ON REDUCIBLE MONODROMY REPRESENTATIONS OF SOME GENERALIZED LAMÉ EQUATION

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ABSTRACT. In this note, we compute the explicit formula of the monodromy data for a generalized Lamé equation when its monodromy is reducible but not completely reducible. We also solve the corresponding Riemann-Hilbert problem.

1. INTRODUCTION

Throughout the paper, we use the notations $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = \tau$, $\omega_3 = 1 + \tau$ and $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H} = \{\tau | \operatorname{Im} \tau > 0\}$. Define $E_\tau := \mathbb{C}/\Lambda_\tau$ to be a flat torus in the plane and $E_\tau[2] := \{\frac{\omega_k}{2} | 0 \leq k \leq 3\} + \Lambda_\tau$ to be the set consisting of the lattice points and 2-torsion points in E_τ.

Let $\wp(z) = \wp(z|\tau)$ be the Weierstrass elliptic function with periods Λ_τ, defined by

$$\wp(z|\tau) := \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

and $e_k = e_k(\tau) := \wp(\frac{\omega_k}{2}|\tau), k \in \{1, 2, 3\}$. Let $\zeta(z) = \zeta(z|\tau) := -\int^z \wp(\xi|\tau)d\xi$ be the Weierstrass zeta function, which is an odd meromorphic function with two quasi-periods:

(1.1) $\qquad \eta_1(\tau) = \zeta(z+1|\tau) - \zeta(z|\tau), \;\; \eta_2(\tau) = \zeta(z+\tau|\tau) - \zeta(z|\tau).$

In this note, we study the following generalized Lamé equation (GLE):

(1.2) $\quad y''(z) = \left[\begin{array}{c} \sum_{k=0}^{3} n_k(n_k+1)\wp(z + \frac{\omega_k}{2}) + \frac{3}{4}(\wp(z+p) \\ + \wp(z-p)) + A(\zeta(z+p) - \zeta(z-p)) + B \end{array} \right] y(z) \;\; \text{in } E_\tau,$

where $n_k \in \mathbb{N} \cup \{0\}$ for all k, $A, B \in \mathbb{C}$ and $\pm p \notin E_\tau[2]$ are always assumed to be *apparent singularities* (i.e. non-logarithmic). Under this assumption, B is determined by (p, A) as follows (see [2]):

(1.3) $\qquad B = A^2 - \zeta(2p)A - \frac{3}{4}\wp(2p) - \sum_{k=0}^{3} n_k(n_k+1)\wp\left(p + \frac{\omega_k}{2}\right).$

We are interested in GLE (1.2), which was studied in [10, 2], because it has a deep relation with the well-known Panlevé VI equation. Indeed, if $(A(\tau), B(\tau), p(\tau))$ depends on τ suitably such that GLE (1.2) preserves the monodromy as τ deforms, then $p(\tau)$ satisfies the elliptic form of Panlevé VI equation. See [2] or Section 3. Note that by letting $x = \wp(z)$, GLE (1.2) can

be projected to a new equation on $\mathbb{CP}^1$, which is a second order Fuchsian equation with five singular points $\{e_1, e_2, e_3, \wp(p), \infty\}$ with $\wp(p)$ being an apparent singularity. Such type of ODEs on $\mathbb{CP}^1$ have been widely studied in the literature. However, the monodromy of this new ODE are not easy to compute. Therefore, it is more convenient for us to study GLE (1.2) in E_τ directly as long as the monodromy is concerned.

The monodromy representation of GLE (1.2) is a homomorphism ρ : $\pi_1\left(E_\tau\backslash(E_\tau[2]\cup(\{\pm p\}+\Lambda_\tau)), q_0\right) \to SL(2,\mathbb{C})$, where $q_0 \notin E_\tau[2]\cup(\{\pm p\}+\Lambda_\tau)$ is a base point. Let $\gamma_\pm \in \pi_1\left(E_\tau\backslash(E_\tau[2]\cup(\{\pm p\}+\Lambda_\tau)), q_0\right)$ be a simple loop encircling $\pm p$ counterclockwise respectively, and ℓ_j, $j = 1,2$, be two fundamental cycles of E_τ connecting q_0 with $q_0 + \omega_j$ such that ℓ_j does not intersect with $L + \Lambda_\tau$ (here L is the straight segment connecting $\pm p$) and satisfies

$$(1.4) \qquad \gamma_+\gamma_- = \ell_1\ell_2\ell_1^{-1}\ell_2^{-1} \text{ in } \pi_1\left(E_\tau\backslash(\{\pm p\}+\Lambda_\tau), q_0\right).$$

Since the local exponents of (1.2) at $\pm p$ are $-\frac{1}{2}$ and $\frac{3}{2}$ and $\pm p \notin E_\tau[2]$ are apparent singularities, we always have

$$(1.5) \qquad \rho(\gamma_\pm) = -I_2.$$

For any $k \in \{0,1,2,3\}$, the local exponents of GLE (1.2) at $\omega_k/2$ are $-n_k$ and $n_k + 1$ with $n_k \in \mathbb{Z}$. Since the potential of GLE (1.2) is even elliptic, the local monodromy matrix of GLE (1.2) at $\omega_k/2$ is I_2 (see e.g. [10, Lemma 2.2]). Therefore, the monodromy group of GLE (1.2) is generated by $\{-I_2, \rho(\ell_1), \rho(\ell_2)\}$. Together with (1.4) and (1.5), we immediately obtain $\rho(\ell_1)\rho(\ell_2) = \rho(\ell_2)\rho(\ell_1)$, which implies that the monodromy group of GLE (1.2) is always *abelian* and hence *reducible*, i.e. *all the monodromy matrices have at least a common eigenfunction*. Clearly there are two cases:

(a) Completely reducible, i.e. all the monodromy matrices have two linearly independent common eigenfunctions. This case has been well studied in [4].

(b) Not completely reducible, i.e. the space of common eigenfunctions is of dimension 1: Up to a common conjugation,

$$(1.6) \qquad \rho(\ell_1) = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix},$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $C \in \mathbb{C} \cup \{\infty\}$. Remark that if $C = \infty$, then (1.6) should be understood as

$$\rho(\ell_1) = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

In this case, C is called the *monodromy data* of GLE (1.2).

Remark that for the projective ODE of GLE (1.2) on $\mathbb{CP}^1$, its monodromy representation is *irreducible* if and only if Case (a) occurs, and *reducible* if and only if Case (b) occurs. Most of the references in the literature are devoted

to the case of irreducible representation on $\mathbb{CP}^1$, but very few are devoted to studying reducible representation.

We are interested in *the explicit formula of the monodromy data C* when Case (b) occurs. This problem is important but challenging for general n_k, which will be studied in a future paper. In this note, we focus on the special case $(n_0, n_1, n_2, n_3) = (1, 0, 0, 0)$, i.e. GLE

$$(1.7) \qquad y''(z) = \left[\begin{array}{c} 2\wp(z) + \frac{3}{4}(\wp(z+p) + \wp(z-p)) \\ +A(\zeta(z+p) - \zeta(z-p)) + B \end{array} \right] y(z) \text{ in } E_\tau.$$

Our first result is following.

Theorem 1.1. *Fix $\tau \in \mathbb{H}$ and $p \notin E_\tau[2]$. If the monodromy of GLE (1.7) is not completely reducible, then the monodromy data C satisfies either*

$$(1.8) \qquad \wp(p|\tau) = \frac{2g_3(C-\tau)^3 - 4(C\eta_1 - \eta_2)^3 - g_2(C\eta_1 - \eta_2)(C-\tau)^2}{(C-\tau)[12(C\eta_1 - \eta_2)^2 - g_2(C-\tau)^2]},$$

with $(\varepsilon_1, \varepsilon_2) = (1, 1)$ or

$$(1.9) \qquad \wp(p|\tau) = \frac{(\frac{g_2}{2} - 3e_k^2)(C\eta_1 - \eta_2) + \frac{g_2}{4}e_k(C-\tau)}{3e_k(C\eta_1 - \eta_2) + (\frac{g_2}{2} - 3e_k^2)(C-\tau)},$$

for some $k \in \{1, 2, 3\}$ with

$$(1.10) \qquad (\varepsilon_1, \varepsilon_2) = \left\{ \begin{array}{ll} (1, -1) & \text{if } k = 1, \\ (-1, 1) & \text{if } k = 2, \\ (-1, -1) & \text{if } k = 3. \end{array} \right.$$

Here $g_2 = g_2(\tau)$ and $g_3 = g_3(\tau)$ are the coefficients of

$$\wp'(z|\tau)^2 = 4\wp(z|\tau)^3 - g_2(\tau)\wp(z|\tau) - g_3(\tau) = 4\prod_{k=1}^{3}(\wp(z|\tau) - e_k(\tau)).$$

The formulas (1.8)-(1.9) first appeared in [10, (3.68)-(3.69)] without detailed proofs and later was obtained in [3] independently, as explicit expressions of Riccati type solutions of Painlevé VI equation. But their connection with the monodromy data seems not be well addressed. Theorem 1.1 can be proved directly without applying Painlevé VI equation; see Section 2.

Conversely, it is natural to consider the following *Riemann-Hilbert problem*: For fixed $\tau \in \mathbb{H}$ and $p \notin E_\tau[2]$, and given any C satisfying (1.8) or (1.9), whether there exists GLE (1.7) (i.e. exist $A, B \in \mathbb{C}$) such that this C is its monodromy data? This problem is fundamental but seems not be settled. Our second main result is to answer this question positively.

Theorem 1.2. *Fix $\tau \in \mathbb{H}$ and $p \notin E_\tau[2]$.*

(1) *If $C \in \mathbb{C} \cup \{\infty\}$ satisfies the cubic equation (1.8), then there exists $A \in \mathbb{C}$ (and B is given by (p, A) via (1.3) with $(n_0, n_1, n_2, n_3) = (1, 0, 0, 0)$) such that for the correpsonding GLE (1.7), up to a common conjugation,*

$$(1.11) \qquad \rho(\ell_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}.$$

(2) *Fix $k \in \{1, 2, 3\}$. If $C \in \mathbb{C} \cup \{\infty\}$ satisfies the equation (1.9), then there exists $A \in \mathbb{C}$ such that for the correpsonding GLE (1.7), up to a common conjugation,*

$$\rho(\ell_1) = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix},$$

where $(\varepsilon_1, \varepsilon_2)$ is given by (1.10).

If we know that the number of A's with the monodromy of the corresponding GLE (1.7) being not completely reducible is 6, then Theorem 1.2 might follow from Theorem 1.1. However, this assumption can not hold for all (p, τ), and the proof of Theorem 1.2 becomes subtle. In Section 3, we will present a proof by applying the connection between GLE (1.7) and Painlevé VI equation. Theorem 1.1 will be proved in Section 2.

2. PROOF OF THEOREM 1.1

The purpose of this section is to prove Theorem 1.1. Let $y_1(z)$ be a common eigenfunction. Then it is known (cf. [4]) that $y_1(z)$ can be expressed as

$$(2.1) \qquad y_1(z) = e^{(r\eta_1 + s\eta_2)z} \frac{\sigma(z - a_1)\sigma(z - a_2)}{\sigma(z)^2} \frac{\sigma(z)}{[\sigma(z - p)\sigma(z + p)]^{\frac{1}{2}}},$$

where (r, s) is determined by

$$(2.2) \qquad r + s\tau = a_1 + a_2, \quad r\eta_1 + s\eta_2 = \zeta(a_1) + \zeta(a_2),$$

and $a_1, a_2 \notin \Lambda_\tau$ satisfy

$$(2.3) \qquad [\zeta(a_i + p) + \zeta(a_i - p) - 2\zeta(a_i)]$$
$$= 2 [\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)], \quad \{i, j\} = \{1, 2\}.$$

Here $\sigma(z) = \sigma(z|\tau)$ is the Weierstrass sigma function defined by $\frac{\sigma'(z)}{\sigma(z)} := \zeta(z)$. It is known that $\sigma(z)$ is an odd entire function with simple zeros only at the lattice points Λ_τ. Remark that (2.3) can be easily obtained by inserting (2.1) into GLE (1.7).

Proof of Theorem 1.1. Suppose the monodromy representation of GLE (1.7) is *not* completely reducible, i.e. the space of common eigenfunctions is of dimension 1. Then $y_1(-z) = \pm y_1(z)$ (because $y_1(-z)$ is also a common eigenfunction), i.e.

$$(2.4) \qquad \{a_1, a_2\} = \{-a_1, -a_2\} \text{ in } E_\tau.$$

Remark that $\frac{\sigma(z)^2}{\sigma(z-p)\sigma(z+p)}$ is even elliptic. Since ℓ_j does not intersect with $L + \Lambda_\tau$, the function $\frac{\sigma(z)}{[\sigma(z-p)\sigma(z+p)]^{\frac{1}{2}}}$ is invariant under analytic continuation

along ℓ_j (see [4]), i.e.

$$(2.5) \qquad \ell_j^* \frac{\sigma(z)}{[\sigma(z-p)\sigma(z+p)]^{\frac{1}{2}}} = \frac{\sigma(z)}{[\sigma(z-p)\sigma(z+p)]^{\frac{1}{2}}}, \quad j = 1, 2.$$

Here we denote by $\ell_j^* f(z)$ to be the function obtained from $f(z)$ under analytic continuation along ℓ_j. By (2.4) there are two cases.

Case 1. $a_j \notin E_\tau[2]$ for $j = 1, 2$, then $a_2 = -a_1$.

Then $(r, s) = (0, 0)$, i.e.

$$y_1(z) = \frac{\sigma(z-a_1)\sigma(z+a_1)}{\sigma(z)^2} \frac{\sigma(z)}{[\sigma(z-p)\sigma(z+p)]^{\frac{1}{2}}}.$$

By (2.5) and the transformation law

$$(2.6) \qquad \sigma(z+\omega_j) = -e^{(z+\omega_j/2)\eta_j}\sigma(z), \quad j = 1, 2,$$

it is easy to prove that

$$(2.7) \qquad \ell_j^* y_1(z) = y_1(z) \text{ for } j = 1, 2.$$

Therefore, $y_1(z)^2$ is even elliptic, i.e. up to a constant,

$$(2.8) \qquad y_1(z)^{-2} = \frac{\wp(z) - \wp(p)}{(\wp(z) - \wp(a_1))^2}.$$

On the other hand, it follows from the addition formulas

$$\zeta(u+v) + \zeta(u-v) - 2\zeta(u) = \frac{\wp'(u)}{\wp(u) - \wp(v)},$$

$$\zeta(2u) - 2\zeta(u) = \frac{\wp''(u)}{2\wp'(u)},$$

and (2.3) that

$$\frac{\wp'(a_1)}{\wp(a_1) - \wp(p)} = 2\left[\zeta(2a_1) - 2\zeta(a_i)\right] = \frac{\wp''(a_1)}{\wp'(a_1)},$$

i.e.

$$(2.9) \qquad \wp(p) = \wp(a_1) - \frac{\wp'(a_1)^2}{\wp''(a_1)}.$$

From here and (2.8), it is easy to see that the *residues* of $y_1(z)^{-2}$ at $\pm a_1$ are both 0, so

$$y_1(z)^{-2} = c_3\left[\wp(z-a_1) + \wp(z+a_1) - 2\wp(a_1)\right],$$

where $c_3 = \frac{\wp(a_1) - \wp(p)}{\wp'(a_1)^2}$. Then

$$\chi(z) := \int_0^z \frac{1}{y_1(\xi)^2} d\xi = -c_3(\zeta(z-a_1) + \zeta(z+a_1) + 2\wp(a_1)z),$$

and so $\chi(z)$ is quasi-periodic with two quasi-periods:

$$\chi_1 = \chi(z+1) - \chi(z) = -2c_3(\eta_1 + \wp(a_1)),$$
$$\chi_2 = \chi(z+\tau) - \chi(z) = -2c_3(\eta_2 + \wp(a_1)\tau).$$

Define

$$(2.10) \qquad\qquad C := \frac{\chi_2}{\chi_1} = \frac{\eta_2 + \wp(a_1)\tau}{\eta_1 + \wp(a_1)},$$

and $y_2(z) := y_1(z)\chi(z)$. Since $y_1(z)$ is a solution of GLE (1.7) and $\chi(z) = \int_0^z \frac{1}{y_1(\xi)^2}d\xi$, it is easy to see that $y_2(z)$ is a *linearly independent* solution of GLE (1.7) with respect to $y_1(z)$. Recalling (2.7) and that $y_2(z)$ is not a common eigenfunction, we know that χ_1, χ_2 can not vanish simultaneously. Consequently, if $\chi_1 = 0$, then $\chi_2 \neq 0$, $C = \infty$ and so

$$\ell_1^* \begin{pmatrix} \chi_2 y_1(z) \\ y_2(z) \end{pmatrix} = \begin{pmatrix} \chi_2 y_1(z) \\ y_2(z) \end{pmatrix}, \quad \text{i.e. } \rho(\ell_1) = I_2,$$

$$\ell_2^* \begin{pmatrix} \chi_2 y_1(z) \\ y_2(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \chi_2 y_1(z) \\ y_2(z) \end{pmatrix}, \quad \text{i.e. } \rho(\ell_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

If $\chi_1 \neq 0$, then $C \neq \infty$ and so

$$\ell_1^* \begin{pmatrix} \chi_1 y_1(z) \\ y_2(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \chi_1 y_1(z) \\ y_2(z) \end{pmatrix}, \quad \text{i.e. } \rho(\ell_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$\ell_2^* \begin{pmatrix} \chi_1 y_1(z) \\ y_2(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} \chi_1 y_1(z) \\ y_2(z) \end{pmatrix}, \quad \text{i.e. } \rho(\ell_2) = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}.$$

Clearly (2.10) gives

$$\wp(a_1) = \frac{C\eta_1 - \eta_2}{\tau - C}.$$

Inserting this into (2.9) and using $\wp'(a_1)^2 = 4\wp(a_1)^3 - g_2\wp(a_1) - g_3$, $\wp''(a_1) = 6\wp(a_1)^2 - g_2/2$, we easily obtain

$$\wp(p) = \frac{2g_3(C - \tau)^3 - 4(C\eta_1 - \eta_2)^3 - g_2(C\eta_1 - \eta_2)(C - \tau)^2}{(C - \tau)[12(C\eta_1 - \eta_2)^2 - g_2(C - \tau)^2]}.$$

This proves (1.8) with $(\varepsilon_1, \varepsilon_2) = (1, 1)$.

   Case 2. $a_l \in E_\tau[2]$ for some $l \in \{1, 2\}$, then $\{a_1, a_2\} = \{\frac{\omega_i}{2}, \frac{\omega_j}{2}\}$ for some $i \neq j \in \{1, 2, 3\}$. Denote $\eta_3 = \eta_1 + \eta_2$.

   Note from (2.2) that $r\eta_1 + s\eta_2 = \frac{1}{2}(\eta_i + \eta_j)$. Let $\{k\} = \{1, 2, 3\}/\{i, j\}$. By (2.5) and (2.6), it is easy to prove that

$$\ell_1^* y_1(z) = (-1)^{k+m(k)} y_1(z), \quad \ell_2^* y_1(z) = (-1)^k y_1(z).$$

Then $y_1(z)^2$ is even elliptic, i.e. up to a constant,

$$y_1(z)^{-2} = \frac{\wp(z) - \wp(p)}{(\wp(z) - e_i)(\wp(z) - e_j)}.$$

Again a direct computation shows that the residue of $y_1(z)^{-2}$ at $\frac{\omega_i}{2}, \frac{\omega_j}{2}$ are both 0, so

$$y_1(z)^{-2} = c_3 \left(\wp(z - \tfrac{\omega_i}{2}) - e_i \right) + c_4 \left(\wp(z - \tfrac{\omega_j}{2}) - e_j \right),$$

where

$$c_3 = \frac{2(e_i - \wp(p))}{(e_i - e_j)\wp''(\frac{\omega_i}{2})}, \quad c_4 = \frac{2(e_j - \wp(p))}{(e_j - e_i)\wp''(\frac{\omega_j}{2})}.$$

By $\wp''(\frac{\omega_i}{2}) = 6e_i^2 - g_2/2$, $g_2 = 4(e_k^2 - e_i e_j)$ and $e_i + e_j + e_k = 0$, a direct computation gives

(2.11)
$$c_3 + c_4 = \frac{2g_2 - 12e_k^2 - 12e_k\wp(p)}{\wp''(\frac{\omega_i}{2})\wp''(\frac{\omega_j}{2})},$$

$$c_3 e_i + c_4 e_j = \frac{g_2 e_k + 12\wp(p)e_k^2 - 2g_2\wp(p)}{\wp''(\frac{\omega_i}{2})\wp''(\frac{\omega_j}{2})}.$$

Define

$$\chi(z) := \int_0^z \frac{1}{y_1(\xi)^2} d\xi = -c_3\left(\zeta(z - \tfrac{\omega_i}{2}) + e_i z\right) - c_4\left(\zeta(z - \tfrac{\omega_j}{2}) + e_j z\right),$$

then it follows from the Lengrede relation $\tau\eta_1 - 2\pi i = \eta_2$ that

$$\chi_1 = \chi(z+1) - \chi(z) = -\eta_1(c_3 + c_4) - (c_3 e_i + c_4 e_j),$$

$$\chi_2 = \chi(z+\tau) - \chi(z) = -\eta_2(c_3 + c_4) - (c_3 e_i + c_4 e_j)\tau$$
$$= \tau\chi_1 + 2\pi i(c_3 + c_4).$$

Again we define

(2.12)
$$C := \frac{\chi_2}{\chi_1} = \tau + \frac{2\pi i(c_3 + c_4)}{\chi_1},$$

and $y_2(z) := y_1(z)\chi(z)$. Then as above, we have the following conclusions: If $\chi_1 = 0$, then $\chi_2 \neq 0$, $C = \infty$ and so

$$\ell_1^*\begin{pmatrix} \chi_2 y_1(z) \\ y_2(z) \end{pmatrix} = (-1)^{k+m(k)}\begin{pmatrix} \chi_2 y_1(z) \\ y_2(z) \end{pmatrix},$$

$$\ell_2^*\begin{pmatrix} \chi_2 y_1(z) \\ y_2(z) \end{pmatrix} = (-1)^k\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \chi_2 y_1(z) \\ y_2(z) \end{pmatrix},$$

i.e. $\rho(\ell_1) = (-1)^{k+m(k)} I_2$ and $\rho(\ell_2) = (-1)^k\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

If $\chi_1 \neq 0$, then $C \neq \infty$ and so

$$\ell_1^*\begin{pmatrix} \chi_1 y_1(z) \\ y_2(z) \end{pmatrix} = (-1)^{k+m(k)}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \chi_1 y_1(z) \\ y_2(z) \end{pmatrix},$$

$$\ell_2^*\begin{pmatrix} \chi_1 y_1(z) \\ y_2(z) \end{pmatrix} = (-1)^k\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}\begin{pmatrix} \chi_1 y_1(z) \\ y_2(z) \end{pmatrix},$$

i.e. $\rho(\ell_1) = (-1)^{k+m(k)}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\rho(\ell_2) = (-1)^k\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$.

By (2.11)-(2.12) we have

$$\frac{C\eta_1 - \eta_2}{\tau - C} = -\eta_1 - \frac{\chi_1}{c_3 + c_4} = \frac{c_3 e_i + c_4 e_j}{c_3 + c_4} = \frac{(\frac{g_2}{2} - 3e_k^2)\wp(p) - \frac{g_2}{4}e_k}{3e_k\wp(p) + 3e_k^2 - \frac{g_2}{2}},$$

and so

$$\wp(p) = \frac{(\frac{g_2}{2} - 3e_k^2)(C\eta_1 - \eta_2) + \frac{g_2}{4}e_k(C - \tau)}{3e_k(C\eta_1 - \eta_2) + (\frac{g_2}{2} - 3e_k^2)(C - \tau)}.$$

This proves (1.9) with $(\varepsilon_1, \varepsilon_2)$ given by (1.10). $\qquad\square$

3. RELATION WITH PAINLEVÉ VI EQUATION

This section is devoted to the proof of Theorem 1.2. First we recall the relation between GLE (1.2) and Panlevé VI equation. The well-known Painlevé VI equation with four parameters $(\alpha, \beta, \gamma, \delta)$ (PVI$(\alpha, \beta, \gamma, \delta)$) is written as

$$
\begin{aligned}
\frac{d^2\lambda}{dt^2} =& \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t}\right)\frac{d\lambda}{dt} \\
(3.1) \qquad & + \frac{\lambda(\lambda - 1)(\lambda - t)}{t^2(t - 1)^2}\left[\alpha + \beta\frac{t}{\lambda^2} + \gamma\frac{t - 1}{(\lambda - 1)^2} + \delta\frac{t(t - 1)}{(\lambda - t)^2}\right].
\end{aligned}
$$

Due to its connection with many different disciplines in mathematics and physics, PVI (3.1) has been extensively studied in the past several decades. See [5, 6, 7, 9] and references therein.

One of the fundamental properties for PVI (3.1) is the so-called *Painlevé property* which says that any solution $\lambda(t)$ of (3.1) has neither movable branch points nor movable essential singularities; in other words, for any $t_0 \in \mathbb{C}\backslash\{0, 1\}$, either $\lambda(t)$ is holomorphic at t_0 or $\lambda(t)$ has a pole at t_0. Then it is reasonable to lift PVI (3.1) to the covering space $\mathbb{H} = \{\tau \,|\, \text{Im}\,\tau > 0\}$ of $\mathbb{C}\backslash\{0, 1\}$ by the following transformation:

$$(3.2) \qquad t = \frac{e_3(\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}, \quad \lambda(t) = \frac{\wp(p(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}.$$

Consequently, $p(\tau)$ satisfies the following *elliptic form* of PVI (cf. [1, 8]):

$$(3.3) \qquad \frac{d^2p(\tau)}{d\tau^2} = \frac{-1}{4\pi^2}\sum_{k=0}^{3}\alpha_k\wp'\left(p(\tau) + \tfrac{\omega_k}{2}\,\middle|\,\tau\right),$$

with parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$. The Painlevé property of PVI (3.1) implies that function $\wp(p(\tau)|\tau)$ is a single-valued meromorphic function in $\mathbb{H}$.

In this note, we only consider the special case $\alpha_0 = \frac{9}{8}$ and $\alpha_k = \frac{1}{8}$ for $k \geq 1$, i.e.

$$(3.4) \qquad \frac{d^2p(\tau)}{d\tau^2} = \frac{-1}{4\pi^2}\left(\frac{9}{8}\wp'\left(p(\tau)|\tau\right) + \sum_{k=1}^{3}\frac{1}{8}\wp'\left(p(\tau) + \tfrac{\omega_k}{2}\,\middle|\,\tau\right)\right),$$

which is the elliptic form of PVI$(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$.

It is well known (cf. [9, 10, 3]) that any solution of PVI$(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ can be obtained from that of PVI$(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ (i.e. the case studied by Hitchin [5]) by the Okamoto transformation [9]. By this way, the following result gives explicit expressions of some solutions of the elliptic from (3.4).

Theorem A. [10, 3] *For any* $k \in \{0, 1, 2, 3\}$ *and* $C \in \mathbb{C} \cup \{\infty\}$, $p_C^{(k)}(\tau)$ *is a solution of the elliptic form (3.4), where* $\wp(p_C^{(k)}(\tau)|\tau)$ *are given by:*

(3.5)

$$
\wp(p_C^{(0)}(\tau)|\tau) =
$$
$$
\frac{2g_3(\tau)(C-\tau)^3 - 4(C\eta_1(\tau)-\eta_2(\tau))^3 - g_2(\tau)(C\eta_1(\tau)-\eta_2(\tau))(C-\tau)^2}{(C-\tau)[12(C\eta_1(\tau)-\eta_2(\tau))^2 - g_2(\tau)(C-\tau)^2]},
$$

and

(3.6)

$$
\wp(p_C^{(k)}(\tau)|\tau) = \frac{(\frac{g_2(\tau)}{2} - 3e_k(\tau)^2)(C\eta_1(\tau)-\eta_2(\tau)) + \frac{g_2(\tau)}{4}e_k(\tau)(C-\tau)}{3e_k(\tau)(C\eta_1(\tau)-\eta_2(\tau)) + (\frac{g_2(\tau)}{2} - 3e_k(\tau)^2)(C-\tau)}
$$

with $k \in \{1, 2, 3\}$. *Equivalently,*

$$
\lambda_C^{(k)}(t) := \frac{\wp(p_C^{(k)}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}, \ t = \frac{e_3(\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}
$$

is a Riccati type solution of $PVI(\frac{9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$.

We recall [10, 2] that the elliptic form (3.4) governs the isomonodromic deformation of GLE (1.7). More precisely,

Theorem B. [2] $p(\tau)$ *is a solution of (3.4) if and only if there exists* $A(\tau)$ *(Consequently, $B(\tau)$ is determined by $(p(\tau), A(\tau))$ via (1.3) with $(n_0, n_1, n_2, n_3) = (1, 0, 0, 0)$) such that GLE (1.7) with $(p, A, B, \tau) = (p(\tau), A(\tau), B(\tau), \tau)$ preserves the monodromy as τ deforms.*

We call such GLE (1.7) with $(p, A, B) = (p(\tau), A(\tau), B(\tau))$ to be the corresponding GLE of the solution $p(\tau)$.

Remark 3.1. The formulas (3.5)-(3.6) were derived in [10, 2] but the monodromy of the corresponding GLE (1.7) was not thoroughly discussed. In [4], it was proved that for $p_C^{(k)}(\tau)$, the monodromy of the corresponding GLE (1.7) is not completely reducible (i.e. the form of (1.6)) but without proving that the monodromy data is precisely the same C. This assertion, which is strongly suggested by Theorem 1.1, will be proved in following.

Proof of Theorem 1.2. Fix $\tau_0 \in \mathbb{H}$ and $p_0 \notin E_{\tau_0}[2]$.

Suppose $C \in \mathbb{C} \cup \{\infty\}$ satisfies the cubic equation (1.8) with $\tau = \tau_0$ and $p = p_0$. Our goal is to prove the existence of $A_0 \in \mathbb{C}$ such that (1.11) holds for the GLE (1.7) with $(p, A, \tau) = (p_0, A_0, \tau_0)$.

Note that $\wp(p_0|\tau_0) = \wp(p_C^{(0)}(\tau_0)|\tau_0)$, where $p_C^{(0)}(\tau)$ is a solution of the elliptic form (3.4) given in Theorem A, so without loss of generality, we may assume $p_0 = p_C^{(0)}(\tau_0)$ in E_{τ_0}. Then by Theorem B and Remark 3.1, there is $A(\tau)$ such that the corresponding GLE (1.7) with $(p, A, \tau) = (p_C^{(0)}(\tau), A(\tau),$

τ) is monodromy preserving with *not* completely monodromy representation as long as $p_C^{(0)}(\tau) \notin E_\tau[2]$. Fix any $\tau \in \mathbb{H}$ such that $p_C^{(0)}(\tau) \notin E_\tau[2]$. For GLE (1.7) with $(p, A, \tau) = (p_C^{(0)}(\tau), A(\tau), \tau)$, by repeating the proof of Theorem 1.1, there exists a monodromy data $C_1 = C_1(\tau)$ such that either

(3.7)

$$\wp(p_C^{(0)}(\tau)|\tau) = $$
$$\frac{2g_3(\tau)(C_1 - \tau)^3 - 4(C_1\eta_1(\tau) - \eta_2(\tau))^3 - g_2(\tau)(C_1\eta_1(\tau) - \eta_2(\tau))(C_1 - \tau)^2}{(C_1 - \tau)[12(C_1\eta_1(\tau) - \eta_2(\tau))^2 - g_2(\tau)(C_1 - \tau)^2]}$$

or
(3.8)

$$\wp(p_C^{(0)}(\tau)|\tau) = \frac{(\frac{g_2(\tau)}{2} - 3e_k(\tau)^2)(C_1\eta_1(\tau) - \eta_2(\tau)) + \frac{g_2(\tau)}{4}e_k(\tau)(C_1 - \tau)}{3e_k(\tau)(C_1\eta_1(\tau) - \eta_2(\tau)) + (\frac{g_2(\tau)}{2} - 3e_k(\tau)^2)(C_1 - \tau)}$$

with some $k \in \{1, 2, 3\}$ holds for any τ satisfying $p_C^{(0)}(\tau) \notin E_\tau[2]$. Since GLE (1.7) with $(p, A, \tau) = (p_C^{(0)}(\tau), A(\tau), \tau)$ is monodromy preserving, so the monodromy data C_1 is a constant dependent of τ. By comparing the RHS of (3.5) and (3.7)-(3.8) which hold for any τ satisfying $p_C^{(0)}(\tau) \notin E_\tau[2]$ (i.e. *any $\tau \in \mathbb{H}$ except a discrete set*), we easily conclude that (3.7) and $C_1 = C$ hold. Then by letting $A_0 = A(\tau_0)$ and using $p_0 = p_C^{(0)}(\tau_0)$, we see that C is precisely the monodromy data of GLE (1.7) with $(p, A, \tau) = (p_0, A_0, \tau_0)$ and $(\varepsilon_1, \varepsilon_2) = (1, 1)$. This proves (1).

The assertion (2) can be proved similarly. □

The above proof highlights the effectiveness of Painlevé VI equation to solve such kind of Riemman-Hilbert problems. We will apply this idea to consider the general case $n_k \in \mathbb{Z}$ for all k in a future work.

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