Kähler-Ricci flow with unbounded curvature

Shaochuang Huang, Luen-Fai Tam

American Journal of Mathematics, Volume 140, Number 1, February 2018,
pp. 189-220 (Article)

Published by Johns Hopkins University Press
DOI: *https://doi.org/10.1353/ajm.2018.0003*

KÄHLER-RICCI FLOW WITH UNBOUNDED CURVATURE

By SHAOCHUANG HUANG and LUEN-FAI TAM

Abstract. Let $g(t)$ be a smooth complete solution to the Ricci flow on a noncompact manifold such that $g(0)$ is Kähler. We prove that if $|\mathrm{Rm}(g(t))|_{g(t)}$ is bounded by a/t for some $a>0$, then $g(t)$ is Kähler for $t>0$. We prove that there is a constant $a(n)>0$ depending only on n such that the following is true: Suppose $g(t)$ is a smooth complete solution to the Kähler-Ricci flow on a noncompact n-dimensional complex manifold such that $g(0)$ has nonnegative holomorphic bisectional curvature and $|\mathrm{Rm}(g(t))|_{g(t)} \le a(n)/t$, then $g(t)$ has nonnegative holomorphic bisectional curvature for $t>0$. These generalize the results by Wan-Xiong Shi. As applications, we prove that (i) any complete noncompact Kähler manifold with nonnegative complex sectional curvature and maximum volume growth is biholomorphic to $\mathbb{C}^n$; and (ii) there is $\epsilon(n)>0$ depending only on n such that if (M^n,g_0) is a complete noncompact Kähler manifold of complex dimension n with nonnegative holomorphic bisectional curvature and maximum volume growth and if $(1+\epsilon(n))^{-1}h \le g_0 \le (1+\epsilon(n))h$ for some Riemannian metric h with bounded curvature, then M is biholomorphic to $\mathbb{C}^n$.

1. Introduction. In [25], Simon proved that there is a constant $\epsilon(n)>0$ depending only on n such that if (M^n,g_0) is a complete n-dimensional Riemannian manifold and if there is another metric h with curvature bounded by k_0 and

$$(1+\epsilon(n))^{-1}h \le g_0 \le (1+\epsilon(n))h,$$

then the so-called h-flow has a smooth short time solution $g(t)$ such that

$$|\mathrm{Rm}(g(t))|_{g(t)} \le C/t. \tag{1.1}$$

Here h-flow is exactly the usual Ricci-DeTurck flow. We call it h-flow as in [25] for emphasizing the background metric h. For the precise definition of h-flow, see Section 5. The method by Schnürer-Schulze-Simon [21] can be carried over to construct Ricci flow using the above solution to the h-flow. On the other hand, in [2], Cabezas-Rivas and Wilking proved that if (M,g_0) is a complete noncompact Riemannian manifold with nonnegative complex sectional curvature, and if the volume of geodesic ball $B(x,1)$ of radius 1 with center at x is uniformly bounded below away from 0, then the Ricci flow has a smooth complete short time solution with nonnegative complex sectional curvature so that (1.1) holds. Recall that a Riemannian manifold is said to have nonnegative complex sectional curvature if $R(X,Y,\bar{Y},\bar{X}) \ge 0$ for any vectors X,Y in the complexified tangent bundle.

Manuscript received June 15, 2015; revised May 15, 2016.

Research of the second author supported in part by Hong Kong RGC General Research Fund #CUHK 1430514.

American Journal of Mathematics 140 (2018), 189–220.

It is natural to ask the following:

QUESTION. *Suppose g_0 is Kähler. Are the above solutions $g(t)$ to the Ricci flow also Kähler for $t > 0$?*

It is well known that if M is compact or if the flow has bounded curvature, the answer to the above question is yes by [11, 24]. On the other hand, the case where the curvature of g_0 may be unbounded has also been studied before. It was proved by Yang and Zheng [28] that for a $U(n)$-invariant initial Kähler metric on $\mathbb{C}^n$, the solution constructed by Cabezas-Rivas and Wilking is Kähler for $t > 0$, under some additional technical conditions. In this paper, we want to prove the following:

THEOREM 1.1. *If (M^n, g_0) is a complete noncompact Kähler manifold with complex dimension n and if $g(t)$ is a smooth complete solution to the Ricci flow on $M \times [0,T]$, $T > 0$, with $g(0) = g_0$ such that*

$$|\mathrm{Rm}(g(t))|_{g(t)} \leq \frac{a}{t}$$

for some $a > 0$, then $g(t)$ is Kähler for all $0 \leq t \leq T$.

This gives an affirmative answer to the above question. The result is related to previous works on the existence of Kähler-Ricci flows without curvature bound, see [3, 4, 10, 28] for example.

We may apply the theorem to the uniformization conjecture by Yau [29] which states that a complete noncompact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{C}^n$. A previous result by Chau and the second author [5] says that the conjecture is true if the Kähler manifold has maximum volume growth and has *bounded curvature*, see also [6, 17]. Combining this with the above theorem, we have:

COROLLARY 1.1. *Let (M^n, g_0) be a complete noncompact Kähler manifold with complex dimension n and nonnegative complex sectional curvature. Suppose M^n has maximum volume growth. Then M^n is biholomorphic to $\mathbb{C}^n$.*

For Kähler surface ($n = 2$), sectional curvature being nonnegative is equivalent to complex sectional curvature being nonnegative [30]. Hence in particular, any complete noncompact Kähler surface with nonnegative sectional curvature and maximum volume growth is biholomorphic to $\mathbb{C}^2$. We should mention that recently Liu [15, 14] proves that a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature and maximum volume growth is diffeomorphic to the Euclidean space and is biholomorphic to an affine algebraic variety, generalizing a result of Mok [17]. Moreover, if the volume of geodesic balls are close to the Euclidean balls with same radii or if the complex dimension is less than or equal to 3, then the manifold is biholomorphic to $\mathbb{C}^n$.

By Theorem 1.1, we know that from the solution to the h-flow constructed by Simon [25] one can construct a solution to the Kähler-Ricci flow if g_0 is Kähler. In view of the conjecture of Yau, we would like to know whether the nonnegativity of holomorphic bisectional curvature will be preserved by the solution $g(t)$ to the Kähler-Ricci flow. The second result in this paper is the following:

THEOREM 1.2. *There is $0 < a(n) < 1$ depending only on n such that if $g(t)$ is a smooth complete solution to the Kähler-Ricci flow on $M \times [0,T]$ with* $\sup_{x\in M} |\mathrm{Rm}(x,t)| \leq \frac{a}{t}$ *and if $g(0)$ has nonnegative holomorphic bisectional curvature, where M is an n-dimensional noncompact complex manifold, then $g(t)$ also has nonnegative holomorphic bisectional curvature for all $t \in [0,T]$.*

We should mention that in [28], Yang and Zheng proved that the nonnegativity of bisectional curvature is preserved under the Kähler-Ricci flow for $U(n)$-invariant solution on $\mathbb{C}^n$ without any condition on the bound of the curvature.

Following exactly the same method as in [25], one can prove that for any $a > 0$, if $\epsilon(n,a) > 0$ is small in the result of Simon, then curvature of the solution to the h-flow will be bounded by a/t. However, given the results in [25], one may also obtain this estimate using an interpolation inequality by Schnürer-Schulze-Simon [22]. The authors would like to thank the referee for pointing out this fact.

Hence as a corollary to Theorem 1.2, using [5] again, we have:

COROLLARY 1.2. *There exists $\epsilon(n) > 0$ depending only on n such that if (M^n, g_0) is a complete noncompact Kähler manifold of complex dimension n with nonnegative holomorphic bisectional curvature and maximum volume growth, and if there is a Riemannian metric h on M with bounded curvature satisfying $(1+\epsilon(n))^{-1}h \leq g_0 \leq (1+\epsilon(n))h$, then M is biholomorphic to $\mathbb{C}^n$.*

By a result of Xu [27], we also have the following corollary which says that the condition that the curvature is bounded in the uniformization result in [5] can be relaxed to the condition that the curvature is bounded in some integral sense. Namely, we have:

COROLLARY 1.3. *Let (M^n, g_0) be a complete noncompact Kähler manifold of complex dimension $n \geq 2$ with nonnegative holomorphic bisectional curvature and maximum volume growth. Suppose there is $r_0 > 0$ and there is $C > 0$ such that*

$$\left(\frac{1}{V_x(r_0)}\int_{B_x(r_0)} |\mathrm{Rm}|^p\right)^{\frac{1}{p}} \leq C$$

for some $p > n$ for all $x \in M$. Then M is biholomorphic to $\mathbb{C}^n$.

The paper is organized as follows: In Section 2, we prove a maximum principle and apply it in Section 3 to prove Theorem 1.1. In Section 4, we prove Theorem 1.2.

In Section 5, we will construct solutions to the Kähler-Ricci flow with nonnegative holomorphic bisectional curvature through the h-flow.

Acknowledgments. The second author would like to thank Albert Chau for some useful discussions and for bringing our attention to the results in [27]. The authors would also like to thank the referee for some useful suggestions so that the proofs of Lemmas 5.1 and 5.2 can be greatly simplified.

Added in proof. For more recent results on Yau's conjecture, see [13, 16].

2. A maximum principle. In this section, we will prove a maximum principle, which will be used in the proof of Theorem 1.1.

Let (M^n, g_0) be a complete noncompact Riemannian manifold. Let $g(t)$ be a smooth complete solution to the Ricci flow on $M \times [0,T]$, $T > 0$ with $g(0) = g_0$, i.e.,

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\mathrm{Ric}, & \text{on } M \times [0,T]; \\ g(0) = g_0. \end{cases} \tag{2.1}$$

Let Γ and $\bar{\Gamma}$ be the Christoffel symbols of $g(t)$ and $\bar{g} = g(T)$ respectively. Let $A = \Gamma - \bar{\Gamma}$. Then A is a $(1,2)$ tensor. In the following, lower case $c, c_1, c_2, \ldots$ will denote positive constants depending only on n.

LEMMA 2.1. *With the above notation and assumptions, suppose the curvature satisfies* $|\mathrm{Rm}(g(t))|_{g(t)} \le at^{-1}$ *for some positive constant* a*. Then there is a constant* $c = c(n) > 0$*, such that*

(i)

$$\left(\frac{T}{t}\right)^{-ca} \bar{g} \le g(t) \le \left(\frac{T}{t}\right)^{ca} \bar{g};$$

(ii) $|\nabla \mathrm{Rm}| \le Ct^{-\frac{3}{2}}$ *for some constant* $C = C(n,a) > 0$ *depending only on* n, a*;*

(iii)

$$|A|_{\bar{g}} \le Ct^{-\frac{1}{2}-ca},$$

for some constant $C = C(n,T,a) > 0$ *depending only on* n, T *and* a*.*

Proof. (i) follows from the Ricci flow equation.

(ii) is a result in [23], see also [9, Theorem 7.1].

To prove (iii), in local coordinates:

$$\frac{\partial}{\partial t} A_{ij}^k = -g^{kl}\left(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}\right).$$

At a point where $\bar{g}_{ij} = \delta_{ij}$ such that $g_{ij} = \lambda_i \delta_{ij}$, by (i) and (ii), we have

$$\begin{aligned}\left|\frac{\partial}{\partial t}|A|^2_{\bar{g}}\right| &\le C_1 t^{-c_1 a} |\nabla \operatorname{Ric}|_{\bar{g}} |A|_{\bar{g}} \\ &\le C_2 t^{-c_2 a - \frac{3}{2}} |A|_{\bar{g}}\end{aligned}$$

for some constants C_1, C_2 depending only on n, T, a and c_1, c_2 depending only on n. From this the result follows. □

Under the assumption of the lemma, since $g(T)$ is complete and the curvature of $\bar{g} = g(T)$ is bounded by a/T, we can find a smooth function ρ on M such that

$$d_{\bar{g}}(x, x_0) + 1 \le \rho(x) \le C'(d_{\bar{g}}(x, x_0) + 1), \quad |\overline{\nabla}\rho|_{\bar{g}} + |\overline{\nabla}^2 \rho|_{\bar{g}} \le C', \tag{2.2}$$

where $\overline{\nabla}$ is covariant derivative with respect to $\bar{g}$ and $C' > 0$ is a constant depending on n and a/T, see [24, 26].

LEMMA 2.2. *With the same assumptions and notation as in the previous lemma, $\rho(x)$ satisfies*

$$|\nabla \rho| \le C_1 t^{-ca}$$

and

$$|\Delta \rho| \le C_2 t^{-\frac{1}{2} - ca}$$

where C_1, C_2 depend only on n, T, a and $c > 0$ depends only on n. Here ∇ and Δ are the covariant derivative and Laplacian of $g(t)$ respectively.

Proof. The first inequality follows from Lemma 2.1(i). To estimate $\Delta \rho$, at a point where $\bar{g}_{ij} = \delta_{ij}$ and g_{ij} is diagonalized, we have

$$\begin{aligned}\left|\Delta \rho - \overline{\Delta} \rho\right| &= \left|g^{ij} \nabla_i \nabla_j \rho - \bar{g}^{ij} \overline{\nabla}_i \overline{\nabla}_j \rho\right| \\ &\le \left|g^{ij}\left(\nabla_i \nabla_j - \overline{\nabla}_i \overline{\nabla}_j\right)\rho\right| + \left|\left(g^{ij} - \bar{g}^{ij}\right)\overline{\nabla}_i \overline{\nabla}_j \rho\right| \\ &\le |g^{ij} A^k_{ij} \rho_k| + C_3 t^{-c_1 a} \\ &\le C_4 t^{-\frac{1}{2} - c_2 a}\end{aligned}$$

for some constants C_3, C_4 depending only on n, T, a, and c_1, c_2 depending only on n. By the estimates of $\overline{\Delta} \rho$, the second result follows. □

LEMMA 2.3. *Let (M^n, g) be a complete noncompact Riemannian manifold with dimension n and let $g(t)$ be a smooth complete solution to the Ricci flow on $M \times [0, T]$, $T > 0$ with $g(0) = g_0$ such that the curvature satisfies $|\operatorname{Rm}| \le a t^{-1}$ for some $a > 0$.*

Let $f \geq 0$ be a smooth function on $M \times [0,T]$ such that
(i)

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \leq \frac{a}{t} f;$$

(ii) $\frac{\partial^k f}{\partial t^k}\Big|_{t=0} = 0$ *for all* $k \geq 0$*;*
(iii) $\sup_{x\in M} f(x,t) \leq Ct^{-l}$, *for some positive integer* l *for some constant* C.
Then $f \equiv 0$ *on* $M \times [0,T]$.

Proof. We may assume that $T \leq 1$. In fact, if we can prove that $f \equiv 0$ on $M \times [0,T_1]$ where $T_1 = \min\{1,T\}$, then it is easy to see that $f \equiv 0$ on $M \times [0,T]$ because f and the curvature of $g(t)$ are uniformly bounded on $M \times [T_1,T]$.

Let $p \in M$ be a fixed point, and let $d(x,t)$ be the distance between p,x with respect to $g(t)$. By [19] (see also [8, Chapter 18]), for all r_0, if $d(x,t) > r_0$, then

$$\frac{\partial_-}{\partial t} d(x,t) - \Delta d(x,t) \geq -C_0\left(t^{-1}r_0 + \frac{1}{r_0}\right) \tag{2.3}$$

in the barrier sense, for some $C_0 = C_0(n,a)$ depending only on n and a. Here

$$\frac{\partial_-}{\partial t} d(x,t) = \liminf_{h\to 0^+} \frac{d(x,t) - d(x,t-h)}{h}. \tag{2.4}$$

The inequality (2.3) means that for any $\epsilon > 0$, there is a function $\sigma(y)$ near x such that $\sigma(x) = d(x,t)$, $\sigma(y) \geq d(y,t)$ near x, such that σ is C^2 and

$$\frac{\partial_-}{\partial t} d(x,t) - \Delta\sigma(x) \geq -C_0\left(t^{-1}r_0 + \frac{1}{r_0}\right) - \epsilon. \tag{2.5}$$

In the following, we always take $\epsilon = T^{-\frac{1}{2}}$.

Let f be as in the lemma. First we want to prove that for any integer $k > a$ there is a constant B_k such that

$$\sup_{x\in M} f(x,t) \leq B_k t^k. \tag{2.6}$$

Let $F = t^{-k} f$, then

$$\left(\frac{\partial}{\partial t} - \Delta\right) F \leq -\frac{k-a}{t} F \leq 0. \tag{2.7}$$

Let $1 \geq \phi \geq 0$ be a smooth function on $[0,\infty)$ such that

$$\phi(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq 1; \\ 0, & \text{if } s \geq 2, \end{cases}$$

and such that $-C_1 \le \phi' \le 0$, $|\phi''| \le C_1$ for some $C_1 > 0$. Let $\Phi = \phi^m$, where $m > 2$ will be chosen later. Then $\Phi = 1$ on $[0,1]$ and $\Phi = 0$ on $[2,\infty)$, $1 \ge \Phi \ge 0$, $-C(m)\Phi^q \le \Phi' \le 0$, $|\Phi''| \le C(m)\Phi^q$, where $C(m) > 0$ depends on m and C_1, and $q = 1 - \frac{2}{m}$.

For any $r >> 1$, let $\Psi(x,t) = \Phi(\frac{d(x,t)}{r})$. Let

$$\theta(t) = \exp(-\alpha t^{1-\beta}),$$

where $\alpha > 0$, $0 < \beta < 1$ will be chosen later.

We claim that one can choose m, α and β such that for all $r >> 1$

$$H(x,t) = \theta(t)\Psi(x,t)F \le 1$$

on $M \times [0,T]$. If the claim is true, then we have that F is bounded. Hence $f(x,t) \le B_k t^k$.

First note that $\Psi(x,t)$ has compact support in $M \times [0,T]$. By assumption (ii) and the fact f is smooth, we conclude that $H(x,t)$ is continuous on $M \times [0,T]$. Moreover, by (ii) again, $H(x,0) = 0$. Suppose $H(x,t)$ attains a positive maximum at (x_0,t_0) for some $x_0 \in M$, $t_0 > 0$. Suppose $d(x_0,t_0) < r$, then there is a neighborhood U of x and $\delta > 0$ such that $d(x,t) < r$ for $x \in U$ and $|t - t_0| < \delta$. For such (x,t), $H(x,t) = \theta(t)F(x,t)$. Since $H(x_0,t_0)$ is a local maximum, we have

$$\begin{aligned} 0 &\le \left(\frac{\partial}{\partial t} - \Delta\right) H \\ &= \theta' F + \theta \left(\frac{\partial}{\partial t} - \Delta\right) F \\ &\le \theta' F \\ &< 0 \end{aligned}$$

which is a contradiction.

Hence we must have $d(x_0,t_0) \ge r$. If $r >> 1$, then $r \ge T^{\frac{1}{2}}$, and at (x_0,t_0),

$$\frac{\partial_-}{\partial t}|_{t=t_0} d(x,t) - \triangle_{t_0}\sigma(x) \ge -C_0 t_0^{-\frac{1}{2}},$$

by taking $r_0 = t_0^{\frac{1}{2}}$. Here $C_0 > 0$ is a constant depending on n and a, $\sigma(x)$ is a barrier function near x_0.

Let $\widetilde{\Psi}(x) = \Phi(\frac{\sigma(x)}{r})$, and let

$$\tilde{H}(x,t) = \theta(t)\tilde{\Psi}(x)F(x,t)$$

which is defined near x_0 for all t. Moreover,

$$\tilde{H}(x_0,t_0) = H(x_0,t_0)$$

and

$$\tilde{H}(x,t_0) \le H(x,t_0)$$

near x_0 because $\sigma(x) \ge d(x,t_0)$ near x_0 and $\Phi' \le 0$. Hence $\widetilde{H}(x,t_0)$ has a local maximum at (x_0,t_0) as a function of x. So we have

$$\nabla \tilde{H}(x_0,t_0) = 0 \tag{2.8}$$

and

$$\Delta \tilde{H}(x_0,t_0) \le 0. \tag{2.9}$$

At (x_0,t_0),

$$\begin{aligned} 0 &\ge \Delta\left(\theta(t)\widetilde{\Psi}(x)F(x,t)\right) \\ &= \theta\widetilde{\Psi}\Delta F + \theta F\Delta\widetilde{\Psi} + 2\theta\langle\nabla F,\nabla\widetilde{\Psi}\rangle \\ &= \theta\widetilde{\Psi}\Delta F + \theta F\left(\frac{1}{r}\Phi'\Delta\sigma + \frac{1}{r^2}\Phi''|\nabla\sigma|^2\right) - 2\theta\frac{|\nabla\widetilde{\Psi}|^2}{\widetilde{\Psi}}F \\ &\ge \theta\Phi\Delta F + \theta F\left(\frac{1}{r}\Phi'\Delta\sigma + \frac{1}{r^2}\Phi''|\nabla\sigma|^2\right) - \frac{2}{r^2}\theta\frac{\Phi'^2}{\Phi}F \end{aligned} \tag{2.10}$$

where we have used the fact that $\sigma(x) \ge d(x,t_0)$ near x_0 and $\sigma(x_0) = d(x_0,t_0)$ so that $|\nabla\sigma(x_0)| \le 1$. Φ and the derivatives Φ' and Φ'' are evaluated at $\frac{d(x_0,t_0)}{r}$.

On the other hand,

$$\begin{aligned} 0 &\le \liminf_{h\to 0^+}\frac{H(x_0,t_0)-H(x_0,t_0-h)}{h} \\ &= \theta'\Psi F + \theta\Psi\frac{\partial}{\partial t}F + \theta F\liminf_{h\to 0^+}\frac{-\Psi(x_0,t_0-h)+\Psi(x_0,t_0)}{h}. \end{aligned}$$

Now

$$\begin{aligned} -\Psi(x_0,t_0-h)+\Psi(x_0,t_0) &= -\Phi\left(\frac{d(x_0,t_0-h)}{r}\right)+\Phi\left(\frac{d(x_0,t_0)}{r}\right) \\ &= \frac{1}{r}\Phi'(\xi)(d(x_0,t_0)-d(x_0,t_0-h)), \end{aligned}$$

for some ξ between $\frac{1}{r}d(x_0,t_0-h)$ and $\frac{1}{r}d(x_0,t_0)$ which implies

$$\begin{aligned} \liminf_{h\to 0^+}\frac{-\Psi(x_0,t_0-h)+\Psi(x_0,t_0)}{h} &\le \limsup_{h\to 0^+}\frac{-\Psi(x_0,t_0-h)+\Psi(x_0,t_0)}{h} \\ &= \frac{1}{r}\Phi'\frac{\partial_-}{\partial t}d(x_0,t)|_{t=t_0} \end{aligned}$$

because $\Phi' \leq 0$, where Φ' is evaluated at $\frac{1}{r}d(x_0,t_0)$. In the following, C_i will denote positive constants independent of α,β. Combining the above inequality with (2.10), we have at (x_0,t_0):

$$\begin{aligned}
0 &\leq \theta'\Phi F + \theta\Phi\frac{\partial}{\partial t}F + \theta F\frac{1}{r}\Phi'\frac{\partial_-}{\partial t}d(x_0,t_0) \\
&\quad - \theta\Psi\Delta F - \theta F\left(\frac{1}{r}\Phi'\Delta\sigma + \frac{1}{r^2}\Phi''|\nabla\sigma|^2\right) + \frac{2}{r^2}\theta\frac{\Phi'^2}{\Phi}F \\
&\leq \theta'\Phi F + C_2\left(t_0^{-\frac{1}{2}}\Phi^q + \Phi^{2q-1}\right)\theta F \\
&\leq -\alpha(1-\beta)t_0^{-\beta}\theta\Phi F \\
&\quad + C_3\theta\left[t_0^{-\frac{1}{2}}t_0^{-(1-q)(k+l)}(\Phi F)^q + t_0^{-\frac{1}{2}}t_0^{-2(1-q)(k+l)}(\Phi F)^{2q-1}\right] \\
&\leq \theta\left[-\alpha(1-\beta)t_0^{-\beta}\Phi F + C_4 t_0^{-\frac{1}{2}-2(1-q)(k+l)}\left((\Phi F)^q + (\Phi F)^{2q-1}\right)\right]
\end{aligned}$$

where Φ,Φ',Φ'' are evaluated at $d(x_0,t_0)/r$. Now first choose m large enough depending only on k,l so that $\frac{1}{2}+2(1-q)(k+l) = \beta < 1$. Then choose α such that $\alpha(1-\beta) > 2C_4$. Then one can see that we must have $\Phi F \leq 1$. Hence $H = \theta\Phi F \leq 1$ at the maximum point of $H(x,t)$. This completes the proof of the claim.

Next, let $F = t^{-a}f$. Then

$$\left(\frac{\partial}{\partial t} - \Delta\right)F \leq 0.$$

Let ρ be the function in Lemma 2.2, we have

$$|\Delta\rho| \leq C_5 t^{-b}$$

for some $b > 1$. Let $\eta(x,t) = \rho(x)\exp(\frac{2C_5}{1-b}t^{1-b})$. Note that $\eta(x,0) = 0$.

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\eta &= \exp\left(\frac{2C_5}{1-b}t^{1-b}\right)\left(2C_5 t^{-b}\rho - \Delta\rho\right) \\
&\geq C_5 t^{-b}\exp\left(\frac{2C_5}{1-b}t^{1-b}\right) \\
&> 0.
\end{aligned}$$

where we have used the fact that $\rho \geq 1$. Since $F \leq C_6 t^2$ in $M\times[0,T]$. In particular it is bounded. Then for any $\epsilon > 0$

$$\left(\frac{\partial}{\partial t} - \Delta\right)(F - \epsilon\eta - \epsilon t) < 0.$$

There is $t_1 > 0$ depending only on ϵ, C_6 such that $F - \epsilon t < 0$ for $t \leq t_1$. For $t \geq t_1$, $F - \epsilon\eta < 0$ outside some compact set. Hence if $F - \epsilon\eta - \epsilon t > 0$ somewhere, then there exist $x_0 \in M$, $t_0 > 0$ such that $F - \epsilon\eta - \epsilon t$ attains maximum. But this is impossible. So $F - \epsilon\eta - \epsilon t \le 0$. Let $\epsilon \to 0$, we have $F = 0$. $\square$

3. Preservation of the Kähler condition. In this section, we want to prove Theorem 1.1 and give some applications. Recall Theorem 1.1 as follows:

THEOREM 3.1. *If (M^n, g_0) is a complete noncompact Kähler manifold with complex dimension n and if $g(t)$ is a smooth complete solution to the Ricci flow* (2.1) *on $M \times [0,T]$, $T > 0$, with $g(0) = g_0$ such that*

$$|\operatorname{Rm}(g(t))|_{g(t)} \le \frac{a}{t}$$

for some $a > 0$, then $g(t)$ is Kähler for all $0 \le t \le T$.

We will use the setup as in [24, Section 5]. Let $T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $T_{\mathbb{R}}M$, where $T_{\mathbb{R}}M$ is the real tangent bundle. Similarly, let $T^*_{\mathbb{C}}M = T^*_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$, where $T^*_{\mathbb{R}}M$ is the real cotangent bundle. Let $z = \{z^1, z^2, \ldots, z^n\}$ be a local holomorphic coordinate on M, and

$$\begin{cases} z^k = x^k + \sqrt{-1}x^{k+n} \\ x^k \in \mathbb{R}, x^{k+n} \in \mathbb{R}, \qquad k = 1, 2, \ldots, n. \end{cases}$$

In the following:

- $i, j, k, l, \ldots$ denote the indices corresponding to real vectors or real covectors;
- $\alpha, \beta, \gamma, \delta, \ldots$ denote the indices corresponding to holomorphic vectors or holomorphic covectors,
- $A, B, C, D, \ldots$ denote both $\alpha, \beta, \gamma, \delta, \ldots$ and $\bar\alpha, \bar\beta, \bar\gamma, \bar\delta, \ldots$.

Extend $g_{ij}(t)$, $R_{ijkl}(t)$ etc. $\mathbb{C}$-linearly to the complexified bundles. We have:

$$\overline{g_{AB}} = g_{\bar A \bar B}, \quad \overline{R_{ABCD}} = R_{\bar A \bar B \bar C \bar D}.$$

In our convention, $R_{1221} = R(e_1, e_2, e_2, e_1)$ is the sectional curvature of the two-plane spanned by orthonormal pair e_1, e_2. R_{ABCD} has the same symmetry as R_{ijkl} and it satisfies the Bianchi identities.

Let $g^{AB} := (g^{-1})^{AB}$, it means $g^{AB} g_{BC} = \delta^A_C$, and let

$$R_{AB} = g^{CD} R_{ACDB}$$

on $M \times [0,T]$. Then we have

$$\frac{\partial}{\partial t} g_{AB} = -2R_{AB} \tag{3.1}$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial t} R_{ABCD} \\
&= \triangle R_{ABCD} - 2g^{EF} g^{GH} R_{EABG} R_{FHCD} \\
&\quad - 2g^{EF} g^{GH} R_{EAGD} R_{FBHC} + 2g^{EF} g^{GH} R_{EAGC} R_{FBHD} \\
&\quad - g^{EF} (R_{EBCD} R_{FA} + R_{AECD} R_{FB} + R_{ABED} R_{FC} + R_{ABCE} R_{FD})
\end{aligned} \tag{3.2}
$$

on $M \times [0, T]$, see [24].

We begin with the following lemma:

LEMMA 3.1. *Let (M, g_0) be a Kähler manifold, and $g(t)$ be a smooth solution to the Ricci flow with $g(0) = g_0$. In the above set up, we have*

$$\frac{\partial^k}{\partial t^k} R_{AB\gamma\delta}|_{t=0} = 0$$

at each point of M for all $k \geq 0$ and for all A, B, γ, δ.

Proof. Let $p \in M$ with holomorphic local coordinate z. In the following, all computations are at $(z, 0)$ unless we have emphasis otherwise. We will prove the lemma by induction. Consider the following statement:

$$
H(k)\begin{cases}
H_1(k): \dfrac{\partial^k}{\partial t^k} R_{AB\gamma\delta} = 0 & \\[2ex]
H_2(k): \dfrac{\partial^k}{\partial t^k} g_{AB} = 0 & \text{if } A, B \text{ are of the same type} \\[2ex]
H_3(k): \dfrac{\partial^k}{\partial t^k} R_{AB} = 0 & \text{if } A, B \text{ are of the same type} \\[2ex]
H_4(k): \dfrac{\partial^k}{\partial t^k} \Gamma^C_{AB} = 0 & \text{unless } A, B, C \text{ are of the same type} \\[2ex]
H_5(k): \dfrac{\partial^k}{\partial t^k} R_{AB\gamma\delta;E} = 0 & \\[2ex]
H_6(k): \dfrac{\partial^k}{\partial t^k} R_{AB\gamma\delta;EF} = 0 & \\[2ex]
H_7(k): \dfrac{\partial^k}{\partial t^k} \triangle R_{AB\gamma\delta} = 0. &
\end{cases}
$$

Here we denote covariant derivative with respect to $g(t)$ by ";" and the partial derivative by ",". If $H_i(k)$ are true for all $i = 1, \ldots, 7$, we will say that $H(k)$ holds. As usual:

$$\Gamma^C_{AB} = \frac{1}{2} g^{CD} (g_{AD,B} + g_{DB,A} - g_{AB,D}).$$

We first consider the case that $k=0$. Since the initial metric is Kähler, it is easy to see that $H(0)$ holds. Now we assume $H(i)$ holds for all $i=0,1,2,\dots,k$. We want to show $H(k+1)$ holds. We first see that

$$\begin{aligned}
&\frac{\partial^{k+1}}{\partial t^{k+1}}R_{AB\gamma\delta}\\
&=\frac{\partial^k}{\partial t^k}(\triangle R_{AB\gamma\delta})-\sum_{\substack{m+n+p+q=k\\0\le m,n,p,q\le k}}2(g^{EF})_m(g^{GH})_n(R_{EABG})_p(R_{FH\gamma\delta})_q\\
&\quad-\sum_{\substack{m+n+p+q=k\\0\le m,n,p,q\le k}}2(g^{EF})_m(g^{GH})_n(R_{EAG\delta})_p(R_{FBH\gamma})_q\\
&\quad+\sum_{\substack{m+n+p+q=k\\0\le m,n,p,q\le k}}2(g^{EF})_m(g^{GH})_n(R_{EAG\gamma})_p(R_{FBH\delta})_q\\
&\quad-\sum_{\substack{m+n+p=k\\0\le m,n,p\le k}}(g^{EF})_m(R_{AF})_n(R_{EB\gamma\delta})_p-\sum_{\substack{m+n+p=k\\0\le m,n,p\le k}}(g^{EF})_m(R_{BF})_n(R_{AE\gamma\delta})_p\\
&\quad-\sum_{\substack{m+n+p=k\\0\le m,n,p\le k}}(g^{EF})_m(R_{\gamma F})_n(R_{ABE\delta})_p-\sum_{\substack{m+n+p=k\\0\le m,n,p\le k}}(g^{EF})_m(R_{\delta F})_n(R_{AB\gamma E})_p.
\end{aligned}$$

Here $(\cdot)_p=\frac{\partial^p}{\partial t^p}(\cdot)$.

Suppose $(g_{AB})_p=0$ at $t=0$ if A,B are of the same type for $p=0,1,\dots,k$, then it is also true that $(g^{AB})_p=0$ if A,B are of the same type for $p=0,1,\dots,k$. On the other hand, in the R.H.S. of the above equality, the derivative of each term with respect to t is only up to order k, by the induction hypothesis, $H_1(k+1)$ holds. Now

$$\frac{\partial}{\partial t}g_{AB}=-2R_{AB},$$

it is easy to see that $H_2(k+1)$ holds because $H_3(k)$ holds.

Since

$$\frac{\partial^{k+1}}{\partial t^{k+1}}R_{\alpha\beta}=\sum_{\substack{m+n=k+1\\0\le m,n\le k+1}}(g^{CD})_m(R_{\alpha CD\beta})_n,$$

and since that $H_1(k+1)$ and $H_2(k+1)$ hold, we conclude that $H_3(k+1)$ holds. Here we have used the symmetries of R_{ABCD}.

Since

$$\begin{aligned}\frac{\partial^{k+1}}{\partial t^{k+1}}\Gamma^{\alpha}_{A\bar\beta} &= -\sum_{\substack{m+n=k\\0\le m,n\le k}}(g^{\alpha D})_m\big(R_{\bar\beta D;A}+R_{AD;\bar\beta}-R_{A\bar\beta;D}\big)_n\\ &= -\sum_{\substack{m+n=k\\0\le m,n\le k}}(g^{\alpha\bar\sigma})_m\big(R_{\bar\beta\bar\sigma;A}+R_{A\bar\sigma;\bar\beta}-R_{A\bar\beta;\bar\sigma}\big)_n,\end{aligned}$$

by the induction hypothesis. If $A=\bar\gamma$, then each term on the R.H.S. is zero by the induction hypothesis. If $A=\gamma$, then

$$(R_{\bar\beta\bar\sigma;\gamma})_n=(R_{\bar\beta\bar\sigma,\gamma})_n-(\Gamma^E_{\gamma\bar\sigma}R_{E\bar\beta})_n-(\Gamma^E_{\gamma\bar\beta}R_{E\bar\sigma})_n,$$

so it vanishes because $n\le k$. On the other hand,

$$\begin{aligned}R_{\gamma\bar\sigma;\bar\beta}-R_{\gamma\bar\beta;\bar\sigma} &= g^{CD}(R_{\gamma CD\bar\sigma;\bar\beta}-R_{\gamma CD\bar\beta;\bar\sigma})\\ &= g^{CD}(R_{\gamma CD\bar\sigma;\bar\beta}+R_{\gamma C\bar\sigma D;\bar\beta}+R_{\gamma C\bar\beta\bar\sigma;D})\\ &= g^{CD}R_{\gamma C\bar\beta\bar\sigma;D}.\end{aligned}$$

So

$$\big(R_{\gamma\bar\sigma;\bar\beta}-R_{\gamma\bar\beta;\bar\sigma}\big)_n=0$$

for $n\le k$ by the induction hypothesis. Thus,

$$\frac{\partial^{k+1}}{\partial t^{k+1}}\Gamma^{\alpha}_{A\bar\beta}=0$$

at $t=0$. Since $\Gamma^C_{AB}=\Gamma^C_{BA}$ and $\overline{\Gamma^C_{AB}}=\Gamma^{\bar C}_{\bar A\bar B}$, it is easy to see that $H_4(k+1)$ holds.

Next,

$$\begin{aligned}R_{AB\gamma\delta;E} &= R_{AB\gamma\delta,E}-\Gamma^G_{EA}R_{GB\gamma\delta}-\Gamma^G_{EB}R_{AG\gamma\delta}\\ &\quad -\Gamma^G_{E\gamma}R_{ABG\delta}-\Gamma^G_{E\delta}R_{AB\gamma G}.\end{aligned}$$

By $H_1(k+1)$, we have

$$\frac{\partial^{k+1}}{\partial t^{k+1}}R_{AB\gamma\delta,E}=\left(\frac{\partial^{k+1}}{\partial t^{k+1}}R_{AB\gamma\delta}\right)_E=0.$$

Since $H_1(i)$ and $H_4(i)$ are true for $0\le i\le k+1$, $H_5(k+1)$ is true. Since $H_1(i)$, $H_4(i)$ and $H_5(i)$ are true for $0\le i\le k+1$, $H_6(k+1)$ is true. Finally $H_6(i)$ is true for $0\le i\le k+1$ implies that $H_7(k+1)$ holds. Therefore, $H(k+1)$ holds. □

Now we use the Uhlenbeck's trick to simplify the evolution equation of the complex curvature tensor. We pick an abstract vector bundle V over M which is isomorphic to $T_{\mathbb{C}}M$ and denote the isomorphism $u_0 : V \to T_{\mathbb{C}}M$. We take $\{e_A := u_0^{-1}(\frac{\partial}{\partial z^A})\}$ as a basis of V. We also consider the metric h on V given by $h := u_0^* g_0$. We let u_0 evolute by

$$\begin{cases} \dfrac{\partial}{\partial t} u(t) = \mathrm{Ric} \circ u(t), \\ \quad u(0) = u_0. \end{cases}$$

In local coordinate, we have

$$\begin{cases} \dfrac{\partial}{\partial t} u_B^A = g^{AC} R_{CD} u_B^D, \\ u_B^A(0) = \delta_B^A. \end{cases}$$

Consider metric $h(t) := u^*(t)g(t)$ on V for each $t \in [0,T]$. It is easy to see that $\frac{\partial}{\partial t} h(t) \equiv 0$ for all t, so $h(t) \equiv h$ for all t. We use $u(t)$ to pull the curvature tensor on $T_{\mathbb{C}}M$ back to V:

$$\widetilde{Rm}(e_A, e_B, e_C, e_D) := R(u(e_A), u(e_B), u(e_C), u(e_D)).$$

In local coordinate, we have

$$\tilde{R}_{ABCD} = R_{EFGH} u_A^E u_B^F u_C^G u_D^H$$

on $M \times [0,T]$. One can also check that

$$\overline{h_{AB}} = h_{\bar{A}\bar{B}}, \quad \overline{\tilde{R}_{ABCD}} = \tilde{R}_{\bar{A}\bar{B}\bar{C}\bar{D}}.$$

Define a connection on V in the following: For any smooth section ξ on V, $X \in T_{\mathbb{C}}M$,

$$D_X^t \xi = u^{-1}(\nabla_X^t (u(\xi))).$$

One can check that $D^t h = 0$ and $D^t u = 0$. We define $\triangle$ acting on any tensor on V by

$$\triangle := g^{EF} D_E^t D_F^t.$$

Then by (3.2), the evolution equation of $\tilde{R}$ is:

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{R}_{ABCD} = {} & \triangle \tilde{R}_{ABCD} - 2h^{EF} h^{GH} R_{EABG} R_{FHCD} \\ & - 2h^{EF} h^{GH} \tilde{R}_{EAGD} \tilde{R}_{FBHC} + 2h^{EF} h^{GH} \tilde{R}_{EAGC} \tilde{R}_{FBHD} \end{aligned} \tag{3.3}$$

where $h^{AB} = (h^{-1})^{AB}$.

LEMMA 3.2. *With the above notations, we have*

$$\frac{\partial^k}{\partial t^k}\tilde{R}_{AB\gamma\delta}=0$$

at $t=0$ *for all* A,B *and* γ,δ.

Proof. Note that we have:

$$\frac{\partial^k}{\partial t^k}\widetilde{R}_{AB\gamma\delta}=\sum_{\substack{m+n+p+q+r=k\\0\le m,n,p,q,r\le k}}(u_A^E)_m(u_B^F)_n(u_\gamma^G)_p(u_\delta^H)_q(R_{EFGH})_r.$$

By Lemma 3.1, in order to prove the lemma, it is sufficient to prove that $\frac{\partial^k}{\partial t^k}u_{\bar\beta}^{\alpha}=0$ and $\frac{\partial^k}{\partial t^k}u_{\beta}^{\bar\alpha}=0$ for all k for all α,β at $t=0$.

Recall that

$$\begin{cases}\dfrac{\partial}{\partial t}u_B^A=g^{AC}R_{CD}u_B^D,\\ u_B^A(0)=\delta_B^A.\end{cases}$$

Hence $u_{\bar\beta}^{\alpha}=0$ and $u_{\beta}^{\bar\alpha}=0$. By induction, Lemma 3.1, and the fact that $u_B^A(0)=\delta_B^A$, one can prove that show $\frac{\partial^k}{\partial t^k}u_{\bar\beta}^{\alpha}=0$ and $\frac{\partial^k}{\partial t^k}u_{\beta}^{\bar\alpha}=0$ for all k. This completes the proof of the lemma. □

Proof of Theorem 3.1. As in [24], define a smooth function φ on $M\times[0,T]$ by

$$\begin{aligned}\varphi(z,t)=&\,h^{\alpha\bar\xi}h^{\beta\bar\zeta}h^{\gamma\bar\sigma}h^{\delta\bar\eta}\tilde{R}_{\alpha\beta\gamma\delta}\tilde{R}_{\bar\xi\bar\zeta\bar\sigma\bar\eta}+h^{\bar\alpha\xi}h^{\bar\beta\zeta}h^{\gamma\bar\sigma}h^{\delta\bar\eta}\tilde{R}_{\bar\alpha\bar\beta\gamma\delta}\tilde{R}_{\xi\zeta\bar\sigma\bar\eta}\\&+h^{\bar\alpha\xi}h^{\beta\bar\zeta}h^{\gamma\bar\sigma}h^{\delta\bar\eta}\tilde{R}_{\bar\alpha\beta\gamma\delta}\tilde{R}_{\xi\bar\zeta\bar\sigma\bar\eta}+h^{\alpha\bar\xi}h^{\bar\beta\zeta}h^{\gamma\bar\sigma}h^{\delta\bar\eta}\tilde{R}_{\alpha\bar\beta\gamma\delta}\tilde{R}_{\bar\xi\zeta\bar\sigma\bar\eta}.\end{aligned}\tag{3.4}$$

One can check φ is well defined (independent of coordinate changes on M) and is nonnegative. The evolution equation of φ is (See [24]):

$$\left(\frac{\partial}{\partial t}-\triangle\right)\varphi=\tilde{R}_{CDEF}*\tilde{R}_{GH\alpha\beta}*\tilde{R}_{AB\gamma\delta}-2g^{EF}\tilde{R}_{AB\gamma\delta;E}\overline{\tilde{R}_{AB\gamma\delta}}_{F}.\tag{3.5}$$

As the real case, define the norm of the complex curvature tensor by:

$$|R_{ABCD}(t)|^2_{g(t)}=g^{AE}g^{BF}g^{CG}g^{DH}R_{ABCD}R_{EFGH}.$$

Then we have

$$|R_{ABCD}(t)|=|R_{ijkl}(t)|\le\frac{a}{t}$$

on $M \times [0,T]$ by assumption. By the definition of $\widetilde{R}_{ABCD}$, we also have:

$$|\widetilde{R}_{ABCD}(t)| = |R_{ABCD}(t)| \le \frac{a}{t}.$$

Combining with (3.5), we have

$$\left(\frac{\partial}{\partial t} - \triangle\right)\varphi \le \frac{C_1}{t}\varphi$$

on $M \times [0,T]$ for some constant C_1. Moreover,

$$\varphi \le |\widetilde{R}_{ABCD}(t)|^2 \le a^2/t^2.$$

On the other hand, by (3.4), Lemma 3.2 and the fact that h is independent of t, we conclude that at $t = 0$,

$$\frac{\partial^k}{\partial t^k}\varphi = 0,$$

for all k. By Lemma 2.3, we conclude that $\varphi \equiv 0$ on $M \times [0,T]$. As in [24], we conclude that $g(t)$ is Kähler for all $t > 0$. $\square$

COROLLARY 3.1. *Let (M^n, g_0) be a complete noncompact Kähler manifold of complex dimension n with nonnegative complex sectional curvature. Suppose*

$$\inf_{p \in M}\{V_p(1)|\, p \in M\} = v_0 > 0.$$

where $V_p(1)$ is the volume of the geodesic ball with radius 1 and center at p with respect to g_0. Then there is $T > 0$ depending only on n, v_0 such that the Kähler-Ricci flow has a smooth complete solution on $M \times [0,T]$ with initial data $g(0) = g_0$ and such that $g(t)$ has nonnegative complex sectional curvature. Moreover the curvature satisfies:

$$|\mathrm{Rm}(g(t))|_{g(t)} \le \frac{c}{t}$$

where c is a constant depending only on n, v_0.

Proof. The corollary follows immediately from the result of Cabezas-Rivas and Wilking [2], and Theorem 3.1. $\square$

COROLLARY 3.2. *Let (M^n, g_0) be a complete noncompact Kähler manifold with complex dimension n and nonnegative complex sectional curvature. Suppose M^n has maximum volume growth. Then M^n is biholomorphic to $\mathbb{C}^n$.*

Proof. By volume comparison, we have

$$\inf_{p \in M}\{V_p(1)|\, p \in M\} = v_0 > 0.$$

Let $g(t)$ be the solution to the Kähler-Ricci flow on $M \times [0,T]$ obtained as in Corollary 3.1. Then for all $t > 0$, $g(t)$ has nonnegative complex sectional curvature and the curvature of $g(t)$ is bounded. We want to prove that $g(t)$ has maximum volume growth.

Let $p \in M$ and let $r > 0$ be fixed. Let $\tilde{g}(s) = r^{-2}g(r^2 s)$, $0 \le s \le r^{-2}T$. Then $\tilde{g}(s)$ is a solution to the Kähler-Ricci flow with initial data $\tilde{g}(0) = r^{-2}g_0$. Since the sectional curvature of $\tilde{g}(s)$ is nonnegative, as in [2], using a result of [20], one can prove that:

$$V_p(\tilde{g}(s),1) - V_p(r^{-2}g_0,1) = V_p(\tilde{g}(s),1) - V_p(\tilde{g}(0),1) \ge -c_n s$$

where $V_p(h,1)$ denotes the volume of the geodesic ball with radius 1 and center at p with respect to h and c_n is a positive constant depending only on n. Now

$$V_p(r^{-2}g_0,1) = \frac{V_p(g_0,r)}{r^{2n}} \ge v_0 > 0$$

because g_0 has maximum volume growth. Hence there is $r_0 > 0$ such that if $r \ge r_0$, then

$$r^{-2n}V_p(g(r^2 s),r) = V_p(\tilde{g}(s),1) \ge C_1$$

for some constant C_1 independent of s and r for all $0 \le s \le r^{-2}T$. Fix $t_0 > 0$, and let s be such that $r^2 s = t_0$. Then $s \le r^{-2}T$. So we have

$$r^{-2n}V_p(g(t_0),r) \ge C_1$$

if r is large enough. That is, $g(t_0)$ has maximum volume growth. By [5], we conclude that M is biholomorphic to $\mathbb{C}^n$. □

4. Preservation of non-negativity of holomorphic bisectional curvature. Let (M^n, g_0) be a complete noncompact Kähler manifold with complex dimension n. We want to study the preservation of non-negativity of holomorphic bisectional curvature under Kähler-Ricci flow, without assuming the curvature is bounded in space and time.

Let us first define a quadratic form for any $(0,4)$-tensor T on $T_{\mathbb{C}}M$ with a metric g by

$$\begin{aligned} Q(T)(X,\bar{X},Y,\bar{Y}) := &\sum_{\mu,\nu=1}^{n} (|T_{X\bar{\mu}\nu\bar{Y}}|^2 - |T_{X\bar{\mu}Y\bar{\nu}}|^2 + T_{X\bar{X}\nu\bar{\mu}}T_{\mu\bar{\nu}Y\bar{Y}}) \\ &- \sum_{\mu=1}^{n} \mathrm{Re}(T_{X\bar{\mu}}T_{\mu\bar{X}Y\bar{Y}} + T_{Y\bar{\mu}}T_{X\bar{X}\mu\bar{Y}}) \end{aligned}$$

for all $X, Y \in T^{1,0}_{\mathbb{C}}M$, where $T_{\alpha\bar{\beta}\gamma\bar{\delta}} = T(e_\alpha, \bar{e}_\beta, e_\gamma, \bar{e}_\delta)$, $T_{\alpha\bar{\beta}} = g^{\gamma\bar{\delta}}T_{\alpha\bar{\beta}\gamma\bar{\delta}}$ and $\{e_1,\ldots,e_n\}$ is a unitary frame with respect to the metric of g, $T_{X\bar{\mu}\nu\bar{Y}} =$

$T(X,\bar{e}_\mu,e_\nu,\bar{Y})$ etc. Here T is a tensor has the following properties:

$$\overline{T(A,B,C,D)} = T(\bar{A},\bar{B},\bar{C},\bar{D});$$
$$T(A,B,C,D) = T(C,D,A,B) = T(B,A,D,C)$$

for all $A,B,C,D \in T_{\mathbb{C}}M$ and

$$T(X,\bar{Y},Z,\bar{W}) = T(X,\bar{W},Z,\bar{Y})$$

for all $X,Y,Z,W \in T_{\mathbb{C}}^{1,0}M$.

Let $g(t)$ be a solution to the Kähler-Ricci flow:

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}}.$$

Recall the evolution equation for holomorphic bisectional curvature: (See [7, Corollary 2.82])

$$\left(\frac{\partial}{\partial t} - \triangle\right) R(X,\bar{X},Y,\bar{Y}) = Q(R)(X,\bar{X},Y,\bar{Y})$$

for all $X,Y \in T_{\mathbb{C}}^{1,0}M$. Here $\triangle$ is with respect to $g(t)$.

Next define a $(0,4)$-tensor B on $T_{\mathbb{C}}M$ (with a metric g) by:

$$B(E,F,G,H) = g(E,F)g(G,H) + g(E,H)g(F,G)$$

for all $E,F,G,H \in T_{\mathbb{C}}M$.

LEMMA 4.1. *In the above notation,* $Q(B)(X,\bar{X},Y,\bar{Y}) \leq 0$ *for all* $X,Y \in T_{\mathbb{C}}^{1,0}M$.

Proof.

$$\begin{aligned} Q(B)(X,\bar{X},Y,\bar{Y}) = &\sum_{\mu,\nu=1}^{n} (|B_{X\bar{\mu}\nu\bar{Y}}|^2 - |B_{X\bar{\mu}Y\bar{\nu}}|^2 + B_{X\bar{X}\nu\bar{\mu}} B_{\mu\bar{\nu}Y\bar{Y}}) \\ &- \sum_{\mu=1}^{n} \mathrm{Re}(B_{X\bar{\mu}} B_{\mu\bar{X}Y\bar{Y}} + B_{Y\bar{\mu}} B_{X\bar{X}\mu\bar{Y}}) \end{aligned}$$

Let $\{e_1,\ldots,e_n\}$ be a unitary frame. $X = \sum_{\mu=1}^{n} X^\mu e_\mu$, $\sum_{\mu=1}^{n} Y = Y^\mu e_\mu$.

We compute it term by term:

$$\begin{aligned}
\sum_{\mu,\nu=1}^{n} |B_{X\bar{\mu}\nu\bar{Y}}|^2 &= \sum_{\mu,\nu=1}^{n} (g_{X\bar{\mu}}g_{\nu\bar{Y}} + g_{X\bar{Y}}g_{\bar{\mu}\nu}) \cdot (g_{\bar{X}\mu}g_{\bar{\nu}Y} + g_{\bar{X}Y}g_{\mu\bar{\nu}}) \\
&= \sum_{\mu,\nu=1}^{n} (X^{\mu}\bar{Y}^{\nu} + g_{X\bar{Y}}g_{\bar{\mu}\nu}) \cdot (\bar{X}^{\mu}Y^{\nu} + g_{\bar{X}Y}g_{\mu\bar{\nu}}) \\
&= |X|^2|Y|^2 + (n+2)|g(\bar{X},Y)|^2, \\
\sum_{\mu,\nu=1}^{n} |B_{X\bar{\mu}Y\bar{\nu}}|^2 &= \sum_{\mu,\nu=1}^{n} (g_{X\bar{\mu}}g_{\bar{\nu}Y} + g_{X\bar{\nu}}g_{\bar{\mu}Y}) \cdot (g_{\bar{X}\mu}g_{\nu\bar{Y}} + g_{\bar{X}\nu}g_{\mu\bar{Y}}) \\
&= \sum_{\mu,\nu=1}^{n} (X^{\mu}Y^{\nu} + X^{\nu}Y^{\mu}) \cdot (\bar{X}^{\mu}\bar{Y}^{\nu} + \bar{X}^{\nu}\bar{Y}^{\mu}) \\
&= 2|X|^2|Y|^2 + 2|g(\bar{X},Y)|^2, \\
\sum_{\mu,\nu=1}^{n} B_{X\bar{X}\nu\bar{\mu}}B_{\mu\bar{\nu}Y\bar{Y}} &= (g_{X\bar{X}}g_{\nu\bar{\mu}} + g_{X\bar{\nu}}g_{\bar{X}\nu}) \cdot (g_{\mu\bar{\nu}}g_{Y\bar{Y}} + g_{\mu\bar{Y}}g_{\bar{\nu}Y}) \\
&= (|X|^2g_{\nu\bar{\mu}} + X^{\mu}\bar{X}^{\nu}) \cdot (g_{\mu\bar{\nu}}|Y|^2 + \bar{Y}^{\mu}Y^{\nu}) \\
&= (n+2)|X|^2|Y|^2 + |g(\bar{X},Y)|^2, \\
\sum_{\mu=1}^{n} B_{X\bar{\mu}}B_{\mu\bar{X}Y\bar{Y}} &= \sum_{\mu=1}^{n} g^{k\bar{l}}B_{X\bar{\mu}k\bar{l}}B_{\mu\bar{X}Y\bar{Y}} \\
&= \sum_{\mu,\nu=1}^{n} B_{X\bar{\mu}\nu\bar{\nu}}B_{\mu\bar{X}Y\bar{Y}} \\
&= \sum_{\mu,\nu=1}^{n} (g_{X\bar{\mu}}g_{\nu\bar{\nu}} + g_{X\bar{\nu}}g_{\bar{\mu}\nu}) \cdot (g_{\bar{X}\mu}g_{\bar{Y}Y} + g_{\bar{X}Y}g_{\mu\bar{Y}}) \\
&= \sum_{\mu,\nu=1}^{n} (X^{\mu}g_{\nu\bar{\nu}} + X^{\nu}g_{\bar{\mu}\nu}) \cdot (\bar{X}^{\mu}g_{Y\bar{Y}} + \bar{Y}^{\mu}g_{\bar{X}Y}) \\
&= (n+1)|X|^2|Y|^2 + (n+1)|g(\bar{X},Y)|^2.
\end{aligned}$$

Similarly, we have

$$\sum_{\mu=1}^{n} B_{Y\bar{\mu}}B_{X\bar{X}\mu\bar{Y}} = (n+1)|X|^2|Y|^2 + (n+1)|g(\bar{X},Y)|^2.$$

Therefore,

$$Q(B)(X,\bar{X},Y,\bar{Y}) = -(n+1)(|X|^2|Y|^2 + |g(\bar{X},Y)|^2) \leq 0. \qquad \square$$

We are ready to prove Theorem 1.2:

THEOREM 4.1. *There is* $0 < a(n) < 1$ *depending only on* n *such that if* $g(t)$ *is a smooth complete solution to the Kähler-Ricci flow on* $M \times [0,T]$ *with* $\sup_{x\in M} |\mathrm{Rm}(x,t)| \le \frac{a}{t}$ *and if* $g(0)$ *has nonnegative holomorphic bisectional curvature, where* M *is an* n*-dimensional noncompact complex manifold, then* $g(t)$ *also has nonnegative holomorphic bisectional curvature for all* $t \in [0,T]$.

Proof. The theorem is known to be true if the curvature is uniformly bounded on space and time [24]. Since $g(t)$ has bounded curvature on $M \times [\tau,T]$ for all $\tau > 0$, it is sufficient to prove that $g(t)$ has nonnegative bisectional curvature on $M \times [0,\tau]$ for some $\tau > 0$. Hence we may assume that $T \le 1$.

In the following, lower case $c_1, c_2, \dots$ will denote constants depending only on n.

Since $g(T)$ has curvature bounded by $\frac{a}{T}$ and is complete, as in Lemma 2.2, there is a smooth function ρ defined on M such that

$$(1 + d_T(x,p)) \le \rho(x) \le D_1(1 + d_T(x,p)) \\ |\overline{\nabla}\rho| + |\overline{\nabla}^2\rho| \le D_1, \tag{4.1}$$

for some constant D_1 depending only on n and g_T, where $d_T(x,p)$ is the distance function with respect to g_T from a fixed point $p \in M$ and $\overline{\nabla}$ is the covariant derivative with respect to g_T.

Suppose $|\mathrm{Rm}(g(t))|_{g(t)} \le a/t$, where a is to be determined later depending only on n. By Lemma 2.2, we have

$$|\nabla\rho| \le D_2 t^{-c_1 a} \tag{4.2}$$

for some constant D_2 depending only on n, g_T. Here and below, ∇ is the covariant derivative of $g(t)$ and hence is time dependent. We may get a better estimate for $\Delta\rho = \Delta_{g(t)}\rho$ than that in Lemma 2.2. Choose a normal coordinate with respect to $g(T)$ which also diagonalizes $g(t)$ with eigenvalues λ_α. Then

$$|\triangle\rho| = |g^{\alpha\bar\beta}\rho_{\alpha\bar\beta}| \le \sum_{\alpha=1}^{n} \lambda_\alpha^{-1} |\overline{\nabla}^2\rho| \le D_3 t^{-c_2 a}, \tag{4.3}$$

by Lemma 2.1.

Let ϕ be a smooth cut-off function from $\mathbb{R}$ to $[0,1]$ such that

$$\phi(x) = \begin{cases} 1, & x \le 1 \\ 0, & x \ge 2 \end{cases}$$

and $|\phi'| + |\phi''| \le D'$, $\phi' \le 0$. Let $\Phi = \phi^m$, where $m > 4$ is an integer to be determined later. Then

$$0 \ge \Phi' \ge -D(m)\Phi^q; \quad |\Phi''| \le D(m)\Phi^q$$

for some positive constant $D(m)$ depending only on D' and m, where $q = 1 - \frac{2}{m}$.

Let $\Psi(x) = \Phi(\frac{\rho(x)}{r})$ on M for $r \geq 1$. Note that Ψ depends on r.

Then we have

$$|\nabla\Psi| \leq \frac{1}{r}D(m)\Psi^q|\nabla\rho| \leq \frac{D_4}{r}\Psi^q t^{-c_1 a} \tag{4.4}$$

by (4.2), and

$$|\triangle\Psi| \leq \frac{1}{r^2}|\Phi''||\nabla\rho|^2 + \frac{1}{r}|\Phi'\triangle\rho| \leq \frac{D_4}{r}\Psi^q t^{-c_2 a} \tag{4.5}$$

by (4.3), where D_4 is a constant depending only on n, g_T, m.

For any $\varepsilon > 0$, we define a tensor A on $M \times (0,T]$: For vectors $X,Y,Z,W \in T_{\mathbb{C}}(M)$,

$$A(X,Y,Z,W) = t^{-\frac{1}{2}}\Psi(x)R(X,Y,Z,W) + \varepsilon B(X,Y,Z,W)$$

where R is the curvature tensor of $g(t)$ and B is evaluated with respect to $g(t)$.

Define the following function on $M \times (0,T]$:

$$H(x,t) = \inf\left\{A_{X\bar{X}Y\bar{Y}}(x,t) \mid |X|_t = |Y|_t = 1,\ X,Y \in T_x^{(1,0)}M\right\}.$$

Here $|\cdot|_t$ is the norm with respect to $g(t)$.

To show the theorem, it suffices to show for all $r \gg 1$, $H(x,t) \geq 0$ for all x and for all $t > 0$. Since Ψ has compact support and $B(X,\bar{X},Y,\bar{Y}) \geq 1$ for all $|X|_t = |Y|_t = 1$, there is a compact set $K \in M$ such that

$$H(x,t) > 0$$

on $(M \setminus K) \times (0,T]$. On the other hand, we claim that there is $T_0 > 0$ such that

$$H(x,t) > 0$$

on $K \times (0,T_0)$. Let $\{e_1, e_2, \ldots, e_n\}$ be a unitary frame near a compact neighborhood U of a point $x_0 \in K$ with respect to g_0. Then at each point $x \in U$,

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}(x,t) = R_{\alpha\bar{\beta}\gamma\bar{\delta}}(x,0) + tE$$

where $|E|$ is uniformly bounded on $U \times [0,T]$. Since $g(t)$ is uniformly equivalent to $g(0)$ on U, for any $X,Y \in T_x^{1,0}(M)$ for $(x,t) \in U \times [0,T]$,

$$R(X,\bar{X},Y,\bar{Y}) \geq -D_5 t|X|_0^2|Y|_0^2$$

for some constant $D_5 > 0$ where we have used the fact that g_0 has nonnegative holomorphic bisectional curvature. Since $g(t)$ and g_0 are uniformly equivalent in K, and K is compact, we conclude that

$$R(X,\bar{X},Y,\bar{Y}) \geq -D_6 t$$

on $K\times[0,T]$ for some constant D_6 for all $X,Y\in T^{1,0}_x(M)$ with $|X|_t=|Y|_t=1$. Since $B(X,\bar X,Y,\bar Y)\ge 1$, it is easy to see the claim is true. To summarize, we have proved that there is a compact set K and there is $T_0>0$, such that $H(x,t)>0$ on $M\setminus K\times(0,T]$ and $K\times(0,T_0)$.

Now we argue by contradiction. Suppose $H(x,t)<0$ for some $t>0$. Then we must have $x\in K$ and $t\ge T_0$. Hence we can find $x_0\in K$, $t_0\ge T_0$ and a neighborhood V of x_0 such that $H(x_0,t_0)=0$, $H(x,t)\ge 0$ for $x\in V$, $t\le t_0$, $H(x_0,t)>0$ for $t<t_0$. This implies that there exist $X_0,Y_0\in T^{(1,0)}_{x_0}M$ with norm $|X_0|_{g(t_0)}=|Y_0|_{g(t_0)}=1$ such that

$$A_{X_0\bar X_0Y_0\bar Y_0}(x_0,t_0)=0.$$

Then we extend X_0,Y_0 near x_0 by parallel translation with respect to $g(t_0)$ to vector fields $\widetilde{X}_0,\widetilde{Y}_0$ such that they are independent of time and

$$\Delta_{g(t_0)}\widetilde{X}_0=\Delta_{g(t_0)}\widetilde{Y}_0=0,$$

at x_0.

Denote $h(x,t):=A_{\widetilde{X}_0\bar{\widetilde{X}}_0\widetilde{Y}_0\bar{\widetilde{Y}}_0}(x,t)$. At (x_0,t_0), we have $h(x_0,t_0)=0$ and $h(x,t)\ge 0$ for $x\in V$, $t\le t_0$, $h(x_0,t)>0$ for $t<t_0$ by the definition of H.

Hence at (x_0,t_0),

(4.6)
$$\begin{aligned}
0&\ge\left(\frac{\partial}{\partial t}-\triangle\right)h\\
&=t_0^{-\frac12}\Psi\left(\left(\frac{\partial}{\partial t}-\Delta\right)R\right)(X_0,\bar X_0,Y_0,\bar Y_0)-t_0^{-\frac12}R(X_0,\bar X_0,Y_0,\bar Y_0)\Delta\Psi\\
&\quad-2t_0^{-\frac12}\langle\nabla R(\widetilde{X}_0,\bar{\widetilde{X}}_0,\widetilde{Y}_0,\bar{\widetilde{Y}}_0),\nabla\Psi\rangle-\frac12 t_0^{-\frac32}\Psi R(X_0,\bar X_0,Y_0,\bar Y_0)\\
&\quad-\varepsilon(\Delta B)(X_0,\bar X_0,Y_0,\bar Y_0)+\varepsilon\big(-\mathrm{Ric}(X_0,\bar X_0)-\mathrm{Ric}(Y_0,\bar Y_0)\\
&\quad-\mathrm{Ric}(X_0,\bar Y_0)g(\bar X_0,Y_0)-\mathrm{Ric}(\bar X_0,Y_0)g(X_0,\bar Y_0)\big)\\
&\ge t_0^{-\frac12}\Psi Q(R)(X_0,\bar X_0,Y_0,\bar Y_0)-D_4r^{-1}t_0^{-\frac12-c_2a}\Psi^q|R(X_0,\bar X_0,Y_0,\bar Y_0)|\\
&\quad-\frac12 t_0^{-\frac32}\Psi R(X_0,\bar X_0,Y_0,\bar Y_0)-c_3\varepsilon a t_0^{-1}-2t_0^{-\frac12}\langle\nabla R(\widetilde{X}_0,\bar{\widetilde{X}}_0,\widetilde{Y}_0,\bar{\widetilde{Y}}_0),\nabla\Psi\rangle,
\end{aligned}$$

where we have used (4.5) and the fact that $\Delta B=0$. On the other hand, at (x_0,t_0)

$$\begin{aligned}
0&=\nabla h\\
&=t_0^{-\frac12}\nabla\left(R(\widetilde{X}_0,\bar{\widetilde{X}}_0,\widetilde{Y}_0,\bar{\widetilde{Y}}_0)\Psi\right)\\
&=t_0^{-\frac12}\left[\Psi\nabla R(\widetilde{X}_0,\bar{\widetilde{X}}_0,\widetilde{Y}_0,\bar{\widetilde{Y}}_0)+R(X_0,\bar X_0,Y_0,\bar Y_0)\nabla\Psi\right]
\end{aligned}$$

where we have used the fact that $\nabla g = 0$ and $\nabla \widetilde{X}_0 = \nabla \widetilde{Y}_0 = 0$ at (x_0, t_0). Hence (4.6) implies

$$\begin{aligned} 0 \geq\ & t_0^{-\frac{1}{2}} \Psi Q(R)(X_0, \bar{X}_0, Y_0, \bar{Y}_0) \\ & - D_7 r^{-1} t_0^{-\frac{1}{2} - c_4 a} |R(X_0, \bar{X}_0, Y_0, \bar{Y}_0)| \left(\Psi^q + \Psi^{2q-1}\right) \\ & - \frac{1}{2} t_0^{-\frac{3}{2}} \Psi R(X_0, \bar{X}_0, Y_0, \bar{Y}_0) - c_3 \varepsilon a t_0^{-1} \end{aligned} \tag{4.7}$$

where we have used (4.4) and $D_7 > 0$ is a constant depending only on g_T, n, m. On the other hand, by the null-vector condition [18, Proposition 1.1] (see also [1]), we have

$$Q(A)(X_0, \bar{X}_0, Y_0, \bar{Y}_0) \geq 0.$$

By a direct computation, one can see that

$$Q(A) = t_0^{-1} \Psi^2 Q(R) + \varepsilon^2 Q(B) + t_0^{-\frac{1}{2}} \Psi \varepsilon R * B,$$

and we have

$$\begin{aligned} 0 & \leq t_0^{-1} \Psi^2 Q(R)(X_0, \bar{X}_0, Y_0, \bar{Y}_0) + \varepsilon^2 Q(B)(X_0, \bar{X}_0, Y_0, \bar{Y}_0) + c_5 \varepsilon \Psi a t_0^{-\frac{3}{2}} \\ & \leq t_0^{-1} \Psi^2 Q(R)(X_0, \bar{X}_0, Y_0, \bar{Y}_0) + c_5 \varepsilon \Psi a t_0^{-\frac{3}{2}} \end{aligned}$$

where we have used Lemma 4.1 and c_5 is a constant depending only on n. That is

$$\Psi Q(R)(X_0, \bar{X}_0, Y_0, \bar{Y}_0) \geq -c_5 \varepsilon a t_0^{-\frac{1}{2}} \tag{4.8}$$

where we have used the fact that $h(x_0, t_0) = 0$ which implies $\Psi(x_0, t_0) > 0$.

Combining this with (4.7), we have

$$\begin{aligned} 0 \geq & -(c_3 + c_5) \varepsilon a t_0^{-1} - D_7 r^{-1} t_0^{-\frac{1}{2} - c_4 a} |R(X_0, \bar{X}_0, Y_0, \bar{Y}_0)| \left(\Psi^q + \Psi^{2q-1}\right) \\ & - \frac{1}{2} t_0^{-\frac{3}{2}} \Psi R(X_0, \bar{X}_0, Y_0, \bar{Y}_0). \end{aligned} \tag{4.9}$$

Since $h(x_0, t_0) = 0$, we also have

$$\Psi(x_0, t_0) R(X_0, \bar{X}_0, Y_0, \bar{Y}_0) = -t_0^{\frac{1}{2}} \varepsilon B(X_0, \bar{X}_0, Y_0, \bar{Y}_0).$$

Hence at (x_0,t_0), (4.9) implies, if $0<a<1$, then

$$\begin{aligned}
0 &\geq -(c_3+c_5)\varepsilon a \\
&\quad -2D_7 r^{-1} t_0^{\frac{1}{2}-c_4 a} |R(X_0,\bar{X_0},Y_0,\bar{Y_0})|\Psi^{2q-1} - \frac{1}{2} t_0^{-\frac{1}{2}} \Psi R(X_0,\bar{X_0},Y_0,\bar{Y_0}) \\
&\geq -(c_3+c_5)\varepsilon a \\
&\quad -2D_7 r^{-1} t_0^{\frac{1}{2}-c_4 a} |R(X_0,\bar{X_0},Y_0,\bar{Y_0})|^{2(1-q)} |\varepsilon t_0^{\frac{1}{2}} B(X_0,\bar{X_0},Y_0,\bar{Y_0})|^{2q-1} \\
&\quad +\frac{1}{2}\varepsilon B(X_0,\bar{X_0},Y_0,\bar{Y_0}) \\
&\geq -(c_3+c_5)\varepsilon a - D_8 r^{-1}\varepsilon^{2q-1} t_0^{\alpha} + \frac{1}{2}\varepsilon
\end{aligned}$$

because $0\leq\Psi\leq 1$, $q=1-\frac{2}{m}<1$, $m>4$, where $D_8>0$ is a constant depending only on g_T,n,m. Here

$$\alpha=\frac{1}{2}-c_4 a-2(1-q)+\frac{1}{2}(2q-1)=3q-c_4 a-2.$$

Hence if $c_4 a<\frac{1}{2}$ and $a<1$, then a depends only on n and $3q-c_4 a-2>0$, provided m is large enough. If a,m are chosen satisfying these conditions, then we have

$$0\geq -(c_3+c_5)\varepsilon a - D_8 r^{-1}\varepsilon^{2q-1} + \frac{1}{2}\varepsilon.$$

If a also satisfies $a(c_3+c_5)<\frac{1}{2}$, then we have a contradiction if r is large enough. Hence if

$$0<a<\min\left\{1,\frac{1}{2}c_4^{-1},\frac{1}{2}(c_3+c_5)^{-1}\right\},$$

then $g(t)$ will have nonnegative holomorphic bisectional curvature. This completes the proof of the theorem. □

As an application, we have the following:

COROLLARY 4.1. *Let (M^n,g_0) is a complete noncompact Kähler manifold with complex dimension $n\geq 2$, nonnegative holomorphic bisectional curvature and maximum volume growth. Suppose there is $r_0>0$ and there is $C>0$ such that*

$$\left(\frac{1}{V_x(r_0)}\int_{B_x(r_0)}|\mathrm{Rm}|^p\right)^{\frac{1}{p}}\leq C$$

for some $p>n$ for all $x\in M$. Then M is biholomorphic to $\mathbb{C}^n$.

Proof. By [27], the Ricci flow with initial data g_0 has a smooth complete short time solution $g(t)$ so that the curvature has the following bound:

$$|\mathrm{Rm}(g(t))| \leq Ct^{-\frac{n}{p}}$$

for some constant C. Since $\frac{n}{p} < 1$, by Theorems 3.1 and 4.1, $g(t)$ is Kähler and has bounded nonnegative bisectional curvature for $t > 0$. Since $\frac{n}{p} < 1$ it is easy to see that $g(t)$ is uniformly equivalent to g_0. Hence $g(t)$ also has maximum volume growth. By [5], M is biholomorphic to $\mathbb{C}^n$. □

5. Producing Kähler-Ricci flow through h-flow. We want to produce solutions to the Kähler-Ricci flow using solutions to the so-called h-flow by M. Simon [25]. Let us recall the set up and some results in [25]. Let M^n be a smooth manifold, and let g and h be two Riemannian metrics on M. For a constant $\delta > 1$, h is said to be δ close to g if

$$\delta^{-1}h \leq g \leq \delta h.$$

Let $g(t)$ be a smooth family of metrics on $M \times [0,T]$, $T > 0$. $g(t)$ is said to be a solution to the h-flow, if $g(t)$ satisfies the following DeTurck flow, see [23, 25]:

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij} + \nabla_i V_j + \nabla_j V_i, \tag{5.1}$$

where

$$V_i = g_{ij} g^{kl} (\Gamma^j_{kl} - {}^h\Gamma^j_{kl}),$$

and Γ^i_{kl}, ${}^h\Gamma^i_{kl}$ are the Christoffel symbols of $g(t)$ and h respectively, and ∇ is the covariant derivative with respect to $g(t)$.

In order to emphasize the background metric h, we call it h-flow as in [25]. We are only interested in the case that M is noncompact and g is complete.

In [25], Simon obtained the following:

THEOREM 5.1. *[Simon] There exists $\epsilon(n) > 0$ depending only on n such that if (M^n, g_0) is a smooth n-dimensional complete noncompact Riemannian manifold and there is a smooth Riemannian metric h on M with $\sup_{x\in M} |\nabla^i \mathrm{Rm}(h)| \leq k_i$ for all i and if h is $(1+\epsilon(n))$ close to g_0, then the h-flow (5.1) has a smooth solution $g(t)$ on $M \times [0,T]$ with initial value $g(0) = g_0$ such that*

$$\sup_{x\in M} |\nabla^i g(t)|^2 \leq \frac{C_i}{t^i}$$

for all i and $g(t)$ is $(1+2\epsilon)$ close to h for all t, where $T(n,k_0) > 0$ depends only on n and k_0, C_i depends only on $n, k_0, \ldots, k_i$. Here and in the following, ∇ and $|\cdot|$ are with respect to h.

As in [21], using solution to the h-flow, one may construct a smooth complete solution $g(t)$ to the Ricci flow with initial metric g_0. Moreover, the curvature of $g(t)$ is bounded by C/t. By Theorem 3.1, we conclude that $g(t)$ is a Kähler-Ricci flow if g_0 is Kähler. However, motivated by the uniformization conjecture of Yau [29], we also want to prove that if g_0 has nonnegative holomorphic bisectional curvature then $g(t)$ also has nonnegative holomorphic bisectional curvature. To achieve this goal, we want to apply Theorem 4.1. To apply the theorem, the constant C in the curvature bound C/t must be small enough. If $\epsilon(n)$ is small enough, then C will be small. This basically follows from the proof of Theorem 5.1. (See [25]) One may also obtain this fact by using Theorem 5.1 together with an interpolation result in [22] as follows:

LEMMA 5.1. *Let (M^n, h) be a complete noncompact Riemannian manifold such that $|\nabla^i \mathrm{Rm}(h)| \le k_i$, $0 \le i \le 3$ with $\sum_{i=0}^3 k_i \le 1$. For any $\alpha > 0$ there is a constant $b(n,\alpha) > 0$ depending only on n and α such that $b \le \epsilon(n)$ where $\epsilon(n)$ is the constant in Theorem 5.1, and such that if $g(t)$ is the solution to the h-flow on $M \times [0,T]$, $T \le 1$, obtained in Theorem 5.1 with $g(0) = g_0$ satisfies $(1+b)^{-1}h \le g_0 \le (1+b)h$, then*

$$|\nabla^i g(t)|^2 \le \left(\frac{\alpha}{t}\right)^i$$

for $i = 1,2$ for all $t \in (0,T]$.

Proof. Let $0 < b \le \epsilon(n)$ be fixed to be determined later and let $g(t)$ be the solution to the h-flow as in the theorem on $M \times [0,T]$, with $T \le 1$. We assume that $\epsilon(n) \le 1$.

Let $p \in M$, by the assumptions on $\nabla^i \mathrm{Rm}(h)$, $\exp_p$ is a local diffeomorphism on $B(2c_1) = \{x \in T_p(M) |\ |x| < 2c_1\}$. Here and below c_i will denote a constant depending only on n. By pulling back $h, g(t)$ via $\exp_p$, in order to prove the claim at p, we may consider $g(t)$ and h as metrics on $B(2c_1)$. In the normal coordinates $x^1, \dots, x^n$, by [12, Corollary 4.11], we have

$$\begin{cases} \frac{1}{2}|\xi|^2 \le h_{ij}\xi^i\xi^j \le 2|\xi|^2, & \text{for } \xi \in \mathbb{R}^n; \\ \left|D_x^\beta h_{ij}\right| \le c_2, & \text{for all } i,j, \end{cases} \tag{5.2}$$

where $h_{ij} = h(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index with $|\beta| \le 3$ and $D_{x^k} = \frac{\partial}{\partial x^k}$. Denote D_{x^k} simply by D_k. Let $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ which can be considered as functions in $B(2c_1)$. Then by (5.2), in $B(2c_1)$, we have

$$\begin{cases} |\nabla^k(g-h)| \le c_2\left(|D^{(k)}(g-h)|_0 + \sum_{i<k}|\nabla^i(g-h)|\right); \\ |D^{(k)}(g-h)|_0 \le c_2\left(|\nabla^k(g-h)| + \sum_{i<k}|\nabla^i(g-h)|\right); \end{cases} \tag{5.3}$$

for $0 \le k \le 3$, where $|\cdot|$ is the norm of tensors with respect to h,

$$|D^{(k)}(g-h)|_0 = \left(\sum_{|\beta|=k} \sum_{i,j} (D^\beta (g_{ij} - h_{ij}))^2 \right)^{\frac{1}{2}}$$

with $(g_{ij} - h_{ij})$ being considered as functions. Note that for $i \ge 1$, $\nabla^i (g-h) = \nabla^i g$.

Fix $T \ge t > 0$ and denote $g(t)$ by g. Let $1 \ge \eta \ge 0$ be a smooth function on $B(2c_1)$ such that $\eta = 1$ on $B(c_1)$ and is zero outside $B(\frac{3}{2}c_1)$ and such that $|\partial_x^\beta \eta|$ is bounded by a constant depending only on n for all β, with $|\beta| \le 3$ because c_1 depends only on n.

For fixed i, j, k, let $\phi = \eta(g_{ij} - h_{ij})$. By the interpolation result [22, Lemma A.5], we have

$$|D_k \phi|^2(p) \le 32 \sup_{B(2c_1)} |\phi| \cdot \sup_{B(2c_1)} |D^{(2)}\phi|.$$

Hence by (5.2), (5.3) and Theorem 5.1,

$$\begin{aligned} |\nabla g|^2(p) &= |\nabla(g-h)|^2 \\ &= |D(g-h)|_0^2(p) \\ &\le c_3 \sup_{B(2c_1)} |g-h| \cdot \sup_{B(2c_1)} |D^{(2)}\phi| \\ &\le c_4 \sup_{B(2c_1)} |g-h| \cdot \sup_{B(2c_1)} \left(|\nabla^2 g| + |\nabla g| + |g-h|\right) \\ &\le c_5 b \cdot \frac{1}{t}, \end{aligned}$$

because $\sum_{i=0}^3 k_i \le 1$, $0 < t \le 1$, $b \le \epsilon(n)$ and $(1+2b)^{-1} h \le g \le (1+2b)h$. Since p is arbitrary, we have:

$$\sup_M |\nabla g|^2 \le c_5 b \cdot \frac{1}{t}. \tag{5.4}$$

Next we want to estimate $|\nabla^2 g|(p)$. As before for fixed i, j, k, l, and $\phi = \eta(g_{ij} - h_{ij})$,

$$\begin{aligned} &|D_k D_l (g_{ij} - h_{ij})|^2(p) \\ &\quad = |D_k D_l \phi|^2(p) \\ &\quad \le 32 \sup_{B(2c_1)} |D\phi| \cdot \sup_{B(2c_1)} |D^{(3)}\phi| \\ &\quad \le c_6 \sup_M (|\nabla g| + |g-h|) \cdot \sup_M \left(|\nabla^3 g| + |\nabla^2 g| + |\nabla g| + |g-h|\right). \end{aligned}$$

By Theorem 5.1, (5.3) and (5.4), we have

$$|\nabla^2 g|^2(p) \leq c_7\sqrt{b}\cdot\frac{1}{t^2}. \tag{5.5}$$

By (5.4) and (5.5), we conclude that the lemma is true. □

LEMMA 5.2. *For any $\alpha > 0$, there exists $\epsilon(n,\alpha) > 0$ depending only on n and α such that if (M^n, g_0) is a complete noncompact Riemannian manifold with real dimension n and if g_0 is $(1+\epsilon(n,\alpha))$ close to a Riemannian metric h with curvature bounded by k_0, then there is a smooth complete Ricci flow $g(t)$ defined on $M\times[0,T]$ with initial value $g(0) = g_0$, where $T > 0$ depends only on n, k_0. Moreover, the curvature of $g(t)$ satisfies:*

$$|\mathrm{Rm}(g(t))|_{g(t)} \leq \frac{\alpha}{t}$$

on $M\times[0,T]$.

Proof. First we remark that by [23], there is a solution to the Ricci flow with initial data h with bounded curvature in space and time. Moreover, for $t > 0$ all order of derivatives of the curvature tensor for a fixed $t > 0$ are uniformly bounded, the solution exists in a time interval depending only on n, k_0, and the bounds of the derivatives of the curvature tensor for a fixed $t > 0$ depend only on n, k_0, and t. Hence without lost of generality, we may assume that $|\widetilde{\nabla}^{(i)}\widetilde{\mathrm{Rm}}|_h \leq k_i < \infty$ for all $i \geq 0$. Here and in the following $\widetilde{\nabla}$ is the covariant derivative with respect to h and $\widetilde{\mathrm{Rm}}$ is the curvature tensor of h and $|\cdot|_h$ is the norm relative to h.

Note that if h is $1+\epsilon$ close to g_0, then λh is also $1+\epsilon$ close to λg_0 for any $\lambda > 0$. Moreover, if $g(t)$ is a solution to the Ricci flow with initial data g_0, then $\lambda g(\lambda^{-1}t)$ is a solution to the Ricci flow with initial data λg_0, and if $s = \lambda t$, then

$$|\mathrm{Rm}(g(t))|_{g(t)} = \lambda|\mathrm{Rm}(\lambda g(\lambda^{-1}s))|_{\lambda g(\lambda^{-1}s)}.$$

Hence we may assume that $k_0 + k_1 + k_2 + k_3 \leq 1$.

Suppose $\epsilon(n,\alpha) < \epsilon(n)$ is small enough, depending only on n, α, where $\epsilon(n)$ is the constant in Theorem 5.1, then the solution $\bar{g}(t)$, $t \in [0,T]$ for some $T > 0$ depending only on n, α, to the h-flow obtained in Theorem 5.1 with initial metric g_0 satisfies

$$|\mathrm{Rm}(\bar{g}(t))|_{\bar{g}(t)} \leq \frac{\alpha}{t} \tag{5.6}$$

by Lemma 5.1 and the fact that

$$|\mathrm{Rm}(\bar{g}(t))|_{\bar{g}(t)} \leq c_1\left(|\widetilde{\mathrm{Rm}}|_h + |\widetilde{\nabla}\bar{g}(t)|_h^2 + |\widetilde{\nabla}^2\bar{g}(t)|_h\right) \tag{5.7}$$

where $\widetilde{\mathrm{Rm}}$ is the curvature tensor of h, $\widetilde{\nabla}$ is the covariant derivative of h, and c_1 is a constant depending only on n.

In order to find the Ricci flow, we proceed as in [21]. We only need to solve the following ODE at each point $x \in M$, $t \in [0,T]$:

$$\begin{cases} \dfrac{d}{dt}\varphi(x,t) = -W(\varphi(x,t),t) \\ \varphi(x,0) = x \end{cases} \tag{5.8}$$

where W is the time-dependent smooth vector field given by

$$W^i(t) = \bar{g}^{jk}(t)(\Gamma^i_{jk} - \widetilde{\Gamma}^i_{jk}).$$

Here Γ^i_{jk} and $\widetilde{\Gamma}^i_{jk}$ are the Christoffel symbols of $g(t)$ and h respectively. If solution exists, then $g(t) = (\phi(t))^*(\bar{g}(t))$ is the required solution to the Ricci flow, see [25] for example.

To solve (5.8), let $\Omega_s \Subset \Omega_{s+1}$ be an increasing sequence of open sets which exhaust M. Let η_s be smooth functions with $0 \le \eta_s \le 1$, $\eta_s = 1$ in Ω_s and $\eta_s = 0$ outside Ω_{s+1}. Let $W_s = \eta_s W$ which is smooth with compact support. Hence

$$\begin{cases} \dfrac{d}{dt}\varphi_s(x,t) = -W_s(\varphi_s(x,t),t) \\ \varphi_s(x,0) = x \end{cases} \tag{5.9}$$

has solution for all $x \in M$ and $t \in [0,T]$. Now $|W_s|_h \le c_2/\sqrt{t}$ by Theorem 5.1, we conclude that

$$d_h(x, \varphi_s(x,t)) \le C_1$$

for some C_1 for all $x \in M$, $t \in [0,T]$ and for all $s = 1,2,3,\ldots$. Let K be any compact set. Then for s large enough $\varphi_s(x,t) \in \Omega_s$ for all $x \in K$, $t \in [0,T]$. Hence for s large enough, for such $x \in K$,

$$\begin{cases} \dfrac{d}{dt}\varphi_s(x,t) = -W(\varphi_s(x,t),t) \\ \varphi_s(x,0) = x. \end{cases}$$

By the uniqueness of solutions of ODE, we have $\varphi_s(x,t) = \varphi_{s+1}(x,t)$ for all s large enough, $x \in K$, $t \in [0,T]$. From this it is easy to see that (5.8) has solution on $[0,T]$. This completes the proof of the lemma. $\square$

Now we can prove the main result of this section:

THEOREM 5.2. *There exists $\epsilon(2n) > 0$ depending only on n such that if (M^n, g_0) is a complete noncompact Kähler manifold with complex dimension n and if there is a smooth Riemannian metric h with curvature bounded by k_0 on M such that g_0 is $(1+\epsilon(n))$ close h, then there is a smooth complete Kähler-Ricci flow $g(t)$ defined on $M \times [0,T]$ with initial value $g(0) = g_0$, where $T > 0$ depends only on n, k_0. Moreover, the curvature of $g(t)$ satisfies:*

$$|\mathrm{Rm}(g(t))|_{g(t)} \le \frac{\alpha}{t}$$

where $\alpha = \alpha(n)$ is the constant in Theorem 4.1. If in addition, g_0 has nonnegative holomorphic bisectional curvature, then $g(t)$ has nonnegative holomorphic bisectional curvature for all $t \in [0,T]$.

Proof. The results follow from Lemma 5.2 and Theorems 3.1 and 4.1. □

COROLLARY 5.1. *Let $\epsilon(2n)$ be as in Theorem 5.2. Suppose (M^n, g_0) is a complete noncompact Kähler manifold of complex dimension n with nonnegative holomorphic bisectional curvature and maximum volume growth. Suppose there is a Riemannian metric h on M with bounded curvature which is $1+\epsilon(2n)$ close to g_0. Then M is biholomorphic to $\mathbb{C}^n$.*

Proof. Let $g(t)$ be the solution to the Kähler-Ricci flow obtained in Theorem 5.2. Then for $t > 0$, $g(t)$ is Kähler with bounded nonnegative holomorphic bisectional curvature. We claim that $g(t)$ has maximum volume growth. Let $x_0 \in M$ be fixed. Let $\bar{g}(t)$ be the solution to the h-flow as in the proof of Lemma 5.2 and φ be the solution to (5.8). Then $\bar{g}(t)$ has nonnegative Ricci curvature because $g(t)$ has nonnegative Ricci curvature. Since $\bar{g}(t)$ is uniformly equivalent to g_0, $\bar{g}(t)$ has maximum volume growth. Hence $g(t)$ also has maximum volume growth. Therefore, M is biholomorphic to $\mathbb{C}^n$ by the result of [5]. □

THE INSTITUTE OF MATHEMATICAL SCIENCES AND DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG, CHINA
E-mail: schuang@math.cuhk.edu.hk

Current address: YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA
E-mail: schuang@mail.tsinghua.edu.cn

THE INSTITUTE OF MATHEMATICAL SCIENCES AND DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG, CHINA
E-mail: lftam@math.cuhk.edu.hk

REFERENCES

---

[1] S. Bando, On the classification of three-dimensional compact Kaehler manifolds of nonnegative bisectional curvature, *J. Differential Geom.* **19** (1984), no. 2, 283–297.

[2] E. Cabezas-Rivas and B. Wilking, How to produce a Ricci flow via Cheeger-Gromoll exhaustion, *J. Eur. Math. Soc. (JEMS)* **17** (2015), no. 12, 3153–3194.

[3] A. Chau, K.-F. Li, and L.-F. Tam, Deforming complete Hermitian metrics with unbounded curvature, *Asian J. Math.* **20** (2016), no. 2, 267–292.

[4] ———, Longtime existence of the Kähler-Ricci flow on $\mathbb{C}^n$, *Trans. Amer. Math. Soc.* **369** (2017), no. 8, 5747–5768.

[5] A. Chau and L.-F. Tam, On the complex structure of Kähler manifolds with nonnegative curvature, *J. Differential Geom.* **73** (2006), no. 3, 491–530.

[6] B.-L. Chen, S.-H. Tang, and X.-P. Zhu, A uniformization theorem for complete non-compact Kähler surfaces with positive bisectional curvature, *J. Differential Geom.* **67** (2004), no. 3, 519–570.

[7] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci Flow: Techniques and Applications. Part I. Geometric Aspects*, *Math. Surveys Monogr.*, vol. 135, Amer. Math. Soc., Providence, RI, 2007.

[8] ———, *The Ricci Flow: Techniques and Applications. Part III. Geometric-Analytic Aspects*, *Math. Surveys Monogr.*, vol. 163, Amer. Math. Soc., Providence, RI, 2010.

[9] B. Chow and D. Knopf, *The Ricci Flow: An Introduction*, *Math. Surveys Monogr.*, vol. 110, Amer. Math. Soc., Providence, RI, 2004.

[10] G. Giesen and P. M. Topping, Existence of Ricci flows of incomplete surfaces, *Comm. Partial Differential Equations* **36** (2011), no. 10, 1860–1880.

[11] R. S. Hamilton, Three-manifolds with positive Ricci curvature, *J. Differential Geom.* **17** (1982), no. 2, 255–306.

[12] ———, A compactness property for solutions of the Ricci flow, *Amer. J. Math.* **117** (1995), no. 3, 545–572.

[13] M.-C. Lee and L.-F. Tam, Chern-Ricci flows on noncompact complex manifolds, preprint, https://arxiv.org/abs/1708.00141.

[14] G. Liu, Gromov-Hausdorff limits of Kähler manifolds and the finite generation conjecture, *Ann. of Math. (2)* **184** (2016), no. 3, 775–815.

[15] ———, Gromov-Hausdorff limits of Kähler manifolds with bisectional curvature lower bound II, preprint.

[16] ———, On Yau's uniformization conjecture, preprint, https://arxiv.org/abs/1606.08958.

[17] N. Mok, An embedding theorem of complete Kähler manifolds of positive bisectional curvature onto affine algebraic varieties, *Bull. Soc. Math. France* **112** (1984), no. 2, 197–250.

[18] ———, The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature, *J. Differential Geom.* **27** (1988), no. 2, 179–214.

[19] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, preprint, https://arxiv.org/abs/math/0211159.

[20] A. M. Petrunin, An upper bound for the curvature integral, *Algebra i Analiz* **20** (2008), no. 2, 134–148, translation in *St. Petersburg Math. J.* **20** (2009), no. 2, 255–265.

[21] O. C. Schnürer, F. Schulze, and M. Simon, Stability of Euclidean space under Ricci flow, *Comm. Anal. Geom.* **16** (2008), no. 1, 127–158.

[22] ———, Stability of hyperbolic space under Ricci flow, *Comm. Anal. Geom.* **19** (2011), no. 5, 1023–1047.

[23] W.-X. Shi, Deforming the metric on complete Riemannian manifolds, *J. Differential Geom.* **30** (1989), no. 1, 223–301.

[24] ———, Ricci flow and the uniformization on complete noncompact Kähler manifolds, *J. Differential Geom.* **45** (1997), no. 1, 94–220.

[25] M. Simon, Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature, *Comm. Anal. Geom.* **10** (2002), no. 5, 1033–1074.

[26] L.-F. Tam, Exhaustion functions on complete manifolds, *Recent Advances in Geometric Analysis*, *Adv. Lect. Math. (ALM)*, vol. 11, Int. Press, Somerville, MA, 2010, pp. 211–215.

[27] G. Xu, Short-time existence of the Ricci flow on noncompact Riemannian manifolds, *Trans. Amer. Math. Soc.* **365** (2013), no. 11, 5605–5654.

[28] B. Yang and F. Zheng, $U(n)$-invariant Kähler-Ricci flow with non-negative curvature, *Comm. Anal. Geom.* **21** (2013), no. 2, 251–294.

[29] S.-T. Yau, A review of complex differential geometry, *Several Complex Variables and Complex Geometry, Part 2 (Santa Cruz, CA, 1989)*, *Proc. Sympos. Pure Math.*, vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 619–625.

[30] F. Zheng, First Pontrjagin form, rigidity and strong rigidity of nonpositively curved Kähler surface of general type, *Math. Z.* **220** (1995), no. 2, 159–169.