}\Bigl\|\Box^{(q)}_{b,k}\Pi^{(q)}_{k,>k^{-N_0}}(s^ke^{k\phi}(\mathcal{S}_ku))\Bigr\|_{h^{L^k}}\\&\leq k^{N_0}\Bigl\|\Box^{(q)}_{b,k}(s^ke^{k\phi}(\mathcal{S}_ku))\Bigr\|_{h^{L^k}}=k^{N_0}\Bigl\|\Box^{(q)}_{s,k}(\mathcal{S}_ku)\Bigr\|.\end{split}\tag{7.62}$$

Here we have used (4.1). In view of Theorem 6.5, we see that $\Box^{(q)}_{s,k}\mathcal{S}_k\equiv 0\mod O(k^{-\infty})$. From this observation and (7.62), we conclude that

$$\hat\Pi^{(q)}_{k,>k^{-N_0},s}\mathcal{S}_k=O(k^{-N}):H^{s_1}_{\rm comp}(D,T^{*0,q}X)\rightarrow H^0_{\rm loc}(D,T^{*0,q}X),\tag{7.63}$$

locally uniformly on D, for all $N\geq 0$, $s_1\in\mathbb{Z}$, $s_1\leq 0$. Since $Y(q)$ holds at each point of D, we can repeat Kohn's L^2 estimates (see [6]) and obtain

Proposition 7.11. *Let $u\in{\rm Dom}\,\Box^{(q)}_{b,k}$. If $(\Box^{(q)}_{b,k})^ju\in{\rm Dom}\,\Box^{(q)}_{b,k}$, for all $j=1,2,\ldots$, then $u|_D\in\Omega^{0,q}(D,L^k)$.*

Moreover, for every $m\in\mathbb{N}_0$ and $D'\Subset D''\Subset D_0$, there are constants $C_m>0$ and $n_m\in\mathbb{N}$ independent of k such that

$$\|u\|_{m,D'}\leq C_mk^{n_m}\Bigl(\,\|u\|_{D''}+\sum^m_{j=1}\Bigl\|(\Box^{(q)}_{s,k})^ju\Bigr\|_{D''}\Bigr),\ \ \forall u\in\Omega^{0,q}(D_0),\tag{7.64}$$

where $\|\cdot\|_{s,D'}$ denote the usual Sobolev norm of order s on D' with respect to $dv_X(x)$ and $\|\cdot\|_{D''}$ denote the L^2 norm on D'' with respect to $dv_X(x)$.

Let $u\in H^{s_1}_{\rm comp}\,(D,T^{*0,q}X)$, $s_1\leq 0$, $s_1\in\mathbb{Z}$. Since $s^ke^{k\phi}\mathcal{S}_ku\in\Omega^{0,q}_0(D,L^k)$, we have

$$(\Box^{(q)}_{b,k})^j(s^ke^{k\phi}\mathcal{S}_ku)\in{\rm Dom}\,\Box^{(q)}_{b,k},\quad\forall j=1,2,\ldots.$$

Hence,

$$(\Box^{(q)}_{b,k})^j\Big(\Pi^{(q)}_{k,\leq k^{-N_0}}(s^ke^{k\phi}\mathcal{S}_ku)\Big)=\Pi^{(q)}_{k,\leq k^{-N_0}}\Big((\Box^{(q)}_{b,k})^j(s^ke^{k\phi}\mathcal{S}_ku)\Big)\in{\rm Dom}\,\Box^{(q)}_{b,k},\quad\forall j=1,2,\ldots.$$

Since $I=\Pi^{(q)}_{k,\leq k^{-N_0}}+\Pi^{(q)}_{k,>k^{-N_0}}$ on $L^2_{(0,q)}(X,L^k)$, we conclude that

$$(\Box^{(q)}_{b,k})^j\Big(\Pi^{(q)}_{k,>k^{-N_0}}(s^ke^{k\phi}\mathcal{S}_ku)\Big)\in{\rm Dom}\,\Box^{(q)}_{b,k},\quad\forall j=1,2,\ldots.$$

From this and Proposition 7.11, we conclude that

$$\begin{aligned}&\Pi^{(q)}_{k,>k^{-N_0}}(s^ke^{k\phi}\mathcal{S}_ku)|_D\in\Omega^{0,q}(D,L^k),\\&\hat\Pi^{(q)}_{k,>k^{-N_0},s}(\mathcal{S}_ku)|_D\in\Omega^{0,q}(D).\end{aligned}\tag{7.65}$$

Moreover, from (7.64), for every $m\in\mathbb{N}_0$ and $D'\Subset D''\Subset D$, it is straightforward to see that

$$\begin{aligned}\Big\|\hat\Pi^{(q)}_{k,>k^{-N_0},s}\mathcal{S}_ku\Big\|_{m,D'}&\leq C_mk^{n_m}\Big(\Big\|\hat\Pi^{(q)}_{k,>k^{-N_0},s}(\mathcal{S}_ku))\Big\|_{D''}+\sum^m_{j=1}\Big\|(\Box^{(q)}_{s,k})^j\big(\hat\Pi^{(q)}_{k,>k^{-N_0},s}(\mathcal{S}_ku)\big)\Big\|_{D''}\Big)\\&\leq C_mk^{n_m}\Big(\Big\|\hat\Pi^{(q)}_{k,>k^{-N_0},s}(\mathcal{S}_ku))\Big\|_{D''}+\sum^m_{j=1}\Big\|\Pi^{(q)}_{k,>k^{-N_0}}(\Box^{(q)}_{b,k})^j(e^{k\phi}s^k\mathcal{S}_ku)\Big\|_{h^{L^k}}\Big)\\&\leq C_mk^{n_m}\Big(\Big\|\hat\Pi^{(q)}_{k,>k^{-N_0},s}(\mathcal{S}_ku))\Big\|_{D''}+\sum^m_{j=1}\Big\|\Pi^{(q)}_{k,>k^{-N_0}}\big(e^{k\phi}s^k(\Box^{(q)}_{s,k})^j\mathcal{S}_ku\big)\Big\|_{h^{L^k}}\Big),\end{aligned}\tag{7.66}$$

where $C_m>0$ and $n_m\in\mathbb{N}$ are constants independent of k. Here we use the facts

$$\begin{aligned}(\Box^{(q)}_{s,k})^j\big(\hat\Pi^{(q)}_{k,>k^{-N_0},s}(\mathcal{S}_ku)\big)&=s^{-k}e^{-k\phi}(\Box^{(q)}_{b,k})^j\big(\Pi^{(q)}_{k,>k^{-N_0}}(e^{k\phi}s^k\mathcal{S}_ku)\big)\\&=s^{-k}e^{-k\phi}\Pi^{(q)}_{k,>k^{-N_0}}\big((\Box^{(q)}_{b,k})^j(e^{k\phi}s^k\mathcal{S}_ku)\big)\end{aligned}$$

and

$$(\Box^{(q)}_{b,k})^j(e^{k\phi}s^k\mathcal{S}_ku)=e^{k\phi}s^k(\Box^{(q)}_{s,k})^j(\mathcal{S}_ku),$$

for all $j=1,2,\ldots$. From $\Box^{(q)}_{s,k}\mathcal{S}_k\equiv0\mod O(k^{-\infty})$, (7.63) and (7.66), we conclude that

$$\hat\Pi^{(q)}_{k,>k^{-N_0},s}\mathcal{S}_k=O(k^{-N}):H^{s_1}_{\rm comp}\,(D,T^{*0,q}X)\rightarrow H^m_{\rm loc}\,(D,T^{*0,q}X),$$

locally uniformly on D, for all $N\geq0$, $s_1\in\mathbb{Z}$, $s_1\leq0$, and $m\in\mathbb{N}_0$. Thus,

$$\hat\Pi^{(q)}_{k,>k^{-N_0},s}\mathcal{S}_k\equiv0\mod O(k^{-\infty}).\tag{7.67}$$

Note that $\mathcal{S}_k=\hat\Pi^{(q)}_{k,\leq k^{N_0},s}\mathcal{S}_k+\hat\Pi^{(q)}_{k,>k^{N_0},s}\mathcal{S}_k$. From this observation, (7.52) and (7.67), we deduce that

$$\hat\Pi^{(q)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}_k-R^*\equiv\mathcal{S}_k\mod O(k^{-\infty})\tag{7.68}$$

on D_0, where $R^*(x,y)$ satisfies (7.54). Hence,

$$\hat{\mathcal{I}}^*_k\hat\Pi^{(q)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}_k-\hat{\mathcal{I}}^*_kR^*\equiv\mathcal{S}^*_k\mathcal{S}_k\mod O(k^{-\infty})\tag{7.69}$$

on D_0, where $(\hat{\mathcal{I}}^*_kR^*)(x,y)$ satisfies (7.56). Here we used (6.39).

Summing up, we obtain one of the main results of this work

Theorem 7.12. *Let s be a local trivializing section of L on an open subset $D\subset X$ and $|s|^2_{h^L}=e^{-2\phi}$. We assume that there exist a $\lambda_0\in\mathbb{R}$ and $x_0\in D$ such that $M^\phi_{x_0}-2\lambda_0\mathcal{L}_{x_0}$ is non-degenerate of constant signature (n_-,n_+). Let $q=n_-$ and assume that $Y(q)$ holds at each point of D. We fix $D_0\Subset D$, D_0 open. Let V be as in* (5.14)*. Let*

$$\hat{\mathcal{I}}_k\equiv\tfrac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta\mod O(k^{-\infty})\ \text{at}\ T^*D_0\bigcap\Sigma$$

be a properly supported classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$, where

$$\alpha(x,\eta,k)\sim\textstyle\sum_{j=0}\alpha_j(x,\eta)k^{-j}\ \text{ in } S^0_{\rm loc}\,(1;T^*D,T^{*0,q}X\boxtimes T^{*0,q}X),$$
$$\alpha_j(x,\eta)\in C^\infty(T^*D,T^{*0,q}D\boxtimes T^{*0,q}D),\quad j=0,1,\ldots,$$

with $\alpha(x,\eta,k)=0$ if $|\eta|>M$, for some large $M>0$ and ${\rm Supp}\,\alpha(x,\eta,k)\bigcap T^*D_0\Subset V$. Then for every $N_0\geq 1$ and every $D'\Subset D_0$, $\alpha,\beta\in\mathbb{N}_0^{2n-1}$, there is a constant $C_{D',\alpha,\beta,N_0}>0$ independent of k, such that

$$\begin{split}&\Big|\partial^\alpha_x\partial^\beta_y\big((\hat\Pi^{(q)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}_k)(x,y)-\int e^{ik\varphi(x,y,s)}a(x,y,s,k)ds\big)\Big|\\&\leq C_{D',\alpha,\beta,N_0}k^{3n+2|\beta|+|\alpha|-N_0-2}\quad\text{on } D'\times D',\\&\Big|\partial^\alpha_x\partial^\beta_y\big((\hat{\mathcal{I}}^*_k\hat\Pi^{(q)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}_k)(x,y)-(\int e^{ik\varphi(x,y,s)}g(x,y,s,k)ds\big)\Big|\\&\leq C_{D',\alpha,\beta,N_0}k^{3n+2|\beta|+|\alpha|-N_0-2}\quad\text{on } D'\times D',\end{split}\tag{7.70}$$

where $\hat\Pi^{(q)}_{k,\leq k^{-N_0},s}$ is the localized spectral projection (7.32), $\varphi(x,y,s)\in C^\infty(\Omega)$ is as in Theorem 5.29, (2.3) and

$$\begin{split}&a(x,y,s,k),\ g(x,y,s,k)\in S^n_{\rm loc}\big(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big)\bigcap C^\infty_0\big(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\\&a(x,y,s,k)\sim\sum^\infty_{j=0}a_j(x,y,s)k^{n-j}\ \text{ in } S^n_{\rm loc}\big(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\\&g(x,y,s,k)\sim\sum^\infty_{j=0}g_j(x,y,s)k^{n-j}\ \text{ in } S^n_{\rm loc}\big(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\\&a_j(x,y,s),\ g_j(x,y,s)\in C^\infty_0\big(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\quad j=0,1,2,\ldots,\end{split}\tag{7.71}$$

with

(7.72)
$$\begin{split}&a_0(x,x,s)=(2\pi)^{-n}\left|\det\big(M^\phi_x-2s\mathcal{L}_x\big)\right|\pi_{(x,s,n_-)}\alpha_0(x,s\omega_0(x)-2{\rm Im}\,\overline\partial_b\phi(x)),\\&g_0(x,x,s)\\&=(2\pi)^{-n}\left|\det\big(M^\phi_x-2s\mathcal{L}_x\big)\right|\alpha^*_0(x,s\omega_0(x)-2{\rm Im}\,\overline\partial_b\phi(x))\pi_{(x,s,n_-)}\alpha_0(x,s_0\omega_0(x)-2{\rm Im}\,\overline\partial_b\phi(x)),\end{split}$$

for every $(x,x,s)\in\Omega$, $x\in D_0$, where

$$\begin{split}\Omega:=\{&(x,y,s)\in D\times D\times\mathbb{R};\ (x,-2{\rm Im}\,\overline\partial_b\phi(x)+s\omega_0(x))\in V\textstyle\bigcap\Sigma,\\&(y,-2{\rm Im}\,\overline\partial_b\phi(y)+s\omega_0(y))\in V\textstyle\bigcap\Sigma,\ |x-y|<\varepsilon,\ \text{for some }\varepsilon>0\},\end{split}$$

$\alpha^*_0(x,\eta):T^{*0,q}_xX\rightarrow T^{*0,q}_xX$ is the adjoint of $\alpha_0(x,\eta)$ with respect to the Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $T^{*0,q}_xX$, $\pi_{(x,s,n_-)}:T^{*0,q}_xX\rightarrow\mathcal{N}(x,s,n_-)$ is the orthogonal projection with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$, $\mathcal{N}(x,s,n_-)$ is given by (5.39), $\left|\det\big(M^\phi_x-2s\mathcal{L}_x\big)\right|=|\lambda_1(s)|\,|\lambda_2(s)|\cdots|\lambda_{n-1}(s)|$, where $\lambda_1(s),\ldots,\lambda_{n-1}(s)$ are eigenvalues of the Hermitian quadratic form $M^\phi_x-2s\mathcal{L}_x$ with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$.

By using Theorem 6.10 and repeat the proof of Theorem 7.12, we deduce

Theorem 7.13. *Let s be a local trivializing section of L on an open subset $D\subset X$ and $|s|^2_{h^L}=e^{-2\phi}$. We assume that there exist a $\lambda_0\in\mathbb{R}$ and $x_0\in D$ such that $M^\phi_{x_0}-2\lambda_0\mathcal{L}_{x_0}$ is non-degenerate of constant signature (n_-,n_+). Let $q\neq n_-$ and assume that $Y(q)$ holds at each point of D. We fix $D_0\Subset D$, D_0 open. Let V be as in (5.14). Let*

$$\hat{\mathcal{I}}_k\equiv\tfrac{k^{2n-1}}{(2\pi)^{2n-1}}\textstyle\int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta\quad\text{mod }O(k^{-\infty})\ \text{ at } T^*D_0\bigcap\Sigma$$

be a properly supported classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$, where $\alpha(x,\eta,k)\in S^0_{\rm loc}\,(1;T^*D,T^{*0,q}X\boxtimes T^{*0,q}X)$ with $\alpha(x,\eta,k)=0$

if $|\eta|>M$, *for some large* $M>0$ *and* $\operatorname{Supp}\alpha(x,\eta,k)\bigcap T^*D_0\Subset V$. *Then for every* $N_0\geq 1$ *and every* $D'\Subset D_0$, $\alpha,\beta\in\mathbb{N}_0^{2n-1}$, *there is a constant* $C_{D',\alpha,\beta,N_0}>0$ *independent of* k, *such that*

$$\begin{aligned}&\left|\partial_x^\alpha\partial_y^\beta\big((\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}_k)(x,y)\big)\right|\leq C_{D',\alpha,\beta,N_0}k^{3n+2|\beta|+|\alpha|-N_0-2}\quad\text{on } D'\times D',\\&\left|\partial_x^\alpha\partial_y^\beta\big((\hat{\mathcal{I}}_k^*\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}_k)(x,y)\right|\leq C_{D',\alpha,\beta,N_0}k^{3n+2|\beta|+|\alpha|-N_0-2}\quad\text{on } D'\times D',\end{aligned}\tag{7.73}$$

where $\hat{\Pi}^{(q)}_{k,\leq k^{-N_0},s}$ *is the localized spectral projection* (7.32).

8. Almost Kodaira embedding Theorems on CR manifolds

In this section, we will use Theorem 7.12 to establish "Almost Kodaira embedding Theorems on CR manifolds" (see Definition 1.3). First, we recall Definition 1.2 for the definition of "positive CR line bundles".

In this section, we assume that X is compact, L is positive and condition $Y(0)$ holds at each point of X. Fix $N_0\gg 2n$, N_0 large. Let s be a local trivializing section of L on an open subset $D\subset X$, $|s|^2_{h^L}=e^{-2\phi}$. As (1.5), we set

$$\begin{aligned}\Sigma'=&\{(x,\lambda\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x))\in T^*D\bigcap\Sigma;\\&M^\phi_x-2\lambda\mathcal{L}_x\text{ is positive definite}\}.\end{aligned}\tag{8.1}$$

Fix $p\in D$. Let $x=(x_1,\ldots,x_{2n-1})$, $z_j=x_{2j-1}+ix_{2j}$, $j=1,\ldots,n-1$, be local coordinates of X defined on D such that (7.1) hold. On D, we write $\omega_0(x)=\sum_{j=1}^{2n-1}\beta_j(x)dx_j$. We take D small enough so that $\beta_{2n-1}(x)\geq\frac{1}{2}$, for every $x\in D$. Let $\eta=(\eta_1,\ldots,\eta_{2n-1})$ denote the dual coordinates of x. Let $\mathbb{R}_p$ be as in (7.24). Take $\psi(\eta)\in C^\infty_0(\mathbb{R}_p,\overline{\mathbb{R}}_+)$ with

$$\int\psi(\eta)\Big(M^\phi_p-2\eta\mathcal{L}_p\Big)d\eta\geq\frac{1}{2}\int_{\mathbb{R}_p}\Big(M^\phi_p-2\eta\mathcal{L}_p\Big)d\eta.\tag{8.2}$$

Let $M>0$ be a large constant so that for every $(x,\eta)\in T^*D$ if $|\eta'|>\frac{M}{2}$ then $(x,\eta)\notin\Sigma$, where $\eta'=(\eta_1,\ldots,\eta_{2n-2})$, $|\eta'|=\sqrt{\sum_{j=1}^{2n-2}|\eta_j|^2}$. Take $D_0\subset D$ be a small open neighbourhood of p so that

$$M^\phi_x-2\langle\,\eta\,|\,\omega_0(x)\,\rangle\mathcal{L}_x\text{ is positive definite, for every } x\in W\text{, every }\eta_{2n-1}\in\Gamma,\ |\eta'|<M,\tag{8.3}$$

where $W\subset D$ and $\Gamma\subset\mathbb{R}_p$ are small open neighbourhoods of D_0 and $\operatorname{Supp}\psi$ respectively. Put

$$V=\{(x,\eta)\in T^*D;\ x\in W,\eta_{2n-1}\in\Gamma,|\eta'|<M\}\,.\tag{8.4}$$

From (8.3), it is straightforward to see that

$$V\bigcap\Sigma\subset\Sigma'.\tag{8.5}$$

Take τ, $\tau_1\in C^\infty_0(D)$, $\tau=1$ on D_0 and $\tau_1=1$ on $\operatorname{Supp}\tau$. Let $\chi\in C^\infty_0(]-1,1[)$, $\chi=1$ on $[-\frac{1}{2},\frac{1}{2}]$. Let

$$\hat{\mathcal{I}}_k=\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\tau(x)\chi(\frac{|\eta'|^2}{M^2})\psi(\eta_{2n-1})\tau_1(y)d\eta.\tag{8.6}$$

It is straightforward to see that $\hat{\mathcal{I}}_k$ satisfies the assumptions in Theorem 6.5. Let

$$\mathcal{S}_k(x,y)=\int e^{ik\varphi(x,y,s)}a(x,y,s,k)ds\in C^\infty(D\times D)$$

be as in Theorem 6.5, Theorem 6.6. From (6.58), it is not difficult to see that

$$a_0(p,p,s)=(2\pi)^{-n}\left|\det\big(M^\phi_p-2s\mathcal{L}_p\big)\right|\psi(s),\tag{8.7}$$

for all $(p,p,s)\in\Omega$. We recall that Ω is as in Theorem 6.6. Put $x'=(x_1,\ldots,x_{2n-2})$ and set $|x'|^2=\sum_{j=1}^{2n-2}|x_j|^2$. Let

$$\begin{aligned}v_k&=\int e^{ik\varphi(x,p,s)}k^{-\frac{n}{2}}a(x,p,s,k)ds\chi(\frac{k}{(\log k)^2}\left|x'\right|^2)\chi(k^{\frac{5}{6}}x_{2n-1})\in C^\infty_0(D_0),\\u_k&=s^ke^{k\phi}v_k\in C^\infty_0(D_0,L^k).\end{aligned}\tag{8.8}$$

Here we assume that k is large enough so that $\operatorname{Supp} v_k \Subset D_0$. First, we need

Lemma 8.1. *With the assumptions and notations above, for every $N > 0$ and $m > 0$, there is a constant $C_{N,m} > 0$ independent of k and the point p such that*

$$\sum_{|\alpha|\leq m, \alpha\in\mathbb{N}_0^{2n-1}} \operatorname{Sup}\left\{\left|\partial_x^\alpha(\Box^{(0)}_{s,k} v_k(x))\right|;\, x\in D_0\right\} \leq C_{N,m}k^{-N}, \tag{8.9}$$

where $\Box^{(0)}_{s,k}$ is given by (4.1) and (4.4).

Moreover, there exist $C > 1$ and $k_0 > 0$ independent of k and the point p such that for every $k \geq k_0$, we have

$$\frac{1}{C} \leq \|u_k\|_{h^{L^k}} \leq C. \tag{8.10}$$

Proof. Put $\widetilde{v}_k = \int e^{ik\varphi(x,p,s)}k^{-\frac{n}{2}}a(x,p,s,k)ds \in C^\infty(D)$. In view of (6.24), we see that

$$\sum_{|\alpha|\leq m, \alpha\in\mathbb{N}_0^{2n-1}} \operatorname{Sup}\left\{\left|\partial_x^\alpha(\Box^{(0)}_{s,k} \widetilde{v}_k(x))\right|;\, x\in D_0\right\} \leq \widetilde{C}_{N,m}k^{-N} \tag{8.11}$$

for every $N > 0$, where $\widetilde{C}_{N,m} > 0$ is a constant independent of k. Since X is assumed to be compact, it is straightforward to see that $\widetilde{C}_{N,m} > 0$ can be taken to be independent of p. Now, we claim that for every $N > 0$ and $m > 0$, there is a constant $\hat{C}_{N,m} > 0$ independent of k and the point p such that

$$\sum_{|\alpha|\leq m, \alpha\in\mathbb{N}_0^{2n-1}} \operatorname{Sup}\left\{\left|\partial_x^\alpha(\widetilde{v}_k(x) - v_k(x))\right|;\, x\in D_0\right\} \leq \hat{C}_{N,m}k^{-N}. \tag{8.12}$$

Note that

$$\operatorname{Supp}(\widetilde{v}_k(x) - v_k(x))\bigcap D_0 \subset \left\{x\in D_0;\, |x'| \geq \frac{1}{2}\frac{\log k}{\sqrt{k}} \text{ or } |x_{2n-1}| \geq \frac{1}{2}k^{-\frac{5}{6}}\right\}.$$

When $|x'| \geq \frac{1}{2}\frac{\log k}{\sqrt{k}}$, we use the fact $k\operatorname{Im}\varphi(x,p,s) \geq ck\left|x'\right|^2 \geq c'(\log k)^2$, where $c > 0$, $c' > 0$ are constants independent of x and s, and conclude that $\widetilde{v}_k(x) - v_k(x) \equiv 0 \mod O(k^{-\infty})$. When $|x_{2n-1}| \geq \frac{1}{2}k^{-\frac{5}{6}}$, we have

$$k\frac{\partial\varphi(x,p,s)}{\partial s} \geq kc_1\left|x_{2n-1}\right| \geq c_1'k^{\frac{1}{6}},$$

where $c_1 > 0$, $c_1' > 0$ are constants independent of x and s. From this we can integrate by parts in s several times and conclude that $\widetilde{v}_k(x) - v_k(x) \equiv 0 \mod O(k^{-\infty})$. The claim (8.12) follows.

From (8.11) and (8.12), the claim (8.9) follows.

On D, we write $dv_X(x) = m(x)dx$, $m(x)\in C^\infty(D)$. Then, $m(0) = 2^{n-1}$. Put

$$\Upsilon(x) = \chi(\frac{k}{(\log k)^2}\left|x'\right|^2)\chi(k^{\frac{5}{6}}x_{2n-1}).$$

We have

$$\begin{aligned}\|u_k\|^2_{h^{L^k}} &= \int\left|\int e^{ik\varphi(x,p,s)}k^{-\frac{n}{2}}a(x,p,s)ds\right|^2\Upsilon^2(x)m(x)dx\\ &= \int\left|\int e^{ik\varphi(F_k^*x,p,s)}k^{-n}a(F_k^*x,p,s)ds\right|^2\Upsilon^2(F_k^*x)m(F_k^*x)dx,\end{aligned} \tag{8.13}$$

where $F_k^*x = (\frac{x'}{\sqrt{k}}, \frac{x_{2n-1}}{k})$. Put

$$\begin{aligned}\varphi(x,p,s) &= -i\sum_{j=1}^{n-1}\alpha_jz_j + i\sum_{j=1}^{n-1}\overline{\alpha}_j\overline{z}_j + s(x_{2n-1} - y_{2n-1})\\ &\quad+\sum_{j,t=1}^{2n-2}\beta_{j,t}(s)x_jx_t + O(\left|x_{2n-1}\right|\left|x'\right|) + O(\left|x_{2n-1}\right|^2) + O(|x|^3)\end{aligned} \tag{8.14}$$

and set

$$\varphi_0(x,s)=\varphi_0(x',s)=\sum_{j,t=1}^{2n-2}\beta_{j,t}(s)x_jx_t. \tag{8.15}$$

From (5.73), it is easy to see that

$$\operatorname{Im}\varphi_0(x,s)\geq c\left|x'\right|^2,\ \ \forall(x,p,s)\in\Omega, \tag{8.16}$$

where $c>0$ is a constant independent of $(x,p,s)\in\Omega$ and c can be take to be independent of the point p. It is easy to see that

$$\operatorname{Supp}\Upsilon(F_k^*x)\subset\Big\{x\in\mathbb{R}^{2n-1};\,|x'|\leq\log k,|x_{2n-1}|\leq k^{\frac{1}{6}}\Big\}. \tag{8.17}$$

From (8.17), it is straightforward to check that on $\operatorname{Supp}\Upsilon(F_k^*x)$,

$$\begin{aligned}&k\varphi(F_k^*x,p,s)=\sqrt{k}(-i\sum_{j=1}^{n-1}\alpha_jz_j+i\sum_{j=1}^{n-1}\overline{\alpha}_j\overline{z}_j)+sx_{2n-1}+\varphi_0(x',s)+\delta_k^0(x,s),\\&k^{-n}a(F_k^*x,p,s)=(2\pi)^{-n}\left|\det\bigl(M_p^\phi-2s\mathcal{L}_p\bigr)\right|\psi(s)+\delta_k^1(x,s),\\&m(F_k^*x)=2^{n-1}+\delta_k^2(x,s),\end{aligned} \tag{8.18}$$

where $\delta_k^0(x,s),\delta_k^1(x,s),\delta_k^2(x,s)\in C^\infty$ and

$$\operatorname{Sup}\left\{\left|\delta_k^0(x,s)\right|+\left|\delta_k^1(x,s)\right|+\left|\delta_k^2(x,s)\right|;\,(x,p,s)\in\Omega\right\}\leq C_0k^{-\frac{1}{3}}\log k, \tag{8.19}$$

where $C_0>0$ is a constant independent of k and the point p. From (8.19) and (8.13), we have

$$\left\|u_k\right\|^2_{h^{L^k}}=2^{n-1}\int\left|\int e^{isx_{2n-1}+i\varphi_0(x',s)}(2\pi)^{-n}\psi(s)\left|\det\bigl(M_p^\phi-2s\mathcal{L}_p\bigr)\right|\bigl(1+\gamma_k(x,s)\bigr)ds\right|^2\Upsilon^2(F_k^*x)dx, \tag{8.20}$$

where $\gamma_k(x,s)\in C^\infty$ and

$$\operatorname{Sup}\left\{\left|\gamma_k(x,s)\right|;\,(x,p,s)\in\Omega\right\}\leq C_1k^{-\frac{1}{3}}\log k. \tag{8.21}$$

Here $C_1>0$ is a constant independent of k and the point p. From (8.21), we can check that

$$\begin{aligned}&2^{n-1}\int\left|\int e^{isx_{2n-1}+i\varphi_0(x',s)}(2\pi)^{-n}\psi(s)\left|\det\bigl(M_p^\phi-2s\mathcal{L}_p\bigr)\right|\left|\gamma_k(x,s)\right|)ds\right|^2\Upsilon^2(F_k^*x)dx\\&\leq C_2k^{-\frac{2}{3}}(\log k)^2(\log k)^{2n-2}k^{\frac{1}{6}}=C_2k^{-\frac{1}{2}}(\log k)^{2n}\to0\ \text{ as } k\to\infty,\end{aligned} \tag{8.22}$$

where $C_2>0$ is a constant independent of k and the point p. Put

$$\begin{aligned}&A=2^{n-1}\int_{|x|\leq1}\left|\int e^{isx_{2n-1}+i\varphi_0(x',s)}(2\pi)^{-n}\psi(s)\left|\det\bigl(M_p^\phi-2s\mathcal{L}_p\bigr)\right|)ds\right|^2dx,\\&B=2^{n-1}\int_{\mathbb{R}^{2n-1}}\left|\int e^{isx_{2n-1}+i\varphi_0(x',s)}(2\pi)^{-n}\psi(s)\left|\det\bigl(M_p^\phi-2s\mathcal{L}_p\bigr)\right|)ds\right|^2dx.\end{aligned}$$

From (8.16) and by using integration by parts with respect to s, it is easy to see that $B<\infty$. From this observation, (8.22) and (8.20), we conclude that if k large then

$$\frac{A}{2}\leq\left\|u_k\right\|^2_{h^{L^k}}\leq2B.$$

(8.10) follows. $\square$

Now, we can prove

Theorem 8.2. *With the assumptions and notations above, fix $N_0>>1$, N_0 large. There exists $k_0>0$ and $C_0>0$ independent of k and the point p such that for every $k\geq k_0$, there is a $\mu_k\in H^0_{b,\leq k^{-N_0}}(X,L^k)$ with $\left\|\mu_k\right\|_{h^{L^k}}=1$ and*

$$\left|\mu_k(p)\right|^2_{h^{L^k}}\geq C_0k^n. \tag{8.23}$$

Proof. Let $u_k \in C^\infty(X, L^k)$ be as in Lemma 8.1. Put $u_k^0 = \Pi^{(0)}_{k,\leq k^{-N_0}} u_k$, $u_k^1 = (I - \Pi^{(0)}_{k,\leq k^{-N_0}})u_k$. We have the orthogonal decomposition

$$u_k = u_k^0 + u_k^1.$$

For every $m \in \mathbb{N}_0$, we have

$$\begin{aligned} \left\| (\Box^{(0)}_{b,k})^m u_k^1 \right\|^2_{h^{L^k}} &\leq k^{N_0} \big((\Box^{(0)}_{b,k})^{m+1} u_k^1 \,|\, u_k^1 \big)_{h^{L^k}} \\ &\leq k^{N_0} \big((\Box^{(0)}_{b,k})^{m+1} u_k \,|\, u_k \big)_{h^{L^k}}. \end{aligned} \tag{8.24}$$

From (8.9) and (8.24), we conclude that for every $N > 0$ and every $m \in \mathbb{N}_0$, there is a constant $C_{N,m} > 0$ independent of k and the point p such that

$$\left\| (\Box^{(0)}_{b,k})^m u_k^1 \right\|^2_{h^{L^k}} \leq C_{N,m} k^{-N}. \tag{8.25}$$

From (8.25), we can repeat the proof of Lemma 7.2 with minor changes and obtain that for every $N > 0$ there is a constant $C_N > 0$ independent of k and the point p such that

$$\left| u_k^1(p) \right|^2_{h^{L^k}} \leq k^{-N}. \tag{8.26}$$

Furthermore, from (8.25) and (8.10), we see that there exist $C > 1$ and $k_0 > 0$ independent of k and the point p such that for every $k \geq k_0$, we have

$$\frac{1}{C} \leq \left\| u_k^0 \right\|_{h^{L^k}} \leq C. \tag{8.27}$$

Put $\mu_k = \frac{u_k^0}{\|u_k^0\|_{h^{L^k}}}$. Then, $\mu_k \in H^0_{b,\leq k^{-N_0}}(X, L^k)$ and $\|\mu_k\|_{h^{L^k}} = 1$. Moreover, from (8.25), (8.26) and (8.27), we deduce that for every $N > 0$ there is a constant $C_N > 0$ independent of k and the point p such that

$$\left| |\mu_k(p)|^2_{h^{L^k}} - \frac{1}{\|u_k^0\|^2_{h^{L^k}}} |v_k(p)|^2 \right| \leq C_N k^{-N}. \tag{8.28}$$

From (8.28) and notice that $|v_k(p)|^2 \geq C_1 k^n$, where $C_1 > 0$ is a constant independent of k and the point p, the theorem follows. $\square$

From now on, we fix $N_0 > 2n + 1$. We assume that k is large enough so that the properties in Theorem 8.2 hold. Let $\left\{ f_1 \in H^0_{b,\leq k^{-N_0}}(X, L^k), \ldots, f_{d_k} \in H^0_{b,\leq k^{-N_0}}(X, L^k) \right\}$ be an orthonormal frame of the space $H^0_{b,\leq k^{-N_0}}(X, L^k)$. From Theorem 8.2, we deduce

Theorem 8.3. *We have*

$$\sum_{j=1}^{d_k} |f_j(x)|^2_{h^{L^k}} \geq C_0 k^n, \quad \forall x \in X, \tag{8.29}$$

where $C_0 > 0$ is the constant as in (8.23). *In particular, there is a constant $c_0 > 0$ such that for every $x \in X$, there exists a $j_0 \in \{1, 2, \ldots, d_k\}$ such that*

$$|f_{j_0}(x)|^2_{h^{L^k}} \geq c_0. \tag{8.30}$$

Proof. We only need to prove (8.30). It is well-known (see [15]) that there is a constant $C_1 > 0$ such that

$$\dim H^0_{b,\leq k^{-N}}(X, L^k) = d_k \leq C_1 k^n, \tag{8.31}$$

where $C_1 > 0$ is a constant independent of k. From (8.31) and (8.29), we have for every $x \in X$,

$$\begin{aligned} &C_1 k^n \mathrm{Sup} \left\{ |f_j(x)|^2_{h^{L^k}} \,;\, j = 1, 2, \ldots, d_k \right\} \\ &\geq d_k \mathrm{Sup} \left\{ |f_j(x)|^2_{h^{L^k}} \,;\, j = 1, 2, \ldots, d_k \right\} \\ &\geq \sum_{j=1}^{d_k} |f_j(x)|^2_{h^{L^k}} \geq C_0 k^n. \end{aligned}$$

From this, (8.30) follows. $\square$

Assume that $X = D_1 \bigcup D_2 \bigcup \cdots D_N$, where D_j is a small open set of X with the properties as in the beginning of section 8, for each j. On D_j, let $\hat{\mathcal{I}}_{k,j}$ be the operator as in (8.6). Fix $N' \gg 1$ be a large constant. The (asymptotic) Kodaira map $\Phi_{N_0,k} : X \to \mathbb{CP}^{d_k-1}$ is given by

$$\begin{aligned}&\Phi_{N_0,k} : x \in X \to \mathbb{CP}^{N_k-1},\\&x \in X \to [f_1(x), \ldots, f_{d_k}(x), k^{-N'}\hat{\mathcal{I}}_{k,1}f_1, \ldots, k^{-N'}\hat{\mathcal{I}}_{k,N}f_{d_k}] \in \mathbb{CP}^{N_k-1},\end{aligned} \tag{8.32}$$

where $N_k = d_k + Nd_k$. In view of (8.30), we see that $\Phi_{N_0,k}$ is well-defined as a smooth map from X to $\mathbb{CP}^{d_k-1}$. Our next goal is to prove

Theorem 8.4. *For k large, the differential map*

$$d\Phi_{N_0,k}(x) : T_xX \to T_{\Phi_{N_0,k}(x)}\mathbb{CP}^{d_k-1}$$

is injective, for every $x \in X$.

To prove Theorem 8.4, we need some preparations. Fix $p \in X$ and let s be a local trivializing section of L on an open neighbourhood $D \subset X$ of p. We take local coordinate $x = (x_1, \ldots, x_{2n-1})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n-1$, and s so that (7.1) hold. Let $R(x) = R(z)$ be as in (7.2). We first need

Lemma 8.5. *With the assumptions and notations above, there exist $v_k^j \in C_0^\infty(D)$, $j = 1, \ldots, n$, with*

$$\frac{1}{C} \leq \left\| s^k e^{k\phi} v_k^j \right\|_{h^{L^k}} \leq C, \quad j = 1, \ldots, n, \tag{8.33}$$

for every k, where $C > 0$ is a constant independent of k and the point p, and

$$\sum_{|\alpha| \leq m, \alpha \in \mathbb{N}_0^{2n-1}} \operatorname{Sup}\left\{\left|\partial_x^\alpha(\Box_{s,k}^{(0)} v_k^j(x))\right| ; x \in D, j = 1, \ldots, n\right\} \leq C_{N,m}k^{-N}, \tag{8.34}$$

for every $N > 0$ and $m \in \mathbb{N}_0$, where $C_{N,m}$ is a constant independent of k and the point p and $\Box_{s,k}^{(0)}$ is given by (4.1) *and* (4.4)*, such that*

$$v_k^j(0) = 0, \quad j = 1, \ldots, n, \tag{8.35}$$

$$\partial_{\overline{z}_t}\big(e^{-2kR}v_k^j\big)(0) = 0, \quad j = 1, \ldots, n, \quad t = 1, \ldots, n-1, \tag{8.36}$$

$$\partial_{z_t}\big(e^{-2kR}v_k^j\big)(0) = 0 \quad \text{if } j \neq t,\ j = 1, \ldots, n,\ t = 1, \ldots, n-1, \tag{8.37}$$

$$\partial_{x_{2n-1}}\big(e^{-2kR}v_k^j\big)(0) = 0, \quad j = 1, \ldots, n-1, \tag{8.38}$$

$$\left|\partial_{x_{2n-1}}\big(e^{-2kR}v_k^n\big)(0)\right|^2 \geq c_1k^{n+2}, \quad \left|\partial_{z_j}\big(e^{-2kR}v_k^j\big)(0)\right|^2 \geq c_1k^{n+1}, \quad j = 1, \ldots, n-1, \tag{8.39}$$

where $c_1 > 0$ is a constant independent of k and the point p.

Proof. From Borel construction, it is clearly that we can find $z_j + \beta_j(x) \in C^\infty(\mathbb{R}^{2n-1})$, $j = 1, \ldots, n-1$, and $x_{2n-1} + \beta_n(x) \in C^\infty(\mathbb{R}^{2n-1})$ such that

$$\begin{aligned}&\overline{\partial}_b(z_j + \beta_j(x)) \text{ vanishes to infinite order at } p,\ j = 1, \ldots, n-1,\\&\overline{\partial}_b(x_{2n-1} + \beta_n(x)) \text{ vanishes to infinite order at } p,\end{aligned} \tag{8.40}$$

where $\beta_j(x) = O(|x|^2)$, $j = 1, \ldots, n$. Let $v_k(x) \in C_0^\infty(D_0)$ be as in (8.8). Recall that $D_0 \Subset D$ be as in the discussion after (8.2). Put

$$\begin{aligned}&v_k^j(x) = \sqrt{k}(z_j + \beta_j(x))v_k(x) \in C_0^\infty(D_0), \quad j = 1, \ldots, n-1,\\&v_k^n(x) = k(x_{2n-1} + \beta_n(x))v_k(x) \in C_0^\infty(D_0).\end{aligned} \tag{8.41}$$

We can repeat the proof of (8.10) with minor changes and deduce (8.33). Moreover, from (8.9), (8.40) and (5.74), we obtain (8.34). Finally, from the constructions of v_k and $v_k^j(x)$, $j = 1, \ldots, n$, we get (8.35), (8.36), (8.37), (8.38) and (8.39). The lemma follows. $\square$

We also need

Lemma 8.6. *With the assumptions and notations above, fix $N_0 > 2n+1$. There exist*

$$\mu^j_k \in H^0_{b,\leq k^{-N_0}}(X, L^k), \quad j = 1, \ldots, n,$$

with $\left\|\mu^j_k\right\|_{h^{L^k}} = 1$, $j = 1, \ldots, n$, such that if we put $\mu^j_k = s^k \widetilde{\mu}^j_k$ on D, $j = 1, \ldots, n$, then

$$\left|\partial_{x_{2n-1}}\big(e^{-2kR}\widetilde{\mu}^n_k\big)(0)\right|^2 \geq c_2 k^{n+2}, \quad \left|\partial_{z_j}\big(e^{-2kR}\widetilde{\mu}^j_k\big)(0)\right|^2 \geq c_2 k^{n+1}, \quad j = 1, \ldots, n-1, \tag{8.42}$$

where $c_2 > 0$ is a constant independent of k and the point p and for every $N > 0$ there is a $C_N > 0$ independent of k and the point p such that

$$\begin{split} &\mathrm{Sup}\,\Big\{\left|\big(e^{-2kR}\widetilde{\mu}^j_k\big)(0)\right|, \left|\partial_{\overline{z}_t}\big(e^{-2kR}\widetilde{\mu}^j_k\big)(0)\right|, \left|(1-\delta_{j,t})\partial_{z_t}\big(e^{-2kR}\widetilde{\mu}^j_k\big)(0)\right|, \left|\partial_{x_{2n-1}}\big(e^{-2kR}\widetilde{\mu}^t_k\big)(0)\right|; \\ &\qquad j = 1, \ldots, n, t = 1, \ldots, n-1\Big\} \leq C_N k^{-N}. \end{split} \tag{8.43}$$

Proof. Fix $j = 1, \ldots, n$. Let $v^j_k \in C^\infty_0(D)$ be as in Lemma 8.5. Put $u^j_k = s^k e^{k\phi} v^j_k$ and set $\beta^j_k = \Pi^{(0)}_{k,\leq k^{-N_0}} u^j_k$, $\gamma^j_k = (I - \Pi^{(0)}_{k,\leq k^{-N_0}}) u^j_k$. We have the orthogonal decomposition

$$u^j_k = \beta^j_k + \gamma^j_k.$$

For every $m \in \mathbb{N}_0$, we have

$$\begin{split} \left\|(\Box^{(0)}_{b,k})^m \gamma^j_k\right\|^2_{h^{L^k}} &\leq k^{N_0}\big(\,(\Box^{(0)}_{b,k})^{m+1}\gamma^j_k \,|\, \gamma^j_k\big)_{h^{L^k}} \\ &\leq k^{N_0}\big(\,(\Box^{(0)}_{b,k})^{m+1} u^j_k \,|\, u^j_k\big)_{h^{L^k}}. \end{split} \tag{8.44}$$

From (8.34) and (8.44), we conclude that for every $N > 0$ and every $m \in \mathbb{N}_0$, there is a constant $C_{N,m} > 0$ independent of k and the point p such that

$$\left\|(\Box^{(0)}_{b,k})^m \gamma^j_k\right\|^2_{h^{L^k}} \leq C_{N,m} k^{-N}. \tag{8.45}$$

From (8.45), we can repeat the proof of Lemma 7.2 with minor changes and obtain that for every $N > 0$ and every $\alpha \in \mathbb{N}^{2n-1}_0$, there is a constant $C_{N,\alpha} > 0$ independent of k and the point p such that

$$\left|\partial^\alpha_x\big(e^{-2kR}\widetilde{\gamma}^j_k\big)(p)\right|^2 \leq k^{-N}, \tag{8.46}$$

where $\gamma^j_k = s^k \widetilde{\gamma}^j_k$ on D. Furthermore, from (8.45) and (8.33), we see that there exist $C > 1$ and $k_0 > 0$ independent of k and the point p such that for every $k \geq k_0$, we have

$$\frac{1}{C} \leq \left\|\beta^j_k\right\|_{h^{L^k}} \leq C. \tag{8.47}$$

Put $\mu^j_k = \frac{\beta^j_k}{\|\beta^j_k\|_{h^{L^k}}}$. Then, $\mu^j_k \in H^0_{b,\leq k^{-N_0}}(X, L^k)$ and $\left\|\mu^j_k\right\|_{h^{L^k}} = 1$. On D, put $\mu^j_k = s^k \widetilde{\mu}^j_k$. From (8.46) and (8.47), we deduce that for every $N > 0$ and every $\alpha \in \mathbb{N}^{2n-1}_0$ there is a constant $C_{N,\alpha} > 0$ independent of k and the point p such that

$$\left|\left|\partial^\alpha_x\big(e^{-2kR}\widetilde{\mu}^j_k\big)(p)\right|^2 - \frac{1}{\left\|\beta^j_k\right\|^2_{h^{L^k}}}\left|\partial^\alpha_x\big(e^{-2kR}v^j_k\big)(p)\right|^2\right| \leq C_{N,\alpha} k^{-N}. \tag{8.48}$$

From (8.48) and Lemma 8.5, the lemma follows. ☐

From Lemma 8.6 and the Gram-Schmidt process, we deduce

Proposition 8.7. *With the assumptions and notations above, fix $N_0 > 2n+1$. There exist $g^j_k \in H^0_{b,\leq k^{-N_0}}(X, L^k)$, $j = 1, \ldots, n$, with $(g^j_k \,|\, g^t_k)_{h^{L^k}} = \delta_{j,t}$, $j, t = 1, \ldots, n$, such that if we put $g^j_k = s^k \widetilde{g}^j_k$ on D, $j = 1, \ldots, n$, then*

$$\begin{split} &\left|\partial_{z_t}\big(e^{-2kR}\widetilde{g}^t_k\big)(0)\right|^2 \geq c_2 k^{n+1}, \quad t = 1, \ldots, n-1, \\ &\left|\partial_{x_{2n-1}}\big(e^{-2kR}\widetilde{g}^n_k\big)(0)\right|^2 \geq c_2 k^{n+2}, \end{split} \tag{8.49}$$

where $c_2 > 0$ is a constant independent of k and the point p and for every $N > 0$ there is a $C_N > 0$ independent of k and the point p such that

$$\begin{aligned}\operatorname{Sup}\{\left|\partial_{x_{2n-1}}\big(e^{-2kR}\widetilde{g}^t_k\big)(0)\right|,\left|\partial_{z_s}\big(e^{-2kR}\widetilde{g}^t_k\big)(0)\right|;\, s,t=1,\ldots,n-1, s>t\}\\ +\operatorname{Sup}\{\left|\big(e^{-2kR}\widetilde{g}^t_k\big)(0)\right|,\left|\partial_{\overline{z}_s}\big(e^{-2kR}\widetilde{g}^t_k\big)(0)\right|;\, s=1,\ldots,n-1, t=1,\ldots,n\}\leq C_Nk^{-N}.\end{aligned}\tag{8.50}$$

From Proposition 8.7 and some straightforward but elementary linear algebra argument, we obtain the following(we omit the proof)

Proposition 8.8. *With the assumptions and notations above, fix $N_0 > 2n+1$. Let*

$$g^j_k \in H^0_{b,\leq k^{-N_0}}(X, L^k),\quad j=1,\ldots,n,$$

be as in Proposition 8.7. We put

$$\begin{aligned}&g^j_k = s^k\widetilde{g}^j_k \text{ on } D,\ j=1,\ldots,n,\\ &\widetilde{g}^j_k = h^{2j-1}_k + ih^{2j}_k,\ j=1,\ldots,n, \text{ where } h^{2j-1}_k=\operatorname{Re}\widetilde{g}^j_k,\ h^{2j}_k=\operatorname{Im}\widetilde{g}^j_k,\ j=1,\ldots,n.\end{aligned}\tag{8.51}$$

There is a $k_0 > 0$ independent of the point p such that for every $k \geq k_0$, the matrix

$$\begin{aligned}H_k:&=\begin{bmatrix}\partial_{x_1}\big(e^{-2kR}h^1_k\big)(p) & \partial_{x_2}\big(e^{-2kR}h^1_k\big)(p) & \cdots & \partial_{x_{2n-1}}\big(e^{-2kR}h^1_k\big)(p)\\ \partial_{x_1}\big(e^{-2kR}h^2_k\big)(p) & \partial_{x_2}\big(e^{-2kR}h^2_k\big)(p) & \cdots & \partial_{x_{2n-1}}\big(e^{-2kR}h^2_k\big)(p)\\ \vdots & \vdots & \vdots & \vdots\\ \partial_{x_1}\big(e^{-2kR}h^{2n}_k\big)(p) & \partial_{x_2}\big(e^{-2kR}h^{2n}_k\big)(p) & \cdots & \partial_{x_{2n-1}}\big(e^{-2kR}h^{2n}_k\big)(p)\end{bmatrix},\\ H_k:&\mathbb{R}^{2n-1}\to\mathbb{R}^{2n},\end{aligned}$$

is injective.

From the proofs of Lemma 7.2 and Proposition 7.6, we conclude that

Lemma 8.9. *With the assumptions and notations above, fix $N_0 > 2n+1$. Let*

$$x_k=(x^1_k,\ldots,x^{2n-1}_k)\in\mathbb{R}^{2n-1},\quad y_k=(y^1_k,\ldots,y^{2n-1}_k)\in\mathbb{R}^{2n-1}$$

with $\lim_{k\to\infty}(\sqrt{k}\sum^{2n-2}_{j=1}\left|x^j_k\right| + k\left|x^{2n-1}_k\right|)=0$, $\lim_{k\to\infty}(\sqrt{k}\sum^{2n-2}_{j=1}\left|y^j_k\right| + k\left|y^{2n-1}_k\right|)=0$. Then, for every $\alpha=(\alpha_1,\ldots,\alpha_{2n-1})\in\mathbb{N}^{2n-1}_0$, $\beta=(\beta_1,\ldots,\beta_{2n-1})$, there are constants $C_\alpha > 0$, $C_{\alpha,\beta} > 0$ independent of k and the point p such that for every $u\in H^0_{b,\leq k^{-N_0}}(X,L^k)$ with $\|u\|_{h^{L^k}}=1$ we have

$$\left|\partial^\alpha_x\big(e^{-2kR}\widetilde{u}\big)(x_k)\right|^2\leq C_\alpha k^{n+|\alpha'|+2\alpha_{2n-1}}\tag{8.52}$$

and

$$\left|\partial^\alpha_x\partial^\beta_y\Big(e^{-kR(x)+k\overline{R}(x)}\hat{\Pi}^{(0)}_{k,\leq k^{-N_0},s}(x,y)e^{-k\overline{R}(y)+kR(y)}\Big)(x_k,y_k)\right|\leq C_{\alpha,\beta}k^{n+\frac{|\alpha'|}{2}+\frac{|\beta'|}{2}+\alpha_{2n-1}+\beta_{2n-1}},\tag{8.53}$$

where $u=s^k\widetilde{u}$ on D, $\hat{\Pi}^{(0)}_{k,\leq k^{-N_0},s}(x,y)$ is the localized Szegö projection (see (7.32) and (7.38)) and $|\alpha'|=\sum^{2n-2}_{j=1}\alpha_j$, $|\beta'|=\sum^{2n-2}_{j=1}\beta_j$

Proof of Theorem 8.4. We are going to prove that if k is large then the map

$$d\Phi_{N_0,k}(x):T_xX\to T_{\Phi_{N_0,k}(x)}\mathbb{CP}^{d_k-1}$$

is injective. Fix $p\in X$ and let s be a local trivializing section of L on an open neighbourhood $D\subset X$ of p. We take local coordinates $x=(x_1,\ldots,x_{2n-1})$, $z_j=x_{2j-1}+ix_{2j}$, $j=1,\ldots,n-1$, and s so that (7.1) hold. We shall use the same notations as above. From Theorem 8.3, we may assume that

$$\left|\big(e^{-2kR}\widetilde{f}_1\big)(p)\right|^2\geq c_0,\tag{8.54}$$

where $c_0 > 0$ is a constant independent of k and the point p. Let $g^1_k,\ldots,g^n_k\in H^0_{b,\leq k^{-N_0}}(X,L^k)$ be as in Proposition 8.7. In view of (8.50), we may assume that

$$\sum^n_{j=1}\left|\big(e^{-2kR}\widetilde{g}^j_k\big)(p)\right|^2\leq\frac{c_0}{2}.\tag{8.55}$$

Now, we claim that $f_1, g_k^1, \ldots, g_k^n$ are linearly independent over $\mathbb{C}$. If $f_1, g_k^1, \ldots, g_k^n$ are linearly dependent then we have $f_1 = \sum_{j=1}^n \lambda_j g_k^j$, where $\lambda_j \in \mathbb{C}$, $j = 1, \ldots, n$. Since $\|f_1\|_{h^{L^k}} = 1$, we have $\sum_{j=1}^n |\lambda_j|^2 = 1$. Thus,

$$\left|(e^{-2kR}\widetilde{f}_1)(p)\right|^2 \leq \Big(\sum_{j=1}^n |\lambda_j|^2\Big)\Big(\sum_{j=1}^n \left|(e^{-2kR}\widetilde{g}_k^j)(p)\right|^2\Big) \leq \frac{c_0}{2}.$$

We get a contradiction. Thus, $f_1, g_k^1, \ldots, g_k^n$ are linearly independent. Put

$$\begin{aligned} &p_k^j = \frac{e^{-2kR}\widetilde{g}_k^j}{e^{-2kR}\widetilde{f}_1}, \quad j = 1, \ldots, n-1, \\ &p_k^j = \alpha_k^{2j-1} + i\alpha_k^{2j}, \quad \alpha_k^{2j-1} = \operatorname{Re} p_k^j, \quad \alpha_k^{2j} = \operatorname{Im} p_k^j, \quad j = 1, \ldots, n-1. \end{aligned} \tag{8.56}$$

From (8.49), (8.50), Lemma 8.9 and (8.54), it is not difficult to see that

$$\left|\partial_{z_t} p_k^t(p)\right|^2 \geq c_2 k^{n+1}, \quad t = 1, \ldots, 2n-1, \quad \left|\partial_{x_{2n-1}} p_k^n(p)\right|^2 \geq c_2 k^{n+2}, \tag{8.57}$$

where $c_2 > 0$ is a constant independent of k and the point p and for every $N > 0$ there is a $C_N > 0$ independent of k and the point p such that

$$\begin{aligned} &\operatorname{Sup}\left\{\left|\partial_{x_{2n-1}} p_k^t(p)\right|, \left|\partial_{z_s} p_k^t(p)\right|;\, s, t = 1, \ldots, n-1, s > t\right\} \\ &\quad + \operatorname{Sup}\left\{\left|p_k^t(p)\right|, \left|\partial_{\overline{z}_s} p_k^t(p)\right|;\, s = 1, \ldots, n-1, t = 1, \ldots, n\right\} \leq C_N k^{-N}. \end{aligned} \tag{8.58}$$

From (8.57), (8.58) and some elementary linear algebra argument, we conclude that there is a $k_0 > 0$ independent of the point p such that for every $k \geq k_0$, the matrix

$$\begin{aligned} &A_k := \begin{bmatrix} \partial_{x_1}\big(e^{-2kR}\alpha_k^1\big)(p) & \partial_{x_2}\big(e^{-2kR}\alpha_k^1\big)(p) & \cdots & \partial_{x_{2n}}\big(e^{-2kR}\alpha_k^1\big)(p) \\ \partial_{x_1}\big(e^{-2kR}\alpha_k^2\big)(p) & \partial_{x_2}\big(e^{-2kR}\alpha_k^2\big)(p) & \cdots & \partial_{x_{2n}}\big(e^{-2kR}\alpha_k^2\big)(p) \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{x_1}\big(e^{-2kR}\alpha_k^{2n-1}\big)(p) & \partial_{x_2}\big(e^{-2kR}\alpha_k^{2n}\big)(p) & \cdots & \partial_{x_{2n}}\big(e^{-2kR}\alpha_k^{2n}\big)(p) \end{bmatrix}, \\ &A_k : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n} \end{aligned}$$

is injective. Hence the differential of the map $x \in X \to (\frac{g_k^1}{f_1}(x), \ldots, \frac{g_k^n}{f_1}(x)) \in \mathbb{C}^n$ at p is injective if $k \geq k_0$. From this and some elementary linear algebra arguments, we conclude that the differential of the map $x \in X \to (\frac{f_2}{f_1}(x), \ldots, \frac{f_{d_k}}{f_1}(x)) \in \mathbb{C}^{d_k}$ at p is injective if $k \geq k_0$. Theorem 8.4 follows. □

Our last goal in this section is to prove that for k large, the map $\Phi_{N_0,k} : X \to \mathbb{CP}^{d_k-1}$ is injective.

Theorem 8.10. *With the assumptions and notations above, fix $N_0 > 2n+1$. For k large, the map $\Phi_{N_0,k} : X \to \mathbb{CP}^{d_k-1}$ is injective.*

Proof. We assume that the claim of the theorem is not true. We can find $x_{k_j}, y_{k_j} \in X$, $x_{k_j} \neq y_{k_j}$, $0 < k_1 < k_2 < \cdots$, $\lim_{j\to\infty} k_j = \infty$, such that $\Phi_{N_0,k_j}(x_{k_j}) = \Phi_{N_0,k_j}(y_{k_j})$, for each j. We may suppose that there are $x_k, y_k \in X$, $x_k \neq y_k$, such that $\Phi_{N_0,k}(x_k) = \Phi_{N_0,k}(y_k)$, for each k. We may assume that $x_k \to p \in X$, $y_k \to q \in X$, as $k \to \infty$. If $p \neq q$. Then, for k large, we have $\operatorname{dist}(x_k, y_k) \geq \frac{1}{2}\operatorname{dist}(p, q)$. In view of the proof of Theorem 8.2, it is not difficult to see that we can find $u_k, v_k \in H^0_{b,\leq k^{-N_0}}(X, L^k)$ such that for k large, we have

$$|u_k(x_k)|^2_{h^{L^k}} \geq C_0 k^n, \quad |u_k(y_k)|^2_{h^{L^k}} \leq \frac{C_0}{2} k^n, \tag{8.59}$$

and

$$|v_k(y_k)|^2_{h^{L^k}} \geq C_0 k^n, \quad |v_k(x_k)|^2_{h^{L^k}} \leq \frac{C_0}{2} k^n, \tag{8.60}$$

where $C_0 > 0$ is a constant independent of k. Now, $\Phi_{N_0,k}(x_k) = \Phi_{N_0,k}(y_k)$ implies that

$$|u_k(x_k)|^2_{h^{L^k}} = r_k\, |u_k(y_k)|^2_{h^{L^k}}\,, \quad |v_k(x_k)|^2_{h^{L^k}} = r_k\, |v_k(y_k)|^2_{h^{L^k}}\,,$$

where $r_k \in \mathbb{R}_+$, for each k. (8.59) implies that $r_k \geq 2$, for k large. But (8.60) implies that $r_k \leq \frac{1}{2}$, for k large. We get a contradiction. Thus, we must have $p = q$.

Let $X=D_1\bigcup D_2\bigcup\cdots D_N$, where D_j is an open set as in the discussion before (8.32). We assume that $p\in D_1=:D$. Let s be a local trivializing section of L on an open subset $D\subset X$ of p, $|s|^2_{h^L}=e^{-2\phi}$. Let $x=(x_1,\ldots,x_{2n-1})$, $z_j=x_{2j-1}+ix_{2j}$, $j=1,\ldots,n-1$, be local coordinates of X defined on D. For simplicity, we assume that (7.1) hold. We shall use the same notations as before. We write $x_k=(x^1_k,\ldots,x^{2n-1}_k)\in\mathbb{R}^{2n-1}$, $y_k=(y^1_k,\ldots,y^{2n-1}_k)\in\mathbb{R}^{2n-1}$.

Case I : $\limsup_{k\to\infty}\sqrt{k}\sum^{2n-2}_{j=1}\left|x^j_k-y^j_k\right|=M>0$ (M can be ∞).

For simplicity, we may assume that

$$\lim_{k\to\infty}\sqrt{k}\sum^{2n-2}_{j=1}\left|x^j_k-y^j_k\right|=M,\quad M\in]0,\infty]. \tag{8.61}$$

Now, $\Phi_{N_0,k}(x_k)=\Phi_{N_0,k}(y_k)$ implies that we can find a sequence $\lambda_k\in\mathbb{C}$ such that for each k,

$$e^{-k\phi(x_k)}\widetilde{u}_k(x_k)=\lambda_ke^{-k\phi(y_k)}\widetilde{u}_k(y_k), \tag{8.62}$$

for every $u_k\in H^0_{b,\leq k^{-N_0}}(X,L^k)$, $u_k=s^k\widetilde{u}_k$ on D. We may assume that

$$\limsup_{k\to\infty}|\lambda_k|\geq1. \tag{8.63}$$

Let $\hat{\mathcal{I}}_k:=\hat{\mathcal{I}}_{k,1}$ be as in (8.6) and (8.32). Let

$$h_k=\sum^{d_k}_{j=1}f_j\big(\hat{\mathcal{I}}_ke^{-k\phi}\widetilde{f}_j\big)(y_k)\in H^0_{b,\leq k^{-N_0}}(X,L^k),$$

where $f_j=s^k\widetilde{f}_j$ on D, $j=1,\ldots,d_k$. On D, we write $h_k=s^k\widetilde{h}_k$. Then, it is easy to see that $(\hat{\Pi}^{(0)}_{k,\leq k^{-N_0},s}\hat{\mathcal{I}}_k)(x,y_k)=e^{-k\phi(x)}\widetilde{h}_k(x)$. From this observation and Theorem 7.12, it is straightforward to check that

$$e^{-k\phi(x)}\widetilde{h}_k(x)=\int e^{ik\varphi(x,y_k,s)}a(x,y_k,s,k)ds+R_k(x), \tag{8.64}$$

where $\varphi(x,y,s)\in C^\infty(\Omega)$ is as in Theorem 7.12, Ω is as in the discussion after (7.72), $a(x,y,s,k)\sim\sum^\infty_{j=0}a_j(x,y,s)k^{n-j}$ in $S^n_{\rm loc}(1;\Omega)$,

$$a_0(p,p,s)=(2\pi)^{-n}\left|\det\big(M^\phi_p-2s\mathcal{L}_p\big)\right|\psi(s), \tag{8.65}$$

and $R_k(x)$ is a smooth function on D such that for every $D'\Subset D$, there is a constant $C_{D'}$ independent of k such that

$$|R_k(x)|\leq C_{D'}k^{3n-N_0-2}\leq C_{D'}k^{n-2}. \tag{8.66}$$

From (8.61), (8.65), (8.66) and (5.73), we have

$$\begin{split}&\limsup_{k\to\infty}k^{-n}\left|e^{-k\phi(x_k)}\widetilde{h}_k(x_k)\right|\\&\leq\limsup_{k\to\infty}k^{-n}\Big(\int e^{-{\rm Im}\,\varphi(x_k,y_k,s)}\left|a(x_k,y_k,s)\right|ds+R_k(x_k)\Big)\\&\leq e^{-cM^2}(2\pi)^{-n}\int\left|\det\big(M^\phi_p-2s\mathcal{L}_p\big)\right|\psi(s)ds,\end{split} \tag{8.67}$$

where $c>0$ is a constant independent of k, and

$$\limsup_{k\to\infty}k^{-n}\left|e^{-k\phi(y_k)}\widetilde{h}_k(y_k)\right|=(2\pi)^{-n}\int\left|\det\big(M^\phi_p-2s\mathcal{L}_p\big)\right|\psi(s)ds. \tag{8.68}$$

From (8.62) and (8.63), we conclude that

$$\limsup_{k\to\infty}k^{-n}\left|e^{-k\phi(x_k)}\widetilde{h}_k(x_k)\right|\geq\limsup_{k\to\infty}k^{-n}\left|e^{-k\phi(y_k)}\widetilde{h}_k(y_k)\right|.$$

From this and (8.67), (8.68), we deduce that

$$e^{-cM^2}(2\pi)^{-n}\int\left|\det\big(M^\phi_p-2s\mathcal{L}_p\big)\right|\psi(s)ds\geq(2\pi)^{-n}\int\left|\det\big(M^\phi_p-2s\mathcal{L}_p\big)\right|\psi(s)ds.$$

But this is impossible. We get a contradiction.

Case II: $\limsup_{k\to\infty}\sqrt{k}\sum_{j=1}^{2n-2}\left|x_k^j-y_k^j\right|=0$, $\limsup_{k\to\infty}k\left|\langle\,\omega_0(x_k),y_k-x_k\,\rangle\right|=M>0$ (M can be ∞).

For simplicity, we may assume that

$$\lim_{k\to\infty}k\left|\langle\,\omega_0(x_k),y_k-x_k\,\rangle\right|=M,\quad M\in]0,\infty]. \tag{8.69}$$

Now, $\Phi_{N_0,k}(x_k)=\Phi_{N_0,k}(y_k)$ implies that we can find a sequence $\lambda_k\in\mathbb{C}$ such that for each k,

$$e^{-k\phi(x_k)}\widetilde{u}_k(x_k)=\lambda_k e^{-k\phi(y_k)}\widetilde{u}_k(y_k), \tag{8.70}$$

for every $u_k\in H^0_{b,\leq k^{-N_0}}(X,L^k)$, $u_k=s^k\widetilde{u}_k$ on D. We may assume that

$$\limsup_{k\to\infty}|\lambda_k|\geq 1. \tag{8.71}$$

We fist assume that $M=\infty$. Let h_k be as above. From the fact that $\left|\frac{\partial\varphi(x,y,s)}{\partial s}\right|_{x=x_k,y=y_k}\geq c\left|\langle\,\omega_0(x_k),y_k-x_k\,\rangle\right|$, where $c>0$ is a constant independent of k, we can integrate by parts with respect to s and conclude that

$$\limsup_{k\to\infty}k^{-n}\left|e^{-k\phi(x_k)}\widetilde{h}_k(x_k)\right|=0.$$

But from (8.71), we have

$$\begin{aligned}0&=\limsup_{k\to\infty}k^{-n}\left|e^{-k\phi(x_k)}\widetilde{h}_k(x_k)\right|\\&\geq\limsup_{k\to\infty}k^{-n}\left|e^{-k\phi(y_k)}\widetilde{h}_k(y_k)\right|=(2\pi)^{-n}\int\left|\det\bigl(M^\phi_p-2s\mathcal{L}_p\bigr)\right|\psi(s)ds.\end{aligned}$$

This is impossible. We get a contradiction. Now, we assume that $M<\infty$. From (8.64) and (8.66), it is not difficult to see that

$$\begin{aligned}\lim_{k\to\infty}k^{-n}\left|e^{-k\phi(x_k)}\widetilde{h}_k(x_k)\right|&=(2\pi)^{-n}\left|\int e^{ikMs}\left|\det\Bigl(M^\phi_p-2s\mathcal{L}_p\Bigr)\right|\psi(s)ds\right|\\&<(2\pi)^{-n}\int\left|\det\Bigl(M^\phi_p-2s\mathcal{L}_p\Bigr)\right|\psi(s)ds=\lim_{k\to\infty}k^{-n}\left|e^{-k\phi(y_k)}\widetilde{h}_k(y_k)\right|.\end{aligned} \tag{8.72}$$

We get a contradiction.

Case III: $\limsup_{k\to\infty}\sqrt{k}\sum_{j=1}^{2n-2}\left|x_k^j-y_k^j\right|=0$, $\limsup_{k\to\infty}k\left|\langle\,\omega_0(x_k),y_k-x_k\,\rangle\right|=0$.

Let $g_j=\hat{\mathcal{I}}_kf_j$ and set $g_j=s^k\widetilde{g}_j$ on D, $j=1,2,\ldots,d_k$. Put

$$\begin{aligned}\alpha_k(t)=&e^{-k\phi(tx_k+(1-t)y_k)-k\phi(y_k)}\times\\&\sum_{j=1}^{d_k}\widetilde{g}_j(tx_k+(1-t)y_k)\overline{\widetilde{g}_j}(y_k)e^{-kR(tx_k+(1-t)y_k)+k\overline{R}(tx_k+(1-t)y_k)-k\overline{R}(y_k)+kR(y_k)},\\A_k(t)=&\left|\alpha_k(t)\right|^2,\\B_k(t)=&e^{-2k\phi(tx_k+(1-t)y_k)-2k\phi(y_k)}\times\\&\sum_{j=1}^{d_k}\left|\widetilde{g}_j(tx_k+(1-t)y_k)e^{-kR(tx_k+(1-t)y_k)+k\overline{R}(tx_k+(1-t)y_k)}\right|^2\sum_{j=1}^{d_k}\left|\widetilde{g}_j(y_k)e^{-kR(y_k)+k\overline{R}(y_k)}\right|^2,\end{aligned} \tag{8.73}$$

where $t\in[0,1]$ and R is as in (7.2). Put $H_k(t)=\frac{A_k(t)}{B_k(t)}$. $H_k(t)$ is a smooth function of $t\in[0,1]$ since $B_k(t)>0$ for every $t\in[0,1]$. Moreover, we can check that $0\leq H_k(t)\leq 1$ and $H_k(1)=H_k(0)=1$. Thus, for each k, there is a $t_k\in[0,1]$ such that

$$H''_k(t_k)\geq 0. \tag{8.74}$$

We now calculate $H''_k(t)$. We first calculate $A''_k(t)$. In view of Theorem 7.12, it is not difficult to see that

$$\begin{aligned}\alpha_k(t)=\int & e^{ik\varphi(tx_k+(1-t)y_k,y_k,s)}e^{-kR(tx_k+(1-t)y_k)+k\overline{R}(tx_k+(1-t)y_k)-k\overline{R}(y_k)+kR(y_k)}\\ &\times a(tx_k+(1-t)y_k,y_k,s,k)ds+\epsilon_k(tx_k+(1-t)y_k,y_k),\end{aligned}\tag{8.75}$$

where $a(x,y,s,k)\sim\sum_{j=0}^{\infty}k^{n-j}a_j(x,y,s)$ in $S^n_{\mathrm{loc}}(1;\Omega)$, $a_j(x,y,s)\in C^\infty_0(\Omega)$, $j=0,1,\ldots$, Ω is as in the discussion after (7.72),

$$a_0(p,p,s)=(2\pi)^{-n}\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2,\tag{8.76}$$

and $\epsilon_{k,\delta}(x,y)$ is a smooth function on $D\times D$ such that for every $D'\Subset D$ and every $\alpha,\beta\in\mathbb{N}_0^{2n-1}$, there is a constant $C_{D',\alpha,\beta,\delta}$ independent of k such that

$$\left|\partial^\alpha_x\partial^\beta_y\epsilon_{k,\delta}(x,y)\right|\leq C_{D',\alpha,\beta,\delta}k^{3n-N_0-2+2|\alpha|+2|\beta|}.\tag{8.77}$$

We can calculate that

$$A''_k(t)=2\left|\alpha'_k(t)\right|^2+\alpha''_k(t)\overline{\alpha}_k(t)+\overline{\alpha}''_k(t)\alpha_k(t).\tag{8.78}$$

From (2.3), (8.75), (8.76) and (8.77), it is straightforward to see that(we omit the computations)

$$\begin{aligned}&2\left|\alpha'_k(t_k)\right|^2+\alpha''_k(t_k)\overline{\alpha}_k(t_k)+\overline{\alpha}''_k(t_k)\alpha_k(t_k)\\ &=2(2\pi)^{-2n}k^{2n+2}\Big(\big(\int s\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2ds\big)^2\\ &-\int s^2\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2ds\int\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2ds\Big)(\left|\langle\,\omega_0(x_k),y_k-x_k\,\rangle\right|)^2\\ &-2(2\pi)^{-2n}k^{2n+1}\int(\sum_{j,l=1}^{2n-2}\frac{\partial^2\mathrm{Im}\,\varphi}{\partial x_j\partial x_l}(p,p,s)(x^j_k-y^j_k)(x^l_k-y^l_k))\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2ds\\ &+o(k^{2n})O\Big(\big(\sqrt{k}\sum_{j=1}^{2n-2}\left|x^j_k-y^j_k\right|+k\left|\langle\,\omega_0(x_k),y_k-x_k\,\rangle\right|\big)^2\Big).\end{aligned}\tag{8.79}$$

Since

$$\begin{aligned}&\Big(\big(\int s\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2ds\big)^2\\ &\quad-\int s^2\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2ds\int\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2ds\Big)<0,\end{aligned}$$

there is a constant $C_1>0$ independent of k such that

$$\begin{aligned}&\Big(\big(\int s\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2ds\big)^2\\ &-\int s^2\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2ds\int\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2ds\Big)\langle\,\omega_0(x_k),y_k-x_k\,\rangle^2\\ &\leq-C_1\left|\langle\,\omega_0(x_k),y_k-x_k\,\rangle\right|^2.\end{aligned}\tag{8.80}$$

Moreover, from (5.73), we can check that there is a constant $C_2>0$ independent of k such that

$$\begin{aligned}&-\int(\sum_{j,l=1}^{2n-2}\frac{\partial^2\mathrm{Im}\,\varphi}{\partial x_j\partial x_l}(p,p,s)(x^j_k-y^j_k)(x^t_k-y^t_k))\left|\det\Big(M^\phi_p-2s\mathcal{L}_p\Big)\right|\left|\psi(s)\right|^2ds\\ &\leq-C_2\sum_{j=1}^{2n-2}(x^j_k-y^j_k)^2.\end{aligned}\tag{8.81}$$

From (8.79), (8.80) and (8.81), we deduce that

$$\limsup_{k\to\infty} k^{-2n}\big(\sqrt{k}\sum_{j=1}^{2n-2}\Big|x^j_k-y^j_k\Big|+k\,|\langle\,\omega_0(x_k),y_k-x_k\,\rangle|\big)^{-2}A''_k(t_k)\leq -C<0, \tag{8.82}$$

where $C>0$ is a constant.

Now, we have

$$H''_k(t_k)=\frac{A''_k(t_k)}{B_k(t_k)}-2\frac{A'_k(t_k)}{B^2_k(t_k)}B'_k(t_k)-\frac{A_k(t_k)}{B^2_k(t_k)}B''_k(t_k)+2\frac{A_k(t_k)(B'_k(t_k))^2}{B^3_k(t_k)}. \tag{8.83}$$

From (1.8), it is easy to see that

$$\begin{split}&\limsup_{k\to\infty}\Big(\big(\sqrt{k}\sum_{j=1}^{2n-2}\Big|x^j_k-y^j_k\Big|+k\,|\langle\,\omega_0(x_k),y_k-x_k\,\rangle|\big)^{-2}\times\\ &\big(-2\frac{A'_k(t_k)}{B^2_k(t_k)}B'_k(t_k)-\frac{A_k(t_k)}{B^2_k(t_k)}B''_k(t_k)+2\frac{A_k(t_k)(B'_k(t_k))^2}{B^3_k(t_k)}\big)\Big)=0.\end{split} \tag{8.84}$$

From (8.84), (8.83) and (8.82), we deduce that

$$\begin{split}&\limsup_{k\to\infty}(\big(\sqrt{k}\sum_{j=1}^{2n-2}\Big|x^j_k-y^j_k\Big|+k\,|\langle\,\omega_0(x_k),y_k-x_k\,\rangle|\big)^{-2}H''_k(t_k)\\ &=\limsup_{k\to\infty}\big(\sqrt{k}\sum_{j=1}^{2n-2}\Big|x^j_k-y^j_k\Big|+k\,|\langle\,\omega_0(x_k),y_k-x_k\,\rangle|\big)^{-2}\frac{A''_k(t_k)}{B_k(t_k)}\leq -C_0<0,\end{split} \tag{8.85}$$

where $C_0>0$ is a constant. But from (8.74), we see that

$$\limsup_{k\to\infty}(\big(\sqrt{k}\sum_{j=1}^{2n-2}\Big|x^j_k-y^j_k\Big|+k\,|\langle\,\omega_0(x_k),y_k-x_k\,\rangle|\big)^{-2}H''_k(t_k)\geq 0.$$

We get a contradiction. The theorem follows. □

Summing up, we obtain one of the main results of this work

Theorem 8.11. *Let $N_0>2n+1$. Then, for k large, the Kodaira map $\Phi_{N_0,k}:X\to\mathbb{CP}^{d_k-1}$ is an embedding.*

From Theorem 8.11, we deduce Theorem 1.4.

9. Asymptotic expansion of the Szegö kernel

We recall some notations we used before. Let s be a local trivializing section of L on an open subset $D\Subset X$ and $|s|^2_{h^L}=e^{-2\phi}$. Let $A_k:L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k)$ be a continuous operator. Let

$$\hat{A}_{k,s}:L^2_{(0,q)}(D)\bigcap\mathscr{E}'(D,T^{*0,q}X)\to L^2_{(0,q)}(D)$$

be the localized operator (with respect to the trivializing section s) of A_k given by (1.6). We write $A_k\equiv 0 \mod O(k^{-\infty})$ on D if $\hat{A}_{k,s}\equiv 0 \mod O(k^{-\infty})$ on D. Until further notice, we assume that $Y(q)$ holds on D. First, we need

Definition 9.1. Fix $q\in\{0,1,\ldots,n-1\}$. Let $A_k:L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k)$ be a continuous operator. Let $D\Subset X$. We say that $\Box^{(q)}_{b,k}$ has $O(k^{-n_0})$ small spectral gap on D with respect to A_k if for every $D'\Subset D$, there exist constants $C_{D'}>0$, $n_0,p\in\mathbb{N}$, $k_0\in\mathbb{N}$, such that for all $k\geq k_0$ and $u\in\Omega^{0,q}_0(D',L^k)$, we have

$$\Big\|A_k(I-\Pi^{(q)}_k)u\Big\|_{h^{L^k}}\leq C_{D'}\,k^{n_0}\sqrt{(\,(\Box^{(q)}_{b,k})^pu\,|\,u\,)_{h^{L^k}}}.$$

Definition 9.2. Let $A_k : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$ be a continuous operator. We say that $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to A_k on D if for any $\chi, \chi_1 \in C^\infty_0(D)$ with $\chi_1 = 1$ on some neighbourhood of $\operatorname{Supp}\chi$, we have

$$\Big(\chi A_k(1-\chi_1)\Big)\Pi^{(q)}_k\Big(\chi A_k(1-\chi_1)\Big)^* \equiv 0 \mod O(k^{-\infty}) \quad \text{on } D,$$

where

$$\Big(\chi A_k(1-\chi_1)\Big)^* : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$$

is the Hilbert space adjoint of $\chi A_k(1-\chi_1)$ with respect to $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$.

It is easy to see that if $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to A_k on D, then for any $\chi, \chi_1 \in C^\infty_0(D)$ with $\chi_1 = 1$ on some neighborhood of $\operatorname{Supp}\chi$, we have

$$\Big(\chi A_k(1-\chi_1)\Big)\Pi^{(q)}_k \equiv 0 \mod O(k^{-\infty}) \ \text{ on } D.$$

Definition 9.3. Let $A_k : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$ be a continuous operator. We say that A_k is a global classical semi-classical pseudodifferential operator of order m on X if for every local trivializing section s of L on an open subset $D \subset X$, the localized operator $\hat{A}_{k,s}$ is a classical semi-classical pseudodifferential operator of order m on D.

Proposition 9.4. *Let $A_k : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$ be a global classical semi-classical pseudo-differential operator on X of order* 0*. If X is compact and $Y(q)$ holds on X then $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to A_k on every local trivialization $D \Subset X$.*

Furthermore, if X is non-compact and A_k is properly supported on $D \Subset X$ and $Y(q)$ holds on D, where D is a local trivialization of X, then $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to A_k on D.

Proof. Let s be a local trivializing section of L on a local trivialization $D \subset X$. From Theorem 7.3, we can repeat the proof of Proposition 7.6 with minor change and conclude that for every $\alpha, \beta \in \mathbb{N}^{2n-1}_0$, and $D' \Subset D$, there is a constant $C_{\alpha,\beta,D'} > 0$ independent of k such that

$$\partial^\alpha_x \partial^\beta_y \Big(\hat{\Pi}^{(q)}_{k,s}(x,y)\Big) \leq C_{\alpha,\beta,D'} k^{n+|\alpha|+|\beta|} \ \text{ on } D' \times D'. \tag{9.1}$$

From (9.1) and by using integration by parts, the proposition can be deduced . We omit the details. $\square$

Now, we can prove

Theorem 9.5. *Let s be a local trivializing section of L on an open subset $D \subset X$ and $|s|^2_{h^L} = e^{-2\phi}$. We assume that there exist a $\lambda_0 \in \mathbb{R}$ and $x_0 \in D$ such that $M^\phi_{x_0} - 2\lambda_0\mathcal{L}_{x_0}$ is non-degenerate of constant signature (n_-, n_+). Let $q = n_-$ and assume that $Y(q)$ holds at each point of D. Let $F_k : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$ be a continuous operator and let $F^*_k : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$ be the Hilbert space adjoint of F_k with respect to $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$. Let $\hat{F}_{k,s}$ and $\hat{F}^*_{k,s}$ be the localized operators of $F_{k,s}$ and $F^*_{k,s}$ respectively. We fix $D_0 \Subset D$, D_0 open. Let V be as in* (5.14)*. Assume that*

$$\hat{F}_{k,s} - A_k = O(k^{-\infty}) : H^s_{\rm comp}(D, T^{*0,q}X) \to H^s_{\rm loc}(D, T^{*0,q}X), \quad \forall s \in \mathbb{N}_0,$$

where

$$A_k \equiv \tfrac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta \mod O(k^{-\infty}) \ \text{at } T^*D_0 \textstyle\bigcap \Sigma$$

is a classical semi-classical pseudodifferential operator on D of order 0 *from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$, where*

$$\alpha(x,\eta,k) \sim \textstyle\sum_{j=0} \alpha_j(x,\eta)k^{-j} \ \text{in } S^0_{\rm loc}(1; T^*D, T^{*0,q}X \boxtimes T^{*0,q}X),$$
$$\alpha_j(x,\eta) \in C^\infty(T^*D, T^{*0,q}D \boxtimes T^{*0,q}D), \quad j = 0, 1, \ldots,$$

with $\alpha(x,\eta,k) = 0$ if $|\eta| > M$, for some large $M > 0$ and $\operatorname{Supp}\alpha(x,\eta,k) \bigcap T^*D_0 \Subset V$. Put $P_k := F_k\Pi^{(q)}_k F^*_k$ and let $\hat{P}_{k,s}$ be the localized operator of P_k. If $\Box^{(q)}_{b,k}$ has $O(k^{-n_0})$ small spectral gap on D

with respect to F_k and $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to F_k on D, then

$$\hat{P}_{k,s}(x,y)\equiv\int e^{ik\varphi(x,y,s)}g(x,y,s,k)ds \mod O(k^{-\infty}) \tag{9.2}$$

on D_0, where $\varphi(x,y,s)\in C^\infty(\Omega)$ is as in Theorem 5.29, (2.3),

$$\begin{aligned}&g(x,y,s,k)\in S^n_{\mathrm{loc}}\big(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big)\bigcap C^\infty_0\big(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\\&g(x,y,s,k)\sim\sum^\infty_{j=0}g_j(x,y,s)k^{n-j}\ \text{in}\ S^n_{\mathrm{loc}}\big(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\\&g_j(x,y,s)\in C^\infty_0\big(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\quad j=0,1,2,\ldots,\end{aligned}\tag{9.3}$$

and for every $(x,x,s)\in\Omega$, $x\in D_0$,

$$\begin{aligned}&g_0(x,x,s)\\&=(2\pi)^{-n}\left|\det\big(M^\phi_x-2s\mathcal{L}_x\big)\right|\alpha_0(x,s\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x))\pi_{(x,s,n_-)}\alpha^*_0(x,s\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x)).\end{aligned}\tag{9.4}$$

Here

$$\begin{aligned}\Omega:=&\{(x,y,s)\in D\times D\times\mathbb{R};\,(x,-2\mathrm{Im}\,\overline{\partial}_b\phi(x)+s\omega_0(x))\in V\textstyle\bigcap\Sigma,\\&(y,-2\mathrm{Im}\,\overline{\partial}_b\phi(y)+s\omega_0(y))\in V\textstyle\bigcap\Sigma,\ |x-y|<\varepsilon,\ \text{for some}\ \varepsilon>0\},\end{aligned}$$

$\alpha^*_0(x,\eta):T^{*0,q}_xX\to T^{*0,q}_xX$ is the adjoint of $\alpha_0(x,\eta)$ with respect to the Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $T^{*0,q}_xX$, $\pi_{(x,s,n_-)}:T^{*0,q}_pX\to\mathcal{N}(x,s,n_-)$ is the orthogonal projection with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$, $\mathcal{N}(x,s,n_-)$ is given by (5.39),

$$\left|\det\big(M^\phi_x-2s\mathcal{L}_x\big)\right|=|\lambda_1(s)|\,|\lambda_2(s)|\cdots|\lambda_{n-1}(s)|\,,$$

$\lambda_1(s),\ldots,\lambda_{n-1}(s)$ are eigenvalues of the Hermitian quadratic form $M^\phi_x-2s_0\mathcal{L}_x$ with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$.

Proof. For simplicity, we assume that A_k is properly supported on D. Take $\chi,\chi_1\in C^\infty_0(D)$ with $\chi=1$ on D_0 and $\chi_1=1$ on some neighbourhood of $\mathrm{Supp}\,\chi$. Put

$$G_k=\chi F_k\chi_1,\quad H_k=\chi F_k(1-\chi_1),\quad B_k=F_k\Pi^{(q)}_k,\quad R_k=H_k\Pi^{(q)}_k,$$

and let $G^*_k,H^*_k:L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k)$ be the Hilbert adjoints of G_k and H_k respectively. Let $\hat{G}^*_{k,s}$, $\hat{G}_{k,s}$, $\hat{H}^*_{k,s}$, $\hat{H}_{k,s}$, $\hat{B}_{k,s}$, $\hat{R}_{k,s}$ be the localized operators of G^*_k, G_k, H^*_k, H_k, B_k, R_k respectively. Since $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to F_k on D, it is not difficult to see that

$$\begin{aligned}\hat{P}_{k,s}&\equiv\hat{G}_{k,s}\hat{\Pi}^{(q)}_{k,s}\hat{G}^*_{k,s}\mod O(k^{-\infty})\ \text{on}\ D_0,\\\hat{P}_{k,s}&\equiv A_k\hat{\Pi}^{(q)}_{k,s}A^*_k\mod O(k^{-\infty})\ \text{on}\ D_0,\end{aligned}\tag{9.5}$$

where A^*_k is the formal adjont of A_k. Let $\mathcal{S}_k$ and $\mathcal{N}_k$ be as in Theorem 6.5. Here we let $\hat{\mathcal{I}}_k=A^*_k$ in Theorem 6.5. Let $\Box^{(q)}_{s,k}$ be as in (4.4). Then,

$$\begin{aligned}&\Box^{(q)}_{s,k}\mathcal{N}_k+\mathcal{S}_k=A^*_k+h_k\ \text{ on }\mathscr{D}'(D_0,T^{*0,q}X),\\&\mathcal{N}^*_k\Box^{(q)}_{s,k}+\mathcal{S}^*_k=A_k+h^*_k\ \text{ on }\mathscr{D}'(D_0,T^{*0,q}X),\end{aligned}\tag{9.6}$$

where $h_k\equiv0\mod O(k^{-\infty})$, $\mathcal{N}^*_k$, $\mathcal{S}^*_k$ and h^*_k are formal adjoints of $\mathcal{N}_k$, $\mathcal{S}_k$ and h_k with respect to $(\,\cdot\,|\,\cdot\,)$ respectively. From (9.6) and notice that $\Box^{(q)}_{s,k}\hat{\Pi}^{(q)}_{k,s}=0$, it is not difficult to see that

$$\begin{aligned}&\mathcal{S}^*_k\hat{\Pi}^{(q)}_{k,s}=(A_k+h^*_k)\hat{\Pi}^{(q)}_{k,s}\ \text{ on }\mathscr{E}'(D_0,T^{*0,q}X),\\&\hat{\Pi}^{(q)}_{k,s}\mathcal{S}_k=\hat{\Pi}^{(q)}_{k,s}(A^*_k+h_k)\ \text{ on }\mathscr{E}'(D_0,T^{*0,q}X).\end{aligned}\tag{9.7}$$

Let $u\in H^m_{\mathrm{comp}}(D_0,T^{*0,q}X)$, $m\le0$, $m\in\mathbb{Z}$. We consider

$$v=s^ke^{k\phi}\mathcal{S}_ku-\Pi^{(q)}_k(s^ke^{k\phi}\mathcal{S}_ku).$$

Since $Y(q)$ holds on D and $\mathcal{S}_k$ is a smoothing operator, we conclude that $v \in L^2_{(0,q)}(X, L^k) \bigcap \Omega^{0,q}(D)$. Moreover, from (4.1), we have

$$\Box^{(q)}_{b,k} v = s^k e^{k\phi} \Box^{(q)}_{s,k} \mathcal{S}_k u. \tag{9.8}$$

In view of Theorem 6.5, we see that $\Box^{(q)}_{s,k} \mathcal{S}_k \equiv 0 \mod O(k^{-\infty})$. Combining this with (9.8), we obtain for every $p \in \mathbb{N}$,

$$\left\| (\Box^{(q)}_{b,k})^p v \right\|_{h^{L^k}} \leq C_{N,p} k^{-N} \left\| u \right\|_m, \tag{9.9}$$

for every $N > 0$, where $C_{N,p} > 0$ is independent of k, u and $\|\cdot\|_m$ denotes the usual Sobolev norm of order m on D_0 with respect to $(\cdot\,|\,\cdot)$. Moreover, from the explicit formula of the kernel of $\mathcal{S}_k$ (see (6.29)), it is straightforward to see that

$$\|v\|_{h^{L^k}} \leq C k^{n+m} \|u\|_m, \tag{9.10}$$

where $C > 0$ is a constant independent of k and u. Note that $\Box^{(q)}_{b,k}$ has $O(k^{-n_0})$ small spectral gap on D with respect to F_k and $\Pi^{(q)}_k v = 0$. From this observation, (9.9) and (9.10), we conclude that $\|F_k v\|_{h^{L^k}} \leq \widetilde{C}_N k^{-N} \|u\|_m$, for every $N > 0$, where $\widetilde{C}_N > 0$ is independent of k. Thus,

$$\hat{F}_{k,s} \mathcal{S}_k - \hat{B}_{k,s} \mathcal{S}_k = O(k^{-N}) : H^m_{\mathrm{comp}}(D_0, T^{*0,q}X) \to L^2_{(0,q)}(D_0), \tag{9.11}$$

for all $N > 0$, $m \in \mathbb{Z}$, $m \leq 0$. Since $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to F_k on D and notice that $\hat{H}_{k,s} \mathcal{S}_k \equiv 0 \mod O(k^{-\infty})$, we conclude that

$$\hat{H}_{k,s} \mathcal{S}_k - \hat{R}_{k,s} \mathcal{S}_k \equiv 0 \mod O(k^{-\infty}) \text{ on } D_0 \tag{9.12}$$

and hence

$$\hat{F}_{k,s} \mathcal{S}_k - \hat{B}_{k,s} \mathcal{S}_k \equiv \hat{G}_{k,s} \mathcal{S}_k - \hat{G}_{k,s} \hat{\Pi}^{(q)}_{k,s} \mathcal{S}_k \mod O(k^{-\infty}) \text{ on } D_0. \tag{9.13}$$

From (9.13) and (9.11), we obtain

$$\begin{aligned} &\hat{G}_{k,s} \mathcal{S}_k - \hat{G}_{k,s} \hat{\Pi}^{(q)}_{k,s} \mathcal{S}_k = O(k^{-N}) : H^m_{\mathrm{comp}}(D_0, T^{*0,q}X) \to L^2_{(0,q)}(D_0), \\ &A_k \mathcal{S}_k - A_k \hat{\Pi}^{(q)}_{k,s} \mathcal{S}_k = O(k^{-N}) : H^m_{\mathrm{comp}}(D_0, T^{*0,q}X) \to L^2_{(0,q)}(D_0), \end{aligned} \tag{9.14}$$

for all $N > 0$, $m \in \mathbb{Z}$, $m \leq 0$. Put

$$\begin{aligned} &A_k \equiv \widetilde{\mathcal{I}}_k + \widetilde{\mathcal{I}}^1_k \mod O(k^{-\infty}) \text{ on } D_0, \\ &\widetilde{\mathcal{I}}_k \equiv \frac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik<x-y,\eta>} \alpha(x,\eta,k) d\eta \mod O(k^{-\infty}) \text{ on } D_0, \\ &\widetilde{\mathcal{I}}^1_k \equiv \frac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik<x-y,\eta>} \beta(x,y,\eta,k) d\eta \text{ on } D_0, \end{aligned} \tag{9.15}$$

where $\widetilde{\mathcal{I}}_k$ and $\widetilde{\mathcal{I}}^1_k$ are properly supported on D_0, $\beta(x,y,\eta,k) \in S^0_{\mathrm{loc,cl}}(1; T^*D, T^{*0,q}X \boxtimes T^{*0,q}X)$ and there is a small neighbourhood Γ of $T^*D_0 \bigcap \Sigma$ such that $\beta(x,y,\eta,k) = 0$ if $(x,\eta) \in \Gamma$. Since $\beta(x,y,\eta,k) = 0$ if (x,η) near T^*D_0 and notice that $\mathcal{F}\hat{\Pi}^{(q)}_{k,s} \equiv 0 \mod O(k^{-\infty})$ on D_0 if $\mathcal{F}$ is a properly supported k-negligible operator on D_0, we deduce that $\widetilde{\mathcal{I}}^1_k \hat{\Pi}^{(q)}_{k,s} \equiv 0 \mod O(k^{-\infty})$ on D_0. Moreover, it is not difficult to see that $\widetilde{\mathcal{I}}^1_k \mathcal{S}_k \equiv 0 \mod O(k^{-\infty})$ on D_0. Combining these with (9.15), we obtain

$$\begin{aligned} &A_k \mathcal{S}_k \equiv \widetilde{\mathcal{I}}_k \mathcal{S}_k \mod O(k^{-\infty}) \text{ on } D_0, \\ &A_k \hat{\Pi}^{(q)}_{k,s} \equiv \widetilde{\mathcal{I}}_k \hat{\Pi}^{(q)}_{k,s} \mod O(k^{-\infty}) \text{ on } D_0, \\ &A_k \mathcal{S}_k - A_k \hat{\Pi}^{(q)}_{k,s} \mathcal{S}_k \equiv \widetilde{\mathcal{I}}_k \mathcal{S}_k - \widetilde{\mathcal{I}}_k \hat{\Pi}^{(q)}_{k,s} \mathcal{S}_k \mod O(k^{-\infty}) \text{ on } D_0. \end{aligned} \tag{9.16}$$

From (9.16) and (9.14), we deduce

$$\widetilde{\mathcal{I}}_k \mathcal{S}_k - \widetilde{\mathcal{I}}_k \hat{\Pi}^{(q)}_{k,s} \mathcal{S}_k = O(k^{-N}) : H^m_{\mathrm{comp}}(D_0, T^{*0,q}X) \to L^2_{(0,q)}(D_0), \tag{9.17}$$

for all $N > 0$, $m \in \mathbb{Z}$, $m \leq 0$. Take

$$\gamma(x,\eta,k) \in S^0_{\mathrm{loc},\mathrm{cl}}(1; T^*D, T^{*0,q}X \boxtimes T^{*0,q}X) \bigcap C^\infty_0(V, T^{*0,q}X \boxtimes T^{*0,q}X)$$

so that $\gamma(x,\eta,k) = 1$ on $\mathrm{Supp}\,\alpha(x,\eta,k) \bigcap T^*D_0$ and let $\Gamma_k \equiv \int e^{i<x-y,\eta>}\gamma(x,\eta,k)d\eta \mod O(k^{-\infty})$ on D_0 be a properly supported classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$. Since $\gamma(x,\eta,k) = 1$ on $\mathrm{Supp}\,\alpha(x,\eta,k) \bigcap T^*D_0$, we have

$$\Gamma_k \widetilde{\mathcal{I}}_k \equiv \widetilde{\mathcal{I}}_k \mod O(k^{-\infty}) \text{ on } D_0 \tag{9.18}$$

and hence

$$\widetilde{\mathcal{I}}_k \mathcal{S}_k - \widetilde{\mathcal{I}}_k \hat{\Pi}^{(q)}_{k,s} \mathcal{S}_k \equiv \Gamma_k \big(\widetilde{\mathcal{I}}_k \mathcal{S}_k - \widetilde{\mathcal{I}}_k \hat{\Pi}^{(q)}_{k,s} \mathcal{S}_k\big) \mod O(k^{-\infty}) \text{ on } D_0. \tag{9.19}$$

Since $\mathrm{Supp}\,\gamma(x,\eta,k) \Subset V$, Γ_k is a smoothing operator and we can check that

$$\Gamma_k = O(k^s) : H^0_{\mathrm{loc}}(D_0, T^{*0,q}X) \to H^s_{\mathrm{loc}}(D_0, T^{*0,q}X), \tag{9.20}$$

for every $s \in \mathbb{N}_0$. Combining (9.20), (9.19) with (9.17), we deduce that

$$\widetilde{\mathcal{I}}_k \mathcal{S}_k - \widetilde{\mathcal{I}}_k \hat{\Pi}^{(q)}_{k,s} \mathcal{S}_k \equiv 0 \mod O(k^{-\infty}) \text{ on } D_0. \tag{9.21}$$

From (9.21), (9.16), (9.7) and note that $\hat{\Pi}^{(q)}_{k,s} h_k \equiv 0 \mod O(k^{-\infty})$, we get

$$A_k \mathcal{S}_k - A_k \hat{\Pi}^{(q)}_{k,s} A^*_k \equiv 0 \mod O(k^{-\infty}) \text{ on } D_0. \tag{9.22}$$

From (9.6), we have $\mathcal{S}^*_k \mathcal{S}_k \equiv A_k \mathcal{S}_k \mod O(k^{-\infty})$ on D_0. From this, (6.40), (6.58), (9.22) and (9.5), the theorem follows. $\square$

By using Theorem 6.10 and repeat the proof of Theorem 9.5, we deduce

Theorem 9.6. *Let s be a local trivializing section of L on an open subset $D \subset X$ and $|s|^2_{h^L} = e^{-2\phi}$. We assume that there exist a $\lambda_0 \in \mathbb{R}$ and $x_0 \in D$ such that $M^\phi_{x_0} - 2\lambda_0 \mathcal{L}_{x_0}$ is non-degenerate of constant signature (n_-, n_+). Let $q \neq n_-$ and assume that $Y(q)$ holds at each point of D. Let $F_k : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$ be a continuous operator and let $F^*_k : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$ be the Hilbert space adjoint of F_k with respect to $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$. Let $\hat{F}_{k,s}$ and $\hat{F}^*_{k,s}$ be the localized operators of $F_{k,s}$ and $F^*_{k,s}$ respectively. We fix $D_0 \Subset D$, D_0 open. Let V be as in (5.14). Assume that*

$$\hat{F}_{k,s} - A_k = O(k^{-\infty}) : H^s_{\mathrm{comp}}(D, T^{*0,q}X) \to H^s(D, T^{*0,q}X), \quad \forall s \in \mathbb{N}_0,$$

where

$$A_k \equiv \tfrac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta \mod O(k^{-\infty}) \text{ at } T^*D_0 \bigcap \Sigma$$

is a classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$, where

$$\alpha(x,\eta,k) \sim \textstyle\sum_{j=0} \alpha_j(x,\eta)k^{-j} \text{ in } S^0_{\mathrm{loc}}(1; T^*D, T^{*0,q}X \boxtimes T^{*0,q}X),$$
$$\alpha_j(x,\eta) \in C^\infty(T^*D, T^{*0,q}D \boxtimes T^{*0,q}D), \quad j = 0, 1, \ldots,$$

with $\alpha(x,\eta,k) = 0$ if $|\eta| > M$, for some large $M > 0$ and $\mathrm{Supp}\,\alpha(x,\eta,k) \bigcap T^*D_0 \Subset V$. Put $P_k := F_k \Pi^{(q)}_k F^*_k$ and let $\hat{P}_{k,s}$ be the localized operator of P_k. If $\Box^{(q)}_{b,k}$ has $O(k^{-n_0})$ small spectral gap on D with respect to F_k and $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to F_k on D, then

$$\hat{P}_{k,s} \equiv 0 \mod O(k^{-\infty}) \text{ on } D_0. \tag{9.23}$$

10. Sezgö kernel asymptotics and Kodairan embedding Theorems on CR manifolds with transversal CR S^1 actions

In this section, we will offer some special classes of CR manifolds and CR line bundles such that the conditions in Theorem 9.5 hold.

Let $(X, T^{1,0}X)$ be a CR manifold. We assume that X admits a S^1 action: $S^1 \times X \to X$. We write $e^{i\theta}$, $0 \leq \theta < 2\pi$, to denote the S^1 action. Let $T \in C^\infty(X, TX)$ be the real vector field given by

$$Tu = \frac{\partial}{\partial\theta}(u(e^{i\theta}x))|_{\theta=0}, \quad u \in C^\infty(X). \tag{10.1}$$

We call T the global vector field induced by the S^1 action. Note that we don't assume that this S^1 action is globally free.

Definition 10.1. We say that the S^1 action $e^{i\theta}$, $0 \leq \theta < 2\pi$, is CR if

$$[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X).$$

Furthermore, we say that the S^1 action $e^{i\theta}$, $0 \leq \theta < 2\pi$, is transversal if for every point $x \in X$,

$$T(x) \oplus T^{1,0}_x X \oplus T^{0,1}_x X = \mathbb{C}T_x X.$$

Until further notice, we assume that $(X, T^{1,0}X)$ is a CR manifold with a transversal CR S^1 action $e^{i\theta}$, $0 \leq \theta < 2\pi$ and we let T be the global vector field induced by the S^1 action.

Fix $\theta_0 \in [0, 2\pi[$. Let

$$de^{i\theta_0} : \mathbb{C}T_x X \to \mathbb{C}T_{e^{i\theta_0}x} X$$

denote the differential of the map $e^{i\theta_0} : X \to X$.

Definition 10.2. Let $U \subset X$ be an open set and let $V \in C^\infty(U, \mathbb{C}TX)$ be a vector field on U. We say that V is T-rigid if

$$de^{i\theta_0} V(x) = V(x), \quad \forall x \in e^{i\theta_0} U \bigcap U,$$

for every $\theta_0 \in [0, 2\pi[$ with $e^{i\theta_0} U \bigcap U \neq \emptyset$.

We also need

Definition 10.3. Let $\langle \cdot \,|\, \cdot \rangle$ be a Hermitian metric on $\mathbb{C}TX$. We say that $\langle \cdot \,|\, \cdot \rangle$ is T-rigid if for T-rigid vector fields V and W on U, where $U \subset X$ is any open set, we have

$$\langle\, V(x) \,|\, W(x) \,\rangle = \langle\, de^{i\theta_0} V(e^{i\theta_0}x) \,|\, de^{i\theta_0} W(e^{i\theta_0}x) \,\rangle, \quad \forall x \in U, \theta_0 \in [0, 2\pi[.$$

We are going to show that there exists a T-rigid Hermitian metric on $\mathbb{C}TX$. We need the following result due to Baouendi-Rothschild-Treves [1, section1]

Theorem 10.4. *For every point $x_0 \in X$, there exists local coordinates $x = (x_1, \ldots, x_{2n-1}) = (z, \theta) = (z_1, \ldots, z_{n-1}, \theta)$, $z_j = x_{2j-1} + i x_{2j}$, $j = 1, \ldots, n-1$, $\theta = x_{2n-1}$, defined in some small neighbourhood U of x_0 such that*

$$\begin{aligned} &T = \frac{\partial}{\partial\theta}, \\ &Z_j = \frac{\partial}{\partial z_j} + i\frac{\partial\varphi}{\partial z_j}(z)\frac{\partial}{\partial\theta}, \quad j = 1, \ldots, n-1, \end{aligned} \tag{10.2}$$

where $Z_j(x)$, $j = 1, \ldots, n-1$, form a basis of $T^{1,0}_x X$, for each $x \in U$, and $\varphi(z) \in C^\infty(U, \mathbb{R})$ independent of θ.

Let x and U be as in Theorem 10.4. We call x canonical coordinates and U canonical coordinate patch.

Theorem 10.5. *There is a T-rigid Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X \perp T^{0,1}X$, $T \perp (T^{1,0}X \oplus T^{0,1}X)$, $\langle\, T \,|\, T \,\rangle = 1$ and $\langle\, u \,|v\, \rangle$ is real if u, v are real tangent vectors.*

Proof. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be any Hermitian metric on $\mathbb{C}TX$ such that $T^{1,0}X \perp T^{0,1}X$, $T \perp (T^{1,0}X \oplus T^{0,1}X)$, $\langle\, T\,,\, T\,\rangle = 1$ and $\overline{\langle\, u\,,\, v\,\rangle} = \langle\, \overline{u}\,,\, \overline{v}\,\rangle$, for all $u, v \in \mathbb{C}TX$. Let x and U be as in Theorem 10.4. On U, define

$$\langle\, Z_j \,|\, Z_t\,\rangle := \int_0^{2\pi} \langle\, de^{i\theta} Z_j\,,\, de^{i\theta} Z_t\,\rangle d\theta, \quad j, t = 1, \ldots, n-1, \tag{10.3}$$

where Z_j, $j = 1, \ldots, n-1$ are as in (10.2). Since $Z_j(x)$, $j = 1, \ldots, n-1$, form a basis of $T^{1,0}_x X$, for each $x \in U$, On U, (10.3) defines a T-rigid Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $T^{1,0}X$. We claim that the definition above is independent of the choice of canonical coordinates. Let $y = (y_1, \ldots, y_{2n-1}) = (w, \gamma)$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \ldots, n-1$, $\gamma = y_{2n-1}$, be another canonical coordinates on U. Then,

$$\begin{aligned} &T = \frac{\partial}{\partial \gamma}, \\ &\widetilde{Z}_j = \frac{\partial}{\partial w_j} + i\frac{\partial \widetilde{\varphi}}{\partial w_j}(w)\frac{\partial}{\partial \gamma}, \quad j = 1, \ldots, n-1, \end{aligned} \tag{10.4}$$

where $\widetilde{Z}_j(y)$, $j = 1, \ldots, n-1$, form a basis of $T^{1,0}_y X$, for each $y \in U$, and $\widetilde{\varphi}(w) \in C^\infty(U, \mathbb{R})$ independent of γ. As (10.3), on U, we define

$$\langle\, \widetilde{Z}_j \,|\, \widetilde{Z}_t\,\rangle_1 := \int_0^{2\pi} \langle\, de^{i\theta} \widetilde{Z}_j\,,\, de^{i\theta} \widetilde{Z}_t\,\rangle d\theta, \quad j, t = 1, \ldots, n-1. \tag{10.5}$$

On U, (10.5) defines a T-rigid Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle_1$ on $T^{1,0}X$. We claim that $\langle\,\cdot\,|\,\cdot\,\rangle_1 = \langle\,\cdot\,|\,\cdot\,\rangle$. From (10.4) and (10.2), it is not difficult to see that

$$\begin{aligned} &w = (w_1, \ldots, w_{n-1}) = (H_1(z), \ldots, H_{n-1}(z)) = H(z), \quad H_j(z) \in C^\infty, \quad \forall j, \\ &\gamma = \theta + G(z), \quad G(z) \in C^\infty, \end{aligned} \tag{10.6}$$

where for each $j = 1, \ldots, n-1$, $H_j(z)$ is holomorphic. From (10.2), (10.4) and (10.6), it is not difficult to see that

$$\begin{aligned} &\widetilde{Z}_j = \sum_{t=1}^{n-1} c_{j,t}(x) Z_t, \quad c_{j,t} \in C^\infty(U), \quad j, t = 1, \ldots, n-1, \\ &(c_{j,t}(x))_{j,t=1}^{n-1} \text{ is invertible at every } x \in U, \\ &Tc_{j,t} = 0, \quad j, t = 1, \ldots, n-1. \end{aligned} \tag{10.7}$$

Let $\Gamma, \Lambda \in C^\infty(U, T^{1,0}X)$. We write

$$\begin{aligned} &\Gamma = \sum_{j=1}^{n-1} a_j(x) Z_j = \sum_{j=1}^{n-1} \widetilde{a}_j(y) \widetilde{Z}_j, \quad a_j, \widetilde{a}_j \in C^\infty(U), \quad j = 1, \ldots, n-1, \\ &\Lambda = \sum_{j=1}^{n-1} b_j(x) Z_j = \sum_{j=1}^{n-1} \widetilde{b}_j(y) \widetilde{Z}_j, \quad b_j, \widetilde{b}_j \in C^\infty(U), \quad j = 1, \ldots, n-1. \end{aligned} \tag{10.8}$$

From (10.8) and (10.7), we can check that

$$\begin{aligned} &a_t = \sum_{j=1}^{n-1} \widetilde{a}_j c_{j,t}, \quad t = 1, \ldots, n-1, \\ &b_t = \sum_{j=1}^{n-1} \widetilde{b}_j c_{j,t}, \quad t = 1, \ldots, n-1. \end{aligned} \tag{10.9}$$

Now, by definition,

$$
\begin{aligned}
\langle\,\Gamma\,|\,\Lambda\,\rangle_1 &= \sum_{j,t=1}^{n-1}\widetilde{a}_j\overline{\widetilde{b}_t}\int_0^{2\pi}\langle\, de^{i\theta}\widetilde{Z}_j\,,\, de^{i\theta}\widetilde{Z}_t\,\rangle d\theta\\
&= \sum_{j,t=1}^{n-1}\widetilde{a}_j\overline{\widetilde{b}_t}\int_0^{2\pi}\langle\, de^{i\theta}(\sum_{s=1}^{n-1}c_{j,s}Z_s)\,,\, de^{i\theta}(\sum_{u=1}^{n-1}c_{t,u}Z_u)\,\rangle d\theta\\
&= \sum_{j,t=1}^{n-1}\sum_{s,u=1}^{n-1}c_{j,s}\overline{c}_{t,u}\widetilde{a}_j\overline{\widetilde{b}_t}\int_0^{2\pi}\langle\, de^{i\theta}Z_s\,,\, de^{i\theta}Z_u\,\rangle d\theta\\
&= \sum_{s,u=1}^{n-1}a_s\overline{b_u}\int_0^{2\pi}\langle\, de^{i\theta}Z_s\,,\, de^{i\theta}Z_u\,\rangle d\theta\\
&= \langle\,\Gamma\,|\,\Lambda\,\rangle.
\end{aligned}
\tag{10.10}
$$

Here we used (10.7), (10.9) and $de^{i\theta}(\sum_{s=1}^{n-1}c_{j,s}Z_s)=\sum_{s=1}^{n-1}c_{j,s}de^{i\theta}Z_s$, $s=1,\ldots,n-1$, since $Tc_{j,s}=0$, $j,s=1,\ldots,n-1$. Thus, (10.3) defines a T-rigid Hermitian metric on $T^{1,0}X$. We extend $\langle\,\cdot\,|\,\cdot\,\rangle$ to a T-rigid Hermitian metric on $\mathbb{C}TX$ by

$$
\begin{aligned}
&\langle\, u\,|\,v\,\rangle=\overline{\langle\,\overline{u}\,|\,\overline{v}\,\rangle},\quad u,v\in T^{0,1}X,\\
&T\perp(T^{1,0}X\oplus T^{0,1}X),\quad \langle\,T\,|\,T\,\rangle=1.
\end{aligned}
$$

The theorem follows. □

Until further notice, we fix a T-rigid Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X\perp T^{0,1}X$, $T\perp(T^{1,0}X\oplus T^{0,1}X)$, $\langle\,T\,|\,T\,\rangle=1$ and $\langle\,u\,|v\,\rangle$ is real if u,v are real tangent vectors. The Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0,q)$ forms $T^{*0,q}X$, $q=0,1,\ldots,n-1$. As before, we denote all these induced metrics by $\langle\,\cdot\,|\,\cdot\,\rangle$.

Definition 10.6. Let U be an open subset of X. A function $u\in C^\infty(U)$ is said to be a T-rigid CR function on U if $Tu=0$ and $Zu=0$ for all $Z\in C^\infty(U,T^{0,1}X)$.

Definition 10.7. Let L be a CR line bundle over $(X,T^{1,0}X)$. We say that L is a T-rigid CR line bundle over $(X,T^{1,0}X)$ if X can be covered with open sets U_j with trivializing sections s_j, $j=1,2,\ldots$, such that the corresponding transition functions are T-rigid CR functions.

Until further notice, we assume that L is a T-rigid CR line bundle over $(X,T^{1,0}X)$. Then, by definition, X can be covered with open sets U_j with trivializing sections s_j, $j=1,2,\ldots$, such that the corresponding transition functions are T-rigid CR functions. In this section, when trivializing sections s are used, we will assume that they are of this special form.

Fix a Hermitian fiber metric h^L on L and we will denote by ϕ the local weights of the Hermitian metric h^L as (3.5). Since the transition functions are T-rigid CR functions, we can check that $T\phi$ is a well-defined global smooth function on X and the Hermitian quadratic form M^ϕ_x is globally defined for every $x\in X$(see Definition 3.4 and Proposition 3.5).

Definition 10.8. h^L is said to be a T-rigid Hermitian fiber metric on L if $T\phi=0$.

Until further notice, we assume that h^L is a T-rigid Hermitian fiber metric on L and X is compact. For $k>0$, as before, we shall consider (L^k,h^{L^k}) and we will use the same notations as before. Since the transition functions are T-rigid CR functions, Tu is well-defined, for every $u\in\Omega^{0,q}(X,L^k)$. For $m\in\mathbb{Z}$, put

$$
A^{0,q}_m(X,L^k):=\big\{u\in\Omega^{0,q}(X,L^k);\,Tu=imu\big\}
\tag{10.11}
$$

and let $\mathcal{A}^{0,q}_m(X,L^k)\subset L^2_{(0,q)}(X,L^k)$ be the completion of $A^{0,q}_m(X,L^k)$ with respect to $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$. It is easy to see that for any $m,m'\in\mathbb{Z}$, $m\neq m'$,

$$
(\,u\,|\,v)_{h^{L^k}}=0,\quad \forall u\in\mathcal{A}^{0,q}_m(X,L^k),v\in\mathcal{A}^{0,q}_{m'}(X,L^k).
\tag{10.12}
$$

For $m\in\mathbb{Z}$, let

$$Q^{(q)}_{m,k}:L^2_{(0,q)}(X,L^k)\to\mathcal{A}^{0,q}_m(X,L^k)\tag{10.13}$$

be the orthogonal projection with respect to $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$. Fix $\delta>0$. Take $\tau_\delta(x)\in C^\infty_0(]-\delta,\delta[)$, $0\leq\tau_\delta\leq 1$ and $\tau_\delta=1$ on $[-\frac{\delta}{2},\frac{\delta}{2}]$. Let $F^{(q)}_{\delta,k}:L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k)$ be the continuous map given by

$$\begin{split}&F^{(q)}_{\delta,k}:L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k),\\&\qquad u\to\sum_{m\in\mathbb{Z}}\tau_\delta(\frac{m}{k})(Q^{(q)}_{m,k}u).\end{split}\tag{10.14}$$

It is easy to see that $F^{(q)}_{\delta,k}$ is well-defined. Moreover, it is not difficult to see that for every $m\in\mathbb{Z}$, we have

$$\begin{split}&\left\|TQ^{(q)}_{m,k}u\right\|_{h^{L^k}}=|m|\left\|Q^{(q)}_{m,k}u\right\|_{h^{L^k}},\ \ \forall u\in L^2_{(0,q)}(X,L^k),\\&\left\|TF^{(q)}_{\delta,k}u\right\|_{h^{L^k}}\leq k\delta\left\|F^{(q)}_{\delta,k}u\right\|_{h^{L^k}},\ \ \forall u\in L^2_{(0,q)}(X,L^k),\end{split}\tag{10.15}$$

and

$$\begin{split}&Q^{(q)}_{m,k}:\Omega^{0,q}(X,L^k)\to A^{0,q}_m(X,L^k),\\&F^{(q)}_{\delta,k}:\Omega^{0,q}(X,L^k)\to\bigcup_{-k\delta\leq m\leq k\delta}A^{0,q}_m(X,L^k).\end{split}\tag{10.16}$$

Since the Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ and h^{L^k} are all T-rigid, it is straightforward to see that (see section 5 in [17])

$$\begin{split}&\Box^{(q)}_{b,k}Q^{(q)}_{m,k}=Q^{(q)}_{m,k}\Box^{(q)}_{b,k}\text{ on }\Omega^{0,q}(X,L^k),\ \forall m\in\mathbb{Z},\\&\Box^{(q)}_{b,k}F^{(q)}_{\delta,k}=F^{(q)}_{\delta,k}\Box^{(q)}_{b,k}\text{ on }\Omega^{0,q}(X,L^k),\\&\overline{\partial}_{b,k}Q^{(q)}_{m,k}=Q^{(q+1)}_{m,k}\overline{\partial}_{b,k}\text{ on }\Omega^{0,q}(X,L^k),\ \forall m\in\mathbb{Z},\ q=0,1,\ldots,n-2,\\&\overline{\partial}_{b,k}F^{(q)}_{\delta,k}=F^{(q+1)}_{\delta,k}\overline{\partial}_{b,k}\text{ on }\Omega^{0,q}(X,L^k),\ q=0,1,\ldots,n-2,\\&\overline{\partial}^*_{b,k}Q^{(q)}_{m,k}=Q^{(q-1)}_{m,k}\overline{\partial}^*_{b,k}\text{ on }\Omega^{0,q}(X,L^k),\ \forall m\in\mathbb{Z},\ q=1,\ldots,n-1,\\&\overline{\partial}^*_{b,k}F^{(q)}_{\delta,k}=F^{(q-1)}_{\delta,k}\overline{\partial}^*_{b,k}\text{ on }\Omega^{0,q}(X,L^k),\ q=1,\ldots,n-1.\end{split}\tag{10.17}$$

By using elementary Fourier analysis, it is straightforward to see that for every $u\in\Omega^{0,q}(X,L^k)$,

$$\begin{split}&\lim_{N\to\infty}\sum^N_{m=-N}Q^{(q)}_{m,k}u\to u\text{ in }C^\infty\text{ Topology},\\&\sum^N_{m=-N}\left\|Q^{(q)}_{m,k}u\right\|^2_{h^{L^k}}\leq\|u\|^2_{h^{L^k}},\ \ \forall N\in\mathbb{N}_0.\end{split}\tag{10.18}$$

Thus, for every $u\in L^2_{(0,q)}(X,L^k)$,

$$\begin{split}&\lim_{N\to\infty}\sum^N_{m=-N}Q^{(q)}_{m,k}u\to u\text{ in }L^2_{(0,q)}(X,L^k),\\&\sum^N_{m=-N}\left\|Q^{(q)}_{m,k}u\right\|^2_{h^{L^k}}\leq\|u\|^2_{h^{L^k}},\ \ \forall N\in\mathbb{N}_0.\end{split}\tag{10.19}$$

Now, we assume that M^ϕ_x is non-degenerate of constant signature (n_-,n_+), for every $x\in X$. The following is essentially follows from Kohn's L^2 estimates (see Chen-Shaw [6]). We omit the proof.

Theorem 10.9. *With the assumptions and notations above, let $q \neq n_-$. For every $u \in \Omega^{0,q}(X, L^k)$, we have*

$$\left\|\Box^{(q)}_{b,k}u\right\|^2_{h^{L^k}} + k\left|(\,Tu\,|\,u\,)_{h^{L^k}}\right| \geq ck^2 \left\|u\right\|^2_{h^{L^k}}\,, \tag{10.20}$$

where $c > 0$ is a constant independent of k and u.

From (10.15) and (10.20), we deduce

Theorem 10.10. *With the assumptions and notations above, let $q \neq n_-$. If $\delta > 0$ is small enough, then for every $u \in \Omega^{0,q}(X, L^k)$, we have*

$$\left\|\Box^{(q)}_{b,k}(F^{(q)}_{\delta,k}u)\right\|^2_{h^{L^k}} \geq c_1k^2 \left\|F^{(q)}_{\delta,k}u\right\|^2_{h^{L^k}}\,, \tag{10.21}$$

where $c_1 > 0$ is a constant independent of k and u.

Now, we assume that $Y(q)$ holds at each point of X. Since X is compact, by the classical result of Kohn [6, Props. 8.4.8-9], condition $Y(q)$ implies that $\Box^{(q)}_{b,k}$ is hypoelliptic, has compact resolvent and the strong Hodge decomposition holds. Let $\operatorname{Spec}\Box^{(q)}_{b,k}$ denote the spectrum of $\Box^{(q)}_{b,k}$. Then $\operatorname{Spec}\Box^{(q)}_{b,k}$ is a discrete subset of $[0,\infty[$, $\operatorname{Spec}\Box^{(q)}_{b,k}$ is the set of all eigenvalues of $\Box^{(q)}_{b,k}$. For $\mu \in \operatorname{Spec}\Box^{(q)}_{b,k}$, put

$$H^q_{b,\mu}(X, L^k) = \left\{u \in L^2_{(0,q)}(X, L^k);\, \Box^{(q)}_{b,k}u = \mu u\right\} \tag{10.22}$$

and let

$$\Pi^{(q)}_{k,\mu} : L^2_{(0,q)}(X, L^k) \to H^q_{b,\mu}(X, L^k) \tag{10.23}$$

be the orthogonal projection.

We notice that $H^q_{b,\mu}(X, L^k) \subset \Omega^{0,q}(X, L^k)$, $\forall \mu \in \operatorname{Spec}\Box^{(q)}_{b,k}$ and for every $\lambda \geq 0$,

$$H^q_{b,\leq\lambda}(X, L^k) = \oplus_{\mu\in\operatorname{Spec}\Box^{(q)}_{b,k},0\leq\mu\leq\lambda}H^q_{b,\mu}(X, L^k). \tag{10.24}$$

Theorem 10.11. *With the assumptions and notations above, let $q = n_-$. If $\delta > 0$ is small enough, then for every $u \in \Omega^{0,q}(X, L^k)$, we have*

$$F^{(q)}_{\delta,k}\Pi^{(q)}_{\mu,k}u = 0, \quad \forall \mu \in \operatorname{Spec}\Box^{(q)}_{b,k},\ 0 < \mu \leq k\delta, \tag{10.25}$$

and

$$\left\|F^{(q)}_{\delta,k}(I - \Pi^{(q)}_k)u\right\|_{h^{L^k}} \leq \frac{1}{k\delta}\left\|\Box^{(q)}_{b,k}u\right\|_{h^{L^k}}. \tag{10.26}$$

In particular, if $\delta > 0$ is small enough then for every $D \Subset X$, $\Box^{(q)}_{b,k}$ has $O(k^{-n_0})$ small spectral gap on D with respect to $F^{(q)}_{\delta,k}$ in the sense of Definition 9.1.

Proof. Let $\delta > 0$ be a small constant. For $u \in \Omega^{0,q}(X, L^k)$, we have

$$(I - \Pi^{(q)}_k)u = \sum_{\mu\in\operatorname{Spec}\Box^{(q)}_k,0<\mu\leq k\delta}\Pi^{(q)}_{k,\mu}u + \Pi^{(q)}_{k,>k\delta}u. \tag{10.27}$$

We claim that for every $\mu \in \operatorname{Spec}\Box^{(q)}_k, 0 < \mu \leq k\delta$ and every $u \in \Omega^{0,q}(X, L^k)$,

$$F^{(q)}_{\delta,k}\Pi^{(q)}_{k,\mu}u = 0 \tag{10.28}$$

if $\delta > 0$ is small enough. Fix $\mu \in \operatorname{Spec}\Box^{(q)}_k, 0 < \mu \leq k\delta$ and $u \in \Omega^{0,q}(X, L^k)$. Since $q + 1 \neq n_-$, from (10.17) and (10.21), we have

$$\left\|\Box^{(q+1)}_{b,k}F^{(q+1)}_{\delta,k}\overline{\partial}_{b,k}\Pi^{(q)}_{k,\mu}u\right\|^2_{h^{L^k}} \geq c_1k^2 \left\|F^{(q+1)}_{\delta,k}\overline{\partial}_{b,k}\Pi^{(q)}_{k,\mu}u\right\|^2_{h^{L^k}}\,, \tag{10.29}$$

where $c_1 > 0$ is a constant independent of k and u. It is easy to see that

$$\Box^{(q+1)}_{b,k}F^{(q+1)}_{\delta,k}\overline{\partial}_{b,k}\Pi^{(q)}_{k,\mu}u = \mu F^{(q+1)}_{\delta,k}\overline{\partial}_{b,k}\Pi^{(q)}_{k,\mu}u.$$

Thus,

$$\left\|\Box^{(q+1)}_{b,k}F^{(q+1)}_{\delta,k}\overline{\partial}_{b,k}\Pi^{(q)}_{k,\mu}u\right\|^2_{h^{L^k}}\leq k^2\delta^2\left\|F^{(q+1)}_{\delta,k}\overline{\partial}_{b,k}\Pi^{(q)}_{k,\mu}u\right\|^2_{h^{L^k}}. \tag{10.30}$$

From (10.29) and (10.30), we conclude that if $\delta>0$ is small enough then

$$F^{(q+1)}_{\delta,k}\overline{\partial}_{b,k}\Pi^{(q)}_{k,\mu}u=\overline{\partial}_{b,k}F^{(q)}_{\delta,k}\Pi^{(q)}_{k,\mu}u=0.$$

Similarly, we have

$$F^{(q-1)}_{\delta,k}\overline{\partial}^*_{b,k}\Pi^{(q)}_{k,\mu}u=\overline{\partial}^*_{b,k}F^{(q)}_{\delta,k}\Pi^{(q)}_{k,\mu}u=0.$$

Hence,

$$F^{(q)}_{\delta,k}\Pi^{(q)}_{k,\mu}u=\frac{1}{\mu}\Box^{(q)}_{b,k}F^{(q)}_{\delta,k}\Pi^{(q)}_{k,\mu}u=0. \tag{10.31}$$

From (10.31), the claim (10.28) follows.

Now, from (10.27) and (10.28), if $\delta>0$ is small enough, then

$$\begin{aligned}\left\|F^{(q)}_{\delta,k}(I-\Pi^{(q)}_k)u)\right\|_{h^{L^k}}&=\left\|F^{(q)}_{\delta,k}\Pi^{(q)}_{k,>k\delta}u\right\|_{h^{L^k}}\leq\left\|\Pi^{(q)}_{k,>k\delta}u\right\|_{h^{L^k}}\\&\leq\frac{1}{k\delta}\left\|\Box^{(q)}_{b,k}\Pi^{(q)}_{k,>k\delta}u\right\|_{h^{L^k}}=\frac{1}{k\delta}\left\|\Pi^{(q)}_{k,>k\delta}\Box^{(q)}_{b,k}u\right\|_{h^{L^k}}\leq\frac{1}{k\delta}\left\|\Box^{(q)}_{b,k}u\right\|_{h^{L^k}},\end{aligned} \tag{10.32}$$

for every $u\in\Omega^{0,q}(X,L^k)$. From (10.32), (10.26) follows. $\Box$

Until further notice, we fix $\delta>0$ and we assume that $\delta>0$ is small enough so that (10.25), (10.26) hold and

(10.33) $\quad M^\phi_x-2\lambda\mathcal{L}_x$ is non-degenerate of constant signature, for every $\lambda\in]-\delta,\delta[$ and $x\in X$.

Let $D\subset X$ be a canonical coordinate patch and let $x=(x_1,\ldots,x_{2n-1})$ be a canonical coordinates on D as in Theorem 10.4. We identify D with $W\times]-\varepsilon,\varepsilon[\subset\mathbb{R}^{2n-1}$, where W is some open set in $\mathbb{R}^{2n-2}$ and $\varepsilon>0$. Until further notice, we work with canonical coordinates $x=(x_1,\ldots,x_{2n-1})$. Let $\eta=(\eta_1,\ldots,\eta_{2n-1})$ be the dual coordinates of x. Let s be a local trivializing section of L on D, $|s|^2_{h^L}=e^{-2\phi}$. Let $M>0$ be a large constant so that for every $(x,\eta)\in T^*D$ if $|\eta'|>\frac{M}{2}$ then $(x,\eta)\notin\Sigma$, where $\eta'=(\eta_1,\ldots,\eta_{2n-2})$, $|\eta'|=\sqrt{\sum^{2n-2}_{j=1}|\eta_j|^2}$. Fix $D_0\Subset D$. Let $D'\Subset D$ be an open neighbourhood of D_0. Put

$$V:=\{(x,\eta)\in T^*D';\,|\eta'|<M,|\eta_{2n-1}|<\delta\}. \tag{10.34}$$

Then $\overline{V}\subset T^*D$ and $\overline{V}\bigcap\Sigma\subset\Sigma'$, where Σ' is given by (1.5). Put

$$\hat{B}_{k,s}=\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\tau_\delta(\eta_{2n-1})d\eta. \tag{10.35}$$

Let $\hat{B}^*_{k,s}$ be the adjoint of $\hat{B}_{k,s}$ with respect to $(\,\cdot\,|\,\cdot\,)$. Then,

$$\hat{B}^*_{k,s}=\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\tau_\delta(\eta_{2n-1})d\eta. \tag{10.36}$$

It is clearly that

$$\hat{B}^*_{k,s}\equiv\tfrac{k^{2n-1}}{(2\pi)^{2n-1}}\textstyle\int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta\mod O(k^{-\infty})\text{ at }T^*D_0\bigcap\Sigma$$

is a classical semi-classical pseudodifferential operator on D of order 0 from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$, where

$$\begin{aligned}&\alpha(x,\eta,k)\sim\textstyle\sum_{j=0}\alpha_j(x,\eta)k^{-j}\text{ in }S^0_{\rm loc}(1;T^*D,T^{*0,q}X\boxtimes T^{*0,q}X),\\&\alpha_j(x,\eta)\in C^\infty(T^*D,T^{*0,q}D\boxtimes T^{*0,q}D),\quad j=0,1,\ldots,\end{aligned}$$

with $\alpha(x,\eta,k)=0$ if $|\eta|>M$, for some large $M>0$ and $\operatorname{Supp}\alpha(x,\eta,k)\bigcap T^*D_0\Subset V$.

Let $\hat{F}^{(q)}_{\delta,k,s}$ be the localized operator of $F^{(q)}_{\delta,k}$.

Lemma 10.12. *$\hat{F}^{(q)}_{\delta,k,s}=\hat{B}^1_{k,s}+\hat{B}_{k,s}$ on D, where*

$$\hat{B}^1_{k,s}=O(k^{-\infty}):H^s_{\rm comp}\,(D,T^{*0,q}X)\to H^s_{\rm loc}\,(D,T^{*0,q}X),\quad\forall s\in\mathbb{N}_0.$$

Proof. Let $u\in\Omega^{0,q}_0(D,L^k)$, $u=s^k\widetilde{u}$, $\widetilde{u}\in\Omega^{0,q}(D)$. We also write $y=(y_1,\ldots,y_{2n-1})$ to denote the canonical coordinates x. It is easy to see that on D,

$$\hat{F}^{(q)}_{\delta,k,s}u(y)=\frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\tau_\delta(\frac{m}{k})e^{imy_{2n-1}}\int^{\pi}_{-\pi}e^{-imt}u(e^{it}\circ y')dt,\quad\forall u\in\Omega^{0,q}_0(D),\tag{10.37}$$

where $y'=(y_1,\ldots,y_{2n-2})$. Fix $D'\Subset D$ and let $\chi(y_{2n-1})\in C^\infty_0(]-\pi,\pi[)$ such that $\chi(y_{2n-1})=1$ for every $(y',y_{2n-1})\in D'$. Let $\hat{B}^1_{k,s}:\Omega^{0,q}_0(D')\to\Omega^{0,q}(D')$ be the continuous operator given by

$$\begin{split}&\hat{B}^1_{k,s}:\Omega^{0,q}_0(D')\to\Omega^{0,q}(D'),\\&u\to\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int_{|t|\leq\pi}e^{i<x_{2n-1}-y_{2n-1},\eta_{2n-1}>}\tau_\delta(\frac{\eta_{2n-1}}{k})\\&\qquad\times(1-\chi(y_{2n-1}))e^{imy_{2n-1}}e^{-imt}u(e^{it}\circ y')dtd\eta_{2n-1}dy_{2n-1}.\end{split}\tag{10.38}$$

By using integration by parts with respect to η_{2n-1}, it is easy to see that the integral (10.38) is well-defined. Moreover, we can integrate by parts with respect to η_{2n-1} and y_{2n-1} several times and conclude that

$$\hat{B}^1_{k,s}=O(k^{-\infty}):H^s_{\rm comp}\,(D,T^{*0,q}X)\to H^s_{\rm loc}\,(D,T^{*0,q}X),\quad\forall s\in\mathbb{N}_0.\tag{10.39}$$

Now, we claim that

$$\hat{B}_{k,s}+\hat{B}^1_{k,s}=\hat{F}^{(q)}_{\delta,k,s}\quad\text{on }\Omega^{0,q}_0(D').\tag{10.40}$$

Let $u\in\Omega^{0,q}_0(D')$. From (10.35) and Fourier inversion formula, it is straightforward to see that

$$\begin{split}\hat{B}_{k,s}u(x)=&\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int_{|t|\leq\pi}e^{i<x_{2n-1}-y_{2n-1},\eta_{2n-1}>}\tau_\delta(\frac{\eta_{2n-1}}{k})\\&\times\chi(y_{2n-1})e^{imy_{2n-1}}e^{-imt}u(e^{it}\circ x')dtd\eta_{2n-1}dy_{2n-1}.\end{split}\tag{10.41}$$

From (10.41) and (10.38), we have

$$\begin{split}&(\hat{B}_{k,s}+\hat{B}^1_{k,s})u(x)\\&=\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int_{|t|\leq\pi}e^{i<x_{2n-1}-y_{2n-1},\eta_{2n-1}>}\tau_\delta(\frac{\eta_{2n-1}}{k})e^{imy_{2n-1}}e^{-imt}u(e^{it}\circ x')dtd\eta_{2n-1}dy_{2n-1}.\end{split}\tag{10.42}$$

From Fourier inversion formula and notice that for every $m\in\mathbb{Z}$,

$$\int e^{imy_{2n-1}}e^{-iy_{2n-1}\eta_{2n-1}}dy_{2n-1}=2\pi\delta_m(\eta_{2n-1}),$$

where the integral above is defined as an oscillatory integral and δ_m is the Dirac measure at m(see Chapter 7.2 in Hörmander [11]), (10.42) becomes

$$\begin{split}&(\hat{B}_{k,s}+\hat{B}^1_{k,s})u(x)\\&=\frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\tau_\delta(\frac{m}{k})e^{ix_{2n-1}m}\int_{|t|\leq\pi}e^{-imt}u(e^{it}\circ x')dt\\&=\hat{F}^{(q)}_{\delta,k,s}u(x).\end{split}\tag{10.43}$$

Here we used (10.37).

From (10.43), the claim (10.40) follows. From (10.40) and (10.39), the lemma follows. □

We need

Lemma 10.13. *Let $D\subset X$ be a canonical coordinate patch of X. Then, $\Pi^{(q)}_k$ is k-negligible away the diagonal with respect to $F^{(q)}_{\delta,k}$ on D.*

Proof. Let $\chi, \chi_1 \in C_0^\infty(D)$, $\chi_1 = 1$ on some neighbourhood of $\operatorname{Supp}\chi$. Let $u \in H_b^q(X, L^k)$ with $\|u\|_{h^{L^k}} = 1$. In view of Theorem 7.3, we see that there is a constant $C > 0$ independent of k and u such that

$$|u(x)|^2_{h^{L^k}} \leq Ck^n, \quad \forall x \in X. \tag{10.44}$$

Let $x = (x_1, \ldots, x_{2n-1}) = (x', x_{2n-1})$ be canonical coordinates on D. Put $v = (1 - \chi_1)u$. It is straightforward to see that on D,

$$\begin{split}\chi F^{(q)}_{\delta,k}(1-\chi_1)u(x) = \frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z},|m|\leq 2k\delta}\int_{|t|\leq\pi} e^{i<x_{2n-1}-y_{2n-1},\eta_{2n-1}>}\chi(x)\tau_\delta(\frac{\eta_{2n-1}}{k})e^{imy_{2n-1}}\\ \times e^{-imt}v(e^{it}\circ x')dtd\eta_{2n-1}dy_{2n-1}.\end{split} \tag{10.45}$$

Let $\varepsilon > 0$ be a small constant so that for every $(x_1, \ldots, x_{2n-1}) \in \operatorname{Supp}\chi$, we have

$$(x_1, \ldots, x_{2n-2}, y_{2n-1}) \in \{x \in D;\ \chi_1(x) = 1\}, \quad \forall\, |y_{2n-1} - x_{2n-1}| < \varepsilon. \tag{10.46}$$

Let $\psi \in C_0^\infty(]-1,1[)$, $\psi = 1$ on $[\frac{1}{2}, \frac{1}{2}]$. Put

$$\begin{split}I_0(x) = \frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z},|m|\leq 2k\delta}\int_{|t|\leq\pi} e^{i<x_{2n-1}-y_{2n-1},\eta_{2n-1}>}(1-\psi(\frac{x_{2n-1}-y_{2n-1}}{\varepsilon}))\chi(x)\tau_\delta(\frac{\eta_{2n-1}}{k})e^{imy_{2n-1}}\\ \times e^{-imt}v(e^{it}\circ x')dtd\eta_{2n-1}dy_{2n-1},\end{split} \tag{10.47}$$

$$\begin{split}I_1(x) = \frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int_{|t|\leq\pi} e^{i<x_{2n-1}-y_{2n-1},\eta_{2n-1}>}\psi(\frac{x_{2n-1}-y_{2n-1}}{\varepsilon})\chi(x)\tau_\delta(\frac{\eta_{2n-1}}{k})e^{imy_{2n-1}}\\ \times e^{-imt}v(e^{it}\circ x')dtd\eta_{2n-1}dy_{2n-1},\end{split} \tag{10.48}$$

and

$$\begin{split}I_2(x) = \frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z},|m|>2k\delta}\int_{|t|\leq\pi} e^{i<x_{2n-1}-y_{2n-1},\eta_{2n-1}>}\psi(\frac{x_{2n-1}-y_{2n-1}}{\varepsilon})\chi(x)\tau_\delta(\frac{\eta_{2n-1}}{k})e^{imy_{2n-1}}\\ \times e^{-imt}v(e^{it}\circ x')dtd\eta_{2n-1}dy_{2n-1}.\end{split} \tag{10.49}$$

It is clearly that on D,

$$\chi F^{(q)}_{\delta,k}(1-\chi_1)u(x) = I_0(x) + I_1(x) - I_2(x). \tag{10.50}$$

By using integration by parts with respect to η_{2n-1} several times and (10.44), we conclude that for every $N > 0$ and $m \in \mathbb{N}$, there is a constant $C_{N,m} > 0$ independent of u and k such that

$$\|I_0(x)\|_{C^m(D)} \leq C_{N,m}k^{-N}. \tag{10.51}$$

Similarly, by using integration by parts with respect to y_{2n-1} several times and (10.44), we conclude that for every $N > 0$ and $m \in \mathbb{N}$, there is a constant $\widetilde{C}_{N,m} > 0$ independent of u and k such that

$$\|I_2(x)\|_{C^m(D)} \leq \widetilde{C}_{N,m}k^{-N}. \tag{10.52}$$

We can check that

$$I_1(x) = \frac{1}{2\pi}\int e^{i<x_{2n-1}-y_{2n-1},\eta_{2n-1}>}\psi(\frac{x_{2n-1}-y_{2n-1}}{\varepsilon})\chi(x)\tau_\delta(\frac{\eta_{2n-1}}{k})v(x', y_{2n-1})d\eta_{2n-1}dy_{2n-1}. \tag{10.53}$$

From (10.46) and (10.53), we deduce that

$$I_1(x) = 0 \text{ on } D. \tag{10.54}$$

From (10.50), (10.51), (10.52) and (10.54), we conclude that for every $N > 0$ and $m \in \mathbb{N}$, there is a constant $\hat{C}_{N,m} > 0$ independent of u and k such that

$$\left\|\chi F^{(q)}_{\delta,k}(1-\chi_1)u(x)\right\|_{C^m(D)} \leq \hat{C}_{N,m}k^{-N}. \tag{10.55}$$

From (10.44) and (10.55), it is not difficult to see that

$$\sum_{j=1}^{d_k}\left|\chi F^{(q)}_{\delta,k}(1-\chi_1)f_j(x)\right|^2_{h^{L^k}}\equiv 0 \mod O(k^{-\infty}) \text{ on } D, \tag{10.56}$$

where $\{f_1,\ldots,f_{d_k}\}$ is an orthonormal basis for $H^q_b(X,L^k)$. From (10.56), the lemma follows. □

From Lemma 10.13, Lemma 10.12 and Theorem 10.11, we see that the operator $F^{(q)}_{\delta,k}:L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k)$ satisfies all the conditions in Theorem 9.5. Summing up, we obtain one of the main results of this work

Theorem 10.14. *Let $(X,T^{1,0}X)$ be a compact CR manifold with a transversal CR S^1 action and let T be the global vector field induced by the S^1 action. Let L be a T-rigid CR line bundle over X with a T-rigid Hermitian fiber metric h^L. We assume that $Y(q)$ holds at each point of X and M^ϕ_x is non-degenerate of constant signature (n_-,n_+), for every $x\in X$. Let s be a local trivializing section of L on an canonical coordinate patch $D\subset X$, $|s|^2_{h^L}=e^{-2\phi}$. Fix $D_0\Subset D$. Let $F^{(q)}_{\delta,k}:L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k)$ be the continuous operator given by* (10.14) *and let $F^{(q),*}_{\delta,k}:L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k)$ be the adjoint of $F^{(q)}_{\delta,k}$ with respect to $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$. Put $P_k:=F^{(q)}_{\delta,k}\Pi^{(q)}_kF^{(q),*}_{\delta,k}$ and let $\hat P_{k,s}$ be the localized operator of P_k. If $q\neq n_-$, then $\hat P_{k,s}\equiv 0 \mod O(k^{-\infty})$ on D_0. If $q=n_-$, then*

$$\hat P_{k,s}(x,y)\equiv\int e^{ik\varphi(x,y,s)}g(x,y,s,k)ds \mod O(k^{-\infty}) \tag{10.57}$$

on D_0, where $\varphi(x,y,s)\in C^\infty(\Omega)$ is as in Theorem 5.29, (2.3),

$$\begin{aligned}&g(x,y,s,k)\in S^n_{\rm loc}\big(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big)\bigcap C^\infty_0\big(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\\&g(x,y,s,k)\sim\sum_{j=0}^\infty g_j(x,y,s)k^{n-j} \text{ in } S^n_{\rm loc}\big(1;\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\\&g_j(x,y,s)\in C^\infty_0\big(\Omega,T^{*0,q}X\boxtimes T^{*0,q}X\big),\quad j=0,1,2,\ldots,\end{aligned} \tag{10.58}$$

and for every $(x,x,s)\in\Omega$,

$$\begin{aligned}&g_0(x,x,s)\\&=(2\pi)^{-n}\left|\det\big(M^\phi_x-2s\mathcal{L}_x\big)\right|\left|\tau_\delta(s)\right|^2\pi_{(x,s,n_-)}.\end{aligned} \tag{10.59}$$

Here

$$\begin{aligned}\Omega:=&\{(x,y,s)\in D\times D\times\mathbb{R};\,(x,-2{\rm Im}\,\overline\partial_b\phi(x)+s\omega_0(x))\in V\bigcap\Sigma,\\&(y,-2{\rm Im}\,\overline\partial_b\phi(y)+s\omega_0(y))\in V\bigcap\Sigma,\,|x-y|<\varepsilon,\text{ for some }\varepsilon>0\},\end{aligned}$$

V is given by (10.34), *$\pi_{(x,s,n_-)}:T^{*0,q}_pX\to\mathcal{N}(x,s,n_-)$ is the orthogonal projection with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$, $\mathcal{N}(x,s,n_-)$ is given by* (5.39),

$$\left|\det\big(M^\phi_x-2s\mathcal{L}_x\big)\right|=\left|\lambda_1(s)\right|\left|\lambda_2(s)\right|\cdots\left|\lambda_{n-1}(s)\right|,$$

$\lambda_1(s),\ldots,\lambda_{n-1}(s)$ are eigenvalues of the Hermitian quadratic form $M^\phi_x-2s_0\mathcal{L}_x$ with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$.

We recall "T-rigid positive CR line bundle"(see Definition 1.10)

Theorem 10.15. *Let $(X,T^{1,0}X)$ be a compact CR manifold with a transversal CR S^1 action and let T be the global vector field induced by the S^1 action. If there is a T-rigid positive CR line bundle over X, then X can be CR embedded into $\mathbb{CP}^N$, for some $N\in\mathbb{N}$.*

Proof. The proof is essentially the same as the the proof of Theorem 1.4. We only give the outline of the proof.

Fix $p\in X$. Let $u_k\in C^\infty(X,L^k)$ be as in Lemma 8.1 and put $u^0_k=\Pi^{(q)}_{k,\leq k^{-N_0}}u_k$. From the proof Theorem 8.2, we see that $u_k\equiv u^0_k \mod O(k^{-\infty})$ and hence

$$F^{(q)}_{\delta,k}u_k\equiv F^{(q)}_{\delta,k}u^0_k \mod O(k^{-\infty}). \tag{10.60}$$

From (10.25) and (10.17), we see that

$$F^{(q)}_{\delta,k}u^0_k \in H^{(0)}_b(X, L^k). \tag{10.61}$$

Moreover, from the construction of u_k(see (8.8)) and (10.60), it is straightforward to see that there exist $C > 1$ and $k_0 > 0$ independent of k and the point p such that for every $k \geq k_0$, we have

$$\begin{aligned} &\frac{1}{C} \leq \left\| F^{(q)}_{\delta,k}u^0_k \right\|_{h^{L^k}} \leq C, \\ &\left| (F^{(q)}_{\delta,k}u^0_k)(p) \right|^2_{h^{L^k}} \geq \frac{1}{C}k^n. \end{aligned} \tag{10.62}$$

From (10.62), we can repeat the procedure in section 8 and conclude that for k large, the Kodaira map

$$\Phi_k : X \to \mathbb{CP}^{d_k-1}$$

is well-defined as a smooth map. Here Φ_k is defined as follows. Let $f_1, \ldots, f_{d_k}$ be orthonormal frame for $H^0_b(X, L^k)$. For $x_0 \in X$, let s be a local trivializing section of L on an open neighbourhood $D \subset X$ of x_0, $|s(x)|^2_{h^{L^k}} = e^{-2\phi}$. On D, put $f_j(x) = s^k \widetilde{f}_j(x)$, $\widetilde{f}_j(x) \in C^\infty(D)$, $j = 1, \ldots, d_k$. Then,

$$\Phi_k(x_0) = [\widetilde{f}_1(x_0), \ldots, \widetilde{f}_{d_k}(x_0)] \in \mathbb{CP}^{d_k-1}.$$

Moreover, with similar modifications, we can repeat the proof of Theorem 8.4 and conclude that for k large, the differential map

$$d\Phi_k(x) : T_xX \to T_{\Phi_k(x)}\mathbb{CP}^{d_k-1}$$

is injective, for every $x \in X$.

Finally, by using Theorem 10.14, we can repeat the proof of Theorem 8.10 and deduce that for k large, Φ_k is injective. $\square$

We now offer some examples of "T-rigid CR line bundles over CR manifolds with transversal CR S^1 actions".

10.1. **CR manifolds in projective spaces.** We consider $\mathbb{CP}^{N-1}$, $N \geq 4$. Let $[z] = [z_1, \ldots, z_N]$ be the homogeneous coordinates of $\mathbb{CP}^{N-1}$. Put

$$X := \left\{ [z_1, \ldots, z_N] \in \mathbb{CP}^{N-1};\ \lambda_1 |z_1|^2 + \cdots + \lambda_m |z_m|^2 + \lambda_{m+1} |z_{m+1}|^2 + \cdots + \lambda_N |z_N|^2 = 0 \right\},$$

where $m \in \mathbb{N}$ and $\lambda_j \in \mathbb{R}$, $j = 1, \ldots, N$. Then, X is a compact CR manifold of dimension $2(N-1)-1$ with CR structure $T^{1,0}X := T^{1,0}\mathbb{CP}^{N-1} \bigcap \mathbb{C}TX$. Now, we assume that $\lambda_1 < 0, \lambda_2 < 0, \ldots, \lambda_m < 0$, $\lambda_{m+1} > 0, \lambda_{m+2} > 0, \ldots, \lambda_N > 0$, where $m \geq 2$, $N - m \geq 2$. Then, it is easy to see that the Levi form has at least one negative and one positive eigenvalues at each point of X. Thus, $Y(0)$ holds at each point of X. X admits a S^1 action:

$$\begin{aligned} &S^1 \times X \to X, \\ &e^{i\theta} \circ [z_1, \ldots, z_m, z_{m+1}, \ldots, z_N] \to [e^{i\theta}z_1, \ldots, e^{i\theta}z_m, z_{m+1}, \ldots, z_N], \quad \theta \in [-\pi, \pi[. \end{aligned} \tag{10.63}$$

Since $(z_1, \ldots, z_m) \neq 0$ on X, this S^1 action is well-defined. Moreover, it is straightforward to check that this S^1 action is CR and transversal. Let T be the global vector field induced by the S^1 action.

Let $E \to \mathbb{CP}^{N-1}$ be the canonical line bundle with respect to the Fubini-Study metric. For $j = 1, 2, \ldots, N$, put $W_j = \{[z_1, \ldots, z_N] \in \mathbb{CP}^{N-1};\ z_j \neq 0\}$. Then, E is trivial on W_j, $j = 1, \ldots, N$, and we can find local trivializing section e_j of E on W_j, $j = 1, \ldots, N$, such that for every $j, t = 1, \ldots, N$,

$$e_j(z) = \frac{z_j}{z_t}e_t(z) \ \text{ on } W_j \bigcap W_t, \quad z = [z_1, \ldots, z_N] \in W_j \bigcap W_t. \tag{10.64}$$

Consider $L := E|_X$. Then, L is a CR line bundle over $(X, T^{1,0}X)$. It is easy to see that X can be covered with open sets $U_j := W_j|_X$, $j = 1, 2, \ldots, m$, with trivializing sections $s_j := e_j|_X$, $j = 1, 2, \ldots, m$, such that the corresponding transition functions are T-rigid CR functions. Thus, L is a T-rigid CR line bundle over $(X, T^{1,0}X)$. Let h^L be the Hermitian fiber metric on L given by

$$|s_j(z_1, \ldots, z_N)|^2_{h^L} := e^{-\log\left(\frac{|z_1|^2 + \cdots + |z_N|^2}{|z_j|^2}\right)}, \quad j = 1, \ldots, m.$$

It is not difficult to check that h^L is well-defined and h^L is a T-rigid positive CR line bindle.

10.2. **Compact Heisenberg groups.** Let $\lambda_1,\ldots,\lambda_{n-1}$ be given non-zero integers. Let $\mathscr{C}H_n = (\mathbb{C}^{n-1}\times\mathbb{R})/_\sim$, where $(z,t)\sim(\widetilde{z},\widetilde{t})$ if

$$\begin{aligned}&\widetilde{z}-z=(\alpha_1,\ldots,\alpha_{n-1})\in\sqrt{2\pi}\mathbb{Z}^{n-1}+i\sqrt{2\pi}\mathbb{Z}^{n-1},\\&\widetilde{t}-t-i\sum_{j=1}^{n-1}\lambda_j(z_j\overline{\alpha}_j-\overline{z}_j\alpha_j)\in 2\pi\mathbb{Z}.\end{aligned}$$

We can check that $\sim$ is an equivalence relation and $\mathscr{C}H_n$ is a compact manifold of dimension $2n-1$. The equivalence class of $(z,t)\in\mathbb{C}^{n-1}\times\mathbb{R}$ is denoted by $[(z,t)]$. For a given point $p=[(z,t)]$, we define $T^{1,0}_p\mathscr{C}H_n$ to be the space spanned by

$$\left\{\tfrac{\partial}{\partial z_j}+i\lambda_j\overline{z}_j\tfrac{\partial}{\partial t},\;\; j=1,\ldots,n-1\right\}.$$

It is easy to see that the definition above is independent of the choice of a representative (z,t) for $[(z,t)]$. Moreover, we can check that $T^{1,0}\mathscr{C}H_n$ is a CR structure. $\mathscr{C}H_n$ admits the natural S^1 action: $e^{i\theta}\circ[z,t]\to[z,t+\theta]$, $0\le\theta<2\pi$. Let T be the global vector field induced by this S^1 action. We can check that this S^1 action is CR and transversal and $T=\frac{\partial}{\partial t}$. We take a Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on the complexified tangent bundle $\mathbb{C}T\mathscr{C}H_n$ such that

$$\left\{\tfrac{\partial}{\partial z_j}+i\lambda_j\overline{z}_j\tfrac{\partial}{\partial t}\,,\tfrac{\partial}{\partial\overline{z}_j}-i\lambda_jz_j\tfrac{\partial}{\partial t}\,,-\tfrac{\partial}{\partial t}\,;\,j=1,\ldots,n-1\right\}$$

is an orthonormal basis. The dual basis of the complexified cotangent bundle is

$$\left\{dz_j\,,\,d\overline{z}_j\,,\,\omega_0:=-dt+\textstyle\sum_{j=1}^{n-1}(i\lambda_j\overline{z}_jdz_j-i\lambda_jz_jd\overline{z}_j);j=1,\ldots,n-1\right\}.$$

The Levi form $\mathcal{L}_p$ of $\mathscr{C}H_n$ at $p\in\mathscr{C}H_n$ is given by $\mathcal{L}_p=\sum_{j=1}^{n-1}\lambda_jdz_j\wedge d\overline{z}_j$.

Now, we construct a T-rigid CR line bundle L over $\mathscr{C}H_n$. Let $L=(\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{C})/_\equiv$ where $(z,\theta,\eta)\equiv(\widetilde{z},\widetilde{\theta},\widetilde{\eta})$ if

$$\begin{aligned}&(z,\theta)\sim(\widetilde{z},\widetilde{\theta}),\\&\widetilde{\eta}=\eta\exp\Big(\sum_{j,t=1}^{n-1}\mu_{j,t}(z_j\overline{\alpha}_t+\frac{1}{2}\alpha_j\overline{\alpha}_t)\Big),\end{aligned}$$

where $\alpha=(\alpha_1,\ldots,\alpha_{n-1})=\widetilde{z}-z$, $\mu_{j,t}=\mu_{t,j}$, $j,t=1,\ldots,n-1$, are given integers. We can check that $\equiv$ is an equivalence relation and L is a T-rigid CR line bundle over $\mathscr{C}H_n$. For $(z,\theta,\eta)\in\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{C}$, we denote $[(z,\theta,\eta)]$ its equivalence class. It is straightforward to see that the pointwise norm

$$\big|[(z,\theta,\eta)]\big|^2_{h^L}:=|\eta|^2\exp\big(-\textstyle\sum_{j,t=1}^{n-1}\mu_{j,t}z_j\overline{z}_t\big)$$

is well-defined. In local coordinates (z,θ,η), the weight function of this metric is

$$\phi=\frac{1}{2}\sum_{j,t=1}^{n-1}\mu_{j,t}z_j\overline{z}_t.$$

Thus, L is a T-rigid CR line bundle over $\mathscr{C}H_n$ with T-rigid Hermitian metric h^L. Note that

$$\overline{\partial}_b=\textstyle\sum_{j=1}^{n-1}d\overline{z}_j\wedge\big(\frac{\partial}{\partial\overline{z}_j}-i\lambda_jz_j\frac{\partial}{\partial\theta}\big)\,,\quad \partial_b=\sum_{j=1}^{n-1}dz_j\wedge\big(\frac{\partial}{\partial z_j}+i\lambda_j\overline{z}_j\frac{\partial}{\partial\theta}\big).$$

Thus $d(\overline{\partial}_b\phi-\partial_b\phi)=\sum_{j,t=1}^{n-1}\mu_{j,t}dz_j\wedge d\overline{z}_t$ and for any $p\in\mathscr{C}H_n$,

$$M^\phi_p=\sum_{j,t=1}^{n-1}\mu_{j,t}dz_j\wedge d\overline{z}_t.$$

Thus, if $(\mu_{j,t})_{j,t=1}^{n-1}$ is positive definite, then L is a T-rigid positive CR line bundle. From this and Theorem 10.15, we conclude that

Theorem 10.16. *Assume that $\lambda_1<0$ and $\lambda_2>0$ (then $Y(0)$ holds on $\mathscr{C}H_n$). Then, $\mathscr{C}H_n$ can be CR embedded into $\mathbb{CP}^N$, for some $N\in\mathbb{N}$.*

10.3. **Holomorphic line bundles over a complex torus.** Let

$$T_n := \mathbb{C}^n/(\sqrt{2\pi}\mathbb{Z}^n + i\sqrt{2\pi}\mathbb{Z}^n)$$

be the flat torus. Let $\lambda = (\lambda_{j,t})_{j,t=1}^n$, where $\lambda_{j,t} = \lambda_{t,j}$, $j,t = 1,\ldots,n$, are given integers. Let L_λ be the holomorphic line bundle over T_n with curvature the $(1,1)$-form $\Theta_\lambda = \sum_{j,t=1}^n \lambda_{j,t} dz_j \wedge d\overline{z}_t$. More precisely, $L_\lambda := (\mathbb{C}^n \times \mathbb{C})/_\sim$, where $(z,\theta) \sim (\widetilde{z},\widetilde{\theta})$ if

$$\widetilde{z} - z = (\alpha_1,\ldots,\alpha_n) \in \sqrt{2\pi}\mathbb{Z}^n + i\sqrt{2\pi}\mathbb{Z}^n\,, \quad \widetilde{\theta} = \exp\big(\textstyle\sum_{j,t=1}^n \lambda_{j,t}(z_j\overline{\alpha}_t + \frac{1}{2}\alpha_j\overline{\alpha}_t\,)\big)\theta\,.$$

We can check that $\sim$ is an equivalence relation and L_λ is a holomorphic line bundle over T_n. For $[(z,\theta)] \in L_\lambda$ we define the Hermitian metric by

$$\big|[(z,\theta)]\big|^2 := |\theta|^2 \exp(-\textstyle\sum_{j,t=1}^n \lambda_{j,t} z_j \overline{z}_t)$$

and it is easy to see that this definition is independent of the choice of a representative (z,θ) of $[(z,\theta)]$. We denote by $\phi_\lambda(z)$ the weight of this Hermitian fiber metric. Note that $\frac{1}{2}\partial\overline{\partial}\phi_\lambda = \Theta_\lambda$.

Let L^*_λ be the dual bundle of L_λ and let $\|\cdot\|_{L^*_\lambda}$ be the norm of L^*_λ induced by the Hermitian fiber metric on L_λ. Consider the compact CR manifold of dimension $2n+1$: $X = \{v \in L^*_\lambda;\ \|v\|_{L^*_\lambda} = 1\}$; this is the boundary of the Grauert tube associated to L^*_λ. The manifold X is equipped with a natural S^1-action. Locally X can be represented in local holomorphic coordinates (z,η), where η is the fiber coordinate, as the set of all (z,η) such that $|\eta|^2 e^{2\phi_\lambda(z)} = 1$. The S^1-action on X is given by $e^{i\theta} \circ (z,\eta) = (z, e^{i\theta}\eta)$, $e^{i\theta} \in S^1$, $(z,\eta) \in X$. Let T be the global vector field on X induced by this S^1 action. We can check that this S^1 action is CR and transversal.

Let $\pi : L^*_\lambda \to T_n$ be the natural projection from L^*_λ onto T_n. Let $\mu = (\mu_{j,t})_{j,t=1}^n$, where $\mu_{j,t} = \mu_{t,j}$, $j,t = 1,\ldots,n$, are given integers. Let L_μ be another holomorphic line bundle over T_n determined by the constant curvature form $\Theta_\mu = \sum_{j,t=1}^n \mu_{j,t} dz_j \wedge d\overline{z}_t$ as above. The pullback line bundle $\pi^* L_\mu$ is a holomorphic line bundle over L^*_λ. If we restrict $\pi^* L_\mu$ on X, then we can check that $\pi^* L_\mu$ is a T-rigid CR line bundle over X.

The Hermitian fiber metric on L_μ induced by ϕ_μ induces a Hermitian fiber metric on $\pi^* L_\mu$ that we shall denote by $h^{\pi^* L_\mu}$. We let ψ to denote the weight of $h^{\pi^* L_\mu}$. The part of X that lies over a fundamental domain of T_n can be represented in local holomorphic coordinates (z,ξ), where ξ is the fiber coordinate, as the set of all (z,ξ) such that $r(z,\xi) := |\xi|^2 \exp(\sum_{j,t=1}^n \lambda_{j,t} z_j \overline{z}_t) - 1 = 0$ and the weight ψ may be written as $\psi(z,\xi) = \frac{1}{2}\sum_{j,t=1}^n \mu_{j,t} z_j \overline{z}_t$. From this we see that $\pi^* L_\mu$ is a T-rigid CR line bundle over X with T-rigid Hermitian fiber metric $h^{\pi^* L_\mu}$. It is straightforward to check that for any $p \in X$, we have $M^\psi_p = d(\overline{\partial}_b\psi - \partial_b\psi)(p)|_{T^{1,0}X} = \sum_{j,t=1}^n \mu_{j,t} dz_j \wedge d\overline{z}_t$. Thus, if $(\mu_{j,t})_{j,t=1}^{n-1}$ is positive definite, then L is a T-rigid positive CR line bundle. From this and Theorem 10.15, we conclude that

Theorem 10.17. *Assume that the matrix $\lambda = (\lambda_{j,t})_{j,t=1}^n$ has at least one negative and one positive eigenvalues. Then, $X = \{v \in L^*_\lambda;\ \|v\|_{L^*_\lambda} = 1\}$ can be CR embedded into $\mathbb{CP}^N$, for some $N \in \mathbb{N}$.*

11. Szegö kernel asymptotics on some non-compact CR manifolds

By using Theorem 9.5, we will establish Szegö kernel asymptotics on some non-compact CR manifolds. Let Γ be an open set in $\mathbb{C}^{n-1}$, $n \geq 2$. Consider $X := \Gamma \times \mathbb{R}$. Let (z,t) be the coordinates of X, where $z = (z_1,\ldots,z_{n-1})$ denote the coordinates of $\mathbb{C}^{n-1}$ and t is the coordinate of $\mathbb{R}$. We write $z_j = x_{2j-1} + ix_{2j}$, $j = 1,\ldots,n-1$. We also write $(z,t) = x = (x_1,\ldots,x_{2n-1})$ and let $\eta = (\eta_1,\ldots,\eta_{2n-1})$ be the dual variables of x. Let $\mu(z) \in C^\infty(\Gamma,\mathbb{R})$. We define $T^{1,0}X$ to be the space spanned by

$$\big\{\tfrac{\partial}{\partial z_j} + i\tfrac{\partial\mu}{\partial z_j}\tfrac{\partial}{\partial t},\quad j = 1,\ldots,n-1\big\}.$$

Then $(X, T^{1,0}X)$ is a non-compact CR manifold of dimension $2n-1$. We take a Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on the complexified tangent bundle $\mathbb{C}TX$ such that

$$\Big\{\tfrac{\partial}{\partial z_j} + i\frac{\partial\mu}{\partial z_j}\tfrac{\partial}{\partial t}\,, \tfrac{\partial}{\partial \overline{z}_j} - i\frac{\partial\mu}{\partial \overline{z}_j}\tfrac{\partial}{\partial t}\,, T := \tfrac{\partial}{\partial t}\,; j = 1,\ldots,n-1\Big\}$$

is an orthonormal basis. The dual basis of the complexified cotangent bundle $\mathbb{C}T^*X$ is

$$\Big\{dz_j\,,\, d\overline{z}_j\,,\, -\omega_0 := dt + \textstyle\sum_{j=1}^{n-1}(-i\frac{\partial\mu}{\partial z_j}dz_j + i\frac{\partial\mu}{\partial \overline{z}_j}d\overline{z}_j); j = 1, \ldots, n-1\Big\}.$$

The Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on the $\mathbb{C}TX$ induces Hermitian metrics on the bundle of $(0,q)$ forms $T^{*0,q}X$, $q = 1, \ldots, n-1$, we shall also denote these Hermitian metrics by $\langle\,\cdot\,|\,\cdot\,\rangle$. For $V \in T^{*0,q}X$, we write $|V|^2 := \langle\, V\,|\,V\,\rangle$.

The Levi form $\mathcal{L}_p$ of X at $p \in X$ is given by

$$\mathcal{L}_p = \sum_{j,\ell=1}^{n-1} \frac{\partial^2\mu}{\partial z_j \partial \overline{z}_\ell}(p) dz_j \wedge d\overline{z}_\ell.$$

Let L be the trivial line bundle over X with non-trivial Hermitian fiber metric $|1|^2_{h^L} = e^{-2\phi}$, where $\phi = \phi(z) \in C^\infty(\Gamma)$ is a real valued function. Then, we can check that

$$M^\phi_p = 2\sum_{j,\ell=1}^{n-1} \frac{\partial^2\phi(z)}{\partial z_j \partial \overline{z}_\ell} dz_j \wedge d\overline{z}_\ell, \quad p = (z,t) \in X. \tag{11.1}$$

We shall consider the k-th power of L and we will use the same notations as before. Let $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$ be the L^2 inner product on $\Omega^{0,q}_0(X)$ given by

$$(\,u\,|\,v\,)_{h^{L^k}} = \int_X \langle\,u\,|\,v\,\rangle e^{-2k\phi(z)} d\lambda(z)dt, \quad u, v \in \Omega^{0,q}_0(X),$$

where $d\lambda(z)dt = m(z)dx_1 \cdots dx_{2n-2}dt$, $m(z) \in C^\infty(\Gamma)$, is the induced volume form. Let $\|\cdot\|_{h^{L^k}}$ denote the corresponding L^2 norm. Let $L^2_{(0,q)}(X, L^k)$ be the completion of $\Omega^{0,q}_0(X)$ with respect to $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$ and let

$$\Box^{(q)}_{b,k} : \mathrm{Dom}\,\Box^{(q)}_{b,k} \subset L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k)$$

be the Gaffney extension of Kohn Laplacian with respect to $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$ (see (3.9)).

11.1. **The partial Fourier transform and the operator $F^{(q)}_{\delta,k}$.** Let $u \in \Omega^{0,q}_0(X, L^k)$. Put

$$(\mathcal{F}u)(z,\eta) = \int_{\mathbb{R}} e^{-i\eta t} u(z,t)dt. \tag{11.2}$$

From Parseval's formula, we have

$$\begin{aligned} \|\mathcal{F}u\|^2_{h^{L^k}} &= \int_X |(\mathcal{F}u)(z,\eta)|^2 e^{-2k\phi(z)} d\eta d\lambda(z) \\ &= (2\pi)\int_X |u(z,t)|^2 e^{-2k\phi(z)} dt d\lambda(z) = (2\pi)\,\|u\|^2_{h^{L^k}}\,. \end{aligned} \tag{11.3}$$

Thus, we can extend the operator $\mathcal{F}$ to $L^2_{(0,q)}(X, L^k)$ and

$$\begin{aligned} &\mathcal{F} : L^2_{(0,q)}(X, L^k) \to L^2_{(0,q)}(X, L^k) \text{ is continuous,} \\ &\|\mathcal{F}u\|_{h^{L^k}} = \sqrt{2\pi}\,\|u\|_{h^{L^k}}\,, \quad \forall u \in L^2_{(0,q)}(X, L^k). \end{aligned} \tag{11.4}$$

For $u \in L^2_{(0,q)}(X, L^k)$, we call $\mathcal{F}u$ the partial Fourier transform of u with respect to t.

Fix $\delta > 0$. Take $\tau_\delta(x) \in C^\infty_0(]-\delta, \delta[)$, $0 \le \tau_\delta \le 1$ and $\tau_\delta = 1$ on $[-\frac{\delta}{2}, \frac{\delta}{2}]$. We also write θ to denote the t variable. Let $F^{(q)}_{\delta,k} : \Omega^{0,q}_0(X, L^k) \to \Omega^{0,q}(X, L^k)$ be the operator given by

$$F^{(q)}_{\delta,k}u(z,t) := \frac{1}{2\pi}\int e^{i<t-\theta,\eta>} u(z,\theta)\tau_\delta(\frac{\eta}{k}) d\eta d\theta \in \Omega^{0,q}(X, L^k), \quad u(z,t) \in \Omega^{0,q}_0(X, L^k). \tag{11.5}$$

From Parseval's formula and (11.3), we have

$$\begin{aligned}\left\|F^{(q)}_{\delta,k}u\right\|^2_{h^{L^k}}&=\frac{1}{4\pi^2}\int_X\left|\int e^{i<t-\theta,\eta>}u(z,\theta)\tau_\delta(\frac{\eta}{k})d\eta d\theta\right|^2e^{-2k\phi(z)}d\lambda(z)dt\\&=\frac{1}{4\pi^2}\int_X\left|\int e^{i<t,\eta>}(\mathcal{F}u)(z,\eta)\tau_\delta(\frac{\eta}{k})d\eta\right|^2e^{-2k\phi(z)}d\lambda(z)dt\\&=\frac{1}{2\pi}\int\left|(\mathcal{F}u)(z,\eta)\right|^2\left|\tau_\delta(\frac{\eta}{k})\right|^2e^{-2k\phi(z)}d\eta d\lambda(z)\\&\leq\frac{1}{2\pi}\int\left|(\mathcal{F}u)(z,\eta)\right|^2e^{-2k\phi(z)}d\eta d\lambda(z)=\left\|u\right\|^2_{h^{L^k}},\end{aligned}\tag{11.6}$$

where $u\in\Omega^{0,q}_0(X,L^k)$. Thus, we can extend $F^{(q)}_{\delta,k}$ to $L^2_{(0,q)}(X,L^k)$ and

$$\begin{aligned}&F^{(q)}_{\delta,k}:L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k)\text{ is continuous,}\\&\left\|F^{(q)}_{\delta,k}u\right\|_{h^{L^k}}\leq\left\|u\right\|_{h^{L^k}},\quad\forall u\in L^2_{(0,q)}(X,L^k).\end{aligned}\tag{11.7}$$

We need

Lemma 11.1. *Let $u\in L^2_{(0,q)}(X,L^k)$. Then,*

$$(\mathcal{F}F^{(q)}_{\delta,k}u)(z,\eta)=(\mathcal{F}u)(z,\eta)\tau_\delta(\frac{\eta}{k}).\tag{11.8}$$

Proof. Let $u_j\in\Omega^{0,q}_0(X,L^k)$, $j=1,2,\ldots$, with $\lim_{j\to\infty}\left\|u_j-u\right\|_{h^{L^k}}=0$. From (11.7) and (11.4), we see that

$$\mathcal{F}F^{(q)}_{\delta,k}u_j\to\mathcal{F}F^{(q)}_{\delta,k}u\text{ in }L^2_{(0,q)}(X,L^k)\text{ as }j\to\infty.\tag{11.9}$$

From Fourier inversion formula, we have

$$(\mathcal{F}F^{(q)}_{\delta,k}u_j)(z,\eta)=(\mathcal{F}u_j)(z,\eta)\tau_\delta(\frac{\eta}{k}),\quad j=1,\ldots.\tag{11.10}$$

Note that $(\mathcal{F}u_j)(z,\eta)\tau_\delta(\frac{\eta}{k})\to(\mathcal{F}u)(z,\eta)\tau_\delta(\frac{\eta}{k})$ in $L^2_{(0,q)}(X,L^k)$ as $j\to\infty$. From this observation, (11.10) and (11.9), we obtain (11.8). □

The following is straightforward. We omit the proofs.

Lemma 11.2. *We have*

$$\begin{aligned}&F^{(q)}_{\delta,k}:\mathrm{Dom}\,\overline{\partial}_{b,k}\to\mathrm{Dom}\,\overline{\partial}_{b,k},\quad q=0,1,\ldots,n-2,\\&F^{(q+1)}_{\delta,k}\overline{\partial}_{b,k}=\overline{\partial}_{b,k}F^{(q)}_{\delta,k}\ \text{on}\ \mathrm{Dom}\,\overline{\partial}_{b,k},\quad q=0,1,\ldots,n-2,\end{aligned}\tag{11.11}$$

and

$$F^{(q)}_{\delta,k}\Pi^{(q)}_k=\Pi^{(q)}_kF^{(q)}_{\delta,k}\ \text{on}\ L^2_{(0,q)}(X,L^k),\ q=0,1,\ldots,n-1.\tag{11.12}$$

Moreover, for $u\in C^\infty_0(X,L^k)$, we have

$$\overline{\partial}_z\big((\mathcal{F}u)(z,\eta)e^{\eta\mu(z)}\big)e^{-\eta\mu(z)}=(\mathcal{F}\overline{\partial}_{b,k}u)(z,\eta),\tag{11.13}$$

where $\mu\in C^\infty(\Gamma,\mathbb{R})$ is as in the beginning of section 11

11.2. **The small spectral gap property for $\Box^{(0)}_{b,k}$ with respect to $F^{(0)}_{\delta,k}$.** We pause and introduce some notations. Let $\Omega^{0,q}(\Gamma)$ be the space of all smooth $(0,q)$ forms on Γ and let $\Omega^{0,q}_0(\Gamma)$ be the subspace of $\Omega^{0,q}(\Gamma)$ whose elements have compact support in Γ. We take the Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $T^{*0,q}\Gamma$ the bundle of $(0,q)$ forms of Γ so that

$$\{d\overline{z}_{j_1}\wedge d\overline{z}_{j_2}\wedge\cdots\wedge d\overline{z}_{j_q};1\leq j_1<j_2\cdots<j_q\leq n-1\}$$

is an orthonormal basis. Let $\Upsilon\in C^\infty(\Gamma,\mathbb{R})$ and let $(\,\cdot\,|\,\cdot\,)_\Upsilon$ be the L^2 inner product on $\Omega^{0,q}_0(\Gamma)$ given by

$$(\,f\,|\,g\,)_\Upsilon=\int\langle\,f\,|\,g\,\rangle e^{-2\Upsilon(z)}d\lambda(z),\quad f,g\in\Omega^{0,q}_0(\Gamma).$$

Let $L^2_{(0,q)}(\Gamma,\Upsilon)$ denote the completion of $\Omega^{0,q}_0(\Gamma)$ with respect to the inner product $(\cdot\,|\,\cdot)_\Upsilon$. We write $L^2(\Gamma,\Upsilon):=L^2_{(0,0)}(\Gamma,\Upsilon)$. Put

$$H^0(\Gamma,\Upsilon):=\left\{f\in L^2(\Gamma,\Upsilon);\,\overline\partial f=0\right\}.$$

Now, we return to our situation. We first consider $\Gamma=\mathbb{C}^{n-1}$.

Theorem 11.3. *Let $\Gamma=\mathbb{C}^{n-1}$. We assume that there are constants $C_0\geq 1$ and $\epsilon_0>0$ such that*

$$\sum_{j,\ell=1}^{n-1}\frac{\partial^2(\phi+\eta\mu)}{\partial z_j\partial\overline z_\ell}(z)w_j\overline w_\ell\geq\frac{1}{C_0}\sum_{j=1}^{n-1}\left|w_j\right|^2,\ \ \forall(w_1,\ldots,w_{n-1})\in\mathbb{C}^{n-1},\ \ z\in\mathbb{C}^{n-1},\ \ |\eta|\leq\epsilon_0,\tag{11.14}$$

and

$$\phi(z)+\eta\mu(z)\geq\frac{1}{C_0}\left|z\right|^2,\ \ \forall\left|z\right|\geq M,\ \ |\eta|\leq\epsilon_0,\tag{11.15}$$

where $\left|z\right|^2=\sum_{j=1}^{n-1}\left|z_j\right|^2$ and $M>0$ is a constant independent of η. Then, for every $0<\delta\leq\epsilon_0$, we have

$$\left\|F^{(0)}_{\delta,k}(I-\Pi^{(0)}_k)u\right\|^2_{h^{L^k}}\leq\frac{C}{k}\left\|\overline\partial_{b,k}u\right\|^2_{h^{L^k}},\ \ \forall u\in C^\infty_0(X,L^k),\tag{11.16}$$

where $C>0$ is a constant independent of k, δ and u.

In particular, $\Box^{(0)}_{b,k}$ has small spectral gap on X.

Proof. Let $u\in C^\infty_0(X,L^k)$. We consider $F^{(0)}_{\delta,k}(I-\Pi^{(0)}_k)u$. In view of (11.12), we see that $F^{(0)}_{\delta,k}(I-\Pi^{(0)}_k)u=(I-\Pi^{(0)}_k)F^{(0)}_{\delta,k}u$. Put

$$v(z,\eta)=\mathcal{F}F^{(0)}_{\delta,k}(I-\Pi^{(0)}_k)u(z,\eta)e^{\eta\mu(z)}.\tag{11.17}$$

From (11.7), (11.4) and (11.8), we see that $\int\left|v(z,\eta)\right|^2e^{-2\eta\mu(z)-2k\phi(z)}d\lambda(z)d\eta<\infty$ and $v(z,\eta)=0$ if $\eta\notin\operatorname{Supp}\tau_\delta(\frac{\eta}{k})$. From Fubini's Theorem and some elementary real analysis, we know that for every $\eta\in\mathbb{R}$, $v(z,\eta)$ is a measurable function of z and for almost every $\eta\in\mathbb{R}$, $v(z,\eta)\in L^2(\mathbb{C}^{n-1},\eta\mu+k\phi)$ and for every $z\in\mathbb{C}^{n-1}$, $v(z,\eta)$ is a measurable function of η and for almost every $z\in\mathbb{C}^{n-1}$, $\int\left|v(z,\eta)\right|^2d\eta<\infty$. Moreover, let $\beta\in L^2(\mathbb{C}^{n-1},\eta\mu+k\phi)$, then the function

$$f(\eta):=\eta\rightarrow\int v(z,\eta)\overline\beta(z)e^{-2\eta\mu(z)-2k\phi(z)}d\lambda(z)$$

is measurable and $f(\eta)$ is finite for almost every $\eta\in\mathbb{R}$, $f(\eta)=0$ if $\eta\notin\operatorname{Supp}\tau_\delta(\frac{\eta}{k})$ and $f(\eta)\in L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$. We claim that

$$\begin{aligned}&\text{if }\delta\leq\epsilon_0\text{, then for almost every }\eta\in\mathbb{R},\ v(z,\eta)\in L^2(\mathbb{C}^{n-1},\eta\mu+k\phi)\text{ and}\\&(\,v(z,\eta)\,|\,\beta\,)_{\eta\mu+k\phi}=0,\ \ \forall\beta\in H^0(\mathbb{C}^{n-1},\eta\mu+k\phi).\end{aligned}\tag{11.18}$$

From the discussion after (11.17), we know that there is a measurable set A_0 in $\mathbb{R}$ with $|A_0|=0$ such that for every $\eta\notin A_0$, $v(z,\eta)\in L^2(\mathbb{C}^{n-1},\eta\mu+k\phi)$, where $|A_0|$ denote the Lebesgue measure of A_0. From (11.15), we see that $\left\{z^\alpha;\,\alpha\in\mathbb{N}^{n-1}_0\right\}$ is a basis for $H^0(\mathbb{C}^{n-1},\eta\mu+k\phi)$, for every $|\eta|\leq\epsilon_0$, where ϵ_0 is as in (11.15). Let $\eta\notin A_0$. Fix $\alpha\in\mathbb{N}^{n-1}_0$. We consider

$$f_\alpha(\eta)=\int v(z,\eta)\overline z^\alpha e^{-2\eta\mu(z)-2k\phi(z)}d\lambda(z).$$

From the discussion after (11.17), we know that $f_\alpha(\eta)\in L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$. We consider the Fourier transform

$$\hat f_\alpha(\xi)=\int e^{-i\xi\eta}f_\alpha(\eta)d\eta$$

of $f_\alpha(\eta)$. Let $g_\ell\in C^\infty_0(X,L^k)$, $\ell=1,2,\ldots$, such that

$$g_\ell\rightarrow(I-\Pi^{(0)}_k)u\text{ in }L^2(X,L^k)\text{ as }\ell\rightarrow\infty.$$

From (11.4), (11.7) and (11.8), we see that for every $\xi\in\mathbb{R}$,

$$\lim_{\ell\to\infty}\int\mathcal{F}F^{(0)}_{\delta,k}g_\ell(z,\eta)\overline{z}^\alpha e^{-\eta\mu(z)-2k\phi(z)}e^{-i\xi\eta}d\lambda(z)d\eta\to\hat{f}_\alpha(\xi).\tag{11.19}$$

From (11.8) and Parseval's formula, we can check that

$$\begin{aligned}&\int\mathcal{F}F^{(0)}_{\delta,k}g_\ell(z,\eta)\overline{z}^\alpha e^{-\eta\mu(z)-2k\phi(z)}e^{-i\xi\eta}d\lambda(z)d\eta\\&=\int\mathcal{F}g_\ell(z,\eta)\tau_\delta(\frac{\eta}{k})\overline{z}^\alpha e^{-\eta\mu(z)-2k\phi(z)}e^{-i\xi\eta}d\lambda(z)d\eta\\&=\int g_\ell(z,t)(\int\overline{z}^\alpha\tau_\delta(\frac{\eta}{k})e^{-\eta\mu(z)-i\xi\eta-i\eta t}d\eta)e^{-2k\phi(z)}d\lambda(z)dt\\&\to\int(I-\Pi^{(0)}_k)u(z,t)(\int\overline{z}^\alpha\tau_\delta(\frac{\eta}{k})e^{-\eta\mu(z)-i\xi\eta-i\eta t}d\eta)e^{-2k\phi(z)}d\lambda(z)dt\ \text{ as }\ell\to\infty.\end{aligned}\tag{11.20}$$

It is straightforward to check that the function

$$\int z^\alpha\tau_\delta(\frac{\eta}{k})e^{-\eta\mu(z)+i\xi\eta+i\eta t}d\eta\in\operatorname{Ker}\overline{\partial}_{b,k}\bigcap L^2(X,L^k).$$

Thus,

$$\int(I-\Pi^{(0)}_k)u(z,t)(\int\overline{z}^\alpha\tau_\delta(\frac{\eta}{k})e^{-\eta\mu(z)-i\xi\eta-i\eta t}d\eta)e^{-2k\phi(z)}d\lambda(z)dt=0.\tag{11.21}$$

From (11.21), (11.20) and (11.19), we conclude that for every $\xi\in\mathbb{R}$, $\hat{f}_\alpha(\xi)=0$. By L^2 Fourier inversion formula, we conclude that $f_\alpha(\eta)=0$ almost everywhere. Thus, there is a measurable set $A_\alpha\supset A_0$ in $\mathbb{R}$ with $|A_\alpha|=0$ such that for every $\eta\notin A_\alpha$,

$$(\,v(z,\eta)\,|\,z^\alpha\,)_{\eta\mu+k\phi}=0.$$

Put $A=\bigcup_{\alpha\in\mathbb{N}_0^{n-1}}A_\alpha$. Then, $|A|=0$. Note that $\{z^\alpha;\,\alpha\in\mathbb{N}_0^{n-1}\}$ is a basis for $H^0(\mathbb{C}^{n-1},\eta\mu+k\phi)$ if $|\eta|\leq\epsilon_0$. From this observation, we conclude that for every $\eta\notin A$,

$$(\,v(z,\eta)\,|\,\beta\,)_{\eta\mu+k\phi}=0,\quad\forall\beta\in H^0(\mathbb{C}^{n-1},\eta\mu+k\phi).$$

The claim (11.18) follows.

Now, we can prove Theorem 11.3. We assume that $\delta\leq\epsilon_0$. Let $u\in C^\infty_0(X,L^k)$. From (11.11) and (11.13), we have

$$\begin{aligned}&\overline{\partial}_{b,k}F^{(0)}_{\delta,k}(I-\Pi^{(0)}_k)u=F^{(1)}_{\delta,k}\overline{\partial}_{b,k}u,\\&(\mathcal{F}F^{(1)}_{\delta,k}\overline{\partial}_{b,k}u)(z,\eta)=\overline{\partial}_z(\mathcal{F}F^{(0)}_{\delta,k}u(z,\eta)e^{\eta\mu(z)})e^{-\eta\mu(z)}.\end{aligned}\tag{11.22}$$

As before, we put $v(z,\eta)=\big(\mathcal{F}F^{(0)}_{\delta,k}(I-\Pi^{(0)}_k)u\big)(z,\eta)e^{\eta\mu(z)}$ and set

$$\overline{\partial}_z\Big(\big(\mathcal{F}F^{(0)}_{\delta,k}(I-\Pi^{(0)}_k)u\big)(z,\eta)e^{\eta\mu(z)}\Big)=\overline{\partial}_zv(z,\eta):=g(z,\eta).$$

It is easy to see that

$$\begin{aligned}&\overline{\partial}_zg(z,\eta)=0,\\&g(z,\eta)=0\text{ if }\eta\notin\operatorname{Supp}\tau_\delta(\tfrac{\eta}{k}),\\&\int|g(z,\eta)|^2\,e^{-2\eta\mu(z)-2k\phi(z)}d\lambda(z)<\infty,\quad\forall\eta\in\operatorname{Supp}\tau_\delta(\frac{\eta}{k}).\end{aligned}\tag{11.23}$$

From (11.14), we see that there is a $C>0$ such that

$$\sum_{j,\ell=1}^{n-1}\frac{\partial^2(k\phi+\eta\mu)}{\partial z_j\partial\overline{z}_\ell}(z)w_j\overline{w}_\ell\geq\frac{k}{C}\sum_{j=1}^{n-1}|w_j|^2\,,\quad\forall(w_1,\ldots,w_{n-1})\in\mathbb{C}^{n-1},z\in\mathbb{C}^{n-1},\eta\in\operatorname{Supp}\tau_\delta(\frac{\eta}{k}).\tag{11.24}$$

From (11.24) and Hörmander's L^2 estimates (see Lemma 4.4.1. in Hörmander [10]), we conclude that for every $\eta \in \operatorname{Supp} \tau_\delta(\frac{\eta}{k})$, we can find a $\beta_\eta(z) \in L^2_{(0,1)}(\mathbb{C}^{n-1}, \eta\tau + k\phi)$ such that

$$\overline{\partial}_z \beta_\eta(z) = g(z,\eta) \tag{11.25}$$

and

$$\int |\beta_\eta(z)|^2 e^{-2\eta\mu(z)-2k\phi(z)} d\lambda(z) \leq \frac{C}{k} \int |g(z,\eta)|^2 e^{-2\eta\mu(z)-2k\phi(z)} d\lambda(z). \tag{11.26}$$

In view of (11.18), we see that there is a measurable set A in $\mathbb{R}$ with Lebesgue measure zero in $\mathbb{R}$ such that for every $\eta \notin A$, $v(z,\eta) \perp H^0(\mathbb{C}^{n-1}, \eta\mu + k\phi)$. Thus, for every $\eta \notin A$, $v(z,\eta)$ has the minimum L^2 norm with respect to $(\,|\,)_{\eta\mu+k\phi}$ of the solutions $\overline{\partial}\alpha = \overline{\partial}_z v(z,\eta)$. From this observation and (11.26), we conclude that

$$\int |v(z,\eta)|^2 e^{-2\eta\mu(z)-2k\phi(z)} d\lambda(z) \leq \frac{C}{k} \int \left|\overline{\partial}_z v(z,\eta)\right|^2 e^{-2\eta\mu(z)-2k\phi(z)} d\lambda(z), \quad \forall \eta \notin A. \tag{11.27}$$

Thus,

$$\int |v(z,\eta)|^2 e^{-2\eta\mu(z)-2k\phi(z)} d\lambda(z) d\eta \leq \frac{C}{k} \int \left|\overline{\partial}_z v(z,\eta)\right|^2 e^{-2\eta\mu(z)-2k\phi(z)} d\lambda(z) d\eta. \tag{11.28}$$

From the definition of $v(z,\eta)$, (11.4), (11.22) and (11.7), it is straightforward to see that

$$\int |v(z,\eta)|^2 e^{-2\eta\mu(z)-2k\phi(z)} d\lambda(z) d\eta = (2\pi) \int \left|F^{(0)}_{\delta,k}(I - \Pi^{(0)}_k) u(z,t)\right|^2 e^{-2k\phi(z)} d\lambda(z) dt \tag{11.29}$$

and

$$\begin{aligned} &\int \left|\overline{\partial}_z v(z,\eta)\right|^2 e^{-2\eta\mu(z)-2k\phi(z)} d\lambda(z) d\eta \\ &= (2\pi) \int \left|F^{(1)}_{\delta,k} \overline{\partial}_{b,k} u(z,t)\right|^2 e^{-2k\phi(z)} d\lambda(z) dt \\ &\leq (2\pi) \int \left|\overline{\partial}_{b,k} u(z,t)\right|^2 e^{-2k\phi(z)} d\lambda(z) dt. \end{aligned} \tag{11.30}$$

From (11.28), (11.29) and (11.30), we conclude that

$$\begin{aligned} &\left\|F^{(0)}_{\delta,k}(I - \Pi^{(0)}_k) u\right\|^2_{h^{L^k}} = \int \left|F^{(0)}_{\delta,k}(I - \Pi^{(0)}_k) u(z,t)\right|^2 e^{-2k\phi(z)} d\lambda(z) dt \\ &\leq \frac{C}{k} \int \left|\overline{\partial}_{b,k} u(z,t)\right|^2 e^{-2k\phi(z)} d\lambda(z) dt = \frac{C}{k} \left\|\overline{\partial}_{b,k} u\right\|^2_{h^{L^k}}. \end{aligned}$$

Theorem 11.3 follows. ☐

Now, we consider Γ is a bounded strongly pseudoconvex domain in $\mathbb{C}^{n-1}$.

Theorem 11.4. *Let Γ be a bounded strongly pseudoconvex domain in $\mathbb{C}^{n-1}$. Assume that $\mu, \phi \in C^\infty(\widetilde{\Gamma}, \mathbb{R})$, where $\widetilde{\Gamma}$ is an open neighbourhood of $\overline{\Gamma}$. Suppose that $\left(\frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_\ell}(z)\right)^{n-1}_{j,\ell=1}$ is positive definite at each point z of $\widetilde{\Gamma}$. If δ is small enough then*

$$\left\|F^{(0)}_{\delta,k}(I - \Pi^{(0)}_k) u\right\|^2_{h^{L^k}} \leq \frac{C}{k} \left\|\overline{\partial}_{b,k} u\right\|^2_{h^{L^k}}, \quad \forall u \in C^\infty_0(X, L^k), \tag{11.31}$$

where $C > 0$ is a constant independent of k, δ and u.

In particular, $\Box^{(0)}_{b,k}$ has small spectral gap on X.

Proof. Let $u \in C^\infty_0(X, L^k)$. We have

$$\begin{aligned} &\overline{\partial}_{b,k} F^{(0)}_{\delta,k}(I - \Pi^{(0)}_k) u = F^{(1)}_{\delta,k} \overline{\partial}_{b,k} u, \\ &(\mathcal{F} F^{(1)}_{\delta,k} \overline{\partial}_{b,k} u)(z,\eta) = \overline{\partial}_z (\mathcal{F} F^{(0)}_{\delta,k} u(z,\eta) e^{\eta\mu(z)}) e^{-\eta\mu(z)}. \end{aligned} \tag{11.32}$$

As before, we put $v(z,\eta) = \big(\mathcal{F} F^{(0)}_{\delta,k}(I - \Pi^{(0)}_k) u\big)(z,\eta) e^{\eta\mu(z)}$ and set

$$\overline{\partial}_z (\mathcal{F} F^{(0)}_{\delta,k} u(z,\eta) e^{\eta\mu(z)}) = \overline{\partial}_z v(z,\eta) := g(z,\eta).$$

Then,

$$\begin{aligned}&\overline{\partial}_z g(z,\eta)=0,\\&g(z,\eta)=0 \text{ if } \eta\notin[-k\delta,k\delta],\\&\int|g(z,\eta)|^2e^{-2\eta\mu(z)-2k\phi(z)}d\lambda(z)<\infty,\quad\forall\eta\in[-k\delta,k\delta].\end{aligned}\tag{11.33}$$

Since $\left(\frac{\partial^2\phi}{\partial z_j\partial\overline{z}_\ell}(z)\right)_{j,\ell=1}^{n-1}$ is positive definite at each point z of $\widetilde{\Gamma}$, if $\delta>0$ is small enough then there is a $C>0$ such that

$$\sum_{j,\ell=1}^{n-1}\frac{\partial^2(k\phi+\eta\mu)}{\partial z_j\partial\overline{z}_\ell}(z)w_j\overline{w}_\ell\geq\frac{k}{C}\sum_{j=1}^{n-1}|w_j|^2,\quad\forall(w_1,\ldots,w_{n-1})\in\mathbb{C}^{n-1},z\in\Gamma,\eta\in\operatorname{Supp}\tau_\delta(\frac{\eta}{k}).\tag{11.34}$$

We assume that $\delta>0$ is small enough so that (11.34) holds. From (11.34) and Hörmander's L^2 estimates, we conclude that for every $\eta\in[-k\delta,k\delta]$, we can find a $\beta_\eta(z)\in L^2_{(0,1)}(\Gamma,\eta\mu+k\phi)$ such that

$$\overline{\partial}_z\beta_\eta(z)=g(z,\eta)\tag{11.35}$$

and

$$\int|\beta_\eta(z)|^2e^{-2\eta\mu(z)-2k\phi(z)}d\lambda(z)\leq\frac{C}{k}\int|g(z,\eta)|^2e^{-2\eta\mu(z)-2k\phi(z)}d\lambda(z).\tag{11.36}$$

Moreover, since $g(z,\eta)$ is smooth, it is well-known that $\beta_\eta(z)$ can be taken to be dependent smoothly on η and z (see the proof of Lemma 2.1 in Berndtsson [3]). Put

$$\alpha(z,t)=\frac{1}{2\pi}\int\beta_\eta(z)e^{-\eta\mu(z)}e^{i\eta t}1_{[-k\delta,k\delta]}(\eta)d\eta.$$

Since $\beta_\eta(z)$ is smooth with respect to η, $\alpha(z,t)$ is well-defined and $\alpha(z,t)\in C^\infty(X,L^k)$. Moreover, from (11.4), (11.7), (11.36), (11.32) and Parseval's formula, we can check that

$$\begin{aligned}\|\alpha\|^2_{h^{L^k}}&=\int|\alpha(z,t)|^2e^{-2k\phi(z)}d\lambda(z)dt\\&=\int\left|\frac{1}{2\pi}\int\beta_\eta(z)e^{-\eta\mu(z)}e^{i\eta t}1_{[-k\delta,k\delta]}(\eta)d\eta\right|^2e^{-2\phi(z)}dtd\lambda(z)\\&=\frac{1}{2\pi}\int|\beta_\eta(z)|^2e^{-2\eta\mu(z)-2k\phi(z)}d\eta d\lambda(z)\\&\leq\frac{C}{(2\pi)k}\int|g(z,\eta)|^2e^{-2\eta\mu(z)-2k\phi(z)}d\eta d\lambda(z)\\&=\frac{C}{(2\pi)k}\int\left|\mathcal{F}F^{(1)}_{\delta,k}\overline{\partial}_{b,k}u(z,\eta)\right|^2e^{-2k\phi(z)}d\eta d\lambda(z)\\&=\frac{C}{k}\int\left|F^{(1)}_{\delta,k}\overline{\partial}_{b,k}u(z,t)\right|^2e^{-2k\phi(z)}dtd\lambda(z)\\&\leq\frac{C}{k}\int\left|\overline{\partial}_{b,k}u(z,t)\right|^2e^{-2k\phi(z)}dtd\lambda(z)=\frac{C}{k}\left\|\overline{\partial}_{b,k}u\right\|^2_{h^{L^k}}<\infty.\end{aligned}\tag{11.37}$$

Furthermore, it is straightforward to see that

$$\begin{aligned}\overline{\partial}_{b,k}\alpha(z,t)&=\frac{1}{2\pi}\int g(z,\eta)e^{i\eta t}1_{[-k\delta,k\delta]}(\eta)e^{-\eta\mu(z)}d\eta\\&=\frac{1}{2\pi}\int\mathcal{F}F^{(1)}_{\delta,k}\overline{\partial}_{b,k}u(z,\eta)e^{i\eta t}d\eta\\&=F^{(1)}_{\delta,k}\overline{\partial}_{b,k}u(z,t).\end{aligned}\tag{11.38}$$

From (11.37) and (11.38), we conclude that $\overline{\partial}_{b,k}\alpha(z,t)=F^{(1)}_{\delta,k}\overline{\partial}_{b,k}u(z,t)$ and $\|\alpha\|^2_{h^{L^k}}\leq\frac{C}{k}\left\|\overline{\partial}_{b,k}u\right\|^2_{h^{L^k}}$. Since $(I-\Pi^{(0)}_k)F^{(0)}_{\delta,k}u$ has the minimum L^2 norm of the solutions of $\overline{\partial}_{b,k}f=F^{(1)}_{\delta,k}\overline{\partial}_{b,k}u(z,t)$, we conclude that

$$\left\|(I-\Pi_k^{(0)})F_{\delta,k}^{(0)}u\right\|_{h^{L^k}}^2=\left\|F_{\delta,k}^{(0)}(I-\Pi_k^{(0)})u\right\|_{h^{L^k}}^2\leq\|\alpha\|_{h^{L^k}}^2\leq\frac{C}{k}\left\|\overline{\partial}_{b,k}u\right\|_{h^{L^k}}^2.$$

The theorem follows. □

11.3. **Szegö kernel asymptotics on $\Gamma\times\mathbb{R}$, where $\Gamma=\mathbb{C}^{n-1}$ or Γ is a bounded strongly pseudoconvex domain in $\mathbb{C}^{n-1}$.** We fix $0<\delta$, δ is small. Let $D\Subset X$ be any open set. Let $M>0$ be a large constant so that for every $(x,\eta)\in T^*D$ if $|\eta'|>\frac{M}{2}$ then $(x,\eta)\notin\Sigma$, where $\eta'=(\eta_1,\ldots,\eta_{2n-2})$, $|\eta'|=\sqrt{\sum_{j=1}^{2n-2}|\eta_j|^2}$. Fix $D_0\Subset D$. Let $D'\Subset D$ be an open neighbourhood of D_0. Put

$$V:=\{(x,\eta)\in T^*D';\,|\eta'|<M,|\eta_{2n-1}|<\delta\}\,.\tag{11.39}$$

Then $\overline{V}\subset T^*D$. Moreover, if $\delta>0$ is small enough, then $\overline{V}\bigcap\Sigma\subset\Sigma'$, where Σ' is given by (1.5). Let $F_{\delta,k}^{(0)}:L^2(X,L^k)\to L^2(X,L^k)$ be as in (11.5) and (11.7). It is clearly that

$$F_{\delta,k}^{(0)}\equiv\tfrac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik<x-y,\eta>}\alpha(x,\eta,k)d\eta\mod O(k^{-\infty})\text{ at }T^*D_0\bigcap\Sigma$$

is a classical semi-classical pseudodifferential operator on D of order 0, where

$$\begin{aligned}&\alpha(x,\eta,k)\sim\textstyle\sum_{j=0}\alpha_j(x,\eta)k^{-j}\text{ in }S^0_{\rm loc}(1;T^*D),\\&\alpha_j(x,\eta)\in C^\infty(T^*D),\quad j=0,1,\ldots,\\&\alpha_0(x,\eta)=\tau_\delta(\eta_{2n-1}),\end{aligned}$$

with $\alpha(x,\eta,k)=0$ if $|\eta|>M$, for some large $M>0$ and $\operatorname{Supp}\alpha(x,\eta,k)\bigcap T^*D_0\Subset V$. Now, we assume that $\partial\overline{\partial}\phi$ and $\partial\overline{\partial}\mu$ are uniformly bounded on Γ, that is, there is a constant $C_0>0$ such that $\left|\frac{\partial^2\phi}{\partial z_j\partial\overline{z}_\ell}(z)\right|\leq C_0$, $\left|\frac{\partial^2\mu}{\partial z_j\partial\overline{z}_\ell}(z)\right|\leq C_0$, $\forall z\in\Gamma$, $j,\ell=1,\ldots,n-1$. By using the technique in section 7.1 and [15], we can show that for every $\alpha,\beta\in\mathbb{N}_0^{2n-1}$, there is a constant $C_{\alpha,\beta}>0$ independent of k such that

$$\left|\partial_x^\alpha\partial_y^\beta\big(e^{-k\phi(x)}\Pi_k^{(0)}(x,y)e^{k\phi(y)}\big)\right|\leq C_{\alpha,\beta}k^{n+|\alpha|+|\beta|},\quad\forall(x,y)\in(\Gamma\times\mathbb{R})\times(\Gamma\times\mathbb{R}).\tag{11.40}$$

By using (11.40) and integration by parts, it is not difficult to see that $\Pi_k^{(0)}$ is k-negligible away the diagonal with respect to $F_{\delta,k}^{(0)}$ on $\Gamma\times\mathbb{R}$.

From the discussion above, Theorem 11.3, Theorem 11.4 and Theorem 9.5, we obtain one of the main results of this work

Theorem 11.5. *With the notations above, we assume that $\Gamma=\mathbb{C}^{n-1}$ or Γ is a bounded strongly pseudoconvex domain in $\mathbb{C}^{n-1}$ and condition $Y(0)$ holds on X. When $\Gamma=\mathbb{C}^{n-1}$, we assume that there are constants $C_0\geq1$ and $\epsilon_0>0$ such that*

$$\begin{aligned}&\sum_{j,\ell=1}^{n-1}\frac{\partial^2(\phi+\eta\mu)}{\partial z_j\partial\overline{z}_\ell}(z)w_j\overline{w}_\ell\geq\frac{1}{C_0}\sum_{j=1}^{n-1}|w_j|^2,\quad\forall(w_1,\ldots,w_{n-1})\in\mathbb{C}^{n-1},\quad z\in\mathbb{C}^{n-1},\quad|\eta|\leq\epsilon_0,\\&\left|\frac{\partial^2\phi}{\partial z_j\partial\overline{z}_\ell}(z)\right|\leq C_0,\quad\left|\frac{\partial^2\mu}{\partial z_j\partial\overline{z}_\ell}(z)\right|\leq C_0,\quad\forall z\in\Gamma,\quad j,\ell=1,\ldots,n-1,\end{aligned}$$

and

$$\phi(z)+\eta\mu(z)\geq\frac{1}{C_0}|z|^2,\quad\forall\,|z|\geq M,\quad|\eta|\leq\epsilon_0,$$

where $M>0$ is a constant independent of η. When Γ is a bounded strongly pseudoconvex domain in $\mathbb{C}^{n-1}$, we assume that $\mu,\phi\in C^\infty(\widetilde{\Gamma},\mathbb{R})$ and $\left(\frac{\partial^2\phi}{\partial z_j\partial\overline{z}_\ell}(z)\right)_{j,\ell=1}^{n-1}$ is positive definite at each point z of $\widetilde{\Gamma}$, where $\widetilde{\Gamma}$ is an open neighbourhood of $\overline{\Gamma}$. Let $F_{\delta,k}^{(0)}:L^2(X,L^k)\to L^2(X,L^k)$ be the continuous operator given by (11.5) and (11.7) and let $F_{\delta,k}^{(0),*}:L^2(X,L^k)\to L^2(X,L^k)$ be the adjoint of $F_{\delta,k}^{(0)}$ with respect *to* $(\cdot|\cdot)_{h^{L^k}}$. *Let* $D \Subset X$ *be any open set and we fix any* $D_0 \Subset D$. *Put* $P_k := F^{(q)}_{\delta,k}\Pi^{(q)}_k F^{(q),*}_{\delta,k}$. *If* δ *is small, then*

$$\hat{P}_{k,s}(x,y) \equiv \int e^{ik\varphi(x,y,s)}g(x,y,s,k)ds \mod O(k^{-\infty})$$

on D_0, *where* $\varphi(x,y,s) \in C^\infty(\Omega)$ *is as in Theorem 5.29,* (2.3),

$$\begin{aligned}
&g(x,y,s,k) \in S^n_{\rm loc}(1;\Omega)\bigcap C^\infty_0(\Omega),\\
&g(x,y,s,k) \sim \sum_{j=0}^{\infty} g_j(x,y,s)k^{n-j} \text{ in } S^n_{\rm loc}(1;\Omega),\\
&g_j(x,y,s) \in C^\infty_0(\Omega),\quad j=0,1,2,\ldots,
\end{aligned}$$

and for every $(x,x,s) \in \Omega$, $x \in D_0$, $x = (x_1,\ldots,x_{2n-2}) = (z,t)$,

$$g_0(x,x,s) = (2\pi)^{-n}2^{n-1}\left|\det\Big(\Big(\frac{\partial^2(\phi - s\mu)}{\partial z_j\partial\overline{z}_\ell}(z)\Big)^{n-1}_{j,\ell=1}\Big)\right| |\tau_\delta(s)|^2 .$$

Here

$$\begin{aligned}
\Omega :=&\{(x,y,s) \in D\times D\times\mathbb{R};\ (x,-2{\rm Im}\,\overline{\partial}_b\phi(x)+s\omega_0(x)) \in V\bigcap\Sigma,\\
&(y,-2{\rm Im}\,\overline{\partial}_b\phi(y)+s\omega_0(y)) \in V\bigcap\Sigma,\ |x-y|<\varepsilon, \text{ for some } \varepsilon>0\},
\end{aligned}$$

V *is given by* (11.39),

12. The proof of Theorem 5.28

We now prove Theorem 5.28. We will use the same notations and assumptions as section 5.3. Fix $(\hat{x}_0,\hat{x}_0,s_0) \in \hat{\Omega}$, $t_0 > 0$. Put

$$\begin{aligned}
&\hat{x}_0 = (x_0, x_{n,0}),\quad x_0 \in \mathbb{R}^{2n-1},\quad (x_0,x_0,s_0) \in \Omega,\\
&(\hat{x}_0,\hat{\xi}_0) = (\hat{x}_0, t_0\frac{\partial\Phi}{\partial\hat{x}}(\hat{x}_0,\hat{x}_0,s_0)) = (\hat{x}_0, t_0\frac{\partial\Phi_1}{\partial\hat{x}}(\hat{x}_0,\hat{x}_0,s_0)) \in \Big(U\bigcap\hat{\Sigma}\Big)\bigcap T^*\hat{D}.
\end{aligned}$$

From Lemma 5.26, we know that there are $\hat{\varphi}(x,y,s)$, $\hat{\varphi}_1(x,y,s) \in C^\infty(\Lambda)$, where Λ is a small neighbourhood of (x_0,x_0,s_0), such that $\hat{\varphi}(x,y,s)$ and $\hat{\varphi}_1(x,y,s)$ satisfy (5.68), (5.69), (5.70), (5.73) and (5.74) and $\frac{\partial\hat{\varphi}}{\partial y_{2n-1}}(x,y,s) - (\alpha_{2n-1}(y)+s\beta_{2n-1}(y))$, $\frac{\partial\hat{\varphi}_1}{\partial y_{2n-1}}(x,y,s) - (\alpha_{2n-1}(y)+s\beta_{2n-1}(y))$ vanish to infinite order at $x=y$, and $t\hat{\Phi}(\hat{x},\hat{y},s) := t(x_{2n}-y_{2n}+\hat{\varphi}(x,y,s))$ and $t\Phi(\hat{x},\hat{y},s)$ are equivalent for classical symbols at every point of

$$\mathrm{diag}'\Big((U\bigcap\hat{\Sigma})\times(U\bigcap\hat{\Sigma})\Big)\bigcap\Big\{(\hat{x},\hat{x},td_{\hat{x}}\Phi(\hat{x},\hat{x},s),-td_{\hat{x}}\Phi(\hat{x},\hat{x},s)) \in T^*\hat{D};\ (x,x,s)\in\Lambda, t>0\Big\},$$

$t\hat{\Phi}_1(\hat{x},\hat{y},s) := t(x_{2n}-y_{2n}+\hat{\varphi}_1(x,y,s))$ and $t\Phi_1(\hat{x},\hat{y},s)$ are equivalent for classical symbols at every point of

$$\mathrm{diag}'\Big((U\bigcap\hat{\Sigma})\times(U\bigcap\hat{\Sigma})\Big)\bigcap\Big\{(\hat{x},\hat{x},td_{\hat{x}}\Phi_1(\hat{x},\hat{x},s),-td_{\hat{x}}\Phi_1(\hat{x},\hat{x},s)) \in T^*\hat{D};\ (x,x,s)\in\Lambda, t>0\Big\}.$$

Let $\hat{W}$ be a small neighbourhood of $(\hat{x}_0,\hat{x}_0,s_0)$ and let I_0 be a neighbourhood of t_0 in $\mathbb{R}_+$. Put

$$\begin{aligned}
\Lambda_{\widetilde{t\hat{\Phi}}} :=&\{(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{t}\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{x}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}),\widetilde{t}\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{y}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})) \in \mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n};\\
&\widetilde{\hat{\Phi}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}) = 0, \frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{s}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}) = 0, (\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}) \in \hat{W}^{\mathbb{C}}, \widetilde{t} \in I_0^{\mathbb{C}}\},\\
\Lambda_{\widetilde{t\hat{\Phi}_1}} :=&\{(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{t}\frac{\partial\widetilde{\hat{\Phi}}_1}{\partial\widetilde{x}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}),\widetilde{t}\frac{\partial\widetilde{\hat{\Phi}}_1}{\partial\widetilde{y}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})) \in \mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n};\\
&\widetilde{\hat{\Phi}}_1(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}) = 0, \frac{\partial\widetilde{\hat{\Phi}}_1}{\partial\widetilde{s}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}) = 0, (\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s}) \in \hat{W}^{\mathbb{C}}, \widetilde{t} \in I_0^{\mathbb{C}}\}.
\end{aligned} \tag{12.1}$$

From global theory of complex Fourier integral operators of Melin-Sjöstrand [21], we know that $t\hat{\Phi}(\hat{x},\hat{y},s)$ and $t\hat{\Phi}_1(\hat{x},\hat{y},s)$ are equivalent for classical symbols at $(\hat{x}_0,\hat{x}_0,\hat{\xi}_0,-\hat{\xi}_0)\in\operatorname{diag}'\Big((U\bigcap\hat{\Sigma})\times(U\bigcap\hat{\Sigma})\Big)$ in the sense of Melin-Sjöstrand [21] if and only if $\Lambda_{\widetilde{t\hat{\Phi}}}$ and $\Lambda_{\widetilde{t\hat{\Phi}_1}}$ are equivalent in the sense that there is a neighbourhood Q of $(\hat{x}_0,\hat{x}_0,\hat{\xi}_0,-\hat{\xi}_0)$ in $\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n}\times\mathbb{C}^{2n}$, such that for every $N>0$, we have

$$
\begin{aligned}
&\operatorname{dist}(z,\Lambda_{\widetilde{t\hat{\Phi}_1}})\le C_N\left|\operatorname{Im}z\right|^N,\quad\forall z\in Q\bigcap\Lambda_{\widetilde{t\hat{\Phi}}},\\
&\operatorname{dist}(z_1,\Lambda_{\widetilde{t\hat{\Phi}}})\le C_N\left|\operatorname{Im}z_1\right|^N,\quad\forall z_1\in Q\bigcap\Lambda_{\widetilde{t\hat{\Phi}_1}},
\end{aligned}
\tag{12.2}
$$

where $C_N>0$ is a constant independent of z and z_1.

We first assume that $\varphi(x,y,s)$ and $\varphi_1(x,y,s)$ are equivalent at each point of Ω in the sense of Definition 5.27. We take almost analytic extensions of $t\hat{\Phi}$ and $t\hat{\Phi}_1$ such that

$$
\begin{aligned}
&\widetilde{t\hat{\Phi}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=\widetilde{t}\widetilde{\hat{\Phi}}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=\widetilde{t}(\widetilde{x}_{2n}-\widetilde{y}_{2n})+\widetilde{t}\widetilde{\hat{\varphi}}(\widetilde{x},\widetilde{y},\widetilde{s}),\\
&\widetilde{t\hat{\Phi}_1}(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=\widetilde{t}\widetilde{\hat{\Phi}}_1(\widetilde{\hat{x}},\widetilde{\hat{y}},\widetilde{s})=\widetilde{t}(\widetilde{x}_{2n}-\widetilde{y}_{2n})+\widetilde{t}\widetilde{\hat{\varphi}}_1(\widetilde{x},\widetilde{y},\widetilde{s})
\end{aligned}
\tag{12.3}
$$

on $\hat{\Lambda}^{\mathbb{C}}$. Here $\hat{\Lambda}=\Lambda\times I_1$, I_1 is a small open neighbourhood of $x_{n,0}$. We are going to prove that $\Lambda_{\widetilde{t\hat{\Phi}}}$ and $\Lambda_{\widetilde{t\hat{\Phi}_1}}$ are equivalent in the sense of (12.2).

Since $\frac{\partial^2\hat{\varphi}}{\partial s\partial x_{2n-1}}(x,x,s)\neq0$, $\frac{\partial^2\hat{\varphi}_1}{\partial s\partial x_{2n-1}}(x,x,s)\neq0$(see (5.78)), from Malgrange preparation theorem (see Theorem 7.57 in [11]), we have

$$
\begin{aligned}
&\frac{\partial\hat{\varphi}}{\partial s}(x,y,s)=(x_{2n-1}-\beta(x',y,s))g(x,y,s),\\
&\frac{\partial\hat{\varphi}_1}{\partial s}(x,y,s)=(x_{2n-1}-\beta_1(x',y,s))g_1(x,y,s)
\end{aligned}
\tag{12.4}
$$

in some small neighbourhood of (x_0,x_0,s_0), where $\beta,\beta_1,g,g_1\in C^\infty$, $\beta(x',x,s)=\beta_1(x',x,s)=x_{2n-1}$, $g(x,x,s)\neq0$, $g_1(x,x,s)\neq0$. We may take Λ small enough so that (12.4) hold on Λ. It is easy to check that $\frac{\partial^2\hat{\varphi}}{\partial x_{2n-1}\partial s}(x,x,s)=g(x,x,s)$, $\frac{\partial^2\hat{\varphi}}{\partial x_j\partial s}(x,x,s)=-\frac{\partial\beta}{\partial x_j}(x',x,s)g(x,x,s)$, $j=1,\ldots,2n-2$. From this observation and notice that $\frac{\partial^2\hat{\varphi}}{\partial x_j\partial s}(x,x,s)$ is real, $j=1,\ldots,2n-1$, we conclude that

$$
\begin{aligned}
&\operatorname{Re}g(x,x,s)\neq0,\quad\operatorname{Im}g(x,x,s)=0,\\
&-\frac{\partial\operatorname{Im}\beta}{\partial x_j}(x',x,s)=0,\quad j=1,\ldots,2n-2.
\end{aligned}
\tag{12.5}
$$

From (12.4), we conclude that for every $N>0$, there is a constant $C_N>0$ such that for all $(\widetilde{x},\widetilde{y},\widetilde{s})\in\hat{W}^{\mathbb{C}}$,

$$
\begin{aligned}
&\left|\frac{\partial\widetilde{\hat{\varphi}}}{\partial\widetilde{s}}(\widetilde{x},\widetilde{y},\widetilde{s})-\big((\widetilde{x}_{2n-1}-\widetilde{\beta}(\widetilde{x}',\widetilde{y},\widetilde{s}))\widetilde{g}(\widetilde{x},\widetilde{y},\widetilde{s})\big)\right|\le C_N\left|\operatorname{Im}(\widetilde{x},\widetilde{y},\widetilde{s})\right|^N,\\
&\left|\frac{\partial\widetilde{\hat{\varphi}}_1}{\partial\widetilde{s}}(\widetilde{x},\widetilde{y},\widetilde{s})-\big((\widetilde{x}_{2n-1}-\widetilde{\beta}_1(\widetilde{x}',\widetilde{y},\widetilde{s}))\widetilde{g}_1(\widetilde{x},\widetilde{y},\widetilde{s})\big)\right|\le C_N\left|\operatorname{Im}(\widetilde{x},\widetilde{y},\widetilde{s})\right|^N.
\end{aligned}
\tag{12.6}
$$

In view of (12.3), we may take $\hat{\Lambda}$ and $\hat{\Lambda}^{\mathbb{C}}$ small enough so that on $\hat{\Lambda}^{\mathbb{C}}$, $\frac{\partial^2\widetilde{\hat{\varphi}}}{\partial\widetilde{x}_{2n-1}\partial\widetilde{s}}\neq0$, $\frac{\partial^2\widetilde{\hat{\varphi}}_1}{\partial\widetilde{x}_{2n-1}\partial\widetilde{s}}\neq0$, and there are $\delta(\widetilde{x}',\widetilde{y},\widetilde{s})\in C^\infty(\hat{\Lambda}^{\mathbb{C}})$, $\delta_1(\widetilde{x}',\widetilde{y},\widetilde{s})\in C^\infty(\hat{\Lambda}^{\mathbb{C}})$ such that

$$
\frac{\partial\widetilde{\hat{\varphi}}}{\partial\widetilde{s}}((\widetilde{x}',\delta(\widetilde{x}',\widetilde{y},\widetilde{s})),\widetilde{y},\widetilde{s})=\frac{\partial\widetilde{\hat{\varphi}}_1}{\partial\widetilde{s}}((\widetilde{x}',\delta_1(\widetilde{x}',\widetilde{y},\widetilde{s})),\widetilde{y},\widetilde{s})=0\ \text{ on }\hat{\Lambda}^{\mathbb{C}}.
\tag{12.7}
$$

From (12.7) and (12.6), we see that for every $N > 0$ there is a constant $D_N > 0$ such that for all $(\widetilde{x}', \widetilde{y}, \widetilde{s}) \in \hat{\Lambda}^{\mathbb{C}}$,

$$
\begin{aligned}
&\left|\delta(\widetilde{x}', \widetilde{y}, \widetilde{s}) - \widetilde{\beta}(\widetilde{x}', \widetilde{y}, \widetilde{s})\right| \leq D_N \left|\mathrm{Im}\,(\widetilde{x}', \delta(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s})\right|^N, \\
&\left|\delta_1(\widetilde{x}', \widetilde{y}, \widetilde{s}) - \widetilde{\beta}_1(\widetilde{x}', \widetilde{y}, \widetilde{s})\right| \leq D_N \left|\mathrm{Im}\,(\widetilde{x}', \delta_1(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s})\right|^N.
\end{aligned} \tag{12.8}
$$

Since $\frac{\partial \widetilde{\varphi}}{\partial \widetilde{s}}(y', y_{2n-1}, y, s) = \frac{\partial \widetilde{\varphi}_1}{\partial \widetilde{s}}(y', y_{2n-1}, y, s) = 0$, $\delta_1(y', y, s) = \delta(y', y, s) = y_{2n-1}$. Hence there is a constant $C_0 > 0$ such that for all $(\widetilde{x}', \widetilde{y}, \widetilde{s}) \in \hat{\Lambda}^{\mathbb{C}}$,

$$
\begin{aligned}
&\left|\mathrm{Im}\,\delta(\widetilde{x}', \widetilde{y}, \widetilde{s})\right| \leq C_0(\left|\mathrm{Im}\,(\widetilde{x}', \widetilde{y}, \widetilde{s})\right| + |x' - y'|), \\
&\left|\mathrm{Im}\,\delta_1(\widetilde{x}', \widetilde{y}, \widetilde{s})\right| \leq C_0(\left|\mathrm{Im}\,(\widetilde{x}', \widetilde{y}, \widetilde{s})\right| + |x' - y'|).
\end{aligned} \tag{12.9}
$$

From (12.3) and (12.7), we can check that

$$
\begin{aligned}
\Lambda_{\widetilde{t\hat{\Phi}}} =&\{(\widetilde{\hat{x}}, \widetilde{y}, \widetilde{t}\frac{\partial \widetilde{\hat{\Phi}}}{\partial \widetilde{x}}(\widetilde{\hat{x}}, \widetilde{y}, \widetilde{s}), \widetilde{t}\frac{\partial \widetilde{\hat{\Phi}}}{\partial \widetilde{y}}(\widetilde{\hat{x}}, \widetilde{y}, \widetilde{s}); \\
&\quad \widetilde{x}_{2n-1} = \delta(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{x}_{2n} = \widetilde{y}_{2n} - \widetilde{\varphi}(\widetilde{x}', \delta(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s}), (\widetilde{\hat{x}}, \widetilde{y}, \widetilde{s}) \in \hat{\Lambda}^{\mathbb{C}}, \widetilde{t} \in I_0^{\mathbb{C}}\}, \\
\Lambda_{\widetilde{t\hat{\Phi}_1}} =&\{(\widetilde{\hat{x}}, \widetilde{y}, \widetilde{t}\frac{\partial \widetilde{\hat{\Phi}}_1}{\partial \widetilde{x}}(\widetilde{\hat{x}}, \widetilde{y}, \widetilde{s}), \widetilde{t}\frac{\partial \widetilde{\hat{\Phi}}_1}{\partial \widetilde{y}}(\widetilde{\hat{x}}, \widetilde{y}, \widetilde{s}); \\
&\quad \widetilde{x}_{2n-1} = \delta_1(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{x}_{2n} = \widetilde{y}_{2n} - \widetilde{\varphi}_1(\widetilde{x}', \delta_1(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s}), (\widetilde{\hat{x}}, \widetilde{y}, \widetilde{s}) \in \hat{\Lambda}^{\mathbb{C}}, \widetilde{t} \in I_0^{\mathbb{C}}\}.
\end{aligned} \tag{12.10}
$$

From Definition 5.27 and (12.4), it is easy to see that

$$
\begin{aligned}
&\beta(x', y, s) - \beta_1(x', y, s), \quad \widetilde{\varphi}(x', \beta(x', y, s), y, s) - \widetilde{\varphi}_1(x', \beta(x', y, s), y, s), \\
&\frac{\partial \widetilde{\varphi}}{\partial x_j}(x', \beta(x', y, s), y, s) - \frac{\partial \widetilde{\varphi}_1}{\partial x_j}(x', \beta_1(x', y, s), y, s), \quad j = 1, \ldots, 2n-1, \\
&\frac{\partial \widetilde{\varphi}}{\partial y_j}(x', \beta(x', y, s), y, s) - \frac{\partial \widetilde{\varphi}_1}{\partial y_j}(x', \beta_1(x', y, s), y, s), \quad j = 1, \ldots, 2n-1
\end{aligned} \tag{12.11}
$$

vanish to infinite order at $x' = y'$. Thus, if $\hat{\Lambda}^{\mathbb{C}}$ is small, then for every $N > 0$, there is a constant $E_N > 0$ such that

$$
\begin{aligned}
&\left|\widetilde{\beta}(\widetilde{x}', \widetilde{y}, \widetilde{s}) - \widetilde{\beta}_1(\widetilde{x}', \widetilde{y}, \widetilde{s})\right| \leq E_N(\left|\mathrm{Im}\,(\widetilde{x}', \widetilde{y}, \widetilde{s})\right|^N + \left|\mathrm{Re}\,\widetilde{x}' - \mathrm{Re}\,\widetilde{y}'\right|^N), \\
&\left|\widetilde{\varphi}(\widetilde{x}', \widetilde{\beta}(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s}) - \widetilde{\varphi}_1(\widetilde{x}', \widetilde{\beta}_1(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s})\right| \\
&\quad \leq E_N(\left|\mathrm{Im}\,(\widetilde{x}', \widetilde{y}, \widetilde{s})\right|^N + \left|\mathrm{Re}\,\widetilde{x}' - \mathrm{Re}\,\widetilde{y}'\right|^N), \\
&\left|\partial_{\widetilde{x}}\widetilde{\varphi}(\widetilde{x}', \widetilde{\beta}(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s}) - \partial_{\widetilde{x}}\widetilde{\varphi}_1(\widetilde{x}', \widetilde{\beta}_1(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s})\right| \\
&\quad \leq E_N(\left|\mathrm{Im}\,(\widetilde{x}', \widetilde{y}, \widetilde{s})\right|^N + \left|\mathrm{Re}\,\widetilde{x}' - \mathrm{Re}\,\widetilde{y}'\right|^N), \\
&\left|\partial_{\widetilde{y}}\widetilde{\varphi}(\widetilde{x}', \widetilde{\beta}(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s}) - \partial_{\widetilde{y}}\widetilde{\varphi}_1(\widetilde{x}', \widetilde{\beta}_1(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s})\right| \\
&\quad \leq E_N(\left|\mathrm{Im}\,(\widetilde{x}', \widetilde{y}, \widetilde{s})\right|^N + \left|\mathrm{Re}\,\widetilde{x}' - \mathrm{Re}\,\widetilde{y}'\right|^N),
\end{aligned} \tag{12.12}
$$

where $\partial_{\widetilde{x}} = (\frac{\partial}{\partial \widetilde{x}_1}, \ldots, \frac{\partial}{\partial \widetilde{x}_{2n-1}})$, $\partial_{\widetilde{y}} = (\frac{\partial}{\partial \widetilde{y}_1}, \ldots, \frac{\partial}{\partial \widetilde{y}_{2n-1}})$. From (12.8), (12.9), (12.10) and (12.12), we see that there is a small neighbourhood Q of $(\hat{x}_0, \hat{x}_0, \hat{\xi}_0, -\hat{\xi}_0)$ in $\mathbb{C}^{2n} \times \mathbb{C}^{2n} \times \mathbb{C}^{2n} \times \mathbb{C}^{2n}$ such that for all $N > 0$ and every $z \in Q \bigcap \Lambda_{\widetilde{t\hat{\Phi}}}$, $z = (\widetilde{\hat{x}}, \widetilde{y}, \widetilde{t}\frac{\partial \widetilde{\hat{\Phi}}}{\partial \widetilde{x}}(\widetilde{\hat{x}}, \widetilde{y}, \widetilde{s}), \widetilde{t}\frac{\partial \widetilde{\hat{\Phi}}}{\partial \widetilde{y}}(\widetilde{\hat{x}}, \widetilde{y}, \widetilde{s}))$, $\widetilde{x}_{2n-1} = \widetilde{\delta}(\widetilde{x}', \widetilde{y}, \widetilde{s})$, $\widetilde{x}_{2n} = \widetilde{y}_{2n} - \widetilde{\varphi}(\widetilde{x}', \widetilde{\delta}(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s})$, we have

$$
\mathrm{dist}\,(z, \Lambda_{\widetilde{t\hat{\Phi}_1}}) \leq F_N(\left|\mathrm{Im}\,(\widetilde{x}', \widetilde{y}, \widetilde{s})\right|^N + \left|\mathrm{Re}\,\widetilde{x}' - \mathrm{Re}\,\widetilde{y}'\right|^N), \tag{12.13}
$$

where $F_N > 0$ is a constant independent of $z \in Q$. We claim that if Q is small enough then there is a constant $C_1 > 0$ independent of $z \in Q$ such that

$$\begin{aligned}&\left|\mathrm{Im}\,(\widetilde{x}', \widetilde{y}, \widetilde{s})\right|^2 + \left|\mathrm{Re}\,\widetilde{x}' - \mathrm{Re}\,\widetilde{y}'\right|^2 \leq C_1 \left|\mathrm{Im}\,z\right|, \\ &\forall z = (\widetilde{\hat{x}}, \widetilde{\hat{y}}, \widetilde{t}\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{x}}(\widetilde{\hat{x}}, \widetilde{\hat{y}}, \widetilde{s}), \widetilde{t}\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{y}}(\widetilde{\hat{x}}, \widetilde{\hat{y}}, \widetilde{s})) \in Q \bigcap \Lambda_{\widetilde{t\hat{\Phi}}}.\end{aligned} \tag{12.14}$$

From Taylor expansion, it is easy to see that

$$\begin{aligned}&\mathrm{Im}\,\widetilde{\hat{\varphi}}(x', \delta(x', y, s), y, s) \\ &= \mathrm{Im}\,\hat{\varphi}(x', \mathrm{Re}\,\delta(x', y, s), y, s) + \frac{\partial \mathrm{Im}\,\hat{\varphi}}{\partial x_{2n-1}}(x', \mathrm{Re}\,\delta(x', y, s), y, s)\mathrm{Im}\,\delta(x', y, s) + (\left|\mathrm{Im}\,\delta(x', y, s)\right|^2).\end{aligned}$$

From this and note that $\mathrm{Im}\,d_x\hat{\varphi}(x', \mathrm{Re}\,\delta(x', y, s), y, s)|_{x'=y'} = 0$, $\delta(x', y, s) = y_{2n-1}$ if $y = (x', y_{2n-1})$, we see that there is a constant $c_0 > 0$ such that

$$c_0 \left|x' - y'\right|^2 \leq \left|\mathrm{Im}\,\widetilde{\hat{\varphi}}(x', \delta(x', y, s), y, s)\right| + \left|\mathrm{Im}\,\delta(x', y, s)\right| \leq \frac{1}{c_0}\left|x' - y'\right|^2, \tag{12.15}$$

for all $(\hat{x}, \hat{y}, s)$ in a small neighbourhood of $(\hat{x}_0, \hat{y}_0, \hat{s}_0)$. Thus, if $\hat{\Lambda}$ and $\hat{\Lambda}^{\mathbb{C}}$ are small, then

$$\begin{aligned}&\left|\mathrm{Im}\,\widetilde{y}_{2n} - \mathrm{Im}\,\widetilde{\hat{\varphi}}(\widetilde{x}', \delta(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s})\right| + \left|\mathrm{Im}\,\widetilde{y}_{2n}\right| + \left|\mathrm{Im}\,\delta(\widetilde{x}', \widetilde{y}, \widetilde{s})\right| + \left|\mathrm{Im}\,(\widetilde{x}', \widetilde{y}, \widetilde{s})\right| \\ &\geq c_1 \left|\mathrm{Re}\,\widetilde{x}' - \mathrm{Re}\,\widetilde{y}'\right|^2 \quad \text{on } \hat{\Lambda}^{\mathbb{C}},\end{aligned} \tag{12.16}$$

where $c_1 > 0$ is a constant.

We consider Taylor expansion of $t\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{x}_{2n-1}}(y', \delta(y', y, \widetilde{s}), y, \widetilde{s})$ at $\mathrm{Im}\,s = 0$:

$$\begin{aligned}&t\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{x}_{2n-1}}(y', \delta(y', y, \widetilde{s}), y, \widetilde{s}) = t\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{x}_{2n-1}}(y', \delta(y', y, \mathrm{Re}\,\widetilde{s}), y, \mathrm{Re}\,\widetilde{s}) \\ &\quad + t\Big(\frac{\partial^2\widetilde{\hat{\Phi}}}{\partial\widetilde{x}_{2n-1}\partial\widetilde{s}}(y', \delta(y', y, \mathrm{Re}\,\widetilde{s}), y, \mathrm{Re}\,\widetilde{s})\Big)i\mathrm{Im}\,\widetilde{s} + O(\left|\mathrm{Im}\,\widetilde{s}\right|^2).\end{aligned} \tag{12.17}$$

Here we used the fact that $\frac{\partial\delta}{\partial\widetilde{s}}(y', y, \mathrm{Re}\,\widetilde{s}) = 0$ since $\delta(y', y, \mathrm{Re}\,\widetilde{s}) = y_{2n-1}$. Since

$$t\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{x}_{2n-1}}(y', \delta(y', y, \mathrm{Re}\,\widetilde{s}), y, \mathrm{Re}\,\widetilde{s}), \quad t\frac{\partial^2\widetilde{\hat{\Phi}}}{\partial\widetilde{x}_{2n-1}\partial\widetilde{s}}(y', \delta(y', y, \mathrm{Re}\,\widetilde{s}), y, \mathrm{Re}\,\widetilde{s})$$

are real and $t\frac{\partial^2\widetilde{\phi}}{\partial\widetilde{x}_{2n-1}\partial\widetilde{s}}(y', \delta(y', y, \mathrm{Re}\,\widetilde{s}), y, \mathrm{Re}\,\widetilde{s}) \neq 0$, we conclude that there is a constant $c_1 > 0$ such that

$$\left|\mathrm{Im}\,\Big(t\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{x}_{2n-1}}(y', \delta(y', y, \widetilde{s}), y, \widetilde{s})\Big)\right| \geq c_1 \left|\mathrm{Im}\,\widetilde{s}\right|$$

for $\left|\mathrm{Im}\,\widetilde{s}\right|$ is small. Thus, if $\hat{\Lambda}$ and $\hat{\Lambda}^{\mathbb{C}}$ are small, then

$$\begin{aligned}&\left|\mathrm{Im}\,\Big(\widetilde{t}\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{x}_{2n-1}}(\widetilde{x}', \delta(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s})\Big)\right| + C_0\left|\mathrm{Im}\,\Big(\widetilde{t}\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{x}_{2n}}(\widetilde{x}', \delta(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s})\Big)\right| \\ &\quad + \left|\mathrm{Im}\,(\widetilde{x}', \widetilde{y})\right| + \left|x' - y'\right| \geq c_2 \left|\mathrm{Im}\,\widetilde{s}\right| \quad \text{on } \hat{\Lambda}^{\mathbb{C}},\end{aligned} \tag{12.18}$$

where $C_0 > 0$ and $c_2 > 0$ are constants. Note that $\mathrm{Im}\,\Big(\widetilde{t}\frac{\partial\widetilde{\hat{\Phi}}}{\partial\widetilde{x}_{2n}}(\widetilde{x}', \delta(\widetilde{x}', \widetilde{y}, \widetilde{s}), \widetilde{y}, \widetilde{s})\Big) = \mathrm{Im}\,\widetilde{t}$.

From (12.9), (12.10), (12.16) and (12.18), the claim (12.14) follows. From (12.14) and (12.13), we conclude that there is a neighbourhood Q of $(\hat{x}_0, \hat{x}_0, \hat{\xi}_0, -\hat{\xi}_0)$ in $\mathbb{C}^{2n} \times \mathbb{C}^{2n} \times \mathbb{C}^{2n} \times \mathbb{C}^{2n}$, such that for every N, we have $\mathrm{dist}\,(z, \Lambda_{\widetilde{t\hat{\Phi}_1}}) \leq C_N \left|\mathrm{Im}\,z\right|^N$, for all $z \in Q \bigcap \Lambda_{\widetilde{t\hat{\Phi}}}$. We can repeat the procedure above and conclude that there is a neighbourhood Q_1 of $(\hat{x}_0, \hat{x}_0, \hat{\xi}_0, -\hat{\xi}_0)$ in $\mathbb{C}^{2n} \times \mathbb{C}^{2n} \times \mathbb{C}^{2n} \times \mathbb{C}^{2n}$, such that for every N, we have $\mathrm{dist}\,(z_1, \Lambda_{\widetilde{t\hat{\Phi}}}) \leq C_N \left|\mathrm{Im}\,z_1\right|^N$, $\forall z_1 \in Q_1 \bigcap \Lambda_{\widetilde{t\hat{\Phi}_1}}$. We obtain that $t\hat{\Phi}(\hat{x}, \hat{y}, s)$ and $t\hat{\Phi}_1(\hat{x}, \hat{y}, s)$ are equivalent for classical symbols at $(\hat{x}_0, \hat{x}_0, \hat{\xi}_0, -\hat{\xi}_0)$ in the sense of Melin-Sjöstrand [21].

Now, we assume that $t\Phi(\hat{x},\hat{y},s)$ and $t\Phi_1(\hat{x},\hat{y},s)$ are equivalent for classical symbols at each point of

$$\operatorname{diag}'\Big((U\bigcap\hat{\Sigma})\times(U\bigcap\hat{\Sigma})\Big)\bigcap\Big\{(\hat{x},\hat{x},\hat{\xi},-\hat{\xi});\ (\hat{x},\hat{\xi})\in T^*\hat{D}\Big\}.$$

From global theory of complex Fourier integral operators of Melin-Sjöstrand [21], we see that $\Lambda_{t\widetilde{\hat{\Phi}}}$ and $\Lambda_{t\widetilde{\hat{\Phi}}_1}$ are equivalent in the sense of (12.2).

Since $\frac{\partial^2\hat{\varphi}}{\partial s\partial x_{2n-1}}(x,x,s)\neq 0$, $\frac{\partial^2\hat{\varphi}_1}{\partial s\partial x_{2n-1}}(x,x,s)\neq 0$, we may assume that (12.4) hold on Λ. We may take $\hat{\Lambda}$ and $\hat{\Lambda}^{\mathbb{C}}$ small enough so that (12.7), (12.8), (12.9) and (12.10) hold on $\hat{\Lambda}^{\mathbb{C}}$. Since $\frac{\partial\hat{\varphi}}{\partial y_{2n-1}}(x,y,s)-(\alpha_{2n-1}(y)+s\beta_{2n-1}(y))$ and $\frac{\partial\hat{\varphi}_1}{\partial y_{2n-1}}(x,y,s)-(\alpha_{2n-1}(y)+s\beta_{2n-1}(y))$ vanish to infinite order at $x=y$, it is straightforward to see that if $\hat{\Lambda}$ is small enough then for every $N>0$, every $z\in\hat{\Lambda}\bigcap\Lambda_{t\widetilde{\hat{\Phi}}}$, $z=(\hat{x},\hat{y},t\frac{\partial\widetilde{\hat{\Phi}}}{\partial x}(\hat{x},\hat{y},s),t\frac{\partial\widetilde{\hat{\Phi}}}{\partial y}(\hat{x},\hat{y},s))\in\Lambda_{t\widetilde{\hat{\Phi}}}$, $x_{2n-1}=\delta(x',y,s)$, $x_{2n}=y_{2n}-\widetilde{\hat{\varphi}}(x',\delta(x',y,s),y,s)$, there is a constant $c_N>0$ such that

$$\begin{aligned}
&\operatorname{dist}(z,\Lambda_{t\widetilde{\hat{\Phi}}_1})+\left|x'-y'\right|^N\\
&\geq c_N\Big(\left|\delta(x',y,s)-\delta_1(x',y,s)\right|+\Big|\widetilde{\hat{\varphi}}(x',\delta(x',y,s),y,s)-\widetilde{\hat{\varphi}}_1(x',\delta_1(x',y,s),y,s)\Big|\\
&\quad+\Big|d_x\Big(\widetilde{\hat{\varphi}}(x',\delta(x',y,s),y,s)-\widetilde{\hat{\varphi}}_1(x',\delta_1(x',y,s),y,s)\Big)\Big|\\
&\quad+\Big|d_y\Big(\widetilde{\hat{\varphi}}(x',\delta(x',y,s),y,s)-\widetilde{\hat{\varphi}}_1(x',\delta_1(x',y,s),y,s)\Big)\Big|\Big).
\end{aligned}\tag{12.19}$$

It is easy to see that there is a constant $c>0$ such that for every

$$\begin{aligned}
&z=(\hat{x},\hat{y},t\frac{\partial\widetilde{\hat{\Phi}}}{\partial y}(\hat{x},\hat{y},s),t\frac{\partial\widetilde{\hat{\Phi}}}{\partial y}(\hat{x},\hat{y},s))\in\Lambda_{t\widetilde{\hat{\Phi}}},\\
&x_{2n-1}=\delta(x',y,s),\quad x_{2n}=y_{2n}-\widetilde{\hat{\varphi}}(x',\delta(x',y,s),y,s),
\end{aligned}$$

$(\hat{x},\hat{y},s)$ is in a small real neighbourhood of $(\hat{x}_0,\hat{x}_0,s_0)$, t is in a small real neighbourhood of t_0, we have

$$\left|\operatorname{Im}z\right|\leq\frac{1}{c_0}\left|x'-y'\right|.\tag{12.20}$$

From (12.2), (12.8), (12.9), (12.11), (12.19) and (12.20), we see that if Λ and $\hat{\Lambda}$ are small enough then for every $N>0$, there is a constant $B_N>0$ such that on $\hat{\Lambda}$,

$$\begin{aligned}
&\left|\beta(x',y,s)-\beta_1(x',y,s)\right|\leq B_N\left|x'-y'\right|^N,\\
&\Big|\widetilde{\hat{\varphi}}(x',\beta(x',y,s),y,s)-\widetilde{\hat{\varphi}}_1(x',\beta_1(x',y,s),y,s)\Big|\leq B_N\left|x'-y'\right|^N,\\
&\Big|d_x\widetilde{\hat{\varphi}}(x',\beta(x',y,s),y,s)-d_x\widetilde{\hat{\varphi}}_1(x',\beta_1(x',y,s),y,s)\Big|\leq B_N\left|x'-y'\right|^N,\\
&\Big|d_y\widetilde{\hat{\varphi}}(x',\beta(x',y,s),y,s)-d_y\widetilde{\hat{\varphi}}_1(x',\beta_1(x',y,s),y,s)\Big|\leq B_N\left|x'-y'\right|^N.
\end{aligned}\tag{12.21}$$

From (12.4) and (12.21), it is not difficult to see that $\varphi(x,y,s)$ and $\varphi_1(x,y,s)$ are equivalent at each point of Ω in the sense of Definition 5.27. Theorem 5.28 follows.

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Institute of Mathematics, Academia Sinica, 6F, Astronomy-Mathematics Building, No.1, Sec.4, Roosevelt Road, Taipei 10617, Taiwan

E-mail address: chsiao@math.sinica.edu.tw or chinyu.hsiao@gmail.com