

UNITARY MONODROMY IMPLIES THE SMOOTHNESS ALONG THE REAL AXIS FOR SOME PAINLEVÉ VI EQUATION, I

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ABSTRACT. In this paper, we study the Painlevé VI equation with parameter $(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$. We prove

(i) An explicit formula to count the number of poles of an algebraic solution with the monodromy group D_N , where D_N is the dihedral group of order $2N$.

(ii) There are only four solutions without poles in $\mathbb{C} \setminus \{0, 1\}$.

(iii) If the monodromy group of the associated linear ODE of a solution $\lambda(t)$ is unitary, then $\lambda(t)$ has no poles in $\mathbb{R} \setminus \{0, 1\}$.

1. INTRODUCTION

In literature, Painlevé VI equation with four free parameters $(\alpha, \beta, \gamma, \delta)$ (PVI $(\alpha, \beta, \gamma, \delta)$) can be written as

$$(1.1) \quad \frac{d^2\lambda}{dt^2} = \frac{1}{2} \left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t} \right) \left(\frac{d\lambda}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t} \right) \frac{d\lambda}{dt} + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2} \left[\alpha + \beta \frac{t}{\lambda^2} + \gamma \frac{t-1}{(\lambda-1)^2} + \delta \frac{t(t-1)}{(\lambda-t)^2} \right].$$

There are two fundamental facts about PVI $(\alpha, \beta, \gamma, \delta)$ (1.1). The first one is the *Painlevé property* which says that both the branch points and essential singularities of any solution $\lambda(t)$ of (1.1) are independent of any particular solution and consist of $0, 1, \infty$ only. Thus $\lambda(t)$ is a multi-valued meromorphic function in $\mathbb{C} \setminus \{0, 1\}$; naturally it can be lifted to the universal covering $\mathbb{H} = \{\tau \mid \text{Im } \tau > 0\}$ of $\mathbb{C} \setminus \{0, 1\}$ through the transformation:

$$(1.2) \quad t(\tau) = \frac{e_3(\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)} \quad \text{and} \quad \lambda(t) = \frac{\wp(p(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}.$$

where $\wp(z|\tau)$ be the Weierstrass elliptic function with periods 1 and τ .

Throughout the paper, we use the notations $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = \tau$, $\omega_3 = 1 + \tau$, $e_k = e_k(\tau) \doteq \wp(\frac{\omega_k}{2}|\tau)$, $k = 1, 2, 3$ and $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H}$. Define $E_\tau \doteq \mathbb{C}/\Lambda_\tau$ to be a flat torus in the plane and $E_\tau[2] \doteq \{\frac{\omega_i}{2} \mid 0 \leq i \leq 3\} + \Lambda_\tau$ to be the set consisting of the lattice points and 2-torsion points in E_τ .

By the transformation (1.2), $p(\tau)$ satisfies the following *elliptic form* of PVI

$$(1.3) \quad \frac{d^2 p(\tau)}{d\tau^2} = \frac{-1}{4\pi^2} \sum_{k=0}^3 \alpha_k \wp' \left(p(\tau) + \frac{\omega_k}{2} \middle| \tau \right),$$

where $\wp'(z|\tau) = \frac{d}{dz}\wp(z|\tau)$ and

$$(1.4) \quad (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \left(\alpha, -\beta, \gamma, \frac{1}{2} - \delta \right).$$

See [2, 17] for a proof. As a solution to (1.3), $p(\tau)$ is considered as a continuous function from \mathbb{H} to \mathbb{C} and holomorphic except $p(\tau) \in E_\tau[2]$. Although $p(\tau)$ has branch points at $p(\tau) \in E_\tau[2] \setminus \Lambda_\tau$, by the Painlevé property, $\wp(p(\tau)|\tau)$ is a *single-valued meromorphic* function in \mathbb{H} and its poles are exactly equal to those τ 's such that $p(\tau) \in \Lambda_\tau$.

Another important feature about (1.1) is that for any solution $\lambda(t)$, there associates with a second order Fuchsian ODE defined on \mathbb{CP}^1 , whose regular singular points are exactly $\{0, 1, t, \lambda(t), \infty\}$ with $\lambda(t)$ being an apparent singularity (cf. [12]) such that the monodromy representation is invariant under the deformation of t . By using the transformation (1.2), the associated ODE could be transformed into a new ODE defined on E_τ , a generalized Lamé equation (GLE) which can be written as follows:

$$(1.5) \quad y''(z) = \left[\sum_{k=0}^3 n_k(n_k + 1) \wp \left(z + \frac{\omega_k}{2} \right) + \frac{3}{4} (\wp(z+p) + \wp(z-p)) + A(\zeta(z+p) - \zeta(z-p)) + B \right] y(z),$$

where A, B are complex numbers and the parameters are related by

$$(1.6) \quad \alpha_k = \frac{1}{2} \left(n_k + \frac{1}{2} \right)^2 \quad \text{for } k \in \{0, 1, 2, 3\}.$$

In (1.5), $\zeta(z) = \zeta(z|\tau) \doteq -\int^z \wp(\xi|\tau) d\xi$ is the Weierstrass zeta function. The function $\zeta(z)$ is an odd but not elliptic function. Indeed, $\zeta(z)$ satisfies

$$(1.7) \quad \zeta(z + \omega_k|\tau) = \zeta(z|\tau) + \eta_k(\tau), \quad k = 1, 2, 3,$$

where $\eta_k(\tau)$ are called the quasi-periods of $\zeta(z)$. The GLE (1.5) has regular singularities at $E_\tau[2] \cup (\{\pm p(\tau)\} + \Lambda_\tau)$. Similar to $\lambda(t)$, $p(\tau)$ is always an *apparent* singularity. The monodromy representation of (1.5) is invariant under the deformation of τ if and only if $p(\tau)$ is a solution of (1.3). This fact might be indirectly proved by the transformation (1.2). In [4], this fact and the associated Hamiltonian system has been directly derived.

Naturally, this isomonodromic feature proposes the following question: *How the monodromy group of the associated linear ODE effects the solution $\lambda(t)$ or $p(\tau)$?*

For certain parameters α_k , the monodromy representation of the GLE (1.5) associated with $p(\tau)$ is easier to compute than that of the associated ODE with $\lambda(t)$. For example, if all n_k are integers, then the monodromy representation of (1.5) is reduced to a homomorphism from $\pi_1(E_\tau)$ to $SL(2, \mathbb{C})$. This fact immediately implies that the monodromy group is always *abelian*,

a significant reduction. See [5]. Thus, there always exists a common eigenfunction to all the monodromy matrices of the GLE (1.5).

Definition 1.1. *Let the parameter $(\alpha, \beta, \gamma, \delta)$ be given by (1.4) and (1.6) with $n_k \in \mathbb{Z}$ for all k .*

(i) *A solution $\lambda(t)$ of (1.1) or $p(\tau)$ of the elliptic form (1.3) is called completely reducible if the monodromy representation of the associated GLE (1.5) is completely reducible (i.e. all monodromy matrices can be diagonalized simultaneously); otherwise, $\lambda(t)$ or $p(\tau)$ is called non-completely reducible.*

(ii) *A solution $\lambda(t)$ or $p(\tau)$ is called an unitary solution if the monodromy group of the associated GLE (1.5) is contained in the unitary group $SU(2)$.*

Obviously, any unitary solution is completely reducible. We remark that an unitary solution is related to the existence of conformal metric with constant curvature +1 (those metrics are with conic singularities in general). See [9].

Now we come back to the problem we are concerned above. For $PVI(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ (i.e. $n_k = 0$ in (1.6), $\forall k = 0, 1, 2, 3$, the well-known case studied by Hitchin [11]), among other things, we proved in [7] the following theorem:

Theorem A. ([7])

- (i) *$PVI(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ has exactly three solutions which are completely reducible and satisfy $\lambda(t) \notin \{0, 1, t, \infty\}$ for any $t \in \mathbb{C} \setminus \{0, 1\}$.*
- (ii) *Any unitary solution $\lambda(t)$ of $PVI(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ has no poles in $\mathbb{R} \setminus \{0, 1\}$.*

By using the Okamoto transformation [18], it is well known that any solution of $PVI(\alpha, \beta, \gamma, \delta)$ with parameters given by (1.6) with $n_k \in \mathbb{Z}$ could be obtained from solution $\lambda(t)$ of $PVI(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ (i.e. $n_k = 0$ for all k). However, the Okamoto transformation is a rational map of $\lambda(t)$ and $\lambda'(t)$. So obviously, the smoothness of solutions can not be preserved by the Okamoto transformation. See [7] for discussions of this issue.

In this article, we extend Theorem A (ii) to the case $(n_0, n_1, n_2, n_3) = (1, 0, 0, 0)$.

Theorem 1.2. *Let $\lambda(t)$ be any unitary solution of $PVI(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$. Then $\lambda(t)$ has no poles in $\mathbb{R} \setminus \{0, 1\}$, namely $\lambda(t)$ is holomorphic in $\mathbb{R} \setminus \{0, 1\}$.*

As discussed above, Theorem 1.2 can not be obtained from Theorem A by the Okamoto transformation. Our proof is based on the generalization of the famous Hitchin theorem for $PVI(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ to $PVI(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$.

Theorem B. ([5]) *$p(\tau)$ is a completely reducible solution to the elliptic form of $PVI(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ if and only if there exists a fixed pair $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ such that*

$$(1.8) \quad \wp(p(\tau)|\tau) = \wp(\alpha) + \frac{3\wp'(\alpha)Z_{r,s}^2 + (12\wp^2(\alpha) - g_2)Z_{r,s} + 3\wp(\alpha)\wp'(\alpha)}{2(Z_{r,s}^3 - 3\wp(\alpha)Z_{r,s} - \wp'(\alpha))},$$

where $\alpha = \alpha(\tau) \doteq r + s\tau$,

$$(1.9) \quad Z_{r,s} = Z_{r,s}(\tau) \doteq \zeta(r + s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau),$$

and $g_2 = g_2(\tau)$ is the coefficient of

$$(1.10) \quad \wp'(z|\tau)^2 = 4\wp(z|\tau)^3 - g_2(\tau)\wp(z|\tau) - g_3(\tau).$$

The formula (1.8) was first obtained by Takemura [20] and also obtained in [5] by a different argument. We will see that (1.8) plays a fundamental role for studying the poles of $\lambda(t)$.

Remark 1.3. Let $p(\tau)$ be the solution given by (1.8) and (1.5) be the associated ODE with this solution. Then it is proved in [5] that there is a pair of independent solution $y_i(z|\tau)$, $i = 1, 2$, such that their analytic continuation along path ℓ_i satisfy

$$\begin{aligned} \ell_1^*(y_1, y_2) &= (y_1, y_2) \begin{pmatrix} e^{2\pi is} & 0 \\ 0 & e^{-2\pi is} \end{pmatrix}, \\ \ell_2^*(y_1, y_2) &= (y_1, y_2) \begin{pmatrix} e^{-2\pi ir} & 0 \\ 0 & e^{2\pi ir} \end{pmatrix}, \end{aligned}$$

where (r, s) is the pair in (1.8) and $\ell_i, i = 1, 2$, are two fundamental cycles on E_τ with the base point $q_0 \notin E_\tau[2] \cup (\{\pm p(\tau)\} + \Lambda_\tau)$. In this paper, the solution given by (1.8) is denoted by $p_{r,s}(\tau)$ and the corresponding $\lambda(t)$ through (1.2) by $\lambda_{r,s}(t)$.

Obviously, $\lambda_{r,s}(t)$ (or $p_{r,s}(\tau)$) is unitary if and only if $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Furthermore, $\lambda_{r,s}(t)$ is an algebraic solution if and only if (r, s) is a N -torsion point for some $N \in \mathbb{N}_{\geq 3}$, that is $(r, s) \in Q_N$ where

$$(1.11) \quad Q_N \doteq \left\{ \left(\frac{k_1}{N}, \frac{k_2}{N} \right) \mid \gcd(k_1, k_2, N) = 1, 0 \leq k_1, k_2 \leq N-1 \right\}.$$

For Painlevé VI equation, all the algebraic solutions have been classified through the Okamoto transformation. For example, the monodromy group of the associated linear ODE of an algebraic solution of PVI($\alpha, \beta, \gamma, \delta$) with parameters given by (1.4) and (1.6) with $n_k \in \mathbb{Z}$ is always a dihedral group D_N of order $2N$ for some $N \in \mathbb{N}_{\geq 3}$. See [3, 8, 15, 16]. From the classification of Theorem B, such a $\lambda(t)$ must be of the form $\lambda_{r,s}(t)$ with $(r, s) \in Q_N, N \geq 3$. Note that the problem concerning the distribution of poles of Painlevé VI solution has been addressed in [10].

Let $\phi(N)$ is the Euler function defined by

$$(1.12) \quad \phi(N) := \#\{k \in \mathbb{Z} \mid \gcd(k, N) = 1, 0 \leq k < N\}.$$

In the following, we shall apply Theorem B to obtain the formula to count the number of poles of algebraic solutions of PVI($\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8}$).

Theorem 1.4.

- (i) If N is odd, then there is only one algebraic solution whose monodromy group¹ is the dihedral group D_N . This algebraic solution is

¹In this paper, for convenience, when we say the monodromy group of an algebraic solution, we always mean the one of the associated linear ODE, but not the one of this algebraic solution as a multi-valued function.

has exactly

$$\frac{3|Q_N|}{4} - 3\phi(N)$$

many poles in $\mathbb{C} \setminus \{0, 1\}$.

- (ii) If N is even, there are exactly three algebraic solutions and each of them has

$$\frac{|Q_N|}{4} - \left(\phi(N) + \phi\left(\frac{N}{2}\right) \right)$$

many poles in $\mathbb{C} \setminus \{0, 1\}$.

Remark 1.5. For any solution $\lambda(t)$, there might be two different branches to have poles at the same t_0 . In this case, we count the number of poles at t_0 by 2. It is our conjecture that different branches should not have common poles. See the discussion before the proof of Theorem 1.4 in §3. This conjecture was proved for $PVI\left(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8}\right)$ in [7].

Finally, we extend Theorem A (i) to $PVI\left(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8}\right)$.

Theorem 1.6. Among all the completely reducible solutions $\lambda_{r,s}(t)$ to $PVI\left(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8}\right)$, there are exactly four solutions which have no poles in $\mathbb{C} \setminus \{0, 1\}$. They are $\lambda_{\frac{1}{3}, 0}(t)$, $\lambda_{\frac{1}{4}, 0}(t)$, $\lambda_{\frac{1}{4}, \frac{1}{4}}(t)$ and $\lambda_{0, \frac{1}{4}}(t)$.

The paper is organized as follows: In §2, we introduce $Z_{r,s}^{(2)}(\tau)$ which is the denominator of (1.8) and study its zeros. By connecting the zeros of $Z_{r,s}^{(2)}(\tau)$ with the poles of $\lambda_{r,s}(t)$ (see Theorem 2.1), we prove Theorem 1.2 in §2. Next, we will count the number of the poles of an algebraic solution and the explicit formulae are obtained in §3. Finally, by applying Theorem 1.4, we prove Theorem 1.6 in §4.

2. POLES OF SOLUTIONS AND ZEROS OF PREMODULAR FORMS

In this section, we are going to prove Theorem 1.2. Define $Z_{r,s}^{(2)}(\tau)$ to be the denominator of (1.8), i.e.

$$(2.1) \quad Z_{r,s}^{(2)}(\tau) \doteq Z_{r,s}(\tau)^3 - 3\wp(r + s\tau|\tau)Z_{r,s}(\tau) - \wp'(r + s\tau|\tau).$$

To study the poles of $\lambda_{r,s}(t)$, it is important to study the zeros of $Z_{r,s}^{(2)}(\tau)$ for $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. It was proved in [7] that

$$(2.2) \quad Z_{r,s}(\tau) = \pm Z_{r',s'}(\tau) \iff (r, s) \equiv \pm (r', s') \pmod{\mathbb{Z}^2},$$

which implies that

$$(2.3) \quad Z_{r,s}^{(2)}(\tau) = \pm Z_{r',s'}^{(2)}(\tau) \iff (r, s) \equiv \pm (r', s') \pmod{\mathbb{Z}^2}.$$

From (2.2)-(2.3) and (1.8), we proved in [5] that (the \implies part is not trivial)

$$(2.4) \quad \wp(p_{r,s}(\tau)|\tau) = \wp(p_{r',s'}(\tau)|\tau) \iff (r, s) \equiv \pm (r', s') \pmod{\mathbb{Z}^2}.$$

In particular, we have

$$(2.5) \quad \lambda_{r,s}(t) = \lambda_{r',s'}(t) \iff (r', s') \equiv \pm (r, s) \pmod{\mathbb{Z}^2}.$$

By (2.3)-(2.5), it is suitable to restrict (r, s) on the set $[0, 1] \times [0, \frac{1}{2}] \setminus \frac{1}{2}\mathbb{Z}^2$. Define the four open rectangles as follows:

$$\begin{aligned}\Delta_0 &:= \{(r, s) \mid 0 < r, s < \frac{1}{2}, r + s > \frac{1}{2}\}, \\ \Delta_1 &:= \{(r, s) \mid \frac{1}{2} < r < 1, 0 < s < \frac{1}{2}, r + s > 1\}, \\ \Delta_2 &:= \{(r, s) \mid \frac{1}{2} < r < 1, 0 < s < \frac{1}{2}, r + s < 1\}, \\ \Delta_3 &:= \{(r, s) \mid r > 0, s > 0, r + s < \frac{1}{2}\}.\end{aligned}$$

Clearly $[0, 1] \times [0, \frac{1}{2}] = \cup_{k=0}^3 \overline{\Delta_k}$. Remark that

$$\begin{aligned}Z_{r,s}^{(2)}(\tau) &\equiv 0 \text{ for } (r, s) \in \{(0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})\} + \mathbb{Z}^2, \\ Z_{r,s}^{(2)}(\tau) &\equiv \infty \text{ for } (r, s) \in \mathbb{Z}^2.\end{aligned}$$

Define

$$(2.6) \quad F_0 := \{\tau \in \mathbb{H} \mid 0 \leq \operatorname{Re} \tau \leq 1, |\tau - \frac{1}{2}| \geq \frac{1}{2}\}.$$

It is known (cf. [7]) that F_0 is a fundamental domain for $\Gamma_0(2) := \{\gamma = (a_{ij}) \in SL(2, \mathbb{Z}) \mid a_{21} \equiv 0 \pmod{2}\}$. The following theorem gives us information about the zeros of $Z_{r,s}^{(2)}(\tau)$ and plays an important role to study the poles of $\lambda_{r,s}(t)$.

Theorem C. ([6]) *Let $(r, s) \in [0, 1] \times [0, \frac{1}{2}] \setminus \frac{1}{2}\mathbb{Z}^2$. Then $Z_{r,s}^{(2)}(\tau) = 0$ has a solution τ in F_0 if and only if $(r, s) \in \Delta_1 \cup \Delta_2 \cup \Delta_3$. Furthermore, for any $(r, s) \in \Delta_1 \cup \Delta_2 \cup \Delta_3$, the solution $\tau \in F_0$ is unique and satisfies $\tau \in \overset{\circ}{F}_0$. In particular, $Z_{r,s}^{(2)}(\tau) \neq 0$ for any $\tau \in \partial F_0$ and $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$.*

We will use Theorem C to prove Theorem 1.2. Our proof is based on the following result to connect the poles of a solution $\lambda_{r,s}(t)$ with zeros of $Z_{r,s}^{(2)}(\tau)$.

Theorem 2.1. *Fix any $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $\tau_0 \in \mathbb{H}$. Then $p_{r,s}(\tau_0) = 0$ in E_{τ_0} , or equivalently $t_0 = t(\tau_0) \notin \{0, 1, \infty\}$ is a pole of $\lambda_{r,s}(t)$, if and only if either $r + s\tau_0 \in \Lambda_{\tau_0}$ or $r + s\tau_0 \notin \Lambda_{\tau_0}$ and $Z_{r,s}^{(2)}(\tau_0) = 0$.*

To prove Theorem 2.1, we have to prove that in the formula (1.8), the numerator and denominator can not vanish simultaneously.

Lemma 2.2. *Fix any $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $\tau \in \mathbb{H}$. Under the above notations, if $Z_{r,s}^{(2)}(\tau) = 0$, then*

$$3\wp'(\alpha)Z_{r,s}^2 + (12\wp^2(\alpha) - g_2)Z_{r,s} + 3\wp(\alpha)\wp'(\alpha) \neq 0.$$

Proof. In the following, we denote $Z_{r,s}(\tau)$ simply by Z . First we recall the well-known result

$$(2.7) \quad g_2^3 - 27g_3^2 = 16(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 \neq 0.$$

Assume by contradiction that

$$(2.8) \quad Z_{r,s}^{(2)}(\tau) = Z^3 - 3\wp(\alpha)Z - \wp'(\alpha) = 0$$

and

$$(2.9) \quad 3\wp'(\alpha)Z^2 + (12\wp^2(\alpha) - g_2)Z + 3\wp(\alpha)\wp'(\alpha) = 0.$$

First we claim:

$$(2.10) \quad Z(\tau) \neq 0 \text{ and } \wp'(\alpha) \neq 0.$$

If $Z(\tau) = 0$, then (2.8) gives $\wp'(\alpha) = 0$, namely

$$(2.11) \quad \alpha = r + s\tau \in E_\tau[2] \setminus \Lambda_\tau \text{ and } Z(\tau) = 0.$$

So we may assume $r + s\tau = \frac{\omega_k}{2} + m + n\tau$ for some $(m, n) \in \mathbb{Z}^2$ and $k \in \{1, 2, 3\}$. Consequently,

$$(r - m) + (s - n)\tau - \frac{\omega_k}{2} = 0,$$

$$(r - m)\eta_1 + (s - n)\eta_2 - \frac{\eta_k}{2} = 0,$$

which implies $(r, s) \in \frac{1}{2}\mathbb{Z}^2$ because the non-degeneracy of $\begin{pmatrix} 1 & \tau \\ \eta_1(\tau) & \eta_2(\tau) \end{pmatrix}$, a contradiction. Similarly, if $\wp'(\alpha) = 0$, then $12\wp^2(\alpha) - g_2 = 2\wp''(\alpha) \neq 0$ and so (2.9) gives $Z(\tau) = 0$, again a contradiction. This proves the claim.

Now we do the Euclidean algorithm for (2.8) and (2.9). Multiplying (2.9) by Z , (2.8) by $3\wp'(\alpha)$ and adding them together, we obtain

$$(2.12) \quad (12\wp^2(\alpha) - g_2)Z^2 + 12\wp(\alpha)\wp'(\alpha)Z + 3\wp'(\alpha)^2 = 0.$$

By (2.12), (2.9) and (1.10), we can eliminate Z^2 term and obtain

$$(2.13) \quad [12g_2\wp(\alpha)^2 + 36g_3\wp(\alpha) + g_2^2]Z = -3\wp'(\alpha)[2g_2\wp(\alpha) + 3g_3].$$

Consequently, multiplying (2.9) by $(12g_2\wp(\alpha)^2 + 36g_3\wp(\alpha) + g_2^2)^2$ and using (2.13) lead to (write $x = \wp(\alpha)$ for convenience)

$$(2.14) \quad 9(4x^3 - g_2x - g_3)(2g_2x + 3g_3)^2 + x(12g_2x^2 + 36g_3x + g_2^2)^2 - (12x^2 - g_2)(2g_2x + 3g_3)(12g_2x^2 + 36g_3x + g_2^2) = 0.$$

A straightforward calculation shows that (2.14) is exactly

$$0 = 3(g_2^3 - 27g_3^2)(4x^3 - g_2x - g_3) = 3\wp'(\alpha)^2(g_2^3 - 27g_3^2),$$

which contradicts to (2.7) and (2.10). \square

Corollary 2.3. *For any $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, any zero of $Z_{r,s}^{(2)}(\tau)$ is simple.*

Proof. The Painlevé property implies that $\wp(p_{r,s}(\tau)|\tau)$ is meromorphic in \mathbb{H} . Therefore, this assertion follows from Lemma 2.2 and the fact that any pole of any solution of PVI $(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ must be simple (see e.g. [12, Proposition 1.4.1]). \square

Now we could prove Theorem 2.1 by Theorem B.

Proof of Theorem 2.1. By the expression (1.8) of $\wp(p_{r,s}(\tau)|\tau)$, we see that $p_{r,s}(\tau_0) = 0$ in E_{τ_0} implies either $r + s\tau_0 \in \Lambda_{\tau_0}$ or $Z_{r,s}^{(2)}(\tau_0) = 0$.

So it suffices to prove the other direction. If $r + s\tau_0 \in \Lambda_{\tau_0}$, without loss of generality we may assume $\alpha(\tau_0) = r + s\tau_0 = 0$. By letting $\alpha = \alpha(\tau) \rightarrow \alpha(\tau_0) = 0$ as $\tau \rightarrow \tau_0$, we have

$$(2.15) \quad \wp(\alpha) = \frac{1}{\alpha^2} + O(\alpha^2), \quad \wp'(\alpha) = \frac{-2}{\alpha^3} + O(\alpha),$$

$$(2.16) \quad Z_{r,s} = \frac{1}{\alpha} (1 - c_0\alpha - c_1\alpha^2 + O(\alpha^3)),$$

where

$$(2.17) \quad c_0 \doteq r\eta_1(\tau_0) + s\eta_2(\tau_0) = -2\pi i s \neq 0$$

and

$$c_1 \doteq \frac{r}{s}\eta_1'(\tau_0) + \eta_2'(\tau_0).$$

Here $r = -s\tau_0$ and the Legendre relation $\tau\eta_1(\tau) - \eta_2(\tau) = 2\pi i$ are used in (2.17). Then we deduce from (2.1) and (2.15)-(2.17) that

$$(2.18) \quad Z_{r,s}^{(2)}(\tau) = \frac{3c_0^2}{\alpha} - (c_0^3 - 6c_0c_1) + O(\alpha),$$

$$\begin{aligned} & 3\wp'(\alpha)Z_{r,s}^2 + (12\wp^2(\alpha) - g_2)Z_{r,s} + 3\wp(\alpha)\wp'(\alpha) \\ &= -\frac{6c_0^2}{\alpha^3} - \frac{12c_0c_1}{\alpha^2} + O(\alpha^{-1}), \end{aligned}$$

and so (1.8) gives

$$(2.19) \quad \wp(p_{r,s}(\tau)|\tau) = -\frac{c_0}{3\alpha(\tau)} + O(1) \rightarrow \infty \text{ as } \tau \rightarrow \tau_0,$$

which implies $p_{r,s}(\tau_0) = 0$ in E_{τ_0} , namely t_0 is a pole of $\lambda_{r,s}(t)$ whenever $r + s\tau_0 \in \Lambda_{\tau_0}$.

If $\alpha(\tau_0) = r + s\tau_0 \notin \Lambda_{\tau_0}$ and $Z_{r,s}^{(2)}(\tau_0) = 0$, then it follows from (1.8) and Lemma 2.2 that $\wp(p_{r,s}(\tau_0)|\tau_0) = \infty$. \square

We need another lemma for the proof of Theorem 1.2.

Lemma 2.4. *Let $\tau \in \mathbb{H}$, $\tau' = \gamma \cdot \tau$ and $(s', r') = (s, r) \cdot \gamma^{-1}$ for some $\gamma \in SL(2, \mathbb{Z})$. Then $Z_{r,s}^{(2)}(\tau) = 0$ if and only if $Z_{r',s'}^{(2)}(\tau') = 0$. In particular, $Z_{r,s}^{(2)}(\tau) \neq 0$ for any $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $\tau \in SL(2, \mathbb{Z}) \cdot i\mathbb{R}^+$.*

Proof. Consider the pair $(z, \tau) \in \mathbb{C} \times \mathbb{H}$ and $z = r + s\tau$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, conventionally γ can act on $\mathbb{C} \times \mathbb{H}$ by

$$\gamma(z, \tau) \doteq \left(\frac{z}{c\tau + d}, \gamma \cdot \tau \right) = \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right).$$

Then

$$(2.20) \quad \frac{z}{c\tau + d} = \frac{r + s\tau}{c\tau + d} = r' + s'\tau', \text{ where } \tau' = \gamma \cdot \tau, (s', r') = (s, r) \cdot \gamma^{-1}.$$

Using

$$(2.21) \quad \wp(r' + s'\tau' | \tau') = (c\tau + d)^2 \wp(r + s\tau | \tau),$$

we proved in [7] that

$$(2.22) \quad Z_{r',s'}(\tau') = (c\tau + d)Z_{r,s}(\tau).$$

Together with (2.21), (2.22) and the fact that $g_2(\tau)$ is a modular form of weight 4, we easily derive from (2.1) that

$$(2.23) \quad Z_{r',s'}^{(2)}(\tau') = (c\tau + d)^3 Z_{r,s}^{(2)}(\tau)$$

and

$$(2.24) \quad \wp(p_{r',s'}(\tau') | \tau') = (c\tau + d)^2 \wp(p_{r,s}(\tau) | \tau).$$

In particular, $Z_{r,s}^{(2)}(\tau) = 0$ if and only if $Z_{r',s'}^{(2)}(\tau') = 0$.

Now fix any $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Suppose $Z_{r,s}^{(2)}(\tau_0) = 0$ for some $\tau_0 \in SL(2, \mathbb{Z}) \cdot i\mathbb{R}^+$, then $\tau_0 = \gamma \cdot \tau$ for some $\tau \in i\mathbb{R}^+$ and $\gamma \in SL(2, \mathbb{Z})$. Let $(s', r') = (s, r) \cdot \gamma$. Then we have $(r', s') \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $(c\tau + d)^3 Z_{r',s'}^{(2)}(\tau) = Z_{r,s}^{(2)}(\tau_0) = 0$. But by Theorem C, $Z_{r',s'}^{(2)}(\tau) \neq 0$, a contradiction. \square

Proof of Theorem 1.2. Let $\lambda(t)$ be an unitary solution of $\text{PVI}(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$, i.e. $\lambda(t) = \lambda_{r,s}(t)$ for some $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Suppose $t_0 \in \mathbb{R} \setminus \{0, 1\}$ is a pole of $\lambda(t)$. Recall $t(\tau)$ in (1.2). A result in the theory of the modular form says that $t(i\mathbb{R}^+) = (0, 1)$, $t(S \cdot i\mathbb{R}^+) = (1, +\infty)$ and $t(U \cdot i\mathbb{R}^+) = (-\infty, 0)$ where $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. See e.g. [1, 13]. Thus there exists $\tau_0 \in SL(2, \mathbb{Z}) \cdot i\mathbb{R}^+$ such that $t_0 = t(\tau_0)$ and $p_{r,s}(\tau_0) = 0$ in E_{τ_0} . By Theorem 2.1 and $r + s\tau_0 \notin \Lambda_{\tau_0}$, $Z_{r,s}^{(2)}(\tau_0) = 0$, which yields a contradiction with Lemma 2.4. Therefore, $\lambda(t)$ has no poles in $\mathbb{R} \setminus \{0, 1\}$. \square

3. POLES OF ALGEBRAIC SOLUTIONS

In this section, we want to find the number of poles of algebraic solutions. For $\text{PVI}(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$, it is well known that a solution is algebraic if and only if its monodromy group is finite, and the monodromy group of an algebraic solution is always the dihedral group D_N of order $2N$ for some $N \in \mathbb{N}_{\geq 3}$. In this case, by Theorem B, any branch of this solution must be one of $\lambda_{r,s}(t)$ where $(r, s) \in Q_N$, the set of N -torsion points. For the classification and related subjects of algebraic solutions of Painlevé VI equation, we refer to [3, 8, 15, 16] and references therein.

To count the number of poles, we have to know how many branches of an algebraic solution might have. A branch of a solution $\lambda(t)$ might be considered a single-valued meromorphic function (still denoted by $\lambda(t)$)

restricted on the simply connected domain $\mathbb{C} \setminus (-\infty, 1]$ or equivalently, the single-valued meromorphic function $\wp(p(\tau)|\tau)$ restricted on a fundamental domain of $\Gamma(2)$ (because $t(\tau)$ is invariant under the action of $\Gamma(2)$). Thus, two solutions $\lambda_{r,s}(t)$ and $\lambda_{r',s'}(t)$, $t \in \mathbb{C} \setminus (-\infty, 1]$, belong to the same solution if $\lambda_{r',s'}(t)$ is the analytic continuation of $\lambda_{r,s}(t)$ along a closed path cross the axis $(-\infty, 1]$. We note that for any algebraic solution, $\lambda_{r,s}(t)$ has no poles on $\mathbb{R} \setminus \{0, 1\}$ by Theorem 1.2. Hence whether a branch is considered as defined on $\mathbb{C} \setminus \{0, 1\}$ or $\mathbb{C} \setminus (-\infty, 1]$ does not affect our calculation below.

Remark 3.1. We recall $\Gamma(N)$ is the N -principal congruence subgroup of $SL(2, \mathbb{Z})$, defined by

$$\Gamma(N) := \{\gamma \in SL(2, \mathbb{Z}) \mid \gamma \equiv I_2 \pmod{N}\}.$$

It is known that $e_k(\tau)$, $k = 1, 2, 3$, are modular form of weight 2 with respect to $\Gamma(2)$. We refer to [13] for the basic theory of modular form.

Proposition 3.2. $\lambda_{r,s}(t)$ and $\lambda_{r',s'}(t)$ belong to the same solution of $PVI(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ if and only if $(s, r) \equiv \pm(s', r') \cdot \gamma \pmod{\mathbb{Z}^2}$ by some $\gamma \in \Gamma(2)$.

Using (1.8), (2.24), and (2.3)-(2.5), the proof of Proposition 3.2 is the same as [7, Proposition 4.4], where $PVI(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ is studied. The same result also holds for Picard solutions. See [16, Theorem 1]. So we omit the details here.

Lemma 3.3. (i) For any N -torsion points (r, s) , $\lambda_{r,s}(t)$ belongs to one of the three solutions $\lambda_{0, \frac{1}{N}}(t)$, $\lambda_{\frac{1}{N}, 0}(t)$ and $\lambda_{\frac{1}{N}, \frac{1}{N}}(t)$.

(ii) If N is odd, then all the three in (i) belong to the same solution; and if N is even, then all the three in (i) represent 3 different solutions.

(iii)

$$(3.1) \quad \lambda_{\frac{1}{N}, 0}(1-t) = 1 - \lambda_{0, \frac{1}{N}}(t), \quad \lambda_{\frac{1}{N}, \frac{1}{N}}\left(\frac{1}{t}\right) = \frac{\lambda_{0, \frac{1}{N}}(t)}{t}.$$

Proof. We divide the proof into three steps.

Step 1. Fix an N -torsion point $(r, s) = (\frac{k_1}{N}, \frac{k_2}{N})$ with $0 \leq k_1, k_2 \leq N-1$ and $\gcd(k_1, k_2, N) = 1$. We show that $\lambda_{r,s}(t)$ belong to the same solution as one of $\{\lambda_{0, \frac{1}{N}}(t), \lambda_{\frac{1}{N}, 0}(t), \lambda_{\frac{1}{N}, \frac{1}{N}}(t)\}$. By Proposition 3.2, it suffices to prove that for some $(r', s') \in \{(0, \frac{1}{N}), (\frac{1}{N}, 0), (\frac{1}{N}, \frac{1}{N})\}$,

$$(3.2) \quad (s, r) \equiv (s', r') \cdot \gamma \pmod{\mathbb{Z}^2} \text{ by some } \gamma \in \Gamma(2).$$

Denote $L = \gcd(k_1, k_2)$ and $k_j = m_j L$. Then we have $\gcd(L, N) = 1$ and $\gcd(m_1, m_2) = 1$.

Case 1. both m_1 and m_2 are odd.

Then there exist $l_1, l_2 \in \mathbb{Z}$ such that $l_1 m_1 + l_2 m_2 = 1$. Letting

$$(3.3) \quad \gamma_1 = \begin{pmatrix} l_1 & -l_2 \\ m_2 - l_1 & m_1 + l_2 \end{pmatrix} \in \Gamma(2) \text{ if } l_1 \text{ odd,}$$

$$(3.4) \quad \gamma_1 = \begin{pmatrix} m_2 + l_1 & m_1 - l_2 \\ -l_1 & l_2 \end{pmatrix} \in \Gamma(2) \text{ if } l_1 \text{ even,}$$

we have

$$(3.5) \quad (s, r) = \left(\frac{Lm_2}{N}, \frac{Lm_1}{N} \right) = \left(\frac{L}{N}, \frac{L}{N} \right) \cdot \gamma_1.$$

If L is odd and N is odd, since $\gcd(L, 2(L - N)) = 1$, there exists $d_1, d_2 \in \mathbb{Z}$ such that $d_1L + 2d_2(L - N) = 1$. Let

$$(3.6) \quad \gamma_2 = \begin{pmatrix} L & L - N \\ -2d_2 & d_1 \end{pmatrix} \in \Gamma(2),$$

then $(\frac{L}{N}, \frac{L}{N}) \in (\frac{1}{N}, 0) \cdot \gamma_2 + \mathbb{Z}^2$. Together with (3.5), we see that (3.2) holds by letting $(r', s') = (0, \frac{1}{N})$ and $\gamma = \gamma_2\gamma_1$. If L is even and N is odd, since $\gcd(2L, L - N) = 1$, there exist $\tilde{d}_1, \tilde{d}_2 \in \mathbb{Z}$ such that $2\tilde{d}_1L + \tilde{d}_2(L - N) = 1$. Let

$$(3.7) \quad \gamma_2 = \begin{pmatrix} \tilde{d}_2 & -2\tilde{d}_1 \\ L & L - N \end{pmatrix} \in \Gamma(2),$$

then $(\frac{L}{N}, \frac{L}{N}) \in (0, \frac{1}{N}) \cdot \gamma_2 + \mathbb{Z}^2$, which implies that (3.2) holds by letting $(r', s') = (\frac{1}{N}, 0)$ and $\gamma = \gamma_2\gamma_1$. If L is odd and N is even, then similarly as (3.3)-(3.4), there exists

$$\gamma_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$$

such that $a + c = L$ and $b + d = L - N$. Clearly $(\frac{L}{N}, \frac{L}{N}) \in (\frac{1}{N}, \frac{1}{N}) \cdot \gamma_2 + \mathbb{Z}^2$ and so (3.2) holds by letting $(r', s') = (\frac{1}{N}, \frac{1}{N})$ and $\gamma = \gamma_2\gamma_1$.

Case 2. m_1 is even and m_2 is odd.

Similarly as (3.6) there exists

$$\gamma_1 = \begin{pmatrix} m_2 & m_1 \\ * & * \end{pmatrix} \in \Gamma(2).$$

Then

$$(3.8) \quad (s, r) = \left(\frac{Lm_2}{N}, \frac{Lm_1}{N} \right) = \left(\frac{L}{N}, 0 \right) \cdot \gamma_1.$$

If L is odd and N is even, there exists

$$\gamma_2 = \begin{pmatrix} L & N \\ * & * \end{pmatrix} \in \Gamma(2).$$

Then $(\frac{L}{N}, 0) \in (\frac{1}{N}, 0) \cdot \gamma_2 + \mathbb{Z}^2$, which implies that (3.2) holds by letting $(r', s') = (0, \frac{1}{N})$ and $\gamma = \gamma_2\gamma_1$. If L is even and N is odd, similarly as (3.7) there exists

$$\gamma_2 = \begin{pmatrix} * & * \\ L & N \end{pmatrix} \in \Gamma(2).$$

Then $(\frac{L}{N}, 0) \in (0, \frac{1}{N}) \cdot \gamma_2 + \mathbb{Z}^2$, which implies that (3.2) holds by letting $(r', s') = (\frac{1}{N}, 0)$ and $\gamma = \gamma_2\gamma_1$. If L is odd and N is odd, then similarly as (3.3)-(3.4) there exists

$$\gamma_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$$

such that $a + c = L$ and $b + d = N$. Then $(\frac{L}{N}, 0) \in (\frac{1}{N}, \frac{1}{N}) \cdot \gamma_2 + \mathbb{Z}^2$ and so (3.2) holds by letting $(r', s') = (\frac{1}{N}, \frac{1}{N})$ and $\gamma = \gamma_2\gamma_1$.

Case 3. m_2 is even and m_1 is odd.

The proof is similar to Case 2, so we omit the details. This completes the proof of Step 1.

Step 2. Suppose $N = 2m+1$ is odd. By choosing $\gamma = -\begin{pmatrix} 4m+1 & 2m \\ 2 & 1 \end{pmatrix} \in \Gamma(2)$, we have

$$\left(\frac{1}{N}, 0\right) \cdot \gamma \equiv \left(\frac{1}{N}, \frac{1}{N}\right) \pmod{\mathbb{Z}^2}.$$

Similarly, by choosing $\tilde{\gamma} = \begin{pmatrix} 1 & 0 \\ 2m & 1 \end{pmatrix}$, we see that

$$\left(\frac{1}{N}, \frac{1}{N}\right) \cdot \tilde{\gamma} \equiv \left(0, \frac{1}{N}\right) \pmod{\mathbb{Z}^2}.$$

Thus when N is odd, all three in (i) belong to the same solution.

Now suppose $N = 2m$ is even. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$, c is even and d is odd. We see that $(0, \frac{1}{N}) \cdot \gamma = (\frac{c}{2m}, \frac{d}{2m})$. Since d is odd, $\frac{d}{2m} \not\equiv 0 \pmod{\mathbb{Z}}$ which implies $(0, \frac{1}{N}) \cdot \gamma \not\equiv (\frac{1}{N}, 0) \pmod{\mathbb{Z}^2}$ for any $\gamma \in \Gamma(2)$. Similarly, any two of $\{(0, \frac{1}{N}), (\frac{1}{N}, 0), (\frac{1}{N}, \frac{1}{N})\}$ can not be connected by $\Gamma(2)$ and $\pmod{\mathbb{Z}^2}$. Thus when N is even, all the three in (i) represent 3 different solutions.

Step 3. We prove (3.1).

Let $\gamma = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $\tau' = \gamma \cdot \tau = \tau - 1$. Since $(\frac{1}{N}, \frac{1}{N}) = (\frac{1}{N}, 0) \cdot \gamma^{-1}$, we see from (2.20) and (2.24) that

$$\wp\left(p_{\frac{1}{N}, \frac{1}{N}}(\tau') | \tau'\right) = \wp\left(p_{0, \frac{1}{N}}(\tau) | \tau\right).$$

This, together with $e_1(\tau') = e_1(\tau)$, $e_2(\tau') = e_3(\tau)$, $e_3(\tau') = e_2(\tau)$ and (1.2), implies $t(\tau') = 1/t(\tau)$ and

$$(3.9) \quad \lambda_{\frac{1}{N}, \frac{1}{N}}(t(\tau')) = \frac{\wp\left(p_{0, \frac{1}{N}}(\tau) | \tau\right) - e_1(\tau)}{e_3(\tau) - e_1(\tau)} = \frac{\lambda_{0, \frac{1}{N}}(t(\tau))}{t(\tau)}.$$

This proves the second formula in (3.1). Similarly, by letting $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, it is easy to prove the first formula in (3.1). \square

Define

$$(3.10) \quad M_N(\tau) := \prod_{(r,s) \in Q_N} Z_{r,s}^{(2)}(\tau).$$

By (2.23), $M_N(\tau)$ is a *modular form* with respect to $SL(2, \mathbb{Z})$ of weight $3|Q_N|$, where $|Q_N| = \#Q_N$.

To obtain the number of zeros of $M_N(\tau)$, we recall the classical formula for counting zeros of modular forms. See [19] for the proof.

Theorem D. *Let $f(\tau)$ be a nonzero modular form with respect to $SL(2, \mathbb{Z})$ of weight k . Then*

$$(3.11) \quad \sum_{\tau \in \mathbb{H} \setminus \{i, \rho\}} \nu_\tau(f) + \nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\rho(f) = \frac{k}{12},$$

where $\rho := e^{\pi i/3}$, $\nu_\tau(f)$ denotes the zero order of f at τ and the summation over τ is performed modulo $SL(2, \mathbb{Z})$ equivalence.

To prove Theorem 1.4, by Theorem D we have to calculate the asymptotics of $Z_{r,s}^{(2)}(\tau)$ as $\text{Im } \tau \rightarrow +\infty$. Now we consider the fundamental domain F of $SL(2, \mathbb{Z})$ defined by

$$F := \{\tau \in \mathbb{H} \mid 0 \leq \text{Re } \tau < 1, |\tau| \geq 1, |\tau - 1| > 1\} \cup \{\rho = e^{\pi i/3}\}.$$

Denote $q = e^{2\pi i\tau}$. We recall the q -expansions of $\wp(z|\tau)$ (cf. [14, p.46] and [7, (5.3)]): for $|q| < |e^{2\pi iz}| < |q|^{-1}$,

$$(3.12) \quad \frac{\wp(z|\tau)}{-4\pi^2} = \frac{1}{12} + \frac{e^{2\pi iz}}{(1 - e^{2\pi iz})^2} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} nq^{nm} (e^{2\pi inz} + e^{-2\pi inz} - 2),$$

$$(3.13) \quad \begin{aligned} Z_{r,s}(\tau) &= 2\pi is - \pi i \frac{1 + e^{2\pi iz}}{1 - e^{2\pi iz}} \\ &\quad - 2\pi i \sum_{n=1}^{\infty} \left(\frac{e^{2\pi iz} q^n}{1 - e^{2\pi iz} q^n} - \frac{e^{-2\pi iz} q^n}{1 - e^{-2\pi iz} q^n} \right), \end{aligned}$$

where $z = r + s\tau$ in (3.13). By using (3.12) and (3.13), we could compute the asymptotics of $Z_{r,s}^{(2)}(\tau)$ as $\text{Im } \tau \rightarrow +\infty$. The calculation is straightforward and was done in [6]. So we skip the calculation and state the final result in the following lemma.

Lemma 3.4. [6] *Let $r \in \mathbb{R}$ and $s \in [0, 1)$ and $0 \leq \text{Re } \tau \leq 1$. Then as $\text{Im } \tau \rightarrow +\infty$,*

$$(3.14) \quad \begin{aligned} (i) \quad &\lim_{\text{Im } \tau \rightarrow +\infty} Z_{r,s}^{(2)}(\tau) \neq 0 \text{ iff } s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\ (ii) \quad &Z_{r,s}^{(2)}(\tau) = -48\pi^3 \sin(2\pi r)q + O(q^2) \text{ if } s = 0, \\ (iii) \quad &Z_{r,s}^{(2)}(\tau) = -12\pi^3 \sin(2\pi r)q^{1/2} + O(q) \text{ if } s = 1/2. \end{aligned}$$

Let $\phi(N)$ be the Euler function defined in (1.12) if N is an integer, and be zero if N is not an integer. Now by applying Theorem D to $M_N(\tau)$, we have the following theorem.

Theorem 3.5. *For $N \geq 3$, the total number $P(N)$ of zeros (counting multiplicity) of $M_N(\tau)$ in the fundamental domain F of $SL(2, \mathbb{Z})$ is given by*

$$(3.15) \quad P(N) = \frac{|Q_N|}{4} - \left[\phi(N) + \phi\left(\frac{N}{2}\right) \right].$$

Proof. First, it is easy to see that $\frac{|Q_N|}{4} \in \mathbb{Z}_{>0}$ for any $N \geq 3$ because

$$|Q_N| = N^2 \prod_{p|N, p \text{ prime}} \left(\frac{p^2 - 1}{p^2} \right).$$

Since $i \in \partial F_0$, by Theorem C we see that $Z_{r,s}^{(2)}(i) \neq 0$ for any $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Thus

$$(3.16) \quad \nu_i(M_N(\tau)) = 0.$$

By [6, Theorem 4.2] where $Z_{r,s}^{(2)}(\rho) \neq 0$ for any $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ is proved, we have

$$(3.17) \quad \nu_\rho(M_N(\tau)) = 0.$$

Then we deduce from (3.16), (3.17) and (3.11) that

$$(3.18) \quad P(N) = \sum_{\tau \in \mathbb{H} \setminus \{i, \rho\}} \nu_\tau(M_N(\tau)) = \frac{|Q_N|}{4} - v_\infty(M_N(\tau)).$$

From Lemma 3.4, we see that

$$(3.19) \quad \begin{aligned} v_\infty(M_N(\tau)) &= \# \{1 \leq k_1 < N \mid \gcd(N, k_1) = 1\} \\ &\quad + \frac{1}{2} \# \{1 \leq k_1 < N \mid \gcd(N/2, k_1) = 1\} \\ &= \phi(N) + \phi\left(\frac{N}{2}\right). \end{aligned}$$

Here $\phi\left(\frac{N}{2}\right) = 0$ whenever N is odd. Combining (3.18) and (3.19), we have

$$P(N) = \frac{|Q_N|}{4} - \left[\phi(N) + \phi\left(\frac{N}{2}\right) \right].$$

where the summation over τ is performed modulo $SL(2, \mathbb{Z})$ equivalence. This completes the proof. \square

For $N = 3, 4$, it is easy to see $P(N) = 0$.

Corollary 3.6. *For $N = 3, 4$ and $(r, s) \in Q_N$, $Z_{r,s}^{(2)}(\tau) \neq 0$ for any $\tau \in \mathbb{H}$.*

Now, we are going to prove Theorem 1.4. Before the proof, we have following three discussions:

(i) It is well-known (cf. [1]) that $\Gamma(2)$ is a normal subgroup of $SL(2, \mathbb{Z})$ and $SL(2, \mathbb{Z})/\Gamma(2) = \{I, S, ST, S^2T, TS^{-1}, STS^{-1}\}$ where $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So a fundamental domain F_2 of $\Gamma(2)$ can be obtained by F :

$$(3.20) \quad F_2 = F \cup SF \cup STF \cup S^2TF \cup TS^{-1}F \cup STS^{-1}F.$$

By a straightforward computation, we have

$$F_2 = \{\tau \in \mathbb{H} \mid 0 \leq \operatorname{Re} \tau < 2, |\tau - \frac{1}{2}| \geq \frac{1}{2}, |\tau - \frac{3}{2}| > \frac{1}{2}\}.$$

Note that for any $\tau \in F_2$, $t(\tau) \in \mathbb{R} \setminus \{0, 1\}$ if and only if $\tau \in i\mathbb{R}^+ \cup \{\tau \in \mathbb{H} \mid |\tau - \frac{1}{2}| = \frac{1}{2}\} \cup \{\tau \in \mathbb{H} \mid \operatorname{Re} \tau = 1\}$. More precisely, $t(i\mathbb{R}^+) = (0, 1)$, $t(\{\tau \in \mathbb{H} \mid |\tau - \frac{1}{2}| = \frac{1}{2}\}) = (-\infty, 0)$ and $t(1 + i\mathbb{R}^+) = (1, +\infty)$. See [1, 13]. Hence the image of $F_2 \setminus (i\mathbb{R}^+ \cup \{\tau \in \mathbb{H} \mid |\tau - \frac{1}{2}| = \frac{1}{2}\})$ is $\mathbb{C} \setminus (-\infty, 1]$. By (3.20) and $M_N(\tau) \neq 0$ at $\tau \in \{i, \rho\}$, we have

$$\#\{\tau \in F_2 \mid M_N(\tau) = 0\} = 6P(N),$$

where the RHS is counted by multiplicity.

(ii) If $(r, s) \in Q_N$, then $Q_N \ni (r', s') \doteq (1-r, 1-s) \pmod{\mathbb{Z}^2}$ and $Z_{r', s'}^{(2)}(\tau) = -Z_{r, s}^{(2)}(\tau)$. Hence if $Z_{r, s}^{(2)}(\tau_0) = 0$ then $Z_{r', s'}^{(2)}(\tau_0) = 0$. In other words, the order of each zero of $M_N(\tau)$ is *even*. Remark that $\lambda_{r, s}(t) \equiv \lambda_{r', s'}(t)$ by (2.5).

(iii) By Corollary 2.3, $Z_{r, s}^{(2)}(\tau)$ has only simple zeros. For two N -torsion points $(r, s) \not\equiv \pm(r', s') \pmod{\mathbb{Z}^2}$, $Z_{r, s}^{(2)}(\tau)$ and $Z_{r', s'}^{(2)}(\tau)$ might simultaneously vanish at the same $\tau_0 \in F_2$. But because (2.5) gives $\lambda_{r, s}(t) \not\equiv \lambda_{r', s'}(t)$, either they belongs to two different algebraic solutions (this might happen if N is even) or they are two different branches of the same algebraic solution $\lambda(t)$ (this must happen if N is odd). In the later case, we count the number of poles of $\lambda(t)$ at $t_0 = t(\tau_0)$ as 2 (as multiplicity).

Proof of Theorem 1.4. If N is odd, then for all $(r, s) \in Q_N$, $\lambda_{r, s}(t)$ belong to the same one algebraic solution $\lambda(t)$. By (i), (ii) and the simple zero property (iii), we obtain

$$\text{the total number of poles of } \lambda(t) = 6P(N) \times \frac{1}{2} = 3P(N).$$

If N is even, then we have three different solutions, namely $\lambda_{\frac{1}{N}, 0}(t)$, $\lambda_{0, \frac{1}{N}}(t)$, $\lambda_{\frac{1}{N}, \frac{1}{N}}(t)$. By (3.1), each of them have the same number of branches and the same number of poles. Let $\lambda(t)$ be any one of them. Then by (i), (ii) and (iii), we have

$$\text{the total number of poles of } \lambda(t) = 6P(N) \times \frac{1}{2} \times \frac{1}{3} = P(N).$$

This completes the proof. \square

4. THE PROOF OF THEOREM 1.6

In this section, we want to prove Theorem 1.6, namely the following result.

Theorem 4.1. *Among all such solutions $\lambda_{r,s}(t)$ of $PVI(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$, where $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, there are exactly four solutions which have no poles in $\mathbb{C} \setminus \{0, 1\}$. They are precisely $\lambda_{0, \frac{1}{3}}(t)$, $\lambda_{0, \frac{1}{4}}(t)$, $\lambda_{\frac{1}{4}, 0}(t)$ and $\lambda_{\frac{1}{4}, \frac{1}{4}}(t)$.*

First we consider the case $(r, s) \notin \mathbb{R}^2$. Since $(r, s) \in \mathbb{C}^2 \setminus \mathbb{R}^2$, there are infinitely many $\tau_0 \in \mathbb{H}$ such that $r + s\tau_0 \in \Lambda_{\tau_0}$. The following result is a direct consequence of Theorem 2.1.

Lemma 4.2. *Let $(r, s) \in \mathbb{C}^2 \setminus \mathbb{R}^2$. Then $\lambda_{r,s}(t)$ has infinitely many poles.*

Proof of Theorem 4.1. Let $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. By Theorem 2.1 and Corollary 3.6, $\lambda_{r,s}(t)$ has no poles in $\mathbb{C} \setminus \{0, 1\}$ for $(r, s) \in Q_3 \cup Q_4$. Therefore, we only need to prove that $\lambda_{r,s}(t)$ has poles in $\mathbb{C} \setminus \{0, 1\}$ whenever $(r, s) \notin Q_3 \cup Q_4$. By Lemma 4.2 and (2.5), we only need to consider $(r, s) \in [0, 1] \times [0, \frac{1}{2}] \setminus \frac{1}{2}\mathbb{Z}^2$. Then by Theorem 1.4, we see that $P(N) > 0$ except $N = 3, 4$. Together with Theorem C, it is enough for us to consider

$$(r, s) \in \Delta_0 \cup \bigcup_{k=0}^3 \partial\Delta_k \setminus \mathbb{Q}^2.$$

For $(r, s) \in \bigcup_{k=0}^3 \partial\Delta_k \setminus \mathbb{Q}^2$, we have $\{r, s, r+s\} \cap \mathbb{Q} \neq \emptyset$. Taking $\gamma = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \in \Gamma(2)$ and letting $(s', r') = (s, r) \cdot \gamma = (4r + 3s, 3r + 2s)$, we deduce from $(r, s) \notin \mathbb{Q}^2$ that $\pm(r', s') \notin \bigcup_{k=0}^3 \partial\Delta_k + \mathbb{Z}^2$. Then by replacing (r', s') by some element in $\pm(r', s') + \mathbb{Z}^2$, we may assume $(r', s') \in \bigcup_{k=0}^3 \Delta_k$. By means of Proposition 3.2, $\lambda_{r,s}(t)$ and $\lambda_{r',s'}(t)$ belong to the same solution. Therefore, we conclude that to prove Theorem 4.1, we only need to prove that $\lambda_{r,s}(t)$ has poles in $\mathbb{C} \setminus \{0, 1\}$ provided that

$$(4.1) \quad (r, s) \in \Delta_0 \setminus \mathbb{Q}^2 = \{(r, s) \mid 0 < r, s < \frac{1}{2}, r + s > \frac{1}{2}\} \setminus \mathbb{Q}^2.$$

Fix any $(r, s) \in \Delta_0 \setminus \mathbb{Q}^2$. The same argument as (3.9) gives

$$(4.2) \quad \lambda_{r+s,s} \left(\frac{1}{t} \right) = \frac{\lambda_{r,s}(t)}{t}.$$

Notice that if $r + 2s \neq 1$, then $(r + s, s) \in \Delta_1 \cup \Delta_2$. Applying Proposition 3.5, it follows that $\lambda_{r+s,s}(t)$ has poles in $\mathbb{C} \setminus \{0, 1\}$ and so does $\lambda_{r,s}(t)$.

So it suffices to consider $r = 1 - 2s$, which implies $s \in (\frac{1}{4}, \frac{1}{2}) \setminus \mathbb{Q}$. If $s \in (\frac{1}{4}, \frac{3}{8})$, then $(r', s') \doteq (2 - 4s, s) \in \Delta_1 \cup \Delta_2$, which implies that $\lambda_{r',s'}(t)$ has poles in $\mathbb{C} \setminus \{0, 1\}$. Since

$$(s', r' - 1) = (s, r - 2s) = (s, r) \cdot \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$

by applying Proposition 3.2 and (2.5), we see that $\lambda_{r,s}(t)$ belong to the same solution with $\lambda_{r',s'}(t)$ and so has poles in $\mathbb{C} \setminus \{0, 1\}$.

So we may assume $r = 1 - 2s$ and $s \in (\frac{3}{8}, \frac{1}{2}) \setminus \mathbb{Q}$. Then there exists $m \in \mathbb{N}$ such that either

$$(4.3) \quad 2mr < s < (2m + 1)r,$$

or

$$(4.4) \quad (2m - 1)r < s < 2mr.$$

Let $\gamma = \begin{pmatrix} 1 & 0 \\ -2m & 1 \end{pmatrix} \in \Gamma(2)$ and $(s', r') = (s, r) \cdot \gamma = (s - 2mr, r)$. If (4.3) holds, then $(r', s') = (r, s - 2mr) \in \Delta_3$. If (4.4) holds, then $(1 - r', -s') = (1 - r, 2mr - s) \in \Delta_2$. In both cases, by means of Proposition 3.5 and (2.5), we see that $\lambda_{r', s'}(t)$ has poles in $\mathbb{C} \setminus \{0, 1\}$. Since Proposition 3.2 says that $\lambda_{r, s}(t)$ and $\lambda_{r', s'}(t)$ belong to the same solution, we conclude that $\lambda_{r, s}(t)$ has poles in $\mathbb{C} \setminus \{0, 1\}$. The proof is complete. \square

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