L_p geominimal surface areas and their inequalities *

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Abstract

In this paper, we introduce the L_p geominimal surface area for all $-n \neq p < 1$, which extends the classical geominimal surface area ($p = 1$) by Petty and the L_p geominimal surface area by Lutwak ($p > 1$). Our extension of the L_p geominimal surface area is motivated by recent work on the extension of the L_p affine surface area – a fundamental object in (affine) convex geometry. We prove some properties for the L_p geominimal surface area and its related inequalities, such as, the affine isoperimetric inequality and a Santaló style inequality. Cyclic inequalities are established to obtain the monotonicity of the L_p geominimal surface areas. Comparison between the L_p geominimal surface area and the p-surface area is also provided.

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1 Introduction and Overview of Results

The classical isoperimetric problem asks: what is the minimal area among all convex bodies (i.e., convex compact subsets with nonempty interior) $K \subset \mathbb{R}^n$ with volume 1? The solution of this old problem is now known as the (classical) isoperimetric inequality, namely, the minimal area is attained at and only at Euclidean balls with volume 1. The isoperimetric inequality is an extremely powerful tool in geometry and related areas. Note that the classical isoperimetric inequality does not have the "affine invariant" flavor, because the area may change under linear transformations (even) with unit (absolute value of) determinant.

However, many objects in (affine) convex geometry are invariant under invertible linear transformations. A typical example is the Mahler volume product $M(K) = |K||K^\circ|$, the product of the volume of K and its polar body K°. That is, $M(K) = M(TK)$ for all invertible linear transformations T on $\mathbb{R}^n$. One can ask a question for $M(K)$ similar

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to the classical isoperimetric problem: what is the maximum of $M(K)$ among all convex bodies K? The celebrated Blaschke-Santaló inequality states that $M(K)$ attains its maximum at and only at origin-symmetric ellipsoids if K is assumed to have its centroid at the origin. Such an inequality is arguably more important than the classical isoperimetric inequality. Consequently, the Blaschke-Santaló inequality has numerous applications in (affine) convex geometry and many other related fields, for instance, in quantum information theory [2, 38]. The family of affine isoperimetric inequalities continues to grow (see, for example [6, 10, 11, 24, 25]). It is well known that the Blaschke-Santaló inequality was first discovered and proved by using the classical affine isoperimetric inequality [32], which bounds the classical affine surface area from above in terms of volume.

The study of affine surface areas goes back to Blaschke in [3] about one hundred years ago, and its L_p counterpart was first introduced by Lutwak in [21]. The notion of L_p affine surface area was further extended to all $p \in \mathbb{R}$ for general convex bodies in, e.g., [27, 35, 36]. In fact, extensions of L_p affine surface area to all $-n \neq p \in \mathbb{R}$ were obtained by their integral expressions (see e.g. Theorem 3.1) and by investigating the asymptotic behavior of the volume of certain families of convex bodies [16, 26, 27, 35, 36, 37, 41, 42] (and even star-shape bodies [43]). The L_p affine surface area is now thought to be at the core of the rapidly developing L_p-Brunn-Minkowski theory. It has many nice properties, such as, affine invariance (i.e., invariance under all linear transformations T with $|det(T)| = 1$), the valuation property, semi-continuity (upper for $p > 0$ and lower for $-n \neq p < 0$), etc. Moreover, the L_p affine surface area for $p > 0$ has been proved to be, roughly speaking, the unique valuation with properties of affine invariance and upper semi-continuity [19, 20]. Recently, it has been connected with the information theory of convex bodies (see e.g., [13, 29, 39, 40]). Affine isoperimetric inequalities related to the L_p affine surface area can be found in, for instance, [21, 42].

Another fundamental concept in convex geometry is the (classical) geominimal surface area introduced by Petty [30] about 40 years ago. As explained by Petty in [30], the (classical) geominimal surface area naturally connects affine geometry, relative geometry and Minkowski geometry. Hence it received a lot of attention (e.g., [30, 31, 33]). In fact, it can be viewed as a sibling concept of the classical affine surface area as their definitions are similar. However, they are in general different from each other; for example the (classical) geominimal surface area is continuous while the classical affine surface area is only upper semi-continuous on the set of all convex bodies (equipped with the Hausdorff metric).

In his seminal paper [21], Lutwak introduced L_p geominimal surface area for $p > 1$. L_p geominimal surface area has many properties very similar to L_p affine surface area, for instance, affine invariance with the same degree of homogeneous. For certain class of convex bodies, the L_p geominimal surface area for $p \geq 1$ is equal to the L_p affine surface area (see [21] or Proposition 3.4). However, the L_p geominimal surface area is different from the L_p affine surface area, because the former one is continuous while the latter one is only upper semi-continuous on the set of all convex bodies (equipped with the Hausdorff metric). Moreover, as mentioned before, there are nice integral expressions for the L_p affine surface area (see e.g. Theorem 3.1), while finding similar integral expressions for the L_p geominimal surface area appears to be impossible. Note that the extension of L_p affine surface area from $p \geq 1$ to all $-n \neq p \in \mathbb{R}$ mainly relied on the integral expression of L_p affine surface area. Recently, the L_p affine surface area was further extended to its Orlicz counterpart, general affine surface areas, involving general (even nonhomogeneous) convex or concave functions [18, 19].

Affine isoperimetric inequalities related to the L_p geominimal surface area were proved for $p = 1$ in [30, 31] and for $p > 1$ in [21]. Roughly speaking, those affine isoperimetric inequalities assert that among all convex bodies with fixed volume and with centroid at the origin (the condition on the centroid can be removed for $p = 1$), the L_p geominimal surface area attains its maximum at and only at origin-symmetric ellipsoids. As mentioned in [21], the affine isoperimetric inequality related to the L_p geominimal surface area for $p \geq 1$ is equivalent to the Blaschke-Santaló inequality in the following sense: either of them can be easily obtained from the other. A Santaló style inequality for the L_p geominimal surface area was proved in [46].

This paper is dedicated to further extend Lutwak's L_p geominimal surface area from $p \geq 1$ to all $-n \neq p \in \mathbb{R}$. Hereafter, we use $\tilde{G}_p(K)$ to denote the L_p geominimal surface area of a convex body K (with the origin in its interior). The following Santaló style inequality will be proved.

Theorem 4.1 *Let K be a convex body with centroid (or the Santaló point) at the origin.*
(i). For $p \geq 0$,

$$\tilde{G}_p(K)\tilde{G}_p(K^\circ) \leq [\tilde{G}_p(B_2^n)]^2,$$

with equality if and only if K is an origin-symmetric ellipsoid.
(ii). For $-n \neq p < 0$,

$$\tilde{G}_p(K)\tilde{G}_p(K^\circ) \geq c^n[\tilde{G}_p(B_2^n)]^2,$$

with c the universal constant from the Bourgain-Milman inverse Santaló inequality [4].

We will prove the following affine isoperimetric inequality for $\tilde{G}_p(K)$. See [21, 30, 31] for $p \geq 1$.

Theorem 4.2 *Let K be a convex body with centroid (or the Santaló point) at the origin.*
(i). If $p \geq 0$, then

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \leq \min\left\{ \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}, \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}} \right\}.$$

Equality holds for $p > 0$ if and only if K is an origin-symmetric ellipsoid. For $p = 0$, equality holds trivially for all K.

(ii). If $-n<p<0$, then

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

with equality if and only if K is an origin-symmetric ellipsoid.

(iii). If $p<-n$, then

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \geq \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}},$$

with equality if and only if K is an origin-symmetric ellipsoid. Moreover,

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \geq c^{\frac{np}{n+p}} \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

where c is the constant in the Bourgain-Milman inverse Santaló inequality [4].

We will show the monotonicity of $\tilde{G}_p(\cdot)$. This result will be used to obtain a stronger version of Theorem 4.2.

Theorem 5.2 *Let K be a convex body with the origin in its interior and $p, q \neq 0$.*
(i). If either $-n<q<p$ or $q<p<-n$, one has

$$\left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}} \leq \left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{n+p}{p}}.$$

(ii). If $q<-n<p$, one has

$$\left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}} \geq \left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{n+p}{p}}.$$

This paper is organized as follows. Basic background and notation for convex geometry are given in Section 2. In Section 3, we will provide our definition for the L_p geominimal surface area of K. Such a definition is motivated by Theorem 3.1. We will prove some properties of the L_p geominimal surface area, such as affine invariance. Important inequalities, e.g., the affine isoperimetric inequality (i.e., Theorem 4.2) and a Santaló style inequality (i.e., Theorem 4.1) are established in Section 4. Comparison between the L_p geominimal surface area and the p-surface area is also provided in Section 4. Cyclic inequalities and monotonicity properties for the L_p geominimal surface area are proved in Section 5. A stronger version of the affine isoperimetric inequality is also established in Section 5.

General references for convex geometry are [5, 9, 17, 34].

2 Background and Notation

Our setting is $\mathbb{R}^n$ with the standard inner product $\langle \cdot, \cdot \rangle$, which induces the Euclidian norm $\|\cdot\|$. We use $B_2^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ to denote the unit Euclidean ball in $\mathbb{R}^n$ and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ to denote the unit sphere in $\mathbb{R}^n$ (i.e., the boundary of B_2^n). The volume of B_2^n is denoted by $\omega_n = |B_2^n|$. A *convex body* $K \subset \mathbb{R}^n$ is a convex compact subset of $\mathbb{R}^n$ with nonempty interior. We use $\mathscr{K}$ to denote the set of all convex bodies in $\mathbb{R}^n$. Through this paper, we will concentrate on the subset $\mathscr{K}_0$ of $\mathscr{K}$, which contains all convex bodies with the origin in their interior. The subset $\mathscr{K}_c$ of $\mathscr{K}_0$ contains all convex bodies in $\mathscr{K}_0$ with centroid at the origin. The boundary of K is denoted by ∂K. The *polar body* of K, denoted by K°, is defined as

$$K^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall x \in K\}.$$

For any convex body $K \in \mathscr{K}_0$, its polar body $K^\circ \in \mathscr{K}_0$ as well. The bipolar theorem (see [34]) states that $(K^\circ)^\circ = K$. We write T for an invertible linear transform from $\mathbb{R}^n$ to $\mathbb{R}^n$. The absolute value of the determinant of T is written as $|det(T)|$. A convex body $\mathscr{E} \in \mathscr{K}$ is said to be an *origin-symmetric ellipsoid* if $\mathscr{E} = TB_2^n$ for some invertible linear transform T on $\mathbb{R}^n$.

The *support function* of $K \in \mathscr{K}_0$ at the direction $u \in S^{n-1}$ is defined as

$$h_K(u) = \max_{x \in K} \langle x, u \rangle.$$

The support function $h_K(\cdot)$ determines a convex body uniquely, and can be used to calculate the volume of K°, namely,

$$|K^\circ| = \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_K^n(u)} \, d\sigma(u),$$

where σ is the usual spherical measure on S^{n-1}. More generally, for a set M, we use $|M|$ to denote the Hausdorff content of the appropriate dimension. The Hausdorff metric d_H is a natural metric for $\mathscr{K}_0$. For two convex bodies $K, K' \in \mathscr{K}_0$, the Hausdorff distance between K and K' is

$$d_H(K, K') = \|h_K - h_{K'}\|_\infty = \sup_{u \in S^{n-1}} |h_K(u) - h_{K'}(u)|.$$

For two convex bodies $K, L \in \mathscr{K}_0$ and $\lambda, \eta \geq 0$ (not both zero), the *Minkowski linear combination* $\lambda K + \eta L$ is the convex body with support function $h_{\lambda K + \eta L}$, such that,

$$h_{\lambda K + \eta L}(u) = \lambda h_K(u) + \eta h_L(u), \quad \forall u \in S^{n-1}.$$

The *mixed volume* of K and L, denoted by $V_1(K, L)$, is defined by

$$V_1(K, L) = \lim_{\epsilon \to 0} \frac{|K + \epsilon L| - |K|}{n\epsilon}.$$

For each convex body $K \in \mathscr{K}$, there is a positive Borel measure $S(K, \cdot)$ on S^{n-1} (see [1, 7]), such that, for all convex bodies L,

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u)\, dS(K, u).$$

As noted in [23], the measure $S(K, \cdot)$ has the following simple geometric description: for any Borel subset A of S^{n-1},

$$S(K, A) = |\{x \in \partial K : \ \exists u \in A, \text{ s.t., } H(x, u) \text{ is a support hyperplane of } \partial K \text{ at } x\}|.$$

If the measure $S(K, \cdot)$ is absolute continuous with respect to the spherical measure σ, then by the Radon-Nikodym theorem, there is a function $f_K : S^{n-1} \to \mathbb{R}$, the *curvature function* of K, such that,

$$dS(K, u) = f_K(u) d\sigma(u).$$

Let $\mathscr{F} \subset \mathscr{K}$ be the set of all convex bodies with curvature function. Respectively, we denote $\mathscr{F}_0 = \mathscr{F} \cap \mathscr{K}_0$ and $\mathscr{F}_c = \mathscr{F} \cap \mathscr{K}_c$. We write $K \in \mathscr{F}_0^+$ for those convex bodies $K \in \mathscr{F}_0$ with continuous and positive curvature function $f_K(\cdot)$ on S^{n-1}.

The Minkowski linear combination and the mixed volume $V_1(\cdot, \cdot)$ can be generalized to all $p \geq 1$. For convex bodies $K, L \in \mathscr{K}_0$, and $\lambda, \eta \geq 0$ (not both zeros), the *Firey* p*-sum* $\lambda K +_p \eta L$ for $p \geq 1$ [8] is the convex body with support function defined by

$$\big(h_{\lambda K +_p \eta L}(u)\big)^p = \lambda \big(h_K(u)\big)^p + \eta \big(h_L(u)\big)^p.$$

The p*-mixed volume* of K and L, denoted by $V_p(K, L)$, was defined by Lutwak [22] as

$$\frac{1}{p}\, V_p(K, L) = \lim_{\epsilon \to 0} \frac{|K +_p \epsilon L| - |K|}{n\epsilon}.$$

Lutwak [22] proved that, for $K \in \mathscr{K}_0$, there is a measure $S_p(K, \cdot)$, such that

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u)\, dS_p(K, u), \tag{2.1}$$

holds for all $L \in \mathscr{K}_0$. The measure $S_p(K, \cdot)$ is absolutely continuous with respect to the measure $S(K, \cdot)$. Moreover, for all $p \geq 1$,

$$dS_p(K, u) = h_K^{1-p}(u)\, dS(K, u). \tag{2.2}$$

Therefore, for all $K \in \mathscr{F}_0$ and all $p \geq 1$, one can define the L_p *curvature function* for K, denoted by $f_p(K, u)$, to be (see [21])

$$f_p(K, u) = h_K(u)^{1-p} f_K(u).$$

We will use $\mathscr{K}_s \subset \mathscr{K}_0$ to denote the subset of $\mathscr{K}_0$ containing all convex bodies with Santaló point at the origin. Here, a convex body K is said to have Santaló point at the origin, if its polar body K° has centroid at the origin, i.e., $K \in \mathscr{K}_s \Leftrightarrow K^\circ \in \mathscr{K}_c$. We also denote by $\mathscr{F}_s = \mathscr{F}_0 \cap \mathscr{K}_s$ the subset of $\mathscr{F}_0$ containing all convex bodies with curvature function and Santaló point at the origin.

A subset $L \subset \mathbb{R}^n$ is *star-shaped* (about $x_0 \in L$) if there exists $x_0 \in L$, such that, the line segment from x_0 to any point $x \in L$ is contained in L. Hereafter, we only work on $\mathscr{S}_0$, the set of all n-dimensional *star bodies* about the origin (i.e., compact star-shaped subsets of $\mathbb{R}^n$ about the origin with continuous and positive radial functions). The *radial function* of $L \in \mathscr{S}_0$ at the direction $u \in S^{n-1}$ is given by

$$\rho_L(u) = \max\{\lambda : \lambda u \in L\}.$$

Note that, the volume of L can be calculated by

$$|L| = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^n \, d\sigma(u).$$

For $K \in \mathscr{K}_0$, $L \in \mathscr{S}_0$, and $p \geq 1$, let $V_p(K, L^\circ)$ be

$$V_p(K, L^\circ) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^{-p} \, dS_p(K, u).$$

When $L \in \mathscr{K}_0$, this is consistent with formula (2.1) because $\rho_L(u) h_{L^\circ}(u) = 1$, $\forall u \in S^{n-1}$.

3 L_p geominimal surface area

For $p \geq 1$ and $K \in \mathscr{K}_0$, Lutwak in [21] defined the L_p geominimal surface area of K, denoted by $G_p(K)$, as

$$\omega_n^{p/n} G_p(K) = \inf_{Q \in \mathscr{K}_0} \{n V_p(K, Q) |Q^\circ|^{p/n}\}.$$

We now propose a definition for the L_p geominimal surface area of a convex body $K \in \mathscr{K}_0$, denoted by $\tilde{G}_p(K)$, for all $-n \neq p \in \mathbb{R}$. Before we state our definition, we need some notation. First, notice that the measure $S_p(K, \cdot)$ in (2.2), the p-mixed volume in (2.1), and the L_p curvature function $f_p(K, \cdot)$ were only defined for $p \geq 1$ in [21]. However, we can extend them to all $p \in \mathbb{R}$. For two convex bodies $K, Q \in \mathscr{K}_0$, we define the p-mixed volume of K and Q by

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h_Q(u)^p \, dS_p(K, u), \quad p \in \mathbb{R},$$

where $S_p(K,\cdot)$ is the p-surface area measure of $K \in \mathscr{K}_0$ defined as

$$dS_p(K,u) = h_K^{1-p}(u)\, dS(K,u), \quad p \in \mathbb{R}.$$

We define the L_p *curvature function* (and denoted by $f_p(K,\cdot)$) for $K \in \mathscr{F}_0$ by

$$f_p(K,u) = h_K(u)^{1-p} f_K(u), \quad p \in \mathbb{R}.$$

Note that, if $K \in \mathscr{F}_0$, then

$$dS_p(K,u) = f_p(K,u)\, d\sigma(u), \quad p \in \mathbb{R}.$$

For $K \in \mathscr{K}_0$ and $L \in \mathscr{S}_0$, let $V_p(K,L^\circ)$ be

$$V_p(K,L^\circ) = \frac{1}{n}\int_{S^{n-1}} \rho_L(u)^{-p}\, dS_p(K,u), \quad p \in \mathbb{R}.$$

Definition 3.1 *Let $K \in \mathscr{K}_0$ be a convex body with the origin in its interior.*
(i). For $p \geq 0$, we define the L_p geominimal surface area of K by

$$\tilde{G}_p(K) = \inf_{Q \in \mathscr{K}_0} \left\{ nV_p(K,Q)^{\frac{n}{n+p}}\ |Q^\circ|^{\frac{p}{n+p}} \right\}. \tag{3.3}$$

(ii). For $-n \neq p < 0$, we define the L_p geominimal surface area of K by

$$\tilde{G}_p(K) = \sup_{Q \in \mathscr{K}_0} \left\{ nV_p(K,Q)^{\frac{n}{n+p}}\ |Q^\circ|^{\frac{p}{n+p}} \right\}. \tag{3.4}$$

Remark. Let $p = 0$ and $Q \in \mathscr{K}_0$ be any fixed convex body, one has $V_p(K,Q) = |K|$ and $nV_p(K,Q)^{\frac{n}{n+p}}|Q^\circ|^{\frac{p}{n+p}} = n|K|$. Therefore,

$$\tilde{G}_p(K) = \inf_{Q \in \mathscr{K}_0} \left\{ nV_p(K,Q)^{\frac{n}{n+p}}|Q^\circ|^{\frac{p}{n+p}} \right\} = n|K|.$$

The case $p = -n$ is not covered mainly because $n + p = 0$ appears in the denominators of $\frac{p}{n+p}$ and $\frac{n}{n+p}$. More general L_p geominimal surface area $\tilde{G}_p(K,\mathscr{M})$ can be defined with the infimum or the supremum taking over $\mathscr{M} \subset \mathscr{K}_0$. This paper only deals with $\tilde{G}_p(K) = \tilde{G}_p(K,\mathscr{K}_0)$. For $p \geq 1$, our L_p geominimal surface area is related to Lutwak's by the following formula: for any $K \in \mathscr{K}_0$,

$$[\tilde{G}_p(K)]^{n+p} = (n\omega_n)^p [G_p(K)]^n. \tag{3.5}$$

Proposition 3.1 *Let $K \in \mathscr{K}_0$. For all $p \neq -n$ and all invertible linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, one has*

$$\tilde{G}_p(TK) = |det(T)|^{\frac{n-p}{n+p}} \tilde{G}_p(K).$$

Proof. First, we note that for all $y \in TK$, there exists a unique $x \in K$, such that $y = Tx$ (as T is invertible). The support function of TK can be calculated as follows

$$h_{TK}(v) = \max_{y \in TK} \langle y, v \rangle = \max_{x \in K} \langle Tx, v \rangle = \max_{x \in K} \langle x, T^* v \rangle = \|T^* v\| \max_{x \in K} \langle x, u \rangle = \|T^* v\| h_K(u),$$

where T^* is the transpose of T and $u = \frac{T^* v}{\|T^* v\|}$. Hence,

$$\frac{h_{TQ}(v)}{h_{TK}(v)} = \frac{h_Q(u)}{h_K(u)}.$$

On the other hand, $\frac{1}{n} h_K(u)\, dS(K, u)$ is the volume element of K and hence,

$$h_{TK}(v)\, dS(TK, v) = |det(T)| h_K(u)\, dS(K, u).$$

Therefore, one has

$$\begin{aligned} nV_p(TK, TQ) &= \int_{S^{n-1}} \left(\frac{h_{TQ}(v)}{h_{TK}(v)} \right)^p h_{TK}(v)\, dS(TK, v) \\ &= |det(T)| \int_{S^{n-1}} \left(\frac{h_Q(u)}{h_K(u)} \right)^p h_K(u)\, dS(K, u) \\ &= n|det(T)| V_p(K, Q). \end{aligned}$$

Recall that $(TQ)^\circ = [T^*]^{-1} Q^\circ$. This further implies that, for $p \geq 0$,

$$\begin{aligned} \tilde{G}_p(TK) &= \inf_{Q \in \mathscr{K}_0} \{ n[V_p(TK, TQ)]^{\frac{n}{n+p}} |(TQ)^\circ|^{\frac{p}{n+p}} \} \\ &= |det(T)|^{\frac{n}{n+p}} |det([T^*]^{-1})|^{\frac{p}{n+p}} \inf_{Q \in \mathscr{K}_0} \{ n[V_p(K, Q)]^{\frac{n}{n+p}} |Q^\circ|^{\frac{p}{n+p}} \} \\ &= |det(T)|^{\frac{n-p}{n+p}} \tilde{G}_p(K). \end{aligned}$$

Similarly, for $-n \neq p < 0$, one has

$$\begin{aligned} \tilde{G}_p(TK) &= \sup_{Q \in \mathscr{K}_0} \{ n[V_p(TK, TQ)]^{\frac{n}{n+p}} |(TQ)^\circ|^{\frac{p}{n+p}} \} \\ &= |det(T)|^{\frac{n}{n+p}} |det([T^*]^{-1})|^{\frac{p}{n+p}} \sup_{Q \in \mathscr{K}_0} \{ n[V_p(K, Q)]^{\frac{n}{n+p}} |Q^\circ|^{\frac{p}{n+p}} \} \\ &= |det(T)|^{\frac{n-p}{n+p}} \tilde{G}_p(K). \end{aligned}$$

Remark. The L_p geominimal surface area is invariant under all invertible linear transformations T with $|det(T)| = 1$, i.e., $\tilde{G}_p(K) = \tilde{G}_p(TK)$. Moreover, if T is just a dilation, say $TK = rK$ with $r > 0$, then,

$$\tilde{G}_p(rK) = r^{\frac{n(n-p)}{n+p}} \tilde{G}_p(K).$$

Note that the classical geominimal surface area $\tilde{G}_1(\cdot)$ is also translation invariant. That is, for all $K \in \mathscr{K}_0$ and all $z_0 \in \mathbb{R}^n$, one has

$$\tilde{G}_1(K - z_0) = \tilde{G}_1(K).$$

This can be easily seen from the translation invariance of the surface area measure $dS(K, \cdot)$, which clearly implies, $\forall Q \in \mathscr{K}_0$,

$$nV_1(K, Q) = \int_{S^{n-1}} h_Q \, dS(K, u) = \int_{S^{n-1}} h_Q \, dS(K - z_0, u) = nV_1(K - z_0, Q).$$

However, one *cannot* expect the translation invariance for $\tilde{G}_p(\cdot)$ for $-n \neq p \in \mathbb{R}$ (expect $p = 1$ and $p = 0$). As an example which will be used in Section 5, we show the following result.

Proposition 3.2 *Let $\mathscr{E} \subset \mathbb{R}^n$ be an origin-symmetric ellipsoid, and $z_0 \neq 0$ be any interior point of $\mathscr{E}$.*
(i). For all $p \in (0, 1)$,

$$\tilde{G}_p(\mathscr{E} - z_0) < \tilde{G}_p(\mathscr{E}).$$

(ii). For all $p \in (-n, 0)$,

$$\tilde{G}_p(\mathscr{E} - z_0) > \tilde{G}_p(\mathscr{E}).$$

Proof. Without loss of generality, one only needs to verify Proposition 3.2 for $\mathscr{E} = B_2^n$ due to Proposition 3.1. In this case, the nonzero interior point $z_0 \in B_2^n$ satisfies $0 < \|z_0\| < 1$. Denote by $B_{z_0} = B_2^n - z_0$ the ball with center z_0 and radius 1.

(i). For $p \in (0, 1)$, the function $g(t) = t^{1-p}$ is strictly concave on $t \in (0, \infty)$. By Jensen's inequality, one has,

$$\frac{V_p(B_{z_0}, B_2^n)}{|B_2^n|} = \frac{1}{n|B_2^n|} \int_{S^{n-1}} [h_{B_{z_0}}(u)]^{1-p} \, d\sigma \leq \left(\frac{1}{n|B_2^n|} \int_{S^{n-1}} h_{B_{z_0}}(u) \, d\sigma \right)^{1-p} = \left(\frac{|B_{z_0}|}{|B_2^n|} \right)^{1-p} = 1.$$

Equality holds if and only if $h_{B_{z_0}}(u)$ is a constant, which is not possible as $z_0 \neq 0$. Therefore, as $p \in (0, 1)$,

$$\tilde{G}_p(B_{z_0}) \quad \leq \quad nV_p(B_{z_0}, B_2^n)^{\frac{n}{n+p}} |B_2^n|^{\frac{p}{n+p}} < n|B_2^n| = \tilde{G}_p(B_2^n),$$

where $\tilde{G}_p(B_2^n) = n|B_2^n|$ will be proved in formula (3.15).

(ii). For $p \in (-n, 0)$, the function $g(t) = t^{1-p}$ is strictly convex on $(0, \infty)$. By Jensen's inequality, one has,

$$\frac{V_p(B_{z_0}, B_2^n)}{|B_2^n|} = \frac{1}{n|B_2^n|} \int_{S^{n-1}} [h_{B_{z_0}}(u)]^{1-p} \, d\sigma \geq \left(\frac{1}{n|B_2^n|} \int_{S^{n-1}} h_{B_{z_0}}(u) \, d\sigma \right)^{1-p} = \left(\frac{|B_{z_0}|}{|B_2^n|} \right)^{1-p} = 1.$$

Equality holds if and only if $h_{B_{z_0}}(u)$ is a constant, which is not possible as $z_0 \neq 0$. Therefore, as $p \in (-n, 0)$,

$$\tilde{G}_p(B_{z_0}) \;\; \geq \;\; nV_p(B_{z_0}, B_2^n)^{\frac{n}{n+p}} |B_2^n|^{\frac{p}{n+p}} > n|B_2^n| = \tilde{G}_p(B_2^n),$$

where $\tilde{G}_p(B_2^n) = n|B_2^n|$ is given by formula (3.15).

Our definition of $\tilde{G}_p(K)$ is motivated by Theorem 3.1 regarding a recent extension of the L_p affine surface area. Recall that, Lutwak in [21] defined the L_p affine surface area of $K \in \mathscr{K}_0$ for $p \geq 1$ by

$$as_p(K) = \inf_{L \in \mathscr{S}_0} \left\{ n \; V_p(K, L^\circ)^{\frac{n}{n+p}} \; |L|^{\frac{p}{n+p}} \right\}. \tag{3.6}$$

This definition generalizes the definition of the classical affine surface area (for $p = 1$) in [15]. It was also proved that, for all $K \in \mathscr{F}_0^+$, $as_p(K)$ for $p \geq 1$ defined in (3.6) can be written as an integral over S^{n-1} (see Theorem 4.4 in [21]),

$$as_p(K) = \int_{S^{n-1}} f_p(K, u)^{\frac{n}{n+p}} d\sigma(u). \tag{3.7}$$

Such an integral expression for $as_p(K)$ has been used to extend the L_p affine surface area from $p \geq 1$ to all $-n \neq p \in \mathbb{R}$ (see e.g. [27, 35, 36]).

Theorem 3.1 *Let $K \in \mathscr{F}_0^+$ be a convex body with continuous and positive curvature function and with the origin in its interior.*
(i). For $p \geq 0$, one has

$$as_p(K) = \inf_{L \in \mathscr{S}_0} \left\{ nV_p(K, L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\}.$$

(ii). For $-n \neq p < 0$, one has

$$as_p(K) = \sup_{L \in \mathscr{S}_0} \left\{ nV_p(K, L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\}.$$

Proof. (i). It is clear that if $p = 0$, then $V_p(K, L^\circ) = |K|$, and hence

$$as_0(K) = n|K| = \inf_{L \in \mathscr{S}_0} \left\{ nV_p(K, L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\}.$$

Let $p \in (0, \infty)$. Recall that, for all $K \in \mathscr{F}_0^+$, the L_p affine surface area of K is given by formula (3.7)

$$as_p(K) = \int_{S^{n-1}} f_p(K, u)^{\frac{n}{n+p}} d\sigma(u).$$

Therefore, for all $L \in \mathscr{S}_0$, one has,

$$\begin{aligned} as_p(K) &= \int_{S^{n-1}} [\rho_L^{-p}(u) f_p(K,u)]^{\frac{n}{n+p}} \rho_L(u)^{\frac{pn}{n+p}} d\sigma(u) \\ &\le \left(\int_{S^{n-1}} \rho_L^{-p}(u) f_p(K,u) d\sigma(u)\right)^{\frac{n}{n+p}} \left(\int_{S^{n-1}} \rho_L(u)^n d\sigma(u)\right)^{\frac{p}{n+p}} \\ &= n[V_p(K,L^\circ)]^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}}, \end{aligned} \tag{3.8}$$

where the inequality follows from the Hölder inequality (see [12]) and $p \in (0,\infty)$ (which implies $\frac{n}{n+p} \in (0,1)$). Taking the infimum over $L \in \mathscr{S}_0$, one gets, for all $p \in (0,\infty)$,

$$as_p(K) \le \inf_{L \in \mathscr{S}_0} \left\{ nV_p(K,L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\}.$$

Note that $K \in \mathscr{F}_0^+$ and hence $f_K(u) > 0$ for all $u \in S^{n-1}$. Equality holds in (3.8) if and only if, for some constant $a > 0$,

$$a^{n+p} \rho_L^{-p}(u) f_p(K,u) = \rho_L(u)^n \Leftrightarrow a^{n+p} f_p(K,u) = \rho_L(u)^{n+p}, \quad \forall u \in S^{n-1}.$$

Thus, one can define the star body $L_0 \in \mathscr{S}_0$ by the radial function, which is clearly continuous and positive,

$$\rho_{L_0}(u) = a[f_p(K,u)]^{\frac{1}{n+p}} > 0, \quad \forall u \in S^{n-1},$$

and hence

$$as_p(K) = n[V_p(K,L_0^\circ)]^{\frac{n}{n+p}} |L_0|^{\frac{p}{n+p}} = \inf_{L \in \mathscr{S}_0} \{ n[V_p(K,L^\circ)]^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \}.$$

(ii). Let $-n \neq p \in (-\infty, 0)$. Similar to (3.8), for all $L \in \mathscr{S}_0$, one has,

$$\begin{aligned} as_p(K) &= \int_{S^{n-1}} [\rho_L^{-p}(u) f_p(K,u)]^{\frac{n}{n+p}} \rho_L(u)^{\frac{pn}{n+p}} d\sigma(u) \\ &\ge \left(\int_{S^{n-1}} \rho_L^{-p}(u) f_p(K,u) d\sigma(u)\right)^{\frac{n}{n+p}} \left(\int_{S^{n-1}} \rho_L(u)^n d\sigma(u)\right)^{\frac{p}{n+p}} \\ &= n[V_p(K,L^\circ)]^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}}, \end{aligned} \tag{3.9}$$

where the inequality follows from the Hölder inequality (see [12]) and $-n \neq p \in (-\infty, 0)$ (which implies either $\frac{n}{n+p} > 1$ or $\frac{n}{n+p} < 0$). Taking the supremum over $L \in \mathscr{S}_0$, one gets, for all $-n \neq p \in (-\infty, 0)$,

$$as_p(K) \ge \sup_{L \in \mathscr{S}_0} \left\{ nV_p(K,L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\}.$$

Note that $K \in \mathscr{F}_0^+$ and hence $f_K(u) > 0$ for all $u \in S^{n-1}$. Equality holds in (3.9) if and only if, for some constant $a > 0$,

$$a^{n+p}\rho_L^{-p}(u)f_p(K,u) = \rho_L(u)^n \Leftrightarrow a^{n+p}f_p(K,u) = \rho_L(u)^{n+p}, \quad \forall u \in S^{n-1}.$$

Thus, one can define the star body $L_0 \in \mathscr{S}_0$ by the radial function, which is clearly continuous and positive,

$$\rho_{L_0}(u) = a[f_p(K,u)]^{\frac{1}{n+p}} > 0, \quad \forall u \in S^{n-1},$$

and hence

$$as_p(K) = n[V_p(K, L_0^\circ)]^{\frac{n}{n+p}}|L_0|^{\frac{p}{n+p}} = \sup_{L\in\mathscr{S}_0}\{n[V_p(K,L^\circ)]^{\frac{n}{n+p}}|L|^{\frac{p}{n+p}}\}.$$

Remark. The proof of Theorem 3.1 implies that, for all $K \in \mathscr{F}_0^+$ and all $-n \neq p \in \mathbb{R}$,

$$[as_p(K)]^{n+p} = n^{n+p}\omega_n^n|\Lambda_p K|^p,$$

where $\Lambda_p K \in \mathscr{S}_0$ is the p-curvature image of K defined by (for $p \geq 1$, see [21])

$$f_p(K,u) = \frac{\omega_n}{|\Lambda_p K|}[\rho_{\Lambda_p K}(u)]^{n+p}. \tag{3.10}$$

Motivated by Theorem 3.1, one may define the L_p affine surface area for all $-n \neq p \in \mathbb{R}$ as follows.

Definition 3.2 *Let $K \in \mathscr{K}_0$ be a convex body with the origin in its interior.*
(i). For $p = 0$, let $as_p(K) = n|K|$. For $p > 0$, let

$$as_p(K) = \inf_{L\in\mathscr{S}_0}\left\{nV_p(K,L^\circ)^{\frac{n}{n+p}}|L|^{\frac{p}{n+p}}\right\}. \tag{3.11}$$

(ii). For $-n \neq p < 0$, let

$$as_p(K) = \sup_{L\in\mathscr{S}_0}\left\{nV_p(K,L^\circ)^{\frac{n}{n+p}}|L|^{\frac{p}{n+p}}\right\}. \tag{3.12}$$

One can easily check: for all $K \in \mathscr{K}_0$, $as_p(K) \leq \tilde{G}_p(K)$ for $p \in (0,\infty)$; while $as_p(K) \geq \tilde{G}_p(K)$ for $-n \neq p \in (-\infty, 0)$. To see this, by formula (3.11), one has for $p > 0$,

$$\begin{aligned} as_p(K) &= \inf_{L\in\mathscr{S}_0}\left\{nV_p(K,L^\circ)^{\frac{n}{n+p}}|L|^{\frac{p}{n+p}}\right\} \\ &\leq \inf_{L\in\mathscr{K}_0}\left\{nV_p(K,L^\circ)^{\frac{n}{n+p}}|L|^{\frac{p}{n+p}}\right\} = \tilde{G}_p(K), \end{aligned} \tag{3.13}$$

where the inequality is due to $\mathscr{K}_0 \subset \mathscr{S}_0$. By formula (3.12), one has for $-n \neq p < 0$,

$$\begin{aligned} as_p(K) &= \sup_{L\in\mathscr{S}_0} \left\{ nV_p(K, L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\} \\ &\geq \sup_{L\in\mathscr{K}_0} \left\{ nV_p(K, L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\} = \tilde{G}_p(K), \end{aligned} \tag{3.14}$$

where the inequality is due to $\mathscr{K}_0 \subset \mathscr{S}_0$.

Let $-n \neq p \in \mathbb{R}$. Define the subset $\mathscr{V}_p$ of $\mathscr{F}_0^+$ as

$$\mathscr{V}_p = \{K \in \mathscr{F}_0^+ : \ \exists Q \in \mathscr{K}_0 \ \ with \ \ f_p(K, u) = h_Q(u)^{-(n+p)}, \ \ \forall u \in S^{n-1}\}.$$

Clearly, $\mathscr{V}_p \neq \emptyset$ as $B_2^n \in \mathscr{V}_p$. The following proposition describes the relation between $\Lambda_p K$ and $\mathscr{V}_p$. See [21] for $p \geq 1$.

Proposition 3.3 *Let $-n \neq p \in \mathbb{R}$ and $K \in \mathscr{F}_0^+$, then*

$$K \in \mathscr{V}_p \quad \textit{if and only if} \quad \Lambda_p K \in \mathscr{K}_0.$$

Proof. Let $-n \neq p \in \mathbb{R}$, and $K \in \mathscr{F}_0^+$, then

$$\begin{aligned} K \in \mathscr{V}_p \ &\Leftrightarrow \ [f_p(K,u)]^{\frac{-1}{n+p}} = h_Q(u), \ \ \exists Q \in \mathscr{K}_0, \forall u \in S^{n-1} \\ &\Leftrightarrow \ [f_p(K,u)]^{\frac{1}{n+p}} = \rho_{Q^\circ}(u), \ \ \exists Q \in \mathscr{K}_0, \forall u \in S^{n-1} \\ &\Leftrightarrow \ \left(\frac{\omega_n}{|\Lambda_p K|}\right)^{\frac{1}{n+p}} \rho_{\Lambda_p K}(u) = \rho_{Q^\circ}(u), \ \ \exists Q \in \mathscr{K}_0, \forall u \in S^{n-1} \\ &\Leftrightarrow \ \left(\frac{\omega_n}{|\Lambda_p K|}\right)^{\frac{1}{n+p}} \Lambda_p K = Q^\circ \in \mathscr{K}_0. \end{aligned}$$

Proposition 3.4 *Let $-n \neq p \in \mathbb{R}$ and $K \in \mathscr{V}_p$, then $\tilde{G}_p(K) = as_p(K)$.*

Proof. The proposition holds for $p = 0$ as $\tilde{G}_0(K) = as_0(K) = n|K|$ for all $K \in \mathscr{K}_0$. Assume $p > 0$. From Proposition 3.3, $K \in \mathscr{V}_p$ implies $\Lambda_p K \in \mathscr{K}_0$. Therefore, formula (3.10) and inequality (3.13) imply

$$\begin{aligned} \tilde{G}_p(K) &\geq as_p(K) = n\omega_n^{\frac{n}{n+p}} |\Lambda_p K|^{\frac{p}{n+p}} \\ &= nV_p(K, (\Lambda_p K)^\circ)^{\frac{n}{n+p}} |\Lambda_p K|^{\frac{p}{n+p}} \\ &\geq \inf_{Q\in\mathscr{K}_0} \{nV_p(K, Q^\circ)^{\frac{n}{n+p}} |Q|^{\frac{p}{n+p}}\} \\ &= \tilde{G}_p(K). \end{aligned}$$

Hence, $\tilde{G}_p(K) = as_p(K)$ for all $K \in \mathscr{V}_p$.

For $-n \neq p < 0$, formula (3.10) and inequality (3.14) imply

$$\begin{aligned}\tilde{G}_p(K) &\leq as_p(K) = nV_p(K, (\Lambda_p K)^\circ)^{\frac{n}{n+p}} |\Lambda_p K|^{\frac{p}{n+p}} \\ &\leq \sup_{Q \in \mathscr{K}_0} \{nV_p(K, Q^\circ)^{\frac{n}{n+p}} |Q|^{\frac{p}{n+p}}\} \\ &= \tilde{G}_p(K).\end{aligned}$$

Hence, $\tilde{G}_p(K) = as_p(K)$ for all $K \in \mathscr{V}_p$.

Remark. Recall that $B_2^n \in \mathscr{V}_p$. Thus,

$$\tilde{G}_p(B_2^n) = as_p(B_2^n) = n\omega_n. \tag{3.15}$$

Hence, for all origin-symmetric ellipsoids $\mathscr{E} = TB_2^n$ with T an invertible linear transform on $\mathbb{R}^n$, one has, by Proposition 3.1,

$$\tilde{G}_p(\mathscr{E}) = \tilde{G}_p(TB_2^n) = |det(T)|^{\frac{n-p}{n+p}} \tilde{G}_p(B_2^n) = |det(T)|^{\frac{n-p}{n+p}} n\omega_n.$$

4 Affine isoperimetric and Santaló style inequalities

This section is mainly dedicated to the affine isoperimetric inequality and a Santaló style inequality for the L_p geominimal surface area.

Proposition 4.1 *Let $K \in \mathscr{K}_0$ be a convex body with the origin in its interior.*
(i). For $p \geq 0$, one has

$$\tilde{G}_p(K) \leq n|K|^{\frac{n}{n+p}} |K^\circ|^{\frac{p}{n+p}}.$$

(ii). For $-n \neq p < 0$, one has

$$\tilde{G}_p(K) \geq n|K|^{\frac{n}{n+p}} |K^\circ|^{\frac{p}{n+p}}.$$

Proof. Note that $V_p(K, K) = |K|$ for all $K \in \mathscr{K}_0$ and for all $-n \neq p \in \mathbb{R}$.

(i). The case $p = 0$ is clear (and in fact "=" always holds). For $p > 0$, by formula (3.3) and $K \in \mathscr{K}_0$, one has

$$\tilde{G}_p(K) = \inf_{Q \in \mathscr{K}_0} \left\{nV_p(K, Q)^{\frac{n}{n+p}} |Q^\circ|^{\frac{p}{n+p}}\right\} \leq n|K|^{\frac{n}{n+p}} |K^\circ|^{\frac{p}{n+p}}.$$

(ii). For $-n \neq p < 0$, by formula (3.4) and $K \in \mathscr{K}_0$, one gets,

$$\tilde{G}_p(K) = \sup_{Q \in \mathscr{K}_0} \left\{nV_p(K, Q)^{\frac{n}{n+p}} |Q^\circ|^{\frac{p}{n+p}}\right\} \geq n|K|^{\frac{n}{n+p}} |K^\circ|^{\frac{p}{n+p}}.$$

Proposition 4.2 *Let $K \in \mathscr{K}_0$ be a convex body with the origin in its interior.*
(i). For $p \geq 0$, one has
$$\tilde{G}_p(K)\tilde{G}_p(K^\circ) \leq n^2|K||K^\circ|.$$
(ii). For $-n \neq p < 0$, one has
$$\tilde{G}_p(K)\tilde{G}_p(K^\circ) \geq n^2|K||K^\circ|.$$

Proof. (i). For $p \geq 0$, using Proposition 4.1 for both K and K°, one gets
$$\tilde{G}_p(K)\tilde{G}_p(K^\circ) \leq (n|K|^{\frac{n}{n+p}}|K^\circ|^{\frac{p}{n+p}})(n|K|^{\frac{p}{n+p}}|K^\circ|^{\frac{n}{n+p}}) = n^2|K||K^\circ|.$$
(ii). For $-n \neq p < 0$, using Proposition 4.1 for both K and K°, one gets
$$\tilde{G}_p(K)\tilde{G}_p(K^\circ) \geq (n|K|^{\frac{n}{n+p}}|K^\circ|^{\frac{p}{n+p}})(n|K|^{\frac{p}{n+p}}|K^\circ|^{\frac{n}{n+p}}) = n^2|K||K^\circ|.$$

The Blaschke-Santaló inequality provides a precise upper bound for $M(K) = |K||K^\circ|$ with $K \in \mathscr{K}_c$ (or $K \in \mathscr{K}_s$). That is, for all $K \in \mathscr{K}_c$ (or $K \in \mathscr{K}_s$),
$$M(K) \leq M(B_2^n) = \omega_n^2,$$
with equality if and only if K is an origin-symmetric ellipsoid. Finding the precise lower bound for $M(K)$ is still an open problem and is known as the Mahler conjecture: the precise lower bound for $M(K)$ is conjectured to be obtained by the cube among all origin-symmetric convex bodies (i.e., $K = -K$), and by the simplex among all convex bodies $K \in \mathscr{K}_c$ (or $K \in \mathscr{K}_s$). A remarkable result by Bourgain and Milman [4] states that, there is a universal constant $c > 0$ (independent of n and K), such that, for all $K \in \mathscr{K}_c$ (or $K \in \mathscr{K}_s$),
$$M(K) \geq c^n M(B_2^n) = c^n \omega_n^2.$$
Nice estimates for the constant c can be found in [14, 28].

We now prove the following Santaló style inequality for the L_p geominimal surface area $\tilde{G}_p$. See [46] for $p \geq 1$.

Theorem 4.1 *Let $K \in \mathscr{K}_c$ or $K \in \mathscr{K}_s$.*
(i). For $p \geq 0$,
$$\tilde{G}_p(K)\tilde{G}_p(K^\circ) \leq [\tilde{G}_p(B_2^n)]^2,$$
with equality if and only if K is an origin-symmetric ellipsoid.
(ii). For $-n \neq p < 0$,
$$\tilde{G}_p(K)\tilde{G}_p(K^\circ) \geq c^n[\tilde{G}_p(B_2^n)]^2,$$
with c the universal constant from the Bourgain-Milman inverse Santaló inequality [4].

Proof. (i). Recall that $\tilde{G}_p(B_2^n) = n|B_2^n|$. Proposition 4.2 implies that

$$\tilde{G}_p(K)\tilde{G}_p(K^\circ) \le n^2|K||K^\circ| \le n^2\omega_n^2 = [\tilde{G}_p(B_2^n)]^2, \tag{4.16}$$

where the second inequality is from the Blaschke-Santaló inequality. Clearly, Proposition 3.1 implies that equality holds in inequality (4.16) if K is an origin-symmetric ellipsoid. On the other hand, the equality holds in inequality (4.16) only if the equality holds in the Blaschke-Santaló inequality, that is, K has to be an origin-symmetric ellipsoid.

(ii). Proposition 4.2 implies

$$\tilde{G}_p(K)\tilde{G}_p(K^\circ) \ge n^2|K||K^\circ| \ge c^n n^2\omega_n^2 = c^n[\tilde{G}_p(B_2^n)]^2,$$

where the second inequality is from the Bourgain-Milman inverse Santaló inequality.

Remark. Note that the Bourgain-Milman inverse Santaló inequality still holds true for all $K \in \mathscr{K}_0$. This is because $M(K) \ge M(K - z_0) \ge c^n\omega_n^2$, where z_0 is the centroid of K (and then $K - z_0 \in \mathscr{K}_c$). Hence, part (ii) still holds for all $K \in \mathscr{K}_0$.

The related affine isoperimetric inequality for the L_p geominimal surface area $\tilde{G}_p(K)$ is stated in the following theorem.

Theorem 4.2 *Let $K \in \mathscr{K}_c$ or $K \in \mathscr{K}_s$.*

(i). If $p \ge 0$, then

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \le \min\left\{\left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},\ \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}}\right\}.$$

Equality holds for $p > 0$ if and only if K is an origin-symmetric ellipsoid. For $p = 0$, equality holds trivially for all K.

(ii). If $-n < p < 0$, then

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \ge \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

with equality if and only if K is an origin-symmetric ellipsoid.

(iii). If $p < -n$, then

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \ge \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}},$$

with equality if and only if K is an origin-symmetric ellipsoid. Moreover,

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \ge c^{\frac{np}{n+p}}\left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

where c is the constant in the Bourgain-Milman inverse Santaló inequality [4].

Proof. (i). The case $p = 0$ is trivial, and we only prove the case $p > 0$. Combining Proposition 4.1, the Blaschke-Santaló inequality, and $\tilde{G}_p(B_2^n) = n|B_2^n|$, one obtains

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \le \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n}{n+p}} \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p}{n+p}} = \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \left(\frac{M(K)}{M(B_2^n)}\right)^{\frac{p}{n+p}} \le \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}.$$

Similarly,

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \le \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n}{n+p}} \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p}{n+p}} = \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}} \left(\frac{M(K)}{M(B_2^n)}\right)^{\frac{n}{n+p}} \le \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}}.$$

Clearly, equality holds if K is an origin-symmetric ellipsoid. On the other hand, equality holds in the above inequalities only if equality holds in the Blaschke-Santaló inequality, that is, K has to be an origin-symmetric ellipsoid.

(ii). Let $-n < p < 0$. Combining Proposition 4.1 and $\tilde{G}_p(B_2^n) = n|B_2^n|$, one obtains

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \ge \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n}{n+p}} \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p}{n+p}} = \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \left(\frac{M(K)}{M(B_2^n)}\right)^{\frac{p}{n+p}} \ge \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

where the last inequality follows from the Blaschke-Santaló inequality and $\frac{p}{n+p} < 0$. Clearly, equality holds if K is an origin-symmetric ellipsoid. On the other hand, equality holds in the above inequalities only if equality holds in the Blaschke-Santaló inequality, that is, K has to be an origin-symmetric ellipsoid.

(iii). Let $p < -n$. Combining Proposition 4.1 and $\tilde{G}_p(B_2^n) = n|B_2^n|$, one obtains

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \ge \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n}{n+p}} \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p}{n+p}} = \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}} \left(\frac{M(K)}{M(B_2^n)}\right)^{\frac{n}{n+p}} \ge \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}},$$

where the last inequality follows from the Blaschke-Santaló inequality and $\frac{n}{n+p} < 0$. Clearly, equality holds if K is an origin-symmetric ellipsoid. On the other hand, equality holds in the above inequalities only if equality holds in the Blaschke-Santaló inequality, that is, K has to be an origin-symmetric ellipsoid.

Moreover, from $\frac{p}{n+p} > 0$ and the Bourgain-Milman inverse Santaló inequality, one has

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \ge \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n}{n+p}} \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p}{n+p}} = \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \left(\frac{M(K)}{M(B_2^n)}\right)^{\frac{p}{n+p}} \ge c^{\frac{np}{n+p}} \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}.$$

Remark. If we assume that $|K| = |B_2^n|$, then part (i) of Theorem 4.2 implies that $\tilde{G}_p(K) \le \tilde{G}_p(B_2^n)$. Let B_K be the origin-symmetric Euclidean ball with the same volume as K, and then the radius of B_K is $r = \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}}$. Proposition 3.1 implies that

$$\tilde{G}_p(B_K) = \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \tilde{G}_p(B_2^n).$$

Therefore, part (i) of Theorem 4.2 implies

$$\tilde{G}_p(K) \le \tilde{G}_p(B_K), \quad p > 0,$$

with equality if and only if K is an origin-symmetric ellipsoid. That is, *among all convex bodies in $\mathscr{K}_c$ (or $\mathscr{K}_s$) with fixed volume, the L_p geominimal surface area for $p > 0$ attains the maximum at and only at origin-symmetric ellipsoids.* Similarly, part (ii) of Theorem 4.2 can be rewritten as

$$\tilde{G}_p(K) \ge \tilde{G}_p(B_K), \quad p \in (-n, 0),$$

with equality if and only if K is an origin-symmetric ellipsoid. That is, *among all convex bodies in $\mathscr{K}_c$ (or $\mathscr{K}_s$) with fixed volume, the L_p geominimal surface area for $-n < p < 0$ attains the minimum at and only at origin-symmetric ellipsoids.* From part (iii) of Theorem 4.2, one gets, for $K \in \mathscr{K}_c$ (or $K \in \mathscr{K}_s$),

$$\tilde{G}_p(K)\tilde{G}_p(B_{K^\circ}) \ge [\tilde{G}_p(B_2^n)]^2, \quad p < -n,$$

with equality if and only if K is an origin-symmetric ellipsoid. From part (iii) of Theorem 4.2 and the remark after Theorem 4.1, one sees also that

$$\tilde{G}_p(K) \ge c^{\frac{np}{n+p}} \tilde{G}_p(B_K), \quad p < -n,$$

for all $K \in \mathscr{K}_0$. For $p = 1$, this is the classical result due to Petty [30, 31], where the condition $K \in \mathscr{K}_c$ or $K \in \mathscr{K}_s$ can be replaced by $K \in \mathscr{K}_0$ due to the translation invariant property of the classical geominimal surface area. The case $p > 1$ is due to Lutwak [21]. A stronger version of Theorem 4.2 for $p \in (-n, 1)$, where the condition on centroid or on Santaló point is removed, will be proved in Section 5.

From Lemma 2.1 in [30] (see page 79) and formula (3.5), one has $\tilde{G}_1(K_1) \le \tilde{G}_1(K_2)$ for any two convex bodies $K_1, K_2 \in \mathscr{K}_0$ such that $K_1 \subset K_2$. For the L_p geominimal surface area, we have the following similar result.

Corollary 4.1 *Let $\mathscr{E}$ be an origin-symmetric ellipsoid and $K \in \mathscr{K}_c$ (or $K \in \mathscr{K}_s$).*
(i). For $p \in (0, n)$ and $K \subset \mathscr{E}$, one has

$$\tilde{G}_p(K) \le \tilde{G}_p(\mathscr{E}),$$

with equality if and only if $K = \mathscr{E}$.
(ii). For $p \in (n, \infty)$ and $\mathscr{E} \subset K$, one has

$$\tilde{G}_p(K) \le \tilde{G}_p(\mathscr{E}),$$

with equality if and only if $K = \mathscr{E}$.
(iii). For $p \in (-n, 0)$ and $\mathscr{E} \subset K$, one has

$$\tilde{G}_p(K) \ge \tilde{G}_p(\mathscr{E}),$$

with equality if and only if $K = \mathscr{E}$.
(iv). For $p < -n$ and $K \subset \mathscr{E}$, one has

$$\tilde{G}_p(K) \geq \tilde{G}_p(\mathscr{E}),$$

with equality if and only if $K = \mathscr{E}$.

Proof. (i). Let $p \in (0, n)$ and hence $\frac{n-p}{n+p} > 0$. Then it follows from $K \subset \mathscr{E}$ that

$$\left(\frac{|K|}{|\mathscr{E}|}\right)^{\frac{n-p}{n+p}} \leq 1.$$

Combined with Theorem 4.2 and Proposition 3.1, one has

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(\mathscr{E})} \leq \left(\frac{|K|}{|\mathscr{E}|}\right)^{\frac{n-p}{n+p}} \leq 1 \quad \Rightarrow \quad \tilde{G}_p(K) \leq \tilde{G}_p(\mathscr{E}).$$

Clearly equality holds if $K = \mathscr{E}$. On the other hand, if $K \subsetneq \mathscr{E}$, then $|K| < |\mathscr{E}|$ and hence $\tilde{G}_p(K) < \tilde{G}_p(\mathscr{E})$.
(ii). Let $p \in (n, \infty)$ and hence $\frac{n-p}{n+p} < 0$. Then it follows from $\mathscr{E} \subset K$ that

$$\left(\frac{|K|}{|\mathscr{E}|}\right)^{\frac{n-p}{n+p}} \leq 1.$$

Combined with Theorem 4.2 and Proposition 3.1, one has

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(\mathscr{E})} \leq \left(\frac{|K|}{|\mathscr{E}|}\right)^{\frac{n-p}{n+p}} \leq 1 \quad \Rightarrow \quad \tilde{G}_p(K) \leq \tilde{G}_p(\mathscr{E}).$$

Clearly equality holds if $K = \mathscr{E}$. On the other hand, if $\mathscr{E} \subsetneq K$, then $|\mathscr{E}| < |K|$ and hence $\tilde{G}_p(K) < \tilde{G}_p(\mathscr{E})$.
(iii). Let $p \in (-n, 0)$ and hence $\frac{n-p}{n+p} > 0$. Then it follows from $\mathscr{E} \subset K$ that

$$\left(\frac{|K|}{|\mathscr{E}|}\right)^{\frac{n-p}{n+p}} \geq 1.$$

Combined with Theorem 4.2 and Proposition 3.1, one has

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(\mathscr{E})} \geq \left(\frac{|K|}{|\mathscr{E}|}\right)^{\frac{n-p}{n+p}} \geq 1 \quad \Rightarrow \quad \tilde{G}_p(K) \geq \tilde{G}_p(\mathscr{E}).$$

Clearly equality holds if $K=\mathscr{E}$. On the other hand, if $\mathscr{E}\subsetneq K$, then $|\mathscr{E}|<|K|$ and hence $\tilde{G}_p(K)>\tilde{G}_p(\mathscr{E})$.

(iv). Let $p<-n$ and hence $\frac{p-n}{n+p}>0$. Then it follows from $K\subset\mathscr{E}$ that

$$\mathscr{E}^\circ\subset K^\circ \quad \Rightarrow \quad \left(\frac{|K^\circ|}{|\mathscr{E}^\circ|}\right)^{\frac{p-n}{n+p}}\geq 1.$$

Combined with Theorem 4.2 and Proposition 3.1, one has

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(\mathscr{E})}\geq\left(\frac{|K^\circ|}{|\mathscr{E}^\circ|}\right)^{\frac{p-n}{n+p}}\geq 1 \quad \Rightarrow \quad \tilde{G}_p(K)\geq\tilde{G}_p(\mathscr{E}).$$

Clearly equality holds if $K=\mathscr{E}$. On the other hand, if $K\subsetneq\mathscr{E}$, then $\mathscr{E}^\circ\subsetneq K^\circ$, which implies $|\mathscr{E}^\circ|<|K^\circ|$ and hence $\tilde{G}_p(K)>\tilde{G}_p(\mathscr{E})$.

The following result compares the L_p geominimal surface area with the p surface area $S_p(K)=nV_p(K,B_2^n)$. Note that $S_p(B_2^n)=\tilde{G}_p(B_2^n)=n|B_2^n|=n\omega_n$.

Proposition 4.3 *Let $K\in\mathscr{K}_0$ be a convex body with the origin in its interior.*
(i). For $p\geq 0$, one has

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)}\leq\left(\frac{S_p(K)}{S_p(B_2^n)}\right)^{\frac{n}{n+p}}.$$

(ii). For $-n\neq p<0$, one has

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)}\geq\left(\frac{S_p(K)}{S_p(B_2^n)}\right)^{\frac{n}{n+p}}.$$

Proof. (i). For $p\geq 0$, by formula (3.3), one gets

$$\begin{aligned}\tilde{G}_p(K) &= \inf_{Q\in\mathscr{K}_0}\left\{n\ V_p(K,Q)^{\frac{n}{n+p}}\ |Q^\circ|^{\frac{p}{n+p}}\right\}\\ &\leq n\ [V_p(K,B_2^n)]^{\frac{n}{n+p}}\ |B_2^n|^{\frac{p}{n+p}}\\ &= (n\omega_n)^{\frac{p}{n+p}}[S_p(K)]^{\frac{n}{n+p}}.\end{aligned}$$

Dividing both sides by $n\omega_n=\tilde{G}_p(B_2^n)=S_p(B_2^n)$, one gets the desired inequality.

(ii). For $-n\neq p<0$, by formula (3.4), one gets

$$\begin{aligned}\tilde{G}_p(K) &= \sup_{Q\in\mathscr{K}_0}\left\{n\ V_p(K,Q)^{\frac{n}{n+p}}\ |Q^\circ|^{\frac{p}{n+p}}\right\}\\ &\geq n\ [V_p(K,B_2^n)]^{\frac{n}{n+p}}\ |B_2^n|^{\frac{p}{n+p}}\\ &= (n\omega_n)^{\frac{p}{n+p}}[S_p(K)]^{\frac{n}{n+p}}.\end{aligned}$$

Dividing both sides by $n\omega_n=\tilde{G}_p(B_2^n)=S_p(B_2^n)$, one gets the desired inequality.

5 Cyclic inequalities and monotonicity of $\tilde{G}_p(\cdot)$

Theorem 5.1 *Let $K \in \mathscr{K}_0$ be a convex body with the origin in its interior.*
(i). If $-n < t < 0 < r < s$ or $-n < s < 0 < r < t$, then

$$\tilde{G}_r(K) \leq \tilde{G}_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \tilde{G}_s(K)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}.$$

(ii). If $-n < t < r < s < 0$ or $-n < s < r < t < 0$, then

$$\tilde{G}_r(K) \leq \tilde{G}_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \tilde{G}_s(K)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}.$$

(iii). If $t < r < -n < s < 0$ or $s < r < -n < t < 0$, then

$$\tilde{G}_r(K) \geq \tilde{G}_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \tilde{G}_s(K)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}.$$

Proof. Let $K, Q \in \mathscr{K}_0$. We claim that, for all $r, s, t \in \mathbb{R}$ satisfying $0 < \frac{t-r}{t-s} < 1$,

$$nV_r(K,Q) \leq [nV_s(K,Q)]^{\frac{t-r}{t-s}} \; [nV_t(K,Q)]^{\frac{r-s}{t-s}}. \tag{5.17}$$

In fact, by Hölder's inequality (see [12]) and $0 < \frac{t-r}{t-s} < 1$,

$$\begin{aligned} nV_r(K,Q) &= \int_{S^{n-1}} h_Q(u)^r h_K(u)^{1-r}\, dS(K,u) \\ &= \int_{S^{n-1}} [h_Q(u)^s h_K(u)^{1-s}]^{\frac{t-r}{t-s}} [h_Q(u)^t h_K(u)^{1-t}]^{\frac{r-s}{t-s}}\, dS(K,u) \\ &\leq \left(\int_{S^{n-1}} h_Q(u)^s h_K(u)^{1-s}\, dS(K,u)\right)^{\frac{t-r}{t-s}} \left(\int_{S^{n-1}} h_Q(u)^t h_K(u)^{1-t}\, dS(K,u)\right)^{\frac{r-s}{t-s}} \\ &= [nV_s(K,Q)]^{\frac{t-r}{t-s}} \; [nV_t(K,Q)]^{\frac{r-s}{t-s}}. \end{aligned}$$

(i). Suppose that $-n < t < 0 < r < s$, which clearly implies $0 < \frac{t-r}{t-s} < 1$. First, we have

$$\begin{aligned} \tilde{G}_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} &= \left\{ \sup_{Q_1 \in \mathscr{K}_0} [nV_t(K,Q_1)^{\frac{n}{n+t}} |Q_1^\circ|^{\frac{t}{n+t}}] \right\}^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \\ &= \sup_{Q_1 \in \mathscr{K}_0} \left\{ [nV_t(K,Q_1)^{\frac{n}{n+t}} |Q_1^\circ|^{\frac{t}{n+t}}]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \right\}, \end{aligned}$$

due to $\frac{(r-s)(n+t)}{(t-s)(n+r)} > 0$. Similarly, as $\frac{(t-r)(n+s)}{(t-s)(n+r)} > 0$, one has

$$\tilde{G}_s(K)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} = \inf_{Q \in \mathscr{K}_0} \left\{ [nV_s(K,Q)^{\frac{n}{n+s}} |Q^\circ|^{\frac{s}{n+s}}]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} \right\}.$$

By inequality (5.17) and $\frac{n}{n+r} > 0$, one has, $\forall Q \in \mathscr{K}_0$,

$$\begin{aligned}\tilde{G}_r(K) &= \inf_{Q_0\in\mathscr{K}_0}\{nV_r(K,Q_0)^{\frac{n}{n+r}}\ |Q_0^\circ|^{\frac{r}{n+r}}\} \le nV_r(K,Q)^{\frac{n}{n+r}}\ |Q^\circ|^{\frac{r}{n+r}} \\ &\le \Big\{\big[nV_s(K,Q)^{\frac{n}{n+s}}|Q^\circ|^{\frac{s}{n+s}}\big]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\Big\}\ \Big\{\big[nV_t(K,Q)^{\frac{n}{n+t}}|Q^\circ|^{\frac{t}{n+t}}\big]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\Big\} \\ &\le \Big\{\big[nV_s(K,Q)^{\frac{n}{n+s}}|Q^\circ|^{\frac{s}{n+s}}\big]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\Big\}\sup_{Q_1\in\mathscr{K}_0}\Big\{\big[nV_t(K,Q_1)^{\frac{n}{n+t}}|Q_1^\circ|^{\frac{t}{n+t}}\big]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\Big\} \\ &= \tilde{G}_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\Big\{\big[nV_s(K,Q)^{\frac{n}{n+s}}|Q^\circ|^{\frac{s}{n+s}}\big]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\Big\}. \end{aligned} \tag{5.18}$$

Taking the infimum over $Q \in \mathscr{K}_0$ in inequality (5.18), one gets

$$\begin{aligned}\tilde{G}_r(K) &\le \tilde{G}_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\inf_{Q\in\mathscr{K}_0}\Big\{\big[nV_s(K,Q)^{\frac{n}{n+s}}|Q^\circ|^{\frac{s}{n+s}}\big]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\Big\} \\ &= \tilde{G}_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\tilde{G}_s(K)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}.\end{aligned}$$

The case $-n < s < 0 < r < t$ follows immediately by switching the roles of t and s.

(ii). Suppose that $-n < t < r < s < 0$, which clearly implies $0 < \frac{t-r}{t-s} < 1$. Then

$$\begin{aligned}\tilde{G}_s(K)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} &= \Big\{\sup_{Q\in\mathscr{K}_0}\big[nV_s(K,Q)^{\frac{n}{n+s}}|Q^\circ|^{\frac{s}{n+s}}\big]\Big\}^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} \\ &= \sup_{Q\in\mathscr{K}_0}\Big\{\big[nV_s(K,Q)^{\frac{n}{n+s}}|Q^\circ|^{\frac{s}{n+s}}\big]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\Big\},\end{aligned}$$

due to $\frac{(t-r)(n+s)}{(t-s)(n+r)} > 0$. Similarly, by $\frac{(r-s)(n+t)}{(t-s)(n+r)} > 0$,

$$\tilde{G}_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} = \sup_{Q\in\mathscr{K}_0}\Big\{\big[nV_t(K,Q)^{\frac{n}{n+t}}|Q^\circ|^{\frac{t}{n+t}}\big]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\Big\}.$$

By inequality (5.17) and $\frac{n}{n+r} > 0$, one has, $\forall Q \in \mathscr{K}_0$,

$$nV_r(K,Q)^{\frac{n}{n+r}}\ |Q^\circ|^{\frac{r}{n+r}} \le \big\{\big[nV_s(K,Q)^{\frac{n}{n+s}}|Q^\circ|^{\frac{s}{n+s}}\big]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\big\}\big\{\big[nV_t(K,Q)^{\frac{n}{n+t}}|Q^\circ|^{\frac{t}{n+t}}\big]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\big\}.$$

Taking the supremum over $Q \in \mathscr{K}_0$, one gets

$$\begin{aligned}\tilde{G}_r(K) &= \sup_{Q\in\mathscr{K}_0}\{nV_r(K,Q)^{\frac{n}{n+r}}\ |Q^\circ|^{\frac{r}{n+r}}\} \\ &\le \sup_{Q\in\mathscr{K}_0}\Big\{\big[nV_s(K,Q)^{\frac{n}{n+s}}|Q^\circ|^{\frac{s}{n+s}}\big]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\Big\}\sup_{Q\in\mathscr{K}_0}\Big\{\big[nV_t(K,Q)^{\frac{n}{n+t}}|Q^\circ|^{\frac{t}{n+t}}\big]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\Big\} \\ &= \tilde{G}_s(K)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\tilde{G}_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}.\end{aligned}$$

The case $-n < s < r < t < 0$ follows immediately by switching the roles of t and s.

(iii). Suppose that $t<r<-n<s<0$, which clearly implies $0<\frac{t-r}{t-s}<1$. Then

$$\begin{aligned}\tilde{G}_s(K)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} &= \left\{\sup_{Q_1\in\mathscr{K}_0}[nV_s(K,Q_1)^{\frac{n}{n+s}}|Q_1^\circ|^{\frac{s}{n+s}}]\right\}^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\\ &= \inf_{Q_1\in\mathscr{K}_0}\left\{[nV_s(K,Q_1)^{\frac{n}{n+s}}|Q_1^\circ|^{\frac{s}{n+s}}]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\right\},\end{aligned}$$

due to $\frac{(t-r)(n+s)}{(t-s)(n+r)}<0$. Similarly, by $\frac{(r-s)(n+t)}{(t-s)(n+r)}>0$, one has,

$$\tilde{G}_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} = \sup_{Q\in\mathscr{K}_0}\left\{[nV_t(K,Q)^{\frac{n}{n+t}}|Q^\circ|^{\frac{t}{n+t}}]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\right\}.$$

By inequality (5.17) and $\frac{n}{n+r}<0$,

$$\begin{aligned}\tilde{G}_r(K) &= \sup_{Q_0\in\mathscr{K}_0}\{nV_r(K,Q_0)^{\frac{n}{n+r}}\ |Q_0^\circ|^{\frac{r}{n+r}}\}\geq nV_r(K,Q)^{\frac{n}{n+r}}\ |Q^\circ|^{\frac{r}{n+r}}\\ &\geq \left\{[nV_s(K,Q)^{\frac{n}{n+s}}|Q^\circ|^{\frac{s}{n+s}}]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\right\}\left\{[nV_t(K,Q)^{\frac{n}{n+t}}|Q^\circ|^{\frac{t}{n+t}}]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\right\}\\ &\geq \left\{[nV_t(K,Q)^{\frac{n}{n+t}}|Q^\circ|^{\frac{t}{n+t}}]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\right\}\left\{\inf_{Q_1\in\mathscr{K}_0}[nV_s(K,Q_1)^{\frac{n}{n+s}}|Q_1^\circ|^{\frac{s}{n+s}}]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\right\}\\ &= \tilde{G}_s(K)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\left\{[nV_t(K,Q)^{\frac{n}{n+t}}|Q^\circ|^{\frac{t}{n+t}}]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\right\}.\end{aligned}$$

Therefore, taking the supremum over $Q\in\mathscr{K}_0$, one gets

$$\begin{aligned}\tilde{G}_r(K) &\geq \tilde{G}_s(K)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\sup_{Q\in\mathscr{K}_0}\left\{[nV_t(K,Q)^{\frac{n}{n+t}}|Q^\circ|^{\frac{t}{n+t}}]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\right\}\\ &= \tilde{G}_s(K)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}\tilde{G}_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}.\end{aligned}$$

The case $s<r<-n<t<0$ follows immediately by switching the roles of t and s.

We now show the monotonicity of $\tilde{G}_p(\cdot)$.

Theorem 5.2 *Let $K\in\mathscr{K}_0$, and $p,q\neq 0$.*
(i). If either $-n<q<p$ or $q<p<-n$, one has

$$\left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}}\leq\left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{n+p}{p}}.$$

(ii). If $q<-n<p$, one has

$$\left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}}\geq\left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{n+p}{p}}.$$

Proof. Recall that $\tilde{G}_0(K) = n|K|$. Note that the statement of Theorem 5.1 does not include the cases $s = 0$ or $r = 0$ or $t = 0$. However, from the proof of Theorem 5.1, one can easily see that cyclic inequalities still hold for (only) one of r, s, t equal to 0.

(i). Note that $-n < q < p$ has three different cases: $0 < q < p$, $-n < q < 0 < p$, and $-n < q < p < 0$.

Case 1: $0 < q < p$. Put $t = 0$, $r = q$ and $s = p$ in part (i) of Theorem 5.1, then

$$\tilde{G}_q(K) \le \tilde{G}_0(K)^{\frac{(p-q)n}{p(n+q)}} \tilde{G}_p(K)^{\frac{q(n+p)}{p(n+q)}} = \big(n|K|\big)^{\frac{(p-q)n}{p(n+q)}} \tilde{G}_p(K)^{\frac{q(n+p)}{p(n+q)}}.$$

Dividing both sides by $n|K|$, one gets, as $q > 0$,

$$\left(\frac{\tilde{G}_q(K)}{n|K|}\right) \le \left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{q(n+p)}{p(n+q)}} \Leftrightarrow \left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}} \le \left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{n+p}{p}}.$$

Case 2: $-n < q < 0 < p$. Put $r = 0$, $t = q$ and $s = p$ in part (i) of Theorem 5.1, then

$$\tilde{G}_0(K) = n|K| \le \tilde{G}_q(K)^{\frac{(n+q)p}{n(p-q)}} \tilde{G}_p(K)^{\frac{q(n+p)}{(q-p)n}}.$$

Dividing both sides by $n|K|$, one gets, as $-n < q < 0 < p$,

$$\left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{-p(n+q)}{(p-q)n}} \le \left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{q(n+p)}{(q-p)n}} \Leftrightarrow \left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}} \le \left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{n+p}{p}}.$$

Case 3: $-n < q < p < 0$. Put $s = 0$, $t = q$ and $r = p$ in part (ii) of Theorem 5.1, then

$$\tilde{G}_p(K) \le \tilde{G}_0(K)^{\frac{(q-p)n}{q(n+p)}} \tilde{G}_q(K)^{\frac{p(n+q)}{q(n+p)}} = \big(n|K|\big)^{\frac{(q-p)n}{q(n+p)}} \tilde{G}_q(K)^{\frac{p(n+q)}{q(n+p)}}.$$

Dividing both sides by $n|K|$, one gets, as $-n < p < 0$,

$$\left(\frac{\tilde{G}_p(K)}{n|K|}\right) \le \left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{p(n+q)}{q(n+p)}} \Leftrightarrow \left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}} \le \left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{n+p}{p}}.$$

Case 4: $q < p < -n$. Put $s = 0$, $t = q$ and $r = p$ in part (iii) of Theorem 5.1, then

$$\tilde{G}_p(K) \ge \tilde{G}_0(K)^{\frac{(q-p)n}{q(n+p)}} \tilde{G}_q(K)^{\frac{p(n+q)}{q(n+p)}} = \big(n|K|\big)^{\frac{(q-p)n}{q(n+p)}} \tilde{G}_q(K)^{\frac{p(n+q)}{q(n+p)}}.$$

Dividing both sides by $n|K|$, one gets, as $p < -n$,

$$\left(\frac{\tilde{G}_p(K)}{n|K|}\right) \ge \left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{p(n+q)}{q(n+p)}} \Leftrightarrow \left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}} \le \left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{n+p}{p}}.$$

(ii). Note that $q<-n<p$ has two different cases: $q<-n<0<p$ and $q<-n<p<0$. First, Proposition 4.1 and $\frac{q}{n+q}>0$ imply that

$$\frac{\tilde{G}_q(K)}{n|K|}\geq\left(\frac{|K^\circ|}{|K|}\right)^{\frac{q}{n+q}}\Leftrightarrow\left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}}\geq\frac{|K^\circ|}{|K|}.$$

Similarly, for $p>0$, Proposition 4.1 and $\frac{p}{n+p}>0$ imply that

$$\frac{\tilde{G}_p(K)}{n|K|}\leq\left(\frac{|K^\circ|}{|K|}\right)^{\frac{p}{n+p}}\Rightarrow\left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{n+p}{p}}\leq\frac{|K^\circ|}{|K|}\leq\left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}},$$

which concludes the case $q<-n<0<p$. For the case $q<-n<p<0$, by Proposition 4.1 and $\frac{p}{n+p}<0$, one has,

$$\frac{\tilde{G}_p(K)}{n|K|}\geq\left(\frac{|K^\circ|}{|K|}\right)^{\frac{p}{n+p}}\Rightarrow\left(\frac{\tilde{G}_p(K)}{n|K|}\right)^{\frac{n+p}{p}}\leq\frac{|K^\circ|}{|K|}\leq\left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}}.$$

Remark. In particular, if $p=1$, then for all $-n<q<0$ or $0<q<1$,

$$\left(\frac{\tilde{G}_q(K)}{n|K|}\right)^{\frac{n+q}{q}}\leq\left(\frac{\tilde{G}_1(K)}{n|K|}\right)^{n+1}. \tag{5.19}$$

Now we can prove the following result which removes the centroid (or Santaló point) condition of K in Theorem 4.2.

Corollary 5.1 *Let $K\in\mathscr{K}_0$ be a convex body with the origin in its interior.*
(i). For $p\in(0,1)$, the L_p geominimal surface area attains the maximum at and only at origin-symmetric ellipsoids, among all convex bodies with fixed volume. More precisely,

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)}\leq\left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

with equality if and only if K is an origin-symmetric ellipsoid.
(ii). For $p\in(-n,0)$, the L_p geominimal surface area attains the minimum at and only at origin-symmetric ellipsoids, among all convex bodies with fixed volume. More precisely,

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)}\geq\left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

with equality if and only if K is an origin-symmetric ellipsoid.

Proof. Recall that $\tilde{G}_p(B_2^n) = n|B_2^n|$ for all $p \neq -n$ and the classical geominimal surface area $\tilde{G}_1(K)$ is translation invariant, namely, for all interior point z_0 of K, one has

$$\tilde{G}_1(K) = \tilde{G}_1(K - z_0).$$

(i). Let $K \in \mathscr{K}_0$ have centroid at $z_0 \in \mathbb{R}^n$. By inequality (5.19), one has, as $p \in (0,1)$,

$$\frac{\tilde{G}_p(K)}{n|K|} \leq \left(\frac{\tilde{G}_1(K)}{n|K|}\right)^{\frac{p(n+1)}{n+p}} = \left(\frac{\tilde{G}_1(K-z_0)}{\tilde{G}_1(B_2^n)}\right)^{\frac{p(n+1)}{n+p}} \left(\frac{|B_2^n|}{|K|}\right)^{\frac{p(n+1)}{n+p}}.$$

Now using Theorem 4.2 for $\tilde{G}_1(K - z_0)$ (as $K - z_0 \in \mathscr{K}_c$), one gets,

$$\begin{aligned} \frac{\tilde{G}_p(K)}{n|K|} &\leq \left(\frac{\tilde{G}_1(K-z_0)}{\tilde{G}_1(B_2^n)}\right)^{\frac{p(n+1)}{n+p}} \left(\frac{|B_2^n|}{|K|}\right)^{\frac{p(n+1)}{n+p}} \leq \left(\frac{|K-z_0|}{|B_2^n|}\right)^{\frac{p(n-1)}{n+p}} \left(\frac{|B_2^n|}{|K|}\right)^{\frac{p(n+1)}{n+p}} \\ &= \left(\frac{|K|}{|B_2^n|}\right)^{\frac{p(n-1)}{n+p}} \left(\frac{|B_2^n|}{|K|}\right)^{\frac{p(n+1)}{n+p}} = \left(\frac{|K|}{|B_2^n|}\right)^{\frac{-2p}{n+p}}. \end{aligned}$$

Hence, one has

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \cdot \frac{|B_2^n|}{|K|} \leq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{-2p}{n+p}} \quad \Leftrightarrow \quad \frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \leq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}.$$

Clearly, equality holds if K is an origin-symmetric ellipsoid. On the other hand, to have equality, one needs to have equality for the affine isoperimetric inequality related to $\tilde{G}_1(\cdot)$. Therefore, as explained in Theorem 4.2, $K - z_0 \in \mathscr{K}_c$ must be an origin-symmetric ellipsoid. Proposition 3.2 implies that z_0 must be equal to 0 and hence K is an origin-symmetric ellipsoid.

(ii). Let $K \in \mathscr{K}_0$ have centroid at $z_0 \in \mathbb{R}^n$. By inequality (5.19), one has, as $p \in (-n, 0)$,

$$\frac{\tilde{G}_p(K)}{n|K|} \geq \left(\frac{\tilde{G}_1(K)}{n|K|}\right)^{\frac{p(n+1)}{n+p}} = \left(\frac{\tilde{G}_1(K-z_0)}{\tilde{G}_1(B_2^n)}\right)^{\frac{p(n+1)}{n+p}} \left(\frac{|B_2^n|}{|K|}\right)^{\frac{p(n+1)}{n+p}}.$$

Now using Theorem 4.2 for $\tilde{G}_1(K - z_0)$ (as $K - z_0 \in \mathscr{K}_c$), one gets, as $p \in (-n, 0)$,

$$\begin{aligned} \frac{\tilde{G}_p(K)}{n|K|} &\geq \left(\frac{\tilde{G}_1(K-z_0)}{\tilde{G}_1(B_2^n)}\right)^{\frac{p(n+1)}{n+p}} \left(\frac{|B_2^n|}{|K|}\right)^{\frac{p(n+1)}{n+p}} \geq \left(\frac{|K-z_0|}{|B_2^n|}\right)^{\frac{p(n-1)}{n+p}} \left(\frac{|B_2^n|}{|K|}\right)^{\frac{p(n+1)}{n+p}} \\ &= \left(\frac{|K|}{|B_2^n|}\right)^{\frac{p(n-1)}{n+p}} \left(\frac{|B_2^n|}{|K|}\right)^{\frac{p(n+1)}{n+p}} = \left(\frac{|K|}{|B_2^n|}\right)^{\frac{-2p}{n+p}}. \end{aligned}$$

Hence, one has

$$\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)} \cdot \frac{|B_2^n|}{|K|} \;\geq\; \left(\frac{|K|}{|B_2^n|}\right)^{\frac{-2p}{n+p}} \quad \Leftrightarrow \quad \left(\frac{\tilde{G}_p(K)}{\tilde{G}_p(B_2^n)}\right) \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}.$$

Clearly, equality holds if K is an origin-symmetric ellipsoid. On the other hand, to have equality, one needs to have equality for the affine isoperimetric inequality related to $\tilde{G}_1(\cdot)$. Therefore, as explained in Theorem 4.2, $K - z_0 \in \mathscr{K}_c$ must be an origin-symmetric ellipsoid. Proposition 3.2 implies that z_0 must be equal to 0 and hence K is an origin-symmetric ellipsoid.

Remark. Comparing the condition on K in Corollary 5.1 with those in Theorem 4.2, here one does not require the centroid (or the Santaló point) of K to be at the origin. Analogous results for the L_p affine surface area were first noticed in [45] by Zhang and further strengthened in [44]. The case $p = 1$ corresponds to the classical affine isoperimetric inequality related to the classical geominimal surface area [30, 31].

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