EXACT NUMBER AND NON-DEGENERACY OF CRITICAL POINTS OF MULTIPLE GREEN FUNCTIONS ON RECTANGULAR TORI

Zhijie Chen & Chang-Shou Lin

Abstract

Let $E_\tau := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ be a flat torus and $G(z;\tau)$ be the Green function on E_τ. Consider the multiple Green function G_n on $(E_\tau)^n$:

$$G_n(z_1,\cdots,z_n;\tau) := \sum_{i<j} G(z_i - z_j;\tau) - n\sum_{i=1}^{n} G(z_i;\tau).$$

We prove that for $\tau \in i\mathbb{R}_{>0}$, i.e. E_τ is a rectangular torus, G_n has exactly $2n+1$ critical points modulo the permutation group S_n and all critical points are non-degenerate. More precisely, there are exactly n (resp. $n+1$) critical points $\boldsymbol{a}$'s with the Hessian satisfying $(-1)^n \det D^2 G_n(\boldsymbol{a};\tau) < 0$ (resp. > 0). This confirms a conjecture in [**4**]. Our proof is based on the connection between G_n and the classical Lamé equation from [**4**, **19**], and one key step is to establish a precise formula of the Hessian of critical points of G_n in terms of the monodromy data of the Lamé equation. As an application, we show that the mean field equation

$$\Delta u + e^u = \rho\delta_0 \quad \text{on } E_\tau$$

has exactly n solutions for $8\pi n - \rho > 0$ small, and exactly $n+1$ solutions for $\rho - 8\pi n > 0$ small.

1. Introduction

Let $E_\tau := \mathbb{C}/\Lambda_\tau$ be a flat torus in the plane, where $\Lambda_\tau = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1 = 1$, $\omega_2 = \tau$ and $\tau \in \mathbb{H} = \{\tau \mid \operatorname{Im}\tau > 0\}$. Denote also $\omega_3 = 1+\tau$. Let $G(z) = G(z;\tau)$ be the Green function on E_τ defined by

$$-\Delta G(z;\tau) = \delta_0 - \frac{1}{|E_\tau|} \quad \text{on } E_\tau, \qquad \int_{E_\tau} G(z;\tau) = 0,$$

where δ_0 is the Dirac measure at 0 and $|E_\tau|$ is the area of the torus E_τ. It is an even function with the only singularity at 0. Let $E_\tau^\times := E_\tau \setminus \{0\}$ and consider the complete diagonal in $(E_\tau^\times)^n$:

$$\Delta_n := \{(z_1,\cdots,z_n) \in (E_\tau^\times)^n | z_i = z_j \text{ for some } i \neq j\}.$$

Let $\boldsymbol{z} = (z_1, \cdots, z_n) \in (E_\tau^\times)^n \backslash \Delta_n$ and define the *multiple Green function* $G_n(\boldsymbol{z}) = G_n(\boldsymbol{z}; \tau)$ on $(E_\tau^\times)^n \backslash \Delta_n$ by

$$G_n(\boldsymbol{z}; \tau) := \sum_{i<j} G(z_i - z_j; \tau) - n \sum_{i=1}^{n} G(z_i; \tau). \tag{1.1}$$

Our motivation of studying G_n comes from its connection (see [**4**, **19**]) with the bubbling phenomena of semilinear elliptic PDEs with exponential nonlinearities in two dimension. A typical example is the curvature equation with parameter $\rho > 0$:

$$\Delta u + e^u = \rho \delta_0 \quad \text{on } E_\tau. \tag{1.2}$$

In conformal geometry, a solution u leads to a metric $ds^2 = \frac{1}{2} e^u (dx^2 + dy^2)$ with constant Gaussian curvature $+1$ acquiring a conic singularity at 0. It also appears in statistical physics as the equation for the *mean field limit* of the Euler flow in Onsager's vortex model (cf. [**3**]), hence also called a *mean field equation*. In the physical model of superconductivity, (1.2) is one of limiting equations of the well-known Chern-Simons-Higgs equation

$$\Delta u + \tfrac{1}{\varepsilon^2} e^u (1 - e^u) = 4\pi \sum \alpha_j \delta_{p_j} \quad \text{on } E_\tau \tag{1.3}$$

as the coupling parameter $\varepsilon \to 0$. We refer to [**2**, **4**, **7**, **18**, **21**, **22**] and references therein for recent developments on this equation.

One important feature of (1.2) is the so-called *bubbling phenomena*. Let u_k be a sequence of solutions of (1.2) with $\rho = \rho_k \to 8\pi n$, $n \in \mathbb{N}$, and $\max_{E_\tau} u_k(z) \to +\infty$ as $k \to +\infty$. We call p a blowup point of $\{u_k\}$ if there is a sequence $\{x_k\}_k$ such that $x_k \to p$ and $u_k(x_k) \to +\infty$ as $k \to +\infty$. Then it was proved in [**5**] that u_k always has exactly n blowup points $\{a_1, \cdots, a_n\}$ in E_τ and $a_i \neq 0$ for all i. The well-known *Pohozaev identity* says that the positions of these blowup points could be determined by equations

$$n \nabla G(a_i) = \sum_{j=1, \neq i}^{n} \nabla G(a_i - a_j), \quad 1 \leq i \leq n. \tag{1.4}$$

Clearly this system (1.4) gives the critical point equations of G_n, i.e. the blowup point set $\{a_1, \cdots, a_n\}$ gives a critical point of G_n. As in [**4**], a critical point $\boldsymbol{a} = \{a_1, \cdots, a_n\}$ of G_n is called *trivial* if

$$\{a_1, \cdots, a_n\} = \{-a_1, \cdots, -a_n\} \quad \text{in } E_\tau.$$

Then [**4**, Theorem 0.7.5] shows that *the blowup set* $\{a_1, \cdots, a_n\}$ *provides a trivial critical point if and only if* $\rho_k \neq 8\pi n$ *for large* k. Moreover,

Theorem A. [**4**, **5**] *Let* u_k *be a sequence of bubbling solutions of (*1.2*) with* $\rho = \rho_k \to 8\pi n$, $n \in \mathbb{N}$*. If* $\rho_k \neq 8\pi n$ *for large* k*, then the followings hold:*

(1) *The blowup set* $\boldsymbol{a} = \{a_1, \cdots, a_n\}$ *gives a trivial critical point of* G_n.
(2) *Let* $\lambda_k := \max_{E_\tau} u_k(z)$, *then there is a constant* $D(\boldsymbol{a}) = D(\boldsymbol{a};\tau)$ *such that*

$$\rho_k - 8\pi n = (D(\boldsymbol{a}) + o(1))e^{-\lambda_k}. \tag{1.5}$$

There are similar bubbling phenomena for the Chern-Simons-Higgs equation (1.3); see [**21**]. From (1.5), the geometric quantity $D(\boldsymbol{a})$ plays a fundamental role in controlling the sign of $\rho_k - 8\pi n$ if we can prove $D(\boldsymbol{a}) \neq 0$. Thus it provides one of the key geometric messages for the bubbling solutions. See Section 5 for the explicit expression of $D(\boldsymbol{a})$.

In view of Theorem A, it is of fundamental importance to study the critical points of G_n and the sign of $D(\boldsymbol{a})$. Notice that $G_n(\boldsymbol{z})$ is invariant under the permutation group S_n. If $\boldsymbol{a} = (a_1, \cdots, a_n)$ is a critical point of G_n, then so does $(a_{j_1}, \cdots, a_{j_n})$, where $(j_1, \cdots, j_n)$ is any permutation of $(1, \cdots, n)$. Of course, we consider these $(a_{j_1}, \cdots, a_{j_n})$'s to be the *same critical point.* Naturally we are concerned with the following fundamental questions:

(a) *How many critical points does* G_n *have*?
(b) *Is any critical point non-degenerate? Or equivalently, does the Hessian* $\det D^2 G_n(\boldsymbol{a}) \neq 0$ *for any critical point* $\boldsymbol{a}$? *Note that* $D^2 G_n(\boldsymbol{a})$ *is a* $2n \times 2n$ *matrix of the second derivatives of* G_n.
(c) *Does* $D(\boldsymbol{a}) \neq 0$ *for any trivial critical point* $\boldsymbol{a}$ *of* G_n?

The case $n = 1$ (note $G_1 = -G$) was first studied by Wang and the second author [**17**]. Since $G(z)$ is even and doubly periodic, it always has three trivial critical points $\frac{\omega_k}{2}$, $k = 1, 2, 3$, and nontrivial critical points must appear in pairs if exist. It was proved in [**17**] that $G(z;\tau)$ has *at most one pair of nontrivial critical points* (depending on τ). Furthermore, if $\tau \in i\mathbb{R}_{>0}$, then $G(z;\tau)$ has exactly three critical points $\frac{\omega_k}{2}$, $k = 1, 2, 3$, which are all non-degenerate, and $D(\frac{\omega_3}{2}) < 0$, $D(\frac{\omega_k}{2}) > 0$ for $k = 1, 2$. Recently, a complete characterization of those τ's such that $G(z;\tau)$ has a pair of nontrivial critical points has been given in [**10**, **20**].

For general $n \geq 2$, G_n was first studied by Chai, Wang and the second author [**4**] from the viewpoint of algebraic geometry. Among other things, they established the *one-to-one correspondence* between trivial critical points of G_n and branch points of a hyperelliptic curve Y_n concerning the classical Lamé equation. See Section 2 for a brief overview. Consequently, they proved that

Theorem B. [**4**] $G_n(\boldsymbol{z};\tau)$ *has at most* $2n+1$ *(resp. exactly* $2n+1$*) trivial critical points for any* τ *(resp. for almost all* τ *including* $\tau \in i\mathbb{R}_{>0}$*).*

The purpose of this paper is to settle the above questions (a)-(c) when $\tau \in i\mathbb{R}_{>0}$. Our main result is as follows.

Theorem 1.1. *Let* $\tau \in i\mathbb{R}_{>0}$. *The the followings hold.*

(a) *$G_n(\boldsymbol{z};\tau)$ has exactly $2n+1$ critical points $\boldsymbol{a}$'s, which are all trivial critical points and satisfy $\boldsymbol{a}=\bar{\boldsymbol{a}}$.*
(b) *The Hessian* $\det D^2G_n(\boldsymbol{a};\tau)\neq 0$ *for any critical point $\boldsymbol{a}$.*
(c) *$D(\boldsymbol{a};\tau)\neq 0$ for any critical point $\boldsymbol{a}$.*

Here by writing $\boldsymbol{a}=\{a_1,\cdots,a_n\}$, $\boldsymbol{a}=\bar{\boldsymbol{a}}$ means

$$\{a_1,\cdots,a_n\}=\{\bar{a}_1,\cdots,\bar{a}_n\}\quad \text{in } E_\tau.$$

In fact, our proof also determines the exact number of critical points satisfying $D(\boldsymbol{a};\tau)<0$. We would like to single out this statement:

Corollary 1.2. *Let $\tau\in i\mathbb{R}_{>0}$. Then there are exactly n (resp. $n+1$) critical points $\boldsymbol{a}$'s of $G_n(\boldsymbol{z};\tau)$ satisfying*

$$D(\boldsymbol{a};\tau)<0 \text{ or equivalently } (-1)^n\det D^2G_n(\boldsymbol{a};\tau)<0$$

(resp. $D(\boldsymbol{a};\tau)>0$ or equivalently $(-1)^n\det D^2G_n(\boldsymbol{a};\tau)>0$).

Corollary 1.2 confirms [**4**, Conjecture 8.7.2], where the importance of Corollary 1.2 was highlighted for potential applications in PDE problems such as (1.2) and (1.3). Here we give an example. A solution $u(z)$ is called *even and symmetric* if

$$u(z)=u(-z)=u(\bar{z}),\quad z=x+iy\in\mathbb{C}.$$

As applications of Theorem 1.1 and Corollary 1.2, we have the following result concerning counting the number of solutions for (1.2).

Theorem 1.3 (=Theorem 5.2)**.** *Let $n\in\mathbb{N}$ and $\tau\in i\mathbb{R}_{>0}$, i.e. E_τ is a rectangular torus. Then there exists a small $\varepsilon_0>0$ such that*
(1) *for $\rho\in(8\pi n,8\pi n+\varepsilon_0)$, (1.2) has exactly $n+1$ solutions, which are all even and symmetric;*
(2) *for $\rho=8\pi n$, (1.2) has no solutions;*
(3) *for $\rho\in(8\pi n-\varepsilon_0,8\pi n)$, (1.2) has exactly n solutions, which are all even and symmetric.*

For $n=1$, Theorem 1.3 is a special case of [**20**, Theorem 0.5] for $\tau\in i\mathbb{R}_{>0}$. For $n\geq 2$, Theorem 1.3-(2) was proved by the authors [**11**], while (1) & (3) are new. Theorem 1.3 partially confirms the following conjecture.

Conjecture C. [**11**, Conjecture 1.1] *Suppose $\tau\in i\mathbb{R}_{>0}$ and $n\in\mathbb{N}$. Then (1.2) has exactly n solutions for $\rho\in(8\pi(n-1),8\pi n)$.*

REMARK 1.4. Conjecture C for $\rho=8\pi(n-\frac{1}{2})$ was already proved by Chai, Wang and the second author in [**4**, Corollary 0.4.2]. Their proof is based on the relation between (1.2) and the Lamé equation

$$y''(z)=[\eta(\eta+1)\wp(z;\tau)+B]y(z),\quad \eta=\tfrac{\rho}{8\pi}. \tag{1.6}$$

Since $\eta=n-\frac{1}{2}$ is a half integer for $\rho=8\pi(n-\frac{1}{2})$, the local exponent difference $2\eta+1$ is an *even integer*, which infers the existence of a polynomial $P_n(B;\tau)$ of degree n in B such that solutions of (1.6) with $\eta = n - \frac{1}{2}$ have no logarithmic singularities if and only if $P_n(B;\tau) = 0$. In [**4**, Corollary 0.4.2] they proved the one-to-one correspondence between solutions of (1.2) with $\rho = 8\pi(n-\frac{1}{2})$ and zeros of $P_n(\cdot;\tau)$, so the conclusion follows from the fact that $P_n(\cdot;\tau)$ has exactly n different roots for $\tau \in i\mathbb{R}_{>0}$. Clearly this idea can not apply for $\rho \neq 8\pi(n-\frac{1}{2})$ due to the essential difference between $\eta \notin \frac{1}{2}\mathbb{Z}$ and $\eta \in \frac{1}{2}\mathbb{Z}$ for the Lamé equation (1.6).

Differently, our approach of this paper essentially relies on the bubbling phenomena for $\rho \to 8\pi n$ as stated in Theorem A and hence only work for ρ close to $8\pi n$. In view of Theorem 1.3 and [**4**, Corollary 0.4.2], Conjecture C follows via the continuity method if one can prove that any solution of (1.2) is non-degenerate for $\tau \in i\mathbb{R}_{>0}$ and $\rho \notin 8\pi\mathbb{N}$, which seems challenging and remains open.

Note that Theorem 1.1 deals with $\tau \in i\mathbb{R}_{>0}$, and does not necessarily hold for general $\tau \in \mathbb{H}$. In view of the theory for $n = 1$ in [**17**], we propose the following conjecture for general $\tau \in \mathbb{H}$.

Conjecture 1.5. *For each $n \geq 2$, $G_n(\boldsymbol{z};\tau)$ has at most n pairs of nontrivial critical points (depending on τ).*

Our proof of the non-degeneracy of trivial critical points relies on its connection with the classical Lamé equation. This was first discovered in [**4**]. Let $\boldsymbol{a} = (a_1, \cdots, a_n)$ be a trivial critical point of G_n, then we will explain in Section 2 that the Hermite-Halphen ansatz

$$y_{\boldsymbol{a}}(z) := e^{z\sum_{i=1}^n \zeta(a_i)} \frac{\prod_{i=1}^n \sigma(z-a_i)}{\sigma(z)^n}, \quad z \in \mathbb{C}$$

satisfies the classical Lamé equation

$$y''(z) = [n(n+1)\wp(z;\tau) + B]y(z), \tag{1.7}$$

with $B \in \mathbb{C}$ given by

$$B = B_{\boldsymbol{a}} := (2n-1)\sum_{i=1}^n \wp(a_i), \tag{1.8}$$

where $\wp, \zeta, \sigma$ are the Weierstrass functions that will be recalled in Section 2. See e.g. [**4**, Theorem 6.2] for the proof. For later usage, we will briefly review it in Section 2.

Since $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$ are linearly dependent, in Section 3 we will choose another solution of the form $y_{\boldsymbol{a}}(z)\chi(z)$, where $\chi(z)$ has two quasi-periods:

$$\chi_1 = \chi(z+1) - \chi(z) \quad \text{and} \quad \chi_2 = \chi(z+\tau) - \chi(z).$$

Define

$$\mathcal{C} := \tfrac{\chi_2}{\chi_1} \in \mathbb{C} \cup \{\infty\}. \tag{1.9}$$

Then the monodromy group of the Lamé equation (1.7) is generated by

$$\varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \varepsilon_2 \begin{pmatrix} 1 & 0 \\ \mathcal{C} & 1 \end{pmatrix} \quad \text{with } \varepsilon_1, \varepsilon_2 = \pm 1. \tag{1.10}$$

Remark that if $\mathcal{C} = \infty$, then (1.10) should be understood as

$$\varepsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varepsilon_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{with } \varepsilon_1, \varepsilon_2 = \pm 1. \tag{1.11}$$

See Section 3 for the proof. Therefore, $\mathcal{C}$ is referred as the *monodromy data* of the Lamé equation (1.7) with $B = B_{\boldsymbol{a}}$ when $\boldsymbol{a}$ is a trivial critical point.

On the other hand, it is known (cf. [**4**, **28**]) that there is a polynomial $\ell_n(B;\tau)$ of degree $2n+1$ in B such that the Lamé equation (1.7) has the monodromy group generated by (1.10) if and only if $\ell_n(B;\tau) = 0$. This $\ell_n(B;\tau)$ is called the *Lamé polynomial* in the literature. See also Section 2 for a brief review.

Our third main result of this paper is the deep relation between the Hessian $\det D^2 G_n(\boldsymbol{a};\tau)$ at a trivial critical point $\boldsymbol{a}$ and the monodromy data $\mathcal{C}$ of the corresponding Lamé equation (1.7) with $B = B_{\boldsymbol{a}}$.

Theorem 1.6 (=Theorem 3.3)**.** *Let $\boldsymbol{a} = \{a_1, \cdots, a_n\}$ be a trivial critical point of G_n, and $\mathcal{C}$ be the monodromy data of the corresponding Lamé equation (1.7) with $B = B_{\boldsymbol{a}}$, i.e. $\mathcal{C}$ is given by (1.9). Then*

$$\det D^2 G_n(\boldsymbol{a};\tau) = (-1)^n P_1(\boldsymbol{a};\tau) P_2(\boldsymbol{a};\tau) \operatorname{Im} \mathcal{C}, \tag{1.12}$$

where $P_2(\boldsymbol{a};\tau) = |\chi_1|^2$ and $P_1(\boldsymbol{a};\tau) \geq 0$. Moreover,

1. *$P_1(\boldsymbol{a};\tau) = 0$ if and only if $B_{\boldsymbol{a}}$ is a multiple zero of the Lamé polynomial $\ell_n(B;\tau)$.*
2. *$P_2(\boldsymbol{a};\tau) = 0$ if and only if the monodromy data $\mathcal{C} = \infty$. If this happens, then $\det D^2 G_n(\boldsymbol{a};\tau) = 0$.*

The formula (1.12) gives a geometric meaning of the monodromy data. Theorem 1.6 is crucial in our proof of the non-degeneracy of trivial critical points. After Theorem 1.6, the next step of proving the non-degeneracy is to prove the following result, which is also interesting from the viewpoint of the monodromy theory.

Theorem 1.7 (=Theorem 4.3)**.** *Let $\tau \in i\mathbb{R}_{>0}$, B be any zero of the Lamé polynomial $\ell_n(\cdot;\tau)$ and $\mathcal{C}$ be the monodromy data of the corresponding Lamé equation (1.7). Then $\mathcal{C} \in i\mathbb{R} \setminus \{0\}$.*

This paper is organized as follows. In Section 2, we briefly review the connection between trivial critical points of G_n and the Lamé equation from [**4**]. Theorem 1.6 will be proved in Section 3, where we will apply a formula concerning $\det D^2 G_n(\boldsymbol{a};\tau)$ from [**19**]. Theorems 1.1 and 1.7 will be proved in Section 4, where we will apply a classical result from the Floquet theory concerning the Lamé equation. In Section 5, we will give applications to the mean field equation (1.2) and prove Corollary 1.2 and Theorem 1.3. Finally in Section 6, Theorem 1.7 will be extended to a generalized Lamé equation with the Treibich-Verdier potential ([**29**]).

2. Trivial critical points and the Lamé equation

In this section, for the reader's convenience and also for later usage, we briefly review the connection between trivial critical points of G_n and the classical Lamé equation from [**4**].

Let $\wp(z) = \wp(z;\tau)$ be the Weierstrass elliptic function with periods Λ_τ, defined by

$$\wp(z;\tau) := \frac{1}{z^2} + \sum_{\omega\in\Lambda_\tau\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right), \tag{2.1}$$

which satisfies the well-known cubic equation

$$\wp'(z;\tau)^2 = 4\wp(z;\tau)^3 - g_2(\tau)\wp(z;\tau) - g_3(\tau). \tag{2.2}$$

Let $\zeta(z) = \zeta(z;\tau) := -\int^z \wp(\xi;\tau)d\xi$ be the Weierstrass zeta function with two quasi-periods $\eta_j = \eta_j(\tau)$, $j = 1, 2$:

$$\eta_j(\tau) = 2\zeta(\tfrac{\omega_j}{2};\tau) = \zeta(z+\omega_j;\tau) - \zeta(z;\tau), \quad j = 1, 2, \tag{2.3}$$

and $\sigma(z) = \sigma(z;\tau)$ be the Weierstrass sigma function defined by $\sigma(z) := \exp\int^z \zeta(\xi)d\xi$. Notice that $\zeta(z)$ is an odd meromorphic function with simple poles at Λ_τ and $\sigma(z)$ is an odd entire function with simple zeros at Λ_τ.

The Green function G on E_τ can be expressed in terms of elliptic functions. In [**17**], Wang and the second author proved that

$$-4\pi\frac{\partial G}{\partial z}(z;\tau) = \zeta(z) - r\eta_1 - s\eta_2 = \zeta(z) - z\eta_1 + 2\pi i s, \tag{2.4}$$

where $z = r + s\tau$ with $r, s \in \mathbb{R}$ and the Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$ is used. By (2.4), the critical point equations (1.4) can be translated into the following equivalent system:

$$\sum_{i=1}^{n} \nabla G(a_i) = 0, \tag{2.5}$$

and

$$\sum_{j\neq i} (\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)) = 0, \ \forall 1 \le i \le n, \tag{2.6}$$

subject to the constraint $\boldsymbol{a} \in (E_\tau^\times)^n\setminus\Delta_n$, i.e.

$$a_i \neq 0, \quad a_i \neq a_j \quad \text{in } E_\tau, \quad \forall i \neq j. \tag{2.7}$$

See [**4**] for the proof. Remark that (2.6) is independent of any representative $\tilde{a}_i \in a_i + \Lambda_\tau$. We will use (2.6)-(2.7) to connect a critical point of G_n with the classical Lamé equation.

Recall the Lamé equation

$$\mathcal{L}_{n,B}: \qquad y''(z) = [n(n+1)\wp(z;\tau) + B]y(z), \tag{2.8}$$

where $n \in \mathbb{R}_{\geq -1/2}$ and $B \in \mathbb{C}$ are called *index* and *accessory parameter* respectively. Generally, a solution $y(z)$ is a multi-valued meromorphic function on $\mathbb{C}$ with branch points at Λ_τ, since any lattice point is a regular singular point with local exponents $-n$ and $n+1$. In this paper, we consider only $n \in \mathbb{N}$. Then any solution is single-valued meromorphic on $\mathbb{C}$.

For $\boldsymbol{a} = (a_1, \cdots, a_n)$, we consider the *Hermite-Halphen ansatz* (cf. [**4**, **28**]):

$$y_{\boldsymbol{a}}(z) := e^{z\sum_{i=1}^n \zeta(a_i)} \frac{\prod_{i=1}^n \sigma(z - a_i)}{\sigma(z)^n}. \tag{2.9}$$

Theorem 2.1. *(cf. [**4**, **28**]) Suppose $\boldsymbol{a} = (a_1, \cdots, a_n) \in (E_\tau^\times)^n \backslash \Delta_n$. Then $y_{\boldsymbol{a}}(z)$ is a solution of $\mathcal{L}_{n,B}$ for some B if and only if $\boldsymbol{a}$ satisfies (2.6) and*

$$B = B_{\boldsymbol{a}} := (2n-1)\sum_{i=1}^n \wp(a_i). \tag{2.10}$$

Note that if $\boldsymbol{a} = (a_1, \cdots, a_n) \in (E_\tau^\times)^n \backslash \Delta_n$ satisfies (2.6), then so does $-\boldsymbol{a} = (-a_1, \cdots, -a_n)$ and then $y_{-\boldsymbol{a}}(z)$ is also a solution of the same Lamé equation because $B_{\boldsymbol{a}} = B_{-\boldsymbol{a}}$. Clearly $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$ are linearly independent if and only if $\{a_1, \cdots, a_n\} \neq \{-a_1, \cdots, -a_n\}$ in E_τ. Furthermore, this condition actually implies

$$\{a_1, \cdots, a_n\} \cap \{-a_1, \cdots, -a_n\} = \emptyset, \tag{2.11}$$

because $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$ can not have common zeros, otherwise the Wronskian of $(y_{\boldsymbol{a}}(z), y_{-\boldsymbol{a}}(z))$ would be identical zero, which infers that $y_{\boldsymbol{a}}(z)$, $y_{-\boldsymbol{a}}(z)$ are linearly dependent.

Definition 2.2. *Suppose $\boldsymbol{a} = (a_1, \cdots, a_n) \in (E_\tau^\times)^n \backslash \Delta_n$ satisfies (2.6). We say that $\boldsymbol{a}$ is a branch point if $\{a_1, \cdots, a_n\} = \{-a_1, \cdots, -a_n\}$ in E_τ.*

Let $\boldsymbol{a}$ satisfy (2.6)-(2.7) and be *not* a branch point. Since it was proved in [**4**, Proposition 5.8.3] that (2.6) is equivalent to

$$\sum_{i=1}^n \wp'(a_i)\wp(a_i)^l = 0, \quad \forall\, 0 \leq l \leq n-2,$$

then

$$g_{\boldsymbol{a}}(z) := \sum_{i=1}^n \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} = \frac{C(\boldsymbol{a})}{\prod_{i=1}^n (\wp(z) - \wp(a_i))} \tag{2.12}$$

for a constant $C(\boldsymbol{a}) \neq 0$. Equivalently,

$$C(\boldsymbol{a}) = \sum_{i=1}^{n} \wp'(a_i) \prod_{j\neq i} (\wp(z) - \wp(a_j)). \tag{2.13}$$

There are various ways to represent $C(\boldsymbol{a})$ by plugging different values of z into (2.13). In particular, for $z = a_i$ we get

$$C(\boldsymbol{a}) = \wp'(a_i) \prod_{j\neq i} (\wp(a_i) - \wp(a_j)) \text{ is independent of } i. \tag{2.14}$$

Remark that if $\boldsymbol{a}$ is a branch point, then $g_{\boldsymbol{a}}(z) \equiv 0$ and so $C(\boldsymbol{a}) = 0$. Here we recall the following important theorem:

Theorem 2.3. [**4**] *There is a so-called spectral polynomial $\ell_n(B) = \ell_n(B;\tau) \in \mathbb{Q}[g_2(\tau), g_3(\tau)][B]$ of degree $2n+1$ such that if $\boldsymbol{a}$ satisfies (2.6)-(2.7), then $C^2 = \ell_n(B)$, where $C = C(\boldsymbol{a})$ and $B = B_{\boldsymbol{a}}$ are given by (2.14) and (2.10) respectively.*

This spectral polynomial $\ell_n(B)$ is called the *Lamé polynomial* in the literature. Define $Y_n(\tau) = Y_n$ by the set of $\boldsymbol{a} = \{a_1, \cdots, a_n\}$ satisfying (2.6)-(2.7), i.e.

$$Y_n := \left\{ \{a_1, \cdots, a_n\} \;\middle|\; \begin{array}{l} a_i \neq 0,\ a_i \neq a_j \text{ in } E_\tau \text{ for any } i \neq j, \\ \sum_{j\neq i}^{n} (\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)) = 0,\ \forall i \end{array} \right\}. \tag{2.15}$$

Clearly $-\boldsymbol{a} := \{-a_1, \cdots, -a_n\} \in Y_n$ if $\boldsymbol{a} \in Y_n$, and $\boldsymbol{a} \in Y_n$ is a *branch point* of Y_n if $\boldsymbol{a} = -\boldsymbol{a}$ in E_τ. Then the map $B : Y_n \to \mathbb{C}$ defined by (2.10) is a ramified covering of degree 2, and Theorem 2.3 implies

$$Y_n(\tau) \cong \left\{ (B, C) \mid C^2 = \ell_n(B;\tau) \right\}.$$

See [**4**, Theorem 7.4] for the proof. Therefore, Y_n is a hyperelliptic curve, known as the *Lamé curve*.

On the other hand, since $G(z)$ is even, (2.5) holds automatically for any branch point $\boldsymbol{a}$ of Y_n. Therefore, the above arguments yield

Theorem 2.4. [**4**]

(1) *$\boldsymbol{a}$ is a trivial critical point of G_n if and only if $\boldsymbol{a}$ is a branch point of Y_n.*
(2) *Y_n is singular at a branch point $\boldsymbol{a}$ if and only if $B_{\boldsymbol{a}}$ is a multiple zero of $\ell_n(B)$.*
(3) *Given any $\boldsymbol{a} \in Y_n$, $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$ are linearly dependent if and only if $B_{\boldsymbol{a}}$ is a zero of $\ell_n(B)$.*

Remark 2.5. As suggested in Theorem 2.4-(3), the Wronskian

$$W(\boldsymbol{a}) := y'_{\boldsymbol{a}}(z) y_{-\boldsymbol{a}}(z) - y'_{-\boldsymbol{a}}(z) y_{\boldsymbol{a}}(z)$$

is a nonzero multiple of $C(\boldsymbol{a})$. Indeed, (2.9) gives

$$y_{\boldsymbol{a}}(z)y_{-\boldsymbol{a}}(z) = \prod_{i=1}^{n} \frac{\sigma(z-a_i)\sigma(z+a_i)}{\sigma(z)^2}$$

$$= (-1)^n \prod_{i=1}^{n} \sigma(a_i)^2 \cdot \prod_{i=1}^{n} (\wp(z) - \wp(a_i)), \tag{2.16}$$

and

$$\frac{W(\boldsymbol{a})}{y_{\boldsymbol{a}}(z)y_{-\boldsymbol{a}}(z)} = \frac{y'_{\boldsymbol{a}}(z)}{y_{\boldsymbol{a}}(z)} - \frac{y'_{-\boldsymbol{a}}(z)}{y_{-\boldsymbol{a}}(z)} = \sum_{i=1}^{n} (\zeta(z-a_i) - \zeta(z+a_i) + 2\zeta(a_i))$$

$$= \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)}.$$

Then it follows from (2.12) and $a_i \neq 0$ in E_τ for all i that

$$W(\boldsymbol{a}) = (-1)^n \prod_{i=1}^{n} \sigma(a_i)^2 \cdot C(\boldsymbol{a}) \quad \text{with} \quad \prod_{i=1}^{n} \sigma(a_i)^2 \neq 0.$$

For later usage, we denote

$$X_n := \{\boldsymbol{a} = \{a_1, \cdots, a_n\} \in Y_n \mid \boldsymbol{a} \text{ is not a branch point}\}.$$

3. The monodromy data $\mathcal{C}$ and the Hessian

The purpose of this section is to prove Theorem 1.6. First we introduce the monodromy representation of the Lamé equation $\mathcal{L}_{n,B}$. Consider two linearly independent solutions $y_1(z)$, $y_2(z)$ and any loop $\ell \in \pi_1(E_\tau)$. Let $\ell^* y_i(z)$ denote the analytic continuation of $y_i(z)$ along ℓ. Then there is a matrix $\rho(\ell) \in SL(2,\mathbb{C})$ such that

$$\ell^* \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = \rho(\ell) \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix},$$

where $\rho(\ell) \in SL(2,\mathbb{C})$ follows from the fact that the Wronskian $y_1'(z)y_2(z) - y_2'(z)y_1(z)$ is a nonzero constant and so invariant under the analytic continuation of $y_i(z)$ along ℓ. Hence the monodromy representation ρ is a homomorphism from $\pi_1(E_\tau)$ to $SL(2,\mathbb{C})$. Since $\pi_1(E_\tau)$ is abelian, so the monodromy group is always abelian and hence *reducible*, which contains two cases:

(a) Completely reducible (i.e. all the monodromy matrices have two linearly independent common eigenfunctions): Let ℓ_1, ℓ_2 be two fundamental cycles in $\pi_1(E_\tau)$, then up to a common conjugation,

$$\rho(\ell_1) = \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}, \quad \rho(\ell_2) = \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{-2\pi i r} \end{pmatrix} \tag{3.1}$$

for some $(r,s) \in \mathbb{C}^2 \backslash \frac{1}{2}\mathbb{Z}^2$. See e.g. [**4**].

(b) Not completely reducible (i.e. the space of common eigenfunctions is of dimension 1): Up to a common conjugation,

$$\rho(\ell_1)=\varepsilon_1\begin{pmatrix}1&0\\1&1\end{pmatrix},\quad \rho(\ell_2)=\varepsilon_2\begin{pmatrix}1&0\\\mathcal{C}&1\end{pmatrix},\tag{3.2}$$

where $\varepsilon_1,\varepsilon_2\in\{\pm1\}$ and $\mathcal{C}\in\mathbb{C}\cup\{\infty\}$. Remark that if $\mathcal{C}=\infty$, then (3.2) should be understood as

$$\rho(\ell_1)=\varepsilon_1\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \rho(\ell_2)=\varepsilon_2\begin{pmatrix}1&0\\1&1\end{pmatrix}.\tag{3.3}$$

This will be clear in Proposition 3.1, where we will explain how to determine $\varepsilon_1,\varepsilon_2$ and the monodromy data $\mathcal{C}$.

It is known (cf. [**4**]) that up to nonzero constants, the common eigenfunctions must be of the form $y_{\pm\boldsymbol{a}}(z)$ for a unique pair $\pm\boldsymbol{a}\in Y_n$, where $y_{\boldsymbol{a}}(z)$ is given in (2.9). Together with Theorem 2.4, we summarize the following conclusions:

Case (a) occurs for $\mathcal{L}_{n,B}$, i.e. monodromy is completely reducible

$$\Longleftrightarrow \ell_n(B;\tau)\neq 0$$

$$\Longleftrightarrow B=B_{\boldsymbol{a}}\ \text{ for some }\ \boldsymbol{a}=\{a_1,\cdots,a_n\}\in X_n$$

$$\Longleftrightarrow y_{\boldsymbol{a}}(z)\text{ and }y_{-\boldsymbol{a}}(z)\text{ are linearly independent;}$$

and

Case (b) occurs for $\mathcal{L}_{n,B}$ with monodromy data $\mathcal{C}$

$$\Longleftrightarrow \ell_n(B;\tau)=0$$

$$\Longleftrightarrow B=B_{\boldsymbol{a}}\ \text{ for some branch point }\ \boldsymbol{a}=\{a_1,\cdots,a_n\}\in Y_n\setminus X_n$$

$$\Longleftrightarrow y_{\boldsymbol{a}}(z)\text{ and }y_{-\boldsymbol{a}}(z)\text{ are linearly dependent.}$$

Now given a branch point $\boldsymbol{a}=\{a_1,\cdots,a_n\}\in Y_n\backslash X_n$, a basic question is how to determine the monodromy data $\mathcal{C}$ of the corresponding Lamé equation (2.8) with $B=B_{\boldsymbol{a}}$? This question is answered by the following result, which will play a key role in proving Theorem 1.6. Define

$$\Lambda_2:=\{i\mid a_i=\tfrac{\omega_k}{2}\text{ in }E_\tau\text{ for some }k\in\{1,2,3\}\}$$

and for $i\notin\Lambda_2$, we define $i^*\notin\Lambda_2$ to be the index so that $a_{i^*}=-a_i$ in E_τ.

Proposition 3.1. *Let $\boldsymbol{a}=\{a_1,\cdots,a_n\}\in Y_n\setminus X_n$ be a branch point. Then the monodromy data $\mathcal{C}$ of the corresponding Lamé equation $\mathcal{L}_{n,B}$ with $B=B_{\boldsymbol{a}}$ is given by*

$$\mathcal{C}=\frac{c_0\tau-\eta_2\sum_{j=1}^n c_j}{c_0-\eta_1\sum_{j=1}^n c_j},\tag{3.4}$$

where c_0, c_j, $1 \le j \le n$, are given by

$$c_0 = -\sum_{j=1}^{n} c_j \wp(a_j), \tag{3.5}$$

$$c_i = 2\wp''(a_i)^{-1} \prod_{j \neq i} (\wp(a_i) - \wp(a_j))^{-1}, \quad \textit{if } \ i \in \Lambda_2, \tag{3.6}$$

$$c_i = c_{i^*} = \wp'(a_i)^{-2} \prod_{j \neq i, i^*} (\wp(a_i) - \wp(a_j))^{-1}, \quad \textit{if } \ i \notin \Lambda_2. \tag{3.7}$$

Proof. Recall (2.9) that

$$y_{\boldsymbol{a}}(z) = e^{z \sum_{i=1}^{n} \zeta(a_i)} \frac{\prod_{i=1}^{n} \sigma(z - a_i)}{\sigma(z)^n}.$$

Note that $\{a_1, \cdots, a_n\} = \{-a_1, \cdots, -a_n\}$ in E_τ implies

$$\sum_{j=1}^{n} a_j = \tfrac{\omega_k}{2} \ \text{ in } E_\tau \text{ for some } k \in \{0, 1, 2, 3\},$$

where $\omega_0 = 0$. Applying the transformation law of $\sigma(z)$:

$$\sigma(z + \omega_j) = -e^{\eta_j (z + \omega_j/2)} \sigma(z), \quad j = 1, 2,$$

and the Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$, a direct computation shows

$$y_{\boldsymbol{a}}(z + \omega_j) = \varepsilon_j y_{\boldsymbol{a}}(z), \quad j = 1, 2, \tag{3.8}$$

where

$$(\varepsilon_1, \varepsilon_2) = \begin{cases} (1, 1) \ \text{ if } \ k = 0, \\ (1, -1) \ \text{ if } \ k = 1, \\ (-1, 1) \ \text{ if } \ k = 2, \\ (-1, -1) \ \text{ if } \ k = 3. \end{cases}$$

Define

$$P_{\boldsymbol{a}}(z) := \frac{1}{\prod_{j=1}^{n} (\wp(z) - \wp(a_j))}.$$

Since $\boldsymbol{a} \in Y_n \setminus X_n$ is a branch point, it was proved in [**19**, Lemma 3.2] that the residue of $P_{\boldsymbol{a}}(z)$ at a_j is zero for all j and then we may rewrite $P_{\boldsymbol{a}}(z)$ as

$$P_{\boldsymbol{a}}(z) = \sum_{j=1}^{n} c_j \wp(z - a_j) + c_0,$$

where c_0, c_j's are given by (3.5)-(3.7). Define

$$\chi(z) := \int_0^z P_{\boldsymbol{a}}(\xi) d\xi = -\sum_{j=1}^{n} c_j (\zeta(z - a_j) + \zeta(a_j)) + c_0 z. \tag{3.9}$$

Then it follows from $\{a_1, \cdots, a_n\} = \{-a_1, \cdots, -a_n\}$ in E_τ that $\chi(z)$ is *odd* and has two quasi-periods χ_1, χ_2:

$$\chi_1 = \chi(z+1) - \chi(z) = c_0 - \eta_1 \sum_{j=1}^{n} c_j, \tag{3.10}$$

$$\chi_2 = \chi(z+\tau) - \chi(z) = c_0\tau - \eta_2 \sum_{j=1}^{n} c_j. \tag{3.11}$$

On the other hand, it follows from (2.16) and the linear dependence of $y_{\pm\boldsymbol{a}}(z)$ that $P_{\boldsymbol{a}}(z) = c(\boldsymbol{a})/y_{\boldsymbol{a}}(z)^2$ for some constant $c(\boldsymbol{a}) \neq 0$. Then a direct computation shows that

$$y_2(z) := y_{\boldsymbol{a}}(z)\chi(z) = c(\boldsymbol{a}) y_{\boldsymbol{a}}(z) \int_0^z \frac{1}{y_{\boldsymbol{a}}(\xi)^2} d\xi. \tag{3.12}$$

is also a solution of the Lamé equation (2.8), which is linearly independent with $y_{\boldsymbol{a}}(z)$. Since Case (b) occurs, $y_2(z)$ can not be a common eigenfunction. Then (3.8) and (3.10)-(3.11) infer that χ_1, χ_2 can not vanish simultaneously. Define

$$\mathcal{C} := \frac{\chi_2}{\chi_1} = \frac{c_0\tau - \eta_2 \sum_{j=1}^{n} c_j}{c_0 - \eta_1 \sum_{j=1}^{n} c_j}. \tag{3.13}$$

We claim that this $\mathcal{C}$ is precisely the *monodromy data*. In fact, if $\chi_1 = 0$, then $\chi_2 \neq 0$, $\mathcal{C} = \infty$ and it follows from (3.8), (3.10)-(3.11) that

$$\begin{pmatrix} \chi_2 y_{\boldsymbol{a}}(z+1) \\ y_2(z+1) \end{pmatrix} = \varepsilon_1 \begin{pmatrix} \chi_2 y_{\boldsymbol{a}}(z) \\ y_2(z) \end{pmatrix}, \tag{3.14}$$

$$\begin{pmatrix} \chi_2 y_{\boldsymbol{a}}(z+\tau) \\ y_2(z+\tau) \end{pmatrix} = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \chi_2 y_{\boldsymbol{a}}(z) \\ y_2(z) \end{pmatrix},$$

which is precisely (3.3). If $\chi_1 \neq 0$, then $\mathcal{C} \neq \infty$ and (3.8), (3.10)-(3.11) give

$$\begin{pmatrix} \chi_1 y_{\boldsymbol{a}}(z+1) \\ y_2(z+1) \end{pmatrix} = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \chi_1 y_{\boldsymbol{a}}(z) \\ y_2(z) \end{pmatrix}, \tag{3.15}$$

$$\begin{pmatrix} \chi_1 y_{\boldsymbol{a}}(z+\tau) \\ y_2(z+\tau) \end{pmatrix} = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ \mathcal{C} & 1 \end{pmatrix} \begin{pmatrix} \chi_1 y_{\boldsymbol{a}}(z) \\ y_2(z) \end{pmatrix}, \tag{3.16}$$

which is precisely (3.2). This completes the proof. q.e.d.

The following result, which is important for our future study, is interesting from the viewpoint of the monodromy theory. It establishes the 1-1 correspondence between the zeros of $\ell_n(B)$ and the monodromy datas $\mathcal{C}$'s.

Proposition 3.2. *Let B_1, B_2 be two zeros of $\ell_n(B)$ such that $B_1 \neq B_2$, and $\mathcal{C}_j$ be the monodromy data of the corresponding Lamé equation $\mathcal{L}_{n,B_j}$. Then $\mathcal{C}_1 \neq \mathcal{C}_2$.*

Proof. Suppose by contradiction that $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$. Without loss of generality, we may assume $\mathcal{C} \neq \infty$ (the case $\mathcal{C} = \infty$ can be treated similarly).

For the Lamé equation $\mathcal{L}_{n,B_1}$, there exists a branch point $\boldsymbol{a} \in Y_n \setminus X_n$ such that $B_1 = B_{\boldsymbol{a}}$. Applying the proof of Proposition 3.1, there exist linearly independent solutions $y_1(z) = \chi_1 y_{\boldsymbol{a}}(z)$ and $y_2(z) = y_{\boldsymbol{a}}(z)\chi(z)$ such that

$$\begin{pmatrix} y_1(z+1) \\ y_2(z+1) \end{pmatrix} = \varepsilon_{1,1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix},$$

$$\begin{pmatrix} y_1(z+\tau) \\ y_2(z+\tau) \end{pmatrix} = \varepsilon_{1,2} \begin{pmatrix} 1 & 0 \\ \mathcal{C} & 1 \end{pmatrix} \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}$$

with $\varepsilon_{1,1}$, $\varepsilon_{1,2} = \pm 1$. Furthermore, since $\chi(z)$ is odd and 0 is a pole of $y_{\boldsymbol{a}}(z)$ with order n, and the local exponent of y_2 at 0 is either $-n$ or $n+1$, we conclude that

0 *is a pole of* $y_1(z)$ *with order* n *and is a zero of* $y_2(z)$ *with order* $n+1$.

Similarly, for the Lamé equation $\mathcal{L}_{n,B_2}$, there exist linearly independent solutions $\hat{y}_1(z)$ and $\hat{y}_2(z)$ such that

$$\begin{pmatrix} \hat{y}_1(z+1) \\ \hat{y}_2(z+1) \end{pmatrix} = \varepsilon_{2,1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \hat{y}_1(z) \\ \hat{y}_2(z) \end{pmatrix},$$

$$\begin{pmatrix} \hat{y}_1(z+\tau) \\ \hat{y}_2(z+\tau) \end{pmatrix} = \varepsilon_{2,2} \begin{pmatrix} 1 & 0 \\ \mathcal{C} & 1 \end{pmatrix} \begin{pmatrix} \hat{y}_1(z) \\ \hat{y}_2(z) \end{pmatrix}$$

with $\varepsilon_{2,1}$, $\varepsilon_{2,2} = \pm 1$, and

0 *is a pole of* $\hat{y}_1(z)$ *with order* n *and is a zero of* $\hat{y}_2(z)$ *with order* $n+1$.

Define

$$\Phi_1(z) := \begin{pmatrix} y_1(z) & y_2(z) \\ y_1'(z) & y_2'(z) \end{pmatrix}, \quad \Phi_2(z) := \begin{pmatrix} \hat{y}_1(z) & \hat{y}_2(z) \\ \hat{y}_1'(z) & \hat{y}_2'(z) \end{pmatrix}.$$

Then

$$\Phi_j(z+1) = \varepsilon_{j,1}\Phi_j(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\Phi_j(z+\tau) = \varepsilon_{j,2}\Phi_j(z) \begin{pmatrix} 1 & \mathcal{C} \\ 0 & 1 \end{pmatrix}.$$

Define

$$\begin{aligned} \Phi(z) &:= \Phi_1(z) \cdot \Phi_2(z)^{-1} \\ &= \frac{1}{W} \begin{pmatrix} y_1\hat{y}_2' - y_2\hat{y}_1' & -y_1\hat{y}_2 + \hat{y}_1 y_2 \\ y_1'\hat{y}_2' - y_2'\hat{y}_1' & -y_1'\hat{y}_2 + y_2'\hat{y}_1 \end{pmatrix} \\ &=: \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{12}(z) & a_{22}(z) \end{pmatrix}, \end{aligned}$$

where $W := \hat{y}_1\hat{y}_2' - \hat{y}_2\hat{y}_1'$ is the Wronskian of $(\hat{y}_2(z), \hat{y}_1(z))$ which is a non-zero constant. Then

$$\Phi(z+1) = \varepsilon_{1,1}\varepsilon_{2,1}\Phi(z), \quad \Phi(z+\tau) = \varepsilon_{1,2}\varepsilon_{2,2}\Phi(z). \tag{3.17}$$

Note that $a_{12}(z) = \frac{1}{W}(-y_1\hat{y}_2 + \hat{y}_1 y_2)$ has a zero at 0 and hence is holomorphic in $\mathbb{C}$. Since (3.17) implies that $a_{12}(z)$ is bounded in $\mathbb{C}$, we conclude from the Liouville theorem that $a_{12}(z) \equiv 0$.

Clearly $a_{11}(z) = \frac{1}{W}(y_1\hat{y}_2' - y_2\hat{y}_1')$ is holomorphic at 0 and hence holomorphic in $\mathbb{C}$. Applying (3.17) we conclude that $a_{11}(z) \equiv d_1$ is a constant. Similarly, $a_{22}(z) \equiv d_2$ is also a constant. Therefore,

$$\begin{pmatrix} y_1(z) & y_2(z) \\ y_1'(z) & y_2'(z) \end{pmatrix} = \begin{pmatrix} d_1 & 0 \\ a_{12}(z) & d_2 \end{pmatrix} \begin{pmatrix} \hat{y}_1(z) & \hat{y}_2(z) \\ \hat{y}_1'(z) & \hat{y}_2'(z) \end{pmatrix},$$

where $d_1 d_2 \neq 0$. It follows that $y_1(z) = d_1\hat{y}_1(z)$, i.e. $y_1(z)$, $\hat{y}_1(z)$ solve the same Lamé equation, which implies $B_1 = B_2$, a contradiction.

Therefore, $\mathcal{C}_1 \neq \mathcal{C}_2$. q.e.d.

We conclude this section by proving Theorem 1.6, i.e. the following theorem.

Theorem 3.3. *Let $\boldsymbol{a} = \{a_1, \cdots, a_n\}$ be a branch point of $Y_n(\tau)$, c_0, c_j's be constants defined in (3.5)-(3.7), and $\mathcal{C}$ be the monodromy data of the corresponding Lamé equation $\mathcal{L}_{n,B}$ with $B = B_{\boldsymbol{a}}$, i.e. $\mathcal{C}$ is given by (3.4). Then*

$$\det D^2 G_n(\boldsymbol{a};\tau) = (-1)^n P_1(\boldsymbol{a};\tau) P_2(\boldsymbol{a};\tau) \operatorname{Im} \mathcal{C}, \tag{3.18}$$

where

$$P_2(\boldsymbol{a};\tau) := |\chi_1|^2 = \left| c_0 - \eta_1 \sum_{j=1}^{n} c_j \right|^2,$$

and $P_1(\boldsymbol{a};\tau) \geq 0$. Moreover,

1. *$P_1(\boldsymbol{a};\tau) = 0$ precisely when the hyperelliptic curve $Y_n(\tau)$ is singular at $\boldsymbol{a}$. There are only finitely many such tori E_τ for each n.*
2. *$P_2(\boldsymbol{a};\tau) = 0$ if and only if the monodromy data $\mathcal{C} = \infty$. If this happens, then* $\det D^2 G_n(\boldsymbol{a};\tau) = 0$.

Proof. Denote $s = \sum_{j=1}^{n} c_j$. In [**19**, Theorems 3.4 and 4.1], Wang and the second author proved that

$$\det D^2 G_n(\boldsymbol{a};\tau) = (-1)^n c_{\boldsymbol{a}}(\tau) D(\boldsymbol{a};\tau), \tag{3.19}$$

with

$$D(\boldsymbol{a};\tau) = e^{\tilde{c}(\boldsymbol{a};\tau)} \operatorname{Im}\tau \cdot \left(|c_0 - s\eta_1|^2 + \frac{2\pi}{\operatorname{Im}\tau} \operatorname{Re} \bar{s}(c_0 - s\eta_1) \right), \tag{3.20}$$

where $\tilde{c}(\boldsymbol{a};\tau) \in \mathbb{R}$, i.e. $e^{\tilde{c}(\boldsymbol{a};\tau)} > 0$, and $c_{\boldsymbol{a}}(\tau) \geq 0$. Moreover, $c_{\boldsymbol{a}}(\tau) = 0$ precisely when the hyperelliptic curve $Y_n(\tau)$ is singular at this branch point $\boldsymbol{a}$. There are only finitely many such tori E_τ for each fixed n. Remark that this $D(\boldsymbol{a};\tau)$ is an important geometric quantity arising from the bubbling phenomena of the mean field equation (1.2) as mentioned in the introduction. We will briefly review it in Section 5.

On the other hand, Proposition 3.1 shows that

$$\mathcal{C}=\frac{\chi_2}{\chi_1}=\frac{c_0\tau-\eta_2\sum_{j=1}^n c_j}{c_0-\eta_1\sum_{j=1}^n c_j}=\tau+\frac{2\pi i}{\frac{c_0}{s}-\eta_1},$$

where we used the Legendre relation $\tau\eta_1-\eta_2=2\pi i$. It follows that

$$\begin{aligned}&\operatorname{Im}\tau\cdot\left(|c_0-s\eta_1|^2+\frac{2\pi}{\operatorname{Im}\tau}\operatorname{Re}\bar{s}(c_0-s\eta_1)\right)\\ =&|c_0-s\eta_1|^2\operatorname{Im}\left(\tau+\frac{2\pi i}{\frac{c_0}{s}-\eta_1}\right)=|c_0-s\eta_1|^2\operatorname{Im}\mathcal{C}.\end{aligned}$$

Together with (3.19) and (3.20), we finally obtain

$$\det D^2G_n(\boldsymbol{a};\tau)=(-1)^n c_{\boldsymbol{a}}(\tau)e^{\tilde{c}(\boldsymbol{a};\tau)}|c_0-s\eta_1|^2\operatorname{Im}\mathcal{C}.$$

Letting $P_1(\boldsymbol{a};\tau)=c_{\boldsymbol{a}}(\tau)e^{\tilde{c}(\boldsymbol{a};\tau)}$ and $P_2(\boldsymbol{a};\tau)=|c_0-s\eta_1|^2=|\chi_1|^2$, the proof is complete. q.e.d.

The explicit expressions of $\det D^2G_n(\boldsymbol{a};\tau)$ at all trivial critical points for $n=1,2$ were calculated in [**17**] and [**19**] respectively. We will recall these formulas for $n=2$ in the next section; see (4.7)-(4.8). Here we compute a new example.

EXAMPLE 3.4. Let $\boldsymbol{a}=(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})$. Since $\nabla G(\frac{\omega_k}{2})=0$, we see that $\boldsymbol{a}$ is a trivial critical point of $G_3(\boldsymbol{z};\tau)$. Note that $\det D^2G_3(\boldsymbol{a};\tau)$ is the determinant of a 6×6 real matrix and is not easy to calculate directly. Here we apply Theorem 3.3 to compute it. Conventionally, we denote $e_k=e_k(\tau):=\wp(\frac{\omega_k}{2};\tau)$. It is well known that

$$\wp'(z)^2=4\prod_{k=1}^3(\wp(z)-e_k)=4\wp(z)^3-g_2\wp(z)-g_3,$$

and $g_2^3-27g_3^2=16(e_1-e_2)^2(e_1-e_3)^2(e_2-e_3)^2\neq 0$. Recalling c_0, c_j's defined in (3.5)-(3.6), we have (write $\{i,j,k\}=\{1,2,3\}$)

$$c_j=\frac{2}{\wp''(\frac{\omega_j}{2})(e_j-e_i)(e_j-e_k)}=\frac{1}{(e_j-e_i)^2(e_j-e_k)^2},$$

and then straightforward computations give

$$\sum_{j=1}^3 c_j=\frac{24g_2}{g_2^3-27g_3^2},\quad c_0=-\sum_{j=1}^3 e_jc_j=\frac{36g_3}{g_2^3-27g_3^2}.$$

Consequently,

$$\chi_1 = c_0 - \eta_1 \sum_{j=1}^{n} c_j = \frac{-12(2\eta_1 g_2 - 3g_3)}{g_2^3 - 27g_3^2},$$

$$\mathcal{C} = \frac{\chi_2}{\chi_1} = \tau + \frac{2\pi i \sum c_i}{\chi_1} = \tau - \frac{4\pi i g_2}{2\eta_1 g_2 - 3g_3}.$$

Applying Theorem 3.3, we finally obtain

$$\det D^2 G_3(\boldsymbol{a};\tau) = -P(\tau)|2\eta_1 g_2 - 3g_3|^2 \operatorname{Im}\left(\tau - \frac{4\pi i g_2}{2\eta_1 g_2 - 3g_3}\right),$$

where $P(\tau) \neq \infty$ is given by

$$P(\tau) := \frac{144 P_1(\boldsymbol{a};\tau)}{|g_2^3 - 27g_3^2|^2} > 0 \quad \forall \tau \in \mathbb{H}.$$

In fact, (2.10) gives $B_{\boldsymbol{a}} = 5\sum_{i=1}^{3} e_i = 0$, which is always a simple zero of the Lamé polynomial $\ell_3(B;\tau)$ (see [**18**, p.569]):

$$\ell_3(B;\tau) = \frac{2^2}{3^4 5^4} B \prod_{i=1}^{3} (B^2 - 6e_i B + 15(3e_i^2 - g_2)).$$

Thus $Y_3(\tau)$ is always smooth at $\boldsymbol{a}$, i.e. $P_1(\boldsymbol{a};\tau) > 0$ for all τ.

4. Non-degeneracy of trivial critical points

The purpose of this section is to prove Theorems 1.1 and 1.7. From now on, we consider only $\tau \in i\mathbb{R}_{>0}$. First we briefly recall the Floquet theory about the Lamé equation restrict on $z = x + \frac{\tau}{2}$ with $x \in \mathbb{R}$ (to avoid the singularities of $\wp(z)$ on $\mathbb{R}$):

$$y''(x) - n(n+1)\wp(x + \tfrac{\tau}{2};\tau)y(x) = By(x), \quad x \in \mathbb{R}. \tag{4.1}$$

See e.g. [**12**] for the Floquet theory for general Hill's equation.

For any $x_0 \in \mathbb{R}$, we let $c(B, x, x_0)$ and $s(B, x, x_0)$ be the special fundamental system of solutions of (4.1) satisfying the initial values

$$c(B, x_0, x_0) = s'(B, x_0, x_0) = 1, \;\; c'(B, x_0, x_0) = s(B, x_0, x_0) = 0.$$

Clearly

$$\begin{pmatrix} c(B, x+1, x_0) \\ s(B, x+1, x_0) \end{pmatrix} = \begin{pmatrix} c(B, x_0+1, x_0) & c'(B, x_0+1, x_0) \\ s(B, x_0+1, x_0) & s'(B, x_0+1, x_0) \end{pmatrix} \begin{pmatrix} c(B, x, x_0) \\ s(B, x, x_0) \end{pmatrix}.$$

Define

$$\Delta(B) := c(B, x_0+1, x_0) + s'(B, x_0+1, x_0)$$

to be the trace of the above monodromy matrix and

$$d(B) := \operatorname{ord}_B(\Delta(\cdot)^2 - 4).$$

Remark that $\Delta(B)$ is independent of the choice of $x_0 \in \mathbb{R}$, and $d(E)$ is known to coincide with *the algebraic multiplicity of (anti)periodic eigenvalues*. Define

$$p(B, x_0) := \mathrm{ord}_B s(\cdot, x_0 + 1, x_0),$$

$$p_i(B) := \min\{p(B, x_0) : x_0 \in \mathbb{R}\}. \tag{4.2}$$

Then $p(B, x_0)$ and $p_i(B)$ are known as the algebraic multiplicity of a Dirichlet eigenvalue and its immovable part respectively. See [**12**].

Notice that the Lamé potential $q(x) := -n(n+1)\wp(x + \frac{\tau}{2}; \tau)$ is real-valued for $x \in \mathbb{R}$ because of $\tau \in i\mathbb{R}_{>0}$. Here we recall the following well-known result for (4.1).

Theorem 4.1. [**13**, **15**] *Let $\tau \in i\mathbb{R}_{>0}$. Then the followings hold.*

(1) *All zeros of the Lamé polynomial $\ell_n(B)$ are real and distinct, denoted by $B_{2n} < B_{2n-1} < \cdots < B_1 < B_0$.*
(2) *For each $j \in [0, 2n]$, $d(B_j) = 1$ and $p_i(B_j) = 0$.*
(3) *The Lamé potential $q(x) = -n(n+1)\wp(x + \frac{\tau}{2}; \tau)$ is a finite-gap potential, i.e. the spectrum of the associated operator $L = \frac{d^2}{dx^2} + q(x)$ in $L^2(\mathbb{R})$ has a finite-gap spectrum of the type*

$$\sigma(L) = (-\infty, B_{2n}] \cup [B_{2n-1}, B_{2n-2}] \cup \cdots \cup [B_1, B_0].$$

Now we turn back to the Lamé equation (2.8).

Lemma 4.2. *Let $\tau \in i\mathbb{R}_{>0}$, B be a zero of $\ell_n(B; \tau)$ and $\mathcal{C}$ be the monodromy data of the corresponding Lamé equation* (2.8)*. Then $\mathcal{C} \in i\mathbb{R} \cup \{\infty\}$.*

Proof. Since $\tau \in i\mathbb{R}_{>0}$, it is easy to see that

$$\overline{\wp(z)} = \wp(\bar{z}), \quad \overline{\zeta(z)} = \zeta(\bar{z}). \tag{4.3}$$

In particular, $\eta_1 = 2\zeta(\frac{1}{2}) \in \mathbb{R}$. Since B is a zero of $\ell_n(B; \tau)$, Theorem 4.1-(1) says $B \in \mathbb{R}$, and it follows from Section 2 that there exists a branch point $\boldsymbol{a} = (a_1, \cdots, a_n)$ of Y_n such that

$$B = B_{\boldsymbol{a}} = (2n-1) \sum_{i=1}^{n} \wp(a_i).$$

Clearly $\bar{\boldsymbol{a}} := (\bar{a}_1, \cdots, \bar{a}_n)$ also satisfies (2.6) and $\{\bar{a}_1, \cdots, \bar{a}_n\} = \{-\bar{a}_1, \cdots, -\bar{a}_n\}$, which implies that $\bar{\boldsymbol{a}}$ is also a branch point of Y_n. Since

$$B_{\bar{\boldsymbol{a}}} = (2n-1) \sum_{i=1}^{n} \wp(\bar{a}_i) = \overline{B_{\boldsymbol{a}}} = \overline{B} = B,$$

we conclude from the hyperelliptic geometry of Y_n (i.e. the map $B : Y_n \to \mathbb{C}$ defined by (2.10) is a ramified covering of degree 2) that $\bar{\boldsymbol{a}} = \boldsymbol{a}$, i.e.

$$\{a_1, \cdots, a_n\} = \{\bar{a}_1, \cdots, \bar{a}_n\} \quad \text{in } E_\tau. \tag{4.4}$$

Given $j \in [1, n]$, we denote $\bar{j}$ to be the unique index such that $a_{\bar{j}} = \bar{a}_j$ in E_τ. Then it follows from (4.3) and (3.6)-(3.7) that

$$\bar{c}_j = c_{\bar{j}}, \quad \forall j \in [1, n].$$

Therefore, $\sum_{j=1}^n c_j \in \mathbb{R}$ and

$$\overline{\chi_1} = -\sum_{j=1}^{n} c_{\bar{j}} \wp(a_{\bar{j}}) - \eta_1 \sum_{j=1}^{n} c_{\bar{j}} = \chi_1,$$

i.e. $\chi_1 \in \mathbb{R}$. Since $\eta_2 = \tau\eta_1 - 2\pi i$ gives

$$\chi_2 = c_0 \tau - \eta_2 \sum_{j=1}^{n} c_j = \tau\chi_1 + 2\pi i \sum_{j=1}^{n} c_j,$$

we obtain $\chi_2 \in i\mathbb{R}$. Therefore, $\mathcal{C} = \frac{\chi_2}{\chi_1} \in i\mathbb{R} \cup \{\infty\}$. q.e.d.

The following result is crucial for us to prove the non-degeneracy of trivial critical points of G_n.

Theorem 4.3. *Let $\tau \in i\mathbb{R}_{>0}$, B be a zero of $\ell_n(B; \tau)$ and $\mathcal{C}$ be the monodromy data of the corresponding Lamé equation (2.8). Then $\mathcal{C} \notin \{0, \infty\}$, i.e. $\mathcal{C} \in i\mathbb{R} \setminus \{0\}$.*

Proof. Since B is a zero of $\ell_n(B; \tau)$, it follows from Section 2 that there exists a branch point $\boldsymbol{a} = (a_1, \cdots, a_n)$ of Y_n such that $B = B_{\boldsymbol{a}}$.

Step 1. We prove that $\mathcal{C} \neq \infty$.

Assume by contradiction that $\mathcal{C} = \infty$. Then the proof of Proposition 3.1 (especially (3.14)) shows that $y_1(z) := \chi_2 y_{\boldsymbol{a}}(z), y_2(z)$ are linearly independent solutions of the Lamé equation (2.8) and

$$y_1(z+1) = \varepsilon_1 y_1(z), \quad y_2(z+1) = \varepsilon_1 y_2(z), \quad \varepsilon_1 = \pm 1.$$

Consequently, $\tilde{y}_j(x) := y_j(x + \frac{\tau}{2})$, $j = 1, 2$, are linearly independent solutions of (4.1) and

$$\tilde{y}_j(x+1) = \varepsilon_1 \tilde{y}_j(x), \quad \varepsilon_1 = \pm 1, \quad j = 1, 2.$$

This implies that all solutions of (4.1) are periodic if $\varepsilon_1 = 1$ (resp. antiperiodic if $\varepsilon_1 = -1$) and so does $s(B, x, x_0)$ for any $x_0 \in \mathbb{R}$. In particular, $s(B, x_0 + 1, x_0) = 0$ for all $x_0 \in \mathbb{R}$ and so (4.2) gives $p_i(B) \geq 1$. However, Theorem 4.1 (1)-(2) say that $p_i(B) = 0$, a contradiction. Therefore, $\mathcal{C} \neq \infty$.

Step 2. We prove that $\mathcal{C} \neq 0$.

Assume by contradiction that $\mathcal{C} = 0$. Recalling the modular property of $\wp(z; \tau)$:

$$\wp(z; \tfrac{-1}{\tau}) = \tau^2 \wp(\tau z; \tau),$$

we immediately see that $y(z)$ is a solution of the Lamé equation (2.8) if and only if $\tilde{y}(z) := y(\tau z)$ is a solution of the Lamé equation

$$\tilde{y}''(z) = \left[n(n+1)\wp\left(z; \tfrac{-1}{\tau}\right) + \tau^2 B\right] \tilde{y}(z). \tag{4.5}$$

Define $\tilde{y}_1(z) := -\chi_1 y_{\boldsymbol{a}}(\tau z)$ and $\tilde{y}_2(z) := y_2(\tau z)$. Since the proof of Proposition 3.1 shows that $\chi_1 y_{\boldsymbol{a}}(z), y_2(z)$ are linearly independent solutions of (2.8) and satisfy (3.15)-(3.16) with $\mathcal{C}=0$, we see that $\tilde{y}_1(z), \tilde{y}_2(z)$ are linearly independent solutions of (4.5) and satisfy

$$\begin{pmatrix}\tilde{y}_1(z+1)\\ \tilde{y}_2(z+1)\end{pmatrix} = \varepsilon_2 \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\tilde{y}_1(z)\\ \tilde{y}_2(z)\end{pmatrix},$$

$$\begin{pmatrix}\tilde{y}_1(z+\frac{-1}{\tau})\\ \tilde{y}_2(z+\frac{-1}{\tau})\end{pmatrix} = \varepsilon_1 \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\tilde{y}_1(z)\\ \tilde{y}_2(z)\end{pmatrix}.$$

Together with (3.3), we conclude that the monodromy data of (4.5), denoted by $\mathcal{C}(\frac{-1}{\tau})$, is ∞. However, since $\frac{-1}{\tau} \in i\mathbb{R}_{>0}$, we can apply Step 1 to (4.5) (i.e. replacing τ by $\frac{-1}{\tau}$) and obtain $\mathcal{C}(\frac{-1}{\tau}) \neq \infty$, a contradiction. This proves $\mathcal{C} \neq 0$. Finally, together with Lemma 4.2, we have $\mathcal{C} \in i\mathbb{R}\setminus\{0\}$. q.e.d.

We are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\tau \in i\mathbb{R}_{>0}$.

Step 1. We prove that G_n has no nontrivial critical points.

This assertion follows directly from the following two statements:

(i) $G_n(\boldsymbol{z};\tau)$ *has nontrivial critical points if and only if the mean field equation*

$$\Delta u + e^u = 8\pi n \delta_0 \quad \text{on } E_\tau \tag{4.6}$$

has solutions. This statement holds for any $\tau \in \mathbb{H}$ and was proved by Chai, Wang and the second author [**4**].

(ii) *If $\tau \in i\mathbb{R}_{>0}$, i.e. E_τ is a rectangular torus, then (*4.6*) has no solutions.* This statement was proved by Wang and the second author [**17**] for $n=1$ and by the authors [**11**] for all $n \in \mathbb{N}$ via a different idea.

Together with Theorem B in Section 1, we conclude that $G_n(\boldsymbol{z};\tau)$ has exactly $2n+1$ critical points which are all trivial. Since the assertion $\boldsymbol{a}=\bar{\boldsymbol{a}}$ has been proved in (4.4), we obtain the assertion (a).

Step 2. We show that $\det D^2 G_n(\boldsymbol{a};\tau) \neq 0$ for each trivial critical point $\boldsymbol{a}$.

Recall Theorem 3.3 that

$$\det D^2 G_n(\boldsymbol{a};\tau) = (-1)^n P_1(\boldsymbol{a};\tau) P_2(\boldsymbol{a};\tau)\,\mathrm{Im}\,\mathcal{C}.$$

Note from Theorem 2.4-(2) and Theorem 4.1-(1) that $P_1(\boldsymbol{a};\tau) > 0$, and Theorem 4.3 implies $P_2(\boldsymbol{a};\tau)\,\mathrm{Im}\,\mathcal{C} \neq 0$. Therefore, $\det D^2 G_n(\boldsymbol{a};\tau) \neq 0$, i.e. any trivial critical point is non-degenerate. This proves (b).

Finally, recall Theorem 3.3 that

$$\det D^2 G_n(\boldsymbol{a};\tau) = (-1)^n c_{\boldsymbol{a}}(\tau) D(\boldsymbol{a};\tau),$$

where $c_{\boldsymbol{a}}(\tau) = e^{-\tilde{c}(\boldsymbol{a};\tau)} P_1(\boldsymbol{a};\tau) > 0$. Thus $D(\boldsymbol{a};\tau) \neq 0$, i.e. (c) holds.

The proof is complete. q.e.d.

We take the special case $n = 2$ as an example. It is known [**19**] that G_2 has only five trivial critical points $\{(q_\pm, -q_\pm) | \wp(q_\pm;\tau) = \pm\sqrt{g_2(\tau)/12}\}$ and $\{(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j) \mid i \neq j\}$, and the Hessian is given by

$$\det D^2 G_2(q_\pm, -q_\pm;\tau) = \frac{3|g_2(\tau)|}{4\pi^4 \operatorname{Im}\tau} |\wp(q_\pm;\tau) + \eta_1(\tau)|^2 \operatorname{Im}\phi_\pm(\tau), \tag{4.7}$$

$$\det D^2 G_2(\tfrac{1}{2}\omega_i, \tfrac{1}{2}\omega_j;\tau) = \frac{4|G_k(\tau)|^2}{(2\pi)^4 \operatorname{Im}\tau} \operatorname{Im}\phi_k(\tau), \ \ \{i,j,k\} = \{1,2,3\}, \tag{4.8}$$

where $e_k(\tau) := \wp(\frac{\omega_k}{2};\tau)$ and

$$\phi_\pm(\tau) := \tau - \frac{2\pi i}{\eta_1(\tau) \pm \sqrt{g_2(\tau)/12}}, \tag{4.9}$$

$$G_k(\tau) := \tfrac{1}{2} g_2(\tau) + 3\eta_1(\tau) e_k(\tau) - 3e_k(\tau)^2. \tag{4.10}$$

$$\phi_k(\tau) := \tau - \frac{6\pi i e_k(\tau)}{\frac{g_2(\tau)}{2} + 3\eta_1(\tau) e_k(\tau) - 3e_k(\tau)^2}. \tag{4.11}$$

We conclude this section by determining the sign of the Hessian.

Theorem 4.4. *Let $\tau = ib$ with $b > 0$. Then*

$$\det D^2 G_2(q_+, -q_+;\tau) > 0, \quad \det D^2 G_2(q_-, -q_-;\tau) > 0,$$

$$\det D^2 G_2(\tfrac{1}{2}\omega_i, \tfrac{1}{2}\omega_j;\tau) \begin{cases} > 0 & \text{if } \{i,j\} = \{1,2\}, \\ < 0 & \text{if } \{i,j\} = \{1,3\} \text{ or } \{2,3\}. \end{cases}$$

Proof. Applying Theorem 1.1 to the special case $n = 2$, we have

$$\det D^2 G_2(z_1, z_2; ib) \neq 0, \quad \forall b > 0,$$

for any $(z_1, z_2) \in \{(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j) | i \neq j\} \cup \{(q_\pm, -q_\pm)\}$. Together with (4.7)-(4.8), we obtain

$$|\wp(q_\pm; ib) + \eta_1(ib)| > 0, \quad |G_k(ib)| > 0,$$

and

$$\operatorname{Im}\phi_\pm(ib) \neq 0, \quad \operatorname{Im}\phi_k(ib) \neq 0, \quad \forall b > 0, \ \ k = 1,2,3. \tag{4.12}$$

Therefore, to prove this theorem, it suffices for us to prove that

$$\operatorname{Im}\phi_+(ib) > 0, \quad \operatorname{Im}\phi_-(ib) > 0, \tag{4.13}$$

$$\operatorname{Im}\phi_1(ib) < 0, \quad \operatorname{Im}\phi_2(ib) < 0, \quad \operatorname{Im}\phi_3(ib) > 0, \tag{4.14}$$

hold *for some $b > 0$ and hence for all $b > 0$.*

Since $\eta_1(ib), e_k(ib) \in \mathbb{R}$ and $g_2(ib) > 0$, (4.9) and (4.11) give

$$\operatorname{Im}\phi_\pm(ib) = b - \frac{2\pi}{\eta_1(ib) \pm \sqrt{g_2(ib)/12}},$$

$$\operatorname{Im} \phi_k(ib) = b - \frac{2\pi e_k(ib)}{e_k(ib)\eta_1(ib) + \frac{g_2(ib)}{6} - e_k(ib)^2}.$$

Denote $q = e^{2\pi i\tau} = e^{-2\pi b}$. Recall the well known q-expansions

$$\eta_1(\tau) = \frac{\pi^2}{3} - 8\pi^2 \sum_{k=1}^{\infty} \sigma_1(k) q^k, \quad \text{where } \sigma_1(k) = \sum_{1\le d|k} d, \tag{4.15}$$

$$g_2(\tau) = \frac{4}{3}\pi^4 + 320\pi^4 \sum_{k=1}^{\infty} \sigma_3(k) q^k, \text{ where } \sigma_3(k) = \sum_{1\le d|k} d^3. \tag{4.16}$$

$$e_1(\tau) = \frac{2\pi^2}{3} + 16\pi^2 \sum_{k=1}^{\infty} a_k q^k, \quad a_k = \sum_{d|k, d \text{ is odd}} d, \tag{4.17}$$

$$e_j(\tau) = -\frac{\pi^2}{3} - 8\pi^2 \sum_{k=1}^{\infty} (-1)^{jk} a_k q^{\frac{k}{2}}, \quad j = 2, 3. \tag{4.18}$$

See e.g. [**10**, (6.17)] for (4.15), [**16**, p.44] for (4.16) and [**23**, p.70] for (4.17)-(4.18). Applying these q-expansions, direct computations show that (4.13)-(4.14) hold for b sufficiently large and hence for all b. q.e.d.

5. Applications to the mean field equation

In this section, we give some applications of Theorem 1.1 to the mean field equation

$$\Delta u + e^u = \rho\delta_0 \text{ on } E_\tau \tag{5.1}$$

and prove Corollary 1.2 and Theorem 1.3.

Recalling the Green function G on E_τ, we denote $G(z, w) := G(z-w)$. Define the regular part $\tilde{G}(z, w)$ of $G(z, w)$ by

$$\tilde{G}(z, w) := G(z, w) + \frac{1}{2\pi} \log |z - w|.$$

Given $\boldsymbol{a} := \{a_1, \cdots, a_n\}$ being a trivial critical point of G_n, we set

$$\begin{aligned} f_{a_i}(z) =& 8\pi\Big(\tilde{G}(z, a_i) - \tilde{G}(a_i, a_i) + \sum_{j\neq i} (G(z, a_j) - G(a_i, a_j)) \\ & - n(G(z) - G(a_i))\Big), \end{aligned}$$

$$\mu_i = \exp\left(8\pi(\tilde{G}(a_i, a_i) + \sum_{j\neq i} G(a_i, a_j) - nG(a_i))\right),$$

and define a quantity $D(\boldsymbol{a}) = D(\boldsymbol{a}; \tau)$ by

$$D(\boldsymbol{a}; \tau) := \lim_{r\to 0} \sum_{i=1}^{n} \mu_i \Big(\int_{\Omega_i \setminus B_r(a_i)} \frac{e^{f_{a_i}(z)} - 1}{|z - a_i|^4} - \int_{\mathbb{R}^2 \setminus \Omega_i} \frac{1}{|z - a_i|^4} \Big), \tag{5.2}$$

where Ω_i is any open neighborhood of a_i in E_τ such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^n \bar{\Omega}_i = E_\tau$. The limit exists since $f_{a_i}(z) = O(|z-a_i|^3)$ plus a quadratic harmonic function for all i. See [**21**] for a proof. This $D(\boldsymbol{a})$ arises in the bubbling phenomena of (5.1) as shown in Theroem A.

From (1.5), the quantity $D(\boldsymbol{a})$ controls the sign of $\rho_k - 8\pi n$ and so provides one of the key geometric messages for the bubbling solutions. On the other hand, the Hessian $\det D^2G_n(\boldsymbol{a})$ can be used to determine the local maximum points of u_k near a_i, $1 \leq i \leq n$, and to provide other useful geometric information for the bubbling solutions (cf. [**5**]). There are many potential applications of these two quantities. For example, $\det D^2G_n(\boldsymbol{a})$ and $D(\boldsymbol{a})$ together imply the existence and *local uniqueness* of bubbling solutions, as described in the following striking theorems.

Theorem 5.A. [**6**, **8**] *Suppose $\boldsymbol{a} = \{a_1, \cdots, a_n\}$ is a trivial critical point such that both $D(\boldsymbol{a})$ and $\det D^2G_n(\boldsymbol{a})$ do not vanish. Then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, equation (5.1) with $\rho = 8\pi n + \varepsilon$ (resp. $\rho = 8\pi n - \varepsilon$) possesses a solution $u_\varepsilon(z)$ if $D(\boldsymbol{a}) > 0$ (resp. $D(\boldsymbol{a}) < 0$). Moreover, $u_\varepsilon(z)$ blows up exactly at $\{a_1, \cdots, a_n\}$ as $\varepsilon \to 0$.*

REMARK 5.1. In [**6**] the non-degenerate conditions $D(\boldsymbol{a}) \neq 0$ and $\det D^2G_n(\boldsymbol{a}) \neq 0$ in Theorem 5.A were replaced by some other non-degenerate conditions. Nevertheless the similar process as there still works for $D(\boldsymbol{a}) \neq 0$ and $\det D^2G_n(\boldsymbol{a}) \neq 0$ (see e.g. the remark in [**8**, Section 4] concerning with the degree counting formula). Indeed, for the Chern-Simons-Higgs equation (1.3), the same non-degenerate conditions $D(\boldsymbol{a}) < 0$ and $\det D^2G_n(\boldsymbol{a}) \neq 0$ were used to construct such kind of bubbling solutions in [**21**].

Theorem 5.B. [**1**] *Suppose the hypothesis of Theorem 5.A holds, and $u_k(z)$, $\tilde{u}_k(z)$ are two sequences of solutions of equation (5.1) with the same parameter $\rho_k \to 8\pi n$ and $\rho_k \neq 8\pi n$ for large k, which blow up at the same points $\{a_1, \cdots, a_n\}$. Then $u_k(z) = \tilde{u}_k(z)$ for large k.*

The above results highlight the importance of determining whether $D(\boldsymbol{a})$ and $\det D^2G_n(\boldsymbol{a})$ are nonzero or not. Theorem 1.1 proves

$$\det D^2G_n(\boldsymbol{a}) = (-1)^n c_{\boldsymbol{a}}(\tau) D(\boldsymbol{a}) \neq 0$$

with $c_{\boldsymbol{a}}(\tau) > 0$ for any trivial critical point $\boldsymbol{a}$ if $\tau \in i\mathbb{R}_{>0}$.

Let $\tau \in i\mathbb{R}_{>0}$. Then the hypothesis of Theorem 5.A holds automatically and so the assertions of Theorems 5.A and 5.B follow. Define

$$\begin{aligned} m :=& \#\{\text{trivial critical point } \boldsymbol{a} \,|\, D(\boldsymbol{a}) > 0\} \\ =& \#\{\text{trivial critical point } \boldsymbol{a} \,|\, (-1)^n \det D^2G_n(\boldsymbol{a}) > 0\}. \end{aligned} \tag{5.3}$$

Then Theorem 1.1 implies

$$2n + 1 - m = \#\{\text{trivial critical point } \boldsymbol{a} \,|\, D(\boldsymbol{a}) < 0\}. \tag{5.4}$$

It is known [**20**] that $m = 2$ for $n = 1$, and Theorem 4.4 proves $m = 3$ for $n = 2$. The following result shows $m = n+1$ for all $n \in \mathbb{N}$ and hence proves Corollary 1.2, which confirms [**4**, Conjecture 8.7.2].

Theorem 5.2. *Let $n \in \mathbb{N}$ and $\tau \in i\mathbb{R}_{>0}$, i.e. E_τ is a rectangular torus. Then there exists a small $\varepsilon_0 > 0$ such that*

(1) *for $\rho \in (8\pi n, 8\pi n + \varepsilon_0)$, (5.1) has exactly $n+1$ solutions, which are all even and symmetric;*
(2) *for $\rho = 8\pi n$, (5.1) has no solutions;*
(3) *for $\rho \in (8\pi n - \varepsilon_0, 8\pi n)$, (5.1) has exactly n solutions, which are all even and symmetric.*

Proof of Theorem 5.2 *and Corollary* 1.2. The assertion (2) is just the statement (ii) recalled in the proof of Theorem 1.1. This implies that if u_ρ is a solution of (5.1) for ρ close to $8\pi n$, then u_ρ must blow up as $\rho \to 8\pi n$ and so the assertions of Theorem A follow. From here, (5.3)-(5.4) and Theorems 5.A-5.B, we easily conclude that there exists a small $\varepsilon_0 > 0$ such that

(i) for $\rho \in (8\pi n, 8\pi n + \varepsilon_0)$, (5.1) has exactly m solutions, which are all even and symmetric;
(ii) for $\rho \in (8\pi n - \varepsilon_0, 8\pi n)$, (5.1) has exactly $2n + 1 - m$ solutions, which are all even and symmetric.

Remark that since $\tau \in i\mathbb{R}_{>0}$, the *even and symmetry* of these solutions follow from Theorem 5.B by applying two facts: (1) if $u(z)$ is a solution of (5.1), then so do $u(-z)$ and $u(\bar{z})$; (2) any trivial critical point $\boldsymbol{a}$ of G_n satisfies $\boldsymbol{a} = -\boldsymbol{a} = \bar{\boldsymbol{a}}$.

Therefore, it suffices to prove $m = n + 1$. It was proved in [**7**] that for $\rho \in (8\pi(l-1), 8\pi l)$, $l \in \mathbb{N}$, the topological Leray-Schauder degree d_ρ for (5.1) is well-defined and $d_\rho = l$. Since each solution constructed in Theorem 5.A is non-degenerate and hence contributes degree 1 or -1 (see [**6**]), we obtain

$$m \geq n + 1 \quad \text{and} \quad 2n + 1 - m \geq n,$$

which immediately implies $m = n + 1$. This proves Theorem 5.2 and Corollary 1.2. q.e.d.

6. Reducible monodromy of generalized Lamé equation

In this final section, we would like to mention that Theorem 4.3 can be extended to the following generalized Lamé equation (GLE)

$$y''(z) = \left[\sum_{k=0}^{3} n_k(n_k+1)\wp\left(z + \tfrac{\omega_k}{2}; \tau\right) + B\right] y(z) =: I(z)y(z), \tag{6.1}$$

where $n_k \in \mathbb{Z}_{\geq 0}$ for all k and $\max_k n_k \geq 1$. Clearly GLE (6.1) is just the Lamé equation (1.7) if $n_1 = n_2 = n_3 = 0$. GLE (6.1) is the elliptic form of Heun's equation and the potential

$$-\sum_{k=0}^{3} n_k(n_k+1)\wp\left(z+\tfrac{\omega_k}{2};\tau\right) \tag{6.2}$$

is the so-called *Treibich-Verdier potential* ([**29**]), which is famous as an algebro-geometric finite-gap potential associated with the stationary KdV hierarchy. We refer the readers to [**14**, **24**, **25**, **26**, **27**, **29**] and references therein for historical reviews and subsequent developments. Our motivation of studying GLE (6.1) comes from its deep connection (cf. [**11**]) with the following mean field equation with four singular sources

$$\Delta u + e^u = 8\pi \sum_{k=0}^{3} n_k \delta_{\frac{\omega_k}{2}} \quad \text{on } E_\tau. \tag{6.3}$$

Like the Lamé equation, all solutions of GLE (6.1) are meromorphic in $\mathbb{C}$ and the monodromy representation is a homomorphism from $\pi_1(E_\tau)$ to $SL(2,\mathbb{C})$. Since $\pi_1(E_\tau)$ is abelian, so the monodromy group is always abelian and hence *reducible*, which contains the same two cases (a) and (b) as in Section 3. Again like the Lamé case, a so-called *spectral polynomial* $Q^{(n_0,n_1,n_2,n_3)}(B;\tau)$ is associated for the Treibich-Verdier potential (6.2) such that Case (b) occurs for GLE (6.1) if and only if $Q^{(n_0,n_1,n_2,n_3)}(B;\tau)=0$. See e.g. [**9**, **14**, **24**].

Lemma 6.1. *Let $\tau \in i\mathbb{R}_{>0}$, B be a zero of $Q^{(n_0,n_1,n_2,n_3)}(B;\tau)$ and $\mathcal{C}$ be the monodromy data of the corresponding GLE (*6.1*) given in (*3.2*). Then $\mathcal{C} \in i\mathbb{R}\cup\{\infty\}$ provided $B\in\mathbb{R}$.*

Proof. This lemma is a generalization of Lemma 4.2 to GLE (6.1). Here we give a different proof comparing to that of Lemma 4.2.

Recalling $I(z)$ in (6.1), we let

$$I_0(z) := \sum_{k=0}^{3} n_k(n_k+1)\wp\left(z+\tfrac{\omega_k}{2};\tau\right) + \bar{B}.$$

Since $\tau \in i\mathbb{R}_{>0}$, it follows from (4.3) that $I_0(z) = \overline{I(\bar{z})}$. Thus, if $y(z)$ is a solution of GLE (6.1), then $\tilde{y}(z) := \overline{y(\bar{z})}$ is a solution of GLE $y'' = I_0(z)y(z)$. Recalling our assumption, there exist linearly independent solutions $y_1(z)$ and $y_2(z)$ of GLE (6.1) such that

$$\begin{pmatrix} y_1(z+1) \\ y_2(z+1) \end{pmatrix} = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix},$$

$$\begin{pmatrix} y_1(z+\tau) \\ y_2(z+\tau) \end{pmatrix} = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ \mathcal{C} & 1 \end{pmatrix} \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}$$

with ε_1, $\varepsilon_2 = \pm 1$. Define $\tilde{y}_j(z) := \overline{y_j(\bar{z})}$, $j = 1, 2$. Then

$$\begin{pmatrix} \tilde{y}_1(z+1) \\ \tilde{y}_2(z+1) \end{pmatrix} = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{pmatrix},$$

$$\begin{pmatrix} \tilde{y}_1(z+\tau) \\ \tilde{y}_2(z+\tau) \end{pmatrix} = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ -\overline{\mathcal{C}} & 1 \end{pmatrix} \begin{pmatrix} \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{pmatrix}.$$

Therefore, the monodromy representation of GLE $y'' = I_0(z)y(z)$ is of the type (3.2) with the same $(\varepsilon_1, \varepsilon_2)$ as that of GLE (6.1) and the monodromy data being $-\overline{\mathcal{C}}$.

Now if $B \in \mathbb{R}$, then $I_0(z) = I(z)$, i.e. GLE $y'' = I_0(z)y(z)$ coincides with GLE (6.1). This implies $-\overline{\mathcal{C}} = \mathcal{C}$ and hence $\mathcal{C} \in i\mathbb{R} \cup \{\infty\}$. q.e.d.

The following result generalizes Theorem 4.3 to GLE (6.1), and we believe that it will also have potential applications to the mean field equation (6.3) and the related multiple Green function. One key step is to establish the relation between the monodromy data of GLE (6.1) and the Hessian of the related multiple Green function at trivial critical points. For this purpose, first we need to generalize the results of [**19**] to (6.3), which is not trivial at all. We will turn back to this project in future.

Theorem 6.2. *Let $\tau \in i\mathbb{R}_{>0}$, B be a zero of $Q^{(n_0,n_1,n_2,n_3)}(B;\tau)$ and $\mathcal{C}$ be the monodromy data of the corresponding GLE (6.1) given in (3.2). Then $\mathcal{C} \in i\mathbb{R} \setminus \{0\}$ provided that (n_0, n_1, n_2, n_3) satisfies neither*

$$\frac{n_1 + n_2 - n_0 - n_3}{2} \geq 1, \quad n_1 \geq 1, \quad n_2 \geq 1 \tag{6.4}$$

nor

$$\frac{n_1 + n_2 - n_0 - n_3}{2} \leq -1, \quad n_0 \geq 1, \quad n_3 \geq 1. \tag{6.5}$$

Proof. Let $\tau \in i\mathbb{R}_{>0}$ and (n_0, n_1, n_2, n_3) satisfy neither (6.4) nor (6.5). Then we proved in [**11**, Theorem 1.7] that all the zeros of $Q^{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ are *real* and *distinct*, denoted by $B_{2g} < B_{2g-1} < \cdots < B_1 < B_0$, where $2g + 1 := \deg_B Q^{(n_0,n_1,n_2,n_3)}(B;\tau)$.

Now we consider GLE (6.1) restricted on $z = x + \frac{\tau}{4}$ with $x \in \mathbb{R}$:

$$y''(x) - \left[\sum_{k=0}^{3} n_k(n_k+1)\wp\left(x + \tfrac{\tau}{4} + \tfrac{\omega_k}{2};\tau\right)\right] y(x) = By(x),\ x \in \mathbb{R}. \tag{6.6}$$

Clearly the potential $q_0(x) := -\sum_{k=0}^{3} n_k(n_k+1)\wp\left(x + \frac{\tau}{4} + \frac{\omega_k}{2};\tau\right)$ is continuous on $\mathbb{R}$. As in the beginning of Section 4, we define $c(B, x, x_0)$, $s(B, x, x_0)$, $\Delta(B)$, $d(B)$, $p(B, x_0)$ and $p_i(B)$ for (6.6). Although its potential $q_0(x)$ is not real-valued on $\mathbb{R}$, we proved in [**11**, Lemma 3.6] that the same conclusions as Theorem 4.1 hold for (6.6):

(1) For each $j \in [0, 2g]$, $d(B_j) = 1$ and $p_i(B_j) = 0$.

(2) The potential $q_0(x)$ is a finite-gap potential, i.e. the spectrum of the associated operator $L_0 = \frac{d^2}{dx^2} + q_0(x)$ in $L^2(\mathbb{R})$ has a finite-gap spectrum of the type

$$\sigma(L_0) = (-\infty, B_{2g}] \cup [B_{2g-1}, B_{2g-2}] \cup \cdots \cup [B_1, B_0].$$

Now let B be a zero of $Q^{(n_0,n_1,n_2,n_3)}(B;\tau)$ and $\mathcal{C}$ be the monodromy data of the corresponding GLE (6.1) given in (3.2). Thanks to the above assertion (1), we can apply the same proof as Theorem 4.3 to obtain $\mathcal{C} \notin \{0, \infty\}$. Together with Lemma 6.1, we conclude $\mathcal{C} \in i\mathbb{R} \setminus \{0\}$. q.e.d.

Acknowledgements The authors thank the anonymous referee very much for careful reading and valuable comments. The research of Z. Chen was supported by Independent Research Program at Tsinghua University (Grant No. 2019Z07L02016) and NSFC (Grant No. 11871123).

References

[1] D. Bartolucci, A. Jevnikar, Y. Lee and W. Yang; *Uniqueness of bubbling solutions of mean field equations.* J. Math. Pure Appl. **123** (2019), 78-126. MR3912647, Zbl07026466.

[2] L. A. Caffarelli and Y. Yang; *Vortex condensation in the Chern-Simons-Higgs model: An existence theorem.* Comm. Math. Phys. **168** (1995), 321-336. MR1324400, Zbl0846.58063.

[3] E. Caglioti, P. L. Lions, C. Marchioro and M. Pulvirenti; *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description.* Comm. Math. Phys. **143** (1992), 501-525. MR1145596, Zbl0745.76001.

[4] C. L. Chai, C. S. Lin and C. L. Wang; *Mean field equations, Hyperelliptic curves, and Modular forms: I.* Camb. J. Math. **3** (2015), 127-274, MR3356357, Zbl1327.35116.

[5] C. C. Chen and C. S. Lin; *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces.* Comm. Pure Appl. Math. **55** (2002), 728-771. MR1885666, Zbl1040.53046.

[6] C. C. Chen and C. S. Lin; *Topological degree for a mean field equation on Riemann surfaces.* Comm. Pure Appl. Math. **56** (2003), 1667-1727. MR2001443, Zbl1032.58010.

[7] C. C. Chen and C. S. Lin; *Mean field equation of Liouville type with singular data: Topological degree.* Comm. Pure Appl. Math. **68** (2015), 887-947. MR3340376, Zbl1319.35283.

[8] C. C. Chen, C. S. Lin and G. Wang; *Concentration phenomena of two-vortex solutions in a Chern-Simons model.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **3** (2004), 367-397. MR2075988, Zbl1170.35413.

[9] Z. Chen, T. J. Kuo and C. S. Lin; *The geometry of generalized Lamé equation, I.* J. Math. Pure Appl. **127** (2019), 89-120. MR3960139.

[10] Z. Chen, T. J. Kuo, C. S. Lin and C. L. Wang; *Green function, Painlevé VI equation and Eisenstein series of weight one.* J. Differ. Geom. **108** (2018), 185-241. MR3960139, Zbl1390.34242.

[11] Z. Chen and C. S. Lin; *Sharp nonexistence results for curvature equations with four singular sources on rectangular tori.* Amer. J. Math. to appear.

[12] F. Gesztesy and R. Weikard; *Picard potentials and Hill's equation on a torus.* Acta Math. **176** (1996), 73-107. MR1395670, Zbl0927.37040.

[13] F. Gesztesy and R. Weikard; *Lamé potentials and the stationary (m)KdV hierarchy.* Math. Nachr. **176** (1995), 73-91. MR1361127, Zbl0843.35100.

[14] F. Gesztesy and R. Weikard; *Treibich-Verdier potentials and the stationary (m)KdV hierarchy.* Math. Z. **219** (1995), 451-476. MR1339715, Zbl0830.35119.

[15] E. Ince; *Further investigations into the periodic Lamé functions.* Proc. Roy. Soc. Edinburgh **60** (1940), 83-99. MR0002400, Zbl0027.21201.

[16] S. Lang; *Elliptic Functions.* Graduate Text in Mathematics **112**, Springer–Verlag 1987.

[17] C. S. Lin and C. L. Wang; *Elliptic functions, Green functions and the mean field equations on tori.* Ann. Math. **172** (2010), 911-954. MR2680484, Zbl1207.35011.

[18] C. S. Lin and C. L. Wang; *Mean field equations, Hyperelliptic curves, and Modular forms: II.* J. Éc. polytech. Math. **4** (2017), 557-593. MR3665608, Zbl06754336.

[19] C. S. Lin and C. L. Wang; *Geometric quantities arising from bubbling analysis of mean field equations.* Comm. Anal. Geom. to appear. arXiv:1609.07204v1 [math. AP].

[20] C. S. Lin and C. L. Wang; *On the minimality of extra critical points of Green functions on flat tori*, Int. Math. Res. Not. **2017** (2017), 5591-5608. MR3704740, Zbl1405.35058.

[21] C. S. Lin and S. Yan; *Existence of bubbling solutions for Chern-Simons model on a torus.* Arch. Ration. Mech. Anal. **207** (2013), 353-392. MR3005320, Zbl1260.35155.

[22] M. Nolasco and G. Tarantello; *Double vortex condensates in the Chern-Simons-Higgs theory.* Calc. Var. PDE. **9** (1999), 31-94. MR1710938, Zbl0951.58030.

[23] G. Shimura; *Elementary Dirichlet Series and Modular Forms.* Springer–Verlag 2007.

[24] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system I: the Bethe Ansatz method.* Comm. Math. Phys. **235** (2003), 467-494. MR1974511, Zbl1086.81051.

[25] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system III: the finite gap property and the monodromy.* J. Nonlinear Math. Phys. **11** (2004), 21-46. MR2031210, Zbl1087.81028.

[26] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system IV: the Hermite-Krichever Ansatz.* Comm. Math. Phys. **258** (2005), 367-403. MR2171700, Zbl1088.81056.

[27] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system V: generalized Darboux transformations.* J. Nonlinear Math. Phys. **13** (2006), 584-611. MR2263090, Zbl1122.34069.

[28] E. T. Whittaker and G. N. Watson; *A course of modern analysis*, *4th edition.* Cambridge University Press, 1927.

[29] A. Treibich and J. L. Verdier; *Revetements exceptionnels et sommes de 4 nombres triangulaires.* Duke Math. J. **68** (1992), 217-236. MR1191559, Zbl0806.14013.

Department of Mathematical Sciences, Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, China

E-mail address: zjchen2016@tsinghua.edu.cn

Taida Institute for Mathematical Sciences (TIMS), Center for Advanced Study in Theoretical Sciences (CASTS), National Taiwan University, Taipei 10617, Taiwan

E-mail address: cslin@math.ntu.edu.tw