Self-dual radial non-topological solutions to a competitive Chern-Simons model *

Zhijie Chen[1], **Chang-Shou Lin**[2]

[1] *Department of Mathematical Sciences, Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China*

[2] *Taida Institute for Mathematical Sciences, Center for Advanced Study in Theoretical Sciences, National Taiwan University, Taipei 106, Taiwan*

Abstract

We study a non-Abelian Chern-Simons system of rank 2:

$$\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}+K\begin{pmatrix}e^{u_1}\\ e^{u_2}\end{pmatrix}-K\begin{pmatrix}e^{u_1} & 0\\ 0 & e^{u_2}\end{pmatrix}K\begin{pmatrix}e^{u_1}\\ e^{u_2}\end{pmatrix}=\begin{pmatrix}4\pi N_1\delta_0\\ 4\pi N_2\delta_0\end{pmatrix}\ \text{ in } \mathbb{R}^2,$$

where $N_1, N_2 \in \mathbb{N}\cup\{0\}$, δ_0 is the Dirac measure at 0, and $K=(a_{ij})$ is a 2×2 matrix satisfying $a_{11}, a_{22} > 0$, $a_{12}, a_{21} < 0$ and $\det K > 0$, including the Cartan matrix $\mathbf{B}_2$. The existence of non-topological solutions has remained a long-standing open problem. Here by applying the degree theory, we prove the existence of radial non-topological solutions (u_1, u_2) satisfying the prescribed asymptotic condition $u_k(x) = -2\alpha_k \ln|x| + O(1)$ as $|x|\to\infty$ for some $\alpha_k > 1$. We also construct bubbling solutions to show that the range of (α_1, α_2) is optimal in some sense. This generalizes a recent work by Choe, Kim and the second author, where the $SU(3)$ case (i.e. K is the Cartan matrix $\mathbf{A}_2$) was investigated.

1 Introduction

In this paper, we study a non-Abelian Chern-Simons system of rank 2:

$$\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}+K\begin{pmatrix}e^{u_1}\\ e^{u_2}\end{pmatrix}-K\begin{pmatrix}e^{u_1} & 0\\ 0 & e^{u_2}\end{pmatrix}K\begin{pmatrix}e^{u_1}\\ e^{u_2}\end{pmatrix}=\begin{pmatrix}4\pi N_1\delta_0\\ 4\pi N_2\delta_0\end{pmatrix}\ \text{ in } \mathbb{R}^2, \tag{1.1}$$

where $N_1, N_2 \in \mathbb{N}\cup\{0\}$, δ_0 is the Dirac measure at 0, and $K=(a_{ij})$ is a 2×2 matrix satisfying

$$a_{11}, a_{22} > 0,\ \ a_{12}, a_{21} < 0 \ \text{ and } \ a_{11}a_{22} - a_{12}a_{21} > 0. \tag{1.2}$$

System (1.1) could be considered as a perturbation of the following Liouville system

$$\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}+\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}e^{u_1}\\ e^{u_2}\end{pmatrix}=\begin{pmatrix}4\pi N_1\delta_0\\ 4\pi N_2\delta_0\end{pmatrix}\ \text{ in } \mathbb{R}^2. \tag{1.3}$$

See [1, 2, 3]. Under the assumption (1.2), system (1.3) is also called *competitive* in the literature, comparing to the *cooperative* case where $a_{12}, a_{21} > 0$.

System (1.1) is motivated by various self-dual gauge field theories in physics. Our first motivation comes from the relativistic non-Abelian Chern-Simons model, which was proposed

*E-mail address: zjchen2016@tsinghua.edu.cn, zjchen@math.tsinghua.edu.cn (Chen); cslin@math.ntu.edu.tw (Lin)

by Kao and Lee [19] and Dunne [9, 10] to explain the physics of high critical temperature superconductivity. Following [9, 10], the relativistic non-Abelian Chern-Simons model is defined in the $(2+1)$ Minkowski space $\mathbb{R}^{1,2}$, and the gauge group is a compact Lie group with a semi-simple Lie algebra $\mathcal{G}$. The Chern-Simons Lagrangian action density $\mathcal{L}$ in $2+1$ dimensional spacetime involves the Higgs field ϕ and the gauge potential $A=(A_0,A_1,A_2)$. We consider the energy minimizers of the Lagrangian functional, which turn out to be the solutions of the following self-dual Chern-Simons equations:

$$\begin{aligned} &D_-\phi=0,\\ &F_{+-}=\frac{1}{\kappa^2}[\phi-[[\phi,\phi^\dagger],\phi],\phi^\dagger], \end{aligned} \tag{1.4}$$

where $D_-=D_1-iD_2$, $\kappa>0$, $F_{+-}=\partial_+A_--\partial_-A_++[A_+,A_-]$ with $A_\pm=A_1\pm iA_2$, $\partial_\pm=\partial_1\pm i\partial_2$ and $[\cdot,\cdot]$ is the Lie bracket over $\mathcal{G}$. In [10], Dunne considered a simplified form of the self-dual system (1.4), in which the fields ϕ and A are algebraically restricted:

$$\phi=\sum_{a=1}^r\phi^aE_a,$$

where r is the rank of the gauge Lie algebra, E_a is the simple root step operator, and ϕ^a are complex-valued functions. Let $u_a=\ln|\phi^a|, a=1,\cdots,r$. Then system (1.4) can be reduced to the following system of nonlinear partial differential equations

$$\Delta u_a+\frac{1}{\kappa^2}\left(\sum_{b=1}^rK_{ab}e^{u_b}-\sum_{b=1}^r\sum_{c=1}^re^{u_b}K_{ab}e^{u_c}K_{bc}\right)=4\pi\sum_{j=1}^{N_a}\delta_{p_j^a},\ 1\le a\le r, \tag{1.5}$$

where $K=(K_{ab})$ is the Cartan matrix of a semi-simple Lie algebra, $\{p_j^a\}_{j=1,\cdots,N_a}$ are zeros of ϕ^a $(a=1,\cdots,r)$, and δ_p denotes the Dirac measure concentrated at p in $\mathbb{R}^2$. See [27] for the derivation of (1.5) from (1.4). For example, there are three types of Cartan matrix of rank 2:

$$\mathbf{A}_2(\text{i.e. } SU(3))=\begin{pmatrix}2&-1\\-1&2\end{pmatrix},\ \mathbf{B}_2=\begin{pmatrix}2&-1\\-2&2\end{pmatrix},\ \mathbf{G}_2=\begin{pmatrix}2&-1\\-3&2\end{pmatrix}. \tag{1.6}$$

Let $(K^{-1})_{ab}$ denote the inverse of the matrix K. Assume that

$$\sum_{b=1}^r(K^{-1})_{ab}>0,\quad a=1,2,\cdots,r. \tag{1.7}$$

A solution $\mathbf{u}=(u_1,\cdots,u_r)$ of (1.5) is called a *topological solution* if

$$u_a(x)\to\ln\left(\sum_{b=1}^r(K^{-1})_{ab}\right)\quad\text{as } |x|\to+\infty,\quad a=1,\cdots,r,$$

a solution $\mathbf{u}$ is called a *non-topological solution* if

$$u_a(x)\to-\infty\quad\text{as } |x|\to+\infty,\quad a=1,\cdots,r.$$

The existence of topological solutions of system (1.5) either in $\mathbb{R}^2$ or a flat torus has been well investigated in the last decades. In 1997, Yang [27] proved the existence of topological solutions to (1.5) for any configuration p_j^a in $\mathbb{R}^2$ via variational methods. Later, for a flat torus with doubly periodic boundary conditions, Nolasco and Tarantello [23] studied system (1.5) with the Cartan matrix $\mathbf{A}_2$ and obtained the existence of a topological solution and a mountain-pass type solution. Recently, Han and Tarantello [14] generalized the results of [23] to the more general competitive case where K is a 2×2 matrix satisfying (1.2). Their proof [14] is based on variational methods and does not seem to work in the cooperative case. This indicates that the cooperative case is generally different from the competitive case. See also [13] for similar results to the Gudnason model.

On the other hand, the existence of non-topological solutions (and mixed-type solutions, see below) seems much more difficult than topological solutions to obtain, and there are very few results concerning the existence of non-topological solutions (and mixed-type solutions) in the literature. Very recently, some existence results of non-topological solutions to (1.5) with $K = \mathbf{A}_2, \mathbf{B}_2$ and $\mathbf{G}_2$ have been proved by Ao, Lin and Wei [1, 2] by a perturbation from the corresponding Liouville system (1.3). However, their results are still very limited toward understanding the general theory of non-topological solutions.

In this paper, we focus on the radially symmetric solutions of (1.5) when all the vortices coincide at the origin. We only consider the rank 2 and competitive case, namely K is a 2×2 matrix satisfying (1.2). Moreover, we may assume, without loss of generality, that $\kappa = 1$. Then system (1.5) turns to be (1.1). In particular, when $K = \mathbf{A}_2$, (1.1) becomes the following $SU(3)$ Chern-Simons system

$$\begin{cases} \Delta u_1 + 2(e^{u_1} - 2e^{2u_1} + e^{u_1+u_2}) - (e^{u_2} - 2e^{2u_2} + e^{u_1+u_2}) = 4\pi N_1\delta_0 \\ \Delta u_2 + 2(e^{u_2} - 2e^{2u_2} + e^{u_1+u_2}) - (e^{u_1} - 2e^{2u_1} + e^{u_1+u_2}) = 4\pi N_2\delta_0 \end{cases} \text{in } \mathbb{R}^2. \tag{1.8}$$

Recently, Huang and Lin made classifications of radially symmetric solutions for the $SU(3)$ system (1.8) in [16] and for the more general system (1.1) in [17] respectively. As in [17], in order to simplify the expression of system (1.1), we consider the transformation

$$(u_1, u_2) \to \left(u_1 + \ln\frac{a_{22} - a_{12}}{a_{11}a_{22} - a_{12}a_{21}},\ u_2 + \ln\frac{a_{11} - a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \right)$$

and let

$$(a_1, a_2) = \left(\frac{-a_{12}(a_{11} - a_{21})}{a_{11}a_{22} - a_{12}a_{21}},\ \frac{-a_{21}(a_{22} - a_{12})}{a_{11}a_{22} - a_{12}a_{21}} \right).$$

Clearly the assumption (1.2) gives $a_1 > 0$ and $a_2 > 0$. Then system (1.1) becomes

$$\begin{cases} \Delta u_1 + (1+a_1)(e^{u_1} - (1+a_1)e^{2u_1} + a_1e^{u_1+u_2}) \\ \qquad - a_1(e^{u_2} - (1+a_2)e^{2u_2} + a_2e^{u_1+u_2}) = 4\pi N_1\delta_0 \\ \Delta u_2 + (1+a_2)(e^{u_2} - (1+a_2)e^{2u_2} + a_2e^{u_1+u_2}) \\ \qquad - a_2(e^{u_1} - (1+a_1)e^{2u_1} + a_1e^{u_1+u_2}) = 4\pi N_2\delta_0 \end{cases} \text{in } \mathbb{R}^2. \tag{1.9}$$

Remark that the above transformation is just the identity for the case $K = \mathbf{A}_2$.

The main goal of this paper is to study the existence of non-topological solutions of the competitive Chern-Simons system (1.1) by seeking non-topological solutions of system (1.9). Therefore, in the sequel, we only need to consider system (1.9). As in [17], we easily see that a solution (u_1, u_2) of (1.9) is a *topological solution* if $(u_1, u_2) \to (0, 0)$ as $|x| \to +\infty$; a *non-topological solution* if $(u_1, u_2) \to (-\infty, -\infty)$ as $|x| \to +\infty$; a *mixed-type solution* if $(u_1, u_2) \to (\ln\frac{1}{1+a_1}, -\infty)$ or $(u_1, u_2) \to (-\infty, \ln\frac{1}{1+a_2})$ as $|x| \to +\infty$.

It is worth to point out that system (1.9) has also applications to the Lozano-Marqués-Moreno-Schaposnik model [22] of bosonic sector of $\mathcal{N} = 2$ supersymmetric Chern-Simons-Higgs theory, and the Gudnason model [11, 12] of $\mathcal{N} = 2$ supersymmetric Yang-Mills-Chern-Simons-Higgs theory. We refer the reader to [17] for details on these applications.

Define a continuous function $J : \mathbb{R}^2 \to \mathbb{R}$ by

$$J(x, y) = \frac{a_2(1+a_2)}{2}x^2 + a_1a_2xy + \frac{a_1(1+a_1)}{2}y^2. \tag{1.10}$$

In [17], among other things, Huang and Lin proved the following interesting result.

Theorem A. [17] *Let $a_1, a_2 > 0$. Suppose that $(u_1, u_2) \neq (0, 0)$ is a radially symmetric solution of system* (1.9)*. Then both $u_1 < 0$ and $u_2 < 0$ in $\mathbb{R}^2$, and one of the following conclusions holds.*

(i) (u_1, u_2) *is a topological solution.*

(ii) (u_1, u_2) *is a mixed-type solution.*

(iii) (u_1, u_2) *is a non-topological solution and there exist constants* $\alpha_1, \alpha_2 > 1$ *such that*

$$u_j(x) = -2\alpha_j \ln |x| + O(1) \quad as\ |x| \to +\infty, \quad j = 1, 2. \tag{1.11}$$

Consequently, $e^{u_1}, e^{u_2} \in L^1(\mathbb{R}^2)$. *Moreover,* (α_1, α_2) *satisfies*

$$J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1). \tag{1.12}$$

Remark that the inequality (1.12) follows from the following Pohozaev identity (see [17] or Lemma 2.2 below):

$$\begin{aligned} & J(\alpha_1 - 1, \alpha_2 - 1) - J(N_1 + 1, N_2 + 1) \\ = & \frac{1 + a_1 + a_2}{4} \int_0^\infty r \left[a_2(1 + a_1)e^{2u_1} + a_1(1 + a_2)e^{2u_2} - 2a_1a_2e^{u_1+u_2} \right] dr. \end{aligned} \tag{1.13}$$

Therefore, (1.12) is a necessary condition for the existence of radially symmetric non-topological solutions satisfying the asymptotic condition (1.11). After Theorem A, it is natural to consider the following long-standing open question (see [27, Section 8] for instance).

Open Question: Let $a_1, a_2 > 0$. Given $\alpha_1, \alpha_2 > 1$ satisfying (1.12). Is there a radial non-topological solution of system (1.9) subject to the asymptotic condition (1.11)?

If we let $N_1 = N_2 = N$, $a_1 = a_2$ and $u_1 = u_2 = u$ in (1.9), then system (1.9) turns to be the following Chern-Simons-Higgs equation

$$\Delta u + e^u(1 - e^u) = 4\pi N \delta_0 \quad \text{in } \mathbb{R}^2. \tag{1.14}$$

Equation (1.14) is known as the $SU(2)$ Chern-Simons equation for the Abelian case, which was proposed by Hong, Kim and Pac [15] and by Jakiw and Weinberg [18] independently. In the past twenty years, the topological solutions and non-topological solutions of (1.14) have been well studied; see [3, 4, 5, 6, 7, 25, 26] and references therein. Remark that the Pohozaev identity plays an important role in studying non-topological solutions of (1.14). Let u be a radially symmetric non-topological solution of (1.14) and satisfies $u(x) = -2\alpha \ln |x| + O(1)$ near ∞. Then the Pohozaev identity gives

$$(\alpha - 1)^2 - (N + 1)^2 = \frac{1}{2} \int_0^\infty re^{2u} dr > 0,$$

which implies $\alpha > N + 2$. In 2002, Chan, Fu and Lin [4] proved that the inequality $\alpha > N + 2$ is also a sufficient condition for the existence of radial non-topological solutions u with $u(x) = -2\alpha \ln |x| + O(1)$ near ∞. However, as pointed out in [8], this might not hold for our problem (1.9) with the asymptotic condition (1.11). The reason is following: there might be a sequence of solutions $(u_{1,n}, u_{2,n})$ such that only one component blows up, but the other does not, i.e. the so-called phenomena of *partial blowup*; see Theorems C and 1.3 for instance. As a result, only one of the L^1 norms of $e^{2u_{1,n}}$ and $e^{2u_{2,n}}$ tends to 0 as $n \to \infty$, which implies that the quantity $J(\alpha_1 - 1, \alpha_2 - 1) - J(N_1 + 1, N_2 + 1)$ might not converge to 0, namely it has a gap. Therefore, roughly speaking, the inequality (1.12) might *not* be a sufficient condition for the existence of radial non-topological solutions satisfying (1.11).

Define $J_{\mathbf{A}_2}(x, y) = x^2 + xy + y^2$ for the $SU(3)$ Chern-Simons system (1.8). Recently, Choe, Kim and the second author [8] gave a sufficient condition for the existence of non-topological solutions to system (1.8) subject to the asymptotic condition (1.11).

Theorem B. [8] *Let N_1, N_2 be non-negative integers. Define*

$$S_{\mathbf{A}_2} = \left\{ (\alpha_1, \alpha_2) \;\middle|\; \begin{array}{l} -2N_1 - N_2 - 3 < \alpha_2 - \alpha_1 < N_1 + 2N_2 + 3 \\ 2\alpha_1 + \alpha_2 > N_1 + 2N_2 + 6 \\ \alpha_1 + 2\alpha_2 > 2N_1 + N_2 + 6 \end{array} \right\}. \tag{1.15}$$

Then

$$S_{\mathbf{A}_2} \subset \Omega_{\mathbf{A}_2} = \left\{ (\alpha_1, \alpha_2) \;\middle|\; \begin{array}{l} \alpha_1, \alpha_2 > 1 \\ J_{\mathbf{A}_2}(\alpha_1 - 1, \alpha_2 - 1) > J_{\mathbf{A}_2}(N_1 + 1, N_2 + 1) \end{array} \right\}, \tag{1.16}$$

and for each fixed $(\alpha_1, \alpha_2) \in S_{\mathbf{A}_2}$, system (1.8) *has a radially symmetric non-topological solution (u_1, u_2) subject to the asymptotic condition* (1.11).

In [8], Theorem B was established via the Leray-Schauder degree theory. To apply the degree theory, they made a deformation from (1.8) to the $SU(3)$ Chern-Simons system without singular sources (i.e. $N_1 = N_2 = 0$ in (1.8)), and then proved a priori estimates for radial solutions satisfying (1.11) under the condition $(\alpha_1, \alpha_2) \in S_{\mathbf{A}_2}$. Furthermore, they also proved the following interesting result, which indicates that the phenomena of partial blowup occurs on some part of $\partial S_{\mathbf{A}_2}$, and so $S_{\mathbf{A}_2}$ is an *optimal* range of (α_1, α_2) in view of the degree theory.

Theorem C. [8] *Let N_1, N_2 be non-negative integers. Let $(\alpha_1, \alpha_2) \in \Omega_{\mathbf{A}_2}$ satisfy*

$$\alpha_2 - \alpha_1 = N_1 + 2N_2 + 3.$$

Then there exists a sequence of radially symmetric bubbling solutions $(u_{1,n}, u_{2,n})$ to system (1.8) *such that $\sup_{\mathbb{R}^2} u_{2,n} \to -\infty$ as $n \to \infty$. Furthermore,*

(*i*) *there is a intersection point $R_{1,n} \gg 1$ of $u_{1,n}$ and $u_{2,n}$ such that $u_{1,n} \to U$ in $C^2_{loc}(B(0, R_{1,n}))$ and $\lim_{n\to\infty} \int_{R_{1,n}}^{\infty} re^{u_{1,n}} dr = 0$, where U is the unique radial solution of*

$$\begin{cases} \Delta U + 2e^U - 4e^{2U} = 4\pi N_1 \delta_0 \quad \text{in } \mathbb{R}^2, \\ U(x) = -2(\alpha_1 + \alpha_2 - 1) \ln |x| + O(1) \quad \text{as } |x| \to \infty. \end{cases}$$

(*ii*) *there exists $(\alpha_{1,n}, \alpha_{2,n}) \in \Omega$ such that*

$$u_{j,n}(x) = -2\alpha_{j,n} \ln |x| + O(1) \quad \text{as } |x| \to +\infty, \qquad j = 1, 2,$$

and $(\alpha_{1,n}, \alpha_{2,n}) \to (\alpha_1, \alpha_2)$ as $n \to \infty$.

The purpose of this paper is to generalize Theorems B and C to the more general system (1.9). Clearly, as pointed out in [14, 17] and seen also in the following proof, this more general situation poses new analytical difficulties compared to the $SU(3)$ case. Define

$$\Omega := \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 > 1 \text{ and } J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1)\}, \tag{1.17}$$

and

$$S := \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 > 0 \text{ and } (\alpha_1, \alpha_2) \text{ satisfies } (1.19) - (1.22)\}, \tag{1.18}$$

where

$$[(1 + a_1)(1 + a_2) - 2a_1 a_2]\alpha_2 - a_2(1 + a_2)\alpha_1$$
$$< a_2(1 + a_2)N_1 + (1 + a_1)(1 + a_2)N_2 + 2(1 + a_1 + a_2), \tag{1.19}$$
$$[(1 + a_1)(1 + a_2) - 2a_1 a_2]\alpha_1 - a_1(1 + a_1)\alpha_2$$
$$< a_1(1 + a_1)N_2 + (1 + a_1)(1 + a_2)N_1 + 2(1 + a_1 + a_2), \tag{1.20}$$
$$[3(1 + a_1)(1 + a_2) - 4a_1 a_2]\alpha_1 + \frac{1 + a_1}{a_2}[(1 + a_1)(1 + a_2) - 2a_1 a_2]\alpha_2$$

$$> (1+a_1)(1+a_2)N_1 + \frac{(1+a_1)^2(1+a_2)}{a_2}N_2 + \left(4+2\frac{1+a_1}{a_2}\right)(1+a_1+a_2), \tag{1.21}$$

$$[3(1+a_1)(1+a_2)-4a_1a_2]\alpha_2 + \frac{1+a_2}{a_1}[(1+a_1)(1+a_2)-2a_1a_2]\alpha_1$$

$$> (1+a_1)(1+a_2)N_2 + \frac{(1+a_2)^2(1+a_1)}{a_1}N_1 + \left(4+2\frac{1+a_2}{a_1}\right)(1+a_1+a_2). \tag{1.22}$$

Then our first result is following.

Theorem 1.1. *Let N_1, N_2 be non-negative integers. Suppose that $a_1, a_2 > 0$ satisfies*

$$(1+a_1)(1+a_2) > \left(6-2\sqrt{5}\right)a_1a_2. \tag{1.23}$$

Let Ω and S be defined in (1.17)-(1.18). Then $S\cap\Omega \neq \emptyset$, and for any fixed $(\alpha_1,\alpha_2)\in S\cap\Omega$, system (1.9) admits a radially symmetric non-topological solution (u_1,u_2) satisfying the asymptotic condition (1.11).

Remark 1.1. *Theorem 1.1 gives a partial answer to an open question raised by Yang [27, Section 8] in 1997. We will prove in Section 2 that $S\neq\emptyset$ if and only if (a_1,a_2) satisfies (1.23). Hence (1.23) is a necessary condition for our result. For the $SU(3)$ system (1.8), we have $(a_1,a_2)=(1,1)$, which yields $S=S_{\mathbf{A}_2}$, $\Omega=\Omega_{\mathbf{A}_2}$ and $J=J_{\mathbf{A}_2}$, so Theorem 1.1 reduces to Theorem B. Differently from the $SU(3)$ case, the statement $S\subset\Omega$ might not hold in general. For example, we will prove in Section 2 that*

$$\begin{cases} S\subset\Omega & \text{if } (1+a_1)(1+a_2)-4a_1a_2\leq 0,\\ S\not\subset\Omega & \text{if } (1+a_1)(1+a_2)-4a_1a_2>0 \text{ small enough.}\end{cases}$$

It is interesting that, when $(1+a_1)(1+a_2)\leq 2a_1a_2$, then (1.19)-(1.20) hold automatically for all $\alpha_1,\alpha_2>0$. For example, let us consider the Cartan matrix $\mathbf{B}_2$ case. Let $K=\mathbf{B}_2$ in (1.1), then system (1.1) becomes the following $\mathbf{B}_2$ Chern-Simons system

$$\begin{cases} \Delta u_1 + 2e^{u_1} - e^{u_2} - 4e^{2u_1} + 2e^{2u_2} = 4\pi N_1\delta_0 \\ \Delta u_2 + 2e^{u_2} - 2e^{u_1} - 4e^{2u_2} + 2e^{u_1+u_2} + 4e^{2u_1} = 4\pi N_2\delta_0 \end{cases} \text{in } \mathbb{R}^2. \tag{1.24}$$

For this $\mathbf{B}_2$ system, we have $(a_1,a_2)=(2,3)$ and so $(1+a_1)(1+a_2)=2a_1a_2$. Then by applying Theorem 1.1 directly, we easily obtain

Theorem 1.2. *Let N_1, N_2 be non-negative integers. Then for any (α_1,α_2) satisfying*

$$\alpha_1 > N_1+N_2+3 \quad \text{and} \quad \alpha_2 > 2N_1+N_2+4, \tag{1.25}$$

the $\mathbf{B}_2$ Chern-Simons system (1.24) has a radially symmetric non-topological solution (u_1,u_2) subject to the asymptotic condition (1.11).

For the Cartan matrix $K=\mathbf{G}_2$ case, we have $(a_1,a_2)=(5,9)$, which does not satisfy (1.23). Therefore, Theorem 1.1 can not be applied to the $\mathbf{G}_2$ case unfortunately. For the case

$$(1+a_1)(1+a_2)\leq\left(6-2\sqrt{5}\right)a_1a_2$$

(such as the $\mathbf{G}_2$ case), the existence of non-topological solutions of system (1.9) subject to the asymptotic condition (1.11) remains open.

As in [8], we will also prove Theorem 1.1 via the degree theory. Differently, here we make a continuous deformation from our original problem (1.9) to the $SU(3)$ Chern-Simons system (1.8). By proving a priori estimates for this deformation problem, we will apply the homotopy invariance of the Leray-Schauder degree to prove Theorem 1.1. Since the variational method does not seem to work for the cooperative case, we believe that the degree theory is a feasible way to treat the cooperative case, which will be considered in a future project.

After Theorem 1.1, we may ask a natural question: *Is the set $S \cap \Omega$ an optimal range of (α_1, α_2) for the existence of radial solutions satisfying* (1.11) *in view of the degree theory?* To answer this question, we prove the following result about the existence of bubbling solutions, which shows that the phenomena of partial blowup also occurs on some part of the boundary of $S \cap \Omega$ just as Theorem C.

Theorem 1.3. *Let N_1, N_2 be non-negative integers. Suppose that $a_1, a_2 > 0$ satisfies*

$$(1 + a_1)(1 + a_2) > 2a_1 a_2. \tag{1.26}$$

Let $(\alpha_1, \alpha_2) \in \Omega$ satisfy $\alpha_1 \neq \alpha_2$ and

$$\begin{aligned}&[(1 + a_1)(1 + a_2) - 2a_1 a_2]\alpha_2 - a_2(1 + a_2)\alpha_1 \\ =&a_2(1 + a_2)N_1 + (1 + a_1)(1 + a_2)N_2 + 2(1 + a_1 + a_2).\end{aligned} \tag{1.27}$$

Then system (1.9) *admits a sequence of radially symmetric bubbling solutions $(u_{1,n}, u_{2,n})$ such that $\sup_{\mathbb{R}^2} u_{2,n} \to -\infty$ as $n \to \infty$. Furthermore,*

(*i*) *there exists a intersection point $R_{1,n} \gg 1$ of $u_{1,n}$ and $u_{2,n}$ such that $u_{1,n} \to U$ in $C^2_{loc}(B(0, R_{1,n}))$, where U is the unique radial solution of*

$$\begin{cases}\Delta U + (1 + a_1)e^U - (1 + a_1)^2 e^{2U} = 4\pi N_1 \delta_0 \ \text{ in } \mathbb{R}^2, \\ U(x) = -2\gamma \ln |x| + O(1) \ \text{ as } |x| \to \infty\end{cases}$$

with $\gamma = \alpha_1 + \frac{2a_1}{1+a_2}(\alpha_2 - 1)$. Besides, $\lim_{n\to\infty} \int_{R_{1,n}}^{\infty} re^{u_{1,n}} dr = 0$.

(*ii*) *there exists $(\alpha_{1,n}, \alpha_{2,n}) \in \Omega$ such that*

$$u_{j,n}(x) = -2\alpha_{j,n} \ln |x| + O(1) \ \text{ as } |x| \to +\infty, \quad j = 1, 2,$$

and $(\alpha_{1,n}, \alpha_{2,n}) \to (\alpha_1, \alpha_2)$ as $n \to \infty$.

Remark 1.2. *Clearly Theorem 1.3 generalizes Theorem C. Observe that* (1.26) *is a necessary condition for Theorem 1.3, otherwise there are no $(\alpha_1, \alpha_2) \in \Omega$ satisfying* (1.27)*. The assumption $(\alpha_1, \alpha_2) \in \Omega$ is also necessary in view of the Pohozaev identity. In the case $(1 + a_1)(1 + a_2) \le 4a_1 a_2$, we can prove that the assumption $(\alpha_1, \alpha_2) \in \Omega$ holds automatically provided that $\alpha_1, \alpha_2 > 1$ satisfy* (1.27)*; see Remark 5.1.*

Remark 1.3. *The assumption $\alpha_1 \neq \alpha_2$ is a technical condition, since we can not treat the case $\alpha_1 = \alpha_2$. In the $SU(3)$ case $(a_1, a_2) = (1, 1)$ studied in [8], $\alpha_2 > \alpha_1$ holds automatically in Theorem C. In Theorem 1.3 here, since we deal with generic (a_1, a_2), the relation $\alpha_2 > \alpha_1$ holds automatically only when $a_2 \ge 1$ (i.e. $(1 + a_1)(1 + a_2) - 2a_1 a_2 \le a_2(1 + a_2)$). Therefore, the assumption $\alpha_1 \neq \alpha_2$ in Theorem 1.3 is equivalent to*

$$\alpha_2 \neq \alpha_0 := \frac{a_2(1 + a_2)N_1 + (1 + a_1)(1 + a_2)N_2}{(1 - a_2)(1 + a_1 + a_2)} + \frac{2}{1 - a_2} \ \text{ if } a_2 \in (0, 1).$$

More precisely, when $a_2 \in (0, 1)$, we have $\alpha_1 < \alpha_2$ if and only if $\alpha_2 < \alpha_0$, and $\alpha_2 < \alpha_1$ if and only if $\alpha_2 > \alpha_0$. We will see in Section 5 that the situation is different with respect to the sign of $\alpha_1 - \alpha_2$. This phenomena provides an evidence that the general problem (1.9) *is more involved than the $SU(3)$ Chern-Simons system* (1.8)*.*

The rest of this paper is organized as follows. In Section 2, we give the deformation of the problem and prove some preliminary lemmas. In Section 3, by a delicate blow up analysis, we prove a priori estimates for radial non-topological solutions satisfying (1.11) under the assumption $(\alpha_1, \alpha_2) \in S \cap \Omega$. In Section 4, we complete the proof of Theorem 1.1 by applying the degree theory. In Section 5, we prove Theorem 1.3 via the shooting method.

2 Preliminaries and a deformation of the problem

In the sequel, we denote

$$A = (1 + a_1)(1 + a_2) \quad \text{and} \quad B = a_1 a_2$$

for convenience. Then $A > B + 1$ since $a_1, a_2 > 0$. First we prove some facts about S and Ω mentioned in Remark 1.1.

Lemma 2.1. *Let Ω and S be in* (1.17)-(1.18).

(*i*) *There holds*

$$S \neq \emptyset \Longleftrightarrow (1 + a_1)(1 + a_2) > \left(6 - 2\sqrt{5}\right) a_1 a_2. \tag{2.1}$$

(*ii*) *Let $(1 + a_1)(1 + a_2) > \left(6 - 2\sqrt{5}\right) a_1 a_2$. Then*

$$\alpha_1 > 1, \alpha_2 > 1, \quad \forall\, (\alpha_1, \alpha_2) \in S. \tag{2.2}$$

Furthermore, $S \cap \Omega \neq \emptyset$ and

$$\begin{cases} S \subset \Omega & \text{if } (1 + a_1)(1 + a_2) - 4a_1 a_2 \leq 0, \\ S \not\subset \Omega & \text{if } (1 + a_1)(1 + a_2) - 4a_1 a_2 > 0 \text{ small enough.} \end{cases} \tag{2.3}$$

Proof. Recalling (1.19)-(1.22), we define

$$h_1(\alpha_1, \alpha_2) = (A - 2B)\alpha_2 - a_2(1 + a_2)\alpha_1 - a_2(1 + a_2)N_1 - AN_2 - 2(A - B), \tag{2.4}$$

$$h_2(\alpha_1, \alpha_2) = (A - 2B)\alpha_1 - a_1(1 + a_1)\alpha_2 - a_1(1 + a_1)N_2 - AN_1 - 2(A - B), \tag{2.5}$$

$$\begin{aligned} h_3(\alpha_1, \alpha_2) =& (3A - 4B)\alpha_1 + \frac{1 + a_1}{a_2}(A - 2B)\alpha_2 \\ & - AN_1 - \frac{1 + a_1}{a_2} AN_2 - \left(4 + 2\frac{1 + a_1}{a_2}\right)(A - B), \end{aligned} \tag{2.6}$$

$$\begin{aligned} h_4(\alpha_1, \alpha_2) =& (3A - 4B)\alpha_2 + \frac{1 + a_2}{a_1}(A - 2B)\alpha_1 \\ & - AN_2 - \frac{1 + a_2}{a_1} AN_1 - \left(4 + 2\frac{1 + a_2}{a_1}\right)(A - B). \end{aligned} \tag{2.7}$$

Clearly

$$S = \left\{ (\alpha_1, \alpha_2) \;\middle|\; \begin{array}{l} \alpha_1 > 0,\ h_1(\alpha_1, \alpha_2) < 0,\ h_2(\alpha_1, \alpha_2) < 0 \\ \alpha_2 > 0,\ h_3(\alpha_1, \alpha_2) > 0,\ h_4(\alpha_1, \alpha_2) > 0 \end{array} \right\}.$$

Case 1. $A - 2B \leq 0$.

In this case, $h_1(\alpha_1, \alpha_2) < 0$ and $h_2(\alpha_1, \alpha_2) < 0$ always hold for all $\alpha_1 > 0, \alpha_2 > 0$. Obviously, $S = \emptyset$ if $3A - 4B \leq 0$, and $S \neq \emptyset$ if $A - 2B = 0$. So we assume $3A - 4B > 0 > A - 2B$. Then it is easy to see that $S \neq \emptyset$ if and only if the slopes of lines $h_3 = 0$ and $h_4 = 0$ satisfy

$$\frac{3A - 4B}{\frac{1 + a_1}{a_2}(2B - A)} > \frac{\frac{1 + a_2}{a_1}(2B - A)}{3A - 4B}. \tag{2.8}$$

A direct calculation shows that (2.8) is equivalent to $(A - B)(A^2 - 12AB + 16B^2) < 0$. So

$$A > (6 - 2\sqrt{5})B \quad (\text{note that } 6 - 2\sqrt{5} > 4/3). \tag{2.9}$$

This proves (2.1) in this case.

Now we let (2.9) holds and take any $(\alpha_1, \alpha_2) \in S$. Then $h_3(\alpha_1, \alpha_2) > 0$ and $A - 2B \leq 0$ yield

$$(3A - 4B)\alpha_1 > AN_1 + 4(A - B) \geq (3A - 4B)(N_1 + 2),$$

namely $\alpha_1 > N_1 + 2$. Similarly, $\alpha_2 > N_2 + 2$. Consequently,

$$J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1),$$

namely $S \subset \Omega$ in this case.

Case 2. $A - 2B > 0$ and $A - 4B \leq 0$.

In this case, we have

$$\frac{a_2(1 + a_2)}{A - 2B} \geq \frac{A - 2B}{a_1(1 + a_1)} > 0.$$

Consequently, the line $h_1 = 0$ can not intersect with the line $h_2 = 0$ in the first quadrant. Since the slopes of $h_3 = 0$ and $h_4 = 0$ are both negative, so it is trivial to see that $S \neq \emptyset$.

To prove (2.2), we take any $(\alpha_1, \alpha_2) \in S$. By $h_1(\alpha_1, \alpha_2) < 0$ we have

$$(A - 2B)\alpha_2 < a_2(1 + a_2)\alpha_1 + a_2(1 + a_2)N_1 + AN_2 + 2(A - B).$$

Substituting this inequality into $h_3(\alpha_1, \alpha_2) > 0$, we easily conclude $\alpha_1 > 1$. Similarly, $\alpha_2 > 1$. Remark that this argument of proving (2.2) also works for Case 3 where $A - 4B > 0$.

It remains to prove $S \subset \Omega$. Note that the intersection point (β_1, β_2) of $h_3 = 0$ and $h_4 = 0$ is

$$\begin{cases} \beta_1 = \frac{A(A-4B)(N_1+1) - \frac{2(1+a_1)}{a_2}AB(N_2+1)}{A^2-12AB+16B^2} + 1, \\ \beta_2 = \frac{A(A-4B)(N_2+1) - \frac{2(1+a_2)}{a_1}AB(N_1+1)}{A^2-12AB+16B^2} + 1. \end{cases} \tag{2.10}$$

Since $A^2 - 12AB + 16B^2 < 0$, we have $\beta_1 > 1$ and $\beta_2 > 1$. Take any $(\alpha_1, \alpha_2) \in S$. Then either $\alpha_1 > \beta_1$ or $\alpha_2 > \beta_2$. Without loss of generality, we assume $\alpha_2 > \beta_2$. In the following argument, we write

$$\tilde{\alpha}_j := \alpha_j - 1, \;\; \tilde{\beta}_j := \beta_j - 1 \;\text{ and }\; \tilde{N}_j := N_j + 1$$

for convenience. Then $\tilde{\alpha}_2 > \tilde{\beta}_2$. By $h_3(\alpha_1, \alpha_2) > 0$ we have

$$\begin{aligned}(3A - 4B)\tilde{\alpha}_1 >& A\tilde{N}_1 + \frac{1 + a_1}{a_2}A\tilde{N}_2 - \frac{1 + a_1}{a_2}(A - 2B)\tilde{\alpha}_2 \\ =&: C - \frac{1 + a_1}{a_2}(A - 2B)\tilde{\alpha}_2. \end{aligned} \tag{2.11}$$

Recalling the definition (1.10) of J, a direct computation shows that

$$J((3A - 4B)\tilde{\alpha}_1, (3A - 4B)\tilde{\alpha}_2) > J\left(C - \frac{1 + a_1}{a_2}(A - 2B)\tilde{\alpha}_2, (3A - 4B)\tilde{\alpha}_2\right)$$

holds if we have

$$\frac{1 + a_2}{2a_1}\left[(3A - 4B)\tilde{\alpha}_1 + A\tilde{N}_1 + \frac{1 + a_1}{a_2}A\tilde{N}_2\right] + \left[3A - 4B - \frac{A}{2B}(A - 2B)\right]\tilde{\alpha}_2 > 0.$$

Since $A - 4B \leq 0$ gives $3A - 4B - \frac{A}{2B}(A - 2B) > 0$, the above inequality holds. So

$$\begin{aligned} J(\tilde{\alpha}_1, \tilde{\alpha}_2) >& \frac{1}{(3A - 4B)^2}J\left(C - \frac{1 + a_1}{a_2}(A - 2B)\tilde{\alpha}_2, (3A - 4B)\tilde{\alpha}_2\right) \\ =& \frac{\frac{a_2(1+a_2)}{2}C^2 + \frac{a_1(1+a_1)A^2(A-B)}{2B}\tilde{\alpha}_2^2 - (A - B)(A - 4B)C\tilde{\alpha}_2}{(3A - 4B)^2} \\ >& \frac{\frac{a_2(1+a_2)}{2}C^2 + \frac{a_1(1+a_1)A^2(A-B)}{2B}\tilde{\beta}_2^2 - (A - B)(A - 4B)C\tilde{\beta}_2}{(3A - 4B)^2}. \end{aligned}$$

Recalling from (2.10)-(2.11) that

$$\tilde{\beta}_2 = \frac{A(A - 4B)\tilde{N}_2 - \frac{2(1+a_2)}{a_1}AB\tilde{N}_1}{A^2 - 12AB + 16B^2}, \quad C = A\tilde{N}_1 + \frac{1 + a_1}{a_2}A\tilde{N}_2.$$

By substituting these two expressions into the last inequality and by a direct calculation, we conclude from $A-4B\le 0$ that

$$\begin{aligned}J(\tilde\alpha_1,\tilde\alpha_2)>&\frac{a_2(1+a_2)}{2}\left[1+\frac{16B(A-B)(A-4B)^2}{(A^2-12AB+16B^2)^2}\right]\tilde N_1^2\\&+a_1a_2\left[1-\frac{4(A-B)(A-4B)(A^2-4AB+16B^2)}{(A^2-12AB+16B^2)^2}\right]\tilde N_1\tilde N_2\\&+\frac{a_1(1+a_1)}{2}\left[1+\frac{16B(A-B)(A-4B)^2}{(A^2-12AB+16B^2)^2}\right]\tilde N_2^2\\\ge&J(\tilde N_1,\tilde N_2).\end{aligned}\tag{2.12}$$

Consequently, $(\alpha_1,\alpha_2)\in\Omega$. The case $\alpha_1>\beta_1$ is similar. Therefore, $S\subset\Omega$.

Case 3. $A-4B>0$.

Then the intersection point (γ_1,γ_2) of $h_1=0$ and $h_2=0$ is

$$\begin{cases}\gamma_1=\frac{A(N_1+1)+2a_1(1+a_1)(N_2+1)}{A-4B}+1>1,\\\gamma_2=\frac{A(N_2+1)+2a_2(1+a_2)(N_1+1)}{A-4B}+1>1.\end{cases}$$

A direct calculation shows that $h_j(\gamma_1,\gamma_2)>0$ holds for $j=3,4$. Therefore, $S\neq\emptyset$ with $(\gamma_1,\gamma_2)\in\partial S$. We also write $\tilde\gamma_j=\gamma_j-1$. Then

$$\begin{aligned}&J(\tilde\gamma_1,\tilde\gamma_2)-J(\tilde N_1,\tilde N_2)\\=&\frac{J(A\tilde N_1+2a_1(1+a_1)\tilde N_2,A\tilde N_2+2a_2(1+a_2)\tilde N_1)-(A-4B)^2J(\tilde N_1,\tilde N_2)}{(A-4B)^2}\\=&4B\frac{2a_2(1+a_2)(A-B)\tilde N_1^2+(A^2+3AB-4B^2)\tilde N_1\tilde N_2+2a_1(1+a_1)(A-B)\tilde N_2^2}{(A-4B)^2}\\>&0,\end{aligned}$$

namely $(\gamma_1,\gamma_2)\in\Omega\cap\partial S$. Since Ω is an open set, we conclude $S\cap\Omega\neq\emptyset$.

Finally, we need to prove that $S\not\subset\Omega$ provided $A-4B>0$ small enough. Let $4B<A<5B$, then $A^2-12AB+16B^2<0$ and a_1,a_2 are uniformly bounded. Note that $A=(1+a_1)(1+a_2)>1$ and so $B>1/5$. Recalling (2.10), we can prove that both $h_1(\beta_1,\beta_2)<0$ and $h_2(\beta_1,\beta_2)<0$ if and only if

$$2AB\tilde N_2>a_2(1+a_2)(A-4B)\tilde N_1\ \text{ and }\ 2AB\tilde N_1>a_1(1+a_1)(A-4B)\tilde N_2,$$

which are always true provided $A-4B>0$ small enough. So $(\beta_1,\beta_2)\in\partial S$ if $A-4B>0$ is small enough. Also by (2.10), we can compute (the expression is the same as (2.12))

$$\begin{aligned}&J(\tilde\beta_1,\tilde\beta_2)-J(\tilde N_1,\tilde N_2)\\=&\frac{J(A(A-4B)\tilde N_1-\frac{2(1+a_1)}{a_2}AB\tilde N_2,A(A-4B)\tilde N_2-\frac{2(1+a_2)}{a_1}AB\tilde N_1)}{(A^2-12AB+16B^2)^2}-J(\tilde N_1,\tilde N_2)\\=&\frac{B(A-B)(A-4B)}{(A^2-12AB+16B^2)^2}\Big[8a_2(1+a_2)(A-4B)\tilde N_1^2\\&\quad-4[(A-4B)^2+4AB)]\tilde N_1\tilde N_2+8a_1(1+a_1)(A-4B)\tilde N_2^2\Big]\\<&0\quad\text{if } A-4B>0\text{ small enough,}\end{aligned}$$

namely $J(\beta_1-1,\beta_2-1)<J(N_1+1,N_2+1)$ if $A-4B>0$ small enough. Since $(\beta_1,\beta_2)\in\partial S$, we conclude that $S\not\subset\Omega$ for $A-4B>0$ small enough. This prove the lemma. $\square$

From now on, we always assume that (1.23) holds. As in [8], we will employ the Leray-Schauder degree theory to prove Theorem 1.1. For $t\in[0,1]$, we define

$$b_k=b_k(t):=1+t(a_k-1),\quad k=1,2.\tag{2.13}$$

Then

$$0 < \min\{1, a_1, a_2\} \le b_1, b_2 \le \max\{1, a_1, a_2\}, \quad \forall\, t \in [0,1], \tag{2.14}$$

and a straightforward computation gives

$$(1+b_1)(1+b_2) > \left(6 - 2\sqrt{5}\right) b_1 b_2, \quad \forall\, t \in [0,1]. \tag{2.15}$$

Consider the following deformation of system (1.9)

$$\begin{cases} \Delta u_1 + (1+b_1)(e^{u_1} - (1+b_1)e^{2u_1} + b_1 e^{u_1+u_2}) \\ \qquad -b_1(e^{u_2} - (1+b_2)e^{2u_2} + b_2 e^{u_1+u_2}) = 4\pi N_1 \delta_0 \\ \Delta u_2 + (1+b_2)(e^{u_2} - (1+b_2)e^{2u_2} + b_2 e^{u_1+u_2}) \\ \qquad -b_2(e^{u_1} - (1+b_1)e^{2u_1} + b_1 e^{u_1+u_2}) = 4\pi N_2 \delta_0 \end{cases} \text{in } \mathbb{R}^2. \tag{2.16}$$

Clearly, if $t=1$, then $b_k = a_k$ and system (2.16) is just our original problem (1.9); if $t=0$, then $b_k = 1$ and system (2.16) is just the $SU(3)$ Chern-Simons system (1.8). As before, we define

$$J_t(x,y) := \frac{b_2(1+b_2)}{2}x^2 + b_1 b_2 xy + \frac{b_1(1+b_1)}{2}y^2, \tag{2.17}$$

$$\Omega_t := \left\{(\alpha_1,\alpha_2) \mid \alpha_1,\alpha_2 > 1 \text{ and } J_t(\alpha_1 - 1, \alpha_2 - 1) > J_t(N_1+1, N_2+1)\right\}, \tag{2.18}$$

and

$$S_t := \left\{(\alpha_1,\alpha_2) \mid \alpha_1,\alpha_2 > 0 \ \text{ and } (\alpha_1,\alpha_2) \text{ satisfies } (2.20) - (2.23)\right\}, \tag{2.19}$$

where

$$\begin{aligned} &[(1+b_1)(1+b_2) - 2b_1b_2]\alpha_2 - b_2(1+b_2)\alpha_1 \\ &\qquad < b_2(1+b_2)N_1 + (1+b_1)(1+b_2)N_2 + 2(1+b_1+b_2), \end{aligned} \tag{2.20}$$

$$\begin{aligned} &[(1+b_1)(1+b_2) - 2b_1b_2]\alpha_1 - b_1(1+b_1)\alpha_2 \\ &\qquad < b_1(1+b_1)N_2 + (1+b_1)(1+b_2)N_1 + 2(1+b_1+b_2), \end{aligned} \tag{2.21}$$

$$\begin{aligned} &[3(1+b_1)(1+b_2) - 4b_1b_2]\alpha_1 + \frac{1+b_1}{b_2}[(1+b_1)(1+b_2) - 2b_1b_2]\alpha_2 \\ &> (1+b_1)(1+b_2)N_1 + \frac{(1+b_1)^2(1+b_2)}{b_2}N_2 + \left(4 + 2\frac{1+b_1}{b_2}\right)(1+b_1+b_2), \end{aligned} \tag{2.22}$$

$$\begin{aligned} &[3(1+b_1)(1+b_2) - 4b_1b_2]\alpha_2 + \frac{1+b_2}{b_1}[(1+b_1)(1+b_2) - 2b_1b_2]\alpha_1 \\ &> (1+b_1)(1+b_2)N_2 + \frac{(1+b_2)^2(1+b_1)}{b_1}N_1 + \left(4 + 2\frac{1+b_2}{b_1}\right)(1+b_1+b_2). \end{aligned} \tag{2.23}$$

Then Lemma 2.1 tell us that $S_t \cap \Omega_t \neq \emptyset$ for all $t \in [0,1]$. Remark that $J_1 = J$, $S_1 = S$, $\Omega_1 = \Omega$ and $J_0 = J_{\mathbf{A}_2}$, $S_0 = S_{\mathbf{A}_2}$, $\Omega_0 = \Omega_{\mathbf{A}_2}$. Moreover, J_t and $S_t \cap \Omega_t$ are both continuous with respect to $t \in [0,1]$.

From now on, we fix any $(\alpha_1,\alpha_2) \in S \cap \Omega$. Our goal is to prove the existence of a radial non-topological solution of system (1.9) subject to the asymptotic condition (1.11). Since $S_t \cap \Omega_t$ is open and continuous, we can take a continuous function $(\beta_1,\beta_2) : [0,1] \to \mathbb{R}^2$ such that

$$(\beta_1,\beta_2)(1) = (\alpha_1,\alpha_2) \ \text{ and } \ (\beta_1,\beta_2)(t) \in S_t \cap \Omega_t \ \ \forall\, t \in [0,1]. \tag{2.24}$$

Obviously, (2.24) implies the existence of a constant $c_0 > 0$ such that

$$\min_{t \in [0,1]} \beta_k(t) \ge 1 + c_0, \quad k = 1,2. \tag{2.25}$$

Then we turn to study the existence of radially symmetric solutions to system (2.16) subject to the following asymptotic condition

$$u_k(x) = -2\beta_k \ln|x| + O(1) \ \text{ as } |x| \to +\infty, \quad k = 1,2. \tag{2.26}$$

By Theorem B we know that, for $t = 0$, system (2.16) has a radial solution satisfying (2.26). In fact, Choe, Kim and Lin [8] proved that the Leray-Schauder degree is not 0. Therefore, our strategy is to prove the uniform boundedness of radially symmetric solutions satisfying (2.26) when t varies in $[0, 1]$; see Section 3. Consequently, the degree theory can be applied and the degree is invariant under this deformation, so Theorem 1.1 follows; see Section 4.

In the sequel, we always denote positive constants independent of $t \in [0, 1]$ (possibly different in different places) by $C, C_0, C_1, \cdots$. Before ending this section, we prove some useful results that are needed in Section 3. First we prove the Pohozaev identity of system (2.16). Clearly, any radially symmetric solution (u_1, u_2) depends on $r = |x|$ as well as t. Also, $b_k(t)$ and $\beta_k(t)$ are continuous functions of t. When there is no confusion arising, we denote (u_1, u_2) by $(u_1(r), u_2(r))$, $b_k(t)$ by b_k and $\beta_k(t)$ by β_k for convenience (i.e. we omit the notation t). We will use $\Delta u = u'' + \frac{1}{r}u'$ frequently, where $u'(r) = \frac{du}{dr}$ and $u''(r) = \frac{d^2u}{dr^2}$.

By (2.24), it is easy to see that

$$\inf_{t\in[0,1]} \left[J_t(\beta_1(t) - 1, \beta_2(t) - 1) - J_t(N_1 + 1, N_2 + 1)\right] > 0. \tag{2.27}$$

Lemma 2.2. *Let (u_1, u_2) be radially symmetric solutions of system (2.16) satisfying (2.26). Then $u_1 < 0$ and $u_2 < 0$ in $\mathbb{R}^2$, and*

$$\begin{aligned} &J_t(\beta_1 - 1, \beta_2 - 1) - J_t(N_1 + 1, N_2 + 1)\\ =&\frac{1 + b_1 + b_2}{4} \int_0^\infty r\left[b_2(1 + b_1)e^{2u_1} + b_1(1 + b_2)e^{2u_2} - 2b_1b_2e^{u_1+u_2}\right] dr. \end{aligned} \tag{2.28}$$

Furthermore, there exist positive constants C_1 and C_2 independent of $t \in [0, 1]$ such that

$$\int_0^\infty r(e^{2u_1} + e^{2u_2})dr \ge C_1, \tag{2.29}$$

$$\int_0^\infty r(e^{u_1} + e^{u_2} + e^{2u_1} + e^{2u_2} + e^{u_1+u_2})dr \le C_2. \tag{2.30}$$

Proof. The fact $u_1, u_2 < 0$ in $\mathbb{R}^2$ was proved in Theorem A. By (2.16) and (2.26), we have

$$\lim_{r\to 0} ru_k'(r) = 2N_k, \quad \lim_{r\to\infty} ru_k'(r) = -2\beta_k, \quad k = 1, 2. \tag{2.31}$$

Since (u_1, u_2) is radially symmetric, system (2.16) can be written as

$$(1 + b_2)(ru_1')' + b_1(ru_2')' = -(1 + b_1 + b_2)r(e^{u_1} - (1 + b_1)e^{2u_1} + b_1e^{u_1+u_2}), \tag{2.32}$$

$$b_2(ru_1')' + (1 + b_1)(ru_2')' = -(1 + b_1 + b_2)r(e^{u_2} - (1 + b_2)e^{2u_2} + b_2e^{u_1+u_2}), \tag{2.33}$$

for $0 < r < \infty$. We multiply (2.32) by b_2ru_1', (2.33) by b_1ru_2' and integrate them over $[\varepsilon, R]$, which yields

$$\begin{aligned} J_t(ru_1', ru_2')\Big|_\varepsilon^R =& \left[\frac{b_2(1 + b_2)}{2}(ru_1')^2 + b_1b_2r^2u_1'u_2' + \frac{b_1(1 + b_1)}{2}(ru_2')^2\right]\Bigg|_\varepsilon^R\\ =& - (1 + b_1 + b_2)r^2\Bigg[b_2\left(e^{u_1} - \frac{1 + b_1}{2}e^{2u_1}\right) + b_1\left(e^{u_2} - \frac{1 + b_2}{2}e^{2u_2}\right)\\ &+ b_1b_2e^{u_1+u_2}\Bigg]\Bigg|_\varepsilon^R + 2(1 + b_1 + b_2)\int_\varepsilon^R r\Bigg[b_2\left(e^{u_1} - \frac{1 + b_1}{2}e^{2u_1}\right)\\ &+ b_1\left(e^{u_2} - \frac{1 + b_2}{2}e^{2u_2}\right) + b_1b_2e^{u_1+u_2}\Bigg]dr. \end{aligned} \tag{2.34}$$

By letting $\varepsilon \to 0$ and $R \to \infty$, it follows from (2.26) and (2.31) that

$$J_t(2\beta_1, 2\beta_2) - J_t(2N_1, 2N_2) = 2(1 + b_1 + b_2)\int_0^\infty r\left[b_2\left(e^{u_1} - \frac{1 + b_1}{2}e^{2u_1}\right)\right.$$

$$+b_1\left(e^{u_2}-\frac{1+b_2}{2}e^{2u_2}\right)+b_1b_2e^{u_1+u_2}\Bigg]dr. \tag{2.35}$$

On the other hand, by integrating (2.32) and (2.33) over $(0,\infty)$, we obtain

$$\begin{aligned}(1+b_1+b_2)&\int_0^\infty r(e^{u_1}-(1+b_1)e^{2u_1}+b_1e^{u_1+u_2})dr\\ &\qquad=2(1+b_2)(\beta_1+N_1)+2b_1(\beta_2+N_2), \end{aligned}\tag{2.36}$$

$$\begin{aligned}(1+b_1+b_2)&\int_0^\infty r(e^{u_2}-(1+b_2)e^{2u_2}+b_2e^{u_1+u_2})dr\\ &\qquad=2b_2(\beta_1+N_1)+2(1+b_1)(\beta_2+N_2). \end{aligned}\tag{2.37}$$

By eliminating the terms $\int_0^\infty re^{u_k}dr$ in (2.35) via (2.36)-(2.37), we obtain the Pohozaev identity (2.28), and (2.29) follows directly from (2.14), (2.27) and (2.28). Since $2e^{u_1+u_2}\le e^{2u_1}+e^{2u_2}$, (2.28) also implies

$$\int_0^\infty r(e^{2u_1}+e^{2u_2}+e^{u_1+u_2})dr\le C$$

for some $C>0$ independent of t. This, together with (2.36)-(2.37), gives (2.30). □

Lemma 2.3. *Let (u_1,u_2) be radially symmetric solutions of system (2.16) satisfying (2.26). Then there exist constants $C_1,C_2>2+2N_1+2N_2$ independent of t such that*

$$|ru_k'(r)-2N_k|\le\begin{cases}C_1r^2 & \text{if } 0<r<1,\\ C_1 & \text{if } r\ge 1,\end{cases}\qquad k=1,2, \tag{2.38}$$

$$u_k(r)\le -2\ln r+C_2\quad \forall\, r\ge 1,\quad k=1,2. \tag{2.39}$$

Proof. Note that $e^{u_k}<1$ in $\mathbb{R}^2$. Obviously, (2.38) follows directly from (2.16), (2.30) and (2.31). By (2.38), we have

$$u_k(s)=u_k(r)+\int_r^s u_k'(\rho)d\rho\ge u_k(r)-(C_1-2N_k)\ln\frac{s}{r},\quad \text{for } s\ge r\ge 1.$$

Consequently,

$$C\ge\int_r^\infty se^{u_k(s)}ds\ge e^{u_k(r)}\int_r^\infty s\left(\frac{r}{s}\right)^{C_1-2N_k}ds=\frac{e^{u_k(r)}r^2}{C_1-2-2N_k},\quad \forall\, r\ge 1,$$

which implies (2.39). □

Lemma 2.4. *Recall J_t in (2.17). There holds*

$$\begin{aligned}J_t(x,y)&=J_t(-x,-y)\\ &=J_t\left(x,\,-\frac{2b_2}{1+b_1}x-y\right)=J_t\left(-x,\,\frac{2b_2}{1+b_1}x+y\right)\\ &=J_t\left(-x-\frac{2b_1}{1+b_2}y,\,y\right)=J_t\left(x+\frac{2b_1}{1+b_2}y,\,-y\right).\end{aligned}$$

Proof. These formulae can be proved via direct calculations, and we omit the details. □

3 A priori estimates

In this section, in order to apply the degree theory, we prove a priori estimates of radially symmetric solutions of (2.16) satisfying (2.26). Assume that (u_1,u_2) are any radially symmetric solutions of (2.16) satisfying (2.26). Define

$$f_k(x)=f_k(x;t):=2N_k\ln|x|-(\beta_k(t)+N_k)\ln(1+|x|^2),\quad k=1,2. \tag{3.1}$$

Clearly $f_k<0$ in $\mathbb{R}^2$. The main result of this section is following

Theorem 3.1. *There exists a constant $C > 0$ independent of $t \in [0,1]$ such that*

$$\|u_1 - f_1\|_{L^\infty(\mathbb{R}^2)} + \|u_2 - f_2\|_{L^\infty(\mathbb{R}^2)} \le C, \quad \forall\, t \in [0,1]. \tag{3.2}$$

To prove this result, first we need to prove the following local uniform boundedness. We denote $B_R := \{x \in \mathbb{R}^2 \mid |x| < R\}$.

Theorem 3.2. *For any $R > 0$, there exists a constant $C_R > 0$ independent of $t \in [0,1]$ such that*

$$\|u_1 - f_1\|_{L^\infty(B_R)} + \|u_2 - f_2\|_{L^\infty(B_R)} \le C_R, \quad \forall\, t \in [0,1]. \tag{3.3}$$

The proof of Theorem 3.2 is quite long and delicate, and will be separated into several steps. Our basic strategy of proving Theorem 3.2 is the same as that in [8]. Roughly speaking, by a careful blow up analysis, we will show that the phenomena of partial blowup could lead to $(\beta_1(t), \beta_2(t)) \notin S_t$. However, as we will see in the following, since we consider generic (a_1, a_2) instead of $(a_1, a_2) = (1, 1)$, the proof is more involved and different ideas are needed. Hence, although our main procedure is close to that in [8], we prefer to provide all the necessary details to make the paper self-contained.

Denote

$$v_k = u_k - f_k, \quad k = 1, 2.$$

Then we easily deduce from Lemma 2.3 that

$$|v_1'(r)| + |v_2'(r)| \le \begin{cases} Cr, & \text{if } 0 < r < 1, \\ \frac{C}{r}, & \text{if } r \ge 1, \end{cases} \tag{3.4}$$

where $C > 0$ is independent of t. In particular, (3.4) indicates that (3.3) holds for all $R > 0$ provided that it holds for some $R > 0$.

To prove Theorem 3.2, we argue by contradiction. Suppose that there exists a sequence $\{t_n\}_{n\ge1} \subset [0,1]$ such that $t_n \to t_* \in [0,1]$ and the corresponding sequence $(v_{1,n}, v_{2,n}) = (u_{1,n} - f_{1,n},\, u_{2,n} - f_{2,n})$ is not uniformly bounded in $L^\infty(B_R) \times L^\infty(B_R)$ for any $R > 0$. Since $v_{k,n}(1) < (\beta_k + N_k)\ln 2$, (3.4) shows that $v_{k,n}$ are uniformly bounded from above in B_R for $k = 1, 2$. Therefore, up to a subsequence, either $v_{1,n} \to -\infty$ uniformly in B_R or $v_{2,n} \to -\infty$ uniformly in B_R. By renumbering if necessary, we assume in the sequel that

$$v_{2,n} \to -\infty \text{ uniformly in } B_R \text{ for any } R > 0, \text{ as } n \to \infty. \tag{3.5}$$

Lemma 3.1. *$\{v_{1,n}\}_{n\ge1}$ is bounded in $L^\infty_{loc}(\mathbb{R}^2)$ and*

$$\limsup_{n\to\infty} \left(\sup_{\mathbb{R}^2} u_{2,n} \right) = -\infty. \tag{3.6}$$

Proof. Assume by contradiction that, up to a subsequence, $v_{1,n} \to -\infty$ uniformly in B_R for any $R > 0$. It follows from $u_{k,n} = v_{k,n} + f_{k,n} \le v_{k,n}$ that $u_{1,n} \to -\infty$ and $u_{2,n} \to -\infty$ uniformly in B_R for any $R > 0$. This, together with (2.39), gives

$$\limsup_{n\to\infty} \left(\sup_{\mathbb{R}^2} u_{k,n} \right) = -\infty, \quad k = 1, 2,$$

which proves (3.6). Consequently, (2.29)-(2.30) yield

$$C_1 \le \int_{\mathbb{R}^2} (e^{2u_{1,n}} + e^{2u_{2,n}})dx \le \sum_{k=1}^{2} \left(\sup_{\mathbb{R}^2} e^{u_{k,n}} \right) \int_{\mathbb{R}^2} e^{u_{k,n}} dx \to 0$$

as $n \to \infty$, a contradiction. □

By (3.4) we know that $\{v_{2,n} - v_{2,n}(0)\}$ is also bounded in $L^\infty_{loc}(\mathbb{R}^2)$. Then, passing to a subsequence, we may assume that

$$v_{1,n} \to v_1^* \text{ and } v_{2,n} - v_{2,n}(0) \to v_2^* \text{ in } C^2_{loc}(\mathbb{R}^2).$$

Recalling $t_n \to t_*$, we denote

$$\beta_k^* = \beta_k(t_*), \quad b_k^* = b_k(t_*), \quad f_k^*(x) = f_k(x; t_*), \quad J^* = J_{t_*}$$

for convenience. Let $u_k^* = v_k^* + f_k^*$. Then (u_1^*, u_2^*) satisfies

$$\Delta u_1^* + (1 + b_1^*) \left[e^{u_1^*} - (1 + b_1^*) e^{2u_1^*} \right] = 4\pi N_1 \delta_0 \text{ in } \mathbb{R}^2, \tag{3.7}$$

$$\Delta u_2^* - b_2^* \left[e^{u_1^*} - (1 + b_1^*) e^{2u_1^*} \right] = 4\pi N_2 \delta_0 \text{ in } \mathbb{R}^2. \tag{3.8}$$

By Fatou lemma, we have

$$\int_{\mathbb{R}^2} e^{u_1^*} dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^2} e^{u_{1,n}} dx \le C,$$

so $u_1^* + \ln(1 + b_1^*)$ is a radially symmetric non-topological solution of the Chern-Simons-Higgs equation (1.14). Consequently, $u_1^* < -\ln(1 + b_1^*)$ in $\mathbb{R}^2$, and

$$u_1^*(|x|) = -2\gamma_1 \ln|x| + O(1) \text{ as } |x| \to \infty \tag{3.9}$$

for some constant $\gamma_1 > N_1 + 2$. Moreover,

$$\int_{\mathbb{R}^2} \left[(1 + b_1^*) e^{u_1^*} - (1 + b_1^*)^2 e^{2u_1^*} \right] dx = 4\pi(\gamma_1 + N_1), \tag{3.10}$$

$$\int_{\mathbb{R}^2} e^{2u_1^*} dx = \frac{4\pi}{(1 + b_1^*)^2} \left[(\gamma_1 - 1)^2 - (N_1 + 1)^2 \right]. \tag{3.11}$$

See [4] for details. On the other hand, since u_2^* is radially symmetric, it follows easily from (3.8) and (3.10) that

$$u_2^*(|x|) = -2\gamma_2 \ln|x| + O(1) \text{ as } |x| \to \infty, \tag{3.12}$$

where

$$\gamma_2 = -\frac{b_2^*}{1 + b_1^*}(\gamma_1 + N_1) - N_2. \tag{3.13}$$

Note that $u_{1,n} \le v_{1,n}$ and $v_{1,n} \to v_1^*$ in $C^2_{loc}(\mathbb{R}^2)$. Combining these with (2.39), we easily deduce from the Lebesgue convergence theorem that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} e^{2u_{1,n}} dx = \int_{\mathbb{R}^2} e^{2u_1^*} dx = \frac{4\pi}{(1 + b_1^*)^2} \left[(\gamma_1 - 1)^2 - (N_1 + 1)^2 \right]. \tag{3.14}$$

Furthermore, (3.6) gives

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} e^{2u_{2,n}} dx = \lim_{n\to\infty} \int_{\mathbb{R}^2} e^{u_{1,n} + u_{2,n}} dx = 0. \tag{3.15}$$

Consequently, we can take a sequence $R_n \to \infty$, and then choose a subsequence which is still denoted by $(u_{1,n}, u_{2,n})$, such that

$$\|v_{1,n} - v_1^*\|_{L^\infty(B_{R_n})} + \|v_{2,n} - v_{2,n}(0) - v_2^*\|_{L^\infty(B_{R_n})} \to 0 \text{ as } n \to \infty$$

and

$$\lim_{n\to\infty} \int_{B_{R_n}} e^{u_{2,n}} dx = 0 \text{ and } \lim_{n\to\infty} \int_{B_{R_n}} e^{l u_{1,n}} dx = \int_{\mathbb{R}^2} e^{l u_1^*} dx, \quad l = 1, 2. \tag{3.16}$$

On the other hand, by (2.37) and (3.15) we have

$$(1+b_1+b_2)\int_0^\infty re^{u_{2,n}}dr = 2b_2(\beta_1+N_1)+2(1+b_1)(\beta_2+N_2)+o(1),$$

so it follows from (3.16) that

$$\lim_{n\to\infty}\int_{R_n}^\infty re^{u_{2,n}}dr = \frac{2b_2^*(\beta_1^*+N_1)+2(1+b_1^*)(\beta_2^*+N_2)}{1+b_1^*+b_2^*}. \tag{3.17}$$

Lemma 3.2. *For $k=1,2$, there hold*

$$R_n u'_{k,n}(R_n) = -2\gamma_k + o(1), \quad \text{as } n\to\infty, \tag{3.18}$$

$$u_{k,n}(R_n) + 2\ln R_n \to -\infty, \quad \text{as } n\to\infty. \tag{3.19}$$

Proof. By (2.16) and (3.14)-(3.16), we easily deduce that

$$R_n u'_{1,n}(R_n) - 2N_1 \to \int_0^\infty r\left[-(1+b_1^*)e^{u_1^*}+(1+b_1^*)^2e^{2u_1^*}\right]dr \ \text{ as } n\to\infty,$$

$$R_n u'_{2,n}(R_n) - 2N_2 \to b_2^*\int_0^\infty r\left[e^{u_1^*}-(1+b_1^*)e^{2u_1^*}\right]dr \ \text{ as } n\to\infty.$$

Hence, (3.10) and (3.13) give (3.18).

Since $v_{1,n}(R_n) - v_1^*(R_n)\to 0$ and

$$f_1(R_n) - f_1^*(R_n) = (\beta_1(t_*)-\beta_1(t_n))\ln(1+R_n^2) = o(\ln R_n),$$

we obtain $u_{1,n}(R_n) = u_1^*(R_n) + o(\ln R_n)$. This, together with (3.9), proves (3.19) for $k=1$. For $k=2$, (2.38) gives $|u'_{2,n}(r)|\le C/r$ for $r\ge 1$, where $C>0$ is independent of n. Consequently, $u_{2,n}(r)\ge u_{2,n}(R_n)+C\ln\frac{r}{R_n}$ for all $R_n/2\le r\le R_n$, and

$$\int_{\frac{R_n}{2}}^{R_n} re^{u_{2,n}(r)}dr \ge e^{u_{2,n}(R_n)}R_n^{-C}\int_{\frac{R_n}{2}}^{R_n} r^{C+1}dr = C_1e^{u_{2,n}(R_n)}R_n^2.$$

Combining this with (3.16), we obtain (3.19) for $k=2$. □

To study the concentration near infinity, we define the Kelvin transform of $(u_{1,n},u_{2,n})$ by

$$w_{k,n}(x) = u_{k,n}\left(\frac{R_n x}{|x|^2}\right) + 2\ln R_n - 4\ln|x|, \quad |x|\le 1. \tag{3.20}$$

Then $(w_{1,n},w_{2,n})$ satisfies

$$\begin{cases}\Delta w_{1,n} = -(1+b_1)e^{w_{1,n}}+b_1e^{w_{2,n}}+G_{1,n}+4\pi(\beta_1-2)\delta_0\\ \Delta w_{2,n} = -(1+b_2)e^{w_{2,n}}+b_2e^{w_{1,n}}+G_{2,n}+4\pi(\beta_2-2)\delta_0\end{cases} \text{in } B_1, \tag{3.21}$$

where

$$G_{1,n}(x) = \frac{|x|^4}{R_n^2}\left[(1+b_1)^2e^{2w_{1,n}}-b_1(1+b_2)e^{2w_{2,n}}-b_1(1+b_1-b_2)e^{w_{1,n}+w_{2,n}}\right],$$

$$G_{2,n}(x) = \frac{|x|^4}{R_n^2}\left[(1+b_2)^2e^{2w_{2,n}}-b_2(1+b_1)e^{2w_{1,n}}-b_2(1+b_2-b_1)e^{w_{1,n}+w_{2,n}}\right].$$

We recall that $b_k=b_k(t_n)$ and $\beta_k=\beta_k(t_n)$ in (3.21) and the expressions of $G_{k,n}$, and we write them by b_k and β_k respectively for convenience.

Lemma 3.2 gives

$$w'_{k,n}(1) = 2\gamma_k - 4 + o(1) \ \text{ and } \ w_{k,n}(1)\to-\infty \ \text{ as } n\to\infty \text{ for } k=1,2. \tag{3.22}$$

Moreover, by (2.39) we have

$$w_{k,n}(x) \le -2\ln|x| + C \ \text{ for } |x|\le 1, \ \ k=1,2, \tag{3.23}$$

which implies that $|G_{k,n}(x)| \le CR_n^{-2}$ uniformly for $|x|\le 1$. Therefore, we can derive from (3.22) and (3.23) that

$$2(\gamma_1-\beta_1) = -(1+b_1)\int_0^1 re^{w_{1,n}}dr + b_1\int_0^1 re^{w_{2,n}}dr + o(1), \tag{3.24}$$

$$2(\gamma_2-\beta_2) = -(1+b_2)\int_0^1 re^{w_{2,n}}dr + b_2\int_0^1 re^{w_{1,n}}dr + o(1). \tag{3.25}$$

Moreover, (2.38) yields $|rw'_{k,n}(r)| \le C$ for all r. Then it follows from (3.22) that $w_{k,n}\to-\infty$ uniformly in any compact subset of $\overline{B_1}\setminus\{0\}$. We note from (3.17) that

$$\int_0^1 re^{w_{2,n}}dr = \int_{R_n}^\infty re^{u_{2,n}}dr = \frac{2b_2^*(\beta_1^*+N_1)+2(1+b_1^*)(\beta_2^*+N_2)}{1+b_1^*+b_2^*} + o(1). \tag{3.26}$$

Lemma 3.3. *For any $0<r\le 1$, there holds*

$$\begin{aligned} &J_{t_n}\left(\frac{1}{2}rw'_{1,n}(r)+1,\ \frac{1}{2}rw'_{2,n}(r)+1\right)\\ &= J_{t_n}(\beta_1-1,\beta_2-1) - \frac{1+b_1+b_2}{4b_1b_2}\sum_{k=1}^2 b_kr^2e^{w_{k,n}(r)} + o(1). \end{aligned} \tag{3.27}$$

In particular,

$$J^*(\gamma_1-1,\gamma_2-1) = J^*(\beta_1^*-1,\beta_2^*-1). \tag{3.28}$$

Proof. The proof is similar to that of Lemma 2.2. Since $(w_{1,n},w_{2,n})$ is radially symmetric, system (3.21) can be written as

$$(1+b_2)(rw'_{1,n})' + b_1(rw'_{2,n})' = -(1+b_1+b_2)re^{w_{1,n}} + (1+b_2)rG_{1,n}(r) + b_1rG_{2,n}(r), \tag{3.29}$$

$$b_2(rw'_{1,n})' + (1+b_1)(rw'_{2,n})' = -(1+b_1+b_2)re^{w_{2,n}} + (1+b_1)rG_{2,n}(r) + b_2rG_{1,n}(r), \tag{3.30}$$

for $0<r<1$. As before, we multiply (3.29) by $b_2rw'_{1,n}$, (3.30) by $b_1rw'_{2,n}$ and plus together, which yields

$$\frac{d}{dr}J_{t_n}(rw'_{1,n},rw'_{2,n}) = -(1+b_1+b_2)r^2\left[b_2\left(e^{w_{1,n}}\right)' + b_1\left(e^{w_{2,n}}\right)'\right] + \mathcal{G}_n, \tag{3.31}$$

where

$$\mathcal{G}_n(r) = \frac{(1+b_1+b_2)r^6}{2R_n^2}\frac{d}{dr}\left[b_2(1+b_1)e^{2w_{1,n}} - 2b_1b_2e^{w_{1,n}+w_{2,n}} + b_1(1+b_2)e^{2w_{2,n}}\right].$$

By (3.23) and $|rw'_{k,n}(r)|\le C$, we easily get $\|\mathcal{G}_n\|_{L^\infty(B_1)}\le CR_n^{-2}$. Note that for each n, $rw'_{k,n}(r)\to 2\beta_k(t_n)-4$ as $r\to 0$. By integrating (3.31) over $(0,r)$, we obtain

$$\begin{aligned} &J_{t_n}(rw'_{1,n}(r),\ rw'_{2,n}(r)) - J_{t_n}(2\beta_1-4,2\beta_2-4)\\ &= -(1+b_1+b_2)\left(b_2r^2e^{w_{1,n}(r)} + b_1r^2e^{w_{2,n}(r)}\right)\\ &\quad + 2(1+b_1+b_2)\int_0^r \rho\left(b_2e^{w_{1,n}} + b_1e^{w_{2,n}}\right)d\rho + o(1). \end{aligned} \tag{3.32}$$

On the other hand, we integrate (3.29)-(3.30) and obtain

$$\begin{aligned} &\int_0^r \rho\left(b_2e^{w_{1,n}} + b_1e^{w_{2,n}}\right)d\rho\\ =&b_2(2\beta_1-4) + b_1(2\beta_2-4) - b_2rw'_{1,n}(r) - b_1rw'_{2,n}(r) + o(1). \end{aligned}$$

This, together with (3.32), yields (3.27). Finally, recalling $t_n\to t_*$ and (3.22), (3.28) follows from (3.27) by taking $r=1$ and letting $n\to\infty$. □

Lemma 3.4.
$$\liminf_{n\to\infty}\int_{|x|\le 1} e^{w_{1,n}}dx = \liminf_{n\to\infty}\int_{|x|\ge R_n} e^{u_{1,n}}dx > 0.$$

Proof. This proof is different from that in [8] and seems simpler. More importantly, this new proof can be applied directly in Section 5, where we will construct bubbling solutions to prove Theorem 1.3; see Lemma 5.6 below. Assume by contradiction that, up to a subsequence, $\int_0^1 re^{w_{1,n}}dr = \int_{R_n}^\infty re^{u_{1,n}}dr \to 0$. Then by (3.24)-(3.26) we obtain

$$\gamma_1 - \beta_1^* = \frac{b_1^*}{1+b_1^*+b_2^*}\left[b_2^*(\beta_1^*+N_1) + (1+b_1^*)(\beta_2^*+N_2)\right], \tag{3.33}$$
$$\gamma_2 - \beta_2^* = -\frac{1+b_2^*}{1+b_1^*+b_2^*}\left[b_2^*(\beta_1^*+N_1) + (1+b_1^*)(\beta_2^*+N_2)\right]. \tag{3.34}$$

As before, for the sake of convenience, we denote

$$\tilde\gamma_k = \gamma_k - 1,\ \ \tilde\beta_k^* = \beta_k^* - 1,\ \ \tilde N_k = N_k + 1.$$

Then (3.33)-(3.34) become

$$(1+b_1^*+b_2^*)\tilde\gamma_1 = b_1^*\big[\underbrace{(1+b_1^*)\tilde N_2 + b_2^*\tilde N_1}_{=:X}\big] + (1+b_1^*)\big[\underbrace{(1+b_2^*)\tilde\beta_1^* + b_1^*\tilde\beta_2^*}_{=:Y}\big],$$
$$(1+b_1^*+b_2^*)\tilde\gamma_2 = -(1+b_2^*)\big[(1+b_1^*)\tilde N_2 + b_2^*\tilde N_1\big] - b_2^*\big[(1+b_2^*)\tilde\beta_1^* + b_1^*\tilde\beta_2^*\big].$$

Recalling the definition (2.17) of J_t and $J^* = J_{t_*}$, a direct calculation gives

$$\begin{aligned} J^*(\tilde\gamma_1,\tilde\gamma_2) &= \frac{1}{(1+b_1^*+b_2^*)^2}J^*\big(b_1^*X + (1+b_1^*)Y,\ -(1+b_2^*)X - b_2^*Y\big) \\ &= \frac{1}{1+b_1^*+b_2^*}\tilde J^*(X,Y), \end{aligned}$$

where $\tilde J^*$ is defined by

$$\tilde J^*(x,y) := \frac{b_1^*(1+b_2^*)}{2}x^2 + b_1^*b_2^*xy + \frac{b_2^*(1+b_1^*)}{2}y^2.$$

Denote $Z = b_2^*\tilde\beta_1^* + (1+b_1^*)\tilde\beta_2^*$. Then we can compute

$$\begin{aligned} \tilde J^*(-Z,Y) &= \tilde J^*\big(-b_2^*\tilde\beta_1^* - (1+b_1^*)\tilde\beta_2^*,\ (1+b_2^*)\tilde\beta_1^* + b_1^*\tilde\beta_2^*\big) \\ &= (1+b_1^*+b_2^*)J(\tilde\beta_1^*,\tilde\beta_2^*). \end{aligned}$$

Combining these with $J(\tilde\gamma_1,\tilde\gamma_2) = J(\tilde\beta_1^*,\tilde\beta_2^*)$ by (3.28), we obtain $\tilde J^*(X,Y) = \tilde J^*(-Z,Y)$, which implies from the definition of $\tilde J^*$ that $(1+b_2^*)(X-Z) + 2b_2^*Y = 0$. Recalling the expressions of X, Y, Z, we finally obtain

$$\begin{aligned} &[(1+b_1^*)(1+b_2^*) - 2b_1^*b_2^*]\beta_2^* - b_2^*(1+b_2^*)\beta_1^* \\ &\quad = b_2^*(1+b_2^*)N_1 + (1+b_1^*)(1+b_2^*)N_2 + 2(1+b_1^*+b_2^*). \end{aligned} \tag{3.35}$$

However, since $(\beta_1^*,\beta_2^*) = (\beta_1(t_*),\beta_2(t_*)) \in S_{t_*}\cap\Omega_{t_*}$, so (β_1^*,β_2^*) satisfies (2.20) with $(b_1,b_2) = (b_1^*,b_2^*)$, which contradicts to (3.35). □

Remark 3.1. *For the application in Section 5 where we will construct bubbling solutions, we point out the following fact. By* (3.35) *we can eliminate the term* $b_2^*N_1 + (1+b_1^*)N_2$ *in the formula* (3.33)*, which yields*

$$\gamma_1 = \beta_1^* + \frac{2b_1^*}{1+b_2^*}(\beta_2^* - 1).$$

This, together with (3.35) *again, gives*

$$\beta_1^* = \frac{(1+b_1^*)(1+b_2^*) - 2b_1^* b_2^*}{(1+b_1^*)(1+b_2^*)}\gamma_1 - \frac{2b_1^* b_2^*}{(1+b_1^*)(1+b_2^*)}N_1 - \frac{2b_1^*}{1+b_2^*}(N_2+1),$$
$$\beta_2^* = \frac{b_2^*}{1+b_1^*}(\gamma_1 + N_1) + N_2 + 2.$$

Therefore, these three formulas hold provided $\int_{R_n}^\infty re^{u_{1,n}}dr \to 0$ *as* $n \to \infty$.

To continue our proof, we need to consider the regular part of $w_{k,n}$. Define

$$V_{k,n}(x) := w_{k,n}(x) - (2\beta_k(t_n) - 4)\ln|x|, \quad k = 1, 2.$$

Then Lemma 3.4 and (3.26) yield

$$\liminf_{n\to\infty} \int_{|x|\le 1} |x|^{2\beta_k - 4} e^{V_{k,n}} dx = \liminf_{n\to\infty} \int_{|x|\le 1} e^{w_{k,n}} dx > 0, \quad k = 1, 2.$$

Since $w_{k,n} \to -\infty$ uniformly in any compact subset of $\overline{B_1} \setminus \{0\}$ and $2\beta_k - 4 \ge -2 + 2c_0$ by (2.25), we must have that $\max_{|x|\le 1} V_{k,n}(x) \to +\infty$ for $k = 1, 2$. Let $z_{k,n} \in B_1$ be a maximum point of $V_{k,n}$ in $\overline{B_1}$, and define

$$\mu_{k,n} := \exp\left(-\frac{1}{2\beta_k - 2}V_{k,n}(z_{k,n})\right), \quad k = 1, 2. \tag{3.36}$$

Then $\mu_{k,n} \to 0$ for $k = 1, 2$. On the other hand, (3.23) gives

$$V_{k,n}(x) \le -(2\beta_k - 2)\ln|x| + C \ \text{ for } |x| \le 1, \quad k = 1, 2, \tag{3.37}$$

so $|z_{k,n}|/\mu_{k,n} \le C$ for some $C > 0$ independent of n.

Lemma 3.5. $\mu_{1,n}/\mu_{2,n} \le C$ *for some constant* $C > 0$ *independent of* n.

Proof. Assume by contradiction that, up to a subsequence, $\mu_{1,n}/\mu_{2,n} \to \infty$ as $n \to \infty$. The most part of the following proof is close to that of [8, Lemma 3.5]. Here due to our general (b_1, b_2), some new ideas are also developed.

Step 1. Consider the scaled function

$$\overline{V}_{k,n}(x) = V_{k,n}(\mu_{2,n}x) + (2\beta_k - 2)\ln\mu_{2,n} \ \text{ for } |x| \le 1/\mu_{2,n}, \quad k = 1, 2.$$

Then $(\overline{V}_{1,n}, \overline{V}_{2,n})$ satisfies

$$\begin{cases} \Delta\overline{V}_{1,n} = -(1+b_1)|x|^{2\beta_1 - 4}e^{\overline{V}_{1,n}} + b_1|x|^{2\beta_2 - 4}e^{\overline{V}_{2,n}} + \mu_{2,n}^2 G_{1,n}(\mu_{2,n}x), \\ \Delta\overline{V}_{2,n} = -(1+b_2)|x|^{2\beta_2 - 4}e^{\overline{V}_{2,n}} + b_2|x|^{2\beta_1 - 4}e^{\overline{V}_{1,n}} + \mu_{2,n}^2 G_{2,n}(\mu_{2,n}x) =: f_n. \end{cases} \tag{3.38}$$

Note that

$$\overline{V}_{1,n}(x) \le V_{1,n}(z_{1,n}) + (2\beta_1 - 2)\ln\mu_{2,n} = (2\beta_1 - 2)\ln\frac{\mu_{2,n}}{\mu_{1,n}} \to -\infty$$

uniformly for $|x| \le 1/\mu_{2,n}$. Moreover, $\overline{V}_{2,n}(x) \le \overline{V}_{2,n}(\frac{z_{2,n}}{\mu_{2,n}}) = 0$ for $|x| \le 1/\mu_{2,n}$. Recalling $|G_{k,n}(\mu_{2,n}x)| \le CR_n^{-2}$ for $|x| \le 1/\mu_{2,n}$, we conclude from $2\beta_k - 4 \ge -2 + 2c_0$ that f_n are uniformly bounded in $L^s_{loc}(B_{1/\mu_{2,n}})$ for some $s > 1$. Consequently, the Harnack inequality yields that $\overline{V}_{2,n}$ are uniformly bounded in $L^\infty_{loc}(B_{1/\mu_{2,n}})$. Up to a subsequence, we may assume that $\overline{V}_{2,n} \to \overline{V}_2$ in $C^2_{loc}(\mathbb{R}^2)$, where $\overline{V}_2$ satisfies

$$\begin{aligned} &\Delta\overline{V}_2 = -(1+b_2^*)|x|^{2\beta_2^* - 4}e^{\overline{V}_2} \ \text{ in } \mathbb{R}^2, \\ &\int_{\mathbb{R}^2} |x|^{2\beta_2^* - 4}e^{\overline{V}_2}dx \le \liminf_{n\to\infty}\int_{|x|\le 1} e^{w_{2,n}}dx \le C. \end{aligned} \tag{3.39}$$

Consequently, we see from [24] that

$$\int_{\mathbb{R}^2} |x|^{2\beta_2^*-4} e^{\overline{V}_2} dx = \frac{8\pi(\beta_2^*-1)}{1+b_2^*}. \tag{3.40}$$

As before, we can take a sequence $r_n \to \infty$, and then choose a subsequence still denoted by $(\overline{V}_{1,n}, \overline{V}_{2,n})$, such that $r_n \le \sqrt{\mu_{1,n}/\mu_{2,n}}$, $\|\overline{V}_{2,n} - \overline{V}_2\|_{L^\infty(B_{r_n})} \to 0$ and

$$\lim_{n\to\infty} \int_{B_{r_n}} |x|^{2\beta_1-4} e^{\overline{V}_{1,n}} dx = 0, \quad \lim_{n\to\infty} \int_{B_{r_n}} |x|^{2\beta_2-4} e^{\overline{V}_{2,n}} dx = \frac{8\pi(\beta_2^*-1)}{1+b_2^*}. \tag{3.41}$$

Step 2. Consider another scaled function:

$$\begin{aligned} U_{k,n}(x) &= V_{k,n}(\mu_{1,n}x) + (2\beta_k - 2)\ln \mu_{1,n} \\ &= \overline{V}_{k,n}\left(\frac{\mu_{1,n}}{\mu_{2,n}}x\right) + (2\beta_k-2)\ln\frac{\mu_{1,n}}{\mu_{2,n}} \quad \text{for } |x| \le 1/\mu_{1,n}. \end{aligned}$$

Then $(U_{1,n}, U_{2,n})$ satisfies

$$\begin{cases} \Delta U_{1,n} = -(1+b_1)|x|^{2\beta_1-4}e^{U_{1,n}} + b_1|x|^{2\beta_2-4}e^{U_{2,n}} + \mu_{1,n}^2 G_{1,n}(\mu_{1,n}x), \\ \Delta U_{2,n} = -(1+b_2)|x|^{2\beta_2-4}e^{U_{2,n}} + b_2|x|^{2\beta_1-4}e^{U_{1,n}} + \mu_{1,n}^2 G_{2,n}(\mu_{1,n}x). \end{cases} \tag{3.42}$$

By (3.37), we have

$$U_{k,n}(x) \le -(2\beta_k-2)\ln|x| + C \ \text{ for } |x| \le 1/\mu_{1,n}, \ \ k=1,2. \tag{3.43}$$

Moreover, $U_{1,n}(x) \le U_{1,n}(\frac{z_{1,n}}{\mu_{1,n}}) = 0$ for $|x| \le 1/\mu_{1,n}$. Recall $\frac{|z_{1,n}|}{\mu_{1,n}} \le C$ and $\mu_{1,n}^2 G_{1,n}(\mu_{1,n}x) = o(1)$ uniformly for $|x| \le 1/\mu_{1,n}$. For any $r > \frac{|z_{1,n}|}{\mu_{1,n}}$, we derive from (3.42) and $e^{U_{1,n}} \le 1$ that

$$\begin{aligned} rU_{1,n}'(r) &= \int_{\frac{|z_{1,n}|}{\mu_{1,n}}}^r (\rho U_{1,n}')' d\rho \ge -(1+b_1)\int_{\frac{|z_{1,n}|}{\mu_{1,n}}}^r \rho^{2\beta_1-3}e^{U_{1,n}}d\rho + o(1)\cdot r^2 \\ &\ge -Cr^{2\beta_1-2} + o(1)\cdot r^2, \end{aligned}$$

for some constant $C>0$. Consequently,

$$\begin{aligned} U_{1,n}(r) &= \int_{\frac{|z_{1,n}|}{\mu_{1,n}}}^r U_{1,n}'(\rho)d\rho \ge -C\int_{\frac{|z_{1,n}|}{\mu_{1,n}}}^r \rho^{2\beta_1-3}d\rho + o(1)\cdot r^2 \\ &\ge -Cr^{2\beta_1-2} + o(1)\cdot r^2, \quad \forall\, r > \frac{|z_{1,n}|}{\mu_{1,n}}. \end{aligned}$$

Hence, $U_{1,n}$ are uniformly bounded in $L^\infty_{loc}(\mathbb{R}^2\setminus\{0\})$.

For any fixed $r>0$, we have $\frac{\mu_{1,n}}{\mu_{2,n}}r > r_n$ for n large because of $r_n \le \sqrt{\mu_{1,n}/\mu_{2,n}}$. Then it follows from (3.41) that

$$\liminf_{n\to\infty}\int_{|x|\le r}|x|^{2\beta_2-4}e^{U_{2,n}}dx = \liminf_{n\to\infty}\int_{|x|\le\frac{\mu_{1,n}}{\mu_{2,n}}r}|x|^{2\beta_2-4}e^{\overline{V}_{2,n}}dx \ge \frac{8\pi(\beta_2^*-1)}{1+b_2^*}. \tag{3.44}$$

Now we claim that $U_{2,n} \to -\infty$ uniformly on any compact subset of $\mathbb{R}^2\setminus\{0\}$. Assume by contradiction that, up to a subsequence, $\sup_{\mathcal{K}} U_{2,n} \ge -C > -\infty$ for some compact subset $\mathcal{K} \subset \mathbb{R}^2\setminus\{0\}$. By (2.38), we easily deduce that $|rU_{2,n}'(r)| \le C$ for all r. Together these with (3.43), it follows that $U_{2,n}$ are uniformly bounded in $L^\infty_{loc}(\mathbb{R}^2\setminus\{0\})$. Up to a subsequence, we may assume that $U_{2,n} \to U_2$ in $C^2_{loc}(\mathbb{R}^2\setminus\{0\})$. For any fixed $r\in(0,1]$, again by (3.42) and $e^{U_{1,n}} \le 1$, we have

$$rU_{2,n}'(r) = b_2\int_0^r \rho^{2\beta_1-3}e^{U_{1,n}}d\rho - (1+b_2)\int_0^r r^{2\beta_2-3}e^{U_{2,n}}d\rho + o(1)\cdot r^2$$

$$\leq Cr^{2\beta_1-2} - (1+b_2)\int_0^r r^{2\beta_2-3}e^{U_{2,n}}d\rho + o(1)\cdot r^2.$$

Letting $n\to\infty$, it follows from (3.44) that $rU_2'(r) \leq Cr^{2\beta_1^*-2} - 4(\beta_2^*-1)$ holds for all $r\leq 1$. Then there exists $r_0>0$ small such that $rU_2'(r)\leq -3(\beta_2^*-1)$ for any $r\leq r_0$. Consequently, $U_2(r)\geq -3(\beta_2^*-1)\ln r + C$ for any $r\leq r_0$, where C is a constant independent of r. Hence

$$\begin{aligned}\infty > \liminf_{n\to\infty}\int_{|x|\leq r_0}|x|^{2\beta_2-4}e^{U_{2,n}}dx &\geq \int_{|x|\leq r_0}|x|^{2\beta_2^*-4}e^{U_2}dx\\ \geq C\int_{|x|\leq r_0}|x|^{-\beta_2^*-1}dx = \infty,\end{aligned}$$

which is a contradiction. This proves the claim.

Consequently, up to a subsequence, we may assume that

$$|x|^{2\beta_2-4}e^{U_{2,n}} \to \frac{8\pi}{1+b_2^*}(\beta_2^*-1+\varepsilon)\delta_0 \ \text{ in } B_r \tag{3.45}$$

in the distribution sense for any $r>0$, where $\varepsilon\geq 0$ is a constant. By the diagonal process, we may further assume that $U_{1,n}\to U_1$ in $C^2_{loc}(\mathbb{R}^2\setminus\{0\})$, where U_1 satisfies

$$\begin{aligned}&\Delta U_1 = -(1+b_1^*)|x|^{2\beta_1^*-4}e^{U_1} + \frac{8b_1^*\pi}{1+b_2^*}(\beta_2^*-1+\varepsilon)\delta_0 \ \text{ in } \mathbb{R}^2,\\ &\int_{\mathbb{R}^2}|x|^{2\beta_1^*-4}e^{U_1}dx \leq \liminf_{n\to\infty}\int_{|x|\leq 1}e^{w_{1,n}}dx \leq C.\end{aligned}\tag{3.46}$$

Again by [24] we have

$$\int_{\mathbb{R}^2}|x|^{2\beta_1^*-4}e^{U_1}dx = \frac{8\pi}{1+b_1^*}\left[\frac{2b_1^*}{1+b_2^*}(\beta_2^*-1+\varepsilon)+\beta_1^*-1\right], \tag{3.47}$$

$$U_1(x) = -4\left[\frac{b_1^*}{1+b_2^*}(\beta_2^*-1+\varepsilon)+\beta_1^*-1\right]\ln|x| + O(1), \ \text{ as } |x|\to\infty. \tag{3.48}$$

Step 3. We claim that $\varepsilon=0$. Recall that $U_{2,n}\to-\infty$ uniformly on any compact subset of $\mathbb{R}^2\setminus\{0\}$ and $|rU_{2,n}'(r)|\leq C$ for all r. Up to a subsequence, we may assume $U_{2,n}-U_{2,n}(1)\to W_2$ in $C^2_{loc}(\mathbb{R}^2\setminus\{0\})$, where

$$\Delta W_2 = b_2^*|x|^{2\beta_1^*-4}e^{U_1} - 8\pi(\beta_2^*-1+\varepsilon)\delta_0 \ \text{ in } \mathbb{R}^2.$$

By (3.47) we can prove

$$W_2(x) = 4\left[\frac{b_2^*(\beta_1^*-1)}{1+b_1^*} - \frac{(1+b_1^*)(1+b_2^*)-2b_1^*b_2^*}{(1+b_1^*)(1+b_2^*)}(\beta_2^*-1+\varepsilon)\right]\ln|x| + O(1) \tag{3.49}$$

as $|x|\to\infty$. Remark that, for any fixed $s>0$, $U_{2,n}(s)+(2\beta_2-2)\ln s\to-\infty$ as $n\to\infty$. Hence, as before, by (3.45) and (3.47), we can take a sequence $s_n\to\infty$ first, and then choose a subsequence still denoted by $(U_{1,2},U_{2,n})$, such that $s_n<1/\mu_{1,n}$, $|U_{1,n}(s_n)-U_1(s_n)|\to 0$, $U_{2,n}(s_n)+(2\beta_2-2)\ln s_n\to-\infty$ and

$$\lim_{n\to\infty}\int_{|x|\leq s_n}|x|^{2\beta_1-4}e^{U_{1,n}}dx = \frac{8\pi}{1+b_1^*}\left[\frac{2b_1^*}{1+b_2^*}(\beta_2^*-1+\varepsilon)+\beta_1^*-1\right], \tag{3.50}$$

$$\lim_{n\to\infty}\int_{|x|\leq s_n}|x|^{2\beta_2-4}e^{U_{2,n}}dx = \frac{8\pi}{1+b_2^*}(\beta_2^*-1+\varepsilon). \tag{3.51}$$

By (3.48), we also have $U_{1,n}(s_n)+(2\beta_1-2)\ln s_n\to-\infty$. Recall that

$$w_{k,n}(|x|)+2\ln|x| = U_{k,n}\left(\frac{|x|}{\mu_{1,n}}\right) + (2\beta_k-2)\ln\frac{|x|}{\mu_{1,n}}, \ \ k=1,2.$$

Hence,

$$w_{k,n}(\mu_{1,n}s_n) + 2\ln(\mu_{1,n}s_n) \to -\infty, \quad k = 1, 2. \tag{3.52}$$

On the other hand, we easily deduce from (3.42) and (3.50)-(3.51) that

$$\begin{aligned} \lim_{n\to\infty} s_n U'_{1,n}(s_n) &= -4\left[\frac{b_1^*}{1+b_2^*}(\beta_2^* - 1 + \varepsilon) + \beta_1^* - 1\right], \\ \lim_{n\to\infty} s_n U'_{2,n}(s_n) &= \frac{4b_2^*}{1+b_1^*}\left[\frac{2b_1^*}{1+b_2^*}(\beta_2^* - 1 + \varepsilon) + \beta_1^* - 1\right] - 4(\beta_2^* - 1 + \varepsilon). \end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \mu_{1,n}s_n w'_{1,n}(\mu_{1,n}s_n) = -2\beta_1^* - \frac{4b_1^*}{1+b_2^*}(\beta_2^* - 1 + \varepsilon), \tag{3.53}$$

$$\begin{aligned} \lim_{n\to\infty} \mu_{1,n}s_n w'_{2,n}(\mu_{1,n}s_n) =& \frac{4b_2^*(\beta_1^* - 1)}{1+b_1^*} + \left[\frac{8b_1^*b_2^*}{(1+b_1^*)(1+b_2^*)} - 2\right](\beta_2^* - 1 + \varepsilon) \\ & - 2 - 2\varepsilon. \end{aligned} \tag{3.54}$$

Recall $\tilde{\beta}_k^* = \beta_k^* - 1$. Combining these with (3.52) and Lemma 3.3, we can compute via Lemma 2.4 that

$$\begin{aligned} & J^*(\tilde{\beta}_1^*, \tilde{\beta}_2^*) \\ =& J^*\Bigg(\underbrace{-\tilde{\beta}_1^* - \frac{2b_1^*}{1+b_2^*}(\tilde{\beta}_2^* + \varepsilon)}_{=:X}, \frac{2b_2^*\tilde{\beta}_1^*}{1+b_1^*} + \left[\frac{4b_1^*b_2^*}{(1+b_1^*)(1+b_2^*)} - 1\right](\tilde{\beta}_2^* + \varepsilon) - \varepsilon\Bigg) \\ =& J^*\left(X, -\frac{2b_2^*}{1+b_1^*}X - (\tilde{\beta}_2^* + 2\varepsilon)\right) = J^*\big(-X, -(\tilde{\beta}_2^* + 2\varepsilon)\big) \\ =& J^*\left(\left(\tilde{\beta}_1^* - \frac{2b_1^*}{1+b_2^*}\varepsilon\right) + \frac{2b_1^*}{1+b_2^*}(\tilde{\beta}_2^* + 2\varepsilon), -(\tilde{\beta}_2^* + 2\varepsilon)\right) \\ =& J^*\left(\tilde{\beta}_1^* - \frac{2b_1^*}{1+b_2^*}\varepsilon, \tilde{\beta}_2^* + 2\varepsilon\right) \\ =& J^*(\tilde{\beta}_1^*, \tilde{\beta}_2^*) + \frac{2b_1^*(1+b_1^*+b_2^*)}{1+b_2^*}(\beta_2^* + \varepsilon)\varepsilon. \end{aligned}$$

Hence $\varepsilon = 0$.

Recalling $\int_{|x|\le\mu_{1,n}s_n} e^{w_{k,n}}dx = \int_{|x|\le s_n} |x|^{2\beta_k - 4}e^{U_{k,n}}dx$, it follows from (3.50)-(3.51) that

$$\lim_{n\to\infty}\int_{|x|\le\mu_{1,n}s_n} e^{w_{1,n}}dx = \frac{8\pi}{1+b_1^*}\left[\frac{2b_1^*}{1+b_2^*}(\beta_2^* - 1) + \beta_1^* - 1\right], \tag{3.55}$$

$$\lim_{n\to\infty}\int_{|x|\le\mu_{1,n}s_n} e^{w_{2,n}}dx = \frac{8\pi}{1+b_2^*}(\beta_2^* - 1). \tag{3.56}$$

Step 4. Recall (3.55)-(3.56) and $\mu_{1,n}s_n < 1$. Up to a subsequence, we assume

$$\begin{aligned} \lim_{n\to\infty}\int_{|x|\le 1} e^{w_{1,n}}dx &= 4\pi\left[\frac{2}{1+b_1^*}\left(\frac{2b_1^*}{1+b_2^*}\tilde{\beta}_2^* + \tilde{\beta}_1^*\right) + \varepsilon_1\right], \\ \lim_{n\to\infty}\int_{|x|\le 1} e^{w_{2,n}}dx &= 4\pi\left(\frac{2}{1+b_2^*}\tilde{\beta}_2^* + \varepsilon_2\right), \end{aligned}$$

where $\varepsilon_1, \varepsilon_2 \ge 0$ are constants. This, together with (3.24), gives

$$\gamma_1 - \beta_1^* = -2\left(\frac{2b_1^*}{1+b_2^*}\tilde{\beta}_2^* + \tilde{\beta}_1^*\right) - (1+b_1^*)\varepsilon_1 + b_1^*\left(\frac{2}{1+b_2^*}\tilde{\beta}_2^* + \varepsilon_2\right).$$

Since $\gamma_1 > N_1 + 2 = \tilde{N}_1 + 1$, we easily conclude that

$$\varepsilon_2 > \frac{\tilde{\beta}_1^* + \tilde{N}_1}{b_1^*} + \frac{2}{1+b_2^*}\tilde{\beta}_2^*. \tag{3.57}$$

On the other hand, (3.26) gives

$$\lim_{n\to\infty}\int_{|x|\le 1} e^{w_{2,n}}dx = 4\pi\frac{b_2^*(\beta_1^*+N_1)+(1+b_1^*)(\beta_2^*+N_2)}{1+b_1^*+b_2^*},$$

so

$$\frac{b_2^*(\beta_1^*+N_1)+(1+b_1^*)(\beta_2^*+N_2)}{1+b_1^*+b_2^*} > \frac{4}{1+b_2^*}\tilde{\beta}_2^* + \frac{\tilde{\beta}_1^*+\tilde{N}_1}{b_1^*},$$

which implies (note $\tilde{N}_k = N_k + 1$ and $\tilde{\beta}_k^* = \beta_k^* - 1$)

$$\begin{aligned}
&\big[3(1+b_1^*)(1+b_2^*) - 4b_1^*b_2^*\big]\beta_2^* + \frac{1+b_2^*}{b_1^*}\big[(1+b_1^*)(1+b_2^*) - 2b_1^*b_2^*\big]\beta_1^*\\
&< (1+b_1^*)(1+b_2^*)N_2 - \frac{1+b_2^*}{b_1^*}\big[(1+b_1^*)(1+b_2^*) - 2b_1^*b_2^*\big]N_1 + 4(1+b_1^*+b_2^*)\\
&< (1+b_1^*)(1+b_2^*)N_2 + \frac{(1+b_1^*)(1+b_2^*)^2}{b_1^*}N_1 + 4(1+b_1^*+b_2^*).
\end{aligned}$$

However, since $(\beta_1^*,\beta_2^*) = (\beta_1(t_*),\beta_2(t_*)) \in S_{t_*}\cap\Omega_{t_*}$, so (β_1^*,β_2^*) satisfies (2.23) with $(b_1,b_2) = (b_1^*,b_2^*)$, which yields a contradiction. This completes the proof. $\square$

Lemma 3.6. *$\mu_{1,n}/\mu_{2,n} \to 0$ as $n\to\infty$.*

Proof. Assume by contradiction that, up to a subsequence, $\mu_{1,n}/\mu_{2,n} \ge C_1$ for some constant $C_1 > 0$. Then Lemma 3.5 gives

$$0 < C_1 \le \frac{\mu_{1,n}}{\mu_{2,n}} \le C < \infty.$$

As in Step 1 of the proof of Lemma 3.5, we consider

$$\overline{V}_{k,n}(x) = V_{k,n}(\mu_{2,n}x) + (2\beta_k - 2)\ln\mu_{2,n} \ \text{ for } \ |x|\le 1/\mu_{2,n}, \ \ k=1,2.$$

Then $(\overline{V}_{1,n},\overline{V}_{2,n})$ satisfies (3.38), and $\overline{V}_{k,n} \le \overline{V}_{k,n}(\frac{z_{k,n}}{\mu_{2,n}}) = O(1)$ for all $|x|\le 1/\mu_{2,n}$. Note that $\frac{|z_{k,n}|}{\mu_{2,n}} \le C$ for $k=1,2$. Then, as in Step 1 of the proof of Lemma 3.5, the Harnack inequality yields that both $\overline{V}_{1,n}$ and $\overline{V}_{2,n}$ are uniformly bounded in $L^\infty_{loc}(\mathbb{R}^2)$. Up to a subsequence, we may assume that $\overline{V}_{k,n} \to \overline{V}_k$ in $C^2_{loc}(\mathbb{R}^2)$, where $(\overline{V}_1,\overline{V}_2)$ satisfies

$$\begin{cases}\Delta\overline{V}_1 = -(1+b_1^*)|x|^{2\beta_1^*-4}e^{\overline{V}_1} + b_1^*|x|^{2\beta_2^*-4}e^{\overline{V}_2}\\ \Delta\overline{V}_2 = -(1+b_2^*)|x|^{2\beta_2^*-4}e^{\overline{V}_2} + b_2^*|x|^{2\beta_1^*-4}e^{\overline{V}_1}\end{cases} \text{in } \mathbb{R}^2, \tag{3.58}$$

and

$$\int_{\mathbb{R}^2}|x|^{2\beta_1^*-4}e^{\overline{V}_1}dx + \int_{\mathbb{R}^2}|x|^{2\beta_2^*-4}e^{\overline{V}_2}dx < \infty.$$

In the $SU(3)$ case $b_1^* = b_2^* = 1$, system (3.58) is the well-known $SU(3)$ Toda system. Remark that the $SU(3)$ Toda system is an integrable system, and so all the solutions can be known (see [21]). In particular, all the solutions of the $SU(3)$ Toda system satisfy the following quantization

$$\int_{\mathbb{R}^2}|x|^{2\beta_1^*-4}e^{\overline{V}_1}dx = \int_{\mathbb{R}^2}|x|^{2\beta_2^*-4}e^{\overline{V}_2}dx = 4\pi(\beta_1^*+\beta_2^*-2).$$

This formula plays an important role in the proof of [8].

However, for our general (b_1^*, b_2^*), since system (3.58) is not necessarily integrable, we do not have such beautiful results for system (3.58). Therefore, we need to develop different techniques.

Define $\widetilde{W}_k = \overline{V}_k + (2\beta_k - 4)\ln|x|$. Then

$$\begin{cases} \Delta \widetilde{W}_1 = -(1+b_1^*)e^{\widetilde{W}_1} + b_1^* e^{\widetilde{W}_2} + 4\pi(\beta_1^* - 2)\delta_0 \\ \Delta \widetilde{W}_2 = -(1+b_2^*)e^{\widetilde{W}_2} + b_2^* e^{\widetilde{W}_1} + 4\pi(\beta_2^* - 2)\delta_0 \end{cases} \text{in } \mathbb{R}^2. \tag{3.59}$$

Denote

$$4\pi d_k := \int_{\mathbb{R}^2} e^{\widetilde{W}_k} dx = \int_{\mathbb{R}^2} |x|^{2\beta_1^* - 4} e^{\overline{V}_k} dx.$$

Assume that $r\widetilde{W}_k'(r) \to -2M_k$ as $r \to \infty$. Then it is easy to deduce from (3.59) that

$$M_1 = (1+b_1^*)d_1 - b_1^* d_2 - \beta_1^* + 2, \;\; M_2 = (1+b_2^*)d_2 - b_2^* d_1 - \beta_2^* + 2. \tag{3.60}$$

Moreover, by $\int_{\mathbb{R}^2} e^{\widetilde{W}_k} dx < \infty$ we can prove $M_k \geq 1$ for $k = 1, 2$. By computing the Pohozaev identity of (3.59) as in the proof of Lemma 2.2, we easily obtain

$$J^*(M_1 - 1, M_2 - 1) = J^*(\beta_1^* - 1, \beta_2^* - 1).$$

By substituting the expressions (3.60) of M_k into the above equality, a direct calculation gives

$$\frac{b_2^*(1+b_1^*)}{2} d_1^2 + \frac{b_1^*(1+b_2^*)}{2} d_2^2 - b_1^* b_2^* d_1 d_2 - b_2^*(\beta_1^* - 1)d_1 - b_1^*(\beta_2^* - 1)d_2 = 0. \tag{3.61}$$

On the other hand, Fatou lemma implies

$$\int_{\mathbb{R}^2} e^{\widetilde{W}_k} dx \leq \liminf_{n\to\infty} \int_{|x|\leq 1} e^{w_{k,n}} dx =: 4\pi e_k,$$

so $d_k \leq e_k$ for $k = 1, 2$. By (3.24)-(3.25) we have

$$\gamma_1 = \beta_1^* - (1+b_1^*)e_1 + b_1^* e_2, \;\; \gamma_2 = \beta_2^* - (1+b_2^*)e_2 + b_2^* e_1.$$

By substituting these two expressions of γ_k into $J^*(\gamma_1 - 1, \gamma_2 - 1) = J^*(\beta_1^* - 1, \beta_2^* - 1)$, we obtain (the same as (3.61))

$$\frac{b_2^*(1+b_1^*)}{2} e_1^2 + \frac{b_1^*(1+b_2^*)}{2} e_2^2 - b_1^* b_2^* e_1 e_2 - b_2^*(\beta_1^* - 1)e_1 - b_1^*(\beta_2^* - 1)e_2 = 0.$$

Denote $\varepsilon_k = e_k - d_k$, then $\varepsilon_k \geq 0$ for $k = 1, 2$. By putting $e_k = d_k + \varepsilon_k$ into the above equality and recalling (3.61) and (3.60), we can obtain

$$\frac{b_2^*(1+b_1^*)}{2} \varepsilon_1^2 + \frac{b_1^*(1+b_2^*)}{2} \varepsilon_2^2 - b_1^* b_2^* \varepsilon_1 \varepsilon_2 + b_2^*(M_1 - 1)\varepsilon_1 + b_1^*(M_2 - 1)\varepsilon_2 = 0.$$

Since $M_k \geq 1$ and $\varepsilon_k \geq 0$, so $\varepsilon_1 = \varepsilon_2 = 0$, namely $e_k = d_k$. Consequently, $M_1 + \gamma_1 = 2$, which is a contradiction with $\gamma_1 > N_1 + 2$ and $M_1 \geq 1$. This completes the proof. $\square$

Remark 3.2. *The equality* (3.61) *implies the existence of a positive constant C, which only depends on b_k^* and β_k^*, such that $d_1 + d_2 \geq C$, namely*

$$\int_{\mathbb{R}^2} |x|^{2\beta_1^* - 4} e^{\overline{V}_1} dx + \int_{\mathbb{R}^2} |x|^{2\beta_2^* - 4} e^{\overline{V}_2} dx \geq C$$

holds for any solution $(\overline{V}_1, \overline{V}_2)$ of system (3.58).

Now we can finish the proof of Theorem 3.2.

Completion of the proof of Theorem 3.2. By Lemma 3.6, we have $\mu_{2,n}/\mu_{1,n} \to \infty$ as $n \to \infty$. The following proof is different from that in [8] and seems simpler. We can repeat Steps 1-3 of the proof of Lemma 3.5. In particular, there exists $s_n' < 1/\mu_{2,n}$ such that (compare with (3.55)-(3.56))

$$\begin{aligned} &\lim_{n\to\infty}\int_{|x|\le \mu_{2,n}s_n'} e^{w_{1,n}}dx = \frac{8\pi}{1+b_1^*}(\beta_1^*-1), \\ &\lim_{n\to\infty}\int_{|x|\le \mu_{2,n}s_n'} e^{w_{2,n}}dx = \frac{8\pi}{1+b_2^*}\left[\frac{2b_2^*}{1+b_1^*}(\beta_1^*-1)+\beta_2^*-1\right]. \end{aligned} \tag{3.62}$$

Recall from (3.26) that

$$\lim_{n\to\infty}\int_{|x|\le 1} e^{w_{2,n}}dx = 4\pi\frac{b_2^*(\beta_1^*+N_1)+(1+b_1^*)(\beta_2^*+N_2)}{1+b_1^*+b_2^*},$$

This, together with (3.62), gives

$$\frac{2}{1+b_2^*}\left[\frac{2b_2^*}{1+b_1^*}(\beta_1^*-1)+\beta_2^*-1\right] \le \frac{b_2^*(\beta_1^*+N_1)+(1+b_1^*)(\beta_2^*+N_2)}{1+b_1^*+b_2^*},$$

which is equivalent to

$$\begin{aligned} &\left[3(1+b_1^*)(1+b_2^*)-4b_1^*b_2^*\right]\beta_1^* + \frac{1+b_1^*}{b_2^*}\left[(1+b_1^*)(1+b_2^*)-2b_1^*b_2^*\right]\beta_2^* \\ &\le (1+b_1^*)(1+b_2^*)N_1 + \frac{(1+b_1^*)^2(1+b_2^*)}{b_2^*}N_2 + \left(4+2\frac{1+b_1^*}{b_2^*}\right)(1+b_1^*+b_2^*). \end{aligned}$$

However, since $(\beta_1^*, \beta_2^*) = (\beta_1(t_*), \beta_2(t_*)) \in S_{t_*} \cap \Omega_{t_*}$, so (β_1^*, β_2^*) satisfies (2.22) with $(b_1, b_2) = (b_1^*, b_2^*)$, which yields a contradiction. This completes the proof. □

To prove Theorem 3.1, we need the following lemma first.

Lemma 3.7. *There exists a constant $C > 0$ independent of $t \in [0,1]$ such that for $k = 1, 2$,*

$$v_k = u_k - f_k \le C \ \ in\ \mathbb{R}^2, \ \ \forall\, t \in [0,1].$$

Proof. The proof is similar to that of [8, Lemma 3.8]. Assume by contradiction that, up to a subsequence, there exists a sequence $t_n \to t_* \in [0,1]$ such that

$$\sup_{\mathbb{R}^2} v_{k,n} = \max\left\{\sup_{\mathbb{R}^2} v_{1,n},\ \sup_{\mathbb{R}^2} v_{2,n}\right\} \to \infty \ \text{ for some } 1 \le k \le 2.$$

For convenience, we use the same notations $b_j^* = b_j(t_*)$ and so on as before. By Theorem 3.2, both $v_{1,n}$ and $v_{2,n}$ are uniformly bounded on any compact set. Hence, if $x_{k,n}$ is a maximum point of $v_{k,n}$, then $|x_{k,n}| \to \infty$.

We claim that there exists a sequence $s_n \to \infty$ such that

$$\liminf_{n\to\infty}\int_{|x|\ge s_n}(e^{u_{1,n}}+e^{u_{2,n}})\,dx \ge C \tag{3.63}$$

for some constant $C > 0$.

To prove this, we define the Kelvin transform of $u_{j,n} = v_{j,n}+f_{j,n}$ by $\tilde u_{j,n}(|x|) = u_{j,n}(1/|x|) = v_{j,n}(1/|x|) + f_{j,n}(1/|x|)$ for $|x| \le 1$ and $j = 1, 2$. Note that $f_{j,n}(1/|x|) = 2\beta_j \ln|x| - (\beta_j + N_j)\ln(1+|x|^2)$. Let

$$\xi_{j,n}(|x|) = \tilde u_{j,n}(|x|) - 2\beta_j \ln|x| \ \text{ for } |x| \le 1,$$

then $\xi_{j,n}(|x|) = v_{j,n}(1/|x|) + O(1)$ for $|x| \le 1$, so $\sup_{|x|\le 1}\xi_{k,n}(|x|) \to \infty$. By (2.39), $\xi_{j,n}(|x|) \le (2-2\beta_j)\ln|x| + C$ for $|x| \le 1$. Let $y_{j,n}$ be a maximum point of $\xi_{j,n}$ in $\overline{B_1}$ and define

$$\mu_n = \min\left\{\mu_{j,n} := \exp\left(-\frac{1}{2\beta_j - 2}\xi_{j,n}(y_{j,n})\right),\ \ j = 1, 2\right\}.$$

Clearly $\mu_n \to 0$ as $n \to \infty$ and $\frac{|y_{j,n}|}{\mu_{j,n}} \le C$ for $j = 1, 2$. Define

$$\bar{\xi}_{j,n}(x) = \xi_{j,n}(\mu_n x) + (2\beta_j - 2)\ln \mu_n, \quad |x| \le \frac{1}{\mu_n}.$$

Up to a subsequence, we may let $l \in \{1, 2\}$ such that $\mu_{l,n} = \mu_n$ for all n. Then for $i \in \{1, 2\} \setminus \{l\}$ we have $\mu_{i,n} \ge \mu_n$. Along a subsequence, there are two cases: (i) $\mu_{i,n}/\mu_n \to +\infty$, then $\bar{\xi}_{i,n} \to -\infty$ uniformly, and as in Step 1 of the proof of Lemma 3.5, the limit equation for $(\bar{\xi}_{1,n}, \bar{\xi}_{2,n})$ is a Liouville type equation (see (3.39)). (ii) $1 \le \mu_{i,n}/\mu_n \le C$, then $\frac{|y_{i,n}|}{\mu_n} \le C$, and as in the proof of Lemma 3.6, the limit equation for $(\bar{\xi}_{1,n}, \bar{\xi}_{2,n})$ is a Toda type system (see (3.58)). In both cases, we can find a sequence $a_n \to 0$ such that (see Remark 3.2 for case (ii))

$$\liminf_{n\to\infty} \int_{|x|\le a_n} \left(|x|^{2\beta_1 - 4} e^{\xi_{1,n}} + |x|^{2\beta_2 - 4} e^{\xi_{2,n}}\right) dx \ge C,$$

where $C > 0$ only depends on b_1^*, b_2^*, β_1^* and β_2^*. Let $s_n = 1/a_n$, then (3.63) holds.

Recall that $v_{1,n}$ and $v_{2,n}$ are locally uniformly bounded. Up to a subsequence, we may assume that $v_{j,n} \to v_j^*$ in $C^2_{loc}(\mathbb{R}^2)$ for $j = 1, 2$. Let $u_j^* = v_j^* + f_j^*$. By (2.39), it turns out that (u_1^*, u_2^*) is a radially symmetric non-topological solution of system (2.16) with (b_1, b_2) replaced by (b_1^*, b_2^*). Consequently, Theorem A implies $u_j^*(x) = -2\gamma_j \ln|x| + O(1)$ for some constants $\gamma_j > 1$ as $|x| \to \infty$, $j = 1, 2$.

As before, we can take a sequence $R_n \to \infty$, and choose a subsequence still denoted by $(v_{1,n}, v_{2,n})$, such that $R_n < s_n$, $\|v_{1,n} - v_1^*\|_{L^\infty(B_{R_n})} + \|v_{2,n} - v_2^*\|_{L^\infty(B_{R_n})} \to 0$ and

$$\lim_{n\to\infty} \int_0^{R_n} r e^{u_{j,n}} dr = \int_0^\infty r e^{u_j^*} dr, \quad j = 1, 2.$$

Moreover, (2.39) implies

$$\lim_{n\to\infty} \int_0^{R_n} r e^{u_{i,n} + u_{j,n}} dr = \int_0^\infty r e^{u_i^* + u_j^*} dr, \quad 1 \le i, j \le 2,$$
$$\lim_{n\to\infty} \int_{R_n}^\infty r e^{u_{i,n} + u_{j,n}} dr = 0, \quad 1 \le i, j \le 2.$$

Consequently, by repeating the proof of the case $k = 1$ in Lemma 3.2, we can prove that $R_n u'_{j,n}(R_n) \to -2\gamma_j$ and $u_{j,n}(R_n) + 2\ln R_n \to -\infty$ for $j = 1, 2$. Now we can modify the proof of Lemma 2.2 by letting $(\varepsilon, R) = (R_n, +\infty)$ in (2.34), which yields by letting $n \to \infty$ that

$$J^*(\beta_1^* - 1, \beta_2^* - 1) = J^*(\gamma_1 - 1, \gamma_2 - 1). \tag{3.64}$$

On the other hand, we denote $\psi_n(r) = (1 + b_2) r u'_{1,n} + b_1 r u'_{2,n}$ and $\zeta_n(r) = (1 + b_1) r u'_{2,n} + b_2 r u'_{1,n}$ for simplicity. Then we easily deduce from (2.32)-(2.33) that

$$\lim_{n\to\infty} \psi_n(R_n) \ge \lim_{n\to\infty} \left(\lim_{r\to\infty} \psi_n(r)\right) = -2(1 + b_2^*)\beta_1^* - 2b_1^*\beta_2^*,$$
$$\lim_{n\to\infty} \zeta_n(R_n) \ge \lim_{n\to\infty} \left(\lim_{r\to\infty} \zeta_n(r)\right) = -2(1 + b_1^*)\beta_2^* - 2b_2^*\beta_1^*.$$

Hence,

$$(1 + b_2^*)\gamma_1 + b_1^*\gamma_2 \le (1 + b_2^*)\beta_1^* + b_1^*\beta_2^*, \tag{3.65}$$
$$(1 + b_1^*)\gamma_2 + b_2^*\gamma_1 \le (1 + b_1^*)\beta_2^* + b_2^*\beta_1^*. \tag{3.66}$$

Moreover, by (3.63) and $R_n < s_n$, at least one of $\le$ in (3.65) and (3.66) must be $<$. Remark that, if L is a line tangent to the ellipse $E_* = \{(x, y) \,|\, J^*(x, y) = J^*(\beta_1^* - 1, \beta_2^* - 1)\}$ in the first quadrant, then the slope a of L satisfies

$$-\frac{1 + b_2^*}{b_1^*} < a < -\frac{b_2^*}{1 + b_1^*}.$$

Therefore, we easily conclude from $\gamma_j, \beta_j^* > 1$ that $J^*(\gamma_1 - 1, \gamma_2 - 1) < J^*(\beta_1^* - 1, \beta_2^* - 1)$, which contradicts to the equality (3.64). This completes the proof. $\square$

Now we can give the proof of Theorem 3.1.

Proof of Theorem 3.1. By (2.16), (v_1, v_2) satisfies

$$\begin{cases} \Delta v_1 = -(1+b_1)(e^{u_1} - (1+b_1)e^{2u_1} + b_1 e^{u_1+u_2}) \\ \qquad\qquad +b_1(e^{u_2} - (1+b_2)e^{2u_2} + b_2 e^{u_1+u_2}) + 4\dfrac{\beta_1+N_1}{(1+|x|^2)^2}, \\ \Delta v_2 = -(1+b_2)(e^{u_2} - (1+b_2)e^{2u_2} + b_2 e^{u_1+u_2}) \\ \qquad\qquad +b_2(e^{u_1} - (1+b_1)e^{2u_1} + b_1 e^{u_1+u_2}) + 4\dfrac{\beta_2+N_2}{(1+|x|^2)^2}. \end{cases} \text{in } \mathbb{R}^2 \tag{3.67}$$

Let $w_k(x) = v_k(\frac{x}{|x|^2})$ for $k = 1, 2$. Then

$$\Delta w_1 = -|x|^{-4}\Big[(1+b_1)\left(e^{u_1(\frac{1}{|x|})} - (1+b_1)e^{2u_1(\frac{1}{|x|})} + b_1 e^{u_1(\frac{1}{|x|})+u_2(\frac{1}{|x|})}\right)$$
$$-\, b_1\left(e^{u_2(\frac{1}{|x|})} - (1+b_2)e^{2u_2(\frac{1}{|x|})} + b_2 e^{u_1(\frac{1}{|x|})+u_2(\frac{1}{|x|})}\right)\Big] + 4\frac{\beta_1+N_1}{(1+|x|^2)^2} =: g_1(x)$$

By Lemma 3.7, we have

$$u_k(x) \le C + f_k(x) \le -2\beta_k \ln|x| + C, \quad k = 1, 2, \tag{3.68}$$

which improves the estimate (2.39). Then it is easy to check that g_1 are uniformly bounded in $L^s(B_1)$ for some $s > 1$. Note that w_1 are uniformly bounded for $|x| \ge 1/2$ by Theorem 3.2. Therefore, the Harnack inequality implies that w_1 are also uniformly bounded in B_1. The same holds for w_2. Consequently, both v_1 and v_2 are uniformly bounded in $\mathbb{R}^2$ for $t \in [0, 1]$. □

4 Existence via the degree theory

In this section, we prove that $v_k = u_k - f_k$ are uniformly bounded for $t \in [0, 1]$ in a suitable Hilbert space and then apply the degree theory. Recalling (2.25), we define

$$\sigma(|x|) := (1+|x|^2)^{-\alpha_0}, \quad \alpha_0 = \min_{t\in[0,1]}\{\beta_1(t), \beta_2(t), 2\} > 1.$$

As in [20], we consider the functional space

$$\mathcal{D} := \left\{ w \in H^1_{loc}(\mathbb{R}^2) \;\Big|\; \|w\|^2_{\mathcal{D}} = \int_{\mathbb{R}^2} |\nabla w|^2 dx + \int_{\mathbb{R}^2} \sigma w^2 dx < \infty \right\},$$

and denote by $\mathcal{D}_r$ the radial subspace of $\mathcal{D}$. We also let $\mathcal{D}_r^2 = D_r \times \mathcal{D}_r$ equipped with the norm $\|(w_1, w_2)\|_{\mathcal{D}_r^2} = \|w_1\|_{\mathcal{D}} + \|w_2\|_{\mathcal{D}}$.

Remark that $rv_k'(r) \to 0$ as $r \to \infty$. By (3.67) and (3.68), we easily obtain $|v_k'(r)| \le Cr^{-s}$ for large r, where $s > 1$ is a constant. Then we have $(v_1, v_2) \in \mathcal{D}_r^2$. Recalling that (v_1, v_2) satisfies (3.67), we denote

$$\begin{aligned} F_k(x) &= (1+b_k)e^{2u_k} - e^{u_k} - b_k e^{u_1+u_2} \\ &= (1+b_k)e^{2v_k+2f_k} - e^{v_k+f_k} - b_k e^{v_1+v_2+f_1+f_2} \end{aligned}$$

and $g(|x|) = (1+|x|^2)^{-2}$ for convenience. Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Note $g(|x|) \le \sigma(|x|)$. By Theorem 3.1, it is easy to check that

$$|F_k(x)| \le Ce^{f_k} \le C(1+|x|^2)^{-\beta_k} \le C\sigma(|x|), \quad k = 1, 2,$$

and $|\sigma v_k| \le C\sigma$. Then by multiplying (3.67) by (v_1, v_2) and integrating by part over $\mathbb{R}^2$, we easily obtain

$$\int_{\mathbb{R}^2} \left(|\nabla v_1|^2 + |\nabla v_2|^2\right) dx \le C \int_{\mathbb{R}^2} \sigma(|x|) dx.$$

Moreover, $\int_{\mathbb{R}^2} \sigma v_k^2 dx \le C \int_{\mathbb{R}^2} \sigma(|x|)dx$ for $k = 1, 2$. Therefore, $\|(v_1, v_2)\|_{\mathcal{D}_r^2} \le M - 1$ for some constant $M > 0$ independent of t.

For $t \in [0, 1]$, we define an operator $G(t, \cdot, \cdot) : \mathcal{D}_r^2 \to \mathcal{D}_r^2$ by

$$G(t, v_1, v_2) := (v_1 - G_1(t, v_1, v_2),\ v_2 - G_2(t, v_1, v_2)),$$

where I is the identity map in $\mathcal{D}_r$ and

$$G_k(t, v_1, v_2) = (\Delta - \sigma I)^{-1}\big((1 + b_k)F_k - b_k F_{3-k} - \sigma v_k + 4(\beta_k + N_k)g\big).$$

For each $t \in [0, 1]$, $(G_1(t, \cdot, \cdot), G_2(t, \cdot, \cdot))$ is a continuous compact operator from $\mathcal{D}_r^2$ into $\mathcal{D}_r^2$ by the Moser-Trudinger inequality in $\mathcal{D}_r$ (see [8, 20]). Moreover, $G(t, v_1, v_2) = 0$ is equivalent to (3.67). Recall that $\|(v_1, v_2)\|_{\mathcal{D}_r^2} \le M-1$ if $G(t, v_1, v_2) = 0$. Therefore, the degree $\deg(G(t, \cdot, \cdot), \Omega_M, 0)$ in D_r^2 is well defined for each $t \in [0, 1]$, where $\Omega_M := \{(v_1, v_2) \in \mathcal{D}_r^2 \mid \|(v_1, v_2)\|_{\mathcal{D}_r^2} \le M\}$. Moreover, $G : [0, 1] \times \mathcal{D}_r^2 \to \mathcal{D}_r^2$ defines a good homotopy. Observe that when $t = 0$, $b_1 = b_2 = 1$ and system (3.67) is the $SU(3)$ Chern-Simons system studied in [8]. In particular, Choe, Kim and Lin [8] proved that $\deg(G(0, \cdot, \cdot), \Omega_M, 0)$ is an odd number. Therefore, $\deg(G(1, \cdot, \cdot), \Omega_M, 0)$ is an odd number, namely (3.67) with $(b_1, b_2) = (a_1, a_2)$ has a solution (v_1, v_2) in $\mathcal{D}_r^2$. Consequently, system (1.9) has a radially symmetric non-topological solution (u_1, u_2) satisfying the asymptotic condition (1.11). This completes the proof. □

5 Bubbling solutions

In this section, we will prove Theorem 1.3 by constructing bubbling solutions via the shooting method. In the sequel, we assume that $a_1, a_2 > 0$ satisfy

$$(1 + a_1)(1 + a_2) > 2a_1 a_2. \tag{5.1}$$

Recall Section 2 that we write $A = (1 + a_1)(1 + a_2)$ and $B = a_1 a_2$ for convenience.

Recalling Ω in (1.17), we fix any $(\alpha_1, \alpha_2) \in \Omega$ such that $\alpha_1 \ne \alpha_2$ and

$$\begin{aligned}&[(1 + a_1)(1 + a_2) - 2a_1 a_2]\alpha_2 - a_2(1 + a_2)\alpha_1 \\ &\qquad = a_2(1 + a_2)N_1 + (1 + a_1)(1 + a_2)N_2 + 2(1 + a_1 + a_2).\end{aligned} \tag{5.2}$$

Clearly, the assumption (5.1) is *necessary* to guarantee the existence of such (α_1, α_2). Remark that the assumption $\alpha_1 \ne \alpha_2$ plays a crucial role in the following proof; see Lemmas 5.6-5.8. In the case $a_2 \ge 1$, since $A - 2B \le a_2(1 + a_2)$, we always have $\alpha_2 > \alpha_1$ for all (α_1, α_2) satisfying (5.2) and $\alpha_1, \alpha_2 > 1$. In the case $a_2 \in (0, 1)$, we define

$$\alpha_0 := \frac{a_2(1 + a_2)N_1 + AN_2 + 2(A - B)}{(1 - a_2)(A - B)} > 2. \tag{5.3}$$

Then for any (α_1, α_2) satisfying (5.2) and $\alpha_1, \alpha_2 > 1$, we have

$$\begin{cases} \alpha_2 > \alpha_1 \text{ if and only if } \alpha_2 < \alpha_0, \\ \alpha_2 < \alpha_1 \text{ if and only if } \alpha_2 > \alpha_0. \end{cases} \tag{5.4}$$

Inspired by Remark 3.1, we define

$$\gamma := \alpha_1 + \frac{2a_1}{1 + a_2}(\alpha_2 - 1). \tag{5.5}$$

Clearly, (5.2) and (5.5) give

$$\alpha_1 = \frac{A - 2B}{A}\gamma - \frac{2B}{A}N_1 - \frac{2a_1}{1 + a_2}(N_2 + 1), \tag{5.6}$$

$$\alpha_2 = \frac{a_2}{1+a_1}(\gamma + N_1) + N_2 + 2. \tag{5.7}$$

By $\alpha_1 > 1$ and (5.6) we obtain

$$\gamma > \frac{2B}{A-2B}N_1 + \frac{2a_1(a_1+1)}{A-2B}(N_2+1) + \frac{A}{A-2B}. \tag{5.8}$$

Moreover, in the case $a_2 \in (0,1)$, since $\alpha_1 \neq \alpha_2$, we see from (5.7) and (5.3)-(5.4) that

$$\zeta_0 := \frac{a_2(1+2a_1+a_2)}{(1+a_1)(A-B)}N_1 + \frac{1+2a_1+a_2}{A-B}N_2 + 2 - \frac{1-a_2}{1+a_1}\gamma \begin{cases} > 0 \text{ if } \alpha_2 > \alpha_1 \\ < 0 \text{ if } \alpha_2 < \alpha_1. \end{cases} \tag{5.9}$$

We also let $\zeta_0 := 1$ in the case $a_2 \geq 1$. As before, we denote $\tilde{\gamma} = \gamma - 1$, $\tilde{\alpha}_k = \alpha_k - 1$ and $\tilde{N}_k = N_k + 1$ for convenience.

Lemma 5.1. *$J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1+1, N_2+1)$ is equivalent to $\gamma > N_1 + 2$.*

Proof. By (5.6)-(5.7) we have

$$\tilde{\alpha}_2 = \frac{a_2}{1+a_1}\tilde{\gamma} + \frac{a_2}{1+a_1}\tilde{N}_1 + \tilde{N}_2, \quad \tilde{\alpha}_1 = \tilde{\gamma} - \frac{2a_1}{1+a_2}\tilde{\alpha}_2.$$

Consequently, by Lemma 2.4 and (1.10) we can derive

$$\begin{aligned} J(\tilde{\alpha}_1, \tilde{\alpha}_2) &= J\left(\tilde{\gamma} - \frac{2a_1}{1+a_2}\tilde{\alpha}_2,\ \tilde{\alpha}_2\right) = J(-\tilde{\gamma}, \tilde{\alpha}_2) \\ &= J\left(-\tilde{\gamma},\ \frac{a_2}{1+a_1}\tilde{\gamma} + \frac{a_2}{1+a_1}\tilde{N}_1 + \tilde{N}_2\right) \\ &= J(\tilde{N}_1, \tilde{N}_2) + \frac{a_2(1+a_1+a_2)}{2(1+a_1)}\left(\tilde{\gamma}^2 - \tilde{N}_1^2\right). \end{aligned}$$

This proves the lemma. □

Remark 5.1. *In the case $A \leq 4B$, (5.8) gives $\gamma > N_1 + 2$. Therefore, $(\alpha_1, \alpha_2) \in \Omega$ holds automatically for any $\alpha_1, \alpha_2 > 1$ satisfying (5.2).*

Since $(\alpha_1, \alpha_2) \in \Omega$, Lemma 5.1 gives $\gamma > N_1 + 2$. Then by [4], there is a unique radially symmetric solution U of the Chern-Simons-Higgs equation

$$\Delta U = -(1+a_1)e^U + (1+a_1)^2 e^{2U} + 4\pi N_1 \delta_0 \text{ in } \mathbb{R}^2, \tag{5.10}$$

satisfying the asymptotic condition

$$U(x) = -2\gamma \ln|x| + O(1) \text{ as } |x| \to \infty.$$

Moreover, $U < -\ln(1+a_1)$ in $\mathbb{R}^2$ and

$$\begin{aligned} &\int_0^\infty r\left[(1+a_1)e^U - (1+a_1)^2 e^{2U}\right] dr = 2(\gamma + N_1), \\ &\lim_{r\to\infty}\left[r^2 e^{U(r)} + |rU'(r) + 2\gamma|\right] = 0. \end{aligned} \tag{5.11}$$

Let $V(|x|) = V(x) := U(x) - 2N_1 \ln|x|$, then $V(0) := \lim_{r\to 0} V(r)$ is well defined; see [4].

To use the shooting method, we consider an initial problem of system (1.9) in a radial variable. As before, we denote

$$F_k(r) := (1+a_k)e^{2u_k(r)} - e^{u_k(r)} - a_k e^{u_1(r)+u_2(r)}, \ k = 1, 2,$$

for convenience. Clearly

$$|F_k| \le (1+a_k)e^{u_k} \le 1+a_k \ \text{ whenever } \ u_1, u_2 \le 0. \tag{5.12}$$

Denote $a = \max\{a_1, a_2\}$. Then it is easy to prove that

$$\begin{cases} F_k < 0 \quad \text{if } u_k < -\ln(1+a), \\ F_{3-k} < F_k < 0 \quad \text{if } u_k < u_{3-k} < -\ln[2(1+a)], \end{cases} \quad k = 1, 2. \tag{5.13}$$

Let $v_k(|x|) = v_k(x) := u_k(x) - 2N_k \ln|x|$ for $k = 1, 2$. We study the following initial problem

$$\begin{cases} v_1''(r) + \frac{1}{r}v_1'(r) = (1+a_1)F_1(r) - a_1F_2(r), & r > 0, \\ v_2''(r) + \frac{1}{r}v_2'(r) = (1+a_2)F_2(r) - a_2F_1(r), & r > 0, \\ v_1(0) = V(0), \ \ v_2(0) = \ln\varepsilon, \ \ v_1'(0) = v_2'(0) = 0, \end{cases} \tag{5.14}$$

where $\varepsilon \in (0,1)$. Clearly, the solution of (5.14) depends on ε and we denote it by $(v_{1,\varepsilon}, v_{2,\varepsilon})$ and consequently $u_{k,\varepsilon}(r) = v_{k,\varepsilon}(r) + 2N_k \ln r$. For the sake of convenience, we also call $(u_{1,\varepsilon}, u_{2,\varepsilon})$ a solution of (5.14), and we will omit the subscript ε when there is no confusion arising. The main result of this section is following, and Theorem 1.3 is a direct corollary.

Theorem 5.1. *Assume that $(\alpha_1, \alpha_2) \in \Omega$ satisfies (5.2) and $\alpha_2 \ne \alpha_1$. Then there exists small $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, system (5.14) has an entire solution $(u_{1,\varepsilon}, u_{2,\varepsilon})$ which satisfies $v_{1,\varepsilon} \to V$ in $C^2_{loc}(\mathbb{R}^2)$ and $\sup_{\mathbb{R}^2} u_{2,\varepsilon} \to -\infty$ as $\varepsilon \to 0$. Furthermore,*

$$u_{k,\varepsilon}(x) = -2\alpha_{k,\varepsilon} \ln|x| + O(1) \ \text{ as } |x| \to \infty, \ \ k = 1, 2,$$

and $(\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}) \to (\alpha_1, \alpha_2)$ as $\varepsilon \to 0$.

Theorem 5.1 indicates that $(u_{1,\varepsilon}, u_{2,\varepsilon})$ is a sequence of bubbling solutions of (1.9) with only the second component blowing up. Theorem 5.1 also shows that the constraints (1.19)-(1.20) in Theorem 1.1 are not superfluous but *necessary* ones in view of applying the degree theory.

In the rest of this section, we give the proof of Thereom 5.1, which is quite long and delicate, and we divide it into several lemmas. The basic strategy is similar to that of proving Theorem C in [8]. However, since we deal with generic (a_1, a_2), even in the case $\alpha_2 > \alpha_1$, some ideas in [8] can not work here, and we need to develop different approaches; see the proof of Lemma 5.6 below for a typical example, which is a key step in the proof of Theorem 5.1. Furthermore, as pointed out in Remark 1.3, the case $\alpha_1 > \alpha_2$, which does not appear in the $SU(3)$ case $(a_1, a_2) = (1, 1)$, is different from the case $\alpha_2 > \alpha_1$, and we need to develop different ideas.

Recalling $\zeta_0 \ne 0$, we let δ be a constant such that

$$0 < \delta < \frac{\min\{1, a_1, A - 2B, B, \alpha_1 - 1\} \min\{|\zeta_0|, 1\}}{1000(\gamma+2)^2(2a+1)^4}. \tag{5.15}$$

By (5.11), there exists a large constant $R_\delta \gg 1$ such that

$$R_\delta^2 e^{U(R_\delta)} + |R_\delta U'(R_\delta) + 2\gamma| < \delta^3, \ \int_{R_\delta}^{\infty} r e^{U(r)} dr < \delta^3,$$
$$\left| \int_0^{R_\delta} r\left[e^U - (1+a_1)e^{2U}\right] dr - \frac{2(\gamma + N_1)}{1+a_1} \right| < \delta^3. \tag{5.16}$$

Clearly, $R_\delta \to +\infty$ as $\delta \downarrow 0$. Unless otherwise stated, we let δ be a fixed constant. Denote

$$a_0 := \min\big\{1, \ 2B/(A-2B)\big\}. \tag{5.17}$$

Lemma 5.2. *There exists $\varepsilon_1 = \varepsilon_1(\delta) > 0$ such that, for any $\varepsilon \in (0, \varepsilon_1)$, problem (5.14) has a solution $(u_{1,\varepsilon}, u_{2,\varepsilon})$ up to $r \le R_\delta$ and*

$$R_\delta^2 e^{u_{1,\varepsilon}(R_\delta)} + \left|R_\delta u_{1,\varepsilon}'(R_\delta) + 2\gamma\right| < \frac{a_0}{1+a}\delta^2, \tag{5.18}$$

$$\|v_{1,\varepsilon} - V\|_{L^\infty([0,R_\delta])} < \delta, \tag{5.19}$$

$$u_{2,\varepsilon}(r) < u_{1,\varepsilon}(R_\delta) < 2\ln\delta, \quad \forall\, r \le R_\delta, \tag{5.20}$$

$$\left|\int_0^{R_\delta} re^{u_{1,\varepsilon}}dr - \int_0^\infty re^U dr\right| + \int_0^{R_\delta} re^{u_{2,\varepsilon}}dr \le \delta, \tag{5.21}$$

$$\left|R_\delta u'_{2,\varepsilon}(R_\delta) - \tfrac{2a_2}{1+a_1}(\gamma+N_1) - 2N_2\right| \le \delta^2. \tag{5.22}$$

More precisely, $|u_{1,\varepsilon} - U| \to 0$ *and* $u_{2,\varepsilon} \to -\infty$ *uniformly in* $(0, R_\delta]$ *as* $\varepsilon \to 0$.

Proof. Since $v_k(0) < \infty$ and $u_k(r) = v_k(r) + 2N_k \ln r$ for $r > 0$, then, for some $R > 0$, there exists a solution of system (5.14) on $(0, R)$ with $u_1(r), u_2(r) \le 0$ up to R. Let $r^* = \sup\{R \,|\, 0 < R \le R_\delta$ and $u_1, u_2 \le 0$ on $[0, R]\}$. Consequently, either $u_1(r^*) = 0$ or $u_2(r^*) = 0$ or $r^* = R_\delta$.

First we claim that $r^* = R_\delta$ for $\varepsilon > 0$ small. By (5.12)-(5.14), we have

$$rv_2'(r) = \int_0^r \rho((1+a_2)F_2 - a_2F_1)d\rho \le \frac{(1+2a)^2}{2}r^2.$$

Consequently, for any $r \le r^* \le R_\delta$,

$$\begin{aligned} u_2(r) &= \ln\varepsilon + 2N_2\ln r + \int_0^r v_2'(\rho)d\rho \le \ln\varepsilon + 2N_2\ln r + \frac{(1+2a)^2}{4}r^2 \\ &\le \ln\varepsilon + C(R_\delta) \to -\infty \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{5.23}$$

Here and in the sequel, we denote by $C(R_\delta)$ various constants depending only on R_δ. Letting $\varepsilon_1 < -e^{C(R_\delta)}$, we obtain $u_{2,\varepsilon}(r^*) < 0$ for all $\varepsilon < \varepsilon_1$. Moreover, for any $r \le r^* \le R_\delta$,

$$\left|(1+a_1)e^{u_1(r)+u_2(r)} + F_2(r)\right| \le 2(1+a)e^{u_2(r)} \le \varepsilon C(R_\delta).$$

Recall that $v_1(0) = V(0)$ and

$$v_1''(r) + \frac{1}{r}v_1'(r) = \left[(1+a_1)^2e^{2u_1} - (1+a_1)e^{u_1}\right] - a_1\left[(1+a_1)e^{u_1+u_2} + F_2\right].$$

Then by the standard continuous dependence on data in the ODE theory, we conclude

$$\|v_{1,\varepsilon} - V\|_{C^1([0,r^*])} \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.24}$$

Consequently, $u_{1,\varepsilon}(r^*) \to U(r^*) < -\ln(1+a_1)$. By taking ε_1 smaller, we have $u_{1,\varepsilon}(r^*) < 0$ for all $\varepsilon < \varepsilon_1$. Therefore, $r^* = R_\delta$ for all $\varepsilon < \varepsilon_1$. In particular, $|u_{1,\varepsilon} - U| \to 0$ and $u_{2,\varepsilon} \to -\infty$ uniformly in $(0, R_\delta]$ as $\varepsilon \to 0$.

Recall (5.15)-(5.17), (5.23) and (5.24). By taking ε_1 smaller if necessary, we easily conclude that (5.18)-(5.21) hold for all $\varepsilon < \varepsilon_1$ (note that $u_{1,\varepsilon}(R_\delta) < 2\ln\delta$ follows from (5.18)). Moreover, $\int_0^{R_\delta} rF_2(r)dr \to 0$ and

$$\int_0^{R_\delta} rF_1(r)dr \to -\int_0^{R_\delta} r\left[e^U - (1+a_1)e^{2U}\right]dr \ \text{ as } \varepsilon \to 0.$$

Then it follows from $R_\delta u_2'(R_\delta) - 2N_2 = (1+a_2)\int_0^{R_\delta} rF_2(r)dr - a_2\int_0^{R_\delta} rF_1(r)dr$ and (5.16) that (5.22) holds for all $\varepsilon < \varepsilon_1$ by taking ε_1 smaller if necessary. This proves the lemma. ☐

Lemma 5.3. *For each* $\varepsilon < \varepsilon_1(\delta)$*, there exists* $R_1 = R_{1,\varepsilon}$ *such that* $u_{1,\varepsilon}(R_{1,\varepsilon}) = u_{2,\varepsilon}(R_{1,\varepsilon}) < 2\ln\delta$ *and*

$$u_{2,\varepsilon}(r) < u_{1,\varepsilon}(r) < 2\ln\delta, \quad \forall\, r \in [R_\delta, R_{1,\varepsilon}), \tag{5.25}$$

$$\left|ru'_{1,\varepsilon}(r) + 2\gamma\right| \le 2(1+a)\delta^2, \quad \forall\, r \in [R_\delta, R_{1,\varepsilon}], \tag{5.26}$$

$$\left|ru'_{2,\varepsilon}(r) - \tfrac{2a_2}{1+a_1}(\gamma+N_1) - 2N_2\right| \le 2(1+a)\delta^2, \quad \forall\, r \in [R_\delta, R_{1,\varepsilon}]. \tag{5.27}$$

Moreover, $R_{1,\varepsilon} \to +\infty$ *and* $R_{1,\varepsilon}^2 e^{u_{1,\varepsilon}(R_{1,\varepsilon})} = R_{1,\varepsilon}^2 e^{u_{2,\varepsilon}(R_{1,\varepsilon})} \to 0$ *as* $\varepsilon \to 0$.

Proof. Fix any $\varepsilon < \varepsilon_1$. Let $R_1 = R_{1,\varepsilon} := \sup\{R \,|\, u_{1,\varepsilon} > u_{2,\varepsilon} \text{ on } [R_\delta, R)\}$. First we claim that (5.25) holds. Lemma 5.2 gives $u_1(R_\delta) < 2\ln\delta$ and $u_1'(R_\delta) < 0$. Assume by contradiction that (5.25) is not true. Then there exists $r^* \in (R_\delta, R_1)$ such that $u_1'(r^*) = 0$ and $u_1'(r) < 0$ for $r \in [R_\delta, r^*)$. Consequently, $u_2(r) < u_1(r) < 2\ln\delta$ for $r \in [R_\delta, r^*]$. This, together with (5.13), gives $F_1(r) < F_2(r) < 0$ for $r \in [R_\delta, r^*]$. So

$$0 = r^* u_1'(r^*) = R_\delta u_1'(R_\delta) + \int_{R_\delta}^{r^*} t\,[(1+a_1)F_1 - a_1F_2]\,dt < 0,$$

a contradiction. Hence (5.25) holds, and then $F_1(r) < F_2(r) < 0$, which implies $ru_1'(r) < R_\delta u_1'(R_\delta)$ for $r \in (R_\delta, R_1)$. Consequently,

$$u_1(r) = u_1(R_\delta) + \int_{R_\delta}^{r} u_1'(t)dt \le u_1(R_\delta) + R_\delta u_1'(R_\delta)\ln\frac{r}{R_\delta}, \quad r \in (R_\delta, R_1).$$

Recalling $\gamma > 2 + N_1$, we note from (5.18) and (5.17) that $R_\delta^2 e^{u_1(R_\delta)} < \frac{a_0}{1+a}\delta^2$ and $R_\delta u_1'(R_\delta) < -2 - a_0$. Then for any $r \in [R_\delta, R_1)$, we have

$$\begin{aligned}
\int_{R_\delta}^{r} t|F_1(t)|dt &\le (1+a)\int_{R_\delta}^{r} te^{u_1(t)}dt \\
&\le (1+a)e^{u_1(R_\delta)}R_\delta^{-R_\delta u_1'(R_\delta)}\int_{R_\delta}^{\infty} t^{1+R_\delta u_1'(R_\delta)}dt \\
&= \frac{1+a}{|2 + R_\delta u_1'(R_\delta)|}R_\delta^2 e^{u_1(R_\delta)} < \delta^2. \qquad (5.28)
\end{aligned}$$

Combining this with (5.22) and $(1+a_2)F_2 - a_2F_1 \ge -|F_1|$ on $[R_\delta, R_1)$, we have

$$\begin{aligned}
ru_2'(r) &= R_\delta u_2'(R_\delta) + \int_{R_\delta}^{r} t\,[(1+a_2)F_2 - a_2F_1]\,dt \\
&\ge \frac{2a_2}{1+a_1}(\gamma + N_1) + 2N_2 - 2\delta^2, \quad \text{for } r \in [R_\delta, R_1).
\end{aligned}$$

This, together with $ru_1'(r) \le R_\delta u_1'(R_\delta) < 0$ for $r \in [R_\delta, R_1)$, gives $R_1 < \infty$ and $u_1(R_1) < 2\ln\delta$.

Finally, recalling $F_1 < F_2 < 0$ on $[R_\delta, R_1)$, we deduce from (5.14) and (5.28) that

$$|ru_k'(r) - R_\delta u_k'(R_\delta)| \le (1+2a)\int_{R_\delta}^{r} t|F_1|dt \le (1+2a)\delta^2$$

for $r \in [R_\delta, R_1]$ and $k = 1, 2$. Then (5.26)-(5.27) follow from (5.18) and (5.22). Consequently, $|ru_k'(r)| \le C$ on $[R_\delta, R_1]$. Since Lemma 5.2 says $|u_1(R_\delta) - u_2(R_\delta)| \to \infty$ as $\varepsilon \to 0$, we immediately obtain $R_1 \to +\infty$ as $\varepsilon \to 0$. By (5.26) and $\gamma > 2$ we have $ru_1'(r) < -3$ for $r \in [R_\delta, R_1]$. This, together with (5.18), implies

$$R_1^2 e^{u_1(R_2)} = R_1^2 e^{u_1(R_1)} \le \frac{1}{R_1}R_\delta^3 e^{u_1(R_\delta)} \to 0 \text{ as } \varepsilon \to 0.$$

The proof is complete. □

Lemma 5.4. *Recall a_0 in (5.17). There exists $\varepsilon_2 = \varepsilon_2(\delta) \in (0, \varepsilon_1(\delta))$, such that for each $\varepsilon < \varepsilon_2$, there exists $R_2 = R_{2,\varepsilon} > R_{1,\varepsilon}$ satisfying $u_{2,\varepsilon}'(R_{2,\varepsilon}) = 0$ and $u_{2,\varepsilon}(R_{2,\varepsilon}) \le \ln\delta$, $u_{2,\varepsilon}' > 0$ on $[R_{1,\varepsilon}, R_{2,\varepsilon})$ and $ru_{1,\varepsilon}'(r) < -2 - a_0$ on $[R_{1,\varepsilon}, R_{2,\varepsilon}]$. Moreover,*

$$\left|R_{2,\varepsilon}u_{1,\varepsilon}'(R_{2,\varepsilon}) + 2\frac{A-B}{A}\gamma - \frac{2B}{A}N_1 - \frac{2a_1}{1+a_2}N_2\right| \le 3(1+a)^2\delta^2, \qquad (5.29)$$

and $u_{2,\varepsilon}(R_{2,\varepsilon}) \to -\infty$ as $\varepsilon \to 0$.

Proof. Let $R_2 = R_{2,\varepsilon} := \sup\{R > R_{1,\varepsilon} \,|\, u'_{2,\varepsilon} > 0 \text{ on } [R_{1,\varepsilon}, R)\}$. We want to show $R_{2,\varepsilon} < \infty$ when $\varepsilon > 0$ sufficiently small.

First we claim that $u_1'(r) < 0$ on $[R_1, R_2)$. If not, there exists $r^* \in [R_1, R_2)$ such that $u_1'(r^*) = 0$ and $u_1'(r) < 0$ for $r \in [R_1, r^*)$. Consequently, $u_1 < u_1(R_1) < 2\ln\delta$ and so $F_1 < 0$ on $(R_1, r^*]$. Since

$$\frac{d}{dr}\big[(1+a_2)ru_1'(r) + a_1 ru_2'(r)\big] = (1+a_1+a_2)rF_1, \tag{5.30}$$

we derive from (5.26)-(5.27) that, for $r \in [R_1, r^*]$,

$$\begin{aligned}
ru_1'(r) &\le ru_1'(r) + \frac{a_1}{1+a_2}ru_2'(r) \le R_1u_1'(R_1) + \frac{a_1}{1+a_2}R_1u_2'(R_1)\\
&\le -2\gamma + 2(1+a)\delta^2 + \frac{a_1}{1+a_2}\left[\frac{2a_2}{1+a_1}(\gamma+N_1) + 2N_2 + 2(1+a)\delta^2\right]\\
&= -2\frac{A-B}{A}\gamma + \frac{2B}{A}N_1 + \frac{2a_1}{1+a_2}N_2 + 2(1+a)^2\delta^2\\
&= -2\alpha_1 - \frac{2B}{A}\gamma - \frac{2B}{A}N_1 - \frac{2a_1}{1+a_2}N_2 - \frac{4a_1}{1+a_2} + 2(1+a)^2\delta^2
\end{aligned}$$

Recalling from (5.15) that $\frac{4a_1}{1+a_2} > 2(1+a)^2\delta^2$, we easily conclude from (5.8) and $\alpha_1 > 1$ that

$$ru_1'(r) < -2\alpha_1 - \frac{2B}{A}\gamma < -2\alpha_1 - \frac{2B}{A-2B} < -2 - a_0$$

for all $r \in [R_1, r^*]$, a contradiction with $u_1'(r^*) = 0$. Therefore, $u_1'(r) < 0$ on $[R_1, R_2)$ and the above argument shows that $ru_1'(r) < -2 - a_0$ on $[R_1, R_2)$. Again, $u_1 < u_2$ on (R_1, R_2).

Now we claim that

$$\lim_{\varepsilon\to 0}\sup_{r\in[R_{1,\varepsilon}, R_{2,\varepsilon})} u_{2,\varepsilon}(r) = -\infty.$$

Suppose not, since Lemma 5.3 says $u_2(R_1) \to -\infty$ as $\varepsilon \to 0$, there exist a constant $c_0 \le \ln\delta$ and $\varepsilon_n \downarrow 0$ such that the corresponding solution u_{2,ε_n} (still write it by u_2 for convenience) satisfies $u_2(r_n) = c_0$ for some $r_n \in (R_1, R_2)$. Then $r_n \to \infty$ by Lemma 5.3. Since $u_1 < u_2 \le c_0 \le \ln\delta$ on $(R_1, r_n]$, we have $F_2 < F_1 < 0$ and $F_2 < -\frac{1}{2}e^{u_2}$ on $(R_1, r_n]$, namely

$$(ru_2'(r))' = r((1+a_2)F_2 - a_2F_1) \le -\frac{1}{2}re^{u_2} \text{ on } (R_1, r_n].$$

Hence, $0 \le ru_2'(r) \le R_1u_2'(R_1)$, that is, by (5.27) we have $|ru_2'(r)| \le C$ for $r \in [R_1, r_n]$, where C independent of n. Since there exists $t_n \in (R_1, r_n)$ such that $u_2(t_n) = 2c_0$, we have $r_n - t_n \ge C > 0$ for some constant C independent of n. Noting $2c_0 \le u_2 \le c_0$ on $[t_n, r_n]$ and $r_n \to \infty$, we conclude

$$-C \le r_nu_2'(r_n) - t_nu_2'(t_n) = \int_{t_n}^{r_n}(ru_2')'dr \le -\frac{1}{2}\int_{t_n}^{r_n} re^{u_2}dr \to -\infty,$$

a contradiction. Therefore, our claim holds and then there exists $\varepsilon_2 = \varepsilon_2(\delta) \in (0, \varepsilon_1)$ such that $u_2 < \ln\delta$ on $[R_1, R_2)$ for any $\varepsilon < \varepsilon_2$. Let $\varepsilon < \varepsilon_2$. Then the above argument shows that $(ru_2'(r))' \le -\frac{1}{2}re^{u_2(r)} \le -\frac{1}{2}R_1e^{u_2(R_1)}$ on (R_1, R_2). Consequently, $0 \le ru_2'(r) \le R_1u_2'(R_1) - \frac{1}{2}R_1e^{u_2(R_1)}(r - R_1)$ holds for any $r \in (R_1, R_2)$. Letting $r \uparrow R_2$ we conclude $R_2 < \infty$ and $u_2(R_2) \le \ln\delta$. Furthermore, $u_2(R_2) \to -\infty$ as $\varepsilon \to 0$.

Finally, it suffices to prove (5.29). By (5.26) we know that $r^2e^{u_1(r)}$ decreases on $[R_\delta, R_1]$. Recalling $ru_1'(r) < -2 - a_0$ on $[R_1, R_2]$, similarly as (5.28) we can obtain

$$\int_{R_1}^{R_2} t|F_1(t)|dt \le \frac{1+a}{a_0}R_1^2e^{u_1(R_1)} \le \frac{1+a}{a_0}R_\delta^2e^{u_1(R_\delta)} \le \delta^2.$$

Then integrating (5.30) over (R_1, R_2) gives

$$\left|R_2u_1'(R_2) - R_1u_1'(R_1) - \frac{a_1}{1+a_2}R_1u_2'(R_1)\right| \le (1+a)\delta^2,$$

and so (5.29) follows from (5.26)-(5.27). ☐

Remark 5.2. *Lemmas 5.2-5.4 indicate that* $\sup_{(0,R_{2,\varepsilon}]} u_{2,\varepsilon}(r) \to -\infty$ *as* $\varepsilon \to 0$.

Lemma 5.4 shows $u_{1,\varepsilon} < u_{2,\varepsilon}$ on $(R_{1,\varepsilon}, R_{2,\varepsilon}]$. Now we define

$$R_3 = R_{3,\varepsilon} := \sup\{R > R_{2,\varepsilon} \mid u_{1,\varepsilon} < u_{2,\varepsilon} \text{ on } [R_{2,\varepsilon}, R)\}. \tag{5.31}$$

Lemma 5.5. *For each* $\varepsilon < \varepsilon_2(\delta)$*, there holds* $u_{2,\varepsilon}' < 0$ *on* $(R_{2,\varepsilon}, R_{3,\varepsilon})$*. Furthermore, there exists* $\varepsilon_3 = \varepsilon_3(\delta) \in (0, \varepsilon_2(\delta))$ *such that for each* $\varepsilon < \varepsilon_3$*,* $ru_{1,\varepsilon}'(r) < -2 - 2\delta$ *for all* $R_{2,\varepsilon} \le r < R_{3,\varepsilon}$.

Proof. First we claim that $u_2'(r) < 0$ on (R_2, R_3). Assume by contradiction that there exists $r^* \in (R_2, R_3)$ such that $u_2'(r^*) = 0$ and $u_2' < 0$ on (R_2, r^*). Then $u_1 < u_2 \le u_2(R_2) \le \ln\delta$ on $[R_2, r^*)$, which implies $F_2 < F_1 < 0$ on $[R_2, r^*)$. Consequently,

$$0 = r^*u_2'(r^*) - R_2u_2'(R_2) = \int_{R_2}^{r^*} r\left[(1+a_2)F_2 - a_2F_1\right] dr < 0,$$

a contradiction. Hence $u_2' < 0$ on (R_2, R_3), which implies $u_1 < u_2 < \ln\delta$ on (R_2, R_3).

Recall from Lemma 5.4 that $ru_1'(r) < -2 - a_0 < -2 - 2\delta$ on $[R_1, R_2]$. Assume that, up to a subsequence of $\varepsilon \downarrow 0$, there exists $r^* = r^*_\varepsilon \in (R_2, R_3)$ such that $r^*u_1'(r^*) = -2 - 2\delta$ and $ru_1'(r) < -2 - 2\delta$ for $r \in [R_1, r^*)$. Then similarly as (5.28) we can obtain

$$\begin{aligned}\int_{R_1}^{r^*} t|F_1|dt &\le (1+a)\int_{R_1}^{r^*} te^{u_1}dt \le \frac{1+a}{2\delta}R_1^2e^{u_1(R_1)} \\ &\le \frac{1+a}{2\delta}R_\delta^2e^{u_1(R_\delta)} \le \frac{\delta}{2}.\end{aligned} \tag{5.32}$$

For convenience, we denote

$$d_k = -\frac{1}{2}r^*u_k'(r^*) - 1 \;\text{ and }\; e_k = -\frac{1}{2}R_1u_k'(R_1) - 1, \;\; k = 1, 2.$$

Then $d_1 = \delta$ and (5.26)-(5.27) give

$$|e_1 - \gamma + 1| \le (1+a)\delta^2, \quad \left|e_2 + \frac{a_2}{1+a_1}(\gamma + N_1) + N_2 + 1\right| \le (1+a)\delta^2. \tag{5.33}$$

Observe that, Lemma 5.4 and $u_2'(r) < 0$ on (R_2, R_3) yield

$$u_1 < u_2 \le u_2(R_2) \le \ln\delta \text{ on } (R_1, r^*). \tag{5.34}$$

Consequently, $F_2 < F_1 < 0$ on (R_1, r^*). Then by (5.30) we easily obtain

$$d_2 > \frac{1+a_2}{a_1}e_1 + e_2 - \frac{1+a_2}{a_1}d_1 > 0, \tag{5.35}$$

where the second inequality follows from (5.33) and $d_1 = \delta$. Moreover, by integrating (5.30) over (R_1, r^*) and using (5.32), we see that $d_2 \le C$ for some constant $C > 0$ independent of ε. Since

$$\frac{d}{dr}\left[(1+a_1)ru_2'(r) + a_2ru_1'(r)\right] = (1 + a_1 + a_2)rF_2$$

and (5.34) gives $|F_2| \ge \frac{1}{2}e^{u_2}$ on (R_1, r^*), by integrating the above formula over (R_1, r^*), we easily conclude

$$\int_{R_1}^{r^*} re^{u_2}dr \le C, \text{ where } C > 0 \text{ independent of } \varepsilon. \tag{5.36}$$

Now we can modify the proof of Lemma 2.2 by letting $(\varepsilon, R) = (R_1, r^*)$ in (2.34), which yields

$$J(d_1, d_2) - J(e_1, e_2) = I_1 + I_2,$$

where

$$I_1 = \frac{A-B}{4} r^2 \left[a_2 e^{u_1} \left(\frac{a_1+1}{2} e^{u_1} - 1 \right) + a_1 e^{u_2} \left(\frac{a_2+1}{2} e^{u_2} - 1 \right) - B e^{u_1+u_2} \right] \Bigg|_{R_1}^{r^*},$$

$$I_2 = \frac{A-B}{4} \int_{R_1}^{r^*} r \left[a_2(1+a_1) e^{2u_1} + a_1(1+a_2) e^{2u_2} - 2B e^{u_1+u_2} \right] dr.$$

Since $\frac{1+a_k}{2} e^{u_k(r^*)} - 1 \le \frac{1+a_k}{2}\delta - 1 < 0$ and $R_1^2 e^{u_2(R_1)} = R_1^2 e^{u_1(R_1)} < \frac{\delta^2}{1+a}$, we have

$$I_1 < \frac{A-B}{4} R_1^2 \left(a_2 e^{u_1(R_1)} + a_1 e^{u_2(R_1)} \right) < \frac{(1+2a)^2}{4(1+a)} \delta^2 < \frac{\delta}{2}.$$

Meanwhile, (5.32), (5.34), (5.36) and $u_2(R_2) \to -\infty$ yield $I_2 \to 0$ as $\varepsilon \to 0$. So for $\varepsilon > 0$ sufficiently small, we have $I_2 < \delta/2$, namely $J(d_1, d_2) - J(e_1, e_2) < \delta$. Then by (5.35) and $d_1 = \delta$ we can prove

$$\begin{aligned} \delta > J(d_1, d_2) - J(e_1, e_2) &\ge J\left(\delta, \frac{1+a_2}{a_1} e_1 + e_2 - \frac{1+a_2}{a_1} \delta \right) - J(e_1, e_2) \\ &= \frac{(1+a_2)(1+a_1+a_2)}{2a_1} (e_1 - \delta) \left(e_1 - \delta + \frac{2a_1}{1+a_2} e_2 \right). \end{aligned} \tag{5.37}$$

Recall from (5.15) that $\alpha_1 - 1 > 102\delta$ and $(1 + \frac{2a_1}{1+a_2})(1+a)\delta^2 < \delta$. By (5.33) we have

$$\begin{aligned} e_1 - \delta + \frac{2a_1}{1+a_2} e_2 &\ge \frac{A-2B}{A} \gamma - \frac{2B}{A} N_1 - \frac{2a_1}{1+a_2}(N_2+1) - 1 - 2\delta \\ &= \alpha_1 - 1 - 2\delta > 100\delta. \end{aligned}$$

Moreover, $\gamma > 2 + N_1$ gives $e_1 - \delta > 1/2$. So for $\varepsilon > 0$ sufficiently small, (5.37) gives $\delta > 25\delta$, a contradiction. This completes the proof. □

To continue our proof, we consider the case $\alpha_2 > \alpha_1$ first. We will prove $R_{3,\varepsilon} < +\infty$ for $\varepsilon > 0$ sufficiently small.

Lemma 5.6. *Assume that $\alpha_2 > \alpha_1$. Then there exists $\varepsilon_4 = \varepsilon_4(\delta) \in (0, \varepsilon_3(\delta))$ such that $R_{3,\varepsilon} < +\infty$ for each $\varepsilon < \varepsilon_4$. Consequently, $u_{1,\varepsilon}(R_{3,\varepsilon}) = u_{2,\varepsilon}(R_{3,\varepsilon})$.*

Proof. Assume by contradiction that $R_{3,\varepsilon_n} = +\infty$ for a sequence $\varepsilon_n \downarrow 0$. Then Lemma 5.5 shows that $u_{1,\varepsilon_n} < u_{2,\varepsilon_n} < \ln\delta$ on $(R_{2,\varepsilon_n}, +\infty)$. Consequently, we easily conclude that $(u_{1,\varepsilon_n}, u_{2,\varepsilon_n})$ is an entire solution of (5.14). By Theorem A, there exists $\beta_{k,n} > 1$ such that $u_{k,n}(|x|) = -2\beta_{k,n} \ln|x| + O(1)$ as $|x| \to \infty$ for $k = 1, 2$. Clearly $u_{1,\varepsilon_n} < u_{2,\varepsilon_n}$ on $(R_{2,\varepsilon_n}, +\infty)$ gives $\beta_{1,n} \ge \beta_{2,n}$.

Recalling (5.26) and Lemmas 5.4-5.5, it turns out that $ru'_{1,\varepsilon_n}(r) < -2 - 2\delta$ for all $r \ge R_\delta$. Then similarly as before, we have

$$\int_{R_\delta}^{\infty} r|F_1| dr \le (1+a) \int_{R_\delta}^{\infty} r e^{u_{1,\varepsilon_n}} dr \le \frac{1+a}{2\delta} R_\delta^2 e^{u_{1,\varepsilon_n}(R_\delta)} < \frac{\delta}{2}. \tag{5.38}$$

By (5.18), (5.22) and (5.38), we can integrate (5.30) over (R_δ, r) to yield

$$\left| (1+a_2) r u'_{1,\varepsilon_n}(r) + a_1 r u'_{2,\varepsilon_n}(r) \right| \le C, \quad \forall\, r \ge R_\delta,$$

where $C > 0$ independent of r, ε_n and δ. Letting $r \to \infty$ we obtain $\beta_{k,n} \le C$ for all n and $k = 1, 2$. Up to a subsequence, we may assume $(\beta_{1,n}, \beta_{2,n}) \to (\beta_1, \beta_2)$. Then $\beta_k \ge 1$ for $k = 1, 2$.

Now, we take a sequence δ_n satisfying (5.15) and $\delta_n \downarrow 0$. Denote $R_n := R_{\delta_n}$. Clearly, $R_n \to +\infty$. Then, up to a subsequence, we may assume that $\varepsilon_n < \varepsilon_3(\delta_n)$. Consequently, we have the following conclusions as $n \to \infty$:

(1) $v_{1,\varepsilon_n} \to V$ in $C^2_{loc}(\mathbb{R}^2)$;

(2) $\int_0^{R_n} re^{u_{1,\varepsilon_n}}dr \to \int_0^\infty re^U dr$ and $\int_0^{R_n} re^{u_{2,\varepsilon_n}}dr \to 0$;

(3) $\sup_{\mathbb{R}^2} u_{2,\varepsilon_n} \to -\infty$;

(4) $\int_{R_n}^\infty re^{u_{1,\varepsilon_n}}dr \to 0$.

Here, (1) and (2) follow from Lemma 5.2, (3) follows from Remark 5.2 and Lemma 5.5 that $u'_{2,\varepsilon} < 0$ on $(R_{2,\varepsilon}, +\infty)$, and (4) follows from (5.38). Therefore, we can repeat the proof of Lemma 3.4 to conclude that (β_1, β_2) satisfies the formula (5.2). Moreover, by Remark 3.1 we actually have

$$\begin{aligned}\beta_1 &= \frac{(1+a_1)(1+a_2)-2a_1a_2}{(1+a_1)(1+a_2)}\gamma - \frac{2a_1a_2}{(1+a_1)(1+a_2)}N_1 - \frac{2a_1}{1+a_2}(N_2+1),\\ \beta_2 &= \frac{a_2}{1+a_1}(\gamma+N_1)+N_2+2.\end{aligned}$$

Hence, (5.6)-(5.7) yield $(\beta_1,\beta_2)=(\alpha_1,\alpha_2)$. In particular, $\beta_2>\beta_1$, which contradicts to $\beta_{1,n}\geq \beta_{2,n}$ for all n. Therefore, there exists $\varepsilon_4=\varepsilon_4(\delta)\in(0,\varepsilon_3(\delta))$ such that $R_{3,\varepsilon}<\infty$ for each $\varepsilon<\varepsilon_4$. This proves the lemma. □

Lemma 5.7. *Assume that $\alpha_2 > \alpha_1$. Then for each $\varepsilon < \varepsilon_4$, there holds $b_{2,\varepsilon} - b_{1,\varepsilon} > \frac{3\zeta_0}{4}$, where $\zeta_0 > 0$ is in* (5.9) *and*

$$b_{k,\varepsilon} := -\frac{1}{2}R_{3,\varepsilon}u'_{k,\varepsilon}(R_{3,\varepsilon}) - 1, \quad k=1,2. \tag{5.39}$$

Proof. Recall from Lemmas 5.4-5.5 that $ru'_1(r) \leq -2-2\delta$ for $r\in[R_1,R_3]$. Similarly as (5.32), we have

$$\int_{R_1}^{R_3} r|F_1|dr \leq (1+a)\int_{R_1}^{R_3} te^{u_1}dt \leq \frac{\delta}{2}. \tag{5.40}$$

Recall $e_2 = -\frac{1}{2}R_1u'_2(R_1)-1$ in Lemma 5.5. By computing the Pohazaev identity of $(ru'_2)' = r[(1+a_2)F_2 - a_2F_1]$ over (R_1,R_3), we can obtain

$$\begin{aligned} b_2^2 - e_2^2 =& \frac{1+a_2}{4}r^2\left[(1+a_2)e^{2u_2} - 2e^{u_2}\right]\Big|_{R_1}^{R_3} - a_2\int_{R_1}^{R_3} rF_1dr\\ &+\frac{1+a_2}{2}\int_{R_1}^{R_3} re^{u_2}\left[(1+a_2)e^{u_2} - 2a_2e^{u_1}\right]dr\\ &-\frac{1}{2}\int_{R_1}^{R_3}(r^2u'_2)\left[a_2(1+a_2)e^{u_1+u_2}+a_2F_1\right]dr\\ =:& P_1+P_2+P_3+P_4. \end{aligned} \tag{5.41}$$

Since $F_2 < F_1 < 0$ on (R_1,R_3), we have

$$0 < P_2 = -a_2\int_{R_1}^{R_3} rF_1dr \leq \frac{a_2}{2}\delta.$$

Noting for $r\in\{R_1,R_3\}$ that $(1+a_2)e^{2u_2(r)} - 2e^{u_2(r)} < 0$ since $u_2(r) < \ln\delta$, we have

$$\begin{aligned} P_1 &> -\frac{1+a_2}{2}R_3^2e^{u_2(R_3)} = -\frac{1+a_2}{2}R_3^2e^{u_1(R_3)} > -\delta^2,\\ P_1 &< \frac{1+a_2}{2}R_1^2e^{u_2(R_1)} = \frac{1+a_2}{2}R_1^2e^{u_1(R_1)} < \delta^2, \end{aligned}$$

where we have used $r^2e^{u_1(r)} < R_\delta^2e^{u_1(R_\delta)}$ for any $r\in(R_\delta,R_3]$ since $ru'_1(r)\leq -2-2\delta$ on $[R_\delta,R_3]$. Since $e^{u_1}\leq e^{u_2}\leq\delta$ on $[R_1,R_3]$, we deduce from (5.40) that

$$P_3 \geq -\frac{a_2^2}{2}\delta\int_{R_1}^{R_3} te^{u_1}dt > -a\delta^2.$$

Again by integrating (5.30) over (R_1, r) for any $r \in (R_1, R_3]$, we derive from $ru_1'(r) \le -2-2\delta$, (5.40) and (5.26)-(5.27) that

$$\begin{aligned} ru_2'(r) &\ge \frac{1+a_2}{a_1}R_1u_1'(R_1) + R_1u_2'(R_1) - \frac{1+a_2}{a_1}ru_1'(r) - \frac{1+a}{a_1}\delta \\ &> -2\frac{1+a_1+a_2}{a_1(1+a_1)}\gamma + \frac{2}{a_1} - \frac{1+a}{a_1}\delta - 2(1+a)\left(1+\frac{1+a_2}{a_1}\right)\delta^2 \\ &> -2\frac{1+a_1+a_2}{a_1(1+a_1)}\gamma, \ \ \forall r \in (R_1, R_3]. \end{aligned} \tag{5.42}$$

On the other hand,

$$-\frac{3}{2}a_2e^{u_1} < a_2(1+a_2)e^{u_1+u_2} + a_2F_1 < 0 \ \text{ on } [R_1, R_3], \tag{5.43}$$

so we can obtain from (5.40) that

$$P_4 \ge -\frac{3a_2(1+a_1+a_2)}{2a_1(1+a_1)}\gamma\int_{R_1}^{R_3} re^{u_1}dr > -\frac{3(1+a)}{4a_1}\gamma\delta.$$

From the above arguments, we obtain

$$b_2^2 - e_2^2 > -(1+a)\delta^2 - \frac{3(1+a)}{4a_1}\gamma\delta.$$

This, together with (5.33), gives

$$\begin{aligned} b_2 &> |e_2| - (1+a)\delta^2 - \frac{3(1+a)}{4a_1}\gamma\delta \\ &> \frac{a_2}{1+a_1}(\gamma+N_1) + N_2 + 1 - 2(1+a)\delta^2 - \frac{3(1+a)}{4a_1}\gamma\delta \\ &> \frac{a_2}{1+a_1}(\gamma+N_1) + N_2 + 1 - \frac{1+a}{a_1}\gamma\delta. \end{aligned}$$

On the other hand, the definition (5.31) of R_3 yields $b_2 > b_1$. By integrating (5.30) over (R_2, R_3) and recalling (5.40) and (5.29), we can prove

$$\begin{aligned} b_1 &< \frac{(1+a_2)b_1 + a_1b_2}{1+a_1+a_2} = -\frac{(1+a_2)R_2u_1'(R_2)}{2(1+a_1+a_2)} + \frac{1}{2}\int_{R_2}^{R_3} r|F_1|dr - 1 \\ &\le \frac{1+a_2}{2(A-B)}\left[2\frac{A-B}{A}\gamma - \frac{2B}{A}N_1 - \frac{2a_1}{1+a_2}N_2 + 3(1+a)^2\delta^2\right] + \frac{\delta}{4} - 1 \\ &< \frac{\gamma}{1+a_1} - \frac{B}{(1+a_1)(A-B)}N_1 - \frac{a_1}{A-B}N_2 - 1 + \delta, \end{aligned} \tag{5.44}$$

and consequently,

$$\begin{aligned} b_2 - b_1 > &\frac{a_2-1}{1+a_1}\gamma + \frac{a_2(1+2a_1+a_2)}{(1+a_1)(A-B)}N_1 + \frac{1+2a_1+a_2}{A-B}N_2 + 2 \\ &- \left(\frac{1+a}{a_1}\gamma + 1\right)\delta. \end{aligned}$$

In the case $a_2 \ge 1$, we immediately obtain $b_2 - b_1 > 1 = \zeta_0$. In the case $a_2 \in (0,1)$, since $\alpha_2 > \alpha_1$, (5.9) gives $\zeta_0 > 0$ and

$$b_2 - b_1 > \zeta_0 - \left(\frac{1+a}{a_1}\gamma + 1\right)\delta.$$

Since (5.15) gives $\delta < \frac{a_1\zeta_0}{4(1+a)(\gamma+1)}$, we also obtain $b_2 - b_1 > \frac{3}{4}\zeta_0$. $\square$

Now let us consider the second case $\alpha_2 < \alpha_1$. Differently, we will prove $R_{3,\varepsilon} = +\infty$ for $\varepsilon > 0$ sufficiently small.

Lemma 5.8. *Assume that $\alpha_2 < \alpha_1$. Then there exists $\varepsilon_4 = \varepsilon_4(\delta) \in (0, \varepsilon_3(\delta))$ such that $R_{3,\varepsilon} = +\infty$ for each $\varepsilon < \varepsilon_4$.*

Proof. Since $\alpha_2 < \alpha_1$, we must have $a_2 \in (0,1)$. Assume by contradiction that there exists a sequence $\varepsilon_n \downarrow 0$ such that $R_{3,\varepsilon_n} < +\infty$. Then $u_{1,\varepsilon_n}(R_{3,\varepsilon_n}) = u_{2,\varepsilon_n}(R_{3,\varepsilon_n})$. We use the same notions b_{k,ε_n} as in Lemma 5.7. Then the definition (5.31) of R_{3,ε_n} yields $b_{2,\varepsilon_n} > b_{1,\varepsilon_n}$. In the following argument, we omit the subscript ε_n for convenience, namely we still write u_{k,ε_n} by u_k and b_{k,ε_n} by b_k respectively.

Recalling from Lemmas 5.4-5.5 that $ru_1'(r) \le -2 - 2\delta$ for $r \in [R_1, R_3]$, we see that (5.40) also holds here. Differently from Lemma 5.7, here we want to get a sharp upper bound of b_2 by (5.41). We have proved in Lemma 5.7 that $P_1 < \delta^2$ and $P_2 < \frac{\delta}{2}$. So it suffices to estimate P_3 and P_4 from above.

By (5.42) we see that $R_3 u_2'(R_3) \ge -C$ for some $C > 0$ independent of ε_n. Since $u_1 < u_2 \le u_2(R_2) \le \ln\delta$ on (R_1, R_3), we have $F_2 < F_1 < 0$ on (R_1, R_3). Consequently, by integrating $(ru_2')' = r[(1+a_2)F_2 - a_2 F_1] < 0$ over (R_1, R_3), we deduce from (5.27) and (5.40) that

$$(1+a_2)\int_{R_1}^{R_3} r|F_2|dr = R_1 u_2'(R_1) - R_3 u_2'(R_3) + a_2 \int_{R_1}^{R_3} r|F_1|dr \le C,$$

where $C > 0$ is independent of ε_n. Since $|F_2| \ge \frac{1}{2}e^{u_2}$ on $[R_1, R_3]$, we conclude that $\int_{R_1}^{R_3} re^{u_2}dx \le C$ for some $C > 0$ independent of ε_n. Combining this with $u_2 \le u_2(R_2)$ on $[R_1, R_3]$ and $u_2(R_2) \to -\infty$ as $\varepsilon_n \to 0$, we conclude that

$$P_3 \le \frac{(1+a_2)^2}{2} e^{u_2(R_2)} \int_{R_1}^{R_3} re^{u_2}dr \to 0 \text{ as } \varepsilon_n \to 0,$$

$$\text{i.e. } \quad P_3 \le a\delta^2 \text{ for } \varepsilon_n \text{ small enough.}$$

By $N_1 + 2 < \gamma$, $2B < A$, (5.27) and (5.8), we have

$$\begin{aligned} R_1 u_2'(R_1) &\le \frac{2a_2}{1+a_1}(\gamma + N_1) + 2(N_2 + (1+a)\delta^2) \\ &< \frac{4a_2}{1+a_1}\gamma + \frac{A-2B}{a_1(1+a_1)}\gamma < \frac{2(1+a)}{a_1}\gamma. \end{aligned} \tag{5.45}$$

Observe that $ru_2'(r)$ decreases on $[R_1, R_3]$. So $0 \le ru_2'(r) \le R_1 u_2'(R_1)$ for $r \in [R_1, R_2]$ and $ru_2'(r) < 0$ for $r \in (R_2, R_3]$. Then it follows from $a_2 < 1$, (5.43), (5.40) and (5.45) that

$$\begin{aligned} P_4 &< -\frac{1}{2}\int_{R_1}^{R_2} (r^2 u_2') \left[a_2(1+a_2)e^{u_1+u_2} + a_2 F_1\right] dr \\ &\le \frac{3}{4} R_1 u_2'(R_1) \int_{R_1}^{R_2} re^{u_1} dr < \frac{3}{4a_1}\gamma\delta. \end{aligned}$$

Combining all the arguments above, we obtain for ε_n sufficiently small that

$$b_2^2 - e_2^2 < \frac{\delta}{2} + (1+a)\delta^2 + \frac{3}{4a_1}\gamma\delta < (1+a)\delta^2 + \frac{3(1+a)}{4a_1}\gamma\delta.$$

This, together with (5.33), gives for ε_n sufficiently small that

$$\begin{aligned} b_2 &< |e_2| + (1+a)\delta^2 + \frac{3(1+a)}{4a_1}\gamma\delta \\ &< \frac{a_2}{1+a_1}(\gamma + N_1) + N_2 + 1 + 2(1+a)\delta^2 + \frac{3(1+a)}{4a_1}\gamma\delta \end{aligned}$$

$$< \frac{a_2}{1+a_1}(\gamma + N_1) + N_2 + 1 + \frac{1+a}{a_1}\gamma\delta.$$

On the other hand, similarly as (5.44), we can prove

$$\begin{aligned}
\frac{(1+a_2)b_1 + a_1 b_2}{1+a_1+a_2} &= -\frac{(1+a_2)R_2 u_1'(R_2)}{2(1+a_1+a_2)} + \frac{1}{2}\int_{R_2}^{R_3} r|F_1|dr - 1 \\
&\geq \frac{1+a_2}{2(A-B)}\left[2\frac{A-B}{A}\gamma - \frac{2B}{A}N_1 - \frac{2a_1}{1+a_2}N_2 - 3(1+a)^2\delta^2\right] - 1 \\
&> \frac{\gamma}{1+a_1} - \frac{B}{(1+a_1)(A-B)}N_1 - \frac{a_1}{A-B}N_2 - 1 - \delta.
\end{aligned}$$

Recalling from (5.9) that $\zeta_0 < 0$ since $\alpha_1 > \alpha_2$. Therefore, we finally obtain for ε_n sufficiently small that

$$\begin{aligned}
\frac{1+a_2}{1+a_1+a_2}(b_2 - b_1) <& \frac{a_2 - 1}{1+a_1}\gamma + \frac{a_2(1+2a_1+a_2)}{(1+a_1)(A-B)}N_1 + \frac{1+2a_1+a_2}{A-B}N_2 \\
&+ 2 + \left(\frac{1+a}{a_1}\gamma + 1\right)\delta \\
=& -|\zeta_0| + \left(\frac{1+a}{a_1}\gamma + 1\right)\delta < 0,
\end{aligned}$$

which contradicts to $b_2 > b_1$. This completes the proof. □

Remark 5.3. *In the case $a_2 \in (0,1)$, there is a remaining situation $\alpha_1 = \alpha_2 = \alpha_0$ that we can not deal with, since we have no idea whether $R_{3,\varepsilon} < +\infty$ or $R_{3,\varepsilon} = +\infty$. This remains as an open question.*

Now we are in a position to finish the proof of Theorem 5.1.

Completion of the proof of Theorem 5.1. Let $\varepsilon < \varepsilon_4(\delta)$. We consider two cases separately.

Case 1. $\alpha_1 < \alpha_2$. Then $R_3 < +\infty$. First we claim that $u_1'(r) > u_2'(r)$ for all $r \geq R_3$. Suppose not, then there exists $r^* > R_3$ such that $u_1'(r^*) = u_2'(r^*)$ and $u_1'(r) > u_2'(r)$ for $r \in [R_3, r^*)$. Clearly, $u_2 < u_1$ on $(R_3, r^*]$. If there exists $t^* \in (R_3, r^*]$ such that $u_1(t^*) = \ln\delta$ and $u_2 < u_1 < \ln\delta$ on $[R_3, t^*)$, then $F_1 < F_2 < 0$ and so $(ru_1')' = r[(1+a_1)F_1 - a_1F_2] < 0$ on $[R_3, t^*)$. Consequently, $0 \leq t^* u_1'(t^*) < R_3 u_1'(R_3) \leq -2 - 2\delta$, a contradiction. Hence, $u_2 < u_1 < \ln\delta$ and $F_1 < F_2 < 0$ on $(R_3, r^*]$, which implies that $ru_1'(r)$ decreases on $[R_3, r^*]$. Recalling $ru_1'(r) \leq -2 - 2\delta$ for $r \in [R_\delta, R_3]$, it follows that $ru_1'(r) \leq -2 - 2\delta$ for $r \in [R_\delta, r^*]$. Then, as before, we can prove

$$\int_{R_3}^{r^*} r|F_2|dr \leq \int_{R_3}^{r^*} r|F_1|dr \leq (1+a)\int_{R_\delta}^{r^*} re^{u_1}dr \leq \frac{\delta}{2}.$$

Consequently, Lemma 5.7 yields

$$\begin{aligned}
0 =& r^* u_2'(r^*) - r^* u_1'(r^*) \\
=& R_3 u_2'(R_3) - R_3 u_1'(R_3) + (1+a_1+a_2)\int_{R_3}^{r^*} r(F_2 - F_1)dr \\
\leq& -2(b_2 - b_1) + (1+a)\delta \leq -\frac{3}{2}\zeta_0 + (1+a)\delta < 0,
\end{aligned}$$

a contradiction. Therefore, $u_1'(r) > u_2'(r)$ for all $r \geq R_3$. By repeating the above argument, we can prove that $u_2 < u_1 < \ln\delta$ on $[R_3, +\infty)$ and $ru_1'(r) \leq -2 - 2\delta$ for all $r \in [R_\delta, +\infty)$. Consequently,

$$\int_{R_\delta}^{\infty} r|F_1|dr \leq (1+a)\int_{R_\delta}^{\infty} re^{u_1}dr \leq \frac{\delta}{2}. \tag{5.46}$$

By $u_{2,\varepsilon} < u_{1,\varepsilon} < \ln \delta$ on $[R_3, +\infty)$, we easily conclude that $(u_{1,\varepsilon}, u_{2,\varepsilon})$ is an entire solution of (5.14) for any $\varepsilon < \varepsilon_4$.

Case 2. $\alpha_1 > \alpha_2$. Then $R_3 = +\infty$. By the definition (5.31) of R_3 and Lemma 5.5, we have $u_1 < u_2 < u_2(R_2) \le \ln \delta$ on $(R_2, +\infty)$, which also implies that $(u_{1,\varepsilon}, u_{2,\varepsilon})$ is an entire solution of (5.14). Moreover, by Lemma 5.5, we also have $ru_1'(r) < -2 - 2\delta$ for any $r \ge R_\delta$, and so (5.46) also holds in this case.

In both cases, we have proved that $(u_{1,\varepsilon}, u_{2,\varepsilon})$ is an entire solution of (5.14) and $u_{2,\varepsilon}'(r) < 0$ in $(R_{2,\varepsilon}, +\infty)$ for any $\varepsilon < \varepsilon_4$. Therefore, it follows from Remark 5.2 that

$$\lim_{\varepsilon \to 0} \left(\sup_{\mathbb{R}^2} u_{2,\varepsilon} \right) = -\infty.$$

On the other hand, Theorem A yields the existence of $\alpha_{k,\varepsilon} > 1$ such that

$$u_{k,\varepsilon}(r) = -2\alpha_{k,\varepsilon} \ln r + O(1) \text{ as } r \to \infty, \quad k = 1, 2.$$

Combining these with Lemma 5.2 and (5.46), and letting $\delta \downarrow 0$, we can repeat the proof of Lemma 5.6 to conclude that $v_{1,\varepsilon} \to V$ in $C^2_{loc}(\mathbb{R}^2)$ and $(\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}) \to (\alpha_1, \alpha_2)$ as $\varepsilon \to 0$. Remark that due to the uniqueness of the limit V, this assertion holds for all $\varepsilon \to 0$ but not only up to a subsequence. This completes the proof. □

Acknowledgement The authors wish to thank the anonymous referee very much for his/her careful reading and valuable comments. The research of the first author was supported by NSFC (No. 11701312).

References

[1] W. Ao, C.-S. Lin, J. Wei, On non-topological solutions of the $\mathbf{A}_2$ and $\mathbf{B}_2$ Chern-Simons system, Mem. Amer. Math. Soc., **239** (2016), no. 1132, v+88 pp.

[2] W. Ao, C.-S. Lin, J. Wei, On non-topological solutions of the $\mathbf{G}_2$ Chern-Simons system, Comm. Anal. Geom., **24** (2016), 717-752.

[3] D. Chae, O. Imanuvilov, The existence of non-topological multivortex solutions in the relativistic self-dual Chern-Simons theory, Comm. Math. Phys., **215** (2000) 119-142.

[4] H. Chan, C. Fu, C.-S. Lin, Non-topological multi-vortex solutions to the self-dual Chern-Simons-Higgs equation, Comm. Math. Phys., **231** (2002) 189-221.

[5] K. Choe, Asymptotic behavior of condensate solutions in the Chern-Simons-Higgs theory, J. Math. Phys., **48** (2007) 103501-103517.

[6] K. Choe, Multiple existence results for the self-dual Chern-Simons-Higgs vortex equation, Comm. Partial Differ. Equ., **34** (2009) 1465-1507.

[7] K. Choe, N. Kim, C.-S. Lin, Existence of self-dual non-topological solutions in the Chern-Simons-Higgs model, Ann. Inst. H. Poincaré Anal. Non Linéaire, **28** (2011) 837-852.

[8] K. Choe, N. Kim, C.-S. Lin, Self-dual symmetric nontopological solutions in the $SU(3)$ model in $\mathbb{R}^2$, Comm. Math. Phys., **334** (2015) 1-37.

[9] G. Dunne, Vacuum mass spectra for $SU(N)$ self-dual Chern-Simons-Higgs systems, Nuclear Phys. B, **433** (1995) 333-348.

[10] G. Dunne, Mass degeneracies in self-dual models, Phys. Lett. B, **345** (1995) 452-457.

[11] S. Gudnason, Non-Abelian Chern-Simons vortices with generic gauge groups, Nuclear Phys. B, **821** (2009) 151-169.

[12] S. Gudnason, Fractional and semi-local non-Abelian Chern-Simons vortices, Nuclear Phys. B, **840** (2010) 160-185.

[13] X. Han, C.-S. Lin, G. Tarentello, Y. Yang, Chern-Simons vortices in the Gudnason model, J. Funct. Anal., **267** (2014) 678-726.

[14] X. Han, G. Tarantello, Doubly periodic self-dual vortices in a relativistic non-Abelian Chern-Simons model, Calc. Var. PDE., **49** (2014) 1149-1176.

[15] J. Hong, Y. Kim, P. Pac, Multivortex solutions of the Abelian Chern-Simons-Higgs theory, Phys. Rev. Lett., **64** (1990) 2230-2233.

[16] H. Huang, C.-S. Lin, On the entire radial solutions of the Chern-Simons $SU(3)$ system, Comm. Math. Phys., **327** (2014) 815-848.

[17] H. Huang, C.-S. Lin, Classification of the entire radial self-dual solutions to non-Abelian Chern-Simons systems, J. Funct. Anal., **266** (2014) 6796-6841.

[18] R. Jackiw, E. Weinberg, Self-dual Chern-Simons vortices, Phys. Rev. Lett., **64** (1990) 2234-2237.

[19] H. Kao, K. Lee, Self-dual $SU(3)$ Chern-Simons Higgs systems, Phys. Rev. D, **50** (1994) 6626-6632.

[20] N. Kim, Existence of vortices in a self-dual gauged linear sigma model and its singular limit, Nonlinearity, **19** (2006) 721-739.

[21] C.-S. Lin, J. Wei, D. Ye, Classification and nondegeneracy of $SU(n+1)$ Toda system with singular sources, Invent. Math., **190** (2012) 169-207.

[22] G. Lozano, D. Marqués, E. Moreno, F. Schaposnik, Non-Abelian Chern-Simons vortices, Phys. Lett. B, **654** (2007) 27-34.

[23] M. Nolasco, G. Tarantello, Vortex condensates for the $SU(3)$ Chern-Simons theory, Comm. Math. Phys., **213** (2000) 599-639.

[24] J. Prajapat, G. Tarantello, On a class of elliptic problems in $\mathbb{R}^2$: symmetry and uniqueness results, Proc. Roy. Soc. Edinburgh Sect. A, **131** (2001) 967-985.

[25] J. Spruck, Y. Yang, The existence of non-topological solutions in the self-dual Chern-Simons theory, Comm. Math. Phys., **149** (1992) 361-376.

[26] J. Spruck, Y. Yang, Topological solutions in the self-dual Chern-Simons theory: existence and approximation, Ann. Inst. H. Poincaré Anal. Non Linéaire, **12** (1995) 75-97.

[27] Y. Yang, The relativistic non-Abelian Chern-Simons equations, Comm. Math. Phys., **186** (1997) 199-218.