

HAMILTONIAN SYSTEM FOR THE ELLIPTIC FORM OF PAINLEVÉ VI EQUATION

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ABSTRACT. In literature, it is known that any solution of Painlevé VI equation governs the isomonodromic deformation of a second order linear Fuchsian ODE on \mathbb{CP}^1 . In this paper, we extend this isomonodromy theory on \mathbb{CP}^1 to the moduli space of elliptic curves by studying the isomonodromic deformation of the generalized Lamé equation. Among other things, we prove that the isomonodromic equation is a new Hamiltonian system, which is equivalent to the elliptic form of Painlevé VI equation for generic parameters. For Painlevé VI equation with some special parameters, the isomonodromy theory of the generalized Lamé equation greatly simplifies the computation of the monodromy group in \mathbb{CP}^1 . This is one of the advantages of the elliptic form.

Résumé: Dans la littérature, il est connu que toute solution de l'équation de Painlevé VI régit la déformation isomonodromique d'une EDO fuchsienne ordinaire sur \mathbb{CP}^1 . Dans cet article, nous étendons cette théorie de isomonodromie sur \mathbb{CP}^1 à l'espace des modules des courbes elliptiques en étudiant la déformation isomonodromique de l'équation de Lamé généralisée. Entre autres choses, nous prouvons que l'équation isomonodromique est un nouveau système hamiltonien, qui est équivalent à la forme elliptique de l'équation de Painlevé VI pour des paramètres génériques. Pour l'équation de Painlevé VI avec certains paramètres spéciaux, la théorie de isomonodromie de l'équation de Lamé généralisée simplifie grandement le calcul du groupe de monodromie dans \mathbb{CP}^1 . Ceci est un des avantages de la forme elliptique.

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1. INTRODUCTION

The isomonodromic deformation plays an universal role to connect many different research areas of mathematics and physics. Our purpose of this paper is to develop an isomonodromy theory for the generalized Lamé equation on the moduli space of elliptic curves.

1.1. Painlevé VI in elliptic form. Historically, the discovery of Painlevé equations was originated from the research on complex ODEs from the middle of 19th century up to early 20th century, led by many famous mathematicians including Painlevé and his school. The aim is to classify those

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nonlinear ODEs whose solutions have the so-called *Painlevé property*. We refer the reader to [1, 5, 6, 9, 10, 11, 12, 13, 15, 17, 19, 20, 21, 22, 23, 25, 26] and references therein for some historic account and the recent developments. Painlevé VI with four free parameters $(\alpha, \beta, \gamma, \delta)$ can be written as

$$(1.1) \quad \frac{d^2\lambda}{dt^2} = \frac{1}{2} \left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t} \right) \left(\frac{d\lambda}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t} \right) \frac{d\lambda}{dt} + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2} \left[\alpha + \beta \frac{t}{\lambda^2} + \gamma \frac{t-1}{(\lambda-1)^2} + \delta \frac{t(t-1)}{(\lambda-t)^2} \right].$$

In the literature, it is well-known that Painlevé VI (1.1) is closely related to the isomonodromic deformation of either a 2×2 linear ODE system of first order (under the non-resonant condition, the isomonodromic equation is known as the Schlesinger system; see [16]) or a second order Fuchsian ODE (under the non-resonant condition, the isomonodromic equation is a Hamiltonian system; see [7, 21]). This associated second order Fuchsian ODE is defined on \mathbb{CP}^1 and has five regular singular points $0, 1, t, \lambda(t)$ and ∞ . Among them, $\lambda(t)$ (as a solution of Painlevé VI) is an apparent singularity (i.e., non-logarithmic). This isomonodromy theory on \mathbb{CP}^1 was first discovered by R. Fuchs [7], and later generalized to the n -dimensional Garnier system by K. Okamoto [21]. We will briefly review this classical isomonodromy theory in Section 4.

Throughout the paper, we use the notations $\omega_0 = 0, \omega_1 = 1, \omega_2 = \tau, \omega_3 = 1 + \tau, \Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$, and $E_\tau \doteq \mathbb{C}/\Lambda_\tau$ where $\tau \in \mathbb{H} = \{\tau \mid \text{Im } \tau > 0\}$ (the upper half plane). We also define $E_\tau[2] \doteq \{\frac{\omega_i}{2} \mid i = 0, 1, 2, 3\}$ to be the set of 2-torsion points in the flat torus E_τ . From the Painlevé property of (1.1), any solution $\lambda(t)$ is a multi-valued meromorphic function in $\mathbb{C} \setminus \{0, 1\}$. To avoid the multi-valueness of $\lambda(t)$, it is better to lift solutions of (1.1) to its universal covering. It is known that the universal covering of $\mathbb{C} \setminus \{0, 1\}$ is \mathbb{H} . Then t and the solution $\lambda(t)$ can be lifted to τ and $p(\tau)$ respectively through the covering map by

$$(1.2) \quad t(\tau) = \frac{e_3(\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)} \quad \text{and} \quad \lambda(t) = \frac{\wp(p(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)},$$

where $\wp(z|\tau)$ is the Weierstrass elliptic function defined by

$$\wp(z|\tau) = \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right],$$

and $e_i = \wp(\frac{\omega_i}{2}|\tau)$, $i = 1, 2, 3$. Then $p(\tau)$ satisfies the following elliptic form

$$(1.3) \quad \frac{d^2 p(\tau)}{d\tau^2} = \frac{-1}{4\pi^2} \sum_{i=0}^3 \alpha_i \wp' \left(p(\tau) + \frac{\omega_i}{2} \mid \tau \right),$$

where $\wp'(z|\tau) = \frac{d}{dz}\wp(z|\tau)$ and

$$(1.4) \quad (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \left(\alpha, -\beta, \gamma, \frac{1}{2} - \delta \right).$$

This elliptic form was already known to Painlevé [24]. For a modern proof, see [1, 20].

The advantage of (1.3) is that $\wp(p(\tau)|\tau)$ is single-valued for $\tau \in \mathbb{H}$, although $p(\tau)$ has a branch point at those τ_0 such that $p(\tau_0) \in E_{\tau_0}[2]$ (see e.g. (1.22) below). We take $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ for an example to explain it. Painlevé VI with this special parameter has connections with some geometric problems; see [4, 11]. In the seminal work [11], N. Hitchin discovered that, for a pair of complex numbers $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, $p(\tau)$ defined by the following formula:

$$(1.5) \quad \wp(p(\tau)|\tau) = \wp(r + s\tau|\tau) + \frac{\wp'(r + s\tau|\tau)}{2(\zeta(r + s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau))},$$

is a solution to (1.3) with $\alpha_k = \frac{1}{8}$ for all k . Here $\zeta(z|\tau) \doteq -\int^z \wp(\xi|\tau)d\xi$ is the Weierstrass zeta function and has the quasi-periods

$$(1.6) \quad \zeta(z + 1|\tau) = \zeta(z|\tau) + \eta_1(\tau) \text{ and } \zeta(z + \tau|\tau) = \zeta(z|\tau) + \eta_2(\tau).$$

By (1.5), Hitchin could construct an Einstein metric with positive curvature if $r \in \mathbb{R}$ and $s \in i\mathbb{R}$, and an Einstein metric with negative curvature if $r \in i\mathbb{R}$ and $s \in \mathbb{R}$. It follows from (1.5) that $\wp(p(\tau)|\tau)$ is a single-valued meromorphic function in \mathbb{H} . However, each τ_0 with $p(\tau_0) \in E_{\tau_0}[2]$ is a branch point of order 2 for $p(\tau)$.

Motivated from Hitchin's solutions, we would like to extend the beautiful formula (1.5) to Painlevé VI with other parameters. But it is not a simple matter because it involves complicated derivatives with respect to the moduli parameter τ . For example, for Hitchin's solutions, it seems not easy to derive (1.3) with $\alpha_k = \frac{1}{8}$ for all k directly from the formula (1.5). We want to provide a systematical way to study this problem. To this goal, the first step is to develop a theory in the moduli space of tori which is analogous to the Fuchs-Okamoto theory on \mathbb{CP}^1 . The purpose of this paper is to derive the Hamiltonian system for the elliptic form (1.3) by developing such an isomonodromy theory in the moduli space of tori. The key issue is *what the linear Fuchsian equation in tori is such that its isomonodromic deformation is related to the elliptic form (1.3)*.

1.2. Generalized Lamé equation. Motivated from our study of the surprising connection of the mean field equation and the elliptic form (1.3) of Painlevé VI in [4], our choice of the Fuchsian equation is the generalized Lamé equation defined by (1.11) below. More precisely, let us consider the following mean field equation

$$(1.7) \quad \Delta u + e^u = 8\pi \sum_{i=0}^3 n_i \delta_{\frac{\omega_i}{2}} + 4\pi (\delta_p + \delta_{-p}) \text{ in } E_\tau,$$

where $n_k > -1$, $\delta_{\pm p}$ and $\delta_{\frac{\omega_k}{2}}$ are the Dirac measure at $\pm p$ and $\frac{\omega_k}{2}$ respectively. By the Liouville theorem, any solution u to equation (1.7) could be written into the following form:

$$(1.8) \quad u(z) = \log \frac{8|f'(z)|^2}{(1 + |f(z)|^2)^2},$$

where $f(z)$ is a meromorphic function in \mathbb{C} . Conventionally $f(z)$ is called a developing map of u . We could see below that there associates a 2nd order complex ODE reducing from the nonlinear PDE (1.7). Indeed, it follows from (1.7) that outside $E_\tau [2] \cup \{\pm p\}$,

$$\left(u_{zz} - \frac{1}{2}u_z^2\right)_{\bar{z}} = (u_{z\bar{z}})_z - u_z u_{z\bar{z}} = \left(-\frac{1}{4}e^u\right)_z + \frac{1}{4}e^u u_z = 0.$$

So $u_{zz} - \frac{1}{2}u_z^2$ is an elliptic function on the torus E_τ with singularities at $E_\tau [2] \cup \{\pm p\}$. Since the behavior of u is fixed by the RHS of (1.7), for example, $u(z) = 2 \log |z - p| + O(1)$ near p , we could compute explicitly the dominate term of $u_{zz} - \frac{1}{2}u_z^2$ near each singular point. Let us further assume that $u(z)$ is *even*, i.e., $u(z) = u(-z)$. Then we have

$$(1.9) \quad \begin{aligned} & u_{zz} - \frac{1}{2}u_z^2 \\ &= -2 \left[\sum_{k=0}^3 n_k (n_k + 1) \wp \left(z + \frac{\omega_k}{2}\right) + \frac{3}{4} (\wp(z+p) + \wp(z-p)) \right] \\ &\doteq -2I(z), \end{aligned}$$

where A, B are two (unknown) complex numbers.

On the other hand, we could deduce from (1.8) that the Schwarzian derivative $\{f; z\}$ of f can be expressed by

$$(1.10) \quad \{f; z\} \doteq \left(\frac{f''}{f'}\right)' - \frac{1}{2} \left(\frac{f''}{f'}\right)^2 = u_{zz} - \frac{1}{2}u_z^2 = -2I(z).$$

From (1.10), we connect the developing maps of an even solution u to (1.7) with the following generalized Lamé equation

$$(1.11) \quad y''(z) = I(z)y(z) \text{ in } E_\tau,$$

where the potential $I(z)$ is given by (1.9). In the classical literature, the 2nd order ODE

$$(1.12) \quad y''(z) = (n(n+1)\wp(z) + B)y(z) \text{ in } E_\tau$$

is called the Lamé equation, and has been extensively studied since the 19th century, particularly for the case $n \in \mathbb{Z}/2$. See [3, 10, 25, 31] and the references therein. In this paper, we will prove that (1.12) appears as a limiting equation of (1.11) under some circumstances (see Theorem 1.5).

From (1.8) and (1.10), any two developing maps f_i , $i = 1, 2$ of the same solution u must satisfy

$$f_2(z) = \alpha \cdot f_1(z) \doteq \frac{af_1(z) + b}{cf_1(z) + d}$$

for some $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSU(2)$. From here, we could define a projective monodromy representation $\rho : \pi_1(E_\tau \setminus (E_\tau[2] \cup \{\pm p\}), q_0) \rightarrow PSU(2)$, where $q_0 \notin E_\tau[2] \cup \{\pm p\}$ is a base point. Indeed, any developing map f might be multi-valued. For any loop $\ell \in \pi_1(E_\tau \setminus (E_\tau[2] \cup \{\pm p\}), q_0)$, $\ell^* f$ denotes the analytic continuation of f along ℓ . Since $\ell^* f$ is also a developing map of the same u , there exists $\rho(\ell) \in PSU(2)$ such that $\ell^* f = \rho(\ell) \cdot f$. Thus, the map $\ell \mapsto \rho(\ell)$ defines a group homomorphism from $\pi_1(E_\tau \setminus (E_\tau[2] \cup \{\pm p\}), q_0)$ to $PSU(2)$. When $n_k \in \mathbb{N} \cup \{0\}$ for all k , the developing map is a single-valued meromorphic function defined in \mathbb{C} , and $\{\pm p\}$ would become apparent singularities. Thus, the projective monodromy representation would be reduced to a homomorphism from $\pi_1(E_\tau, q_0)$ to $PSU(2)$. This could greatly simplify the computation of the monodromy group. The deep connection of the mean field equation (1.7) and the elliptic form (1.3) has been discussed in detail in [4]. This is our motivation to study the isomonodromic deformation of the generalized Lamé equation (1.11) in this paper.

1.3. Isomonodromic deformation and Hamiltonian system. We are now in a position to state our main results. Recall the generalized Lamé equation (1.11):

$$(1.13) \quad y'' = \left[\begin{array}{l} \sum_{k=0}^3 n_k (n_k + 1) \wp(z + \frac{\omega_k}{2}) + \frac{3}{4} (\wp(z + p) + \wp(z - p)) \\ + A(\zeta(z + p) - \zeta(z - p)) + B \end{array} \right] y.$$

Equation (1.13) has no solutions with logarithmic singularity at $\frac{\omega_k}{2}$ unless $n_k \in \frac{1}{2} + \mathbb{Z}$. Therefore, we need to assume the non-resonant condition: $n_k \notin \frac{1}{2} + \mathbb{Z}$ for all k . Observe that the exponent difference of (1.13) at $\pm p$ is 2. Here the singular points $\pm p$ are always assumed to be *apparent* (i.e., non-logarithmic). Under this assumption, the coefficients A and B together satisfy (1.17) below. Our first main result is following.

Theorem 1.1. *Let*

$$(1.14) \quad \alpha_k = \frac{1}{2} \left(n_k + \frac{1}{2} \right)^2 \quad \text{with } n_k \notin \frac{1}{2} + \mathbb{Z}, \quad k = 0, 1, 2, 3.$$

Then $p(\tau)$ is a solution of the elliptic form (1.3) if and only if there exist $A(\tau)$ and $B(\tau)$ such that the generalized Lamé equation (1.13) with apparent singularities at $\pm p(\tau)$ preserves the monodromy while τ is deforming.

Our method to prove Theorem 1.1 consists of two steps: the first is to derive the isomonodromic equation, a Hamiltonian system, in the moduli space of tori for (1.13) under the non-resonant condition $n_k \notin \frac{1}{2} + \mathbb{Z}$. The

second is to prove that this isomonodromic equation (the Hamiltonian system) is equivalent to the elliptic form (1.3). To describe the isomonodromic equation for (1.13), we let the Hamiltonian $K(p, A, \tau)$ be defined by

$$(1.15) \quad K(p, A, \tau) = \frac{-i}{4\pi} (B + 2p\eta_1(\tau) A) \\ = \frac{-i}{4\pi} \left(A^2 + (-\zeta(2p|\tau) + 2p\eta_1(\tau)) A - \frac{3}{4}\wp(2p|\tau) \right) \\ - \sum_{k=0}^3 n_k(n_k+1)\wp\left(p + \frac{\omega_k}{2}|\tau\right).$$

Consider the Hamiltonian system

$$(1.16) \quad \begin{cases} \frac{dp(\tau)}{d\tau} = \frac{\partial K(p, A, \tau)}{\partial A} = \frac{-i}{4\pi} (2A - \zeta(2p|\tau) + 2p\eta_1(\tau)) \\ \frac{dA(\tau)}{d\tau} = -\frac{\partial K(p, A, \tau)}{\partial p} = \frac{i}{4\pi} \left((2\wp(2p|\tau) + 2\eta_1(\tau)) A - \frac{3}{2}\wp'(2p|\tau) \right) \\ - \sum_{k=0}^3 n_k(n_k+1)\wp'\left(p + \frac{\omega_k}{2}|\tau\right) \end{cases}.$$

Then our first step leads to the following result:

Theorem 1.2 (=Theorem 2.3). *Let $n_k \notin \frac{1}{2} + \mathbb{Z}$, $k = 0, 1, 2, 3$. Then $(p(\tau), A(\tau))$ satisfies the Hamiltonian system (1.16) if and only if equation (1.13) with $(p(\tau), A(\tau), B(\tau))$ preserves the monodromy, where*

$$(1.17) \quad B = A^2 - \zeta(2p) A - \frac{3}{4}\wp(2p) - \sum_{k=0}^3 n_k(n_k+1)\wp\left(p + \frac{\omega_k}{2}\right).$$

And the second step is to prove

Theorem 1.3 (=Theorem 2.5). *The elliptic form (1.3) is equivalent to the Hamiltonian system (1.16), where $\alpha_k = \frac{1}{2}(n_k + \frac{1}{2})^2$, $k = 0, 1, 2, 3$.*

Clearly Theorem 1.1 follows from Theorems 1.2 and 1.3 directly.

Remark 1.1. *In [20], Manin rewrote the elliptic form (1.3) into an obvious time-dependent Hamiltonian system:*

$$(1.18) \quad \frac{dp(\tau)}{d\tau} = \frac{\partial H}{\partial q}, \quad \frac{dq(\tau)}{d\tau} = -\frac{\partial H}{\partial p},$$

where

$$H = H(\tau, p, q) \doteq \frac{q^2}{2} + \frac{1}{4\pi^2} \sum_{i=0}^3 \alpha_i \wp\left(p(\tau) + \frac{\omega_i}{2}|\tau\right).$$

However, it is not clear whether the Hamiltonian system (1.18) governs isomonodromic deformations of any Fuchsian equations in E_τ or not. Different from (1.18), our Hamiltonian system (1.16) governs isomonodromic deformations of the generalized Lamé equation for generic parameters.

Both Theorems 1.2 and 1.3 are proved in Section 2. It seems that the generalized Lamé equation (1.13) looks simpler than the corresponding Fuchsian ODE on \mathbb{CP}^1 , and it is the same for the Hamiltonian system (1.16), compared to the corresponding one on \mathbb{CP}^1 . From the second equation of (1.16), $A(\tau)$ can be integrated so that we have the following theorem:

Theorem 1.4. *Suppose $(p(\tau), A(\tau))$ satisfies the Hamiltonian system (1.16). Define*

$$(1.19) \quad F(\tau) \doteq A(\tau) + \frac{1}{2}(\zeta(2p(\tau)|\tau) - 2\zeta(p(\tau)|\tau)).$$

Then

$$(1.20) \quad F(\tau) = \theta'_1(\tau)^{\frac{2}{3}} \exp \left\{ \frac{i}{2\pi} \int^\tau (2\wp(2p(\hat{\tau})|\hat{\tau}) - \wp(p(\hat{\tau})|\hat{\tau})) d\hat{\tau} \right\} \times \\ \left(\int^\tau \frac{-\frac{i}{4\pi}\theta'_1(\hat{\tau})^{-\frac{2}{3}} \left(\sum_{k=0}^3 n_k(n_k+1)\wp'(p(\hat{\tau}) + \frac{\omega_k}{2}|\hat{\tau}) \right)}{\exp \left\{ \frac{i}{2\pi} \int^{\hat{\tau}} (2\wp(2p(\tau')|\tau') - \wp(p(\tau')|\tau')) d\tau' \right\}} d\hat{\tau} + c_1 \right)$$

for some constant $c_1 \in \mathbb{C}$, where $\theta'_1(\tau) = \frac{d\vartheta_1(z;\tau)}{dz}|_{z=0}$ and $\vartheta_1(z;\tau)$ is the odd theta function defined in (2.30). In particular, for $n_k = 0, \forall k$, we have

$$(1.21) \quad F(\tau) = c\theta'_1(\tau)^{\frac{2}{3}} \exp \left\{ \frac{i}{2\pi} \int^\tau (2\wp(2p(\hat{\tau})|\hat{\tau}) - \wp(p(\hat{\tau})|\hat{\tau})) d\hat{\tau} \right\}$$

for some constant $c \in \mathbb{C} \setminus \{0\}$.

Remark 1.2. *Let $\eta(\tau)$ be the Dedekind eta function: $\eta(\tau) \doteq q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$, where $q = e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$. Then $\theta'_1(\tau) = 2\pi\eta^3(\tau)$.*

The elliptic form (1.3) and our results above could be applied to understand the phenomena of collapsing two singular points $\pm p(\tau)$ to $\frac{\omega_k}{2}$ in the generalized Lamé equation (1.13). In general, when $p(\tau) \rightarrow \frac{\omega_k}{2}$ as $\tau \rightarrow \tau_0$, the generalized Lamé equation might not be well-defined. However, when $p(\tau)$ is a solution of the elliptic form (1.3), by using the behavior of $p(\tau)$ near τ_0 , the following result shows that the generalized Lamé equation will converge to the classical Lamé equation (1.12).

Observe that if $p(\tau)$ is a solution of the elliptic form (1.3), then $p(\tau) - \frac{\omega_k}{2}$ is also a solution of (1.3) (maybe with different parameters). Therefore, we only need to study the case $p(\tau) \rightarrow 0$. More precisely, we have:

Theorem 1.5 (=Theorem 3.1). *Suppose that $n_k \notin \frac{1}{2} + \mathbb{Z}$, $k = 0, 1, 2, 3$, and (1.17) holds. Let $(p(\tau), A(\tau))$ be a solution of the Hamiltonian system (1.16) such that $p(\tau_0) = 0$ for some $\tau_0 \in \mathbb{H}$. Then*

$$(1.22) \quad p(\tau) = c_0(\tau - \tau_0)^{\frac{1}{2}}(1 + \tilde{h}(\tau - \tau_0) + O((\tau - \tau_0)^2)) \text{ as } \tau \rightarrow \tau_0,$$

where $c_0^2 = \pm i \frac{n_0 + \frac{1}{2}}{\pi}$ and $\tilde{h} \in \mathbb{C}$ is some constant. Moreover, the generalized Lamé equation (1.13) as $\tau \rightarrow \tau_0$ converges to

$$(1.23) \quad y'' = \left[\sum_{j=1}^3 n_j(n_j+1)\wp\left(z + \frac{\omega_j}{2}\right) + m(m+1)\wp(z) + B_0 \right] y \text{ in } E_{\tau_0}$$

where

$$m = \begin{cases} n_0 + 1 & \text{if } c_0^2 = i \frac{n_0 + \frac{1}{2}}{\pi}, \\ n_0 - 1 & \text{if } c_0^2 = -i \frac{n_0 + \frac{1}{2}}{\pi}, \end{cases}$$

$$B_0 = 2\pi ic_0^2 \left(4\pi i \tilde{h} - \eta_1(\tau_0) \right) - \sum_{j=1}^3 n_j(n_j + 1)e_j(\tau_0).$$

Remark 1.3. (1) Equation (1.23) is an elliptic form of Heun equation, and the potential of equation (1.23) as the Schrödinger operator is also called the Treibich-Verdier potential, which has the finite gap property when $n_k \in \mathbb{N} \cup \{0\}$ for all k . See [27, 30].

(2) In Section 4, we will see that the generalized Lamé equation (1.13) is an elliptic form of the Fuchsian equation (4.1) on \mathbb{CP}^1 ; see also [28], where for the case $n_k \in \mathbb{N} \cup \{0\}$ for all k , Takemura obtained the connection between the generalized Lamé equation (1.13) and Painlevé VI equation by means of equation (4.1). Equation (4.1) has five regular singular points (see (4.2)) and can be seen as a one-parameter deformation of Heun equation. Due to this reason, equation (4.1) is also called Heun1 equation in [18]. The collapse of two singular points of Heun1 equation was studied in [29], where μ is fixed (i.e. independent of λ) during the collapsing process $\lambda \rightarrow$ one of $\{0, 1, t, \infty\}$ (see (4.2)-(4.6) for (t, λ, μ)). Here our study of $p(\tau) \rightarrow \frac{\omega_k}{2}$ is different from those in [29] because we require that $(p(\tau), A(\tau))$ always satisfies the Hamiltonian system (1.16) during the collapsing process.

Theorem 1.5 will be proven in Section 3. In Section 4, we will give another application of our isomonodromy theory (see Corollary 4.1). More precisely, we will establish a one to one correspondence between the generalized Lamé equation and the Fuchsian equation (4.1) on \mathbb{CP}^1 . Furthermore, we will prove that if one of them is monodromy preserving then so is the other one. We remark that all the results above have important applications in our coming paper [4]. For example, Theorem 1.5 can be used to study the converge of even solutions of the mean field equation (1.7) as $p(\tau) \rightarrow 0$ when $\tau \rightarrow \tau_0$.

We conclude this section by comparing our result Theorem 1.1 with the paper [17] by Kawai. Define the Fuchsian equation in E_τ by

$$(1.24) \quad y''(z) = q(z)y(z),$$

where

$$(1.25) \quad q(z) = L + \sum_{i=0}^m \left[H_i \zeta(z - t_i|\tau) + \frac{1}{4} (\theta_i^2 - 1) \wp(z - t_i|\tau) \right] \\ + \sum_{\alpha=0}^m \left[-\mu_\alpha \zeta(z - b_\alpha|\tau) + \frac{3}{4} \wp(z - b_\alpha|\tau) \right]$$

with

$$(1.26) \quad \sum_{i=0}^m H_i - \sum_{\alpha=0}^m \mu_\alpha = 0.$$

Here $L, H_i, t_i, \theta_i, \mu_\alpha, b_\alpha$ are complex parameters with $t_0 = 0$. The isomonodromic deformation of equation (1.24) was first treated by Okamoto [23] without varying the underlying elliptic curves and then generalized by Iwasaki [14] to the case of higher genus. Let \mathcal{R} be the space of conjugacy classes of the monodromy representation of $\pi_1((E_\tau \setminus S), q_0)$, where S denotes the set of singular points. Then it was known that \mathcal{R} is a complex manifold. It was proved by Iwasaki [14] that there exists a natural symplectic structure Ω on the space \mathcal{R} . In [17], Kawai considered the same Fuchsian equation (1.24) but allowed the underlying elliptic curves to vary as well. By using the pull-back principle to the symplectic 2-form Ω , Kawai studied isomonodromic deformations for equation (1.24) which are described as a completely integrable Hamiltonian system: for $1 \leq i \leq m$,

$$(1.27) \quad \frac{\partial b_\alpha}{\partial t_i} = \sum_{i=1}^m \frac{\partial H_i}{\partial \mu_\alpha}, \quad \frac{\partial b_\alpha}{\partial \tau} = \frac{\partial \mathcal{H}}{\partial \mu_\alpha}, \quad \frac{\partial \mu_\alpha}{\partial t_i} = - \sum_{i=1}^m \frac{\partial H_i}{\partial b_\alpha}, \quad \frac{\partial \mu_\alpha}{\partial \tau} = - \frac{\partial \mathcal{H}}{\partial b_\alpha},$$

where

$$\mathcal{H} = \frac{1}{2\pi i} \left[L + \eta_1(\tau) \left(\sum_{\alpha=0}^m b_\alpha \mu_\alpha - \sum_{i=1}^m t_i H_i \right) \right].$$

Now considering the simplest case $m = 0$ and by using (1.26) and $t_0 = 0$, the potential $q(z)$ takes the simple form (the subscript 0 is dropped for simplicity)

$$(1.28) \quad q(z) = L + \mu \zeta(z|\tau) + \frac{1}{4}(\theta^2 - 1) \wp(z|\tau) - \mu \zeta(z - b|\tau) + \frac{3}{4} \wp(z - b|\tau).$$

Consequently, the Hamiltonian system (1.27) is reduced to

$$(1.29) \quad \begin{cases} \frac{db}{d\tau} = \frac{-i}{2\pi} [2\mu - \zeta(b|\tau) + b\eta_1], \\ \frac{d\mu}{d\tau} = \frac{i}{2\pi} [\mu \wp(b|\tau) + \mu \eta_1 - \frac{1}{4}(\theta^2 - 1) \wp'(b|\tau)]. \end{cases}$$

Furthermore, the Hamiltonian system (1.29) is equivalent to

$$\frac{d^2}{d\tau^2} \left(\frac{b}{2} \right) = - \frac{1}{4\pi^2} \sum_{k=0}^3 \frac{\theta^2}{32} \wp' \left(\frac{b}{2} + \frac{\omega_k}{2} | \tau \right),$$

which implies that $\frac{b}{2}$ satisfies the elliptic form (1.3) with $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\frac{\theta^2}{32}, \frac{\theta^2}{32}, \frac{\theta^2}{32}, \frac{\theta^2}{32})$; see [17, Theorem 3]. It is clear that our potential $I(z)$ is *different from* (1.28) except for $n_k = 0, k = 0, 1, 2, 3$ in $I(z)$ and $\theta = \pm 2$ in (1.28). Notice that the linear ODE (1.24) with (1.28) only has the apparent singularity at b . Thus it seems that the monodromy representation for (1.24) could not be reduced to $\pi_1(E_\tau)$ when $\theta \neq \pm 2$. However, when $n_k \in \mathbb{N} \cup \{0\}$ for all k , the monodromy representation for (1.13) could be simplified. We remark it is an advantage when we study the elliptic form (1.3) with $\alpha_k = \frac{1}{2} (n_k + \frac{1}{2})^2, k = 0, 1, 2, 3$. From Kawai's result in [17] and ours, it can be seen that the elliptic form (1.3) governs isomonodromic deformations of different linear ODEs (e.g. (1.24) with (1.28) and (1.13)). Therefore, it

is important to choose a suitable linear ODE when generic parameters are considered.

2. PAINLEVÉ VI AND HAMILTONIAN SYSTEM ON THE MODULI SPACE

In this section, we want to develop an isomonodromy theory on the moduli space of elliptic curves. For this purpose, there are two fundamental issues needed to be discussed: (i) to derive the Hamiltonian system for the isomonodromic deformation of the generalized Lamé equation (1.13) with $n_i \notin \frac{1}{2} + \mathbb{Z}$, $i = 0, 1, 2, 3$; (ii) to prove the equivalence between the Hamiltonian system and the elliptic form (1.3). We remark that the (ii) part holds true without any condition. Recall the generalized Lamé equation defined by

$$(2.1) \quad y'' = I(z; \tau) y,$$

where

$$(2.2) \quad \begin{aligned} I(z; \tau) = & \sum_{i=0}^3 n_i (n_i + 1) \wp \left(z + \frac{\omega_i}{2} | \tau \right) + \frac{3}{4} (\wp(z + p | \tau) + \wp(z - p | \tau)) \\ & + A (\zeta(z + p | \tau) - \zeta(z - p | \tau)) + B, \end{aligned}$$

and $p(\tau) \notin E_\tau[2]$. By replacing n_i by $-n_i - 1$ if necessary, we always assume $n_i \geq -\frac{1}{2}$ for all i . Remark that, since we assume $n_i \notin \frac{1}{2} + \mathbb{Z}$, the exponent difference of (2.1) at $\frac{\omega_i}{2}$ is $2n_i + 1 \notin 2\mathbb{Z}$, implying that (2.1) has no logarithmic singularity at $\frac{\omega_i}{2}$, $0 \leq i \leq 3$.

For equation (2.1), the necessary and sufficient condition for apparent singularity at $\pm p$ is given by

Lemma 2.1. *$\pm p$ are apparent singularities of (2.1) iff A and B satisfy*

$$(2.3) \quad B = A^2 - \zeta(2p)A - \frac{3}{4}\wp(2p) - \sum_{i=0}^3 n_i (n_i + 1) \wp \left(p + \frac{\omega_i}{2} \right).$$

Proof. It suffices to prove this lemma for the point p . Let y_i , $i = 1, 2$, be two linearly independent solutions to (2.1). Define $f \doteq \frac{y_1}{y_2}$ as a ratio of two independent solutions and $v \doteq \log f'$. Then

$$(2.4) \quad \{f; z\} = v'' - \frac{1}{2}(v')^2 = -2I(z).$$

It is obvious that (2.1) has no solutions with logarithmic singularity at p iff $f(z)$ has no logarithmic singularity at p . First we prove the necessary part. Without loss of generality, we may assume $f(z)$ is holomorphic at p . The local expansion of f at p is:

$$(2.5) \quad \begin{aligned} f(z) &= c_0 + c_2(z-p)^2 + \cdots, \\ v(z) &= \log f'(z) = \log 2c_2 + \log(z-p) + \sum_{j \geq 1} d_j (z-p)^j, \end{aligned}$$

$$(2.6) \quad v'(z) = \frac{1}{z-p} + \sum_{j \geq 0} \tilde{e}_j (z-p)^j,$$

$$v''(z) = \frac{-1}{(z-p)^2} + \sum_{j \geq 0} (j+1) \tilde{e}_{j+1} (z-p)^j,$$

where $\tilde{e}_j = (j+1) d_{j+1}$. Thus,

$$(2.7) \quad v'' - \frac{1}{2} (v')^2 = \frac{-1}{(z-p)^2} + \sum_{j \geq 0} (j+1) \tilde{e}_{j+1} (z-p)^j$$

$$- \frac{1}{2} \left[\frac{1}{z-p} + \sum_{j \geq 0} \tilde{e}_j (z-p)^j \right]^2.$$

Recalling $I(z)$ in (2.2), we compare both sides of (2.4). The $(z-p)^{-2}$ terms match automatically. For the $(z-p)^{-1}$ term, we get

$$(2.8) \quad -\tilde{e}_0 = 2A.$$

For the $(z-p)^0$, i.e. the constant term, we have

$$(2.9) \quad \tilde{e}_1 - \frac{2}{2} \tilde{e}_1 - \frac{1}{2} \tilde{e}_0^2$$

$$= -2 \sum_{i=0}^3 n_i (n_i + 1) \wp \left(p + \frac{\omega_i}{2} \right) - \frac{3}{2} \wp(2p) - 2A\zeta(2p) - 2B.$$

Then (2.3) follows from (2.8) and (2.9) immediately.

For the sufficient part, if (2.3) holds, then \tilde{e}_0 is given by (2.8). By any choice of \tilde{e}_1 and comparing (2.4) and (2.7), \tilde{e}_j is determined for all $j \geq 2$. Then it follows from (2.5)-(2.6) that $f(z)$ is holomorphic at p . Since its Schwarzian derivative satisfies (2.4), f is a ratio of two linearly independent solutions of (2.1). This implies that (2.1) has no solutions with logarithmic singularity at p , namely p is an apparent singularity. \square

2.1. Isomonodromic equation and Hamiltonian system. The 2nd order generalized Lamé equation (2.1) can be written into a 1st order linear system

$$(2.10) \quad \frac{d}{dz} Y = Q(z; \tau) Y \quad \text{in } E_\tau,$$

where

$$(2.11) \quad Q(z; \tau) = \begin{pmatrix} 0 & 1 \\ I(z; \tau) & 0 \end{pmatrix}.$$

The isomonodromic deformation of the generalized Lamé equation (2.1) is equivalent to the isomonodromic deformation of the linear system (2.10). Let $y_1(z; \tau)$ and $y_2(z; \tau)$ be two linearly independent solutions of (2.1), then $Y(z; \tau) = \begin{pmatrix} y_1(z; \tau) & y_2(z; \tau) \\ y_1'(z; \tau) & y_2'(z; \tau) \end{pmatrix}$ is a fundamental system of solutions to (2.10). In general, $Y(z; \tau)$ is multi-valued with respect to z and for each

$\tau \in \mathbb{H}$, $Y(z; \tau)$ might have branch points at $S \doteq \{\pm p, \frac{\omega_k}{2} \mid k = 0, 1, 2, 3\}$. The fundamental solution $Y(z; \tau)$ is called M -invariant (M stands for monodromy) if there is some $q_0 \in E_\tau \setminus S$ and for any loop $\ell \in \pi_1(E_\tau \setminus S, q_0)$, there exists $\rho(\ell) \in SL(2, \mathbb{C})$ independent of τ such that

$$\ell^* Y(z; \tau) = Y(z; \tau) \rho(\ell)$$

holds for z near the base point q_0 . Here $\ell^* Y(z; \tau)$ denotes the analytic continuation of $Y(z; \tau)$ along ℓ . Let $\gamma_k \in \pi_1(E_\tau \setminus S, q_0)$, $k = 0, 1, 2, 3, \pm$, be simple loops which encircle the singularities $\frac{\omega_k}{2}$, $k = 0, 1, 2, 3$ and $\pm p$ once respectively, and $\ell_j \in \pi_1(E_\tau \setminus S, q_0)$, $j = 1, 2$, be two fundamental cycles of E_τ such that its lifting in \mathbb{C} is a straight line connecting q_0 and $q_0 + \omega_j$. We also require these lines do not pass any singularities. Of course, all the paths do not intersect with each other except at q_0 . We note that when τ varies in a neighborhood of some τ_0 , γ_k and ℓ_1 can be chosen independent of τ .

Clearly the monodromy group with respect to $Y(z; \tau)$ is generated by $\{\rho(\ell_j), \rho(\gamma_k) \mid j = 1, 2 \text{ and } k = 0, 1, 2, 3, \pm\}$. Thus, $Y(z; \tau)$ is M -invariant if and only if the matrices $\rho(\ell_j), \rho(\gamma_k)$ are independent of τ . Notice that $I(\cdot; \tau)$ is an elliptic function, so we can also treat (2.10) as a equation defined in \mathbb{C} , i.e.,

$$(2.12) \quad \frac{d}{dz} Y = Q(z; \tau) Y \quad \text{in } \mathbb{C}.$$

Furthermore, we can identify solutions of (2.10) and (2.12) in an obvious way. For example, after analytic continuation, any solution $Y(\cdot; \tau)$ of (2.10) can be extended to be a solution of (2.12) as a multi-valued matrix function defined in \mathbb{C} (still denote it by $Y(\cdot; \tau)$). In the sequel, we always identify solutions of (2.10) and (2.12). Then we have the following theorem:

Theorem 2.1. *System (2.10) is monodromy preserving as τ deforms if and only if there exists a single-valued matrix function $\Omega(z; \tau)$ defined in $\mathbb{C} \times \mathbb{H}$ satisfying*

$$(2.13) \quad \begin{cases} \Omega(z+1; \tau) = \Omega(z; \tau) \\ \Omega(z+\tau; \tau) = \Omega(z; \tau) - Q(z; \tau), \end{cases}$$

such that the following Pfaffian system

$$(2.14) \quad \begin{cases} \frac{\partial}{\partial z} Y(z; \tau) = Q(z; \tau) Y(z; \tau) \\ \frac{\partial}{\partial \tau} Y(z; \tau) = \Omega(z; \tau) Y(z; \tau) \end{cases} \quad \text{in } \mathbb{C} \times \mathbb{H}$$

is completely integrable.

Remark 2.1. *The classical isomonodromy theory in \mathbb{C} (see e.g. [15, Proposition 3.1.5]) says that system (2.12) is monodromy preserving if and only if there exists a single-valued matrix function $\Omega(z; \tau)$ defined in $\mathbb{C} \times \mathbb{H}$ such that (2.14) is completely integrable. Theorem 2.1 is the counterpart of this classical theory in the torus E_τ . The property (2.13) comes from the preserving of monodromy matrices $\rho(\ell_j)$, $j = 1, 2$ during the deformation (see*

from the proof of Theorem 2.1 below). Notice that $\rho(\ell_j)$ can be considered as connection matrices along the straight line ℓ_j connecting q_0 and $q_0 + \omega_j$ for system (2.12).

Notice that system (2.14) is completely integrable if and only if

$$(2.15) \quad \frac{\partial}{\partial \tau} Q(z; \tau) = \frac{\partial}{\partial z} \Omega(z; \tau) + [\Omega(z; \tau), Q(z; \tau)], \text{ and}$$

$$(2.16) \quad d(\Omega(z; \tau) d\tau) = [\Omega(z; \tau) d\tau] \wedge [\Omega(z; \tau) d\tau],$$

where d denotes the exterior differentiation with respect to τ in (2.16). See Lemma 3.14 in [15] for the proof. Clearly (2.16) holds automatically since there is only one deformation parameter. We need the following lemma to prove Theorem 2.1.

Lemma 2.2. *Let $Y(z; \tau)$ be an M -invariant fundamental solution of system (2.10) and define a 2×2 matrix-valued function $\Omega(z; \tau)$ in E_τ by*

$$(2.17) \quad \Omega(z; \tau) = \frac{\partial}{\partial \tau} Y \cdot Y^{-1}.$$

Then $\Omega(z; \tau)$ can be extended to be a globally defined matrix-valued function in $\mathbb{C} \times \mathbb{H}$ by analytic continuation (still denote it by $\Omega(z; \tau)$). In particular, (2.17) holds in $\mathbb{C} \times \mathbb{H}$ by considering $Y(z; \tau)$ as a solution of system (2.12).

Proof. The proof is the same as that in the classical isomonodromy theory in \mathbb{C} . Indeed, since $Y(z; \tau)$ is M -invariant, we have

$$(2.18) \quad \begin{aligned} \gamma_k^* \Omega(z; \tau) &= \gamma_k^* \left(\frac{\partial}{\partial \tau} Y \cdot Y^{-1} \right) = \frac{\partial}{\partial \tau} \gamma_k^* Y \cdot \gamma_k^* Y^{-1} \\ &= \frac{\partial}{\partial \tau} (Y \rho(\gamma_k)) \cdot \rho(\gamma_k)^{-1} Y^{-1} \\ &= \frac{\partial}{\partial \tau} Y \cdot Y^{-1} = \Omega(z; \tau) \end{aligned}$$

for $k = 0, 1, 2, 3, \pm$, namely $\Omega(\cdot; \tau)$ is invariant under the analytic continuation along γ_k . Thus, $\Omega(\cdot; \tau)$ is single-valued in any fundamental domain of E_τ for each τ . Then for each $\tau \in \mathbb{H}$, we could extend $\Omega(z; \tau)$ to be a globally defined matrix-valued function in \mathbb{C} by analytic continuation. \square

From now on, we consider equation (2.10) defined in \mathbb{C} , i.e., (2.12). The analytic continuation along any curve in \mathbb{C} always keep the relation (2.17) between $Y(z; \tau)$ and $\Omega(z; \tau)$.

Proof of Theorem 2.1. First we prove the necessary part. Let $Y(z; \tau)$ be an M -invariant fundamental solution of system (2.10) and define $\Omega(z; \tau)$ by $Y(z; \tau)$. By Lemma 2.2, $\Omega(z; \tau)$ is a single-valued matrix function in $\mathbb{C} \times \mathbb{H}$ and $Y(z; \tau)$ is a solution of (2.14), which implies (2.15). Hence the Pfaffian system (2.14) is completely integrable.

It suffices to prove that $\Omega(z; \tau)$ satisfies (2.13). Note that $\Omega(z; \tau)$ is single-valued in $\mathbb{C} \times \mathbb{H}$. Therefore, to prove (2.13), we only need to prove its

validity in a small neighborhood $U_{q_0} \times V_{\tau_0}$ of some (q_0, τ_0) , where q_0 is the base point. By considering $Y(z; \tau)$ as a solution of system (2.12), we see from Remark 2.1 and Lemma 2.2 that, for $(z, \tau) \in U_{q_0} \times V_{\tau_0}$,

$$(2.19) \quad Y(z + \omega_i; \tau) = Y(z; \tau) \rho(\ell_i),$$

$$(2.20) \quad \Omega(z; \tau) = \frac{\partial}{\partial \tau} Y(z; \tau) \cdot Y(z; \tau)^{-1},$$

$$(2.21) \quad \Omega(z + \omega_i; \tau) = \frac{\partial}{\partial \tau} Y(z + \omega_i; \tau) \cdot Y(z + \omega_i; \tau)^{-1}.$$

Therefore, (2.19) and (2.21) give

$$\begin{aligned} & \Omega(z + \omega_i; \tau) \\ &= \left[\frac{d}{d\tau} Y(z + \omega_i; \tau) - \frac{\partial}{\partial z} Y(z + \omega_i; \tau) \frac{d}{d\tau} \omega_i \right] \cdot Y(z + \omega_i; \tau)^{-1} \\ &= \left[\frac{d}{d\tau} (Y(z; \tau) \rho(\ell_i)) - \frac{\partial}{\partial z} (Y(z; \tau) \rho(\ell_i)) \frac{d}{d\tau} \omega_i \right] \cdot (Y(z; \tau) \rho(\ell_i))^{-1}. \end{aligned}$$

Since $\rho(\ell_i)$, $i = 1, 2$, are independent of τ , we have

$$\Omega(z + 1; \tau) = \frac{d}{d\tau} Y(z; \tau) \cdot Y(z; \tau)^{-1} = \Omega(z; \tau),$$

and

$$\begin{aligned} \Omega(z + \tau; \tau) &= \frac{d}{d\tau} Y(z; \tau) \cdot Y(z; \tau)^{-1} - \frac{\partial}{\partial z} Y(z; \tau) \cdot Y(z; \tau)^{-1} \\ &= \Omega(z; \tau) - Q(z; \tau). \end{aligned}$$

This proves (2.13).

Conversely, suppose there exists a single-valued matrix function $\Omega(z; \tau)$ in $\mathbb{C} \times \mathbb{H}$ satisfying (2.13) such that (2.14) is completely integrable. Let $Y(z; \tau)$ be a solution of the Pfaffian system (2.14). Then (2.19)-(2.21) hold and $Y(z; \tau)$ satisfies system (2.10) in E_τ . Hence

$$\frac{\partial}{\partial \tau} Y(z + \omega_i; \tau) = \frac{d}{d\tau} Y(z + \omega_i; \tau) - \frac{\partial}{\partial z} Y(z + \omega_i; \tau) \frac{d}{d\tau} \omega_i,$$

which implies

$$\begin{aligned} (2.22) \quad \frac{\partial}{\partial \tau} Y(z + 1; \tau) &= \frac{d}{d\tau} (Y(z; \tau) \rho(\ell_1)) \\ &= \frac{\partial}{\partial \tau} Y(z; \tau) \cdot \rho(\ell_1) + Y(z; \tau) \frac{d}{d\tau} \rho(\ell_1) \\ &= \Omega(z; \tau) Y(z; \tau) \rho(\ell_1) + Y(z; \tau) \frac{d}{d\tau} \rho(\ell_1) \end{aligned}$$

and

$$\begin{aligned} (2.23) \quad \frac{\partial}{\partial \tau} Y(z + \tau; \tau) &= \frac{d}{d\tau} (Y(z; \tau) \rho(\ell_2)) - \frac{\partial}{\partial z} (Y(z; \tau) \rho(\ell_2)) \end{aligned}$$

$$\begin{aligned}
 &= \frac{\partial}{\partial \tau} Y(z; \tau) \cdot \rho(\ell_2) + Y(z; \tau) \frac{d}{d\tau} \rho(\ell_2) - \frac{\partial}{\partial z} Y(z; \tau) \cdot \rho(\ell_2) \\
 &= [\Omega(z; \tau) - Q(z; \tau)] Y(z; \tau) \rho(\ell_2) + Y(z; \tau) \frac{d}{d\tau} \rho(\ell_2).
 \end{aligned}$$

On the other hand, by (2.19) and (2.21), we also have

$$(2.24) \quad \frac{\partial}{\partial \tau} Y(z + \omega_i; \tau) = \Omega(z + \omega_i; \tau) Y(z; \tau) \rho(\ell_i).$$

Then by (2.22), (2.23), (2.24) and (2.13), we have

$$Y(z; \tau) \frac{d}{d\tau} \rho(\ell_1) = Y(z; \tau) \frac{d}{d\tau} \rho(\ell_2) = 0.$$

Also, by the same argument as (2.18), we could prove

$$Y(z; \tau) \frac{d}{d\tau} \rho(\gamma_k) = 0, \quad k = 0, 1, 2, 3, \pm.$$

Because of $\det Y \neq 0$, we conclude that

$$\frac{d}{d\tau} \rho(\ell_j) = \frac{d}{d\tau} \rho(\gamma_k) = 0.$$

Thus, Y is an M -invariant solution of (2.10). That is, system (2.10) is monodromy preserving. This completes the proof. \square

Write $\Omega(z; \tau) = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$. Since $Q(z; \tau)$ has the special form (2.11), by a straightforward computation, the integrability condition (2.15) is equivalent to

$$(2.25) \quad \Omega_{12}''' - 4I\Omega_{12}' - 2I'\Omega_{12} + 2\frac{\partial}{\partial \tau} I = 0 \quad \text{in } \mathbb{C} \times \mathbb{H},$$

where we denote $' = \frac{\partial}{\partial z}$ to be the partial derivative with respect to the variable z . This computation is the same as the case in \mathbb{C} (see e.g. [15, Proposition 3.5.1]), so we omit the details. Then we have the following fundamental theorem for isomonodromic deformations of (2.10) in the moduli space of elliptic curves:

Theorem 2.2. *System (2.10) is monodromy preserving as τ deforms if and only if there exists a single-valued solution $\Omega_{12}(z; \tau)$ to (2.25) satisfying*

$$(2.26) \quad \begin{aligned} \Omega_{12}(z+1; \tau) &= \Omega_{12}(z; \tau), \\ \Omega_{12}(z+\tau; \tau) &= \Omega_{12}(z; \tau) - 1. \end{aligned}$$

Proof. By Theorem 2.1, it suffices to prove the sufficient part. Suppose there exists a single-valued solution $\Omega_{12}(z; \tau)$ to (2.25) satisfying (2.26). Then we

define $\Omega(z; \tau) = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$ by setting

$$(2.27) \quad \begin{aligned} \Omega_{11}(z; \tau) &= -\frac{1}{2}\Omega_{12}'(z; \tau), \\ \Omega_{21}(z; \tau) &= \Omega_{11}'(z; \tau) + \Omega_{12}(z; \tau) I(z; \tau), \end{aligned}$$

$$\Omega_{22}(z; \tau) = \Omega'_{12}(z; \tau) + \Omega_{11}(z; \tau).$$

By (2.25), it is easy to see that $\Omega(z; \tau)$ satisfies the integrability condition (2.15) (see e.g. [15, Proposition 3.5.1]), namely (2.14) is completely integrable. Finally, (2.13) follows from (2.26). This completes the proof. \square

The first main result of this section is as follows:

Theorem 2.3. *Let $n_k \notin \frac{1}{2} + \mathbb{Z}$, $k = 0, 1, 2, 3$ and $p(\tau)$ is an apparent singular point of the generalized Lamé equation (2.1) with (2.2). Then (2.1) with $(p, A) = (p(\tau), A(\tau))$ is an isomonodromic deformation with respect to τ if and only if $(p(\tau), A(\tau))$ satisfies the Hamiltonian system:*

$$(2.28) \quad \frac{dp(\tau)}{d\tau} = \frac{\partial K(p, A, \tau)}{\partial A}, \quad \frac{dA(\tau)}{d\tau} = -\frac{\partial K(p, A, \tau)}{\partial p},$$

where

$$(2.29) \quad K(p, A, \tau) = \frac{-i}{4\pi} \left(A^2 + (-\zeta(2p|\tau) + 2p\eta_1(\tau))A - \frac{3}{4}\wp(2p|\tau) - \sum_{k=0}^3 n_k(n_k + 1)\wp(p + \frac{\omega_k}{2}|\tau) \right).$$

To prove Theorem 2.3, we need the following formulae for theta functions and functions in Weierstrass elliptic function theory.

Lemma 2.3. *The following formulae hold:*

(i)

$$\frac{\partial}{\partial \tau} \ln \sigma(z|\tau) = \frac{i}{4\pi} \left[\wp(z|\tau) - \zeta^2(z|\tau) + 2\eta_1(z\zeta(z|\tau) - 1) - \frac{1}{12}g_2z^2 \right],$$

(ii)

$$\frac{\partial}{\partial \tau} \zeta(z|\tau) = \frac{i}{4\pi} \left[\wp'(z|\tau) + 2(\zeta(z|\tau) - z\eta_1(\tau))\wp(z|\tau) + 2\eta_1\zeta(z|\tau) - \frac{1}{6}zg_2(\tau) \right],$$

(iii)

$$\frac{\partial}{\partial \tau} \wp(z|\tau) = \frac{-i}{4\pi} \left[2(\zeta(z|\tau) - z\eta_1(\tau))\wp'(z|\tau) + 4(\wp(z|\tau) - \eta_1)\wp(z|\tau) - \frac{2}{3}g_2(\tau) \right],$$

(iv)

$$\frac{\partial}{\partial \tau} \wp'(z|\tau) = \frac{-i}{4\pi} \left[6(\wp(z|\tau) - \eta_1)\wp'(z|\tau) + (\zeta(z|\tau) - z\eta_1(\tau))(12\wp^2(z|\tau) - g_2(\tau)) \right],$$

(v)

$$\frac{d}{d\tau} \eta_1(\tau) = \frac{i}{4\pi} \left[2\eta_1^2 - \frac{1}{6}g_2(\tau) \right],$$

(vi)

$$\frac{d}{d\tau} \ln \theta'_1(\tau) = \frac{3i}{4\pi} \eta_1,$$

where

$$g_2(\tau) = -4(e_1(\tau)e_2(\tau) + e_1(\tau)e_3(\tau) + e_2(\tau)e_3(\tau)),$$

$$\theta'_1(\tau) \doteq \frac{d}{dz} \vartheta_1(z; \tau)|_{z=0}, \quad \frac{d}{dz} \ln \sigma(z|\tau) \doteq \zeta(z|\tau),$$

$$(2.30) \quad \vartheta_1(z; \tau) \doteq -i \sum_{n=-\infty}^{\infty} (-1)^n e^{(n+\frac{1}{2})^2 \pi i \tau} e^{(2n+1)\pi i z}.$$

Those formulae in Lemma 2.3 are known in the literature; see e.g. [2] and references therein for the proofs.

To give a motivation for our proof of Theorem 2.3, we first consider the simplest case $n_k = 0, \forall k$: Let $a_1 = r + s\tau$ where $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ is a fixed pair and $\pm p(\tau)$, $A(\tau)$, $B(\tau)$ be defined by

$$(2.31) \quad \zeta(a_1(\tau) + p(\tau)) + \zeta(a_1(\tau) - p(\tau)) - 2(r\eta_1(\tau) + s\eta_2(\tau)) = 0,$$

$$(2.32) \quad A = \frac{1}{2} [\zeta(p + a_1) + \zeta(p - a_1) - \zeta(2p)],$$

$$(2.33) \quad B = A^2 - \zeta(2p)A - \frac{3}{4}\wp(2p),$$

respectively. In [4] we could prove that under (2.31)-(2.33), the two functions

$$y_{\pm a_1}(z; \tau) = e^{\pm \frac{z}{2}(\zeta(a_1+p) + \zeta(a_1-p))} \frac{\sigma(z \mp a_1)}{[\sigma(z+p)\sigma(z-p)]^{\frac{1}{2}}}$$

are two linearly independent solutions to the generalized Lamé equation (2.1) with $n_k = 0$, $k = 0, 1, 2, 3$, i.e.,

$$(2.34) \quad y'' = \left[\frac{3}{4}(\wp(z+p) + \wp(z-p)) + A(\zeta(z+p) - \zeta(z-p)) + B \right] y.$$

Observe that (2.34) has singularities only at $\pm p$. Thus, the monodromy representation of (2.34) is a group homomorphism $\rho : \pi_1(E_\tau \setminus \{\pm p\}, q_0) \rightarrow SL(2, \mathbb{C})$. Then we could also compute the monodromy group of (2.34) with respect to $(y_{a_1}(z; \tau), y_{-a_1}(z; \tau))^t$ as following [4]:

$$(2.35) \quad \rho(\gamma_{\pm}) \begin{pmatrix} y_{a_1}(z; \tau) \\ y_{-a_1}(z; \tau) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_{a_1}(z; \tau) \\ y_{-a_1}(z; \tau) \end{pmatrix},$$

$$(2.36) \quad \rho(\ell_1) \begin{pmatrix} y_{a_1}(z; \tau) \\ y_{-a_1}(z; \tau) \end{pmatrix} = \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix} \begin{pmatrix} y_{a_1}(z; \tau) \\ y_{-a_1}(z; \tau) \end{pmatrix},$$

$$(2.37) \quad \rho(\ell_2) \begin{pmatrix} y_{a_1}(z; \tau) \\ y_{-a_1}(z; \tau) \end{pmatrix} = \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{-2\pi i r} \end{pmatrix} \begin{pmatrix} y_{a_1}(z; \tau) \\ y_{-a_1}(z; \tau) \end{pmatrix}.$$

By (2.33) and Lemma 2.1, $\pm p(\tau)$ are apparent singularities. Since the pair (r, s) is fixed, we see from (2.35)-(2.37) that the generalized Lamé equation (2.34) is monodromy preserving. Thus $Y = \begin{pmatrix} y_{a_1}(z; \tau) & y_{-a_1}(z; \tau) \\ y'_{a_1}(z; \tau) & y'_{-a_1}(z; \tau) \end{pmatrix}$ is an

M -invariant fundamental solution for the system (2.10). Then by Theorem 2.1, the single-valued matrix $\Omega(z; \tau)$ could be defined by

$$\begin{aligned}\Omega(z; \tau) &= \frac{\partial}{\partial \tau} Y \cdot Y^{-1} \\ &= \frac{1}{\det Y} \begin{pmatrix} \frac{\partial}{\partial \tau} y_{a_1} & \frac{\partial}{\partial \tau} y_{-a_1} \\ \frac{\partial}{\partial \tau} y'_{a_1} & \frac{\partial}{\partial \tau} y'_{-a_1} \end{pmatrix} \begin{pmatrix} y'_{-a_1} & -y_{-a_1} \\ -y'_{a_1} & y_{a_1} \end{pmatrix},\end{aligned}$$

which gives us

$$\Omega_{12} = \frac{y_{a_1} \frac{\partial}{\partial \tau} y_{-a_1} - y_{-a_1} \frac{\partial}{\partial \tau} y_{a_1}}{y_{a_1} y'_{-a_1} - y_{-a_1} y'_{a_1}} = \frac{\frac{\partial}{\partial \tau} \ln \frac{y_{a_1}}{y_{-a_1}}}{\frac{\partial}{\partial z} \ln \frac{y_{a_1}}{y_{-a_1}}} = \frac{\frac{\partial}{\partial \tau} \ln f(z; \tau)}{\frac{\partial}{\partial z} \ln f(z; \tau)},$$

where $f \doteq \frac{y_{a_1}}{y_{-a_1}}$ is given by

$$f(z; \tau) = e^{z(\zeta(a_1+p)+\zeta(a_1-p))} \frac{\sigma(z-a_1)}{\sigma(z+a_1)}.$$

Using (2.31) and Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$, we have

$$(2.38) \quad f(z; \tau) = e^{2za_1\eta_1 - 4\pi isz} \frac{\sigma(z-a_1)}{\sigma(z+a_1)}.$$

In order to compute Ω_{12} , we compute $\frac{\partial}{\partial \tau} \ln f(z; \tau)$ and $\frac{\partial}{\partial z} \ln f(z; \tau)$, respectively. By Lemma 2.3 and (2.38), we have

$$\begin{aligned}(2.39) \quad & \frac{\partial}{\partial \tau} \ln f(z; \tau) \\ &= 2zs\eta_1 + 2za_1 \frac{d\eta_1}{d\tau} - (\zeta(z-a_1|\tau) + \zeta(z+a_1|\tau))s \\ & \quad + \frac{\partial}{\partial \tau} \ln \sigma(z-a_1|\tau) - \frac{\partial}{\partial \tau} \ln \sigma(z+a_1|\tau) \\ &= \frac{i}{4\pi} [\zeta(z+a_1) - \zeta(z-a_1) - 2\eta_1 a_1 + 4\pi is] \\ & \quad \times [\zeta(z-a_1) + \zeta(z+a_1) - 2z\eta_1] + \frac{i}{4\pi} [\wp(z-a_1) - \wp(z+a_1)],\end{aligned}$$

and

$$(2.40) \quad \frac{\partial}{\partial z} \ln f(z; \tau) = 2a_1\eta_1 - 4\pi is + \zeta(z-a_1) - \zeta(z+a_1).$$

Thus from (2.39), (2.40) and (2.31), we have

$$\begin{aligned}(2.41) \quad \Omega_{12}(z; \tau) &= -\frac{i}{4\pi} [\zeta(z-a_1) + \zeta(z+a_1) - 2z\eta_1] \\ & \quad + \frac{i}{4\pi} \frac{\wp(z-a_1) - \wp(z+a_1)}{2a_1\eta_1 - 4\pi is + \zeta(z-a_1) - \zeta(z+a_1)} \\ &= -\frac{i}{4\pi} [\zeta(z-a_1) + \zeta(z+a_1) - 2z\eta_1] \\ & \quad + \frac{i}{4\pi} \frac{\wp(z-a_1) - \wp(z+a_1)}{\zeta(a_1+p) + \zeta(a_1-p) + \zeta(z-a_1) - \zeta(z+a_1)}.\end{aligned}$$

From (2.41), we see that $\pm a_1$ are not poles of $\Omega_{12}(z; \tau)$. In fact, $\pm p$ are the only simple poles and 0 is a zero of $\Omega_{12}(z; \tau)$. Furthermore, we have

$$(2.42) \quad \operatorname{Res}_{z=\pm p} \Omega_{12}(z; \tau) = \frac{-i}{4\pi}.$$

By (2.41), it is easy to see that

$$(2.43) \quad \Omega_{12}(-z; \tau) = -\Omega_{12}(z; \tau).$$

By (2.42), (2.43) and (2.26), $\Omega_{12}(z; \tau)$ has a simpler expression as follows:

$$\Omega_{12}(z; \tau) = -\frac{i}{4\pi} (\zeta(z-p) + \zeta(z+p) - 2z\eta_1).$$

For the general case, we do not have the explicit expression of the two linearly independent solutions. But the discussion above motivates us to find the explicit form of Ω_{12} . For example, we might ask whether there exists Ω_{12} satisfying the property (2.43) or not. Thus, we need to study it via a different way. More precisely, we prove the following theorem:

Theorem 2.4. *Under the assumption of Theorem 2.3, suppose the generalized Lamé equation (2.1) with $(p, A) = (p(\tau), A(\tau))$ is an isomonodromic deformation with respect to τ . Then there exists an M -invariant fundamental solution $Y(z; \tau)$ of system (2.10) such that $\Omega_{12}(z; \tau)$ is of the form:*

$$(2.44) \quad \Omega_{12}(z; \tau) = -\frac{i}{4\pi} (\zeta(z-p(\tau)) + \zeta(z+p(\tau)) - 2z\eta_1),$$

where $\Omega_{12}(z; \tau)$ is the (1,2) component of $\Omega(z; \tau)$ which is defined by $Y(z; \tau)$.

We remark that Theorem 2.4 is a result locally in τ . In the following, we always assume that V_0 is a small neighborhood of τ_0 such that $p(\tau) \notin E_\tau[2]$ and $A(\tau), B(\tau)$ are finite for $\tau \in V_0$. First, we study the singularities of $\Omega_{12}(z; \tau)$:

Lemma 2.4. *Under the assumption and notations of Theorem 2.3, suppose $Y(z; \tau)$ is an M -invariant fundamental solution of (2.10) with $(p, A) = (p(\tau), A(\tau))$ and $\Omega(z; \tau)$ is defined by $Y(z; \tau)$. Then*

- (i) $\Omega_{12}(\cdot; \tau)$ is meromorphic in \mathbb{C} and holomorphic for all $z \notin \{\pm p(\tau), \frac{\omega_i}{2}, i = 0, 1, 2, 3\} + \Lambda_\tau$.
- (ii) If there exist $i \in \{0, 1, 2, 3\}$ and $(b_1, b_2) \in \mathbb{Z}^2$ such that $\frac{\omega_i}{2} + b_1 + b_2\tau$ is a pole of $\Omega_{12}(\cdot; \tau)$ with order m_i , then $m_i = 2n_i$, and any point in $\frac{\omega_i}{2} + \Lambda_\tau$ is also a pole of $\Omega_{12}(\cdot; \tau)$ with the same order m_i . Consequently, if $\Omega_{12}(\cdot; \tau)$ has a pole at $\frac{\omega_i}{2} + \Lambda_\tau$, then $n_i \in \mathbb{N}$.
- (iii) $\Omega_{12}(\cdot; \tau)$ has poles at $\{\pm p\} + \Lambda_\tau$ of order at most one.

Proof. (i) Since equation (2.25) has singularities only at $\{\pm p(\tau), \frac{\omega_i}{2}, i = 0, 1, 2, 3\} + \Lambda_\tau$, $\Omega_{12}(\cdot; \tau)$ is holomorphic for all $z \notin \{\pm p(\tau), \frac{\omega_i}{2}, i = 0, 1, 2, 3\} + \Lambda_\tau$. On the other hand, if $z_0 \in \{\pm p(\tau), \frac{\omega_i}{2}, i = 0, 1, 2, 3\} + \Lambda_\tau$ is a singularity

of $\Omega_{12}(\cdot; \tau)$, then by using (2.17) and the local behavior of $Y(\cdot; \tau)$ at z_0 , it is easy to prove

$$\Omega_{12}(z; \tau) = \frac{c(\tau)}{(z - z_0)^m} (1 + \text{higher order term}) \text{ near } z_0$$

for some $c(\tau) \neq 0$ and $m \in \mathbb{C}$. Since $\Omega_{12}(\cdot; \tau)$ is single-valued, we conclude that $m \in \mathbb{N}$, namely z_0 must be a pole of $\Omega_{12}(\cdot; \tau)$. This proves (i).

The proof of (ii) and (iii) are similar, so we only prove (ii) for $i = 0$. Without loss of generality, we may assume $b_1 = b_2 = 0$. Suppose 0 is a pole of $\Omega_{12}(z; \tau)$ with order $m_0 \in \mathbb{N}$. By (2.26), it is obvious that for any $(b_1, b_2) \in \mathbb{Z}^2$, $b_1 + b_2\tau$ is also a pole with the same order m_0 . Suppose

$$(2.45) \quad \Omega_{12}(z; \tau) = z^{-m_0} \left(\sum_{k=0}^{\infty} c_k z^k \right) = \frac{c_0}{z^{m_0}} + O\left(\frac{1}{z^{m_0-1}}\right) \text{ near } 0,$$

where $c_0 \neq 0$. Then we have

$$(2.46) \quad \Omega'_{12}(z; \tau) = -m_0 \frac{c_0}{z^{m_0+1}} + O\left(\frac{1}{z^{m_0}}\right),$$

$$(2.47) \quad \Omega'''_{12}(z; \tau) = -m_0(m_0+1)(m_0+2) \frac{c_0}{z^{m_0+3}} + O\left(\frac{1}{z^{m_0+2}}\right),$$

and

$$(2.48) \quad \begin{aligned} I(z; \tau) &= n_0(n_0+1) \frac{1}{z^2} \\ &+ \left[\sum_{i=1}^3 n_i(n_i+1) \wp\left(\frac{\omega_i}{2}\right) + \frac{3}{2} \wp(p) + 2A\zeta(p) + B \right] + O(z) \\ &= n_0(n_0+1) \frac{1}{z^2} + D(\tau) + O(z), \end{aligned}$$

where $D(\tau)$ is a constant depending on τ . Thus

$$(2.49) \quad I'(z; \tau) = -2n_0(n_0+1) \frac{1}{z^3} + O(1), \quad \frac{\partial I}{\partial \tau}(z; \tau) = O(1).$$

Substituting (2.45)-(2.49) into (2.25), we easily obtain

$$m_0(m_0+1)(m_0+2)c_0 = 4n_0(n_0+1)(m_0+1)c_0.$$

Since $n_0 \geq -\frac{1}{2}$, we have $m_0 = 2n_0 \in \mathbb{N}$. Together with the assumption that $n_0 \notin \frac{1}{2} + \mathbb{Z}$, we have $n_0 \in \mathbb{N}$. This completes the proof. \square

For the isomonodromic deformation of the 2nd order Fuchsian equation (4.1) on \mathbb{CP}^1 , if the non-resonant condition $n_i \notin \frac{1}{2} + \mathbb{Z}$ holds, then Ω_{12} is independent of the choice of M -invariant fundamental solutions. See [15]. However, the same conclusion is not true in our study of equations defined in tori; see Remark 2.2 below. The following lemma is to classify the structure of solutions of (2.25).

Lemma 2.5. *Under the assumption and notations of Lemma 2.4. Then*

- (i) If $\tilde{Y}(z; \tau)$ is another M -invariant fundamental solution of (2.10), then $\Omega_{12}(z; \tau) - \tilde{\Omega}_{12}(z; \tau)$ is an elliptic function with periods 1 and τ , and satisfies the following second symmetric product equation of (2.1):

$$(2.50) \quad \Phi''' - 4I\Phi' - 2I'\Phi = 0.$$

- (ii) Let $\Phi(z; \tau)$ be an elliptic solution of (2.50). For any $c \in \mathbb{C}$, define $\tilde{\Omega}_{12}(z; \tau)$ by

$$\tilde{\Omega}_{12}(z; \tau) \doteq \Omega_{12}(z; \tau) + c\Phi(z; \tau).$$

Then there exists an M -invariant fundamental solution $\tilde{Y}(z; \tau)$ of system (2.10) such that $\tilde{\Omega}_{12}(z; \tau)$ is the (1,2) component of $\tilde{\Omega}(z; \tau)$ which is defined by $\tilde{Y}(z; \tau)$.

Proof. (i) This follows directly from that $\Omega_{12}(z; \tau)$ and $\tilde{\Omega}_{12}(z; \tau)$ are both single-valued and satisfy (2.25) and (2.26).

(ii) It is trivial to see that $\tilde{\Omega}_{12}(z; \tau)$ satisfies (2.25) and (2.26). Moreover, since both $\Phi(z; \tau)$ and $\Omega_{12}(z; \tau)$ are single-valued, $\tilde{\Omega}_{12}(z; \tau)$ is single-valued. By Theorem 2.2, there exists an M -invariant fundamental solution $\tilde{Y}(z; \tau)$ of system (2.10) such that $\tilde{\Omega}(z; \tau)$ is defined by $\tilde{Y}(z; \tau)$. \square

Lemma 2.6. *Under the assumption and notations of Lemma 2.4. Then there exists an M -invariant fundamental solution $\tilde{Y}(z; \tau)$ such that*

$$\tilde{\Omega}_{12}(z; \tau) = -\Omega_{12}(-z; \tau).$$

Proof. Recall that $Y(z; \tau) = \begin{pmatrix} y_1(z; \tau) & y_2(z; \tau) \\ y_1'(z; \tau) & y_2'(z; \tau) \end{pmatrix}$ is an M -invariant fundamental solution of (2.10) in a neighborhood U_{q_0} of q_0 . Then for $z \in -U_{q_0}$, a neighborhood of $-q_0$, we define

$$\tilde{Y}(z; \tau) := \begin{pmatrix} y_1(-z; \tau) & y_2(-z; \tau) \\ -y_1'(-z; \tau) & -y_2'(-z; \tau) \end{pmatrix}.$$

It is easy to see that $\tilde{Y}(z; \tau)$ is a fundamental solution to (2.10) in $-U_{q_0}$. Define $\tilde{\Omega}(z; \tau)$ by $\tilde{Y}(z; \tau)$, then we have

$$\det \tilde{Y}(z; \tau) \cdot \tilde{\Omega}_{12}(z; \tau) = y_1(-z; \tau) \frac{\partial}{\partial \tau} y_2(-z; \tau) - y_2(-z; \tau) \frac{\partial}{\partial \tau} y_1(-z; \tau),$$

and since $\det \tilde{Y}(z; \tau) = -\det Y(-z; \tau)$, we obtain

$$(2.51) \quad \tilde{\Omega}_{12}(z; \tau) = -\Omega_{12}(-z; \tau)$$

for $z \in -U_{q_0}$. Since Ω_{12} is globally defined and single-valued, by analytic continuation, (2.51) holds true globally. Thus, $\tilde{\Omega}_{12}$ is globally defined and single-valued. Moreover, $\tilde{\Omega}_{12}(z; \tau)$ satisfies (2.25) and (2.26) which implies that $\tilde{Y}(z; \tau)$ is M -invariant. This completes the proof. \square

Proof of Theorem 2.4. Since the generalized Lamé equation (2.1) with (2.2) is monodromy preserving as τ deforms, by Theorem 2.2 and Lemma 2.4, there exists a single-valued meromorphic function $\hat{\Omega}_{12}(z; \tau)$ satisfying (2.25) and (2.26). Define $\Omega_{12}(z; \tau)$ by

$$(2.52) \quad \Omega_{12}(z; \tau) \doteq \frac{1}{2} \left[\hat{\Omega}_{12}(z; \tau) - \hat{\Omega}_{12}(-z; \tau) \right].$$

To prove Theorem 2.4, we divide it into three steps:

Step 1. We prove that there exists an M -invariant fundamental solution $Y(z; \tau)$ of system (2.10) such that

$$(2.53) \quad \Omega(z; \tau) = \frac{\partial}{\partial \tau} Y(z; \tau) \cdot Y^{-1}(z; \tau)$$

and $\Omega_{12}(z; \tau)$ is the (1,2) component of $\Omega(z; \tau)$.

Let

$$\Phi(z; \tau) = -\frac{1}{2} \left[\hat{\Omega}_{12}(z; \tau) + \hat{\Omega}_{12}(-z; \tau) \right].$$

By Lemmas 2.6 and 2.5, Φ is an elliptic solution of equation (2.50) and

$$\Omega_{12}(z; \tau) = \hat{\Omega}_{12}(z; \tau) + \Phi(z; \tau).$$

By Lemma 2.5 (ii), there exists an M -invariant fundamental solution $Y(z; \tau)$ of system (2.10) such that (2.53) holds.

Step 2. We prove that $\Omega_{12}(z; \tau)$ is an odd meromorphic function and only has poles at $\{\pm p\} + \Lambda_\tau$ of order at most one. Furthermore, $\Omega'_{12}(z; \tau)$ is an even elliptic function.

Clearly (2.52) and Lemma 2.4 imply that $\Omega_{12}(z; \tau)$ is an odd meromorphic function. Now we claim that:

$$(2.54) \quad \Omega_{12}(z; \tau) \text{ only has poles at } \{\pm p\} + \Lambda_\tau \text{ of order at most one.}$$

By Lemma 2.4 (i), $\Omega_{12}(z; \tau)$ is holomorphic for all $z \notin \{\pm p, \frac{\omega_i}{2}, i = 0, 1, 2, 3\} + \Lambda_\tau$. If $\Omega_{12}(z; \tau)$ has a pole at $\frac{\omega_i}{2} + \Lambda_\tau$, then the order of the pole is $2n_i \in 2\mathbb{N}$ by Lemma 2.4 (ii), which yields a contradiction to the fact that $\Omega_{12}(z; \tau)$ is odd and satisfies (2.26).

Step 3. We prove that $\Omega_{12}(z; \tau)$ is of the form (2.44):

$$\Omega_{12}(z; \tau) = -\frac{i}{4\pi} (\zeta(z-p) + \zeta(z+p) - 2z\eta_1).$$

By Step 2 and (2.54), we know that $\Omega'_{12}(z; \tau)$ must be of the following form

$$\Omega'_{12}(z; \tau) = -C(\wp(z+p) + \wp(z-p)) + D$$

for some constants $C, D \in \mathbb{C}$. Thus by integration, we get

$$\Omega_{12}(z; \tau) = C(\zeta(z+p) + \zeta(z-p)) + Dz + E$$

for some $E \in \mathbb{C}$. Since $\Omega_{12}(z; \tau)$ is odd, we have $E = 0$. Furthermore,

$$\Omega_{12}(z+1; \tau) = \Omega_{12}(z; \tau) + 2C\eta_1 + D,$$

and

$$\Omega_{12}(z+\tau; \tau) = \Omega_{12}(z; \tau) + 2C\eta_2 + D\tau.$$

By (2.13), we have

$$2C\eta_1 + D = 0, \quad 2C\eta_2 + D\tau = -1.$$

By Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$, we have

$$C = \frac{-i}{4\pi} \quad \text{and} \quad D = \frac{i}{2\pi}\eta_1,$$

which implies (2.44). This completes the proof. \square

Corollary 2.1. *Under the assumption and notations of Lemma 2.4 and assume $n_i \notin \mathbb{Z}$ for some $i \in \{0, 1, 2, 3\}$. Then $\Omega_{12}(z; \tau)$ is unique, i.e., $\Omega_{12}(z; \tau)$ is independent of the choice of M -invariant solution $Y(z; \tau)$ of system (2.10).*

Proof. For any M -invariant solution $Y(z; \tau)$ of system (2.10), by Theorem 2.2, there exists a single-valued function $\Omega_{12}(z; \tau)$ satisfying (2.25) and (2.13). Let

$$\Phi(z; \tau) = \Omega_{12}(z; \tau) + \Omega_{12}(-z; \tau).$$

If $\Phi(z; \tau) \not\equiv 0$, then $\Phi(z; \tau)$ is an even elliptic solution of (2.50). Without loss of generality, we may consider the case $n_1 \notin \mathbb{Z}$. Then $2n_1 \notin \mathbb{Z}$ since $n_1 \notin \frac{1}{2} + \mathbb{Z}$. Since the local exponents of (2.50) at $\frac{\omega_1}{2}$ are $-2n_1, 1, 2n_1 + 2$ and $\Phi(z; \tau)$ is elliptic, the local exponent of $\Phi(z; \tau)$ at $z = \frac{\omega_1}{2}$ must be 1, i.e., $\frac{\omega_1}{2}$ is a simple zero. But again by $\Phi(z; \tau)$ is even elliptic, we have $\Phi'(\frac{\omega_1}{2}; \tau) = 0$, which leads to a contradiction. Thus, $\Phi(z; \tau) \equiv 0$, i.e., $\Omega_{12}(z; \tau)$ is odd. Then by Theorem 2.4, $\Omega_{12}(z; \tau)$ is of the form (2.44). \square

Remark 2.2. *When $n_i \in \mathbb{Z}$ for all $i = 0, 1, 2, 3$, $\Omega_{12}(z; \tau)$ might not be unique. For example, when $n_i = 0$ for all $i = 0, 1, 2, 3$, we define*

$$\Phi(z; \tau) \doteq \zeta(z + p|\tau) - \zeta(z - p|\tau) - \zeta(2p|\tau) - 2A,$$

then $\Phi(z; \tau)$ is an even elliptic solution of (2.50). So for any $c \in \mathbb{C}$,

$$\tilde{\Omega}_{12}(z; \tau) \doteq \frac{-i}{4\pi} (\zeta(z - p) + \zeta(z + p) - 2z\eta_1) + c\Phi(z; \tau)$$

satisfies (2.25) and (2.26). By Lemma 2.5, there exists an M -invariant solution $\tilde{Y}(z; \tau)$ such that $\tilde{\Omega}(z; \tau)$ is defined by $\tilde{Y}(z; \tau)$.

Define $U(z; \tau)$ by

$$U(z; \tau) \doteq \Omega_{12}'''(z; \tau) - 4I(z; \tau)\Omega_{12}'(z; \tau) - 2I'(z; \tau)\Omega_{12}(z; \tau) + 2\frac{\partial}{\partial\tau}I(z; \tau),$$

where $\Omega_{12}(z; \tau)$ is given in Theorem 2.4 (2.44), i.e.,

$$\Omega_{12}(z; \tau) = -\frac{i}{4\pi} (\zeta(z - p) + \zeta(z + p) - 2z\eta_1).$$

In order to prove Theorem 2.3, we need the following local expansions for $\Omega_{12}(z; \tau)$ and $I(z; \tau)$ at p and $\frac{\omega_k}{2}$, $k = 0, 1, 2, 3$, respectively.

Lemma 2.7. *$\Omega_{12}(z; \tau)$ and $I(z; \tau)$ have local expansions at p and $\frac{\omega_k}{2}$, $k = 0, 1, 2, 3$ as follows:*

(i) Near p , let $u = z - p$. Then we have

$$(2.55) \quad \Omega_{12}(z; \tau) = \frac{-i}{4\pi} \left(\begin{array}{l} u^{-1} + (\zeta(2p) - 2p\eta_1) - (\wp(2p) + 2\eta_1)u \\ -\frac{1}{2}\wp'(2p)u^2 - \frac{1}{6}\left(\frac{g_2}{10} + \wp''(2p)\right)u^3 + O(u^4) \end{array} \right),$$

and

$$(2.56) \quad I(z; \tau) = \frac{3}{4}u^{-2} - Au^{-1} + A^2 + H_1(\tau)u + H_2(\tau)u^2 + O(u^3),$$

where

$$(2.57) \quad H_1(\tau) = \sum_{k=0}^3 n_k(n_k + 1)\wp'\left(p + \frac{\omega_k}{2}\right) + \frac{3}{4}\wp'(2p) - A\wp(2p),$$

and

$$H_2(\tau) = \frac{1}{2} \left[\sum_{k=0}^3 n_k(n_k + 1)\wp''\left(p + \frac{\omega_k}{2}\right) + \frac{3}{4}\wp''(2p) + \frac{3}{40}g_2 - A\wp'(2p) \right].$$

(ii) Near $\frac{\omega_k}{2}$, $k \in \{0, 1, 2, 3\}$, let $u_k = z - \frac{\omega_k}{2}$. Then we have

$$(2.58) \quad \Omega_{12}(z; \tau) = \frac{i}{4\pi} \left[\begin{array}{l} -\left(\zeta\left(\frac{\omega_k}{2} + p\right) + \zeta\left(\frac{\omega_k}{2} - p\right) - \omega_k\eta_1\right) \\ + 2\left(\wp\left(\frac{\omega_k}{2} + p\right) + \eta_1\right)u_k + O(u_k^3) \end{array} \right],$$

and

$$(2.59) \quad I(z; \tau) = n_k(n_k + 1)u_k^{-2} + \Lambda_k(\tau) + O(u_k^2),$$

where

$$(2.60) \quad \begin{aligned} \Lambda_k(\tau) &= \sum_{j \neq k}^3 n_j(n_j + 1)\wp\left(\frac{\omega_k + \omega_j}{2}\right) + \frac{3}{2}\wp\left(\frac{\omega_k}{2} + p\right) \\ &\quad + A(\tau)\left(\zeta\left(\frac{\omega_k}{2} + p\right) - \zeta\left(\frac{\omega_k}{2} - p\right)\right) + B(\tau). \end{aligned}$$

Proof. Recall the following expansions:

$$(2.61) \quad \zeta(u) = \frac{1}{u} - \frac{g_2}{60}u^3 - \frac{g_3}{140}u^5 + O(u^7),$$

$$(2.62) \quad \wp(u) = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + O(u^6).$$

The proof follows from a direct computation by using (2.61) and (2.62). \square

By using Lemma 2.7, we have

Lemma 2.8. $U(\cdot; \tau)$ is an even elliptic function and has poles only at $\pm p$ of order at most 3. More precisely, $U(z; \tau)$ is expressed as follows:

$$(2.63) \quad \begin{aligned} U(z; \tau) &= L(\tau)\left(\wp'(z-p) - \wp'(z+p) + \wp'(2p)\right) \\ &\quad + M(\tau)\left(\wp(z-p) + \wp(z+p) - \wp(2p)\right) \\ &\quad + N(\tau)\left(\zeta(z-p) - \zeta(z+p) + \zeta(2p)\right) + C(\tau), \end{aligned}$$

where the coefficients $L(\tau)$, $M(\tau)$, $N(\tau)$ and $C(\tau)$ are given by

$$(2.64) \quad L(\tau) = -\frac{1}{2} \left(3 \frac{dp}{d\tau} + \frac{i}{4\pi} [6A - 3(\zeta(2p) - 2p\eta_1)] \right),$$

$$(2.65) \quad M(\tau) = -2A \frac{dp}{d\tau} + \frac{i}{4\pi} [-4A^2 + 2A(\zeta(2p) - 2p\eta_1)],$$

$$(2.66) \quad N(\tau) = -2 \frac{dA}{d\tau} + \frac{i}{4\pi} \left[\begin{array}{l} 4A(\wp(2p) + \eta_1) - 3\wp'(2p) \\ -2 \sum_{k=0}^3 n_k(n_k + 1) \wp'(p + \frac{\omega_k}{2}) \end{array} \right],$$

$$(2.67) \quad C(\tau) = 4A \frac{dA}{d\tau} - 2H_1(\tau) \frac{dp}{d\tau} + \frac{i}{4\pi} \left[\begin{array}{l} -4A^2(\wp(2p) + 2\eta_1) + 3A\wp'(2p) \\ + 2H_1(\tau)(\zeta(2p) - 2p\eta_1) \end{array} \right].$$

Here $H_1(\tau)$ is given in (2.57).

Proof. Since $I(\cdot; \tau)$ is elliptic, we have $I(z; \tau) = I(z + \tau; \tau)$. Thus

$$(2.68) \quad \begin{aligned} \frac{\partial}{\partial \tau} I(z; \tau) &= I'(z + \tau; \tau) + \frac{\partial}{\partial \tau} I(z + \tau; \tau) \\ &= I'(z; \tau) + \frac{\partial}{\partial \tau} I(z + \tau; \tau). \end{aligned}$$

By using (2.68) and the translation property (2.26) of $\Omega_{12}(z; \tau)$, we have

$$U(z + \omega_k; \tau) = U(z; \tau), \quad k = 1, 2,$$

that is, $U(\cdot; \tau)$ is elliptic. Moreover, since $I(\cdot; \tau)$ is even, we have

$$(2.69) \quad \frac{\partial}{\partial \tau} I(z; \tau) = \frac{\partial}{\partial \tau} I(-z; \tau).$$

By using (2.69) and $\Omega_{12}(\cdot; \tau)$ is odd, we see that $U(\cdot; \tau)$ is even.

Next we claim that:

$$(2.70) \quad U(\cdot; \tau) \text{ is holomorphic at } \frac{\omega_k}{2}, \quad k = 0, 1, 2, 3.$$

Since the proof is similar, we only give the proof for $k = 2$. In this case, by (2.58) and (2.59) in Lemma 2.7, near $\frac{\tau}{2}$ we have

$$(2.71) \quad \Omega_{12}(z; \tau) = \frac{i}{4\pi} [2\pi i + 2(\wp(\frac{\tau}{2} + p) + \eta_1)u_2 + O(u_2^3)],$$

and

$$(2.72) \quad I(z; \tau) = n_2(n_2 + 1)u_2^{-2} + \Lambda_2(\tau) + O(u_2^2),$$

Then near $\frac{\tau}{2}$, we have

$$(2.73) \quad \Omega'_{12}(z; \tau) = \frac{i}{4\pi} [2(\wp(\frac{\tau}{2} + p) + \eta_1) + O(u_2^2)],$$

$$(2.74) \quad I'(z; \tau) = -2n_2(n_2 + 1)u_2^{-3} + O(u_2),$$

and

$$(2.75) \quad \frac{\partial}{\partial \tau} I(z; \tau) = n_2 (n_2 + 1) u_2^{-3} + \frac{\partial}{\partial \tau} \Lambda_2(\tau) + O(u_2).$$

By (2.71)-(2.75) and $\Omega_{12}(z; \tau)$ is holomorphic at $\frac{\omega_k}{2}$, $k = 0, 1, 2, 3$, we have

$$(2.76) \quad \begin{aligned} U(z; \tau) &= \Omega_{12}'''(z; \tau) - 4I(z; \tau) \Omega_{12}'(z; \tau) - 2I'(z; \tau) \Omega_{12}(z; \tau) + 2 \frac{\partial}{\partial \tau} I(z; \tau) \\ &= \Omega_{12}'''(z; \tau) - 4 [n_2 (n_2 + 1) u_2^{-2} + \Lambda_2(\tau) + O(u_2^2)] \\ &\quad \times \left(\frac{i}{4\pi} \right) \left[2 \left(\wp \left(\frac{\tau}{2} + p \right) + \eta_1 \right) + O(u_2^2) \right] \\ &\quad - 2 [-2n_2 (n_2 + 1) u_2^{-3} + O(u_2)] \\ &\quad \times \left(\frac{i}{4\pi} \right) \left[2\pi i + 2 \left(\wp \left(\frac{\tau}{2} + p \right) + \eta_1 \right) u_1 + O(u_2^3) \right] \\ &\quad + 2 \left[n_2 (n_2 + 1) u_2^{-3} + \frac{\partial}{\partial \tau} \Lambda_2(\tau) + O(u_2^2) \right]. \end{aligned}$$

From (2.76), it is easy to see that the coefficients of u_2^{-3} , u_2^{-2} , u_2^{-1} are all vanishing which implies that $U(z; \tau)$ is holomorphic at $\frac{\tau}{2}$.

Now we prove $U(z; \tau)$ can be written as (2.63). To compute the coefficients $L(\tau)$, $M(\tau)$, $N(\tau)$ and $C(\tau)$, we only need to compute near p . By (2.55) and (2.56), near p , we have

$$(2.77) \quad \Omega_{12}'(z; \tau) = \frac{-i}{4\pi} \begin{pmatrix} -u^{-2} - (\wp(2p) + 2\eta_1) - \wp'(2p)u \\ -\frac{1}{2} \left(\frac{g_2}{10} + \wp''(2p) \right) u^2 + O(u^3) \end{pmatrix},$$

$$(2.78) \quad \Omega_{12}'''(z; \tau) = \frac{-i}{4\pi} \left(-6u^{-4} - \left(\frac{g_2}{10} + \wp''(2p) \right) + O(u) \right),$$

$$(2.79) \quad I'(z; \tau) = -\frac{3}{2}u^{-3} + Au^{-2} + H_1(\tau) + 2H_2(\tau)u + O(u^2),$$

$$(2.80) \quad \begin{aligned} \frac{\partial}{\partial \tau} I(z; \tau) &= \frac{3}{2} \frac{dp}{d\tau} u^{-3} - A \frac{dp}{d\tau} u^{-2} - \frac{dA}{d\tau} u^{-1} \\ &\quad + \left(2A \frac{dA}{d\tau} - H_1(\tau) \frac{dp}{d\tau} \right) + O(u). \end{aligned}$$

By (2.55), (2.56) and (2.77)-(2.80), near p , after computations we have

$$(2.81) \quad \begin{aligned} U(z; \tau) &= \left(3 \frac{dp}{d\tau} + \frac{i}{4\pi} [6A - 3(\zeta(2p) - 2p\eta_1)] \right) u^{-3} \\ &\quad + \left(-2A \frac{dp}{d\tau} + \frac{i}{4\pi} [-4A^2 + 2A(\zeta(2p) - 2p\eta_1)] \right) u^{-2} \end{aligned}$$

$$\begin{aligned}
 & + \left(-2 \frac{dA}{d\tau} + \frac{i}{4\pi} \left[4A (\wp(2p) + \eta_1) - 3\wp'(2p) \right. \right. \\
 & \quad \left. \left. - 2 \sum_{k=0}^3 n_k (n_k + 1) \wp' \left(p + \frac{\omega_k}{2} \right) \right] \right) u^{-1} \\
 & + \left(\begin{aligned} & 4A \frac{dA}{d\tau} - 2H_1(\tau) \frac{dp}{d\tau} \\ & + \frac{i}{4\pi} \left[-4A^2 (\wp(2p) + 2\eta_1) + 3A\wp'(2p) \right] \\ & + 2H_1(\tau) (\zeta(2p) - 2p\eta_1) \end{aligned} \right) + O(u).
 \end{aligned}$$

Obviously, (2.81) implies that $U(z; \tau)$ has pole at p with order at most 3. Since $U(z; \tau)$ is an even elliptic function, $U(z; \tau)$ also has pole at $-p$ with order at most 3. From here and (2.70), we conclude that $U(z; \tau)$ has poles only at $\pm p$ with order at most 3. Moreover, from (2.81), it is easy to see that the coefficients $L(\tau)$, $M(\tau)$, $N(\tau)$ and $C(\tau)$ are given by (2.64)-(2.67). \square

Proof of Theorem 2.3. By Theorem 2.2 and Lemma 2.8, the generalized Lamé equation (2.1) with $(p, A) = (p(\tau), A(\tau))$ is monodromy preserving as τ deforms if and only if

$$U(z; \tau) = 0,$$

if and only if

$$(2.82) \quad L(\tau) = M(\tau) = N(\tau) = C(\tau) = 0.$$

By (2.64)-(2.67), a straightforward computation shows that (2.82) is equivalent to that $(p, A) = (p(\tau), A(\tau))$ satisfies the Hamiltonian system (2.28) (see (2.83) below). \square

2.2. Hamiltonian system and Painlevé VI. Next, we will study the Hamiltonian structure for the elliptic form (1.3) with α_i defined by (1.14). Our second main theorem is the following:

Theorem 2.5. *The elliptic form (1.3) with $\alpha_k = \frac{1}{2} (n_k + \frac{1}{2})^2$, $k = 0, 1, 2, 3$ is equivalent to the Hamiltonian system defined by (2.28) and (2.29).*

Proof. Suppose $(p(\tau), A(\tau))$ satisfies the Hamiltonian system (2.28), i.e.,

$$(2.83) \quad \begin{cases} \frac{dp(\tau)}{d\tau} = \frac{\partial K(p, A, \tau)}{\partial A} = \frac{-i}{4\pi} (2A - \zeta(2p|\tau) + 2p\eta_1(\tau)) \\ \frac{dA(\tau)}{d\tau} = -\frac{\partial K(p, A, \tau)}{\partial p} = \frac{i}{4\pi} \left((2\wp(2p|\tau) + 2\eta_1(\tau)) A - \frac{3}{2}\wp'(2p|\tau) \right. \\ \quad \left. - \sum_{k=0}^3 n_k (n_k + 1) \wp' \left(p + \frac{\omega_k}{2} | \tau \right) \right) \end{cases}.$$

Then we compute the second derivative $\frac{d^2 p(\tau)}{d\tau^2}$ of $p(\tau)$ as follows:

$$(2.84) \quad \begin{aligned} \frac{d^2 p(\tau)}{d\tau^2} = & \frac{-i}{4\pi} \left[2 \frac{dA(\tau)}{d\tau} + 2\wp(2p|\tau) \frac{dp(\tau)}{d\tau} - \frac{\partial}{\partial \tau} \zeta(2p|\tau) \right. \\ & \left. + 2\eta_1(\tau) \frac{dp(\tau)}{d\tau} + 2p(\tau) \frac{d\eta_1(\tau)}{d\tau} \right]. \end{aligned}$$

By Lemma 2.3, we have

$$(2.85) \quad -\frac{\partial}{\partial \tau} \zeta(2p|\tau) = \frac{-i}{4\pi} \left[\wp'(2p|\tau) + 2(\zeta(2p|\tau) - 2p\eta_1(\tau)) \wp(2p|\tau) \right. \\ \quad \left. + 2\eta_1 \zeta(2p|\tau) - \frac{1}{3} p g_2(\tau) \right],$$

$$(2.86) \quad 2p(\tau) \frac{d\eta_1(\tau)}{d\tau} = \frac{i}{2\pi} p(\tau) \left[2\eta_1^2 - \frac{1}{6}g_2(\tau) \right].$$

Substituting (2.83), (2.85) and (2.86) into (2.84), we have

$$(2.87) \quad \begin{aligned} \frac{d^2p(\tau)}{d\tau^2} &= \frac{-1}{4\pi^2} \left(\wp'(2p|\tau) + \frac{1}{2} \sum_{k=0}^3 n_k (n_k + 1) \wp' \left(p + \frac{\omega_k}{2} | \tau \right) \right) \\ &= \frac{-1}{4\pi^2} \left[\frac{1}{8} \sum_{k=0}^3 \wp' \left(p + \frac{\omega_k}{2} | \tau \right) + \frac{1}{2} \sum_{k=0}^3 n_k (n_k + 1) \wp' \left(p + \frac{\omega_k}{2} | \tau \right) \right] \\ &= \frac{-1}{4\pi^2} \sum_{k=0}^3 \frac{1}{2} \left(n_k + \frac{1}{2} \right)^2 \wp' \left(p + \frac{\omega_k}{2} | \tau \right), \end{aligned}$$

implying that $p(\tau)$ is a solution of the elliptic form (1.3) with $\alpha_k = \frac{1}{2} (n_k + \frac{1}{2})^2$, $k = 0, 1, 2, 3$.

Conversely, suppose $p(\tau)$ is a solution of the elliptic form (1.3) with $\alpha_k = \frac{1}{2} (n_k + \frac{1}{2})^2$, $k = 0, 1, 2, 3$. We define $A(\tau)$ by the first equation of (2.83), i.e.,

$$(2.88) \quad A(\tau) \doteq 2\pi i \frac{dp(\tau)}{d\tau} + \frac{1}{2} (\zeta(2p|\tau) - 2p\eta_1(\tau)).$$

Then

$$\begin{aligned} \frac{dA(\tau)}{d\tau} &= 2\pi i \frac{d^2p(\tau)}{d\tau^2} + \frac{1}{2} \left(-2\wp(2p|\tau) \frac{dp(\tau)}{d\tau} + \frac{\partial}{\partial \tau} \zeta(2p|\tau) \right) \\ &\quad - \left(\eta_1(\tau) \frac{dp(\tau)}{d\tau} + p(\tau) \frac{d\eta_1(\tau)}{d\tau} \right), \end{aligned}$$

and by using (2.83), (2.85) and (2.86), we have

$$\begin{aligned} \frac{dA(\tau)}{d\tau} &= \frac{i}{4\pi} \left[2(\wp(2p|\tau) + \eta_1(\tau)) A - \frac{3}{2} \wp'(2p|\tau) \right. \\ &\quad \left. - \sum_{k=0}^3 n_k (n_k + 1) \wp' \left(p + \frac{\omega_k}{2} | \tau \right) \right]. \end{aligned}$$

Thus, $(p(\tau), A(\tau))$ is a solution to the Hamiltonian system (2.83). \square

Moreover, from (2.83), we could obtain the integral formula for $A(\tau)$ in Theorem 1.4.

Proof of Theorem 1.4. Let us consider $F(\tau) = A + \frac{1}{2} (\zeta(2p) - 2\zeta(p))$ and compute $\frac{d}{d\tau} F(\tau)$. By (2.83) and Lemma 2.3, we have

$$\begin{aligned} &\frac{d}{d\tau} F(\tau) \\ &= \frac{dA}{d\tau} + \frac{1}{2} \frac{d}{d\tau} (\zeta(2p) - 2\zeta(p)) \end{aligned}$$

$$\begin{aligned}
 &= \frac{i}{4\pi} \left(2(\wp(2p) + \eta_1(\tau))A - \frac{3}{2}\wp'(2p) - \sum_{k=0}^3 n_k(n_k + 1)\wp'(p(\tau) + \frac{\omega_k}{2}) \right) \\
 &\quad - (\wp(2p) - \wp(p)) \frac{dp}{d\tau} + \frac{1}{2} \left(\frac{\partial}{\partial \tau} \zeta(2p) - 2 \frac{\partial}{\partial \tau} \zeta(p) \right) \\
 &= \frac{i}{2\pi} (2\wp(2p) - \wp(p) + \eta_1) F(\tau) - \frac{i}{4\pi} \sum_{k=0}^3 n_k(n_k + 1)\wp'(p(\tau) + \frac{\omega_k}{2}).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (2.89) \quad &F(\tau) \\
 &= \exp \left\{ \frac{i}{2\pi} \int^\tau (2\wp(2p(\hat{\tau})|\hat{\tau}) - \wp(p(\hat{\tau})|\hat{\tau}) + \eta_1(\hat{\tau})) d\hat{\tau} \right\} \cdot J(\tau),
 \end{aligned}$$

where

$$J(\tau) = \int^\tau \frac{-\frac{i}{4\pi} \left(\sum_{k=0}^3 n_k(n_k + 1)\wp'(p(\hat{\tau}) + \frac{\omega_k}{2}|\hat{\tau}) \right)}{\exp \left\{ \frac{i}{2\pi} \int^{\hat{\tau}} (2\wp(2p(\tau')|\tau') - \wp(p(\tau')|\tau') + \eta_1(\tau')) d\tau' \right\}} d\hat{\tau} + c_1$$

for some constant $c_1 \in \mathbb{C}$. By Lemma 2.3, we have

$$(2.90) \quad \frac{3i}{4\pi} \int^\tau \eta_1(\hat{\tau}) d\hat{\tau} = \ln \theta'_1(\tau).$$

Then (1.20) follows from (2.89) and (2.90). \square

3. COLLAPSE OF TWO SINGULAR POINTS

In this section, we study the phenomena of collapsing two singular points $\pm p(\tau)$ to 0 in the generalized Lamé equation (1.13) when $p(\tau)$ is a solution of the elliptic form (1.3). As an application of Theorems 2.3 and 2.5, it turns out that the classical Lamé equation

$$(3.1) \quad y''(z) = (n(n+1)\wp(z) + B)y(z) \quad \text{in } E_\tau$$

appears as a limiting equation if $n_k = 0$ for $k = 1, 2, 3$ (see Theorem 3.1 below). First we recall the following classical result.

Theorem A. [15, Proposition 1.4.1] *Assume $\theta_4 = n_0 + \frac{1}{2} \neq 0$. Then for any $t_0 \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$, there exist two 1-parameter families of solutions $\lambda(t)$ of Painlevé VI (1.1) such that*

$$(3.2) \quad \lambda(t) = \frac{\beta}{t - t_0} + h + O(t - t_0) \quad \text{as } t \rightarrow t_0,$$

where $h \in \mathbb{C}$ can be taken arbitrary and

$$(3.3) \quad \beta = \beta(\theta_4, t_0) \in \left\{ \pm \frac{t_0(t_0 - 1)}{\theta_4} \right\}.$$

Furthermore, these two 1-parameter families of solutions give all solutions of Painlevé VI (1.1) which has a pole at t_0 .

In this paper, we always identify the solutions $p(\tau)$ and $-p(\tau)$ of the elliptic form (1.3). As a consequence of Theorem A and the transformation (1.2), we have the following result.

Lemma 3.1. *Assume $n_0 + \frac{1}{2} \neq 0$. Then for any $\tau_0 \in \mathbb{H}$, by the transformation (1.2) solutions $\lambda(t)$ in Theorem A give two 1-parameter families of solutions $p(\tau)$ of the elliptic form (1.3) such that*

$$(3.4) \quad p(\tau) = c_0(\tau - \tau_0)^{\frac{1}{2}}(1 + \tilde{h}(\tau - \tau_0) + O(\tau - \tau_0)^2) \text{ as } \tau \rightarrow \tau_0,$$

where $\tilde{h} \in \mathbb{C}$ can be taken arbitrary,

$$(3.5) \quad c_0^2 = \begin{cases} i \frac{n_0 + \frac{1}{2}}{\pi} & \text{if } \beta = -\frac{t_0(t_0-1)}{\theta_4} \\ -i \frac{n_0 + \frac{1}{2}}{\pi} & \text{if } \beta = \frac{t_0(t_0-1)}{\theta_4} \end{cases},$$

and $t_0 = t(\tau_0)$. Furthermore, these two 1-parameter families of solutions give all solutions $p(\tau)$ of the elliptic form (1.3) such that $p(\tau_0) = 0$.

Proof. It suffices to prove (3.5), which follows readily from

$$(3.6) \quad t'(\tau_0) = -i \frac{t_0(t_0-1)}{\pi} (e_2(\tau_0) - e_1(\tau_0)).$$

Remark that $t_0 \notin \{0, 1\}$, so (3.6) implies $t'(\tau_0) \neq 0$.

The formula (3.6) is known in the literature. Here we give a proof for the reader's convenience. Recalling theta functions $\vartheta_2(\tau), \vartheta_3(\tau)$ and $\vartheta_4(\tau)$, it is well-known that (cf. see [2] for a reference)

$$(3.7) \quad e_3(\tau) - e_2(\tau) = \pi^2 \vartheta_2(\tau)^4, \quad e_1(\tau) - e_3(\tau) = \pi^2 \vartheta_4(\tau)^4,$$

$$e_1(\tau) - e_2(\tau) = \pi^2 \vartheta_3(\tau)^4,$$

$$\frac{d}{d\tau} \ln \vartheta_4(\tau) = \frac{i}{12\pi} [3\eta_1(\tau) - \pi^2(2\vartheta_2(\tau)^4 + \vartheta_4(\tau)^4)],$$

$$\frac{d}{d\tau} \ln \vartheta_3(\tau) = \frac{i}{12\pi} [3\eta_1(\tau) + \pi^2(\vartheta_2(\tau)^4 - \vartheta_4(\tau)^4)].$$

Therefore, $t = \vartheta_4^4 / \vartheta_3^4$ and then

$$(3.8) \quad \begin{aligned} t'(\tau) &= 4t \left(\frac{d}{d\tau} \ln \vartheta_4 - \frac{d}{d\tau} \ln \vartheta_3 \right) = -i\pi t \cdot \vartheta_2^4 \\ &= -i\pi \frac{\vartheta_2^4 \vartheta_4^4}{\vartheta_3^4} = -i \frac{t(t-1)}{\pi} (e_2 - e_1). \end{aligned}$$

This completes the proof. \square

Theorem 3.1. *Assume that $n_k \notin \mathbb{Z} + \frac{1}{2}$, $k \in \{0, 1, 2, 3\}$, and (2.3) hold. Let $(p(\tau), A(\tau))$ be a solution of the Hamiltonian system (4.9) such that $p(\tau_0) = 0$ for some $\tau_0 \in \mathbb{H}$. Then*

$$(3.9) \quad p(\tau) = c_0(\tau - \tau_0)^{\frac{1}{2}}(1 + \tilde{h}(\tau - \tau_0) + O(\tau - \tau_0)^2) \text{ as } \tau \rightarrow \tau_0,$$

for some $\tilde{h} \in \mathbb{C}$ and the generalized Lamé equation (1.13) converges to

$$(3.10) \quad y'' = \left[\sum_{j=1}^3 n_j (n_j + 1) \wp \left(z + \frac{\omega_j}{2} \right) + m(m+1)\wp(z) + B_0 \right] y \text{ in } E_{\tau_0},$$

where c_0 is seen in (3.5),

$$(3.11) \quad m = \begin{cases} n_0 + 1 & \text{if } c_0^2 = i \frac{n_0 + \frac{1}{2}}{\pi} \text{ i.e., } \beta = -\frac{t_0(t_0-1)}{n_0 + \frac{1}{2}}, \\ n_0 - 1 & \text{if } c_0^2 = -i \frac{n_0 + \frac{1}{2}}{\pi} \text{ i.e., } \beta = \frac{t_0(t_0-1)}{n_0 + \frac{1}{2}}, \end{cases}$$

$$(3.12) \quad B_0 = 2\pi i c_0^2 \left(4\pi i \tilde{h} - \eta_1(\tau_0) \right) - \sum_{j=1}^3 n_j (n_j + 1) e_j(\tau_0).$$

Proof. Clearly (3.9) follows from Lemma 3.1, by which we have (write $p = p(\tau)$)

$$(\tau - \tau_0)^{\frac{1}{2}} = \frac{1}{c_0} p \left(1 - \frac{1}{c_0^2} \tilde{h} p^2 + O(p^4) \right) \text{ as } \tau \rightarrow \tau_0.$$

Consequently,

$$\begin{aligned} p'(\tau) &= \frac{1}{2} c_0 (\tau - \tau_0)^{-\frac{1}{2}} [1 + 3\tilde{h}(\tau - \tau_0) + O((\tau - \tau_0)^2)] \\ &= \frac{c_0^2}{2p} \left[1 + \frac{4}{c_0^2} \tilde{h} p^2 + O(p^4) \right] \text{ as } \tau \rightarrow \tau_0. \end{aligned}$$

This, together with the first equation of the Hamiltonian system (1.16)-(1.15), gives

$$(3.13) \quad \begin{aligned} A(\tau) &= \frac{1}{2} [4\pi i p'(\tau) + \zeta(2p(\tau)) - 2p(\tau)\eta_1(\tau)] \\ &= \frac{\pi i c_0^2}{p} \left[1 + \frac{4}{c_0^2} \tilde{h} p^2 + O(p^4) \right] + \frac{1}{4p} - \eta_1(\tau_0)p + O(p^3) \\ &= \frac{c}{p} + ep + O(p^3) \text{ as } \tau \rightarrow \tau_0, \end{aligned}$$

where $e = 4\pi i \tilde{h} - \eta_1(\tau_0)$ and

$$(3.14) \quad c = \pi i c_0^2 + \frac{1}{4} = \begin{cases} -n_0 - \frac{1}{4} & \text{if } c_0^2 = i \frac{n_0 + \frac{1}{2}}{\pi}, \\ n_0 + \frac{3}{4} & \text{if } c_0^2 = -i \frac{n_0 + \frac{1}{2}}{\pi}. \end{cases}$$

Clearly c satisfies

$$c^2 - \frac{c}{2} - \frac{3}{16} - n_0(n_0 + 1) = 0.$$

Consequently, we have

$$B(\tau) = A^2 - \zeta(2p) A - \frac{3}{4} \wp(2p) - \sum_{j=0}^3 n_j (n_j + 1) \wp \left(p + \frac{\omega_j}{2} \right)$$

$$\begin{aligned}
&= \left(\frac{c}{p} + ep + O(p^3) \right)^2 - \left(\frac{1}{2p} + O(p^3) \right) \left(\frac{c}{p} + ep + O(p^3) \right) \\
&\quad - \frac{\left(\frac{3}{16} + n_0(n_0 + 1) \right)}{p^2} - \sum_{j=1}^3 n_j (n_j + 1) e_j(\tau_0) + O(p^2) \\
&= \frac{4c-1}{2} e - \sum_{j=1}^3 n_j (n_j + 1) e_j(\tau_0) + O(p^2) = B_0 + O(p^2)
\end{aligned}$$

as $p = p(\tau) \rightarrow 0$ since $\tau \rightarrow \tau_0$, where B_0 is given by (3.12). Furthermore, (3.13) implies

$$A(\zeta(z+p) - \zeta(z-p)) = A(-2p\wp(z) + O(p^2)) \rightarrow -2c\wp(z)$$

uniformly for z bounded away from the lattice points. Therefore, the potential of the generalized Lamé equation (1.13) converges to

$$\sum_{j=1}^3 n_j (n_j + 1) \wp \left(z + \frac{\omega_j}{2} \right) + \left[n_0(n_0 + 1) + \frac{3}{2} - 2c \right] \wp(z) + B_0$$

uniformly for z bounded away from the lattice points as $\tau \rightarrow \tau_0$. Using (3.14) we easily obtain (3.10)-(3.11). \square

4. CORRESPONDENCE BETWEEN GENERALIZED LAMÉ EQUATION AND FUCHSIAN EQUATION

In this section, we want to establish a one to one correspondence between the generalized Lamé equation (1.13) and a type of Fuchsian equations on \mathbb{CP}^1 . After the correspondence, naturally we ask the question: *Is the isomonodromic deformation for the generalized Lamé equation in E_τ equivalent to the isomonodromic deformation for the corresponding Fuchsian equation on \mathbb{CP}^1 ?* Notice that we establish the correspondence by using the transformation $x = \frac{\wp(z) - e_1}{e_2 - e_1}$ (see (4.12) below) which is a double cover from E_τ onto \mathbb{CP}^1 . Hence, it is clear that the isomonodromic deformation for the Fuchsian equation could imply the the isomonodromic deformation for the generalized Lamé equation. However, the converse assertion is not easy at all, because the lifting of a closed loop in \mathbb{CP}^1 via $x = \frac{\wp(z) - e_1}{e_2 - e_1}$ is not necessarily a closed loop in E_τ . As an application of Theorems 2.3 and 2.5, we could give a positive answer.

First we review the Fuchs-Okamoto theory. Consider a second order Fuchsian equation defined on \mathbb{CP}^1 as follows:

$$(4.1) \quad y'' + p_1(x) y' + p_2(x) y = 0,$$

which has five regular singular points at $\{t, 0, 1, \infty, \lambda\}$ and $p_j(x) = p_j(x; t, \lambda, \mu)$, $j = 1, 2$, are rational functions in x such that the Riemann scheme of

(4.1) is

$$(4.2) \quad \begin{pmatrix} t & 0 & 1 & \infty & \lambda \\ 0 & 0 & 0 & \hat{\alpha} & 0 \\ \theta_t & \theta_0 & \theta_1 & \hat{\alpha} + \theta_\infty & 2 \end{pmatrix},$$

where $\hat{\alpha}$ is determined by the Fuchsian relation, that is,

$$\hat{\alpha} = -\frac{1}{2}(\theta_t + \theta_0 + \theta_1 + \theta_\infty - 1).$$

Throughout this section we always assume that

$$(4.3) \quad \lambda \notin \{0, 1, t\} \text{ and } \lambda \text{ is an apparent singular point.}$$

Since one exponent at any one of $0, 1, \lambda, t$ is 0 (see (4.2)), $p_2(x)$ has only simple poles at $0, 1, \lambda, t$. The residue of $p_1(x)$ at $x = \lambda$ is -1 because another exponent at $x = \lambda$ is 2. Define μ and K as follows:

$$(4.4) \quad \mu \doteq \operatorname{Res}_{x=\lambda} p_2(x), \quad K \doteq -\operatorname{Res}_{x=t} p_2(x).$$

By (4.2)-(4.4), we have

$$(4.5) \quad p_1(x) = \frac{1 - \theta_t}{x - t} + \frac{1 - \theta_0}{x} + \frac{1 - \theta_1}{x - 1} - \frac{1}{x - \lambda},$$

$$(4.6) \quad p_2(x) = \frac{\hat{\kappa}}{x(x-1)} - \frac{t(t-1)K}{x(x-1)(x-t)} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)},$$

where

$$(4.7) \quad \hat{\kappa} = \hat{\alpha}(\hat{\alpha} + \theta_\infty) = \frac{1}{4} \left\{ (\theta_0 + \theta_1 + \theta_t - 1)^2 - \theta_\infty^2 \right\}.$$

By the condition (4.3), i.e., λ is apparent, K can be expressed explicitly by

$$(4.8) \quad K(\lambda, \mu, t) = \frac{1}{t(t-1)} \left\{ \begin{array}{l} \lambda(\lambda-1)(\lambda-t)\mu^2 + \hat{\kappa}(\lambda-t) \\ - \left[\begin{array}{l} \theta_0(\lambda-1)(\lambda-t) + \theta_1\lambda(\lambda-t) \\ + (\theta_t-1)\lambda(\lambda-1) \end{array} \right] \mu \end{array} \right\}.$$

For all details about (4.5)-(4.8), we refer the reader to [15].

Now let t be the deformation parameter, and assume that (4.1) with $(\lambda(t), \mu(t))$ preserves the monodromy representation. In [7, 21], it was discovered that under the non-resonant condition, $(\lambda(t), \mu(t))$ must satisfy the following Hamiltonian system:

$$(4.9) \quad \frac{d\lambda(t)}{dt} = \frac{\partial K}{\partial \mu}, \quad \frac{d\mu(t)}{dt} = -\frac{\partial K}{\partial \lambda}.$$

Indeed, the following theorem was proved in [7, 21].

Theorem B.[7, 21] *Suppose that $\theta_t, \theta_0, \theta_1, \theta_\infty \notin \mathbb{Z}$ (i.e. the non-resonant condition) and λ is an apparent singular point. Then the second order ODE (4.1) preserves the monodromy as t deforms if and only if $(\lambda(t), \mu(t))$ satisfies the Hamiltonian system (4.9).*

It is well-known in the literature that a solution of Painlevé VI (1.1) can be obtained from the Hamiltonian system (4.9) with the Hamiltonian $K(\lambda, \mu, t)$ defined in (4.8). Let $(\lambda(t), \mu(t))$ be a solution to the Hamiltonian system (4.9). Then $\lambda(t)$ satisfies the Painlevé VI (1.1) with parameters

$$(4.10) \quad (\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}\theta_\infty^2, -\frac{1}{2}\theta_0^2, \frac{1}{2}\theta_1^2, \frac{1}{2}(1 - \theta_t^2) \right).$$

Conversely, if $\lambda(t)$ is a solution to Painlevé VI (1.1), then we define $\mu(t)$ by the first equation of (4.9), where $(\theta_0, \theta_1, \theta_t, \theta_\infty)$ and $\hat{\kappa}$ are given by (4.10) and (4.7), respectively. Consequently, $(\lambda(t), \mu(t))$ is a solution to (4.9). The above facts can be proved directly. For details, we refer the reader to [8, 15]. Together with this fact and Theorem B, we have

Theorem C. *Assume the same hypotheses of Theorem B. Then the second order ODE (4.1) preserves the monodromy as t deforms if and only if $\lambda(t)$ satisfies Painlevé VI (1.1) with parameters (4.10).*

Now let us consider the following generalized Lamé equation in E_τ :

$$(4.11) \quad y'' = \left[\begin{array}{l} \sum_{i=0}^3 n_i(n_i + 1) \wp(z + \frac{\omega_i}{2}) + \frac{3}{4}(\wp(z+p) + \wp(z-p)) \\ + A(\zeta(z+p) - \zeta(z-p)) + B \end{array} \right] y,$$

and suppose that p is an apparent singularity of (4.11). Then we shall prove that the generalized Lamé equation (4.11) is 1-1 correspondence to the 2nd order Fuchsian equation (4.1) with λ being an apparent singularity. To describe the 1-1 correspondence between (4.1) and (4.11), we set

$$(4.12) \quad x = \frac{\wp(z) - e_1}{e_2 - e_1} \text{ and } \mathbf{p}(x) = 4x(x-1)(x-t).$$

Then we have the following theorem:

Theorem 4.1. *Given a generalized Lamé equation (4.11) defined in E_τ . Suppose $p \notin E_\tau[2]$ is an apparent singularity of (4.11). Then by using $x = \frac{\wp(z) - e_1}{e_2 - e_1}$, there is a corresponding 2nd order Fuchsian equation (4.1) satisfying (4.2) and (4.3) whose coefficients $p_1(x)$ and $p_2(x)$ are expressed by (4.5)-(4.8), where*

$$(4.13) \quad t = \frac{e_3 - e_1}{e_2 - e_1}, \quad \lambda = \frac{\wp(p) - e_1}{e_2 - e_1},$$

$$(4.14) \quad (\theta_0, \theta_1, \theta_t, \theta_\infty) = \left(n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, n_3 + \frac{1}{2}, n_0 + \frac{1}{2} \right),$$

$$(4.15) \quad \hat{\alpha} = -\frac{1}{2}(1 + n_0 + n_1 + n_2 + n_3),$$

$$(4.16) \quad \mu = \frac{2n_3 - 1}{4(\lambda - t)} + \frac{2n_2 - 1}{4(\lambda - 1)} + \frac{2n_1 - 1}{4\lambda} + \frac{3\mathbf{p}'(\lambda)}{8\mathbf{p}(\lambda)} + \frac{A\wp'(p)}{b^2\mathbf{p}(\lambda)},$$

$$(4.17) \quad K = -\frac{2n_2n_3 - n_2 - n_3}{4(t-1)} - \frac{2n_1n_3 - n_1 - n_3}{4t} - \frac{2n_3 - 1}{4(t-\lambda)} \\ + \frac{1}{4t(t-1)} \left[\begin{array}{l} \frac{3}{2} \frac{\lambda(\lambda-1)}{(\lambda-t)} - \frac{3}{2} \frac{\wp(p)+e_3}{e_2-e_1} + \frac{A\wp'(p)}{(\lambda-t)(e_2-e_1)^2} \\ + \frac{n_0(n_0+1)e_3}{e_2-e_1} + \frac{n_1(n_1+1)e_2}{e_2-e_1} + \frac{n_2(n_2+1)e_1}{e_2-e_1} \\ - \frac{2n_3(n_3+1)e_3}{e_2-e_1} + \frac{2}{e_2-e_1} A\wp(p) + \frac{B}{e_2-e_1} \end{array} \right].$$

Conversely, given a 2nd order Fuchsian equation (4.1) satisfying (4.2) and (4.3), there is a corresponding generalized Lamé equation (4.11) defined in E_τ where $\tau, \pm p, n_i$ are defined by (4.13)-(4.14), the constant A is defined by solving (4.16), and the constant B is defined by (2.3). In particular, p is an apparent singularity of (4.11).

Remark 4.1. For the second part of Theorem 4.1, the condition $\lambda \notin \{0, 1, t\}$ is equivalent to $p \notin E_\tau[2]$, which implies $\wp'(p) \neq 0$. Thus, A is well-defined via (4.16). The proof of Theorem 4.1 will be given after Corollary 4.1.

Let $\rho : \pi_1(E_\tau \setminus (E_\tau[2] \cup \{\pm p\}), q_0) \rightarrow SL(2, \mathbb{C})$, $\tilde{\rho} : \pi_1(\mathbb{CP}^1 \setminus \{0, 1, t, \infty\}, \lambda_0) \rightarrow GL(2, \mathbb{C})$ where $\lambda_0 = x(q_0)$, be the monodromy representations of the generalized Lamé equation (4.11) and the corresponding Fuchsian equation (4.1) respectively. Let $Y(z) = (y_1(z), y_2(z))$ be a fixed fundamental solution of (4.11). We denote \mathcal{N} and \mathcal{M} to be the monodromy groups of (4.11) and (4.1) with respect to $Y(z)$ and $\hat{Y}(x)$ respectively. Here $\hat{Y}(x) = (\hat{y}_1(x), \hat{y}_2(x))$ with $\hat{y}_j(x)$ defined by

$$y_j(z) = \psi(x)\hat{y}_j(x) \\ \doteq (x-\lambda)^{-\frac{1}{2}} x^{-\frac{n_1}{2}} (x-1)^{-\frac{n_2}{2}} (x-t)^{-\frac{n_3}{2}} \hat{y}_j(x), \quad j = 1, 2,$$

and $x = \frac{\wp(z)-e_1}{e_2-e_1}$ is a fundamental solution of equation (4.1); see the proof of Theorem 4.1 below. Let $\gamma_1 \in \pi_1(E_\tau \setminus (E_\tau[2] \cup \{\pm p\}), q_0)$ be a loop which encircles the singularity $\frac{\omega_1}{2}$ once. Then $x(\gamma_1) \in \pi_1(\mathbb{CP}^1 \setminus \{0, 1, t, \infty\}, \lambda_0)$. Since $x = \frac{\wp(z)-e_1}{e_2-e_1}$ is a double cover, the loop $x(\gamma_1)$ encircles the singularity 0 twice. Thus, $x(\gamma_1) = \beta^2$ for some $\beta \in \pi_1(\mathbb{CP}^1 \setminus \{0, 1, t, \infty\}, \lambda_0)$. Let $\rho(\gamma_1) = N_1$ and $\tilde{\rho}(\beta) = M_0$. Then

$$(4.18) \quad Y(z)N_1 = \gamma_1^* Y(z) = (\beta^2)^* (\psi(x)\hat{Y}(x)) \\ = C(\beta^2) \psi(x)\hat{Y}(x)M_0^2 \\ = Y(z)C(\beta^2)M_0^2$$

for some constant $C(\beta^2) \in \mathbb{C}$ which comes from the analytic continuation of $\psi(x)$ along β^2 . From (4.18), we see that $N_1 = C(\beta^2)M_0^2$. By the same argument, we know that any element $N \in \mathcal{N}$ could be written as

$$(4.19) \quad N = CM_1M_2$$

for some $M_i \in \mathcal{M}$, $i = 1, 2$ and some constant $C \in \mathbb{C}$ coming from the gauge transformation $\psi(x)$. In general, \mathcal{N} is not a subgroup of \mathcal{M} because of $\psi(x)$. By (4.19), the isomonodromic deformation of (4.1) implies the isomonodromic deformation of (4.11). However, it is not clear to see whether the converse assertion is true or not from (4.19). Here we can give a confirmative answer. In fact, by (1.14), (1.4) and (4.10), we have (4.14) holds. Since $n_i \notin \frac{1}{2} + \mathbb{Z}$ for $i \in \{0, 1, 2, 3\}$, we have $\theta_0, \theta_1, \theta_t, \theta_\infty \notin \mathbb{Z}$, i.e., non-resonant. Then as a consequence of Theorems 4.1, 1.1 and C, we have

Corollary 4.1. *Suppose $n_i \notin \frac{1}{2} + \mathbb{Z}$ for $i = 0, 1, 2, 3$. If the generalized Lamé equation (4.11) in E_τ reverses the monodromy, then so does the corresponding Fuchsian equation (4.1) on \mathbb{CP}^1 , and vice versa.*

Proof of Theorem 4.1. Let us first consider the generalized Lamé equation (4.11). By applying

$$x = \frac{\wp(z) - e_1}{e_2 - e_1}, \quad t = \frac{e_3 - e_1}{e_2 - e_1}, \quad \lambda = \frac{\wp(p) - e_1}{e_2 - e_1},$$

and the addition formula

$$(4.20) \quad \wp(z+p) + \wp(z-p) = \frac{\wp'(z)^2 + \wp'(p)^2}{2(\wp(z) - \wp(p))^2} - 2\wp(z) - 2\wp(p),$$

the equation (4.11) becomes the following second order Fuchsian equation defined on \mathbb{CP}^1 :

$$(4.21) \quad y''(x) + \frac{1}{2} \frac{\mathbf{p}'(x)}{\mathbf{p}(x)} y'(x) - \frac{q(x)}{\mathbf{p}(x)} y(x) = 0,$$

where $\mathbf{p}(x)$ is defined in (4.12), $b \doteq e_2 - e_1$ and

$$q(x) = \left[\begin{array}{l} n_0(n_0+1) \left(x + \frac{e_1}{b} \right) + \frac{n_1(n_1+1)}{2} \left(\frac{\mathbf{p}(x)}{2x^2} - 2x + \frac{4e_1}{b} \right) + \frac{B}{b} \\ + \frac{n_2(n_2+1)}{2} \left(\frac{\mathbf{p}(x)}{2(x-1)^2} - 2x + \frac{2e_3}{b} \right) + \frac{n_3(n_3+1)}{2} \left(\frac{\mathbf{p}(x)}{2(x-t)^2} - 2x + \frac{2e_2}{b} \right) \\ + \frac{3}{4} \left(\frac{\mathbf{p}(x)+\mathbf{p}(\lambda)}{2(x-\lambda)^2} - 2x - \frac{2}{b}(\wp(p) + e_1) \right) + A \left(\frac{2}{b}\zeta(p) - \frac{\wp'(p)}{b^2(x-\lambda)} \right) \end{array} \right].$$

Since p is an apparent singularity of (4.11), equation (4.21) has no logarithmic solutions at λ . The Riemann scheme of (4.21) is as follows

$$(4.22) \quad \left(\begin{array}{ccccc} 0 & 1 & t & \infty & \lambda \\ -\frac{n_1}{2} & -\frac{n_2}{2} & -\frac{n_3}{2} & -\frac{n_0}{2} & -\frac{1}{2} \\ \frac{n_1+1}{2} & \frac{n_2+1}{2} & \frac{n_3+1}{2} & \frac{n_0+1}{2} & \frac{3}{2} \end{array} \right).$$

Now consider a gauge transformation $y(x) = (x-\lambda)^{-\frac{1}{2}} x^{-\frac{n_1}{2}} (x-1)^{-\frac{n_2}{2}} (x-t)^{-\frac{n_3}{2}} \hat{y}(x)$. Then the Riemann scheme for $\hat{y}(x)$ is

$$(4.23) \quad \left(\begin{array}{ccccc} 0 & 1 & t & \infty & \lambda \\ 0 & 0 & 0 & \hat{\alpha} & 0 \\ n_1 + \frac{1}{2} & n_2 + \frac{1}{2} & n_3 + \frac{1}{2} & \hat{\alpha} + n_0 + \frac{1}{2} & 2 \end{array} \right),$$

where $\hat{\alpha} = -\frac{1}{2}(1 + n_0 + n_1 + n_2 + n_3)$. Moreover, $\hat{y}(x)$ satisfies the second order Fuchsian equation

$$(4.24) \quad \hat{y}''(x) + \hat{p}_1(x, t) \hat{y}'(x) + \hat{p}_2(x, t) \hat{y}(x) = 0,$$

where

$$(4.25) \quad \hat{p}_1(x, t) = \frac{\frac{1}{2} - n_1}{x} + \frac{\frac{1}{2} - n_2}{x-1} + \frac{\frac{1}{2} - n_3}{x-t} - \frac{1}{x-\lambda}$$

and

$$(4.26) \quad \begin{aligned} \hat{p}_2(x, t) = & \frac{3}{4(x-\lambda)^2} \\ & + \frac{2n_1n_2 - n_1 - n_2}{4x(x-1)} + \frac{2n_2n_3 - n_2 - n_3}{4(x-1)(x-t)} + \frac{2n_1n_3 - n_1 - n_3}{4x(x-t)} \\ & + \frac{2n_1 - 1}{4x(x-\lambda)} + \frac{2n_2 - 1}{4(x-1)(x-\lambda)} + \frac{2n_3 - 1}{4(x-t)(x-\lambda)} \\ & - \frac{1}{\mathfrak{p}(x)} \left[\begin{array}{l} n_0(n_0+1)\left(x + \frac{e_1}{b}\right) - n_1(n_1+1)\left(x + \frac{2e_1}{b}\right) \\ - n_2(n_2+1)\left(x - \frac{e_3}{b}\right) - n_3(n_3+1)\left(x - \frac{e_2}{b}\right) \\ + \frac{3}{4} \left(\frac{\mathfrak{p}(x) + \mathfrak{p}(\lambda)}{2(x-\lambda)^2} - 2x - 2\frac{\wp(p) + e_1}{b} \right) \\ + A \left(\frac{2}{b} \zeta(p) - \frac{\wp'(p)}{b^2(x-\lambda)} \right) + \frac{B}{b} \end{array} \right]. \end{aligned}$$

Since $p \notin E_\tau$ [2] and equation (4.21) has no logarithmic solutions at λ , it follows that $\lambda \notin \{0, 1, t, \infty\}$ and λ is an apparent singularity of (4.24). Thus, $\hat{p}_2(x, t)$ can be written into the form of (4.6) with

$$(4.27) \quad \hat{\kappa} = -\frac{1}{4}(n_0 - n_1 - n_2 - n_3)(1 + n_0 + n_1 + n_2 + n_3),$$

$$(4.28) \quad \begin{aligned} \mu = & \operatorname{Res}_{x=\lambda} \hat{p}_2(x, t) \\ = & \frac{2n_3 - 1}{4(\lambda - t)} + \frac{2n_2 - 1}{4(\lambda - 1)} + \frac{2n_1 - 1}{4\lambda} + \frac{3\mathfrak{p}'(\lambda)}{8\mathfrak{p}(\lambda)} + \frac{A\wp'(p)}{b^2\mathfrak{p}(\lambda)}, \end{aligned}$$

$$K = -\operatorname{Res}_{x=t} \hat{p}_2(x, t) = \tilde{K},$$

where

$$(4.29) \quad \begin{aligned} \tilde{K} \doteq & -\frac{2n_2n_3 - n_2 - n_3}{4(t-1)} - \frac{2n_1n_3 - n_1 - n_3}{4t} - \frac{2n_3 - 1}{4(t-\lambda)} \\ & + \frac{1}{4t(t-1)} \left[\begin{array}{l} \frac{3}{2} \frac{\lambda(\lambda-1)}{(\lambda-t)} - \frac{3}{2} \frac{\wp(p) + e_3}{b} + \frac{A\wp'(p)}{(\lambda-t)b^2} \\ + \frac{n_0(n_0+1)e_3}{b} + \frac{n_1(n_1+1)e_2}{b} + \frac{n_2(n_2+1)e_1}{b} \\ - \frac{2n_3(n_3+1)e_3}{b} + \frac{2}{b} A\wp(p) + \frac{B}{b} \end{array} \right]. \end{aligned}$$

Since λ is an apparent singularity, we conclude from (4.8) that

$$(4.30) \quad \tilde{K} = \frac{1}{t(t-1)} \left\{ \begin{array}{l} \lambda(\lambda-1)(\lambda-t)\mu^2 + \hat{\kappa}(\lambda-t) \\ - \left[\begin{array}{l} \theta_0(\lambda-1)(\lambda-t) + \theta_1\lambda(\lambda-t) \\ + (\theta_t-1)\lambda(\lambda-1) \end{array} \right] \mu \end{array} \right\}.$$

Conversely, for a given second order Fuchsian equation (4.1) satisfying (4.2) and (4.3), we know that $p_1(x)$, $p_2(x)$ and K are given by (4.5)-(4.8), where

$$(4.31) \quad \hat{\kappa} = \hat{\alpha} (\hat{\alpha} + \theta_\infty).$$

Define $\pm p$, n_i ($i = 0, 1, 2, 3$), A , and B by

$$(4.32) \quad \lambda = \frac{\wp(p) - e_1}{e_2 - e_1},$$

$$(4.33) \quad (\theta_0, \theta_1, \theta_t, \theta_\infty) = \left(n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, n_3 + \frac{1}{2}, n_0 + \frac{1}{2} \right),$$

$$(4.34) \quad \mu = \frac{2n_3 - 1}{4(\lambda - t)} + \frac{2n_2 - 1}{4(\lambda - 1)} + \frac{2n_1 - 1}{4\lambda} + \frac{3\mathfrak{p}'(\lambda)}{8\mathfrak{p}(\lambda)} + \frac{A\wp'(p)}{b^2\mathfrak{p}(\lambda)},$$

and

$$(4.35) \quad B = A^2 - \zeta(2p)A - \frac{3}{4}\wp(2p) - \sum_{i=0}^3 n_i(n_i + 1)\wp\left(p + \frac{\omega_i}{2}\right).$$

Since $\lambda \notin \{0, 1, t, \infty\}$, $p \notin E_\tau[2]$. Thus $\wp'(p) \neq 0$ and A is well-defined by (4.34). In order to obtain the corresponding generalized Lamé equation (4.11), it suffices to prove that $p_1(x, t)$ and $p_2(x, t)$ can be expressed in the form of (4.25) and (4.26). By (4.5) and (4.33), it is easy to see that $p_1(x, t)$ is of the form (4.25). By (4.29) and (4.30), we see that K can be written into (4.29), so $p_2(x, t)$ can also be expressed in the form of (4.26). Finally, the assertion that p is an apparent singularity follows from the assumption that λ is an apparent singularity of (4.1) (or follows from (4.35) and Lemma 2.1). This completes the proof. \square

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