THE GEOMETRY OF GENERALIZED LAMÉ EQUATION, II: EXISTENCE OF PRE-MODULAR FORMS AND APPLICATION

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ABSTRACT. In this paper, the second in a series, we continue to study the generalized Lamé equation with the Treibich-Verdier potential

$$y''(z) = \left[\sum_{k=0}^{3} n_k(n_k+1)\wp(z+\tfrac{\omega_k}{2}|\tau) + B\right] y(z), \quad n_k \in \mathbb{Z}_{\geq 0}$$

from the monodromy aspect. We prove the existence of a pre-modular form $Z^{\mathbf{n}}_{r,s}(\tau)$ of weight $\frac{1}{2}\sum n_k(n_k+1)$ such that the monodromy data (r,s) is characterized by $Z^{\mathbf{n}}_{r,s}(\tau) = 0$. This generalizes the result in [17], where the Lamé case (i.e. $n_1 = n_2 = n_3 = 0$) was studied by Wang and the third author. As applications, we prove among other things that the following two mean field equations

$$\Delta u + e^u = 16\pi\delta_0 \quad \text{and} \quad \Delta u + e^u = 8\pi \sum_{k=1}^{3} \delta_{\frac{\omega_k}{2}}$$

on a flat torus has the same number of even solutions. This result is quite surprising from the PDE point of view.

Résumé: Dans cet article, le second d'une série, nous continuons à étudier les équation généralisée de Lamé avec le potentiel de Treibich-Verdier

$$y''(z) = \left[\sum_{k=0}^{3} n_k(n_k+1)\wp(z+\tfrac{\omega_k}{2}|\tau) + B\right] y(z), \quad n_k \in \mathbb{Z}_{\geq 0}$$

de l'aspect monodromie. Nous prouvons l'existence d'une forme pré-modulaire $Z^{\mathbf{n}}_{r,s}(\tau)$ de poids $\frac{1}{2}\sum n_k(n_k+1)$ de sorte que les données de monodromie (r,s) sont caractérisées par $Z^{\mathbf{n}}_{r,s}(\tau) = 0$. Cela généralise le résultat dans [17], où le cas Lamé (c.-à-d. $n_1 = n_2 = n_3 = 0$) a été étudié par Wang et le troisième auteur. En tant qu'applications, nous prouvons entre autres que les deux équations de champ moyennes suivantes

$$\Delta u + e^u = 16\pi\delta_0 \quad \text{and} \quad \Delta u + e^u = 8\pi \sum_{k=1}^{3} \delta_{\frac{\omega_k}{2}}$$

sur un tore plat a le même nombre de solutions même. Ce résultat est assez surprenant du point de vue de la PDE.

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1. Introduction

Throughout the paper, as in Part I [4], we use the notations $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = \tau$, $\omega_3 = 1+\tau$ and $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H} = \{\tau | \operatorname{Im}\tau > 0\}$. Define $E_\tau := \mathbb{C}/\Lambda_\tau$ to be a flat torus and $E_\tau[2] := \{\frac{\omega_k}{2} | k = 0,1,2,3\} + \Lambda_\tau$ to be the set consisting of the lattice points and 2-torsion points in E_τ. For $z \in \mathbb{C}$ we denote $[z] := z \pmod{\Lambda_\tau} \in E_\tau$. For a point $[z]$ in E_τ we often write z instead of $[z]$ to simplify notations when no confusion arises.

Let $\wp(z) = \wp(z|\tau)$ be the Weierstrass elliptic function with periods Λ_τ and define $e_k(\tau) := \wp(\frac{\omega_k}{2}|\tau)$, $k = 1,2,3$. Let $\zeta(z) = \zeta(z|\tau) := -\int^z \wp(\xi|\tau)d\xi$ be the Weierstrass zeta function with two quasi-periods $\eta_k(\tau)$, $k = 1,2$:

$$\eta_k(\tau) := 2\zeta(\tfrac{\omega_k}{2}|\tau) = \zeta(z+\omega_k|\tau) - \zeta(z|\tau), \quad k = 1,2, \tag{1.1}$$

and $\sigma(z) = \sigma(z|\tau)$ be the Weierstrass sigma function defined by $\sigma(z) := \exp \int^z \zeta(\xi)d\xi$. Notice that $\zeta(z)$ is an odd meromorphic function with simple poles at Λ_τ and $\sigma(z)$ is an odd entire function with simple zeros at Λ_τ.

This is the second in a series of papers to study the generalized Lamé equation (denoted by $\mathrm{H}(\mathbf{n}, B, \tau)$)

$$y''(z) = I_{\mathbf{n}}(z; B, \tau)y(z), \quad z \in \mathbb{C}, \tag{1.2}$$

with

$$I_{\mathbf{n}}(z; B, \tau) := \sum_{k=0}^{3} n_k(n_k+1)\wp(z + \tfrac{\omega_k}{2}|\tau) + B, \tag{1.3}$$

where $\mathbf{n} = (n_0, n_1, n_2, n_3)$ with $n_k \in \mathbb{Z}_{\geq 0}$ and $\max n_k \geq 1$. By changing variable $z \to z + \frac{\omega_k}{2}$ if necessary, we always assume $n_0 \geq 1$.

$\mathrm{H}(\mathbf{n}, B, \tau)$ is the elliptic form of the well-known Heun's equation and the potential $\sum_{k=0}^{3} n_k(n_k+1)\wp(z + \frac{\omega_k}{2}|\tau)$ is the so-called *Treibich-Verdier potential* ([28]), which is known as an algebro-geometric finite-gap potential associated with the stationary KdV hierarchy [13, 28]. See also a series of papers [23, 24, 25, 26, 27] by Takemura, where $\mathrm{H}(\mathbf{n}, B, \tau)$ was studied as the eigenvalue problem for the Hamiltonian of the BC_1 (one particle) Inozemtsev model. When $\mathbf{n} = (n, 0, 0, 0)$, the potential $n(n+1)\wp(z|\tau)$ is the well-known Lamé potential and (1.2) becomes the Lamé equation

$$y''(z) = [n(n+1)\wp(z|\tau) + B]y(z), \quad z \in \mathbb{C}. \tag{1.4}$$

Ince [15] first discovered that the Lamé potential is a finite-gap potential. See also the classic texts [14, 22, 29] and recent works [3, 9, 17, 20] for more details about (1.4).

In this paper, we continue our study, initiated in Part I [4], on $\mathrm{H}(\mathbf{n}, B, \tau)$ from the monodromy aspect. Since the local exponents of $\mathrm{H}(\mathbf{n}, B, \tau)$ at $\frac{\omega_k}{2}$ are $-n_k$, $n_k + 1$ and $I_{\mathbf{n}}(z; B, \tau)$ is even elliptic, it is easily seen (cf. [13, 23]) that any solution is meromorphic in $\mathbb{C}$, i.e. the local monodromy matrix at $\frac{\omega_k}{2}$ is I_2. Thus the monodromy representation is a group homeomorphism $\rho : \pi_1(E_\tau) \to SL(2, \mathbb{C})$, which is abelian and hence reducible. Let ℓ_j, $j =$

$1, 2$, be two fundamental cycles of E_τ. Then there are two cases (see Part I [4]):

(a) Completely reducible (i.e. all the monodromy matrices have two linearly independent common eigenfunctions). Up to a common conjugation, $\rho(\ell_1)$ and $\rho(\ell_2)$ can be expressed as

$$\rho(\ell_1) = \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}, \quad \rho(\ell_2) = \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{-2\pi i r} \end{pmatrix} \tag{1.5}$$

for some $(r, s) \in \mathbb{C}^2 \backslash \frac{1}{2}\mathbb{Z}^2$. In particular,

$$(\mathrm{tr}\rho(\ell_1), \mathrm{tr}\rho(\ell_2)) = (2\cos 2\pi s, 2\cos 2\pi r) \notin \{\pm(2, 2), \pm(2, -2)\}. \tag{1.6}$$

(b) Not completely reducible (i.e. the space of common eigenfunctions is of dimension 1). Up to a common conjugation, $\rho(\ell_1)$ and $\rho(\ell_2)$ can be expressed as

$$\rho(\ell_1) = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ \mathcal{C} & 1 \end{pmatrix}, \tag{1.7}$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $\mathcal{C} \in \mathbb{C} \cup \{\infty\}$. In particular,

$$(\mathrm{tr}\rho(\ell_1), \mathrm{tr}\rho(\ell_2)) = (2\varepsilon_1, 2\varepsilon_2) \in \{\pm(2, 2), \pm(2, -2)\}. \tag{1.8}$$

Remark that if $\mathcal{C} = \infty$, then (1.7) should be understood as

$$\rho(\ell_1) = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{1.9}$$

In view of Case (a), a natural question that interest us is *how to characterize the monodromy data* (r, s). For the Lamé equation (1.4), Wang and the third author [17] proved the existence of a *pre-modular form* $Z^n_{r,s}(\tau)$ such that the monodromy matrices $\rho(\ell_j)$'s of (1.4) at $\tau = \tau_0$ are given by (1.5) for some B if and only if $Z^n_{r,s}(\tau_0) = 0$. This $Z^n_{r,s}(\tau)$ is holomorphic in τ if $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Moreover, $Z^n_{r,s}(\tau)$ is a modular form of weight $\frac{1}{2}n(n+1)$ with respect to the principal congruence subgroup

$$\Gamma(m) := \{\gamma \in SL(2, \mathbb{Z}) | \gamma \equiv I_2 \mod m\}$$

if (r, s) is a m-torsion point; see [17]. Due to this property, $Z^n_{r,s}(\tau)$ is called a *pre-modular form* in [17].

Definition 1.1. *A function $f_{r,s}(\tau)$ on $\mathbb{H}$, which depends meromorphically on two parameters $(r, s) (\mathrm{mod}\, \mathbb{Z}^2) \in \mathbb{C}^2$, is called a pre-modular form if the followings hold:*

(1) *If $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, then $f_{r,s}(\tau) \not\equiv 0, \infty$ and is meromorphic in τ. Furthermore, it is holomorphic in τ if $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$.*

(2) *There is $k \in \mathbb{N}$ independent of (r, s) such that if (r, s) is any m-torsion point for some $m \geq 3$, then $f_{r,s}(\tau)$ is a modular form of weight k with respect to $\Gamma(m)$.*

One main purpose of this paper is to extend the above result in [17] to include the Trebich-Verdier potential. Here is our first main result.

Theorem 1.2. *There exists a pre-modular form $Z^{\mathbf{n}}_{r,s}(\tau)$ defined in $\tau \in \mathbb{H}$ for any pair $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ such that the followings hold.*

(a) *If $(r,s) = (\frac{k_1}{m}, \frac{k_2}{m})$ with $m \in 2\mathbb{N}_{\geq 2}$, $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ and $\gcd(k_1,k_2,m) = 1$, then $Z^{\mathbf{n}}_{r,s}(\tau)$ is a modular form of weight $\sum_{k=0}^{3} n_k(n_k+1)/2$ with respect to $\Gamma(m)$.*

(b) *For $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $\tau_0 \in \mathbb{H}$ such that $r + s\tau_0 \notin \Lambda_{\tau_0}$, $Z^{\mathbf{n}}_{r,s}(\tau_0) = 0$ if and only if there is $B \in \mathbb{C}$ such that $H(\mathbf{n}, B, \tau_0)$ has its monodromy matrices $\rho(\ell_1)$ and $\rho(\ell_2)$ given by* (1.5).

To explain our construction of the pre-modular form, we have to briefly recall the hyperelliptic curve associated with $\mathrm{H}(\mathbf{n}, B, \tau)$. It is classical that the product of two solutions of $\mathrm{H}(\mathbf{n}, B, \tau)$ solves the second symmetric product equation of $\mathrm{H}(\mathbf{n}, B, \tau)$:

$$\Phi'''(z;B) - 4I_{\mathbf{n}}(z;B,\tau)\Phi'(z;B) - 2I'_{\mathbf{n}}(z;B,\tau)\Phi(z;B) = 0. \tag{1.10}$$

It is known (see e.g. [23]) that (1.10) has a unique even elliptic solution (still denoted by $\Phi(z;B)$). Multiplying Φ and integrating (1.10), we see that

$$Q_{\mathbf{n}}(B;\tau) := \Phi'(z;B)^2 - 2\Phi(z;B)\Phi''(z;B) + 4I_{\mathbf{n}}(z;B,\tau)\Phi(z;B)^2$$

is independent of z. The fact that the Treibich-Verdier potential is an algebro-geometric solution of the KdV hierarchy follows from that $Q_{\mathbf{n}}(B;\tau)$ is a monic polynomial of B up to a multiplicity of $\Phi(z;B)$; see [12]. In this case, $Q_{\mathbf{n}}(B;\tau)$ is known as the *spectral polynomial* and $\Gamma_{\mathbf{n}}(\tau) := \{(B,W) | W^2 = Q_{\mathbf{n}}(B;\tau)\}$ is called the *spectral curve* of the Treibich-Verdier potential.

In Part I [4], we proved that the spectral curve $\Gamma_{\mathbf{n}}(\tau)$ can be embedded into $\mathrm{Sym}^N E_\tau := E_\tau^N / S_N$, where $N := \sum_{k=0}^{3} n_k$. Since $\mathrm{Sym}^N E_\tau$ has a natural addition map to $E_\tau : \{a_1, \cdots, a_N\} \mapsto \sum_{i=1}^{N} a_i - \sum_{k=1}^{3} n_k \frac{\omega_k}{2}$, the composition gives arise to a finite morphism $\sigma_{\mathbf{n}}(\cdot|\tau) : \overline{\Gamma_{\mathbf{n}}(\tau)} \to E_\tau$, still called the *addition map*. See Section 2 for a brief overview. The main result of Part I [4] is to determine the degree of $\sigma_{\mathbf{n}}$.

Theorem 1.3. [4] *The addition map $\sigma_{\mathbf{n}} : \overline{\Gamma_{\mathbf{n}}(\tau)} \to E_\tau$ has degree $\sum_{k=0}^{3} n_k(n_k + 1)/2$.*

As we will see, Theorem 1.3 determines the weight of the pre-modular form $Z^{\mathbf{n}}_{r,s}(\tau)$ in Theorem 1.2. After Theorem 1.3, the field $K(\overline{\Gamma_{\mathbf{n}}(\tau)})$ of rational functions on $\overline{\Gamma_{\mathbf{n}}(\tau)}$ is a finite extension over $K(E_\tau)$ of degree $\sum_{k=0}^{3} n_k(n_k+1)/2$ via this addition map (or covering map) $\sigma_{\mathbf{n}}$. The second step is to find the primitive generator of this extension, for which we need to prove the uniqueness of $\mathrm{H}(\mathbf{n}, B, \tau)$ with respect to the monodromy data in the completely reducible case; see Lemma 2.3.

Our original motivation of investigating the pre-modular forms comes from the following Liouville equation with *four singular sources*:

$$\Delta u + e^u = 8\pi \sum_{k=0}^{3} n_k \delta_{\frac{\omega_k}{2}} \quad \text{on } E_\tau, \tag{1.11}$$

where $\delta_{\omega_k/2}$ is the Dirac measure at $\frac{\omega_k}{2}$. Not surprisingly, (1.11) is related to various research areas. Geometrically, a solution u of (1.11) leads to a metric $g = \frac{1}{2}e^u(dx^2 + dy^2)$ with constant Gaussian curvature $+1$ acquiring *conic singularities at* $\frac{\omega_k}{2}$*'s*. It also appears in statistical physics as the equation for the *mean field limit* of the Euler flow in Onsager's vortex model (cf. [2]), hence also called a *mean field equation*. Recently (1.11) was shown to be related to the self-dual condensates of the Chern-Simons-Higgs model in superconductivity. We refer the readers to [11, 19, 21] and references therein for recent developments of related subjects of (1.11).

The existence of solutions of (1.11) is very challenging from the PDE point of view. In fact, *the solvability of (1.11) essentially depends on the moduli* τ *in a sophisticated manner*. This phenomenon was first discovered by Wang and the third author [16] when they studied the case $n_0 = 1$ and $n_1 = n_2 = n_3 = 0$, i.e.

$$\Delta u + e^u = 8\pi\delta_0 \text{ on } E_\tau. \tag{1.12}$$

Among other things, they proved that

- if $\tau \in i\mathbb{R}_{>0}$, i.e. E_τ is a rectangular torus, then (1.12) has *no* solution;
- if $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, i.e. E_τ is a rhombus torus, then (1.12) has solutions.

Furthermore, by studying the Green function $G(z;\tau)$ on E_τ, they proved that (1.12) has solutions if and only if $Z_{r,s}(\tau) := \zeta(r + s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau) = 0$ for some $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Recently, (1.12) has been thoroughly investigated in [6, 18]. In particular, we found some interesting applications of the zero structure of $Z_{r,s}(\tau)$ to Painlevé VI equation in [6].

Therefore, a natural question is *how to give a precise characterization of those* τ*'s such that (1.11) has solutions on such* E_τ. Here we give an answer to this question in the even solution case.

Theorem 1.4. *The mean field equation (1.11) has even solutions on* E_τ *if and only if there is* $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ *such that* τ *is a zero of this pre-modular form* $Z^{\mathbf{n}}_{r,s}(\cdot)$, *i.e.* $Z^{\mathbf{n}}_{r,s}(\tau) = 0$.

Theorem 1.4 generalizes the result in [17] where the Lamé case $n_1 = n_2 = n_3 = 0$ was studied. In this case, Wang and the third author [17] also proved that *once* $\Delta u + e^u = 8n_0\pi\delta_0$ *on* E_τ *has solutions, then it also has an even solution*. We believe that this statement should also hold for (1.11) with general $n_k \in \mathbb{Z}_{\geq 0}$, which seems challenging and remains open.

Now let us consider two special cases of (1.11):

$$\Delta u + e^u = 16\pi\delta_0 \text{ on } E_\tau \tag{1.13}$$

and

$$\Delta u + e^u = 8\pi \sum_{k=1}^{3} \delta_{\frac{\omega_k}{2}} \text{ on } E_\tau. \tag{1.14}$$

There seems no obvious relations between these two equations. Therefore, the following result is quite surprising from the PDE aspect.

Theorem 1.5. *The mean field equations (1.13) and (1.14) has the same number of even solutions.*

The rest of the paper is organized as follows. In Section 2, we recall the theory concerning $\mathrm{H}(\mathbf{n}, B, \tau)$ from Part I [4] and then give the proof of Theorem 1.2. In Section 3, we apply Theorem 1.2 to prove Theorem 1.4. We will also prove a general result which contains Theorem 1.5 as a special case. Appendix A is devoted to the proof of Theorem 2.1 which is needed in the construction of the pre-modular form.

2. Existence of pre-modular forms

The purpose of this section is to construct the pre-modular form and prove Theorem 1.2. First we recall some basic theory concerning $\mathrm{H}(\mathbf{n}, B, \tau)$ from Part I [4]. As mentioned before, by changing variable $z \to z + \frac{\omega_k}{2}$ if necessary, we always assume $n_0 \geq 1$.

(i) Any solution of $H(\mathbf{n}, B, \tau)$ is meromorphic in $\mathbb{C}$. The corresponding second symmetric product equation

$$\Phi'''(z; B) - 4I_{\mathbf{n}}(z; B, \tau)\Phi'(z; B) - 2I'_{\mathbf{n}}(z; B, \tau)\Phi(z; B) = 0$$

has a *unique* even elliptic solution $\Phi_e(z; B)$ expressed by

$$\Phi_e(z; B) = C_0(B) + \sum_{k=0}^{3} \sum_{j=0}^{n_k - 1} b_j^{(k)}(B) \wp(z + \tfrac{\omega_k}{2})^{n_k - j} \tag{2.1}$$

where $C_0(B), b_j^{(k)}(B)$ are all polynomials in B with $\deg C_0 > \max_{j,k} \deg b_j^{(k)}$ and the leading coefficient of $C_0(B)$ being $\frac{1}{2}$. Moreover, $\Phi_e(z; B) = y_1(z; B)$ $y_1(-z; B)$, where $y_1(z; B)$ is a common eigenfunction of the monodromy matrices of $H(\mathbf{n}, B, \tau)$ and, up to a constant, can be written as

$$y_1(z; B) = y_{\boldsymbol{a}}(z) := e^{c(\boldsymbol{a})z} \frac{\prod_{i=1}^{N} \sigma(z - a_i)}{\prod_{k=0}^{3} \sigma(z - \frac{\omega_k}{2})^{n_k}} \tag{2.2}$$

with some $\boldsymbol{a} = (a_1, \cdots, a_N)$ and $c(\boldsymbol{a}) \in \mathbb{C}$. See (2.4) for the expression of $c(\boldsymbol{a})$ in the completely reducible case. From (2.2) and the transformation law (let $\eta_3 = 2\zeta(\frac{\omega_3}{2}) = \eta_1 + \eta_2$)

$$\sigma(z + \omega_k) = -e^{\eta_k(z + \frac{\omega_k}{2})}\sigma(z), \quad k = 1, 2, 3,$$

it is easy to see that $y_1(-z; B) = y_{-\boldsymbol{a}}(z)$ up to a sign $(-1)^{n_1+n_2+n_3}$.

(ii) Let W be the Wroskian of $y_1(z; B)$ and $y_1(-z; B)$, then $W^2 = Q_{\mathbf{n}}(B; \tau)$, where

$$Q_{\mathbf{n}}(B; \tau) := \Phi'_e(z; B)^2 - 2\Phi_e(z; B)\Phi''_e(z; B) + 4I_{\mathbf{n}}(z; B, \tau)\Phi_e(z; B)^2$$

is a monic polynomial in B with *odd degree* and independent of z. Define the hyperelliptic curve $\Gamma_{\mathbf{n}}(\tau)$ by

$$\Gamma_{\mathbf{n}}(\tau) := \{(B, W) \mid W^2 = Q_{\mathbf{n}}(B; \tau)\}.$$

Then the map $i_{\mathbf{n}} : \Gamma_{\mathbf{n}}(\tau) \to \mathrm{Sym}^N E_\tau := E_\tau^N / S_N$ defined by

$$i_{\mathbf{n}}(B, W) := \{[a_1], \cdots, [a_N]\}$$

is an embedding, where $\{[a_1], \cdots, [a_N]\}$ is uniquely determined by $y_1(z; B)$ via (2.2). Since $-W$ be the Wroskian of $y_1(-z; B) = y_{-\boldsymbol{a}}(z)$ and $y_1(z; B)$, we have

$$i_{\mathbf{n}}(B, -W) = \{-[a_1], \cdots, -[a_N]\}.$$

(iii) The monodromy of $\mathrm{H}(\mathbf{n}, B, \tau)$ is completely reducible if and only if $y_1(z; B) = y_{\boldsymbol{a}}(z)$ and $y_1(-z; B) = y_{-\boldsymbol{a}}(z)$ are linearly independent, which is also equivalent to

$$\{[a_1], \cdots, [a_N]\} \cap \{-[a_1], \cdots, -[a_N]\} = \emptyset. \tag{2.3}$$

In this case, since $a_j \neq 0$ in E_τ for all j and $n_0 \neq 0$, we have

$$c(\pm \boldsymbol{a}) = \sum_{i=1}^{N} \zeta(\pm a_i) - \sum_{k=1}^{3} \frac{n_k \eta_k}{2}, \tag{2.4}$$

which follows by inserting (2.2) into $\mathrm{H}(\mathbf{n}, B, \tau)$ and computing the leading terms at the singularity 0; see Theorem A.1. Besides, the (r, s) defined by

$$\begin{cases} \sum_{i=1}^{N} a_i - \sum_{k=1}^{3} \frac{n_k \omega_k}{2} = r + s\tau \\ \sum_{i=1}^{N} \zeta(a_i) - \sum_{k=1}^{3} \frac{n_k \eta_k}{2} = r\eta_1 + s\eta_2 \end{cases} \tag{2.5}$$

satisfies $(r, s) \notin \frac{1}{2}\mathbb{Z}^2$. Furthermore, with respect to $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$, the monodromy matrices are given by

$$\rho(\ell_1) = \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}, \ \rho(\ell_2) = \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{-2\pi i r} \end{pmatrix}. \tag{2.6}$$

(iv) Let $Y_{\mathbf{n}}(\tau)$ be the image of $\Gamma_{\mathbf{n}}(\tau)$ in $\mathrm{Sym}^N E_\tau$ under $i_{\mathbf{n}}$, i.e.

$$Y_{\mathbf{n}}(\tau) = \left\{ \begin{array}{r} [\boldsymbol{a}] = \{[a_1], \cdots, [a_N]\} \ \in \mathrm{Sym}^N E_\tau |\ y_{\boldsymbol{a}}(z) \text{ defined in} \\ (2.2) \text{ is a solution of } \mathrm{H}(\mathbf{n}, B, \tau) \text{ for some } B \end{array} \right\}, \tag{2.7}$$

and $X_{\mathbf{n}}(\tau)$ be the image of $\{(B, W) \in \Gamma_{\mathbf{n}} | W \neq 0\}$ under $i_{\mathbf{n}}$, i.e.

$$X_{\mathbf{n}}(\tau) = \{[\boldsymbol{a}] \in Y_{\mathbf{n}}(\tau) |\ (2.3) \text{ holds}\}. \tag{2.8}$$

Clearly $Y_{\mathbf{n}}(\tau) \setminus X_{\mathbf{n}}(\tau)$ consists of those finite branch points, i.e. those $\boldsymbol{a}$'s such that $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$ are linearly dependent, which is equivalent to

$$\{[a_1], \cdots, [a_N]\} = \{-[a_1], \cdots, -[a_N]\}. \tag{2.9}$$

The number of finite branch points is at most $\deg Q_{\mathbf{n}}(B)$. Besides, it was proved in Part I [4] that $\overline{X_{\mathbf{n}}(\tau)} = \overline{Y_{\mathbf{n}}(\tau)} = Y_{\mathbf{n}}(\tau) \cup \{\infty_0\}$, where

$$\infty_0 := \Big(\overbrace{0, \cdots, 0}^{n_0}, \overbrace{\tfrac{\omega_1}{2}, \cdots, \tfrac{\omega_1}{2}}^{n_1}, \overbrace{\tfrac{\omega_2}{2}, \cdots, \tfrac{\omega_2}{2}}^{n_2}, \overbrace{\tfrac{\omega_3}{2}, \cdots, \tfrac{\omega_3}{2}}^{n_3}\Big). \tag{2.10}$$

(v) The first formula of (2.5) motivates us to study the addition map $\sigma_{\mathbf{n}}$: $\overline{X_{\mathbf{n}}(\tau)} \to E_\tau$ (also called a covering map in [26, Section 4]):

$$\sigma_{\mathbf{n}}([\boldsymbol{a}]) := \sum_{i=1}^{N}[a_i] - \sum_{k=1}^{3}[\tfrac{n_k\omega_k}{2}].$$

Since $2\sum_{k=1}^{3}[\frac{n_k\omega_k}{2}] = [0]$, we have

$$\sigma_{\mathbf{n}}([-\boldsymbol{a}]) = -\sum_{i=1}^{N}[a_i] - \sum_{k=1}^{3}[\tfrac{n_k\omega_k}{2}] = -\sigma_{\mathbf{n}}([\boldsymbol{a}]).$$

Since the algebraic curve $\overline{X_{\mathbf{n}}(\tau)}$ is irreducible, $\sigma_{\mathbf{n}}$ is a finite morphism and $\deg\sigma_{\mathbf{n}}$ is well-defined. Theorem 1.3 says that

$$\deg\sigma_{\mathbf{n}} = \frac{1}{2}\sum_{k=0}^{3} n_k(n_k+1). \tag{2.11}$$

The above theories can be found in Part I [4]. Here we also need the following result, which will give a precise characterization of $X_{\mathbf{n}}(\tau)$.

Theorem 2.1. *Suppose $\boldsymbol{a} = \{a_1, \cdots, a_N\}$ satisfies*

$$[a_i] \notin E_\tau[2],\ [a_i] \neq \pm[a_j],\ \forall i \neq j. \tag{2.12}$$

Then $y_{\boldsymbol{a}}(z)$ is a solution of $H(\mathbf{n}, B, \tau)$ for some B if and only if $\boldsymbol{a}$ satisfies

$$\sum_{i=1}^{N}\wp'(a_i)\wp(a_i)^l = 0 \text{ for } 0 \leq l \leq n_0 - 2, \tag{2.13}$$

$$\sum_{i=1}^{N}\wp'(a_i)\prod_{j=1,\neq i}^{N}(\wp(a_j)-e_k)^l = 0 \text{ for } 1 \leq l \leq n_k,\ k \in \{1,2,3\}. \tag{2.14}$$

Theorem 2.1 in the Lamé case $n_1 = n_2 = n_3 = 0$ was proved in [3]. The proof of Theorem 2.1 for general n_k is technical and long, and will be given in Appendix A. Note that if $[\boldsymbol{a}] \in X_{\mathbf{n}}(\tau)$, then (2.3) implies $[a_i] \notin E_\tau[2]$ and $[a_i] \neq -[a_j]$ for all i, j, so a_i is a zero of $y_{\boldsymbol{a}}(z)$ which must be simple, i.e. $[a_i] \neq [a_j]$ for any $i \neq j$ and so (2.12) holds. In conclusion,

$$X_{\mathbf{n}}(\tau) = \{[\boldsymbol{a}] \in \mathrm{Sym}^N E_\tau | \boldsymbol{a} \text{ satisfies (2.12)-(2.14)}\}. \tag{2.15}$$

Now we proceed to construct a pre-modular form $Z^{\mathbf{n}}_{r,s}(\tau)$. Let $K(E_\tau)$ and $K(\overline{X_{\mathbf{n}}(\tau)})$ be the field of rational functions on E_τ and $\overline{X_{\mathbf{n}}(\tau)}$, respectively. Then (2.11) indicates that $K(\overline{X_{\mathbf{n}}(\tau)})$ is a finite extension over $K(E_\tau)$ and

$$\left[K(\overline{X_{\mathbf{n}}(\tau)}) : K(E_\tau)\right] = \deg\sigma_{\mathbf{n}} = \frac{1}{2}\sum_{k=0}^{3} n_k(n_k+1). \tag{2.16}$$

A basic question is *how to find a primitive generator*?

Motivated by [17] and (2.5), we consider the function

$$\mathbf{z_n}(a_1,\cdots,a_N) := \zeta\left(\sum_{i=1}^N a_i - \sum_{k=1}^3 \frac{n_k\omega_k}{2}\right) - \sum_{i=1}^N \zeta(a_i) + \sum_{k=1}^3 \frac{n_k\eta_k}{2},$$

which is meromorphic and periodic in each a_i and hence defines a rational function on E_τ^N. By symmetry, it descends to a rational function on $\mathrm{Sym}^N E_\tau$. We denote the restriction $\mathbf{z_n}|_{\overline{X_\mathbf{n}(\tau)}}$ also by $\mathbf{z_n}$, which is a rational function on $\overline{X_\mathbf{n}(\tau)}$.

Lemma 2.2. *The poles of* $\mathbf{z_n}$ *on* $\overline{X_\mathbf{n}(\tau)}$ *are precisely the fiber* $\sigma_\mathbf{n}^{-1}([0])$.

Proof. Fix any $\boldsymbol{a} = \{a_1,\cdots,a_N\} \in \overline{X_\mathbf{n}(\tau)} \setminus \{\infty_0\} = Y_\mathbf{n}(\tau)$. It suffices to prove that

$$\sum_{i=1}^N \zeta(a_i) \neq \infty. \tag{2.17}$$

If $\boldsymbol{a} \in X_\mathbf{n}(\tau)$, i.e. not a branch point, then (2.17) follows from (2.4). So it suffices to consider that $\boldsymbol{a} \in Y_\mathbf{n}(\tau) \setminus X_\mathbf{n}(\tau)$ is a finite branch point. Then it might happen that $a_i = 0$ for some i's. Since the number of branch points is finite, we can take a sequence $X_\mathbf{n}(\tau) \ni \boldsymbol{a}^m \to \boldsymbol{a}$. Denote $\boldsymbol{a}^m = \{a_1^m,\cdots,a_N^m\}$. Note from (2.7) that $y_{\boldsymbol{a}^m}(z)$ given by (2.2) is a solution of $\mathrm{H}(\mathbf{n},B_m,\tau)$ for some $B_m \in \mathbb{C}$. Since $\boldsymbol{a}^m \to \boldsymbol{a} \neq \infty_0$, we proved in Part I [4] that B_m are uniformly bounded and so do $c(\boldsymbol{a}^m)$. Consequently, we see from (2.4) that

$$\sum_{i=1}^N \zeta(a_i) = \lim_{m\to\infty} \sum_{i=1}^N \zeta(a_i^m) \neq \infty.$$

The proof is complete. □

Lemma 2.3. *Let* $\boldsymbol{a},\boldsymbol{b} \in X_\mathbf{n}(\tau)$ *be not branch points. Suppose*

$$\sigma_\mathbf{n}(\boldsymbol{a}) = \sigma_\mathbf{n}(\boldsymbol{b}) \ \text{ and } \ \mathbf{z_n}(\boldsymbol{a}) = \mathbf{z_n}(\boldsymbol{b}). \tag{2.18}$$

Then $\boldsymbol{a} = \boldsymbol{b}$.

Proof. Under our assumption (2.18), we can take $(a_1,\cdots,a_N),(b_1,\cdots,b_N) \in \mathbb{C}^N$ to be representatives of $\boldsymbol{a},\boldsymbol{b}$ such that

$$\sum_{i=1}^N a_i = \sum_{i=1}^N b_i, \quad \sum_{i=1}^N \zeta(a_i) = \sum_{i=1}^N \zeta(b_i). \tag{2.19}$$

By (2.7), there exist B_1, B_2 such that $y_{\boldsymbol{a}}(z)$ (resp. $y_{\boldsymbol{b}}(z)$) given by (2.2) is a solution of $\mathrm{H}(\mathbf{n},B_1,\tau)$ (resp. $\mathrm{H}(\mathbf{n},B_2,\tau)$). Then (2.5), (2.6) and (2.19) imply that $\mathrm{H}(\mathbf{n},B_1,\tau)$ and $\mathrm{H}(\mathbf{n},B_2,\tau))$ have the same global monodromy data $(r,s) \notin \frac{1}{2}\mathbb{Z}^2$, namely $y_{\boldsymbol{a}}(z)$ and $y_{\boldsymbol{b}}(z)$, which are solutions of $\mathrm{H}(\mathbf{n},B_1,\tau)$ and $\mathrm{H}(\mathbf{n},B_2,\tau)$ respectively, satisfy the same transformation law:

$$y(z+\omega_1) = e^{-2\pi i s} y(z), \quad y(z+\omega_2) = e^{2\pi i r} y(z). \tag{2.20}$$

Now we use the following interesting observation from [17, Lemma 3.5]: Denote $I=\sum_{k=0}^{3}n_k(n_k+1)\wp(z+\frac{\omega_k}{2})$ and $I_j=I+B_j$ for $j=1,2$. Suppose $w_j''=I_jw_j$ for $j=1,2$. Then w_1w_2 satisfies the following forth order ODE:

$$q''''-2(I_1+I_2)q''-6I'q'+((B_1-B_2)^2-2I'')q=0. \tag{2.21}$$

This statement can be proved by direct computations. Furthermore, it is easy to see that the local exponents of (2.21) at $\frac{\omega_k}{2}$ are

$$-2n_k,\quad 1,\quad 3,\quad 2n_k+2. \tag{2.22}$$

Recalling $y_{-\boldsymbol{a}}(z)=(-1)^{n_1+n_2+n_3}y_{\boldsymbol{a}}(-z)$ and $y_{-\boldsymbol{b}}(z)=(-1)^{n_1+n_2+n_3}y_{\boldsymbol{b}}(-z)$, it follows from (2.20) that

$$q(z):=y_{\boldsymbol{a}}(z)y_{-\boldsymbol{b}}(z)-y_{-\boldsymbol{a}}(z)y_{\boldsymbol{b}}(z)$$

is *an odd elliptic solution* of (2.21). Consequently, (2.22) infers that $\frac{\omega_k}{2}$ must be a zero of $q(z)$ (with order 1 or 3) for any k. This implies that $q(z)$ has no poles and so $q(z)\equiv 0$, i.e. $y_{\boldsymbol{a}}(z)y_{-\boldsymbol{b}}(z)$ is even. This implies the zero set $\boldsymbol{a}\cup(-\boldsymbol{b})=(-\boldsymbol{a})\cup\boldsymbol{b}$, and it follows from (2.3) (i.e. $[a_i]\neq-[a_j]$, $[b_i]\neq-[b_j]$ for any i,j) that $\boldsymbol{a}=\boldsymbol{b}$. Clearly this also infers $B_1=B_2$. □

Theorem 2.4. *There is a weighted homogeneous polynomial*

$$W_{\mathbf{n}}(\mathbf{z})\in\mathbb{Q}[e_1(\tau),e_2(\tau),e_3(\tau),\wp(\sigma|\tau),\wp'(\sigma|\tau)][\mathbf{z}] \tag{2.23}$$

of $\mathbf{z}$*-degree* $d_{\mathbf{n}}=\deg\sigma_{\mathbf{n}}$ *such that for* $\sigma=\sigma_{\mathbf{n}}(\boldsymbol{a})$*, we have*

$$W_{\mathbf{n}}(\mathbf{z}_{\mathbf{n}})(\boldsymbol{a})=0. \tag{2.24}$$

Here, the weights of $\mathbf{z}$, $\wp(\sigma)$, e_k*'s,* $\wp'(\sigma)$ *are* 1, 2, 2, 3 *respectively.*

Indeed, $\mathbf{z}_{\mathbf{n}}(\boldsymbol{a})$ *is a primitive generator of the finite extension of rational function field* $K(\overline{X_{\mathbf{n}}(\tau)})$ *over* $K(E_\tau)$ *with* $W_{\mathbf{n}}(\mathbf{z})$ *being its minimal polynomial.*

Proof. Thanks to Lemmas 2.2-2.3, the proof is similar to [17, Theorem 3.2]. Here we give the proof for completeness.

Recall (2.16). Since $\mathbf{z}_{\mathbf{n}}\in K(\overline{X_{\mathbf{n}}(\tau)})$, its minimal polynomial $W_{\mathbf{n}}(\mathbf{z})\in K(E_\tau)[\mathbf{z}]$ exists with degree $d_{\mathbf{n}}:=\deg W_{\mathbf{n}}|\deg\sigma_{\mathbf{n}}$.

Note that if $\boldsymbol{a}$ is a branch point, then it follows from (2.9) that $\sigma_{\mathbf{n}}(\boldsymbol{a})\in E_\tau[2]$. To prove $d_{\mathbf{n}}=\deg\sigma_{\mathbf{n}}$, i.e. $\mathbf{z}_{\mathbf{n}}(\boldsymbol{a})$ is a primitive generator, we take $\sigma_0\in E_\tau\setminus E_\tau[2]$ outside the branch loci of $\sigma_{\mathbf{n}}:\overline{X_{\mathbf{n}}(\tau)}\to E_\tau$. Then there are precisely $\deg\sigma_{\mathbf{n}}$ different points $\boldsymbol{a}\in X_{\mathbf{n}}(\tau)$ with $\sigma_{\mathbf{n}}(\boldsymbol{a})=\sigma_0$, and Lemma 2.3 shows that these $\deg\sigma_{\mathbf{n}}$ different points $\boldsymbol{a}$ give $\deg\sigma_{\mathbf{n}}$ different values $\mathbf{z}_{\mathbf{n}}(\boldsymbol{a})$. Therefore, for $\sigma=\sigma_0$, the polynomial $W_{\mathbf{n}}(\mathbf{z})$ of degree $d_{\mathbf{n}}|\deg\sigma_{\mathbf{n}}$ has $\deg\sigma_{\mathbf{n}}$ distinct zeros, which implies $d_{\mathbf{n}}=\deg\sigma_{\mathbf{n}}$.

Since Lemma 2.2 shows that $\mathbf{z}_{\mathbf{n}}$ has no poles over $E_\tau^\times:=E_\tau\setminus\{[0]\}$, it is indeed *integral* over the affine Weierstrass model of $E_\tau^\times$ with the coordinate ring (let $x=\wp(\sigma)$, $y=\wp'(\sigma)$)

$$R(E_\tau^\times)=\mathbb{C}[x,y]/(y^2-4(x-e_1)(x-e_2)(x-e_3)),$$

i.e. the minimal polynomial $W_{\mathbf{n}}$ is monic in $R(E_\tau^\times)[\mathbf{z}]$. This implies (2.23) and the homogeneity of $W_{\mathbf{n}}$, where as in [17], the coefficients lie in $\mathbb{Q}$, instead of just in $\mathbb{C}$, follows from standard elimination theory and two facts: (i) The equations (2.13)-(2.14) of $X_{\mathbf{n}}(\tau)$ (see (2.15)) are defined over $\mathbb{Q}[e_1, e_2, e_3]$, and (ii) the addition map $E_\tau^N \to E_\tau$ is defined over $\mathbb{Q}$ which, together with the addition formulas of elliptic functions, infers that

$$\begin{aligned}\mathbf{z}_{\mathbf{n}}(\boldsymbol{a}) &= \zeta\left(\sum_{i=1}^{N} a_i - \sum_{k=1}^{3} \frac{n_k \omega_k}{2}\right) - \sum_{i=1}^{N} \zeta(a_i) + \sum_{k=1}^{3} \frac{n_k \eta_k}{2} \\ &= \sum_{j_1 < \cdots < j_m;\, m \text{ odd}} f^{(1)}_{j_1, \cdots, j_m}(\wp(a_{j_1}), \cdots, \wp(a_{j_m})) \prod_{i=1}^{m} \wp'(a_{j_i}),\end{aligned}$$

$$\begin{aligned}\wp(\sigma_{\mathbf{n}}(\boldsymbol{a})) &= \wp\left(\sum_{i=1}^{N} a_i - \sum_{k=1}^{3} \frac{n_k \omega_k}{2}\right) \\ &= \sum_{j_1 < \cdots < j_m;\, m \text{ even}} f^{(2)}_{j_1, \cdots, j_m}(\wp(a_{j_1}), \cdots, \wp(a_{j_m})) \prod_{i=1}^{m} \wp'(a_{j_i}),\end{aligned}$$

$$\wp'(\sigma_{\mathbf{n}}(\boldsymbol{a})) = \sum_{j_1 < \cdots < j_m;\, m \text{ odd}} f^{(3)}_{j_1, \cdots, j_m}(\wp(a_{j_1}), \cdots, \wp(a_{j_m})) \prod_{i=1}^{m} \wp'(a_{j_i}),$$

where $f^{(k)}_{j_1, \cdots, j_m}(x_1, \cdots, x_m) \in \mathbb{Q}(e_1, e_2, e_3)(x_1, \cdots, x_m)$. The minimal polynomial $W_{\mathbf{n}}$ is obtained by eliminating the terms $\wp(a_j), \wp'(a_j)$'s via these formulas and the equations (2.13)-(2.14) of $X_{\mathbf{n}}(\tau)$. The proof is complete. □

As in [17], for any $(r, s) \in \mathbb{C}^2 \backslash \frac{1}{2}\mathbb{Z}^2$, we define

$$Z_{r,s}(\tau) := \zeta(r + s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau), \tag{2.25}$$

(note $Z_{r,s}(\tau) \equiv \infty$ if $(r, s) \in \mathbb{Z}^2$ and $Z_{r,s}(\tau) \equiv 0$ if $(r, s) \in \frac{1}{2}\mathbb{Z}^2 \setminus \mathbb{Z}^2$) and

$$Z^{\mathbf{n}}_{r,s}(\tau) := W_{\mathbf{n}}(Z_{r,s})(r + s\tau; \tau), \tag{2.26}$$

i.e. by letting $\sigma = r + s\tau$ and $\mathbf{z} = Z_{r,s}(\tau)$ in (2.23). Clearly this $Z^{\mathbf{n}}_{r,s}(\tau)$ is holomorphic in τ for given $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. We show that this $Z^{\mathbf{n}}_{r,s}(\tau)$ is precisely the pre-modular form in Theorem 1.2.

Theorem 2.5 (=Theorem 1.2)**.** *Let $Z^{\mathbf{n}}_{r,s}(\tau)$ be defined in (2.26). Then the followings hold.*

(a) *$Z^{\mathbf{n}}_{r,s}(\tau)$ is a pre-modular form in the sense that, if $(r, s) = (\frac{k_1}{m}, \frac{k_2}{m})$ with $m \in 2\mathbb{N}_{\geq 2}$, $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ and $\gcd(k_1, k_2, m) = 1$, then $Z^{\mathbf{n}}_{r,s}(\tau)$ is a modular form of weight $\sum_{k=0}^{3} n_k(n_k + 1)/2$ with respect to the principal congruence subgroup $\Gamma(m)$.*

(b) *For $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $\tau_0 \in \mathbb{H}$ such that $r + s\tau_0 \notin \Lambda_{\tau_0}$, $Z^{\mathbf{n}}_{r,s}(\tau_0) = 0$ if and only if there is $B \in \mathbb{C}$ such that $H(\mathbf{n}, B, \tau_0)$ has its monodromy matrices $\rho(\ell_1)$ and $\rho(\ell_2)$ given by* (2.6).

Proof. (a) It is well-known that $e_k(\tau)$, $k=1,2,3$, are all modular forms of weight 2 with respect to $\Gamma(2)$. Since $Z_{r,s}(\tau)$, $\wp(r+s\tau|\tau)$ and $\wp'(r+s\tau|\tau)$ are modular forms of weight 1, 2, 3 respectively, with respect to $\Gamma(m)$ for any m-torsion point $(r,s)=(\frac{k_1}{m},\frac{k_2}{m})$, $m\geq 3$, the assertion (a) follows directly from the homogeneity of $W_{\mathbf{n}}$ in Theorem 2.4.

(b) First we prove the sufficient part. Suppose for some $\mathrm{H}(\mathbf{n},B,\tau_0)$, its monodromy matrices $\rho(\ell_1)$ and $\rho(\ell_2)$ are given by (2.6), i.e. the monodromy is completely reducible. Then there exists $\boldsymbol{a}\in X_{\mathbf{n}}(\tau_0)$ such that $y_{\boldsymbol{a}}(z)$ is a solution of $\mathrm{H}(\mathbf{n},B,\tau_0)$ and (2.5) holds, i.e.

$$\begin{cases}\sum_{i=1}^N a_i-\sum_{k=1}^3\frac{n_k\omega_k}{2}=r+s\tau_0\\ \sum_{i=1}^N\zeta(a_i)-\sum_{k=1}^3\frac{n_k\eta_k}{2}=r\eta_1+s\eta_2.\end{cases}\tag{2.27}$$

From here we have $\sigma_{\mathbf{n}}(\boldsymbol{a})=[r+s\tau_0]\neq[0]$ and

$$\mathbf{z}_{\mathbf{n}}(\boldsymbol{a})=\zeta(r+s\tau_0|\tau_0)-r\eta_1(\tau_0)-s\eta_2(\tau_0)=Z_{r,s}(\tau_0).$$

Then it follows from (2.24) and (2.26) that $Z_{r,s}^{\mathbf{n}}(\tau_0)=0$.

Conversely, suppose $Z_{r,s}^{\mathbf{n}}(\tau_0)=0$. Then it follows from (2.26) that for $\sigma=r+s\tau_0\notin\Lambda_{\tau_0}$, $Z_{r,s}(\tau_0)$ is a zero of $W_{\mathbf{n}}(\mathbf{z})$. This, together with Theorem 2.4, implies the existence of $\boldsymbol{a}\in\overline{X_{\mathbf{n}}(\tau_0)}\setminus\{\infty_0\}=Y_{\mathbf{n}}(\tau_0)$ such that

$$\sum_{i=1}^N a_i-\sum_{k=1}^3\frac{n_k\omega_k}{2}=\sigma=r+s\tau_0,\quad \mathbf{z}_{\mathbf{n}}(\boldsymbol{a})=Z_{r,s}(\tau_0),$$

which is equivalent to (2.27). The definition (2.7) of $Y_{\mathbf{n}}(\tau_0)$ yields that $y_{\boldsymbol{a}}(z)$ is a solution of some $\mathrm{H}(\mathbf{n},B,\tau_0)$. By (2.27), we see that with respect to $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$, the monodromy matrices $\rho(\ell_1)$ and $\rho(\ell_2)$ are given by (2.6).

The proof is complete. □

We conclude this section by the following remark.

After Theorem 1.2, a further question arises: *What are the explicit expressions of these pre-modular forms*? This question is very difficult because the weight $\frac{1}{2}\sum n_k(n_k+1)$ is large for general $\mathbf{n}$. It is known from [9, 17] that (write $Z=Z_{r,s}(\tau)$, $\wp=\wp(r+s\tau|\tau)$ and $\wp'=\wp'(r+s\tau|\tau)$ for convenience):

$$Z_{r,s}^{(1,0,0,0)}=Z,\quad Z_{r,s}^{(2,0,0,0)}=Z^3-3\wp Z-\wp',$$

$$\begin{aligned}Z_{r,s}^{(3,0,0,0)}=&Z^6-15\wp Z^4-20\wp' Z^3+\left(\tfrac{27}{4}g_2-45\wp^2\right)Z^2\\&-12\wp\wp' Z-\tfrac{5}{4}(\wp')^2.\end{aligned}$$

$$\begin{aligned}Z_{r,s}^{(4,0,0,0)}=&Z^{10}-45\wp Z^8-120\wp' Z^7+(\tfrac{399}{4}g_2-630\wp^2)Z^6-504\wp\wp' Z^5\\&-\tfrac{15}{4}(280\wp^3-49g_2\wp-115g_3)Z^4+15(11g_2-24\wp^2)\wp' Z^3\\&-\tfrac{9}{4}(140\wp^4-245g_2\wp^2+190g_3\wp+21g_2^2)Z^2\\&-(40\wp^3-163g_2\wp+125g_3)\wp' Z+\tfrac{3}{4}(25g_2-3\wp^2)(\wp')^2.\end{aligned}$$

The above formulas are all for the Lamé case. For $n \geq 5$, the explicit expression of $Z_{r,s}^{(n,0,0,0)}(\tau)$ is not known so far. See [6, 9, 17] for applications of the above formulas of $Z_{r,s}^{(n,0,0,0)}(\tau)$, $n \leq 4$.

Here are new examples of $Z_{r,s}^{\mathbf{n}}(\tau)$ for the Treibich-Verdier potential case:

$$Z_{r,s}^{(1,1,0,0)} = Z^2 - \wp + e_1,$$

$$Z_{r,s}^{(1,0,1,0)} = Z^2 - \wp + e_2, \quad Z_{r,s}^{(1,0,0,1)} = Z^2 - \wp + e_3,$$

$$Z_{r,s}^{(2,1,0,0)} = Z^4 + 3(e_1 - 2\wp)Z^2 - 4\wp' Z - 3(\wp^2 + e_1\wp + e_1^2 - \tfrac{g_2}{4}),$$

and similarly, the expression of $Z_{r,s}^{(2,0,1,0)}$ (resp. $Z_{r,s}^{(2,0,0,1)}$) is obtained by replacing e_1 in $Z_{r,s}^{(2,1,0,0)}$ with e_2 (resp. e_3). The proof of these new formulas will be given in a forthcoming work, where we will also study further properties of these pre-modular forms, such as the following interesting formula:

$$Z_{r,s}^{(1,1,0,0)}(\tau) = 4Z_{r,\frac{s}{2}}(2\tau)Z_{r,\frac{s+1}{2}}(2\tau).$$

We believe that these new formulas will have interesting applications.

3. Application to the mean field equation

This section is devoted to the proof of Theorems 1.4 and 1.5. Recall from (1.5)-(1.9) that the monodromy group of $\mathrm{H}(\mathbf{n}, B, \tau)$ is conjugate to a subgroup of $SU(2)$, i.e. the monodromy of $\mathrm{H}(\mathbf{n}, B, \tau)$ is *unitary*, if and only if Case (a) occurs with some $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Applying Theorem 1.2, this is equivalent to that τ is a zero of $Z_{r,s}^{\mathbf{n}}(\cdot)$ for some $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Therefore, Theorem 1.4 is a direct corollary of Theorem 1.2 and the following result.

Theorem 3.1. *The mean field equation*

$$\Delta u + e^u = 8\pi \sum_{k=0}^{3} n_k \delta_{\frac{\omega_k}{2}} \quad \text{on } E_\tau, \tag{3.1}$$

has an even solution if and only if there exists $B \in \mathbb{C}$ such that the monodromy of $H(\mathbf{n}, B, \tau)$ is unitary.

Furthermore, the number of even solutions equals to the number of those B's such that the monodromy of $H(\mathbf{n}, B, \tau)$ is unitary.

For the Lamé case $n_1 = n_2 = n_3 = 0$, Theorem 3.1 was proved in [3]. For general case $n_k \in \mathbb{Z}_{\geq 0}$ as considered here, the necessary part of the first assertion in Theorem 3.1 was already proved in [8, 10]. Here we sketch the proof from [8] for later usage.

Proof of the necessary part of the first assertion in Theorem 3.1 [8]. Let $u(z)$ be a solution of (3.1). Then the Liouville theorem (cf. [3, Section 1.1]) says that there is a local meromorphic function $f(z)$ away from $E_\tau[2] = \{\frac{\omega_k}{2} \mid 0 \leq k \leq 3\} + \Lambda_\tau$ such that

$$u(z) = \log \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}. \tag{3.2}$$

This $f(z)$ is called a developing map. By differentiating (3.2), we have

$$u_{zz} - \frac{1}{2}u_z^2 = \{f;z\} := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{3.3}$$

Conventionally, the RHS of this identity is called the Schwarzian derivative of $f(z)$, denoted by $\{f;z\}$. Note that outside the singularities $E_\tau[2]$,

$$\left(u_{zz} - \tfrac{1}{2}u_z^2\right)_{\bar z} = (u_{z\bar z})_z - u_z u_{z\bar z} = -\tfrac{1}{4}\left(e^u\right)_z + \tfrac{1}{4}e^u u_z = 0.$$

On the other hand, since $v(z) := u(z) - 4n_k \ln|z - \frac{\omega_k}{2}|$ solves $\Delta v + |z - \frac{\omega_k}{2}|^{4n_k} e^v = 0$ near $\frac{\omega_k}{2}$, it is standard (cf. [1]) to see that $v(z)$ is regular at $\frac{\omega_k}{2}$, i.e. $u(z) = 4n_k \ln|z - \frac{\omega_k}{2}| + O(1)$ near $\frac{\omega_k}{2}$. From here we conclude that $u_{zz} - \frac{1}{2}u_z^2$ is an *elliptic function* with at most *double poles* at $E_\tau[2]$.

Now suppose $u(z)$ is even, i.e. $u(z) = u(-z)$. Then

$$u_{zz} - \frac{1}{2}u_z^2 = -2\left[\sum_{k=0}^{3} n_k(n_k+1)\wp(z + \tfrac{\omega_k}{2}|\tau) + B\right] = -2I_{\mathbf{n}}(z;B,\tau) \tag{3.4}$$

for some constant $B = B(u)$, because due to the evenness, $u_{zz} - \frac{1}{2}u_z^2$ has no residues at $z \in E_\tau[2]$. Since $\{f;z\} = -2I_{\mathbf{n}}(z;B,\tau)$, a classical result says that there are linearly independent solutions $y_1(z), y_2(z)$ of $\mathrm{H}(\mathbf{n},B,\tau)$ such that

$$f(z) = \frac{y_1(z)}{y_2(z)}. \tag{3.5}$$

Recalling Section 2 that $y_1(z), y_2(z)$ are both meromorphic, we see that the developing map $f(z)$ is single-valued near each $\frac{\omega_k}{2}$ and then can be extended to be an entire meromorphic function in $\mathbb{C}$.

Define the Wronskian

$$W := y_1'(z)y_2(z) - y_1(z)y_2'(z).$$

Then W is a nonzero constant. By inserting (3.5) into (3.2), a direct computation leads to

$$2\sqrt{2}We^{-\frac{1}{2}u(z)} = |y_1(z)|^2 + |y_2(z)|^2.$$

Note that as a solution of (3.1), $u(z)$ is single-valued and doubly periodic, so $e^{-\frac{1}{2}u(z)}$ is invariant under analytic continuation along any element $\gamma \in \pi_1(E_\tau)$. This implies that for any monodromy matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{C})$ with respect to $(y_1(z), y_2(z))$, there holds

$$|ay_1(z) + cy_2(z)|^2 + |by_1(z) + dy_2(z)|^2 = |y_1(z)|^2 + |y_2(z)|^2,$$

which easily infers $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SU(2)$. Therefore, the monodromy of $\mathrm{H}(\mathbf{n},B,\tau)$ is unitary. $\square$

The main task of this section is to prove the sufficient part and the $1-1$ correspondence between even solutions and those B's such that the monodromy of $\mathrm{H}(\mathbf{n}, B, \tau)$ is unitary. The following proof is also not difficult by applying the monodromy theory of $\mathrm{H}(\mathbf{n}, B, \tau)$ in Section 2.

Proof of Theorem 3.1. First we prove the sufficient part of the first assertion.

Suppose there is $B \in \mathbb{C}$ such that the monodromy of $\mathrm{H}(\mathbf{n}, B, \tau)$ is unitary. Then by (1.5) and (2.3)-(2.6), there exist $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $\boldsymbol{a} \in Y_{\mathbf{n}}(\tau)$ such that

$$y_{\pm \boldsymbol{a}}(z) = e^{c(\pm \boldsymbol{a})z} \frac{\prod_{i=1}^{N} \sigma(z \mp a_i)}{\prod_{k=0}^{3} \sigma(z - \frac{\omega_k}{2})^{n_k}}$$

are linearly independent solutions of $\mathrm{H}(\mathbf{n}, B, \tau)$ and

$$\begin{pmatrix} y_{\boldsymbol{a}}(z+\omega_1) \\ y_{-\boldsymbol{a}}(z+\omega_1) \end{pmatrix} = \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix} \begin{pmatrix} y_{\boldsymbol{a}}(z) \\ y_{-\boldsymbol{a}}(z) \end{pmatrix}, \tag{3.6}$$

$$\begin{pmatrix} y_{\boldsymbol{a}}(z+\omega_2) \\ y_{-\boldsymbol{a}}(z+\omega_2) \end{pmatrix} = \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{-2\pi i r} \end{pmatrix} \begin{pmatrix} y_{\boldsymbol{a}}(z) \\ y_{-\boldsymbol{a}}(z) \end{pmatrix}.$$

Now we define

$$f(z) := \frac{y_{\boldsymbol{a}}(z)}{y_{-\boldsymbol{a}}(z)} = e^{2z\sum_{i=1}^{N}\zeta(a_i)} \frac{\prod_{i=1}^{N} \sigma(z - a_i)}{\prod_{i=1}^{N} \sigma(z + a_i)} \tag{3.7}$$

and

$$u(z) := \log \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}.$$

We claim that this $u(z)$ is an even solution of (3.1).

Clearly (3.6) and $(r,s) \in \mathbb{R}^2$ yield that $u(z)$ is doubly periodic and hence well-defined on E_τ. Furthermore, $f(-z) = \frac{1}{f(z)}$ infers that $u(z) = u(-z)$.

Since (2.3), (2.8) and (2.15) imply that $a_i \notin E_\tau[2]$ and a_i's (resp. $-a_i$'s) are all simple zeros (resp. simple poles) of $f(z)$, we have that (i) $f(z)$ are holomorphic at $\frac{\omega_k}{2}$ for all k, and (ii) $u(\pm a_i) \neq \infty$ for all i and so

$$\Delta u + e^u = 0 \quad \text{on } E_\tau \setminus \{z \in E_\tau \setminus \cup_i \{\pm a_i\} \,|\, f'(z) = 0\}.$$

Since $f(z) = \frac{y_a(z)}{y_{-a}(z)}$ gives

$$\begin{aligned} \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 &= -2I_{\mathbf{n}}(z; B, \tau) \\ &= -2\left[\sum_{k=0}^{3} n_k(n_k+1)\wp(z + \tfrac{\omega_k}{2}|\tau) + B\right], \end{aligned}$$

we easily conclude that $f'(z) \neq 0$ for any $z \notin E_\tau[2]$ and $\mathrm{ord}_{z=\frac{\omega_k}{2}} f'(z) = 2n_k$, i.e. $u(z) = 4n_k \ln|z - \frac{\omega_k}{2}| + O(1)$ near $\frac{\omega_k}{2}$. In conclusion, $u(z)$ is an even solution of (3.1).

Next we prove that the even solution is unique for this given B. Suppose $\tilde{u}(z)$ is an even solution of (3.1) corresponding to the same B with $u(z)$, i.e. (3.4) holds. Our goal is to prove $\tilde{u}(z) = u(z)$.

By the proof of the necessary part of Theorem 3.1, we may let $\hat{f}(z) = \frac{y_1(z)}{y_2(z)}$ be a developing map of $\tilde{u}(z)$ such that $y_1(z), y_2(z)$ are linearly independent solutions of $\mathrm{H}(\mathbf{n}, B, \tau)$ and satisfy

$$\begin{pmatrix} y_1(z+\omega_j) \\ y_2(z+\omega_j) \end{pmatrix} = M_j \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}, \quad M_j \in SU(2), \quad j = 1, 2.$$

Since $M_1 M_2 = M_2 M_1$, there is a matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ such that $\tilde{M}_j := P M_j P^{-1}$ are both diagonal matrices for $j = 1, 2$. Define

$$\begin{pmatrix} \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{pmatrix} := P \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}$$

to be another pair of linearly independent solutions of $\mathrm{H}(\mathbf{n}, B, \tau)$ and

$$\tilde{f}(z) := \frac{\tilde{y}_1(z)}{\tilde{y}_2(z)} = \frac{a\hat{f}(z) + b}{c\hat{f}(z) + d} =: P\hat{f}(z).$$

The fact $P \in SU(2)$ implies that $\tilde{f}(z)$ is also a developing map of $\tilde{u}(z)$, i.e.

$$\tilde{u}(z) = \log \frac{8|\hat{f}'(z)|^2}{(1 + |\hat{f}(z)|^2)^2} = \log \frac{8|\tilde{f}'(z)|^2}{(1 + |\tilde{f}(z)|^2)^2}. \tag{3.8}$$

Clearly

$$\begin{pmatrix} \tilde{y}_1(z+\omega_j) \\ \tilde{y}_2(z+\omega_j) \end{pmatrix} = \tilde{M}_j \begin{pmatrix} \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{pmatrix}, \quad \tilde{M}_j \text{ is diagonal for } j = 1, 2.$$

Together with (3.6), $(r, s) \notin \frac{1}{2}\mathbb{Z}^2$ and the fact that $\tilde{y}_j(z)$'s are linear combinations of $y_{\pm \boldsymbol{a}}(z)$, we easily conclude that, by reordering $\tilde{y}_1(z), \tilde{y}_2(z)$ if necessary,

$$\tilde{y}_1(z) = c_1 y_{\boldsymbol{a}}(z), \quad \tilde{y}_2(z) = c_2 y_{-\boldsymbol{a}}(z)$$

for some constants $c_1, c_2 \neq 0$. So (3.7) gives $\tilde{f}(z) = cf(z)$ for $c = c_1/c_2 \neq 0$ and then

$$\tilde{f}(-z) = cf(-z) = \frac{c}{f(z)} = \frac{c^2}{\tilde{f}(z)}.$$

Inserting this into (3.8), it follows from $\tilde{u}(z) = \tilde{u}(-z)$ that $|c| = 1$. In conclusion, $\tilde{f}(z) = cf(z)$ with $|c| = 1$, which clearly infers that $\tilde{u}(z) = u(z)$. This proves the uniqueness of the even solution with respect to the given B. Therefore, the number of even solutions equals to the number of those B's such that the monodromy of $\mathrm{H}(\mathbf{n}, B, \tau)$ is unitary.

The proof is complete. □

As an application of Theorem 3.1, we have the following result, which contains Theorem 1.5 as a special case.

Theorem 3.2. *Given $n \in \mathbb{N}$, we define*

$$n_0 := \frac{n}{2} - 1,\ n_1 = n_2 = n_3 := \frac{n}{2} \quad \text{if } n \text{ is even},$$

$$n_0 := \frac{n+1}{2},\ n_1 = n_2 = n_3 := \frac{n-1}{2} \quad \text{if } n \text{ is odd}.$$

Then the mean field equations

$$\Delta u + e^u = 8\pi n \delta_0 \ \text{ on } E_\tau \tag{3.9}$$

and

$$\Delta u + e^u = 8\pi \sum_{k=0}^{3} n_k \delta_{\frac{\omega_k}{2}} \ \text{ on } E_\tau \tag{3.10}$$

has the same number of even solutions.

Proof. It was proved by Takemura [27, Section 4] that $\mathrm{H}((n,0,0,0),B,\tau)$ and $\mathrm{H}((n_0,n_1,n_2,n_3),B,\tau)$ are isomonodromic (i.e. their monodromy representations are the same) for any (B,τ). Therefore, this theorem follows from Theorem 3.1. □

In Theorem 3.2, the case $n = 1$ is trivial, and the first nontrivial case $n = 2$ gives Theorem 1.5, which has the following consequence for rhombus tori.

Corollary 3.3. *There exists $b^* \in (\frac{\sqrt{3}}{2}, \frac{6}{5})$ such that for $\tau = \frac{1}{2} + ib$ with $b > b^*$, (1.14) on E_τ has even solutions; while for $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, (1.14) on E_τ has no even solutions.*

Proof. It was proved in [7, Theorem A.2] that there exists $b^* \in (\frac{\sqrt{3}}{2}, \frac{6}{5})$ such that for $\tau = \frac{1}{2} + ib$ with $b > b^*$, (1.13) on E_τ has even solutions. Furthermore, we proved in [5, Theorem 3.1] that (1.13) on E_τ has no solutions for $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Therefore, this assertion follows from Theorem 1.5. □

Remark 3.4. In Theorem 3.2, it is easy to see that $n(n+1) = \sum_{k=0}^{3} n_k(n_k + 1)$, namely the pre-modular forms $Z_{r,s}^{(n,0,0,0)}(\tau)$ and $Z_{r,s}^{(n_0,n_1,n_2,n_3)}(\tau)$ have the same weight. Theorem 3.2 strongly suggests $Z_{r,s}^{(n,0,0,0)}(\tau) = Z_{r,s}^{(n_0,n_1,n_2,n_3)}(\tau)$, which will be studied in a future work.

Remark 3.5. Theorem 3.2 and Corollary 3.3 indicate that the mean field equations with multiple singularities

$$\Delta u + e^u = 8\pi \sum n_k \delta_{p_k} \ \text{ on } E_\tau \tag{3.11}$$

might be studied by establishing relations with some other mean field equations with less singularities. We will apply this idea to study (3.11) in future works.

APPENDIX A. PROOF OF THEOREM 2.1

This appendix is devoted to the proof of Theorem 2.1. First we need the following result which was essentially proved in [13, 23].

Theorem A.1. *Suppose* $\boldsymbol{a} = \{a_1, \cdots, a_N\}$ *satisfies*

$$[a_i] \notin E_\tau[2],\ [a_i] \neq [a_j],\ \forall i \neq j. \tag{A.1}$$

Then

$$y_{\boldsymbol{a}}(z) = e^{c(\boldsymbol{a})z} \frac{\prod_{i=1}^N \sigma(z - a_i)}{\prod_{k=0}^3 \sigma(z - \frac{\omega_k}{2})^{n_k}}$$

is a solution to $H(\mathbf{n}, B, \tau)$ *with some* B *if and only if* $\boldsymbol{a}$ *satisfies*

$$\begin{aligned}&\sum_{k=1}^3 n_k \left[\zeta(a_i + \tfrac{\omega_k}{2}) + \zeta(a_i - \tfrac{\omega_k}{2}) - 2\zeta(a_i)\right]\\ =&2\sum_{j\neq i}^N \left[\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)\right],\ \ 1 \leq i \leq N,\end{aligned} \tag{A.2}$$

$$\begin{aligned}&\sum_{i=1}^N \left(\zeta(a_i + \tfrac{\omega_l}{2}) + \zeta(a_i - \tfrac{\omega_l}{2})\right)\\ &= 2\sum_{i=1}^N \zeta(a_i),\ \textit{whenever } n_l \neq 0, l \in \{1,2,3\},\end{aligned} \tag{A.3}$$

and $c(\boldsymbol{a})$, B *are determined by*

$$c(\boldsymbol{a}) = \sum_{i=1}^N \zeta(a_i) - \frac{1}{2}\sum_{k=1}^3 n_k \eta_k, \tag{A.4}$$

$$B = (2n_0 - 1)\sum_{i=1}^N \wp(a_i) - \sum_{k=1}^3 n_k(n_k + 2n_0)e_k. \tag{A.5}$$

Proof. We sketch the proof here for the reader's convenience. Note that

$$\frac{y_{\boldsymbol{a}}'(z)}{y_{\boldsymbol{a}}(z)} = c(\boldsymbol{a}) + \sum_{i=1}^N \zeta(z - a_i) - \sum_{k=0}^3 n_k \zeta(z - \tfrac{\omega_k}{2}),$$

$$\left(\frac{y_{\boldsymbol{a}}'(z)}{y_{\boldsymbol{a}}(z)}\right)' = -\sum_{i=1}^N \wp(z - a_i) + \sum_{k=0}^3 n_k \wp(z - \tfrac{\omega_k}{2}),$$

are both elliptic functions. Consider the elliptic function

$$h(z) := \left(\frac{y_{\boldsymbol{a}}'(z)}{y_{\boldsymbol{a}}(z)}\right)' + \left(\frac{y_{\boldsymbol{a}}'(z)}{y_{\boldsymbol{a}}(z)}\right)^2 - \sum_{k=0}^3 n_k(n_k+1)\wp(z + \tfrac{\omega_k}{2}) - B.$$

Clearly $y_{\boldsymbol{a}}(z)$ is a solution of $\mathrm{H}(\mathbf{n}, B, \tau)$ if and only if $h(z) \equiv 0$ if and only if *none of* $\frac{\omega_k}{2}$*'s and* a_i*'s are poles of* $h(z)$ and the constant term of the Laurent expansion at $z = 0$ is 0. By computing leading terms of the Laurent expansions at $z = \frac{\omega_k}{2}, a_i$, we easily obtain the conditions (A.2)-(A.5). $\square$

Clearly Theorem 2.1 is a consequence of Theorem A.1 and the following result.

Theorem A.2. *Suppose* $\boldsymbol{a} = \{a_1, \cdots, a_N\}$ *satisfies*

$$[a_i] \notin E_\tau[2],\ [a_i] \neq \pm[a_j],\ \forall i \neq j. \tag{A.6}$$

Then (A.2)-(A.3) are equivalent to

$$\sum_{i=1}^{N} \wp'(a_i)\wp(a_i)^l = 0 \text{ for } 0 \leq l \leq n_0 - 2, \tag{A.7}$$

$$\sum_{i=1}^{N} \wp'(a_i) \prod_{j=1,\neq i}^{N} (\wp(a_j) - e_k)^l = 0 \text{ for } 1 \leq l \leq n_k,\ k \in \{1, 2, 3\}. \tag{A.8}$$

Proof. Define

$$f(z) := e^{2z \sum_{i=1}^{N} \zeta(a_i)} \frac{\prod_{i=1}^{N} \sigma(z - a_i)}{\prod_{i=1}^{N} \sigma(z + a_i)}.$$

First we claim that

$$\text{(A.2)-(A.3) hold} \iff \underset{z=\frac{\omega_k}{2}}{ord} f'(z) = 2n_k \text{ for all } k. \tag{A.9}$$

With the help of Theorem A.1, this claim can be proved by similar arguments as the sufficient part of Theorem 3.1. We leave the details to the interested reader. Here we would like to give an elementary proof by using only the elliptic function theory but without using the ODE $\mathrm{H}(\mathbf{n}, B, \tau)$ and the Schwarzian derivative.

Recalling the addition formula

$$\zeta(u + v) + \zeta(u - v) - 2\zeta(u) = \frac{\wp'(u)}{\wp(u) - \wp(v)}, \tag{A.10}$$

we have

$$\begin{aligned} g(z) := \frac{f'}{f}(z) &= \sum_{i=1}^{N} (\zeta(z - a_i) - \zeta(z + a_i) + 2\zeta(a_i)) \\ &= \sum_{i=1}^{N} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} = \frac{\psi(\wp(z))}{\prod_{i=1}^{N}(\wp(z) - \wp(a_i))}, \end{aligned} \tag{A.11}$$

where

$$\psi(x) := \sum_{h=1}^{N} \wp'(a_h) \prod_{j \neq h}^{N} (x - \wp(a_j)). \tag{A.12}$$

Clearly (A.3) is equivalent to

$$\psi(e_l) = g(\tfrac{\omega_l}{2}) = 0 \text{ whenever } n_l \neq 0, l = 1, 2, 3. \tag{A.13}$$

Notice from (A.6) that $g(z)$ is even elliptic with $2N$ simple poles $\pm a_i, 1 \leq i \leq N$. Since $f(\frac{\omega_k}{2}) \notin \{0, \infty\}$, it is easy to see that $\text{ord}_{z=\frac{\omega_k}{2}} f'(z) = 2n_k$ for all k is equivalent to saying that $\frac{\omega_k}{2}$ is a zero of $g(z)$ with order $2n_k$ for all k and so $g(z)$ has no other zeros (in particular, $f'(z)$ *has no other zeros*), namely

$$g(z) = d \frac{\prod_{k=1}^{3} (\wp(z) - e_k)^{n_k}}{\prod_{i=1}^{N} (\wp(z) - \wp(a_i))} \tag{A.14}$$

for some constant $d \neq 0$. For convenience, we define

$$H(x) := \prod_{k=1}^{3} (x - e_k)^{n_k}. \tag{A.15}$$

First we prove the sufficient part of the claim (A.9). Suppose (A.14) hold-s. Then (A.13) gives (A.3). By comparing (A.11) and (A.14)-(A.15) we have

$$dH(x) = \psi(x) = \sum_{h=1}^{N} \wp'(a_h) \prod_{j \neq h}^{N} (x - \wp(a_j)), \ \ x = \wp(z).$$

Taking derivative with respect to x leads to

$$dH'(x) = \sum_{h=1}^{N} \wp'(a_h) \sum_{l \neq h}^{N} \prod_{j \neq h,l}^{N} (x - \wp(a_j)), \ \forall x \in \mathbb{C}, \tag{A.16}$$

where

$$H'(x) = H(x) \sum_{k=1}^{3} \frac{n_k}{x - e_k}. \tag{A.17}$$

Now we fix any i. Then by letting $x = \wp(a_i)$ in (A.16), the RHS becomes

$$\begin{aligned}
&\sum_{h=1}^{N} \wp'(a_h) \sum_{l \neq h}^{N} \prod_{j \neq h,l}^{N} (\wp(a_i) - \wp(a_j)) \\
=&\wp'(a_i) \sum_{l \neq i}^{N} \prod_{j \neq i,l}^{N} (\wp(a_i) - \wp(a_j)) + \sum_{h \neq i}^{N} \wp'(a_h) \sum_{l \neq h}^{N} \prod_{j \neq h,l}^{N} (\wp(a_i) - \wp(a_j)) \\
=&\wp'(a_i) \sum_{l \neq i}^{N} \prod_{j \neq i,l}^{N} (\wp(a_i) - \wp(a_j)) + \sum_{h \neq i}^{N} \wp'(a_h) \prod_{j \neq h,i}^{N} (\wp(a_i) - \wp(a_j)) \\
=&\sum_{l \neq i}^{N} \left(\wp'(a_i) + \wp'(a_l)\right) \prod_{j \neq i,l}^{N} (\wp(a_i) - \wp(a_j)),
\end{aligned}$$

which gives

$$\frac{\sum_{h=1}^{N} \wp'(a_h) \sum_{l \neq h}^{N} \prod_{j \neq h,l}^{N} (\wp(a_i) - \wp(a_j))}{\prod_{j \neq i}^{N} (\wp(a_i) - \wp(a_j))} = \sum_{j \neq i}^{N} \frac{\wp'(a_i) + \wp'(a_j)}{\wp(a_i) - \wp(a_j)}. \tag{A.18}$$

Together with (A.16) and the addition formula

$$\zeta(u+v)-\zeta(u)-\zeta(v)=\frac{1}{2}\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}, \tag{A.19}$$

we obtain

$$\begin{aligned}\frac{dH'(\wp(a_i))}{\prod_{j\neq i}^{N}(\wp(a_i)-\wp(a_j))} &= \sum_{j\neq i}^{N}\frac{\wp'(a_i)+\wp'(a_j)}{\wp(a_i)-\wp(a_j)} \\ &= 2\sum_{j\neq i}^{N}\left[\zeta(a_i-a_j)+\zeta(a_j)-\zeta(a_i)\right].\end{aligned} \tag{A.20}$$

By (A.11) and (A.14)-(A.15) again, we have

$$1=\operatorname*{Res}_{z=a_i}g(z)=\frac{1}{\wp'(a_i)}\frac{dH(\wp(a_i))}{\prod_{j\neq i}^{N}(\wp(a_i)-\wp(a_j))}.$$

This, together with (A.20), (A.17) and the addition formula (A.10), gives

$$\begin{aligned}&2\sum_{j\neq i}^{N}\left[\zeta(a_i-a_j)+\zeta(a_j)-\zeta(a_i)\right] \\ =&\frac{\wp'(a_i)H'(\wp(a_i))}{H(\wp(a_i))}=\sum_{k=1}^{3}n_k\frac{\wp'(a_i)}{\wp(a_i)-e_k} \\ =&\sum_{k=1}^{3}n_k\left[\zeta\left(a_i+\tfrac{\omega_k}{2}\right)+\zeta\left(a_i-\tfrac{\omega_k}{2}\right)-2\zeta(a_i)\right],\end{aligned} \tag{A.21}$$

namely (A.2) holds.

Conversely, suppose (A.2)-(A.3) hold, then (A.13) and (A.21) hold. By (A.18), the second equality of (A.20) and (A.21), it is easy to see that

$$\frac{\sum_{h=1}^{N}\wp'(a_h)\sum_{l\neq h}^{N}\prod_{j\neq h,l}^{N}(\wp(a_i)-\wp(a_j))}{\prod_{j\neq i}^{N}(\wp(a_i)-\wp(a_j))}\frac{H(\wp(a_i))}{\wp'(a_i)H'(\wp(a_i))}=1 \tag{A.22}$$

holds for all i. Define a polynomial

$$Q(x):=\psi'(x)\cdot H(x)-\psi(x)\cdot H'(x).$$

Recalling (A.12), (A.13) and (A.15), we easily obtain

$$\deg Q(x)\leq N-1+\sum_{k=1}^{3}n_k \text{ and } \prod_{k=1}^{3}(x-e_k)^{n_k}\,\Big|\, Q(x).$$

Since (A.22) just says $Q(\wp(a_i))=0$ for all i, we see that $Q(x)$ has at least $N+\sum_{k=1}^{3}n_k$ zeros, so $Q(x)\equiv 0$. Consequently, $\psi(x)=dH(x)$ for some constant $d\neq 0$, i.e. (A.14) holds, which infers $\operatorname{ord}_{z=\frac{\omega_k}{2}}f'(z)=2n_k$ for all k. This proves the claim (A.9).

Thanks to (A.9), it suffices for us to prove the equivalence between (A.7)-(A.8) and $\mathrm{ord}_{z=\frac{\omega_k}{2}} f'(z) = 2n_k$ for all k. Clearly (A.11) gives

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{N} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} = \frac{1}{\wp(z)} \sum_{i=1}^{N} \frac{\wp'(a_i)}{1 - \frac{\wp(a_i)}{\wp(z)}}$$
$$= \sum_{l=0}^{\infty} \frac{\sum_{i=1}^{N} \wp'(a_i)\wp(a_i)^l}{\wp(z)^{l+1}}.$$

Therefore, $\mathrm{ord}_{z=0} f'(z) = 2n_0$ if and only if (A.7) holds. Similarly, for $k \in \{1, 2, 3\}$, by using the addition formula

$$\wp(z - \tfrac{\omega_k}{2}) - e_k = \frac{\mu_k}{\wp(z) - e_k}, \quad \mu_k = \frac{1}{2}\wp''(\tfrac{\omega_k}{2}) \neq 0,$$

we easily obtain

$$\frac{f'(z)}{f(z)} - \sum_{i=1}^{N} \frac{\wp'(a_i)}{e_k - \wp(a_i)}$$
$$= \sum_{i=1}^{N} \left(\frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} + \frac{\wp'(a_i)}{\wp(a_i) - e_k} \right) = \sum_{i=1}^{N} \frac{\wp'(a_i)}{(\wp(a_i) - e_k)(1 - \frac{\wp(a_i) - e_k}{\wp(z) - e_k})}$$
$$= \sum_{i=1}^{N} \frac{\wp'(a_i)}{(\wp(a_i) - e_k)(1 - \frac{\wp(a_i) - e_k}{\mu_k}(\wp(z - \frac{\omega_k}{2}) - e_k))}$$
$$= -\mu_k \sum_{i=1}^{N} \frac{\wp'(a_i)}{(\wp(a_i) - e_k)^2(\wp(z - \frac{\omega_k}{2}) - e_k)(1 - \frac{\mu_k}{(\wp(a_i) - e_k)(\wp(z - \frac{\omega_k}{2}) - e_k)})}$$
$$= -\sum_{l=1}^{\infty} \frac{\mu_k^l \sum_{i=1}^{N} \frac{\wp'(a_i)}{(\wp(a_i) - e_k)^{l+1}}}{(\wp(z - \frac{\omega_k}{2}) - e_k)^l}$$
$$= -\sum_{l=1}^{\infty} \frac{\mu_k^l \frac{\sum_{i=1}^{N} \wp'(a_i) \prod_{j=1, \neq i}^{N} (\wp(a_j) - e_k)^{l+1}}{\prod_{i=1}^{N} (\wp(a_i) - e_k)^{l+1}}}{(\wp(z - \frac{\omega_k}{2}) - e_k)^l}.$$

Therefore, $\mathrm{ord}_{z=\frac{\omega_k}{2}} f'(z) = 2n_k$ if and only if (A.8) holds.

The proof is complete. □

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